Obstructions for Local Tournament Orientation Completions

by

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ABSTRACT

The orientation completion problem for a hereditary class $\mathcal{C}$ of oriented graphs asks whether a given partially oriented graph can be completed to a graph belonging to $\mathcal{C}$. This problem was introduced recently and is a generalization of several existing problems, including the recognition problem for certain classes of graphs and the representation extension problem for proper interval graphs. A local tournament is an oriented graph in which the in-neighbourhood as well as the out-neighbourhood of each vertex induces a tournament. Local tournaments are a well-studied class of oriented graphs that generalize tournaments and their underlying graphs are intimately related to proper circular-arc graphs. Proper interval graphs are precisely those which can be oriented as acyclic local tournaments. The orientation completion problems for the class of local tournaments and the class of acyclic local tournaments have been shown to be polynomial-time solvable. In this thesis, we characterize the partially oriented graphs that can be completed to local tournaments by finding a complete list of obstructions. These are in a sense the minimal partially oriented graphs that cannot be completed to local tournaments. We also determine the minimal partially oriented graphs that cannot be completed to acyclic local tournaments.

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Chapter 1

Introduction

We consider graphs, digraphs and partially oriented graphs in this thesis. For graphs we assume that they do not contain loops or multiple edges (i.e., they are *simple*), and for digraphs we assume they do not contain loops or two arcs joining the same pair of vertices (i.e., they are *oriented* graphs).

A *partially oriented graph* is a mixed graph H obtained from some graph G by orienting the edges in a subset of the edge set of G. The graph G is called the *underlying graph* of H. We denote H by $(V, E \cup A)$ where E is the set of (non-oriented) edges and A is the set of arcs in H. We use uv to denote an edge in E with endvertices u, v and use (u, v) to denote an arc in A with *tail* u and *head* v. In either case we say that u, v are *adjacent* in H. We say the partially oriented graph H is *connected* if its underlying graph G is.

A class $\mathcal{C}$ of graphs is called *hereditary* if it is closed under taking induced subgraphs, that is, if $G \in \mathcal{C}$ and G' is an induced subgraph of G then $G' \in \mathcal{C}$. Similarly, a class of digraphs is *hereditary* if it is closed under taking induced subdigraphs. We extend this concept to partially oriented graphs.

Let $H = (V, E \cup A)$ and $H' = (V', E' \cup A')$ be partially oriented graphs. We says that H *critically contains* H' (or H' is *critically contained* in H) if $V' \subseteq V$ and for all $u, v \in V'$,

- u and v are adjacent in H' if and only if they are adjacent in H;
- if $(u, v) \in A'$ then $(u, v) \in A$;
- if $uv \in E'$, then $uv \in E$, or $(u, v) \in A$, or $(v, u) \in A$.

. Equivalently, H' is critically contained H if and only if it is obtained from H by deleting some vertices, followed by replacing some arcs (u, v) with edges uv.

We note that, in case when H and H' are both graphs or both digraphs, H critically contains H' if and only if H contains H' as an induced subgraph or as an induced subdigraph. We call a class $\mathcal{C}$ of partially oriented graphs *hereditary* if $H \in \mathcal{C}$ and H' is critically contained in H then $H' \in \mathcal{C}$.

1.1 Overview

Let $\mathcal{C}$ be a hereditary class of oriented graphs. The *orientation completion problem* for $\mathcal{C}$ asks whether a given partially oriented graph $H = (V, E \cup A)$ can be completed to an oriented graph in $\mathcal{C}$ by orienting the edges in E. The hereditary property of $\mathcal{C}$ ensures that if a partially oriented graph H can be completed to an oriented graph in $\mathcal{C}$ then every partially oriented graph that is critically contained in H can also be completed to an oriented graph in $\mathcal{C}$. Therefore the partially oriented graphs which can be completed to oriented graphs in $\mathcal{C}$ form a hereditary class.

Orientation completion problems were introduced recently and are a generalization of several existing problems, cf. [3]. Many graph classes can be defined in terms of the existence of certain orientations, cf. [6, 7, 11, 14, 16, 18, 23]. Deciding whether a graph admits such an orientation is a special orientation completion problem, cf. [13]. An oriented graph $D = (V, A)$ is called *transitive* if for any three vertices u, v, w, $(u, v) \in A$ and $(v, w) \in A$ imply $(u, w) \in A$, cf. [6]. The underlying graphs of transitive oriented graphs are known as *comparability graphs*, cf. [8]. When $\mathcal{C}$ is the class of transitive oriented graphs, the orientation completion problem for $\mathcal{C}$ asks whether a partially oriented graph can be completed to a transitive oriented graph. If the input is restricted to unoriented graphs, the orientation completion problem for $\mathcal{C}$ is exactly the recognition problem for comparability graphs. Finding a linear time recognition algorithm for comparability graphs is a long standing open problem in the structural graph theory. The current best known algorithm runs in $O(n^2)$ time, cf. [21].

A *local tournament* is an oriented graph in which the in-neighbourhood as well as the out-neighbourhood of each vertex induces a tournament. Local tournaments are a well-studied class of oriented graphs that generalize tournaments, cf. [1, 9, 10, 12, 17]. The underlying graphs of acyclic local tournaments are precisely the *proper interval graphs*, cf. [10]. These are the graphs which can be represented by intervals where no interval contains another. Such representations can be obtained from acyclic local tournament orientations of the graphs. Thus the orientation completion problem for the class of acyclic local tournaments corresponds to a representation extension problem for proper interval graphs which has been studied in [15].

Orientation completion problems have been studied for several classes of oriented graphs, including local tournaments, local transitive tournaments, and acyclic local tournaments, cf. [3, 13]. A *local transitive tournament* is an oriented graph in which the in-neighbourhood as well as the out-neighbourhood of each vertex induces a transitive tournament [1]. These three classes of oriented graphs are nested; the class of local tournaments properly contains local transitive tournaments, which in turn as a class properly

[1]Locally transitive local tournaments have been previously used for local transitive tournaments in [3]

contains acyclic local tournaments. It has been proved in [3] that the orientation completion problem is polynomial-time solvable for local tournaments and for acyclic local tournaments, but NP-complete for locally transitive local tournaments.

Any hereditary class of graphs or digraphs admits a characterization by forbidden subgraphs or subdigraphs. The forbidden subgraphs or subdigraphs consists of minimal graphs or digraphs which do not belong to the class. This is also the case for a hereditary class of partially oriented graphs and in particular for the class of partially oriented graphs which can be completed to local tournaments and the class of partially oriented graphs which can be completed to acyclic local tournaments. We call a partially oriented graph $X = (V, E \cup A)$ an *obstruction for local tournament orientation completions* (or simply, an *obstruction*) if the following three properties hold:

1. X cannot be completed to a local tournament;
2. For each $v \in V$, $X - v$ can be completed to a local tournament;
3. For each $(u, v) \in A$, the partially oriented graph obtained from X by replacing (u, v) with the edge uv can be completed to a local tournament.

Thus an obstruction X is a partially oriented graph which cannot be completed to a local tournament and is minimal in the sense that if X' is critically contained in X and $X' \neq X$ then X' can be completed to a local tournament.

The *dual* of an obstruction X is obtained from X by reversing the arcs in X (if any). Clearly, the dual of an obstruction is again an obstruction. Obstructions are present in any partially oriented graph that cannot be completed to a local tournament, as justified by the following proposition.

Proposition 1.1. *A partially oriented graph H cannot be completed to a local tournament if and only if it critically contains an obstruction.*

Proof: If H can be completed to a local tournament, then every partially oriented graph critically contained in H can also be completed to a local tournament so H does not contain an obstruction. On the other hand, suppose that H cannot be completed to a local tournament. By deleting vertices and replacing arcs with edges in H as long as the resulting partially oriented graph still cannot be completed to a local tournament we obtain an obstruction that is critically contained in H. □

Obstructions for acyclic local tournament orientations completions can be defined in a similar way as for local tournament orientation completions (see Chapter 4). An analogous version of Proposition 1.1 can also be obtained in the same way (see Proposition 4.1). The main results of this thesis are the following two theorems.

Theorem 1.2. *Let X be an obstruction for local tournament orientation completions. Then either X or its dual is a graph in Figures 2.1–2.6, or $\overline{C_{2k}}$ ($k \geq 3$), or $\overline{C_{2k+1} + K_1}$ ($k \geq 1$), or the complement of a graph in Figures 1.1, 3.1–3.7 (with arcs being specified in the figures).*

Theorem 1.3. *Let X be an obstruction for acyclic local tournament orientation completions. Then X or its dual is a C_k ($k \geq 4$) or one of the graphs in Figures 4.1–4.3.*

The thesis is organized as follows. In the remainder of Chapter 1 we will give preliminary results on local tournaments and their underlying graphs. We will give a general description of obstructions whose underlying graphs are local tournament orientable. In Chapters 2 and 3, we will determine all obstructions for local tournament orientation completions. In Chapter 4, we will find all obstructions for acyclic local tournament orientation completions. In Chapter 5, we will explain how Theorems 1.2 and 1.3 follow from the results obtained in Chapters 2 - 4 and provide algorithms for recognizing obstructions and finding obstructions critically contained in partially oriented graphs that cannot be completed to local tournaments.

1.2 Preliminary results

A graph $G = (V, E)$ is said to be a *proper circular-arc graph* if there is a family of circular-arcs $I_v, v \in V$ on a circle where no circular-arc contains another such that $uv \in E$ if and only if I_u and I_v intersect. Skrien [20] proved that a connected graph is a proper circular-arc graph if and only if it can be oriented as a local tournament. Thus, if a partially oriented graph H can be completed to a local tournament, then every component of the underlying graph of H must be a proper circular-arc graph.

Tucker [24] found all minimal graphs which are not proper circular-arc graphs.

Theorem 1.4 ([24])**.** *A graph G is a proper circular-arc graph if and only if $\overline{G}$ does not contain C_{2k} ($k \geq 3$), $C_{2k+1} + K_1$ ($k \geq 1$), or any of graphs in Figure 1.1 as an induced subgraph.* □

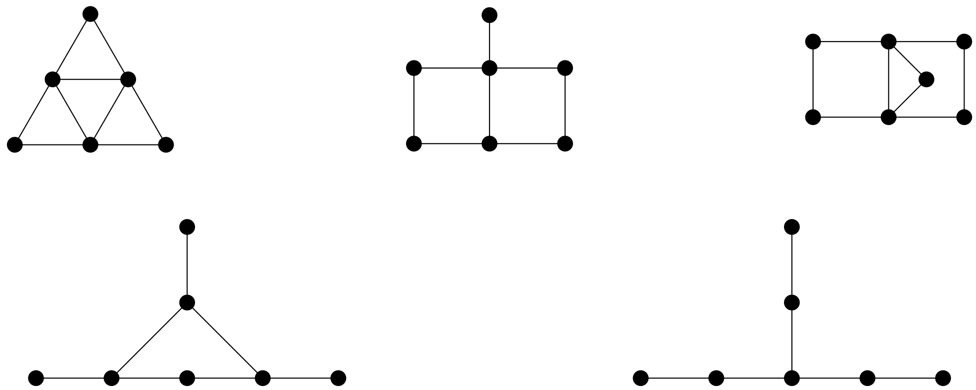


Figure 1.1: Complements of forbidden induced subgraphs for proper circular-arc graphs.

It follows that the complements of C_{2k} $(k \geq 3)$, $C_{2k+1} + K_1$ $(k \geq 1)$, and the graphs in Figure 1.1 are precisely the obstructions for local tournament orientation completions which do not contain arcs. Hence we only need to find obstructions that contain arcs. By definition the underlying graph of any obstruction that contains arcs is a proper circular-arc graph and hence local tournament orientable.

Let $G = (V, E)$ be a graph and $Z(G) = \{(u, v) : uv \in E\}$ be the set of all ordered pairs (u, v) such that $uv \in E$. Note that each edge $uv \in E$ gives rise to two ordered pairs $(u, v), (v, u)$ in $Z(G)$. Suppose that (u, v) and (x, y) are two ordered pairs of $Z(G)$. We say (u, v) *forces* (x, y) and write $(u, v)\Gamma(x, y)$ if one of the following conditions is satisfied:

- $u = x$ and $v = y$;
- $u = y$, $v \neq x$, and $vx \notin E$;
- $v = x$, $u \neq y$, and $uy \notin E$.

We say that (u, v) *implies* (x, y) and write $(u, v)\Gamma^*(x, y)$ if there exists a sequence of pairs $(u_1, v_1), (u_2, v_2), \ldots, (u_k, v_k) \in Z(G)$ such that

$$(u, v) = (u_1, v_1)\Gamma(u_2, v_2)\Gamma \cdots \Gamma(u_k, v_k) = (x, y).$$

We will call such a sequence a Γ-*sequence* from (u, v) to (x, y). It is easy to verify that Γ^* is an equivalence relation on $Z(G)$.

We say a path P *avoids* a vertex u if P does not contain u or any neighbour of u.

Proposition 1.5. *Let G be a graph and u, v, w be vertices Suppose that P is a path of length k connecting v, w that avoids u in $\overline{G}$. If k is even, then $(u, v)\Gamma^*(u, w)$. Otherwise, $(u, v)\Gamma^*(w, u)$.*

Proof: Denote $P : p_0p_1 \dots p_k$ where $p_0 = v$ and $p_k = w$. Since P avoids u in $\overline{G}$, $(u, p_i)\Gamma(p_{i+1}, u)$ for each $0 \le i \le k-1$. If k is even, then

$$(u, v) = (u, p_0)\Gamma(p_1, u)\Gamma(u, p_2)\Gamma \cdots \Gamma(u, p_k) = (u, w).$$

Otherwise,

$$(u, v) = (u, p_0)\Gamma(p_1, u)\Gamma(u, p_2)\Gamma \cdots \Gamma(p_k, u) = (w, u).$$

□

Proposition 1.6 ([12])**.** *Let G be a graph and $D = (V, A)$ be a local tournament orientation of G. Suppose that $(u, v)\Gamma^*(x, y)$ for some $(u, v), (x, y) \in Z(G)$. Then $(u, v) \in A$ if and only if $(x, y) \in A$.* □

Regardless whether or not G is local tournament orientable, the relation Γ^* on $Z(G)$ induces a partition of the edge set of G into *implication classes* as follows: two edges uv, xy of G are in the same implication class if and only if $(u, v)\Gamma^*(x, y)$ or $(u, v)\Gamma^*(y, x)$. An implication class is called *trivial* if it has only one edge and *non-trivial* otherwise. An edge uv of G is called *balanced* if $N[u] = N[v]$ and *unbalanced* otherwise. Clearly, any balanced edge forms a trivial implication class and the unique edge in any trivial implication class is balanced.

The following theorem characterizes the implication classes of a local tournament orientable graph and describes all possible local tournament orientations of such a graph.

Theorem 1.7 ([12])**.** *Let $G = (V, E)$ be a connected graph and let $H_1, H_2, \dots, H_k$ be the components of $\overline{G}$. Suppose that G is local tournament orientable and F is an implication class of G. Then F is one of the following types:*

- *F is trivial;*
- *F consists of all unbalanced edges of G within H_i for some i;*
- *F consists of all edges of G between H_i and H_j for some $i \neq j$.*

Moreover, suppose that $F_1, F_2, \dots, F_\ell$ are the implication classes of G. For each $1 \le i \le \ell$, let A_i be the equivalence class of Γ^* containing (u, v) for some $uv \in F_i$ and let $A = \cup_{i=1}^{\ell} A_i$. Then $D = (V, A)$ is a local tournament orientation of G. □

Let $H = (V, E \cup A)$ be a partially oriented graph and $(a, b), (c, d)$ be arcs of H. We say that the two arcs $(a, b), (c, d)$ are *opposing* in H if $(a, b)\Gamma^*(d, c)$. For convenience we also call an arc of H *balanced* if the corresponding edge is balanced. Clearly, if $(a, b), (c, d)$ are opposing then neither of them is balanced.

Proposition 1.8. *Suppose that H is a partially oriented graph whose underlying graph $U(H)$ is local tournament orientable. Then H can be completed to a local tournament if and only if it does not contain opposing arcs.*

Proof: If H contains opposing arcs, then by Proposition 1.6 it cannot be completed to a local tournament. On the other hand, suppose that H does not contain opposing arcs. Let $F_1, F_2, \dots, F_\ell$ be implication classes of $U(H)$. For each $1 \le i \le \ell$, if no edge in F_i is oriented then let A_i be an equivalence class of Γ^* containing (u, v) for some $uv \in F_i$; otherwise let A_i be the equivalence class of Γ^* containing (u, v) where $uv \in F_i$ and (u, v) is an arc. With $A = \cup_{i=1}^{\ell} A_i$, Theorem 1.7 ensures that $D = (V, A)$ is a local tournament completion of H. □

The next theorem is fundamental in determining whether a partially oriented graph whose underlying graph is local tournament orientable is an obstruction.

Theorem 1.9. *Let X be a partially oriented graph whose underlying graph $U(X)$ is local tournament orientable. Then X is an obstruction if and only if X contains exactly two arcs (say $(a, b), (c, d)$) which are opposing and, for every vertex $v \in V(X) \setminus \{a, b, c, d\}$, the arcs $(a, b), (c, d)$ are not opposing in $X - v$ (that is, the edges ab, cd belong to different implication classes in $U(X - v)$). Moreover, any Γ-sequence connecting (a, b) and (d, c) must include all vertices of X.*

Proof: For sufficiency, suppose that $(a, b), (c, d)$ are the only arcs and they are opposing in X and that, for every vertex $v \in V(X) \setminus \{a, b, c, d\}$, the arcs $(a, b), (c, d)$ are not opposing in $X - v$. Since X contains opposing arcs, it cannot be completed to a local tournament by Proposition 1.8. Let v be a vertex in X. Since $U(X)$ is local tournament orientable, $U(X - v)$ is also local tournament orientable. If $v \in \{a, b, c, d\}$, then $X - v$ contains at most one arc and hence no opposing arcs. If $v \notin \{a, b, c, d\}$, then the only two arcs in $X - v$ are not opposing by assumption. Hence $X - v$ can be completed to a local tournament Proposition 1.8. Therefore X is an obstruction.

Conversely, suppose that X is an obstruction. By Proposition 1.8 X must contain opposing arcs. Let $(a, b), (c, d)$ be opposing arcs in X. If X contains an arc (x, y) that is distinct from $(a, b), (c, d)$, then replacing the arc (x, y) by the edge xy gives a partially orientable graph in which $(a, b), (c, d)$ are still opposing and hence cannot be completed to a local tournament. This contradicts the assumption that X is an obstruction. So $(a, b), (c, d)$ are the only arcs in X. Since X is an obstruction, for every every $v \in V(X)$, $X - v$ can be completed to a local tournament and hence by Proposition 1.8 contains no opposing arcs. This implies in particular that if $v \in V(X) \setminus \{a, b, c, d\}$, the arcs $(a, b), (c, d)$ are not opposing in $X - v$.

The second part of the theorem follows from the fact deleting any vertex results in a graph that contains no Γ-sequence connecting (a, b) and (d, c). □

Let v be a vertex and (x, y) be an arc in a partially oriented graph H where $v \notin \{x, y\}$. We call v the (x, y)-*balancing* vertex if v is the only vertex adjacent to exactly one of x, y; when the arc (x, y) does not need to be specified, we simply call v an *arc-balancing* vertex. Each obstruction has at most two arc-balancing vertices as it contains at most two arcs.

A vertex of a graph G is called a *cut-vertex* of G if $G - v$ has more components than G. For a partially oriented graph H, a cut-vertex of $U(H)$ is also called a *cut-vertex* of H.

Proposition 1.10. *Let X be an obstruction with opposing arcs $(a, b), (c, d)$ and let $v \notin \{a, b, c, d\}$. Then v is an arc-balancing vertex, or a cut-vertex of $U(X)$, or a cut-vertex of $\overline{U(X)}$.*

Proof: Assume that v is not a cut-vertex of $U(X)$ or of $\overline{U(X)}$ as otherwise we are done. We show that v must be an arc-balancing vertex. Since ab, cd are in the same implication of $U(X)$, by Theorem 1.7 ab, cd are unbalanced edges either contained in a component or between two components of $\overline{U(X)}$. Since v is not a cut-vertex of $\overline{U(X)}$, each component of $\overline{U(X - v)}$ is a component of $\overline{U(X)}$ except possibly missing v. It follows that ab, cd are contained in some component or between two components of $\overline{U(X - v)}$. Since v is not a cut-vertex of $U(X)$, $U(X - v)$ is connected. If ab, cd are both unbalanced edges in $U(X - v)$, then they remain in the same implication class of $U(X - v)$ and hence $(a, b), (c, d)$ are still opposing in $X - v$, which contradicts the assumption that X is an obstruction. So one of ab, cd is balanced in $U(X - v)$, which means that v is (a, b)-balancing or (c, d)-balancing. □

An *arc-balancing triple* in a partially oriented graph H is a set of three vertices in which one balances an arc between the other two.

Corollary 1.11. *Let X be an obstruction with opposing arcs $(a, b), (c, d)$. Suppose that $U(X)$ has no cut-vertices. Then $\overline{U(X)}$ contains at most six non-cut-vertices. In the case when $\overline{U(X)}$ has six non-cut-vertices, the six non-cut-vertices form two disjoint arc-balancing triple.*

Proof: Let v be a non-cut-vertex of $\overline{U(X)}$. By assumption v is not a cut-vertex of $U(X)$ and thus, by Proposition 1.10, it is either in $\{a, b, c, d\}$ or an arc-balancing vertex. There are at most two arc-balancing vertices so $\overline{U(X)}$ contains at most six non-cut-vertices. When $\overline{U(X)}$ has six non-cut-vertices, among the six non-cut-vertices two are arc-balancing vertices and the other four are incident with arcs. Hence the six non-cut-vertices form two disjoint arc-balancing triple. □

A *proper interval graph* is the intersection graph of a family of intervals in a line where no interval contains another. Proper interval graphs form a prominent subclass of proper circular-arc graphs and play an important role in the orientation completion problem for local tournaments. It is proved in [10] that a graph is proper interval graph if and only if it can be oriented as an acyclic local tournament

A *straight enumeration* of a graph G is a vertex ordering $\prec$ such that for all $u \prec v \prec w$, if uw is an edge of G, then both uv and vw are edges. This property is referred to as the *umbrella property* of the vertex ordering. A graph is a proper interval graph if and only if it has a straight enumeration, cf. [12].

Proposition 1.12. *Let $G = (V, E)$ be a connected proper interval graph and let $\prec$ be a straight enumeration of G. Suppose that $(u, v)\Gamma^*(x, y)$. Then $u \prec v$ if and only if $x \prec y$.*

Proof: It suffices to show that if $u \prec v$ and $(u, v)\Gamma(x, y)$ then $x \prec y$. So assume that $(u, v)\Gamma(x, y)$. Then one of the following holds:

- $u = x$ and $v = y$;
- $u = y$, $v \neq x$, and $vx \notin E$;
- $v = x$, $u \neq y$, and $uy \notin E$.

Clearly, $x \prec y$ when $u = x$ and $v = y$. Suppose that $u = y$, $v \neq x$, and $vx \notin E$. If $u \prec x \prec v$, then it violates the umbrella property because $uv \in E$ but $xv \notin E$. If $u \prec v \prec x$, then it again violates the umbrella property because $ux \in E$ but $vx \notin E$. Hence we must have $x \prec u = y$. The proof for the case when $v = x$, $u \neq y$, and $uy \notin E$ is similar. □

Let H be a partially oriented graph whose underlying graph $U(H)$ is a proper interval graph. Suppose that $\prec$ is a straight enumeration of $U(H)$. We call an arc (u, v) of H *positive* (with respect to $\prec$) if $u \prec v$ and *negative* otherwise. If H does not contain negative arcs, then H can be completed to an acyclic local tournament by replacing all edges of H with positive arcs. Similarly, if H does not contain positive arcs then it can also be completed to an acyclic local tournament. It follows that if X is an obstruction such that $U(X)$ is a proper interval graph, then the two arcs in X must be *opposite* (i.e., one is positive and the other is negative).

A vertex in a graph is *universal* if it is adjacent to every other vertex.

Proposition 1.13 ([12])**.** *Suppose that $G = (V, E)$ is a connected proper interval graph that is not a complete graph. Then $\overline{G}$ has a unique non-trivial component H. If F is an implication class of G, then F is one of the following types:*

- *F is trivial;*
- *F consists of all unbalanced edges within H;*

- *F consists of all edges of G between H and a universal vertex of G.*

In particular, if G contains no universal vertex, then G has a unique non-trivial implication class. □

Proposition 1.14. *Let G be a connected proper interval graph and let $v_1, v_2, \ldots, v_n$ be a straight enumeration of G. Suppose that v_α is a cut-vertex of $\overline{G}$. Then $\alpha \in \{1, n\}$ and $G - v_\alpha$ contains a vertex that is adjacent to every vertex except v_α in G.*

Proof: Since $\overline{G}$ has a cut-vertex, G is not a complete graph and by Theorem 1.13, $\overline{G}$ has a unique non-trivial component H. Thus the cut-vertex v_α of $\overline{G}$ is in fact a cut-vertex of H. Again by Theorem 1.13, $H - v_\alpha$ has at most one non-trivial component. Hence H contains a vertex v_β that is only adjacent to v_α in $\overline{G}$, that is, in G it is adjacent to every vertex except v_α. If $\alpha < \beta$, then $\alpha = 1$ as otherwise we have $1 < \alpha < \beta$ and v_β is adjacent to v_1 but not to v_α, a contradiction to the umbrella property of the straight enumeration Similarly, if $\beta < \alpha$, then $\alpha = n$ as otherwise $\beta < \alpha < n$ and v_β is adjacent to v_n but not to v_α, also a contradiction to the umbrella property of the straight enumeration Therefore, $\alpha \in \{1, n\}$. □

Chapter 2

Obstructions with cut-vertices

Our goal is to find all obstructions for local tournament orientation completions. In view of Theorem 1.4 we only need to find those which contain arcs and whose underlying graphs are connected and local tournament orientable (i.e., proper circular-arc graphs). By Theorem 1.9 each of them contains exactly two arcs which are opposing. So from now on we assume that all obstructions have a pair of opposing arcs.

In this chapter we examine the obstructions that contain cut-vertices. It is easy to see that their underlying graphs are proper interval graphs and thus have straight enumerations.

Let X be an obstruction that contains arcs and let $\prec$ be a straight enumeration of $U(X)$. If v is a cut-vertex of X, then the umbrella property implies v is neither the first nor the last vertex in $\prec$ and moreover, for all u, w with $u \prec v \prec w$, uw is not an edge in $U(X)$. A cut-vertex v of X is called *dividing* with respect to $\prec$ if one of the two arcs in X is incident with a vertex preceding v and the other is incident with a vertex succeeding v. A cut-vertex that is not dividing is called *non-dividing*. An obstruction may or may not contain dividing cut-vertices.

2.1 Obstructions containing dividing cut-vertices

In this section, we focus on the obstructions that contain dividing cut-vertices. We will show that they consist of the three infinite classes in Figure 2.1 and their duals. In each of these graphs, the dots in the middle represent a path of length ≥ 0; when the length of the path is 0 the two vertices beside the dots are the same vertex.

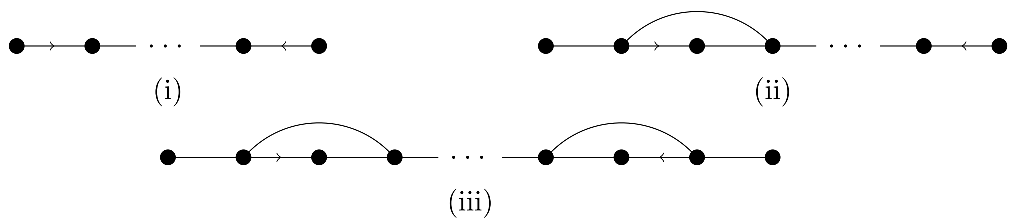
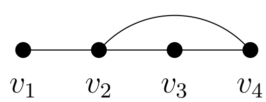


Figure 2.1: Obstructions with dividing cut-vertices.

Lemma 2.1. *Let X be an obstruction that contains a dividing cut-vertex and let $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_c is the first dividing cut-vertex in $\prec$. Then, either $c = 2$ and v_1, v_2 are the endvertices of an arc, or $c = 4$ and v_2, v_3 are the end vertices of an arc. In the case when $c = 4$, v_1, v_2, v_3, v_4 induce in $U(X)$ the following graph:*



Proof: By considering the dual of X if necessary we may assume that (v_j, v_k) and (v_s, v_t) are the two arcs in X where $j < k \leq c \leq t < s$. By Theorem 1.9, there is a Γ-sequence from (v_t, v_s) to (v_j, v_k) that include all vertices of X. Let

$$(v_t, v_s) = (u_1, w_1)\Gamma(u_2, w_2)\Gamma \cdots \Gamma(u_q, w_q) = (v_j, v_k)$$

be the shortest such a sequence. Since $v_t \prec v_s$, we have $u_i \prec w_i$ for each i by Proposition 1.12. Let ℓ be the smallest subscript such that $u_{\ell+1} \prec w_{\ell+1} = v_c = u_\ell \prec w_\ell$. Such ℓ exists because v_c is a cut-vertex dividing (v_j, v_k) and (v_s, v_t). We distinguish two cases depending on whether or not $k = c$. Suppose first $k = c$. Note that $(v_j, v_k)\Gamma(u_\ell, w_\ell)$. Thus the choice of the Γ-sequence implies $(u_{\ell+1}, w_{\ell+1}) = (u_q, w_q) = (v_j, v_k)$. Since the Γ-sequence includes all vertices of X, v_j is the only vertex preceding v_c in $\prec$, that is, $c = 2$ (and (v_1, v_2) is an arc in X).

Suppose now that $k < c$. Thus $j < k < c$. We claim that v_j, v_k, v_c are consecutive vertices in $\prec$ (i.e., $j + 1 = k = c - 1$). Suppose that $k > j + 1$. Since v_j, v_k are adjacent, v_{j+1} cannot be a cut-vertex of $U(X)$. Since v_{j+1} is not the first or the last vertex in $\prec$, Proposition 1.14 ensures that v_{j+1} cannot be a cut-vertex of $\overline{U(X)}$. By Proposition 1.10, v_{j+1} is an arc-balancing vertex. Clearly, v_{j+1} is not (v_j, v_k)-balancing. So it must be (v_s, v_t)-balancing. Since $j+1 < c$ and v_c is a cut-vertex, v_{j+1} has no neighbours succeeding v_c. It follows that $v_s = v_c$. Since $v_{j+1}v_c$ is an edge and $j + 1 < k < c$, $v_k v_c$ is an edge by the umbrella property. Again, since v_c is a cut-vertex, v_k cannot be adjacent to v_t. This contradicts the fact that v_{j+1} is arc-balancing for the arc between v_s, v_t. Hence $j + 1 = k$, i.e., v_j and v_k are consecutive vertices in $\prec$.

Suppose $c > k + 1$. Neither of v_k, v_{k+1} can be a cut-vertex of $U(X)$ as otherwise it would be a dividing cut-vertex preceding v_c, a contradiction to the choice of v_c. Since v_{k+1} is not the first or the last vertex in $\prec$, it is not a cut-vertex of $\overline{U(X)}$ according to Proposition 1.14. By Proposition 1.10, v_{k+1} is an arc-balancing vertex. Since v_k is not a cut-vertex of $U(X)$, $v_{k-1} = v_j$ is adjacent to v_{k+1}. So v_{k+1} is adjacent to both v_j, v_k and hence not arc-balancing for the the arc between them. So v_{k+1} is arc-balancing for the arc between v_s, v_t. Similarly as above we have $v_c = v_s$ and v_{k+1} is adjacent to v_s but not to v_t. If $c > k + 2$, then v_{k+2} is adjacent to v_c by the umbrella property and the fact v_{k+1} is adjacent to v_c. Thus v_{k+2} is adjacent to $v_c = v_s$ but not to v_t, a contradiction to that v_{k+1} is arc-balancing to the arc between v_s, v_t. If $c = k + 2$, since v_{k+1} is not a cut-vertex of $U(X)$, v_k is adjacent to $v_{k+2} = v_c$. Thus v_k is adjacent to $v_s = v_c$ but not to v_t, a contradiction again to the fact that v_{k+1} is arc-balancing to the arc between v_s, v_t. Hence $c = k + 1$, i.e., v_k, v_c are consecutive vertices in $\prec$. Therefore v_j, v_k, v_c are consecutive in $\prec$.

Since v_c is the first dividing cut-vertex in $\prec$, v_k cannot be a cut-vertex of $U(X)$ and hence v_j, v_c are adjacent in X. We claim that there exists a vertex preceding v_j in $\prec$ which is adjacent to v_j but not to v_k. First, observe that if no vertex is adjacent to exactly one of v_j, v_k, then v_j and v_k would share the same closed neighbourhood. In this case, the arc between v_j and v_k would be balanced, a contradiction. Hence, there is at least one vertex adjacent to exactly one of v_j, v_k. Clearly, such a vertex must precede v_j in $\prec$ and hence is adjacent to v_j but not to v_k. Assume that v_p is such a vertex closest to v_j.

We show that v_p and v_j are consecutive in $\prec$, that is, $p = j - 1$. If $p < j - 1$, then v_{j-1} cannot be a cut-vertex of $U(X)$ because v_p is adjacent to v_j. On the other hand, by Proposition 1.14, v_{j-1} is not a cut-vertex of $\overline{U(X)}$. It follows from Theorem 1.10 that v_{j-1} is an arc-balancing vertex. The choice of v_p implies that v_{j-1} is adjacent to both v_j, v_k so it does not balance the arc between v_j and v_k. Hence, v_{j-1} is an arc-balancing vertex for the arc between v_s and v_t. By definition it is the unique vertex adjacent to exactly one of v_s and v_t. This also implies $v_c = v_s$. But then v_k is also a vertex adjacent to v_s but not to v_t, a contradiction. Hence $p = j - 1$.

Since $(u_{\ell+1}, w_{\ell+1})\Gamma(u_\ell, w_\ell)$ and $u_{\ell+1} \prec w_{\ell+1} = v_c = u_\ell \prec w_\ell$ (i.e., $u_{\ell+1}$ is a vertex preceding and adjacent to v_c but not adjacent to w_ℓ), $u_{\ell+1}$ can only be v_{c-1} or v_{c-2}, Since Γ-sequence is chosen to be the shortest from (v_t, v_s) to (v_j, v_k), we must have $u_{\ell+1} = v_{c-2}$. It follows that

$$(v_t, v_s) = (u_1, w_1)\Gamma(u_2, w_2)\Gamma \cdots (u_\ell, w_\ell)\Gamma(v_{c-2}, v_c)\Gamma(v_{c-3}, v_{c-2})\Gamma(v_{c-2}, v_{c-1}) = (v_j, v_k)$$

is the shortest Γ-sequence. The Γ-sequence must contain all vertices of X, which means $v_{c-3}, v_{c-2}, v_{c-1}$ are all the vertices preceding v_c. Therefore $c = 4$ and v_1, v_2, v_3, v_4 induce in $U(X)$ the graph in the statement. □

We can now apply Lemma 2.1 to prove the following:

Theorem 2.2. *Let X be an obstruction that contains a dividing cut-vertex with respect to a straight enumeration. Then X or its dual belongs to one of the three infinite classes in Figure 2.1.*

Proof: Let $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$ and let v_c and v_d be the first and last dividing cut-vertices respectively with respect to $\prec$. By considering the dual of X if necessary assume that (v_j, v_k) and (v_s, v_t) are the arcs in X where $j < k \leq c \leq d \leq t < s$.

Suppose $c = 2$ and $d = n-1$. Since $v_c = v_2$ is a cut-vertex, v_2 is the only neighbour of v_1. Similarly, v_{n-1} is the only neighbour of v_n. If v_p is adjacent to v_q for some $2 \leq p < q-1 \leq n-1$, then it is easy to see that the partially oriented graph obtained from X by deleting $v_{p+1}, \ldots, v_{q-1}$ cannot be completed to local tournament orientation, a contradiction to the assumption X is an obstruction. Hence X belongs to Figure 2.1(i).

Suppose that $c \neq 2$. Then $c = 4$ by Lemma 2.1. If $d = n-1$, then a similar proof as above shows that X belongs to Figure 2.1(ii). On the other hand if $d \neq n-1$, then again by Lemma 2.1 we must have $d = n-3$. In this case X belongs to Figure 2.1(iii). □

2.2 Obstructions containing only non-dividing cut-vertices

In this section, we will determine the rest of obstructions that contain cut-vertices, i.e., those contain only non-dividing cut-vertices.

Lemma 2.3. *Let X be an obstruction and $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_c is a non-dividing cut-vertex. Then $c = 2$ or $c = n-1$. Moreover, if v_c is incident with both arcs then $n = 4$.*

Proof: Let (v_j, v_k) $(j < k)$ and (v_s, v_t) $(s > t)$ be the arcs in X. Since v_c is non-dividing, either $c \leq \min\{j, t\}$ or $c \geq \max\{k, s\}$. Suppose that $c \leq \min\{j, t\}$.

Let $(v_j, v_k) = (u_1, w_1), \ldots, (u_q, w_q) = (v_t, v_s)$ be a Γ-sequence of $U(X)$ between (v_j, v_k) and (v_t, v_s). By Theorem 1.9, the sequence must include all vertices of X. Let α be the smallest subscript such that one of u_α, w_α precedes v_c (and hence the other vertex is v_c since v_c is a cut-vertex). Similarly, let β be the largest subscript such that one of u_β, w_β precedes v_c (and hence the other vertex is v_c). Then it is easy to verify that $(u_1, w_1), \ldots, (u_\alpha, w_\alpha), (u_{\beta+1}, w_{\beta+1}), \ldots, (u_q, w_q)$ is a Γ-sequence between (v_j, v_k) and (v_t, v_s). Since this sequence contains a unique vertex preceding v_c and includes all vertices of X, we must have $c = 2$. A similar argument shows that if $c \geq \max\{k, s\}$ then $c = n-1$.

Suppose v_c is incident with both arcs. Then either $c = j = t = 2$ or $c = k = s = n-1$. If $c = j = t = 2$, then $(v_j, v_k)\Gamma(v_1, v_j)\Gamma(v_t, v_s)$ and by Theorem 1.9, $v_1, v_j = v_t, v_k, v_s$ are all the vertices of X so $n = 4$. A similar argument shows that X has exactly four vertices if $c = k = s = n - 1$. □

The following theorem deals with the case when v_2 and v_{n-1} are both non-dividing cut-vertices of $U(X)$.

Theorem 2.4. *Let X be an obstruction and $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 and v_{n-1} are the two cut-vertices of $U(X)$, both non-dividing. Then X or its dual is one of the two graphs in Figure 2.2.*

Figure 2.2: Obstructions with two non-dividing cut-vertices.

Proof: Since both v_2 and v_{n-1} are non-dividing cut-vertices, $n \geq 5$. Hence by Lemma 2.3, each of v_2 and v_{n-1} is incident with at most one arc.

We show that v_1 and v_n are arc-balancing vertices. By symmetry we only prove that v_1 is arc-balancing. Clearly v_1 is not a cut-vertex of $U(X)$ and is not incident with an arc. By Proposition 1.10, it can only be an arc-balancing vertex or a cut-vertex of $\overline{U(X)}$. Assume that v_1 is a cut-vertex of $\overline{U(X)}$. By Proposition 1.14, some vertex v is adjacent to every vertex in X except v_1. Since v_{n-1} is the only neighbour of v_n in $U(X)$. It follows that $v = v_{n-1}$. Since the vertex $v = v_{n-1}$ is adjacent to v_2, by the umbrella property, the vertices v_i with $2 \leq i \leq n-1$ induce a clique in $U(X)$. Thus the vertices v_i with $3 \leq i \leq n-2$ have the same closed neighbourhood in $U(X)$ and hence cannot contain both endvertices of any arc. It follows that each arc is incident with v_2 or v_{n-1}. From the above we know that each of v_2 and v_{n-1} is incident with at most one arc. It is not possible that v_2 and v_{n-1} are incident with the same arc (as otherwise the endvertices of the other arc have the same closed neighbourhood). Hence v_2 and v_{n-1} are incident with different arcs. We see that v_1 is an arc-balancing vertex.

By taking the dual of X if necessary we assume (v_2, v_k) and (v_{n-1}, v_t) are the two arcs in X where $3 \leq k, t \leq n-2$. Then v_1 is the (v_2, v_k)-balancing vertex and v_n is the (v_{n-1}, v_t)-balancing vertex. No vertex v_i with $2 < i < n-1$ is a cut-vertex of $U(X)$ or $\overline{U(X)}$ and hence each must be incident with an arc of X by Proposition 1.10. Hence v_k and v_t are the only vertices between v_2 and v_{n-1} in $\prec$. It is now easy to verify that X is one of the two graphs in Figure 2.2. □

It remains to consider the case when X has only one cut-vertex and it is non-dividing. By Lemma 2.3 and reversing the straight enumeration $\prec$ if necessary we will assume v_2 is this vertex.

Lemma 2.5. *Let X be an obstruction and $\prec: v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 is the only cut-vertex and it is non-dividing. Then, the following statements hold:*

(a) For each $i \geq 3$, v_i is an arc-balancing vertex or incident with an arc;

(b) For some $i \geq 3$, v_i is adjacent to every vertex except for v_1. Moreover, there are at most two such vertices, each incident with exactly one arc;

(c) The number of vertices in X is between 4 and 8 (i.e., $4 \leq n \leq 8$).

Proof: For (a), if each v_i with $i \geq 3$ is an arc-balancing vertex or incident with an arc then we are done. Otherwise, by Proposition 1.10, some v_i with $i \geq 3$ is a cut-vertex of $\overline{U(X)}$. According to Proposition 1.14, $v_i = v_n$ and there is a vertex adjacent to every vertex except v_n in $U(X)$. Such a vertex can only be v_2. Since v_{n-1} is not a cut-vertex of $U(X)$, v_n is adjacent to v_{n-2}. Since v_2 is not adjacent to v_n, $n-2 > 2$ (i.e., $n > 4$) and hence by Lemma 2.3, there is an arc which is not incident with v_2. This arc must have endvertices strictly between v_2 and v_n in $\prec$. Therefore v_n is an arc-balancing vertex, which contradicts our assumption.

Statement (b) holds if v_1 is a cut-vertex of $\overline{U(X)}$. Indeed, by Proposition 1.14 there is a vertex v_i which is adjacent to every vertex except v_i and it is clear that $i \geq 3$. So assume v_1 is not a cut-vertex of $\overline{U(X)}$. Since v_2 is the only cut-vertex and it is non-dividing, v_1 is neither a cut-vertex of $U(X)$ nor incident with an arc, and hence must be an arc-balancing vertex by Proposition 1.10. Without loss of generality, assume v_1 balances an arc between v_2 and v_j for some $j > 2$. If $v_j = v_n$ or v_j is adjacent to v_n, then v_j is adjacent to every vertex except v_1 and we are done. Otherwise, $j < n$ and v_j is not adjacent to v_n. For each $j < k < n$, v_k is a not cut-vertex of $U(X)$ by assumption so v_{k-1} must be adjacent to v_{k+1}. Since v_j is not adjacent to v_n, $j < n-2$ and thus $n > j+2 > 5$. By statement (a), each vertex v_i with $i \geq 3$ is an arc-balancing vertex or incident with an arc. Since v_1 is arc-balancing and v_2 is incident with an arc, there are at most four vertices v_i with $i \geq 3$. Hence $n \leq 6$ and therefore $n = 6$. It is now easy to see that v_4 is adjacent to every vertex except v_1.

Suppose v_i with $i \geq 3$ is a vertex adjacent to every vertex except v_1. Clearly v_i is not an arc-balancing vertex and hence by (a) it is incident with an arc. We show by contradiction that v_i is incident with exactly one arc. So suppose that v_i is incident with both arcs of X. Let v_s and v_t denote the other endvertices of the two arcs. We first show that either $s = 2$ or $t = 2$. By Theorem 1.9, the edges v_iv_s, v_iv_t belong to different implication classes in $U(X - v_1)$. Since v_i is an isolated vertex in $\overline{U(X-v_1)}$, each of v_s, v_t, v_i belongs to a different component of $\overline{U(X-v_1)}$ by Proposition 1.13 In particular, one of v_s, v_t is an isolated vertex in $\overline{U(X-v_1)}$. Without loss of generality, assume v_s is such a vertex. Thus, v_s is adjacent to every vertex except possibly v_1 in X. If v_s is not adjacent to v_1, then v_s and v_i share the same closed neighbourhood, so the arc between v_s and v_i is balanced, a contradiction. Hence, v_s is adjacent to v_1 and $v_s = v_2$. Consider v_t. Suppose $t < i$. Since $2 = s < t < i$ and v_i is adjacent to v_s, the umbrella property implies v_t is adjacent to v_s. If v_t is also adjacent to v_n, then v_i and v_t have the same closed neighbourhood so the arc between them is balanced, a contradiction. Hence, v_t is not adjacent to v_n. Since $s < t < n$, the umbrella property implies that v_s and v_n are not adjacent. Thus $(v_t, v_i)\Gamma(v_i, v_n)\Gamma(v_s, v_i)$ is a Γ-sequence between the arcs and not containing v_1, a contradiction by Theorem 1.9. It follows that $i < t$. If v_t is non-adjacent to v_s, then $(v_i, v_t)\Gamma(v_s, v_i)$ is a Γ-sequence between the arcs and not containing v_1, a contradiction. Hence, v_t is adjacent to $v_s = v_2$. If $t = n$, then the arc between v_i and v_t is balanced by the umbrella property, a contradiction. If $t < n$, then v_t is adjacent to v_n because $i < t < n$ and v_i is adjacent to v_n, leading to a similar contradiction. Therefore v_i is incident with exactly one arc. Suppose v_i, v_j are two such vertices. By the above, each of them is incident with an arc. Moreover, they cannot be incident with the same arc because they share the same neighbourhood. Hence, they are each incident with a different arc. Since X contains two arcs, there are at most two such vertices.

Finally we prove (c). Clearly, $n \geq 4$. Since there are at most four vertices incident with arcs and at most two arc-balancing vertices in X, there can be at most six vertices v_i with $i \geq 3$ by (a). Therefore $n \leq 8$. □

Theorem 2.6. *Let X be an obstruction and $\prec: v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 is the only cut-vertex and it is non-dividing. If $n = 4$ or 5, then X or its dual is one of the graphs in Figure 2.3.*

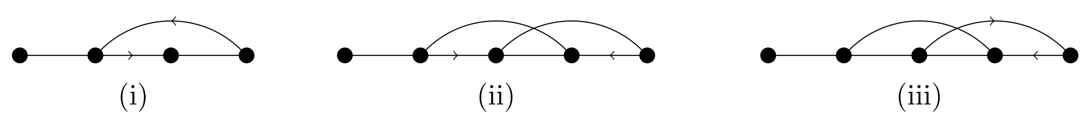


Figure 2.3: Obstructions with a unique non-dividing cut-vertex on 4 or 5 vertices.

Proof: Suppose $n = 4$. Since v_3 is not a cut-vertex, v_2 and v_4 are adjacent. Both v_3 and v_4 are adjacent to every vertex except for v_1 and by Lemma 2.5(b) they are incident with different arcs. It is easy to see that X is Figure 2.3(i).

Suppose $n = 5$. For each $i = 3, 4$, v_i is not a cut-vertex, so v_{i-1} and v_{i+1} are adjacent. On the other hand if v_2 is adjacent to v_5, then the umbrella property implies v_3, v_4, v_5 are all adjacent to every vertex except for v_1, contradicting Lemma 2.5(b). So v_2 and v_5 are not adjacent. Each of v_3, v_4 is adjacent to every vertex except v_1 and by Lemma 2.5(b) they are incident with different arcs. Since $n \neq 4$, v_2 is not incident with both arcs according to Lemma 2.3. It follows that v_5 must be incident with at least one arc. If v_5 is incident with exactly one arc, then X is or its dual is Figure 2.3(ii). Otherwise v_5 is incident with both arcs and X or its dual is Figure 2.3(iii). □

Lemma 2.7. *Let X be an obstruction and $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose v_k is a (v_i, v_j)-balancing vertex. Then, either $k < min\{i, j\}$ or $k > max\{i, j\}$. Moreover,*

- *If $k < min\{i, j\}$, then no v_p with $p < k$ is adjacent to either one of v_i, v_j, and any v_q with $q > max\{i, j\}$ is adjacent to either both or neither of v_i, v_j;*
- *If $k > max\{i, j\}$, then no v_p with $p > k$ is adjacent to either one of v_i, v_j, and any v_q with $q < min\{i, j\}$ is adjacent to either both or neither of v_i, v_j.*

Proof: First we show that either $k < \min\{i, j\}$ or $k > \max\{i, j\}$. Otherwise, v_k is between v_i and v_j. Since $v_i v_j$ is an edge of $U(X)$, the umbrella property implies that both v_i and v_j are adjacent to v_k, a contradiction to the fact that v_k is a (v_i, v_j)-balancing vertex. Thus, either $k < \min\{i, j\}$ or $k > \max\{i, j\}$.

By symmetry, it suffices to consider the first case. Suppose $k < \min\{i, j\}$. If v_p with $p < k$ is adjacent to either one of v_i, v_j, then it must also be adjacent to v_k by the umbrella property. Since v_k is the only vertex adjacent to exactly one of v_i, v_j, v_p must be adjacent to both v_i and v_j. By the umbrella property, both v_i and v_j are adjacent to v_k, a contradiction. On the other hand, since v_k is the only vertex adjacent to exactly one of v_i, v_j, it is clear that any v_q with $q > \max\{i, j\}$ is adjacent to either both or neither of v_i, v_j. □

Theorem 2.8. *Let X be an obstruction and $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 is the only cut-vertex and it is non-dividing. If $n = 6$, then X or its dual is one of the graphs in Figure 2.4.*

Proof: For each $3 \leq i \leq 5$, v_i is not a cut-vertex, so v_{i-1} and v_{i+1} are adjacent. Now $v_2 v_5$ and $v_3 v_6$ cannot both be edges in $U(X)$ as otherwise v_3, v_4 and v_5 each is adjacent to every vertex except for v_1, contradicting Lemma 2.5(b).

We claim that v_4 and v_5 is each incident with an arc. Since v_4 is adjacent to every vertex except for v_1, it is incident with exactly one arc by Lemma 2.5(b). On the other hand, suppose v_5 is not incident with an arc. By Lemma 2.5(a), v_5 is an arc-balancing vertex for some arc. Thus v_5 is adjacent to exactly one endvertex of the arc. It is easy to see that the other endvertex can only be v_2. Since v_2 is a cut-vertex, v_1 is adjacent to exactly one endvertex (i.e., v_2) of the arc, a contradiction to that v_5 is arc-balancing for the arc. Hence v_5 is incident with an arc.

Suppose v_3 and v_6 are also incident with arcs. Then v_3, v_4, v_5, v_6 are endvertices of the two arcs. Suppose that the two arcs are between v_3 and v_4 and between v_5 and v_6 respectively. Then $v_3 v_6$ is not an edge of $U(X)$ as otherwise the arc between v_5 and v_6 is balanced, a contradiction. If $v_2 v_5$ is not an edge of $U(X)$ then X or its dual is

Figure 2.4(i); otherwise, X or its dual is Figure 2.4(ii). Suppose that the two arcs are between v_3 and v_5 and between v_4 and v_6 respectively. Then X or its dual is Figure 2.4(iii), (iv) or (v) depending whether or not v_2v_5 and v_3v_6 are edges of $U(X)$. Suppose the two arcs are between v_3 and v_6 and between v_4 and v_5 respectively. Then X or its dual is again Figure 2.4(v) (with v_5 and v_6 being switched).

Suppose v_3 is not incident with an arc. By Lemma 2.5(a), v_3 is an arc-balancing vertex. By Lemma 2.7, v_3 balances an arc between v_5 and v_6. Since v_4 is incident with an arc and v_3 is not, the arc incident with v_4 has the other endvertex being v_2, v_5, v_6. These three cases are represented by Figure 2.4(vi), (vii) and (viii).

It follows from the above that at least one v_3 and v_6 is incident with an arc. Thus it remains to consider the case that v_3 is incident with an arc but v_6 is not. By Lemma 2.5(a), v_6 is an arc-balancing vertex for some arc. By Lemma 2.7, v_2 cannot be an endvertex of this arc, so the arc must be between v_3 and one of v_4, v_5. In particular, this implies v_3v_6 is not an edge of $U(X)$. It is now easy to verify that X or its dual is Figure 2.4(ix), (x) or (xi). □

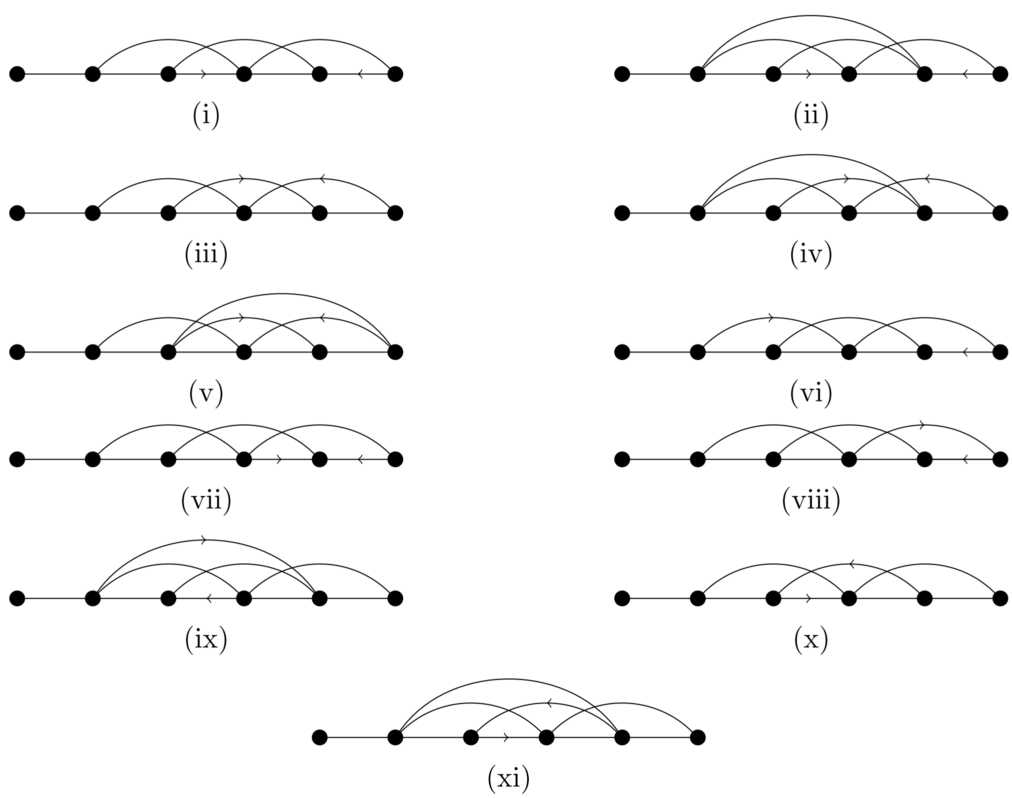


Figure 2.4: Obstructions with a unique non-dividing cut-vertex on 6 vertices.

Lemma 2.9. *Let X be an obstruction and $\prec\colon v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 is the only cut-vertex and it is non-dividing. If $n \geq 7$, then v_2 is not incident with an arc and the subgraph of $U(X)$ induced by the vertices v_i with $i \geq 3$*

cannot contain a copy of K_5.

Proof: Suppose that v_2 is incident with an arc. Then v_1 is adjacent to exactly one endvertex of this arc so this arc cannot be balanced by any vertex v_i with $i \geq 3$. It follows that there is at most one arc-balancing vertex v_i with $i \geq 3$. By Lemma 2.5(a) and the assumption $n \geq 7$ there are at least four vertices v_i with $i \geq 3$ which are incident with arcs, which is impossible because v_2 is such a vertex.

By Lemma 2.5(a) and (c), $n \leq 8$ and each v_i with $i \geq 3$ is an arc-balancing vertex or incident with an arc. Since neither of v_1, v_2 is incident with an arc, any set of five vertices v_i with $i \geq 3$ must contain an arc-balancing triple and hence cannot induce a copy of K_5 in $U(X)$. □

Theorem 2.10. *Let* X *be an obstruction and* $\prec: v_1, v_2, \ldots, v_n$ *be a straight enumeration of* $U(X)$. *Suppose that* v_2 *is the only cut-vertex and it is non-dividing. If* $n = 7$, *then* X *or its dual is one of the graphs in Figure 2.5.*

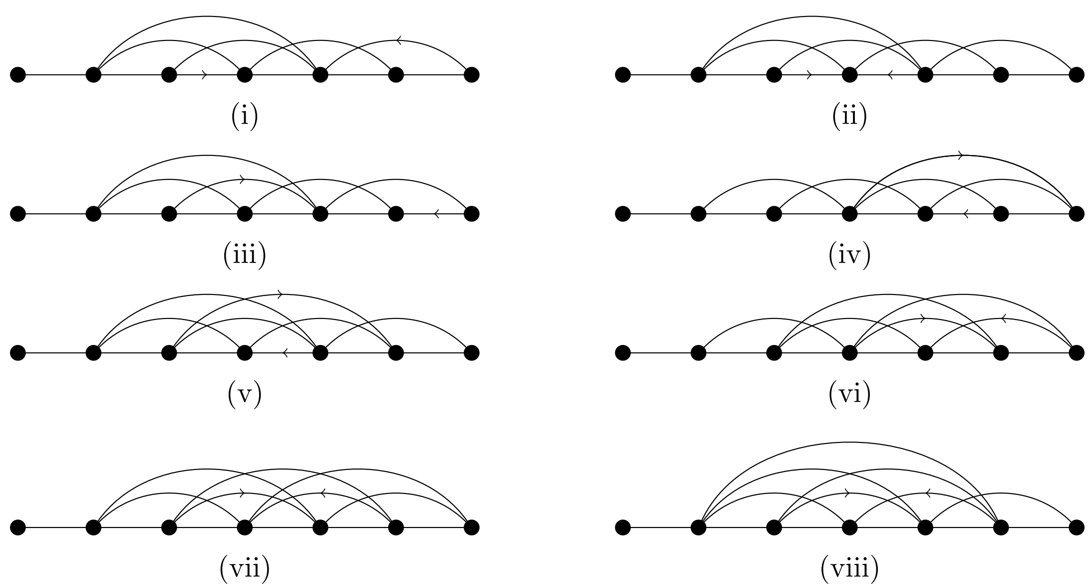


Figure 2.5: Obstructions with a unique non-dividing cut-vertex on 7 vertices.

Proof: First, note that v_3v_7 is not an edge in $U(X)$ as otherwise the vertices v_i with $i \geq 3$ induce a K_5 in $U(X)$, a contradiction to Lemma 2.9. If v_2v_6 and v_4v_7 are both edges of $U(X)$, then each of v_4, v_5, v_6 is adjacent to every vertex except for v_1, contradicting Lemma 2.5(c). So, v_2v_6 and v_4v_7 cannot both be edges in $U(X)$.

By Lemma 2.5(b), there exists a vertex v_i with $i \geq 3$ adjacent to every vertex except for v_1. Since v_3v_7 is not an edge in $U(X)$, neither v_3 nor v_7 is such a vertex. It is easy to see that if v_6 is such a vertex, then v_5 is also such a vertex. Hence, at least one of v_4, v_5 is adjacent to every vertex except for v_1.

Suppose v_4 is adjacent to every vertex except for v_1. This implies in particular that v_4v_7 is an edge of $U(X)$ and thus v_2v_6 is not an edge of $U(X)$. By Lemma 2.5(b), v_4 is incident with exactly one arc. Lemma 2.9 implies the other endvertex of this arc is one of v_3, v_5, v_6, and v_7. Suppose that the other endvertex is v_3. If no v_i with $i \geq 5$ is an arc-balancing vertex for this arc, then $\{v_5, v_6, v_7\}$ must be an arc-balancing triple, a contradiction because these vertices induce a clique. Hence for some $i \geq 5$, v_i is an arc-balancing vertex for the arc between v_4 and v_3. By Lemma 2.7, it must be v_7. Since v_7 balances the arc between v_4 and v_3, we see that v_3 must be adjacent to v_6. Both v_5 and v_6 are adjacent to v_i for each $i \geq 3$ so they cannot be arc-balancing vertices. Hence by Lemma 2.5(a), both v_5 and v_6 are incident with arcs. This means there is an arc between v_5 and v_6, which implies that v_5 is adjacent to v_2 (as otherwise v_5 and v_6 have the same closed neighbourhood in $U(X)$). Since v_3v_7 is not an edge in $U(X)$, X or its dual is Figure 2.5(vii). Suppose next that there is an arc between v_4 and v_5. Since v_4 and v_5 cannot have the same closed neighbourhood in $U(X)$, v_2v_5 is not an edge in $U(X)$. Clearly, the arc between v_4 and v_5 is not balanced by any of v_3, v_6, v_7, so $\{v_3, v_6, v_7\}$ is an arc-balancing triple. By Lemma 2.7, the arc is between v_6 and v_7. It follows that v_3v_6 is an edge in $U(X)$. Hence X or its dual is Figure 2.5(vi).

Suppose next that there is an arc between v_4 and v_6. By Lemma 2.7, the arc between v_4 and v_6 cannot be balanced by v_3, v_5, v_7. Similarly as above, $\{v_3, v_5, v_7\}$ is an arc-balancing triple. If v_7 balances an arc between v_3 and v_5, then v_2v_5 and $v_3v_6$are edges in $U(X)$, and X or its dual is Figure 2.5(vii). Suppose that v_3 balances an arc between v_5 and v_7. Each vertex except v_3 is either adjacent to both v_5, v_7 or neither. Since v_2 is not adjacent to v_7, it is not adjacent to v_5. Hence, X or its dual is Figure 2.5(iv) or (vi) depending whether or not v_3v_6 is an edge of $U(X)$. Finally, suppose there is an arc between v_4 and v_7. By Lemma 2.7, none of v_3, v_5, v_6 is an arc-balancing vertex for this arc. Hence, $\{v_3, v_5, v_6\}$ is an arc-balancing triple. By Lemma 2.7, v_3 balances an arc between v_5 and v_6. It follows that neither v_2v_5 nor v_3v_6 can be an edge in $U(X)$. So X or its dual is Figure 2.5(iv).

Suppose now v_4 is not adjacent to one of $v_2, v_3, \ldots, v_7$. From the above we know that v_5 must be adjacent to every vertex except for v_1. So v_5 is incident with exactly one arc, and the other endvertex of this arc is one of v_3, v_4, v_6, and v_7. We claim that it cannot be v_6. Suppose to the contrary that there is an arc between v_5 and v_6. By Lemma 2.7, none of v_3, v_4, v_7 can be an arc-balancing vertex for this arc. Hence, $\{v_3, v_4, v_7\}$ is an arc-balancing triple. Since neither v_4v_7 nor v_3v_7 is an edge of $U(X)$, the second arc can only be between v_3 and v_4 but it is not balanced by v_7, a contradiction. Hence, there is an arc between v_5 and one of v_3, v_4, and v_7.

Suppose first that there is an arc between v_5 and v_3. Assume that this arc is balanced by a vertex. By Lemma 2.7, it is balanced by v_7. It follows that v_3v_6 is an edge in $U(X)$. Since v_6 is adjacent to every vertex v_i with $i \geq 3$, which are where all endvertices of arcs are, it cannot be an arc-balancing vertex. It follows that v_6 is incident with an arc. We claim that the other endvertex of this arc is v_4. Indeed, if it is not v_4, then v_4 would be the arc-balancing vertex for this arc, a contradiction by Lemma 2.7. Thus, X or its dual is Figure 2.5(v) or (viii) depending whether or not v_2v_6 is an edge of $U(X)$. Assume now that the arc between v_5 and v_3 is not balanced by any vertex. In this case, $\{v_4, v_6, v_7\}$ is an arc-balancing triple. By Lemma 2.7, either v_4 balances an arc between v_6 and v_7, or v_7 balances an arc between v_4 and v_6. In the first case, v_3v_6 cannot be an edge of $U(X)$, as that would imply v_3v_7 is also an edge, a contradiction. Hence, X or its dual is Figure 2.5(iii). In the second case, v_2v_6 must be an edge of $U(X)$ and X or its dual is Figure 2.5(viii).

Suppose there is an arc between v_5 and v_4. We claim that v_3 is not arc-balancing vertex. Indeed, if it is, then it must balance an arc between v_6 and v_7. Thus, v_4v_7 is an edge of $U(X)$, contradicting the fact that v_4 is not adjacent to one of $v_2, v_3, \ldots, v_7$. Hence, v_3 is incident with an arc. The other endvertex of this arc is v_4, v_6, or v_7. Clearly it cannot be v_7 because that would imply v_4v_7 is an edge of $U(X)$, a contradiction. Suppose the second arc is between v_3 and v_4. Then, v_6 must be an arc-balancing vertex. Clearly, v_6 cannot balance the arc between v_5 and v_4, so it must balance the arc between v_3 and v_4. It follows that v_3v_6 is not an edge of $U(X)$, so X or its dual is Figure 2.5(ii). On the other hand, suppose the second arc is between v_3 and v_6. In this case, X or its dual is Figure 2.5(v) or (viii) depending whether v_2v_6 is an edge of $U(X)$.

Finally, suppose there is an arc between v_5 and v_7. Clearly, none of v_3, v_4, v_6 can be an arc-balancing vertex for this arc. Hence $\{v_3, v_4, v_6\}$ is an arc-balancing triple. By Lemma 2.7, v_6 must balance an arc between v_3 and v_4. It follows that v_3v_6 is not an edge of $U(X)$, so X or its dual is Figure 2.5(i). □

Theorem 2.11. *Let X be an obstruction and $\prec: v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. Suppose that v_2 is the only cut-vertex and it is non-dividing. If $n = 8$, then X or its dual is one of the graphs in Figure 2.6.*

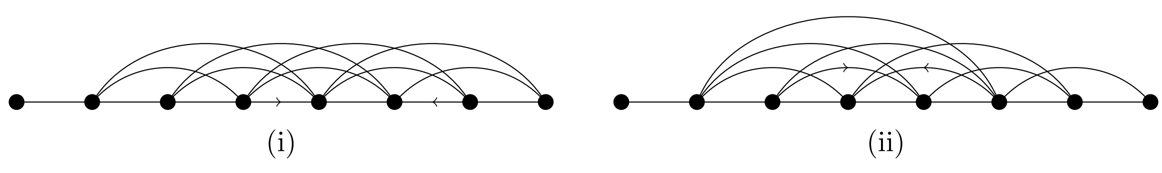


Figure 2.6: Obstructions with a unique non-dividing cut-vertex on 8 vertices.

Proof: Since there are six vertices succeeding v_2, exactly two of them are arc-balancing vertices and the other four are incident with arcs by Lemma 2.5(a). By Lemma 2.5(b), there exists a vertex succeeding v_2 that is adjacent to every vertex except for v_1. If any of v_3, v_4, v_7, v_8 is adjacent to every vertex except for v_1, then $U(X)$ contains a copy of K_5 among the vertices v_i with $i \geq 3$, contradicting Lemma 2.9. Hence, only v_5 and v_6 can be adjacent to every vertex except for v_1.

Suppose v_5 is adjacent to every vertex except for v_1. By Lemma 2.5(b) again, v_5 is incident with exactly one arc. By Lemma 2.7, v_8 balances an arc between v_5 and one of v_3, v_4. If v_3 is an endvertex of this arc, then v_3v_7 would be an edge in $U(X)$, contradicting Lemma 2.9. Hence, v_8 balances an arc between v_5 and v_4. It follows that v_4v_7 is an edge of $U(X)$. Moreover, there is an arc with both endvertices and arc-balancing vertex among v_3, v_6, v_7. If v_7 balances an arc between v_3 and v_6, then v_3v_8 is an edge of $U(X)$, contradiction Lemma 2.5(b). Hence v_3 balances an arc between v_6 and v_7. It follows that X or its dual is Figure 2.6(i).

On the other hand, suppose v_5 is not adjacent to one of $v_2, v_3, \ldots, v_8$. By the previous discussion, v_6 must be the unique such vertex. By Lemma 2.5(b), v_6 is incident with an arc. By Lemma 2.7, v_8 balances an arc between v_6 and one of v_3, v_4, v_5. If v_3 is an endvertex of this arc, then v_3v_7 is an edge of $U(X)$, contradicting Lemma 2.9. Suppose v_8 balances an arc between v_6 and v_4. Then, v_4v_7 is an edge of $U(X)$. Moreover, $\{v_3, v_5, v_7\}$ is an arc-balancing triple. By Lemma 2.7, v_7 balances an arc between v_3 and v_5. If v_5v_8 is an edge of $U(X)$, then v_3v_8 is also an edge, contradicting Lemma 2.9. Hence v_5v_8 is not an edge and so X or its dual is Figure 2.6(ii). Suppose instead that v_8 balances an arc between v_6 and v_5. In this case, $\{v_3, v_4, v_7\}$ is an arc-balancing triple. By Lemma 2.7, v_7 balances an arc between v_3 and v_4. Since v_5v_8 and v_3v_7 are not edges of $U(X)$, X or its dual is Figure 2.6(ii). □

Chapter 3

Obstructions without cut-vertices

We now examine obstructions X that do not contain cut-vertices. We shall consider the complements $\overline{U(X)}$ of the underlying graphs $U(X)$. All theorems and proofs, including the drawings of obstructions X, in this chapter will be presented in terms of $\overline{U(X)}$ instead of $U(X)$.

Lemma 3.1. *Suppose that X is an obstruction that contains no cut-vertices. Then, in $\overline{U(X)}$, each vertex has at least two non-neighbours.*

Proof: Note that X has at least vertices. Since X has no cut-vertices and $U(X)$ is connected, in $U(X)$ each vertex has at least two neighbours and hence in $\overline{U(X)}$ each vertex has at least two non-neighbours. □

Recall from Corollary 1.11 that if an obstruction X has no cut-vertices then $\overline{U(X)}$ has at most six non-cut-vertices. We show this holds for every connected subgraph of $\overline{U(X)}$.

Lemma 3.2. *Let X be an obstruction that contains no cut-vertices and H be a connected subgraph of $\overline{U(X)}$. Then $\overline{U(X)}$ contains at least as many non-cut-vertices as H. In particular, H has at most six non-cut-vertices.*

Proof: Since adding edges does not decrease the number of non-cut-vertices, we may assume H is an induced subgraph of $\overline{U(X)}$. Thus H can be obtained from $\overline{U(X)}$ by successively deleting non-cut-vertices. Since each deletion of a non-cut-vertex does not increase the number of non-cut-vertices, $\overline{U(X)}$ contains at least as many non-cut-vertices as H. By Corollary 1.11, $\overline{U(X)}$ has at most six non-cut-vertices. So H has at most six non-cut-vertices. □

Lemma 3.3. *If X is an obstruction that contains no cut-vertices, then $\overline{U(X)}$ contains no induced cycle of length at least* 6.

Proof: By Lemma 3.2, any connected subgraph of $\overline{U(X)}$ has at most six non-cut-vertices. Thus $\overline{U(X)}$ contains no induced cycle of length at least 7. Theorem 1.4 ensures that $\overline{U(X)}$ does not contain an induced cycle of length 6. Therefore $\overline{U(X)}$ contains no induced cycle of length ≥ 6. □

Lemma 3.3 implies that any induced cycle in $\overline{U(X)}$ has length 3, 4 or 5. We show that $\overline{U(X)}$ contains at most one C_3 and at most one induced C_5 and moreover, if $\overline{U(X)}$ contains an induced C_5, then it does not contain an induced C_3 or C_4.

Lemma 3.4. *Let X be an obstruction. Suppose C is an odd cycle (not necessarily induced) in $\overline{U(X)}$. Then in $\overline{U(X)}$ each vertex is either on C or adjacent to a vertex of C. In particular, each cut-vertex of $\overline{U(X)}$ is on C.*

Proof: Since C is an odd cycle, $\overline{U(X)}$ contains an induced odd cycle C_{2k+1} on some vertices on C. By Theorem 1.4, $\overline{U(X)}$ does not contain $C_{2k+1}+K_1$ as an induced subgraph. Thus each vertex is either in C_{2k+1} or adjacent to a vertex of C_{2k+1}. Since the vertices of C_{2k+1} are all on C, each vertex is either on C or adjacent to a vertex of C. Consequently, each cut-vertex of $\overline{U(X)}$ is on C. □

Lemma 3.5. *Suppose that X is an obstruction that contains no cut-vertices. Then $\overline{U(X)}$ contains at most one C_3.*

Proof: Suppose that C and C' are two copies of C_3 in $\overline{U(X)}$. If C and C' share no common vertex, then every vertex of $\overline{U(X)}$ is either not on C or not on C' and hence by by Lemma 3.4 is a non-cut-vertex. But $\overline{U(X)}$ has at most six non-cut-vertices by Corollary 1.11, so $\overline{U(X)}$ is a union of C and C'. According to Proposition 1.10 each vertex of $\overline{U(X)}$ is an endvertex of an arc or an arc-balancing vertex. There are at most four endvertices of arcs and at most two arc-balancing vertices. So among the six vertices of $\overline{U(X)}$ four are the endvertices of arcs and the remaining two are arc-balancing vertices. Suppose that (a, b) is an arc (of X) and u is its balancing vertex that is adjacent to a but not to b in $\overline{U(X)}$. Then each of the remaining three vertices is adjacent to a or b and thus to both a, b. Hence b is the only non-neighbour of a in $\overline{U(X)}$, a contradiction to Lemma 3.1. Therefore any two copies of C_3 in $\overline{U(X)}$ must share a common vertex.

Suppose that C and C' share exactly one common vertex. Denote $C : v_1v_2v_3$ and $C' : v_1v_4v_5$. Let u, w be two non-neighbours of v_1 in $\overline{U(X)}$ guaranteed by by Lemma 3.1. Each vertex except v_1 is not on C or C' and hence by Lemma 3.4 is a non-cut-vertex. Since $\overline{U(X)}$ has at most six non-cut-vertices, it consists of C, C' and u, v. A similar argument as above among the six non-cut-vertices u, w, v_2, v_3, v_4, v_5 four are the endvertices of arcs and the remaining two are arc-balancing vertices. We claim that the two arc-balancing vertices are u, w. Indeed, since v_1 is not an arc-balancing vertex, there is no arc between u, w and v_2, v_3, v_4, v_5. Suppose that there is an arc between u and w. Assume without loss of generality that this arc is balanced by v_2 which is adjacent to u but not w. By Lemma 3.4, w is adjacent to a vertex on C. Since w is not adjacent to v_1 or v_2, it is adjacent to v_3. Since v_3 does not balance the arc between u and w, v_3 is adjacent to u.

But then uv_2v_3 and C' are vertex-disjoint copies of C_3, a contradiction. Hence neither of u, w is an endvertex of an arc so both are arc-balancing vertices. Without loss of generality assume that u balances an arc between v_2 and v_4 and is adjacent to v_2 but not v_4. Since v_3 is adjacent to v_2, it must be adjacent to v_4. Similarly, v_5 must be adjacent to v_2. By Lemma 3.4, u is adjacent to a vertex on C' which can only be v_5. Hence uv_2v_5 and $v_1v_3v_4$ are vertex-disjoint copies of C_3, a contradiction. Therefore any two copies of C_3 in $\overline{U(X)}$ must share at least two common vertices.

Suppose that C and C' share exactly two vertices. Denote $C : v_1v_2v_3$ and $C' : v_1v_2v_4$. We claim that in $\overline{U(X)}$ any vertex $v \notin C \cup C'$ that is adjacent one of v_3, v_4 must be adjacent to both v_3, v_4 and neither of v_1, v_2. Without loss of generality, suppose $v \notin C \cup C'$ is adjacent to v_3. If it is also adjacent to v_1, then v_1v_3v and $v_1v_2v_4$ would be two distinct copies of C_3 in $\overline{U(X)}$ that share exactly one common vertex, a contradiction to the above. Hence, v is not adjacent to v_1. Similarly, v is not adjacent to v_2. By Lemma 3.4 v must be adjacent to a vertex on C' so it is adjacent to v_4.

Now, we show that v_3 and v_4 are incident with different arcs. Suppose v_3 balances an arc between a and b and is adjacent to a but not b. If $a = v_1$, then since v_2 and v_4 are adjacent to v_1, they must also be adjacent to b. So, bv_2v_4 and C are two distinct copies of C_3 in $\overline{U(X)}$ that share exactly one common vertex, a contradiction. Thus, $a \neq v_1$. Similarly, $a \neq v_2$. Suppose $a = v_4$. Since v_1 and v_2 are adjacent to a, they must be adjacent to b as well. Thus v_1v_2b and $v_3v_4v_2$ are two copies of C_3 in $\overline{U(X)}$ that share exactly one common vertex, a contradiction.

It follows that $a \notin C \cup C'$. Since $a \notin C \cup C'$ and a is adjacent to v_3, it is adjacent to both of v_3, v_4 by the above claim. Since v_4 is adjacent to a, it must also be adjacent to b. Moreover, $b \notin C \cup C'$ because it is adjacent to v_4 but not a. Since $b \notin C \cup C'$ and b is adjacent to v_4, the above claim implies b is adjacent to both of v_3, v_4, a contradiction because v_3 balances the arc (a, b). Thus, v_3 is not an arc-balancing vertex. Since v_3 is not on C', it is not a cut-vertex of $\overline{U(X)}$ by Lemma 3.4. By Proposition 1.10, v_3 is incident with an arc. Similarly, v_4 is incident with an arc. If v_3 and v_4 are incident with the same arc, then there must be a vertex u that is adjacent to exactly one of v_3, v_4 because arcs in X are not balanced. Clearly, $u \notin C \cup C'$. This is a contradiction because any vertex not in $C \cup C'$ is adjacent to either both of v_3, v_4 or neither, by above claim. Thus, v_3 and v_4 are each incident with a different arc.

Suppose there exists a vertex $v \notin C \cup C'$ that is adjacent to either of v_3, v_4. By the above claim, we know that v is adjacent to both of v_3, v_4 and neither of v_1, v_2. Since v is adjacent to both of v_3, v_4, which are each incident with a different arc, v is not incident with an arc. Moreover, since v is not on the odd cycle C, Lemma 3.4 implies v is not a cut-vertex of $\overline{U(X)}$. So by Proposition 1.10, v is an arc-balancing vertex. Since v_3 and v_4 are each incident with a different arc, we may assume without loss of generality that v balances an arc incident with v_3. Let w denote the other endvertex of this arc. Then, w

is adjacent to v_1 and v_2, so v_1v_2w is a triangle. By Lemma 3.4, v is adjacent to a vertex on v_1v_2w, which must be w, a contradiction because v balances the arc between v_3 and w. It follows $\overline{U(X)}$ does not contain a vertex $v \notin C \cup C'$ that is adjacent to either of v_3, v_4.

By Lemma 3.1, v_1 has at least two non-neighbours, say u and w. Clearly, $u, w \notin C \cup C'$. So by the above, neither of u, w is adjacent to either of v_3, v_4. By Lemma 3.4, each of u, w is adjacent to a vertex on C, which must be v_2. Similarly, v_2 has at least two non-neighbours, say x and y, and each is adjacent to v_1. By Lemma 3.4, each of v_3, v_4, u, w, x, y is a non-cut-vertex of $\overline{U(X)}$ and so by Corollary 1.11 they form two disjoint arc-balancing triples. Since v_3 and v_4 are each incident with a different arc, exactly two of u, w, x, y are arc-balancing vertices. Without loss of generality, assume u is an arc-balancing vertex for an arc incident with v_3. Since v_3 is adjacent to both v_1 and v_2, the other endvertex must also be adjacent to both v_1 and v_2. This is a contradiction because none of w, x, y is adjacent to both v_1 and v_2 by assumption. It follows that C and C' cannot share two common vertices. Therefore $\overline{U(X)}$ contains at most one C_3. □

Lemma 3.6. *Suppose that X is an obstruction that contains no cut-vertices. Then $\overline{U(X)}$ contains at most one induced C_5.*

Proof: Suppose that C and C' are induced copies of C_5 contained in $\overline{U(X)}$. By Lemma 3.4, any vertex not on C or C' is a non-cut-vertex of $\overline{U(X)}$ and hence by Corollary 1.11 there can be at most six such vertices. Thus C and C' must share at least two common vertices. If C and C' share less two or three common vertices, then the subgraph of $\overline{U(X)}$ induced by $C \cup C'$ is connected and has at least seven non-cut-vertices, contradicting Lemma 3.2. Hence, C and C' must share exactly four vertices.

Denote $C : v_1v_2v_3v_4v_5$ and $C' : v_2v_3v_4v_5v_6$. Then v_1v_6 is not an edge in $\overline{U(X)}$ as otherwise $v_1v_2v_6$ and $v_1v_5v_6$ are two copies of C_3 in $\overline{U(X)}$, a contradiction to Lemma 3.5. We claim that v_1, v_6 are endvertices of arcs in X. By symmetry we only prove that v_1 is an endvertex of an arc in X. We prove it by contradiction. So assume that v_1 is not an endvertex of an arc in X. Since v_1 is not in C', by Lemma 3.4 it is not a cut-vertex of $\overline{U(X)}$. Hence v_1 is an arc-balancing vertex for some arc according to Proposition 1.10. Suppose that v_1 balances the arc between vertices a, b and is adjacent to a but not to b in $\overline{U(X)}$. If a is not on C', then a must be adjacent to a vertex of C' by Lemma 3.4. But then the subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup \{a\}$ is connected and has seven non-cut-vertices, contradicting Lemma 3.2. Hence a is a vertex of C' and therefore it is v_2 or v_5. Assume by symmetry $a = v_2$. Since v_1 balances the arc between a, b, every vertex not in $\{v_1, a, b\}$ is either adjacent to both a, b or neither. It follows that b cannot be in $C \cup C'$. Thus the subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup \{b\}$ is connected and has seven non-cut-vertices, a contradiction to Lemma 3.2. Therefore v_1, v_6 are both endvertices of arcs of X. We claim that there is no arc between v_1, v_6. Suppose not; there is an arc between v_1, v_6. Then there must exist a vertex u adjacent to exactly one of v_1, v_6. Then there must exist a vertex u adjacent to exactly one of v_1, v_6. A similar argument as above shows that u is not in $C \cup C'$ but adjacent to a vertex in $C \cup C'$. Thus the subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup \{u\}$ is connected and contains seven non-cut-vertices, a contradiction. Thus, v_1, v_6 are endvertices of different arcs.

The subgraph of $\overline{U(X)}$ induced by $C \cup C'$ contains six non-cut-vertices, so $\overline{U(X)}$ contains six non-cut-vertices by Lemma 3.2. It follows from Proposition 1.10 that X contains exactly four vertices incident to arcs and exactly two arc-balancing vertices. In particular, both arcs have an arc-balancing vertex.

By Proposition 1.10, v_3 is a cut-vertex of $\overline{U(X)}$, an arc-balancing vertex, or is incident with an arc. We claim it must be a cut-vertex of $\overline{U(X)}$. Suppose instead v_3 is an arc-balancing vertex. Without loss of generality, assume it balances the arc incident with v_1. Then, the other endvertex must be adjacent to each of v_2, v_5. Clearly, the subgraph of $\overline{U(X)}$ induced by $C \cup C'$ together with this endvertex contains seven non-cut-vertices, a contradiction. On the other hand, suppose v_3 is incident with an arc. The other endvertex is one of v_1, v_6. Without loss of generality, assume it is v_1. Then, v_4 and v_5 are both vertices adjacent to exactly one of the endvertices of this arc, so the arc between v_1 and v_3 has no arc-balancing vertex, a contradiction. Thus, v_3 is a cut-vertex of $\overline{U(X)}$.

Let v_7 be a neighbour of v_3 belong to a different component of $\overline{U(X - v_3)}$ as the vertices in $(C \cup C') \setminus \{v_3\}$. By Lemma 3.4, v_7 cannot be a cut-vertex of $\overline{U(X)}$. On the other hand, suppose v_7 is incident with an arc. Without loss of generality, assume the other endvertex is v_1. Since v_2 and v_3 are both vertices adjacent to exactly one of v_1, v_7, there is no corresponding arc-balancing vertex for this arc, a contradiction. Thus, v_7 cannot be incident with an arc. By Proposition 1.10, v_7 is an arc-balancing vertex for either the arc incident with v_1 or the arc incident with v_6. In either case, the other endvertex must be adjacent to both v_2 and v_5. Clearly, the subgraph of $\overline{U(X)}$ induced by $C \cup C'$ together with this endvertex contains seven non-cut-vertices, a contradiction. □

Lemma 3.7. *Let X be an obstruction that contains no cut-vertices. If $\overline{U(X)}$ contains an induced C_5, then it contains neither C_3 nor an induced C_4.*

Proof: Let $C : v_1v_2v_3v_4v_5$ be an induced C_5 in $\overline{U(X)}$. We first show that $\overline{U(X)}$ does not contain C_3. Suppose otherwise and let C' be a C_3 in $\overline{U(X)}$. A similar argument as the one in Lemma 3.6 shows that C and C' have exactly two common vertices. Without loss of generality let $C' : v_1v_2v_6$. By Lemma 3.4, v_4 must be adjacent to a vertex on C', which clearly must be v_6. The subgraph induced by $C \cup C'$ contains six non-cut-vertices, so $\overline{U(X)}$ contains six non-cut-vertices by Lemma 3.2. Each of these six non-cut-vertices is an arc-balancing vertex or incident with an arc by Proposition 1.10. Hence, each arc has a arc-balancing vertex. If both endvertices of some arc are on C, then C contains two other vertices which are both adjacent to exactly one of the endvertices, contradicting the fact that each arc has a unique arc-balancing vertex. It follows that each arc has at most one endvertex on C. In particular, at most two vertices on C are incident with arcs. On the other hand, at most two vertices on C are arc-balancing. It follows from Proposition 1.10 that C has a cut-vertex. By Lemma 3.4, each cut-vertex belongs to $C \cap C'$, so only v_1 and v_2 can be cut-vertices.

We claim that if v_1 is a cut-vertex, then there exists a vertex v_7 that is adjacent only to v_1 and an arc between v_5 and v_7 that is balanced by v_4. Suppose v_1 is a cut-vertex. Let v_7 be a vertex adjacent to v_1 that belongs to a different component of $\overline{U(X - v_1)}$ as the vertices in $(C \cup C') \setminus \{v_1\}$. If v_7 is adjacent to a vertex other than v_1, then that vertex must be adjacent to a vertex in $C \cup C'$ by Lemma 3.4, contradicting the choice of v_7. Hence, v_7 is adjacent only to v_1. By Lemma 3.4, v_7 is not a cut-vertex. Suppose v_7 is an arc-balancing vertex. Then, v_1 is incident with an arc, and the other endvertex of this arc must be adjacent to v_2, v_5, v_6. Clearly, this endvertex is none of the vertices in $C \cup C'$, so the subgraph of $\overline{U(X)}$ induced by $C \cup C'$ together with this endvertex contains seven non-cut-vertices, contradicting Lemma 3.2. Hence, v_7 is not an arc-balancing vertex. By Proposition 1.10, v_7 is incident with an arc. Since v_7 is adjacent only to v_1, the other endvertex has degree 2 and is adjacent to v_1. Clearly, it must be v_5. The corresponding arc-balancing vertex is adjacent to v_5 and not to v_7, so it must be v_4. This proves our claim.

Recall that at least one of v_1, v_2 is a cut-vertex. Without loss of generality, assume v_1 is a cut-vertex. By the above, there exist a vertex v_7 that is adjacent only to v_1 and an arc between v_5 and v_7 that is balanced by v_4. In particular, v_6 is not an arc-balancing vertex for this arc. By Lemma 3.4, v_6 is not a cut-vertex. Hence by Proposition 1.10, v_6 is arc-balancing for the other arc or incident with it. By symmetry, if v_2 is also a cut-vertex, then there exist a vertex v_8 that is adjacent only to v_2 and an arc between v_3 and v_8, a contradiction because v_6 is neither arc-balancing for this arc nor incident with this arc. Thus, v_2 is not a cut-vertex. Since v_4 balances an arc between v_5 and v_7 and v_2, v_3, v_6 are non-cut-vertices, $\{v_2, v_3, v_6\}$ is an arc-balancing triple. It follows that v_2 balances an arc between v_3 and v_6, a contradiction. Thus, $\overline{U(X)}$ does not contain an induced C_3.

It remains to show that $\overline{U(X)}$ does not contain an induced C_4. Suppose otherwise, and let C' be such a cycle. A similar argument as the one in Lemma 3.6 shows that C and C' have exactly three common vertices. Let $C' : v_1v_2v_3v_6$. If v_6 is adjacent to neither v_4 or v_5, then $v_1v_6v_3v_4v_5$ is an induced C_5, contradicting Lemma 3.6. Hence, v_6 is adjacent to one of v_4, v_5. It follows that $\overline{U(X)}$ contains an induced C_3, a contradiction. □

3.1 $\overline{U(X)}$ is disconnected

We first examine obstructions X that do not contain cut-vertices for which $\overline{U(X)}$ is disconnected. These obstructions have a simple structure as described in the following theorem.

Theorem 3.8. *Let X be an obstruction that does not contain cut-vertices. Suppose that $U(X)$ is disconnected. Then the following statements hold:*

- $\overline{U(X)}$ *is the union of two disjoint paths* $P : p_1p_2 \dots p_k$ *and* $Q : q_1q_2 \dots q_\ell$*;*
- X *or its dual contains the arcs* $(p_1, q_1), (q_\ell, p_k)$ *if* $k + \ell$ *is even, and* $(p_1, q_1), (p_k, q_\ell)$ *otherwise.*

That is, $U(X)$ *is one of the graphs in Figure 3.1 and* X *or its dual contains the dotted arcs.*

Figure 3.1: Obstructions X for which $\overline{U(X)}$ is disconnected.

Proof: Let (a, b) and (c, d) be the two arcs of X. Then ab and cd belong to the same implication class of $U(X)$. By Theorem 1.7, either ab and cd are unbalanced edges of $U(X)$ within a component of $\overline{U(X)}$ or they are edges between two components of $\overline{U(X)}$. Since $\overline{U(X)}$ is disconnected, it has at least two components. If some component of $\overline{U(X)}$ does not contain any of a, b, c, d, then any non-cut-vertex of that component is not a cut-vertex of $U(X)$ by assumption, and is also not an arc-balancing vertex because it is not adjacent to any of a, b, c, d in $\overline{U(X)}$. This contradicts Proposition 1.10. Thus $\overline{U(X)}$ has exactly two components and ab, cd are edges between them.

Consider a component H of $\overline{U(X)}$ and let P be a shortest path in H between some two of a, b, c, d. If H contains a vertex v that is not in P then it follows from Proposition 1.5 that ab and cd are still in the same implication class of $U(X - v)$, which is a contradiction to Theorem 1.9. This shows that each component of $\overline{U(X)}$ is a path connecting two vertices of a, b, c, d and $\overline{U(X)}$ is the union of two disjoint paths.

Let $P : p_1 \dots p_k$ and $Q : q_1 \dots q_\ell$ be the two paths in $\overline{U(X)}$. The two arcs are between p_1 and q_1 and between p_k and q_ℓ respectively. Without loss of generality, assume (p_1, q_1) is an arc. Suppose $k+\ell$ is even. If k, ℓ are both even, then $(p_1, q_1)\Gamma^*(q_\ell, p_1)$ and $(q_\ell, p_1)\Gamma^*(p_k, q_\ell)$ by Proposition 1.5. Since the arcs must be opposing, the other arc is (q_ℓ, p_k). Otherwise, k, ℓ are both odd. In this case, we have $(p_1, q_1)\Gamma^*(p_1, q_\ell)$ and $(p_1, q_\ell)\Gamma^*(p_k, q_\ell)$, so the other arc is (q_ℓ, p_k). Hence X or its dual is Figure 3.1(i). A similar proof shows that, when $k + \ell$ is odd, X or its dual is Figure 3.1(ii). □

We remark that if $k = 1$ and $2 \le \ell \le 3$, then the graph X is an obstruction that contains cut-vertices, and thus does not belong to this case. In particular, the dual of X is Figure 2.1(i) if $k = 1$ and $\ell = 2$, and X is Figure 2.3(i) if $k = 1$ and $\ell = 3$.

Corollary 3.9. *If X is an obstruction that does not contain cut-vertices and for which $\overline{U(X)}$ is disconnected, then $\overline{U(X)}$ is acyclic.* □

3.2 $\overline{U(X)}$ is a tree

We next examine obstructions X that do not contain cut-vertices and for which $\overline{U(X)}$ is a tree. We begin with a useful lemma.

Lemma 3.10. *Let X be an obstruction that contains no cut-vertices. If $\overline{U(X)}$ is a tree, then it is a caterpillar and has at most four leaves. Moreover, suppose $P : p_1p_2 \dots p_k$ is a longest path in $\overline{U(X)}$. If p_1 is an arc-balancing vertex, then p_2 has only two neighbours (namely, p_1, p_3) and p_1 balances an arc between p_2 and a leaf adjacent to p_3 but not in P.*

Proof: Since $U(X)$ is a proper circular-arc graph, $\overline{U(X)}$ does not contain the fifth graph in Figure 1.1 by Theorem 1.4 and hence is a caterpillar. If v is a leaf of $\overline{U(X)}$ that is not incident with an arc of X, then by Proposition 1.10 v is an arc-balancing vertex and hence adjacent to a vertex that is incident with an arc. Clearly, the vertex adjacent to v cannot be adjacent to any other leaf. Since there are at most four vertices incident with arcs, $\overline{U(X)}$ has at most four leaves.

Since P is a longest path, p_1 is a leaf. If p_1 is an arc-balancing vertex, then p_2 is incident with an arc balanced by p_1. Let u be the other endvertex of the arc. Every vertex other than p_1 is adjacent either to both p_2, u or neither. Since p_3 is adjacent to p_2, it is adjacent to u. Since $\overline{U(X)}$ is a tree, p_3 is the only neighbour of p_2 other than p_1 and the only neighbour of u. It follows that p_1, p_3 are the only neighbours of p_2. If u is in P then $u = p_4$ and $k = 4$. Thus each vertex not in P can only be adjacent to p_3 in $\overline{U(X)}$, which implies that p_1 is a cut-vertex of $U(X)$, a contradiction. Therefore u is a leaf of $\overline{U(X)}$ adjacent to p_3 but not in P. □

Theorem 3.11. *Let X be an obstruction that contains no cut-vertices and for which $\overline{U(X)}$ is a tree. Let $P : p_1p_2 \dots p_k$ be a longest path in $\overline{U(X)}$. Then $\overline{U(X)}$ consists of P and u, v (possibly $u = v$) where u is either a leaf adjacent to some p_ℓ but not in P or $u = p_\ell$ and v is either a leaf adjacent to some p_j but not in P or $v = p_j$, and one of the following statements holds:*

(i) u is not in P and $\ell = 3$, v is not in P and $j = k - 2$, and X or its dual has arcs $(p_2, u), (p_{k-1}, v)$ (See Figure 3.2(i));

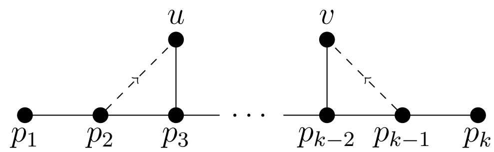
(ii) u is not in P and $\ell = 3$*,* $1 \le j \le k-2$ *with* $j > 2$ *when v is not in P, and X or its dual has arcs* $(p_2, u), (v, p_k)$ *if either* $k+j$ *is even and v is not in P or* $k+j$ *is odd and v is in P; otherwise X or its dual has arcs* $(p_2, u), (p_k, v)$ *(See Figure 3.2(ii));*

(iii) u is not in P and $2 \le \ell \le k-2$*,* $j = \ell + 1$*, X or its dual has arcs* $(p_1, p_k), (v, u)$ *if either k is even and v is not in P or k is odd and v is in P; otherwise X or its dual has arcs* $(p_1, p_k), (u, v)$ *(See Figure 3.2(iii));*

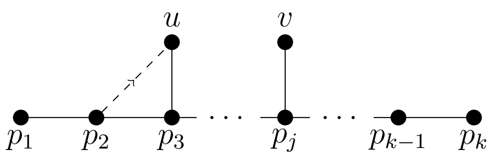
(iv) $3 \le \ell \le k-1$*,* $\ell - 1 \le j \le k-2$*, and X or its dual has arcs* $(p_1, u), (p_k, v)$ *if either* $k + \ell + j$ *is even and P contains both u, v or neither, or* $k + \ell + j$ *is odd and P contains exactly one of u, v; otherwise X or its dual has arcs* $(p_1, u), (v, p_k)$ *(See Figure 3.2(iv)).*

Proof: Suppose both p_1, p_k are arc-balancing vertices. By Lemma 3.10, p_1 balances an arc between p_2 and a leaf u adjacent to p_3 but not in P, and p_k balances an arc between p_{k-1} and a leaf v adjacent to p_{k-2} but not in P. In the tree $\overline{U(X)}$ the unique (u, p_k)-path avoids p_1 and the unique (p_1, v)-path avoids p_k. The lengths of these two paths have the same parity so by Proposition 1.5, we have either $(u, p_1)\Gamma^*(p_k, p_1)\Gamma^*(p_k, v)$ or $(u, p_1)\Gamma^*(p_1, p_k)\Gamma^*(p_k, v)$. In both cases, we have $(u, p_1)\Gamma^*(p_k, v)$ and therefore it follows that $(p_2, u)\Gamma(u, p_1)\Gamma^*(p_k, v)\Gamma(v, p_{k-1})$. Since the two arcs of X are opposing, X or its dual contains arcs $(p_2, u), (p_{k-1}, v)$. Since the subgraph of X induced by P together with u, v is an obstruction, and proper induced subgraphs of obstructions are not obstructions, the minimality of X ensures that $\overline{U(X)}$ consists of P and u, v and thus statement (i) holds.

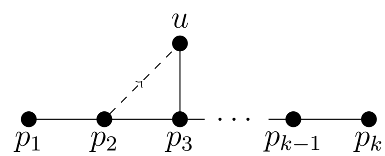
Suppose next that p_1 is an arc-balancing vertex but p_k is not. By Lemma 3.10 p_1 balances an arc between p_2 and a leaf u adjacent to p_3 but not in P. Since p_k is not an arc-balancing vertex, it is an endvertex of an arc. Let v be the other endvertex. Then either $v = p_j$ for some $1 \le j \le k-2$ or a leaf adjacent to some vertex in P. Suppose that v is a leaf adjacent to p_j. Then $j \notin \{1, k\}$ because P is the longest path in $\overline{U(X)}$, and $j \ne 2$ because p_2 has no neighbour other than p_1, p_3 according to Lemma 3.10. Moreover, $j \ne k-1$ as otherwise the arc between p_j and p_k is balanced, which is not possible. So $2 < j < k-1$. In the tree $\overline{U(X)}$ the unique (u, p_k)-path avoids p_1 and the unique (p_1, v)-path avoids p_k. If $k+j$ is even and v is not in P or $k+j$ is odd and v is in P, then the lengths of these two paths have the same parity. By Proposition 1.5, $(u, p_1)\Gamma^*(p_k, v)$ and so $(p_2, u)\Gamma(u, p_1)\Gamma^*(p_k, v)$. Since the two arcs of X are opposing, X or its dual contains arcs $(p_2, u), (v, p_k)$. Otherwise, the lengths of the two paths have the opposite parities and we have $(p_2, u)\Gamma(u, p_1)\Gamma^*(v, p_k)$. Hence X or its dual contains arcs $(p_2, u), (p_k, v)$. The minimality of X ensures that $\overline{U(X)}$ consists of P and u, v and thus statement (ii) holds.



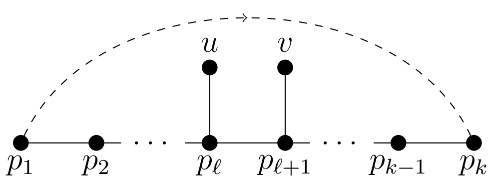
(i)



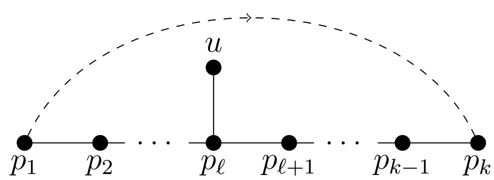
(ii.a): The second arc is (v, p_k) if $k + j$ is even and (p_k, v) otherwise where $2 < j < k - 1$.



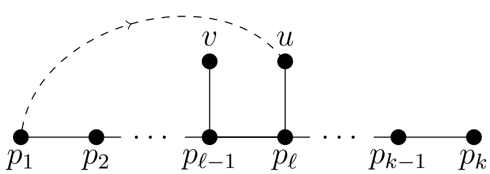
(ii.b): The second arc is (p_j, p_k) if $k + j$ is odd and (p_k, p_j) otherwise where $1 \leq j \leq k - 2$.



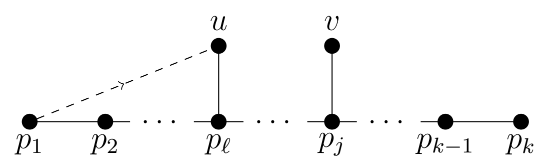
(iii.a): The second arc is (v, u) if k is even and (u, v) otherwise.



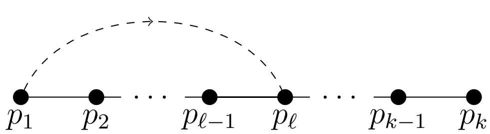
(iii.b): The second arc is $(p_{\ell+1}, u)$ if k is odd and $(u, p_{\ell+1})$ otherwise.



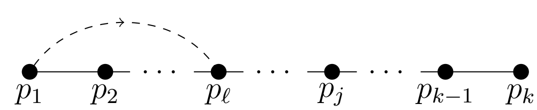
(iv.a): The second arc is (p_k, v) if $k + \ell + j$ is even and (v, p_k) otherwise.



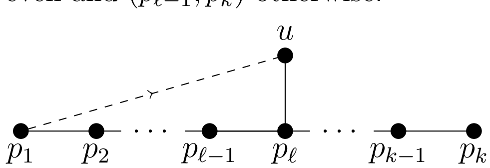
(iv.b): The second arc is (p_k, v) if $k + \ell + j$ is even and (v, p_k) otherwise.



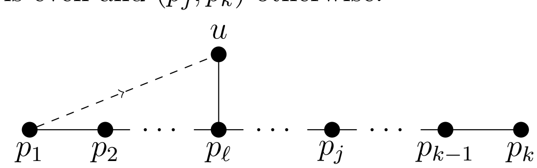
(iv.c): The second arc is $(p_k, p_{\ell-1})$ if $k+\ell+j$ is even and $(p_{\ell-1}, p_k)$ otherwise.



(iv.d): The second arc is (p_k, p_j) if $k + \ell + j$ is even and (p_j, p_k) otherwise.



(iv.e): The second arc is $(p_k, p_{\ell-1})$ if $k+\ell+j$ is odd and $(p_{\ell-1}, p_k)$ otherwise.



(iv.f): The second arc is (p_k, p_j) if $k + \ell + j$ is odd and (p_j, p_k) otherwise.

Figure 3.2: Obstructions X for which $\overline{U(X)}$ is a tree.

It remains to consider the case when neither of p_1, p_k is an arc-balancing vertex. Suppose first that X contains an arc between p_1 and p_k. Let u, v be the endvertices of the other arc. We claim that at least one of u, v is not in P. Indeed, if they are both in P (say $u = p_i$ and $v = p_j$ where $i < j$) then $j > i + 1$. It is easy to check that $X - p_{i+1}$ cannot be completed to a local tournament, which contradicts the minimality of X. So at least one of u, v is not in P. Assume without loss of generality that u is not in P. Since $\overline{U(X)}$ is a caterpillar, u is a leaf adjacent to some p_ℓ in P, and $v = p_j$ or v is a leaf adjacent to some p_j in P. By reversing $\prec$ if needed we assume that $j \geq \ell$. Since uv is an unbalanced edge of $U(X)$, $j \neq \ell$. In the tree $\overline{U(X)}$ the unique (p_1, u)-path avoids p_k and the unique (v, p_k)-path avoids u. If k is even and v is not in P or k is odd and v is in P, then the lengths of these two paths have the same parity. By Proposition 1.5, $(p_1, p_k)\Gamma^*(u, v)$ and hence X or its dual contains arcs $(p_1, p_k), (v, u)$. Otherwise, the lengths of the two paths have opposite parities and we have $(p_1, p_k)\Gamma^*(v, u)$ and X or its dual contains arcs $(p_1, p_k), (u, v)$. If $j > \ell+1$, then the two arcs are still opposing in $X - p_{\ell+1}$, a contradiction to the assumption that X is an obstruction. So $j = \ell + 1$. The minimality of X ensures $\overline{U(X)}$ contains no other vertices. Therefore statement (iii) holds.

Suppose now that X does not contain an arc between p_1, p_k. Then p_1, p_k are incident with different arcs. Let u, v be the other endvertices of the arcs incident with p_1, p_k respectively. Then $u = p_\ell$ or is a leaf adjacent to some p_ℓ in P and $v = p_j$ or is a leaf adjacent to some p_j in P. Since X has no arc between p_1, p_k and $p_1 u$ is an unbalanced edge of $U(X)$, $3 \leq \ell \leq k - 1$. Similarly, $2 \leq j \leq k - 2$. In $\overline{U(X)}$ the unique (p_1, v)-path avoids p_k and the unique (u, p_k)-path avoids p_1. If $k + \ell + j$ is even and P contains either both u, v or neither, or $k + \ell + j$ is odd and P contains exactly one of u, v, then the lengths of these two paths have opposite parities. By Proposition 1.5, $(p_1, u)\Gamma^*(v, p_k)$ and hence X or its dual contains arcs $(p_1, u), (p_k, v)$. Otherwise, the lengths of the two paths have the same parity and $(p_1, u)\Gamma^*(p_k, v)$ and X or its dual contains arcs $(p_1, u), (v, p_k)$. If $j < \ell - 1$, then the two arcs are opposing in $X - p_{\ell-1}$, contradicting that X is an obstruction. So $j \geq \ell - 1$. The minimality of X ensures that $\overline{U(X)}$ contains no other vertices. Therefore statement (iv) holds. □

3.3 $\overline{U(X)}$ contains a C_3 but no induced C_4

We now examine obstructions X that do not contain cut-vertices and for which $\overline{U(X)}$ contains cycles. By Corollary 3.9, $\overline{U(X)}$ is connected. We know from Lemma 3.3 that any induced cycle in $\overline{U(X)}$ is of length 3, 4 or 5, and also from Lemma 3.7 that if $\overline{U(X)}$ contains an induced cycle of length 5 then it does not contain an induced cycle of length 3 or 4.

We divide our discussion into four cases: $\overline{U(X)}$ contains a C_3 but no induced C_4;

$\overline{U(X)}$ contains an induced C_4 but no C_3; $\overline{U(X)}$ contains both C_3 and an induced C_4; and $\overline{U(X)}$ contains an induced C_5. These four cases will be treated separately.

Lemma 3.12. *Let X be an obstruction that contains no cut-vertices. Suppose $\overline{U(X)}$ contains a C_3 but no induced C_4. Then the C_3 is the only cycle in $\overline{U(X)}$ and any vertex not on C_3 is a leaf adjacent to a vertex on C_3 and incident with an arc. Moreover, any vertex on C_3 is adjacent to a vertex not on it.*

Proof: Since $\overline{U(X)}$ contains a C_3 but no induced C_4, by Lemmas 3.3, 3.5, and 3.7, the C_3 is the unique cycle in $\overline{U(X)}$. Let $C_3 : v_1v_2v_3$ the unique cycle. Consider a vertex u that is not on C_3. By Lemma 3.4, u is adjacent to a vertex on C_3. Since the C_3 is the unique cycle in $\overline{U(X)}$, u must be a leaf. Clearly, u is not a cut-vertex of $\overline{U(X)}$ and by assumption is not a cut-vertex of $U(X)$. If u is an arc-balancing vertex, then u balances an arc incident with a vertex in the C_3. Thus the other two vertices of the C_3 must be adjacent to both endvertices of the arc, a contradiction to the fact the C_3 is the unique cycle in $\overline{U(X)}$. So u is not an arc-balancing vertex and therefore by Lemma 1.10 it is incident with an arc.

It remains to show that each vertex on the C_3 is adjacent to a vertex not on it. Suppose on the contrary that v_1 is not adjacent to a vertex not on the C_3. By Lemma 3.1, v_2 and v_3 each has two non-neighbours. Clearly, the non-neighbours of v_2 and of v_3 are not in the C_3. We know from the above they are endvertices of arcs. Since v_1 is adjacent to none of them, v_1 is not an arc-balancing vertex. By assumption v_1 is not a cut-vertex of $U(X)$. It cannot be a cut-vertex of $\overline{U(X)}$ because it is adjacent only to v_2, v_3 (which are adjacent). This is a contradiction to Lemma 1.10. □

Theorem 3.13. *Let X be an obstruction that contains no cut-vertices. Suppose $\overline{U(X)}$ contains a C_3 but no induced C_4. Then $\overline{U(X)}$ is one of the graphs in Figure 3.3 and X or its dual contains the dotted arcs.*

Figure 3.3: Obstructions X for which $\overline{U(X)}$ contains a C_3 but no induced C_4.

Proof: Suppose X is an obstruction. Let C_3 : $v_1v_2v_3$ be the unique C_3 in $\overline{U(X)}$. By Lemma 3.12, each vertex of the C_3 is adjacent to a vertex not on it and each vertex not on the C_3 is a leaf adjacent to a vertex of the C_3. Let u, v, w be vertices adjacent to v_1, v_2, v_3

respectively but not on the C_3. By Lemma 3.12, each of u, v, w is incident with an arc. Since X contains exactly two arcs, there must be an arc with both endvertices among u, v, w. Without loss of generality, assume there is an arc between u and v. By possibly considering the dual of X, let (u, v) be an arc. On the other hand, let z denote the other endvertex of the arc incident with w. First suppose z is on the C_3. Then $z \in \{v_1, v_2\}$. Without loss of generality, assume $z = v_1$. Then $(u, v)\Gamma(v_2, u)\Gamma(u, v_3)\Gamma(w, u)\Gamma(v_1, w) = (z, w)$. Since the two arcs in X are opposing, the second arc must be $(w, z) = (w, v_1)$. Thus $\overline{U(X)}$ is Figure 3.3(i). Otherwise, z is not on the C. By Lemma 3.12, z is a leaf adjacent to a vertex on C. Clearly, z cannot be adjacent to v_3 because otherwise the arc between w and z would be balanced. Hence, assume without loss of generality that z is adjacent to v_1. If $z = u$, then u, z belong to one component of $\overline{U(X - v_1)}$ and v, w belong to another, so uv and wz belong to the same implication class of $U(X - v_1)$, contradicting Theorem 1.9. Hence, $z \neq u$. In this case, we have $(u, v)\Gamma^*(w, u)\Gamma(v_1, w)\Gamma(w, z)$. Hence, the second arc is (z, w). Thus $\overline{U(X)}$ is Figure 3.3(ii). □

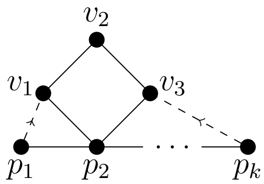
3.4 $\overline{U(X)}$ contains an induced C_4 but no C_3

We consider next the case when $\overline{U(X)}$ contains an induced C_4 but no C_3. Since $U(X)$ is a proper circular-arc graph, by Theorem 1.4 any induced C_4 in $\overline{U(X)}$ contains at most two cut-vertices of $\overline{U(X)}$.

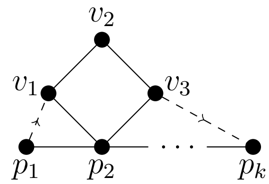
Theorem 3.14. *Let X be an obstruction that contains no cut-vertices. Suppose $\overline{U(X)}$ contains a unique induced C_4 but no C_3. Then $\overline{U(X)}$ is one of the graphs in Figure 3.4 and X or its dual contains the dotted arcs.*

Proof: Let $C : v_1v_3v_3v_4$ be the unique induced C_4 in $\overline{U(X)}$. Since $U(X)$ and $\overline{U(X)}$ are both connected, at least one vertex on C is adjacent to a vertex not on C. Moreover, since C is the unique cycle in $\overline{U(X)}$, any vertex on C that is adjacent to a vertex not on C is a cut-vertex of $\overline{U(X)}$. So C contains at least one cut-vertex.

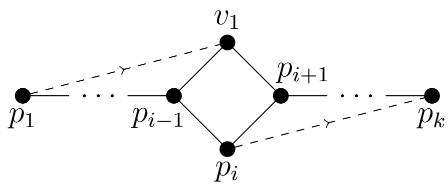
Suppose that only one vertex on C is a cut-vertex of $\overline{U(X)}$. Without loss of generality assume v_4 is such a vertex. We claim that v_1, v_3 are incident with different arcs. Indeed, since v_1, v_3 are not cut-vertices, by Proposition 1.10 they are either incident with arcs or arc-balancing vertices. If v_1 is an arc-balancing vertex, then it balances an arc incident with v_2 or v_4. Note that v_3 is adjacent to both v_2 and v_4 so v_3 must be adjacent to the other endvertex of the arc balanced by v_1, which is not possible. Hence v_1 is not an arc-balancing vertex. By symmetry v_3 is not an arc-balancing vertex either. Therefore each of v_1, v_3 is incident with an arc. Since v_1, v_3 have the same neighbourhood, there cannot be an arc between v_1, v_3, which implies that v_1, v_3 are incident with different arcs as claimed.



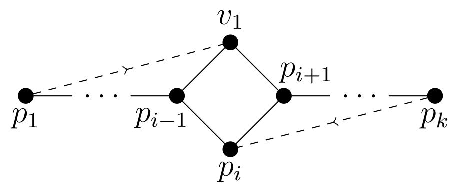
(i): k is even.



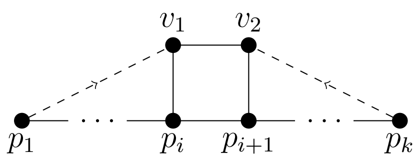
(ii): k is odd.



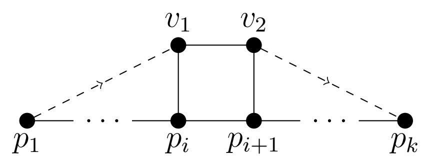
(iii): k is even.



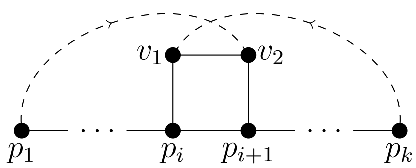
(iv): k is odd.



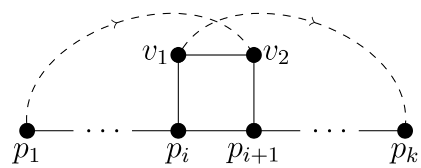
(v): k is even.



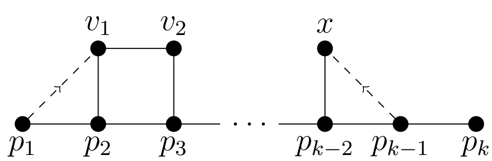
(vi): k is odd.



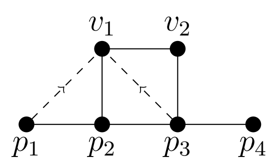
(vii): k is even.



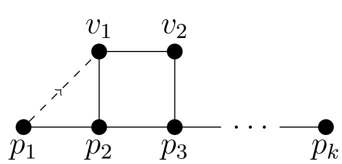
(viii): k is odd.



(ix): $k > 4$.



(x): $k = 4$.



(xi): The second arc is between p_k and any vertex $x \notin \{v_2, p_1, p_{k-1}, p_k\}$. If $k + d(p_1, x)$ is even, then (x, p_k) is an arc. Otherwise, (p_k, x) is an arc.

Figure 3.4: Obstructions X for which $\overline{U(X)}$ contains a unique induced C_4 but no C_3.

Let u, w denote the other endvertices of the arcs incident with v_1, v_3 respectively. Clearly, u, w are not on C. Since v_4 is the unique cut-vertex on C, each of u, w belongs to a component of $\overline{U(X - v_4)}$ that does not contain a vertex of C. According to Theorem 1.9, v_1u and v_3w belong to different implication classes of $U(X - v_4)$. Since v_1, v_3 are in the same component of $\overline{U(X - v_4)}$, u, w are in different components of $\overline{U(X - v_4)}$ by Theorem 1.7. The vertex v_2 is not a cut-vertex so it is an arc-balancing vertex by Proposition 1.10. Without loss of generality, assume v_2 balances the arc between v_1 and u. Then u must be a leaf adjacent to v_4, as otherwise there is a vertex adjacent to u but not to v_1, a contradiction to the fact v_2 balances the arc between v_1 and u. Let $P : u = p_1, p_2, \ldots, p_k = w$ be a shortest (u, w)-path. Such a path exists because $\overline{U(X)}$ is connected. It is easy to see that $p_2 = v_4$. By possibly considering the dual of X, assume (p_1, v_1) is an arc of X. If k is even, then $(p_1, v_1)\Gamma(v_2, p_1)\Gamma(p_2, v_2)\Gamma^*(p_k, v_2)\Gamma(v_3, p_k)$ by Proposition 1.5. The two arcs of X are opposing, so the second arc is (p_k, v_3). The minimality of X ensures $\overline{U(X)}$ is Figure 3.4(i) and X contains the dotted arcs. Otherwise, k is odd and the second arc is (v_3, p_k), so $\overline{U(X)}$ is Figure 3.4(ii) and X contains the dotted arcs.

Suppose that exactly two vertices of C are cut-vertices of $\overline{U(X)}$. We consider first the case when the two cut-vertices of $\overline{U(X)}$ on C are non-consecutive, say v_2 and v_4. We claim that v_1, v_3 are incident with different arcs. Since v_1, v_3 are not cut-vertices, neither of them is adjacent to any vertex not on C. In particular, if v_1 is an arc-balancing vertex, it must balance an arc incident with v_2 or v_4, and the other endvertex is adjacent to v_3 but not to v_1. Such a vertex does not exist, so v_1 is not an arc-balancing vertex. Similarly, v_3 is not an arc-balancing vertex. By Proposition 1.10, v_1, v_3 are incident with arcs. Moreover, since v_1, v_3 share the same neighbourhood, they must be incident with different arcs as claimed. Let H_1 denote a component of $\overline{U(X - v_2)}$ not containing vertices on C, and H_2 denote a component of $\overline{U(X - v_4)}$ not containing vertices on C. Since C is the unique cycle in $\overline{U(X)}$, H_1, H_2 are vertex-disjoint trees. Let u, w be leaves of $\overline{U(X)}$ in H_1, H_2 respectively. Clearly, neither u nor w can balance the arc incident with v_1 because otherwise the other endvertex would be adjacent to both of v_2, v_4 and thus would be v_3, a contradiction to the fact that v_1, v_3 are incident with different arcs. Similarly, neither u nor w can balance the arc incident with v_3. Hence each of u, w is incident with an arc by Proposition 1.10. Without loss of generality, assume there is an arc between u, v_3 and an arc between w, v_1. By the choice of u and w, there is a (w, u)-path that contains v_3 but not v_1. Let $P : w = p_1, \ldots, p_k = u$ be a shortest (w, u)-path where $p_{i-1} = v_4$, $p_i = v_3$, and $p_{i+1} = v_2$ for some i. By possibly considering the dual of X, assume $(w, v_1) = (p_1, v_1)$ is an arc. Suppose k is even. If i is even, $(p_1, v_1)\Gamma^*(p_1, p_k)\Gamma^*(p_k, p_i)$ by Proposition 1.5. So the second arc is $(p_i, p_k) = (v_3, u)$. If instead i is odd, then $(p_1, v_1)\Gamma^*(p_k, p_1)\Gamma^*(p_k, p_i)$. The second arc is again $(p_i, p_k) = (v_3, u)$. The minimality of X ensures $\overline{U(X)}$ is Figure 3.4(iii) and X contains the dotted arcs. Otherwise, k is odd and $\overline{U(X)}$ is Figure 3.4(iv) and X contains the dotted arcs.

We now consider the case when the two cut-vertices of $\overline{U(X)}$ on C are consecutive, say v_3 and v_4. First suppose both v_1 and v_2 are incident with arcs. Clearly, v_1, v_2 are incident with different arcs. Let u, w be the other two endvertices of the arcs. By a similar argument as above, u, w are leaves in components of $\overline{U(X-p_3)}, \overline{U(X-p_4)}$ respectively. Let $P : w = p_1 \dots p_k = u$ be a shortest (w, u)-path where $p_i = v_4$ and $p_{i+1} = v_3$ for some i. There are two possibilities: either w or u is the endvertex of the arc incident with v_1. Suppose there is an arc between w and v_1. By possibly considering the dual of X, assume $(p_1, v_1) = (w, v_1$ is an arc in X. Suppose k is even. If i is odd, then Proposition 1.5 implies $(p_1, v_1)\Gamma^*(p_1, p_k)\Gamma^*(v_2, p_k)$. If i is even, then $(p_1, v_1)\Gamma^*(p_k, p_1)\Gamma^*(v_2, p_k)$. In either case, the second arc is (p_k, v_2), so $\overline{U(X)}$ is Figure 3.4(v) and X contains the dotted arcs. Otherwise, k is odd and $\overline{U(X)}$ is Figure 3.4(vi) and X contains the dotted arcs. On the other hand, suppose (p_1, v_2) is an arc in X. Suppose k is even. Then we have $(p_1, v_2)\Gamma^*(p_1, p_k)\Gamma^*(v_1, p_k)$ if i is even, and $(p_1, v_2)\Gamma^*(p_k, p_1)\Gamma^*(v_1, p_k)$ if i is odd. In either case, the second arc is (p_k, v_1), so $\overline{U(X)}$ is Figure 3.4(vii) and X contains the dotted arcs. Otherwise, k is odd and $\overline{U(X)}$ is Figure 3.4(viii) and X contains the dotted arcs.

Suppose that one of v_1, v_2 is not incident with an arc. Without loss of generality, assume it is v_2. Then v_2 is an arc-balancing vertex by Proposition 1.10. Since v_3 is a cut-vertex, it is adjacent to a vertex x not on C. So, if v_2 balances an arc incident with v_3, then the other endvertex must be adjacent to both v_4 and x, contradicting the fact that C is the unique cycle. Hence v_2 balances an arc incident with v_1. Since v_1 is adjacent only to v_2 and v_4, the other endvertex w is a leaf adjacent to v_4. Without loss of generality, assume (w, v_1) is an arc. Since v_3 is a cut-vertex, there is a component H of $\overline{U(X-v_3)}$ not containing the vertices on C. Let u be a vertex of maximal distance from v_3 in H, and let $P : w = p_1 \dots p_k = u$ be a shortest (w, u)-path in $\overline{U(X)}$. Clearly, $p_2 = v_4$ and $p_3 = v_3$. Moreover, since C is the unique cycle and u is of maximal distance from v_3 in H, u is a leaf. First suppose u balances an arc incident with p_{k-1}. There are two cases depending on whether or not $k > 4$. If $k > 4$, then $p_{k-1} \neq v_3$, so the other endvertex is a leaf x adjacent to p_{k-2}. If $k = 4$, then $p_{k-1} = v_3$, so the other endvertex is v_1, because it must be adjacent to both v_2 and v_4 and C is the unique cycle. In either case, we have $d(v_1, p_k) + d(p_1, x) = 2k-3$, so one of $d(v_1, p_k), d(p_1, x)$ is even and the other is odd. Otherwise if $d(v_1, p_k)$ is even and $d(p_1, x)$ is odd, then Proposition 1.5 implies $(p_1, v_1)\Gamma^*(p_1, p_k)\Gamma^*(p_k, x)\Gamma(x, p_{k-1})$. If $d(v_1, p_k)$ is odd and $d(p_1, x)$ is even, then $(p_1, v_1)\Gamma^*(p_k, p_1)\Gamma^*(p_k, x)\Gamma(x, p_{k-1})$. In either case, the second arc must be (p_{k-1}, x). Thus, $\overline{U(X)}$ is Figure 3.4(ix) if $k > 4$ and is Figure 3.4(x) if $k = 4$, and X contains the dotted arcs.

Otherwise, u is incident with an arc by Proposition 1.10. Let x denote the other endvertex. Since v_2 is not incident with an arc, $x \neq v_2$. Theorem 1.9 implies p_1v_1 and p_kx belong to different implication classes of $U(X-p_2)$, so $x \neq p_1$ by Theorem 1.7. Thus, $x \notin \{v_2, p_1, p_{k-1}, p_k\}$. Suppose $k+d(p_1, x)$ is even. Since $d(v_1, p_k)+d(p_1, x) = (k-1)+d(p_1, x)$, one of $d(v_1, p_k)$ and $d(p_1, x)$ is even and the other is odd. If $d(v_1, p_k)$ is even and $d(p_1, x)$ is odd, then Proposition 1.5 implies $(p_1, v_1)\Gamma^*(p_1, p_k)\Gamma^*(p_k, x)$. Otherwise if $d(v_1, p_k)$ is odd and $d(p_1, x)$ is even, then $(p_1, v_1)\Gamma^*(p_k, p_1)\Gamma^*(p_k, x)$. In either case, the second arc is (x, p_k). Otherwise, $k + d(p_1, x)$ is odd and the second arc is (p_k, x). So, $\overline{U(X)}$ is Figure 3.4(xi) and X contains the dotted arcs. □

Theorem 3.15. *Let X be an obstruction that has no cut-vertices. Suppose that $\overline{U(X)}$ contains two induced C_4's but no C_3. Then $\overline{U(X)}$ is one of the graphs in Figure 3.5 and X or its dual contains the dotted arcs.*

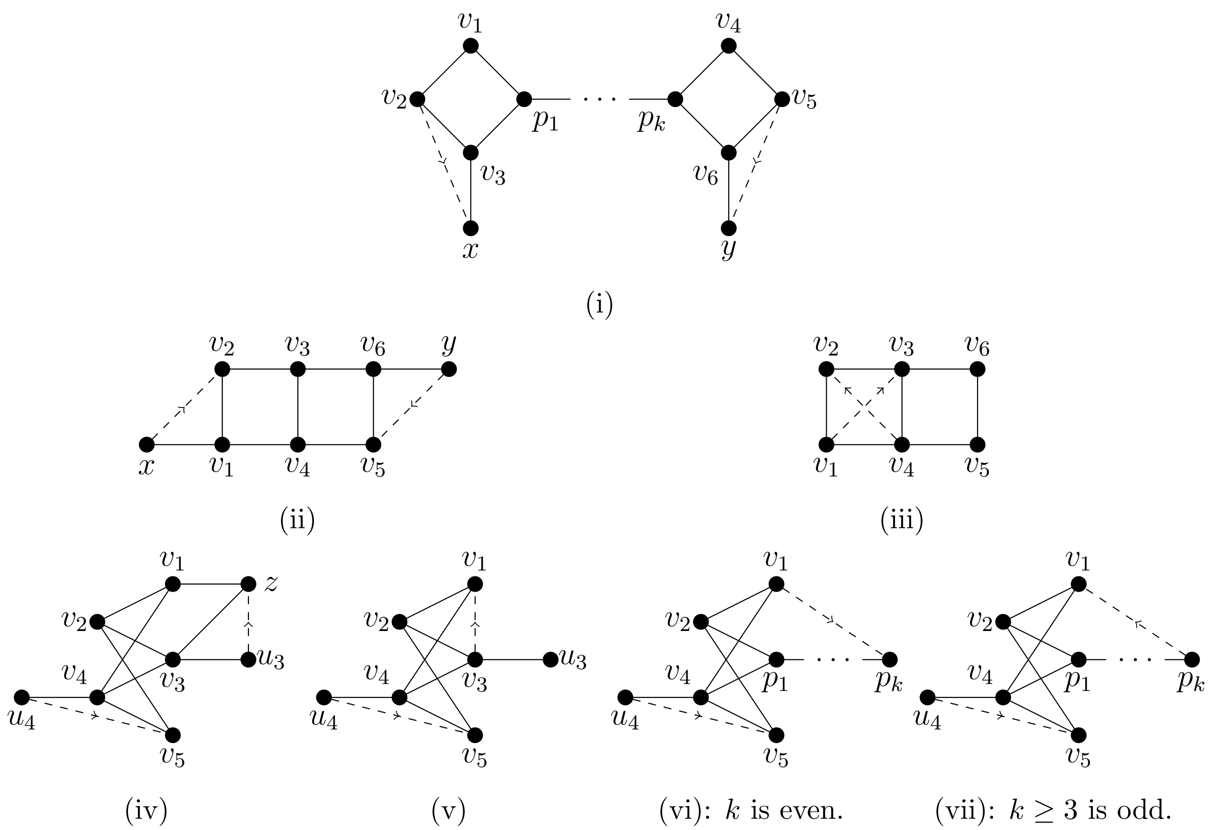


Figure 3.5: Obstructions X for which $\overline{U(X)}$ contains two induced C_4 but no C_3.

Proof: Suppose there are two induced C_4's in $\overline{U(X)}$ which share at most one common vertex. Let C and C' be such induced C_4's and let $P : p_1p_2 \dots p_k$ be a shortest path between a vertex of C and a vertex of C'. By Lemma 3.2 any connected subgraph of $\overline{U(X)}$ has at most six non-cut-vertices. The (connected) subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup P$ has at least six non-cut-vertices and thus has exactly six non-cut-vertices. This implies that P is the unique path between C and C' and each p_i of P is a cut-vertex of $\overline{U(X)}$. Since the subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup P$ has six non-cut-vertices, $\overline{U(X)}$ also has six non-cut-vertices according to Lemma 3.2. Thus by Corollary 1.11 the six non-cut-vertices of $\overline{U(X)}$ form two disjoint arc-balancing triples.

Denote $C : v_1v_2v_3p_1$ and $C' : v_4v_5v_6p_k$. We first show that v_1 is not incident with an arc. Suppose there is an arc between v_1 and a vertex z. Since p_1 is a cut-vertex of $\overline{U(X)}$, it does not balance the arc between v_1 and z. Since p_1 adjacent to v_1, it is adjacent to z. If z is not in $C \cup C' \cup P$, then the subgraph induced by $C \cup C' \cup P \cup \{z\}$ contains seven non-cut-vertices (i.e., $v_1, v_2, v_3, v_4, v_5, v_6, z$), which contradicts Lemma 3.2. So z is in $C \cup C' \cup P$. Note that z is adjacent to p_1. If $z \neq v_3$, then v_2 is adjacent to v_1 but not z and there is a vertex in $C \cup C' \cup P$ adjacent to z but not v_1, a contradiction to the fact that v_1 and z are in a an arc-balancing triple. Thus $z = v_3$. But then the vertex v which balances the arc between v_1 and z cannot be in $C \cup C' \cup P$. Assume without loss of generality that v is adjacent to v_1 but not to z. Since v_1 is incident with an arc, it is not a cut-vertex of $\overline{U(X)}$. So $\overline{U(X)} - v_1$ has a (v, v_3)-path Q. The connected subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup P \cup Q$ contains seven non-cut-vertices (i.e., $v_1, v_2, v_3, v_4, v_5, v_6, v$), a contradiction to Lemma 3.2. Therefore v_1 is not incident with an arc. By symmetry, none of v_3, v_4, v_6 is incident with an arc.

Since p_1 is a cut-vertex and any induced C_4 in $\overline{U(X)}$ contains at most two cut-vertices of $\overline{U(X)}$, v_1, v_3 cannot both be cut-vertices of $\overline{U(X)}$. Moreover, we know from above that neither of v_1, v_3 is incident with an arc so Proposition 1.10 implies that one of v_1, v_3 is an arc-balancing vertex. Similarly, one of v_4, v_6 is an arc-balancing vertex. Hence one of v_1, v_3 is an arc-balancing vertex and the other is a cut-vertex of $\overline{U(X)}$. Without loss of generality, assume v_1 is an arc-balancing vertex and v_3 is a cut-vertex of $\overline{U(X)}$. The vertex v_1 is adjacent to exactly one endvertex u of the arc it balances. We claim that $u = v_2$. If u is not in $C \cup C' \cup P$, then $\overline{U(X)}$ contains a (u, v_2)-path Q not containing v_1 because v_1 is a non-cut-vertex. So the connected subgraph induced by $C \cup C' \cup P \cup Q$ contains seven non-cut-vertices, contradicting Lemma 3.2. So u is in $C \cup C' \cup P$. Since p_1 is a cut-vertex and no cut-vertex of $\overline{U(X)}$ is arc-balancing or incident with an arc, $u \neq p_1$. Hence $u = v_2$ as claimed, and v_1 balances an arc incident with v_2. The other endvertex x must therefore be outside of $C \cup C' \cup P$ and adjacent to v_3.

By symmetry, v_4 balances an arc between v_5 and a vertex y outside of $C \cup C' \cup P$ and adjacent to v_6. By possibly taking the dual of X assume (v_2, x) is an arc in X. If k is even, then Proposition 1.5 implies $(p_1, x)\Gamma^*(x, p_k)$ and $(p_1, y)\Gamma^*(y, p_k)$. Hence

$$(v_2, x)\Gamma(x, v_1)\Gamma(p_1, x)\Gamma^*(x, p_k)\Gamma(v_6, x)\Gamma(x, y)\Gamma(y, v_3)\Gamma(p_1, y)\Gamma^*(y, p_k)\Gamma(v_4, y)\Gamma(y, v_5)$$

and so the second arc is (v_5, y). Otherwise, k is odd and in this case, we have

$$(v_2, x)\Gamma(x, v_1)\Gamma(p_1, x)\Gamma^*(p_k, x)\Gamma(x, v_6)\Gamma(y, x)\Gamma(v_3, y)\Gamma(y, p_1)\Gamma^*(y, p_k)\Gamma(v_4, y)\Gamma(y, v_5)$$

and the second arc is again (v_5, y). The minimality of X ensure that $\overline{U(X)}$ is the graph in Figure 3.5(i) and X contains the dotted arcs.

Suppose next there are two induced C_4's in $\overline{U(X)}$ which share two common vertices but no two induced C_4's in $\overline{U(X)}$ share three common vertices. Then such two C_4's must share an edge. Let $C : v_1v_2v_3v_4$ and $C' : v_3v_4v_5v_6$ be such induced C_4's in $\overline{U(X)}$. Since $\overline{U(X)}$ contains no C_3 and no two induced C_4's share three vertices, the subgraph induced by $C \cup C'$ has exactly seven edges belonging to the two C_4's. We claim that no vertex outside of $C \cup C'$ is adjacent to v_3 or v_4. Indeed, if some vertex z outside of $C \cup C'$ is adjacent to v_3 or v_4, then it must be adjacent to at least two vertices in $C \cup C'$, because otherwise Theorem 1.4 would imply that $U(X)$ is not a proper circular-arc graph. But then $C \cup C' \cup \{z\}$ would induced a connected subgraph in $\overline{U(X)}$ having seven non-cut-vertices, a contradiction to Lemma 3.2.

Suppose both v_3 and v_4 are arc-balancing vertices. If v_3 balances an arc incident with v_4, then the other endvertex z of the arc is not in $C \cup C'$ that is adjacent to both v_1 and v_5. Thus the (connected) subgraph of $\overline{U(X)}$ induced by $C \cup C' \cup \{z\}$ has seven non-cut-vertices, contradicting Lemma 3.2. Hence v_3 does not balance an arc incident with v_4. Moreover, since no vertex outside of $C \cup C'$ is adjacent to v_3, v_3 must balance an arc incident with v_2 or v_6. Similarly, v_4 must balance an arc incident with v_1 or v_5. Without loss of generality, assume v_3 balances an arc incident with v_2. Thus the other endvertex x is a vertex whose only neighbour in $C \cup C'$ is v_1. We claim v_4 balances an arc incident with v_5. Otherwise, v_4 balances an arc incident with v_1, so the other endvertex of the arc has v_2 as the only neighbour in $C \cup C'$. Clearly, either v_5 is a non-cut-vertex or $\overline{U(X)}$ has a non-cut-vertex in a component of $\overline{U(X - v_5)}$ not containing vertices in $C \cup C'$. In either case, $\overline{U(X)}$ contains a non-cut-vertex that is neither an endvertex of an arc nor an arc-balancing vertex, a contradiction by Proposition 1.10. Hence v_4 balances an arc incident with v_5 as claimed. The other endvertex y of the arc has v_6 as the only neighbour in $C \cup C'$. By possibly considering the dual of X, assume (x, v_2) is an arc in X. Since $(x, v_2)\Gamma(v_3, x)\Gamma(x, v_6)\Gamma(y, x)\Gamma(v_1, y)\Gamma(y, v_4)\Gamma(v_5, y)$, the second arc is (y, v_5). The minimality of X ensures that $\overline{U(X)}$ is Figure 3.5(ii) and X contains the dotted arcs.

Suppose at least one of v_3, v_4 is not an arc-balancing vertex. Without loss of generality assume that v_3 is not an arc-balancing vertex. Then by Proposition 1.10, v_3 must be incident with an arc. The subgraph of $\overline{U(X)}$ induced by $C \cup C'$ has six non-cut-vertices so by Lemma 3.2 $\overline{U(X)}$ has six non-cut-vertices. Corollary 1.11 implies that the six non-cut-vertices of $\overline{U(X)}$ form two disjoint arc-balancing triples. Since v_3 has three neighbours in $C \cup C'$ and the arc incident with v_3 has an arc-balancing vertex, the other endvertex must be adjacent to at least two of the three neighbours of v_3 in $C \cup C'$. Since any connected subgraph of $\overline{U(X)}$ has at most six non-cut-vertices by Lemma 3.2, so any vertex not in $C \cup C'$ is adjacent to at most one vertex in $C \cup C'$. It follows that the other endvertex of the arc incident with v_3 is in $C \cup C'$. Without loss of generality, assume (v_1, v_3) is an arc in X. Clearly, v_6 is the (v_1, v_3)-balancing vertex. We claim that v_4 is incident with an arc. Otherwise, Proposition 1.10 would imply v_4 is an arc-balancing vertex and hence balances an arc incident with v_5. By the above, the other endvertex

x of the arc incident with v_5 is adjacent to v_6. Clearly, either v_2 is a non-cut-vertex or $\overline{U(X)}$ has a non-cut-vertex in a component of $\overline{U(X-v_2)}$ not containing vertices in $C \cup C'$. In either case, $\overline{U(X)}$ contains a non-cut-vertex that does not belong to either arc-balancing triple, a contradiction by Proposition 1.10. Therefore v_4 must be incident with an arc. By a similar argument as above, the other endvertex of the arc incident with v_4 is in $C \cup C'$. Since $\overline{U(X)}$ contains two disjoint arc-balancing triples and v_6 is the (v_1, v_3)-balancing vertex, the other endvertex cannot be v_6 and hence must be v_2. Since $(v_1, v_3)\Gamma(v_6, v_1)\Gamma(v_2, v_6)\Gamma(v_5, v_2)\Gamma(v_2, v_4)$, the second arc is (v_4, v_2). The minimality of X ensures that $\overline{U(X)}$ is Figure 3.5(iii) and X contains the dotted arcs.

Suppose now that there are two induced C_4's in $\overline{U(X)}$ which share three common vertices. Let $C : v_1v_2v_3v_4$ and $C' : v_2v_3v_4v_5$ be such induced C_4's. Note that $C \cup C'$ induces a $K_{2,3}$ in $\overline{U(X)}$. Each v_i with $1 \leq i \leq 5$ may or may not be a cut-vertex of $\overline{U(X)}$. If v_i is a cut-vertex of $\overline{U(X)}$, then $\overline{U(X-v_i)}$ must contain a non-cut-vertex of $\overline{U(X)}$ that is not in $C \cup C'$. Let u_i be such a vertex in $\overline{U(X-v_i)}$ when v_i is a cut-vertex; otherwise let $u_i = v_i$ for each $1 \leq i \leq 5$. First note that u_2, u_4 are non-adjacent and that u_1, u_3, u_5 are pairwise non-adjacent. Moreover, if $u_i \neq v_i$ then u_i is not adjacent to u_j for all $j \neq i$. Since each u_i is a non-cut-vertex, it is an endvertex of an arc or an arc-balancing vertex by Proposition 1.10. This implies that there is an arc-balancing triple T contained in $\{u_1, u_2, \ldots, u_5\}$. Since there is exactly one edge in T, we know from the above observation the only edge in T has one endvertex in $\{u_2, u_4\}$ and the other in $\{u_1, u_3, u_5\}$. Without loss of generality assume that u_2u_5 is the edge in T. Then we must have $u_2 = v_2$ and $u_5 = v_5$ and thus neither v_2 nor v_5 is a cut-vertex of $\overline{U(X)}$. It is easy to see that the third vertex of T is u_4 and v_2 balances the arc between u_4 and v_5. Without loss of generality assume (u_4, v_5) is an arc in X. Clearly, $u_4 \neq v_4$ and v_4 is a cut-vertex of $\overline{U(X)}$.

Since C has at most two cut-vertices of $\overline{U(X)}$, at most one of v_1, v_3 can be a cut-vertex. If neither of v_1, v_3 is a cut-vertex, then one of them is an endvertex of an arc which is balanced by the other vertex. Since v_2 is adjacent to both v_1, v_3, it is adjacent to the other endvertex of the arc, which implies $U(X)$ contains a C_3, a contradiction to assumption. So exactly one of v_1, v_3 is a cut-vertex of $\overline{U(X)}$ and we assume it is v_3. Suppose that there is an arc between v_1 and u_3. Let $P : v_3 = p_1 \ldots p_k = u_5$ be the shortest (v_3, u_5)-path in $\overline{U(X)}$. Suppose k is even. Then by Proposition 1.5,

$$(u_4, v_5)\Gamma(v_2, u_4)\Gamma(u_4, p_1)\Gamma^*(p_k, u_4)\Gamma(v_4, p_k)\Gamma(p_k, v_5)\Gamma(v_2, p_k)\Gamma(p_k, v_1)$$

and hence the second arc is (v_1, p_k). The minimality of X ensures $\overline{U(X)}$ is Figure 3.5(vi) and X contains the dotted arcs. Otherwise, k is odd and $\overline{U(X)}$ is Figure 3.5(vii) and X contains the dotted arcs.

Suppose that there is no arc between v_1 and u_3. Then either u_3 is incident with an arc balanced by v_1 or v_1 is incident with an arc balanced by u_3. Suppose it is the the former. Let z denote the other endvertex. Since v_1 is not adjacent to u_3, it is adjacent to z. Moreover, since u_3 is not adjacent to v_5, z is also not adjacent to v_5. In particular, $z \notin C \cup C'$. If a vertex other than v_1 is adjacent to z, then it must also be adjacent to u_3. In particular, the choice of u_3 implies that v_3 is the only vertex that is possibly adjacent to u_3. Since $\overline{U(X)}$ is connected, v_3 must be adjacent to u_3, so v_3 is adjacent to z as well. Since $(u_4, v_5)\Gamma(v_2, u_4)\Gamma(u_4, v_3)\Gamma(u_3, u_5)\Gamma(v_4, u_3)\Gamma(u_3, v_1)\Gamma(z, u_3)$, the second arc is (u_3, z). The minimality of X ensures $\overline{U(X)}$ is Figure 3.5(iv) and X contains the dotted arcs.

Suppose instead that v_1 is incident with an arc balanced by u_3. Since u_3 is not adjacent to v_1, it is adjacent to the other endvertex. Moreover, since v_1 is adjacent to v_2 and v_4, the other endvertex is also adjacent to v_2 and v_4. By the choice of u_3, the only vertex that can be adjacent to all of u_3, v_2, v_4 is v_3, so the other endvertex is v_3 and u_3 is adjacent to v_3. Since $(u_4, v_5)\Gamma(v_2, u_4)\Gamma(u_4, v_3)\Gamma(u_3, u_4)\Gamma(v_4, u_3)\Gamma(u_3, v_1)\Gamma(v_1, v_3)$, the second arc is (v_3, v_1). The minimality of X ensures $\overline{U(X)}$ is Figure 3.5(v) and X contains the dotted arcs. □

3.5 $\overline{U(X)}$ contains a C_3 and an induced C_4

Lemma 3.16. *Let X be an obstruction which has no no cut-vertices. Suppose $\overline{U(X)}$ contains a C_3 and an induced C_4. Then $\overline{U(X)}$ contains a unique C_3 and a unique induced C_4, which share two common vertices. Moreover, each vertex not in any of the cycles is a leaf adjacent to a vertex on the C_3 and is incident with an arc.*

Proof: By Lemmas 3.4 and 3.5, $\overline{U(X)}$ contains a unique C_3 and each vertex not on the C_3 is adjacent to a vertex in the C_3. It follows that each vertex not on the C_3 is adjacent to exactly one vertex on the C_3. We show that if C is an induced C_4 in $\overline{U(X)}$ then C shares exactly two vertices with the C_3. Clearly, C share at most two vertices with the C_3. The fact that every vertex not on the C_3 is adjacent to a vertex in the C_3 implies that C cannot share exactly one vertex with the C_3. If C shares no vertex with the C_3, then $C \cup C_3$ induces a connected subgraph in $\overline{U(X)}$ with seven non-cut-vertices, a contradiction to Lemma 3.2.

Denote the unique C_3 in $\overline{U(X)}$ by $v_1v_2v_3$ and without loss of generality assume that $v_2v_3v_4v_5$ is an induced C_4 in the graph. Let $u \notin \{v_1, v_2, \ldots, v_5\}$. From the above we know that u is adjacent to exactly one vertex in the C_3. Suppose that u is adjacent to v_1. Then u cannot be adjacent to both v_4, v_5 as otherwise uv_4v_5 is another C_3 in $\overline{U(X)}$, a contradiction. If u is adjacent to one of v_4, v_5 then $uv_1v_2v_5v_4$ or $uv_1v_3v_4v_5$ is an induced C_5 in $\overline{U(X)}$, which contradicts Lemma 3.7. If u is adjacent to a vertex $w \notin \{v_1, v_2, \ldots, v_5\}$, then w is not adjacent to v_1 due to the uniqueness of the C_3 and so is adjacent to v_2 or v_3. But then $\{u, w, v_1, v_2, \ldots, v_5\}$ induces a connected subgraph of $\overline{U(X)}$ with seven non-cut-vertices, a contradiction to Lemma 3.2. Hence u is a leaf in $\overline{U(X)}$. Suppose that u is not adjacent to v_1. Then it is adjacent to v_2 or v_3. By symmetry we assume u is adjacent to v_2. It is not adjacent to v_5 as otherwise uv_2v_5 is another C_3 in $\overline{U(X)}$. It is not adjacent to v_4 as otherwise $uv_4v_5v_2$ is an induced C_4 which share just one vertex (namely, v_2) with the C_3. Suppose that u is adjacent to a vertex $w \notin \{v_1, v_2, \ldots, v_5\}$. Then w is not adjacent to v_2 due to the uniqueness of the C_3. It is not adjacent to v_1 because from the above any such vertex is a leaf. So w is adjacent to v_3. But then $\{u, w, v_1, v_2, \ldots, v_5\}$ induces a connected subgraph of $\overline{U(X)}$ with seven non-cut-vertices, a contradiction to Lemma 3.2. Therefore any vertex $u \notin \{v_1, v_2, \ldots, v_5\}$ is a leaf adjacent to a vertex in the C_3. It follows that $v_2v_3v_4v_4$ is the unique induced C_4 in $\overline{U(X)}$. Moreover, such a u is not a cut-vertex of $U(X)$ or of $\overline{U(X)}$ and cannot be an arc-balancing vertex. So by Proposition 1.10 u is incident with an arc. □

Theorem 3.17. *Let X be an obstruction that contains no cut-vertices. Suppose $\overline{U(X)}$ contains a C_3 and an induced C_4. Then $\overline{U(X)}$ is one of the graphs in Figure 3.6 and X or its dual contains the dotted arcs.*

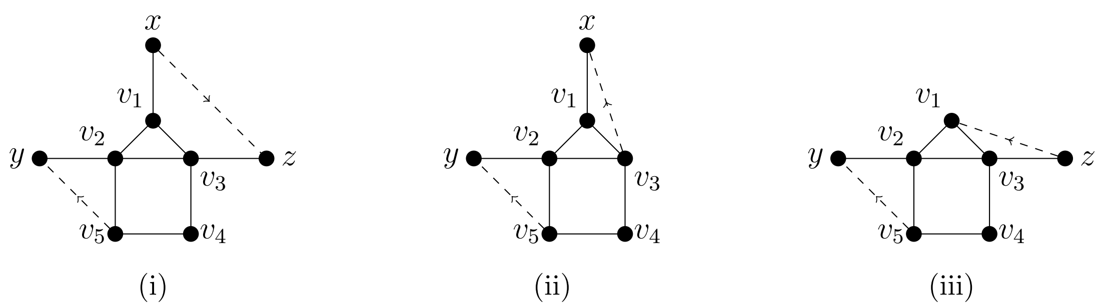


Figure 3.6: Obstructions X for which $\overline{U(X)}$ contains a C_3 and an induced C_4.

Proof: By Lemma 3.16, $\overline{U(X)}$ contains a unique C_3 and a unique C_4 sharing two vertices. Denote the C_3 and the induced C_4 by $v_1v_2v_3$ and $v_2v_3v_4v_5$ respectively. We also know from the lemma that every vertex not in any of the cycles is a leaf adjacent to a vertex in the C_3 and is incident with an arc. We claim that at least two vertices of the C_3 are neighbours of leaves. Indeed, since $U(X)$ does not contain cut-vertices and $v_1, v_2, \ldots, v_5$ induce a path in $U(X)$, there must be at least one vertex not on the cycles, which implies at least one vertex on the C_3 is adjacent to a leaf. If v_2 is the only vertex in the C_3 adjacent to a leaf. Then v_2 is adjacent to every vertex except v_4 in $\overline{U(X)}$, a contradiction to Lemma 3.1. So v_2 cannot be the only vertex in the C_3 adjacent to a leaf. By symmetry, v_3 cannot be the only vertex in the C_3 adjacent to a leaf.

Suppose v_1 is the only vertex in the C_3 adjacent to a leaf. Let u be a leaf of $\overline{U(X)}$ adjacent to v_1. By Lemma 3.16 u is incident with an arc. The other endvertex of this arc cannot be another leaf v as otherwise uv is a balanced edge in $U(X)$, a contradiction. So the other endvertex of this arc must be among v_2, v_3, v_4, v_5. Note first that none of

v_2, v_3, v_4, v_5 is a cut-vertex. If the arc is between u and v_2, then v_4 does not balance this arc as it is adjacent to neither of the endvertices. So v_4 either balances or is incident with the second arc. If v_4 is incident with the second arc then v_3 and v_5 are arc-balancing vertices, which is not possible. If v_4 balances the second arc, then the second arc must be incident with exactly one of v_3, v_5. But then v_2 is also a vertex adjacent to exactly one of the endvertices of the second arc, which is again impossible. This shows there is no arc between u and v_2. A similar argument shows that there is no arc between u and any of v_3, v_4, v_5. Therefore at least two vertices of the C_3 are neighbours of leaves.

Suppose that all three vertices of the C_3 are neighbours of leaves. Let x, y, z be leaves adjacent to v_1, v_2, v_3 respectively. By Lemma 3.16 each of x, y, z is incident with an arc. So there is an arc between two of x, y, z. Suppose there is an arc between y and z. By Lemma 3.4, neither of v_4, v_5 can be a cut-vertex so each of them is an arc-balancing vertex or incident with an arc. At least one of v_4, v_5 must be an arc-balancing vertex because otherwise x, y, z, v_4, v_5 would be five vertices incident with arcs. Without loss of generality, assume v_4 is an arc-balancing vertex. Since v_4 is adjacent to neither of y, z, it cannot balance the arc between y and z. Thus v_4 balances an arc between x and x_3 or between x and v_5. This is a contradiction because v_2 is another vertex adjacent to exactly one of endvertices of the arc balanced by v_4. Thus there is no arc between y and z. So there is an arc between x and y or between x and z. By symmetry and taking the dual of X if necessary we may assume that (x, z) is an arc. Since v_1 and v_3 are two vertices adjacent to exactly one of x, z, there cannot be an (x, z)-balancing vertex. By Lemma 3.4, v_4, v_5 are non-cut-vertices of $\overline{U(X)}$ so each of them is arc-balancing or incident with an arc by Proposition 1.10. Clearly, none of them can be an (x, z)-balancing vertex. So either v_4 balances the arc between y and v_5 or v_5 balances the arc between y and v_4. The latter case is not possible because v_3 is adjacent to v_4 but not to y. Hence there is an arc between y and v_5. Since the two arcs of X are opposing and $(x, z)\Gamma(z, v_1)\Gamma(v_2, z)\Gamma(z, y)\Gamma(y, v_3)\Gamma(v_4, y)\Gamma(y, v_5)$, (v_5, y) is an arc. The minimality of X ensure that $\overline{U(X)}$ is the graph in Figure 3.6(i).

Suppose now that exactly two of v_1, v_2, v_3 are neighbours of leaves. First consider the case when v_1 and v_2 are neighbours of leaves. Let x, y be leaves adjacent to v_1, v_2 respectively. Clearly, v_3 is not a cut-vertex. Since v_4, v_5 are not the C_3, by Lemma 3.4 they are not cut-vertices. Hence, each of v_3, v_4, v_5 is an arc-balancing vertex or incident with an arc by Proposition 1.10. By Lemma 3.16, x, y are incident with arcs. So at least one of v_3, v_4, v_5 is an arc-balancing vertex. If v_3 is an arc-balancing vertex, then it must balance an arc incident with v_2 or v_4. But then v_5 is another vertex adjacent to exactly one endvertex of this arc, a contradiction. Hence v_3 is not the arc-balancing vertex. For a similar reason, v_5 is also not an arc-balancing vertex. Thus v_4 is an arc-balancing vertex. It is easy check an arc balanced by v_4 cannot be incident with v_3. So v_4 balances an arc incident with v_5. The other endvertex of this arc cannot be a leaf adjacent to v_1. Hence v_4 balances an arc between v_5 and a leaf adjacent to v_2. Without loss of generality assume it is between v_5 and y. By taking the dual of X if necessary we may assume (v_5, y) is an arc. Thus the second arc is between x and v_3. Since $(v_5, y)\Gamma(y, v_4)\Gamma(v_4, v_2)\Gamma(v_1, v_4)\Gamma(v_4, x)\Gamma(x, v_3)$ and the two arcs of X are opposing, the second arc is (v_3, x). So $\overline{U(X)}$ is Figure 3.6(ii).

The case when v_1 and v_3 are neighbours of leaves is symmetric to the case when v_1 and v_2 are cut-vertices. So we now consider the case where v_2 and v_3 are neighbours of leaves. Let y, z be leaves adjacent to v_2, v_3 respectively. By assumption, v_1 is not a cut-vertex. Since v_4, v_5 are not on the C_3, they are not cut-vertices by Lemma 3.4. Thus each of v_1, v_4, v_5 is an arc-balancing vertex or incident with an arc by Proposition 1.10. It follows that at least one of them is an arc-balancing vertex. A similar proof as above shows that v_4 balances an arc between v_5 and y. Without loss of generality, assume (v_5, y) is an arc. It is easy to see that v_1 is not an arc-balancing vertex so it is incident with an arc. So the second arc is between v_1 and z. Since $(v_5, y)\Gamma(y, v_4)\Gamma(v_3, y)\Gamma(y, z)\Gamma(z, v_2)\Gamma(v_1, z)$ and the two arcs are opposing, the second arc is (z, v_1). Hence $\overline{U(X)}$ is Figure 3.6(iii). □

3.6 $\overline{U(X)}$ contains an induced C_5

Lemma 3.18. *Let X be an obstruction which has no cut-vertices. If C is an induced C_5 in $\overline{U(X)}$, then the following statements hold:*

(a) C is the unique cycle in $\overline{U(X)}$;

(b) Each vertex not on C is a leaf adjacent to a vertex on C and incident with an arc;

(c) C contains an arc-balancing vertex that is not a cut-vertex of $\overline{U(X)}$;

(d) If v is an arc-balancing vertex on C, then v balances an arc between a neighbour of v on C and a leaf.

Proof: Let $C : v_1v_2v_3v_4v_5$ be an induced C_5 in $\overline{U(X)}$. By Lemma 3.3, C is a longest induced cycle in $\overline{U(X)}$. According to Lemmas 3.6 and 3.7, $\overline{U(X)}$ contains at most one induced C_5 and no C_3 nor induced C_4. Thus C is the unique cycle in $\overline{U(X)}$.

Suppose that u is a vertex not on C. Then by Lemma 3.4 u is adjacent to a vertex on C. Since C the unique cycle in $\overline{U(X)}$, u is a leaf and hence not a cut-vertex of $\overline{U(X)}$. Let v_i be the neighbour of u. If u is an arc-balancing vertex, then it balances an arc between v_i and some vertex w. Since v_i is adjacent to both v_{i-1}, v_{i+1} which do not balance the arc between v_i and w, w must be adjacent to both v_{i-1}, v_{i+1}. Thus $v_iv_{i+1}wv_{i-1}$ is a C_4, a contradiction to the fact C is the unique cycle in $\overline{U(X)}$. Hence u is incident with an arc by Proposition 1.10.

Suppose there are k cut-vertices on C. We know from above that each such vertex is adjacent to a leaf that is incident with an arc. Since there are at most four vertices incident with arcs, $k \leq 4$ and at most $4 - k$ vertices on C are incident with arcs. So there are at least $5 - (4 - k) = k + 1$ vertices on C which are not incident with arcs. It follows that C contains at least one vertex that is not a cut-vertex of $\overline{U(X)}$ and not incident with an arc. Such a vertex must be an arc-balancing vertex by Proposition 1.10. Hence C contains an arc-balancing vertex that is not a cut-vertex of $\overline{U(X)}$. Without loss of generality, assume v_1 is an arc-balancing vertex and it balances an arc incident with v_2. Clearly the other endvertex cannot be on C so it is a leaf of $\overline{U(X)}$.

Suppose that v a vertex C which balances an arc between u and w. If one of u, w is a leaf neighbour of v, then the other vertex has a neighbour that is not v but is adjacent to exactly one of u, w, a contradiction to the assumption that v balances the arc between u, w. So neither of u, w can be a leaf neighbour of v. Since v is adjacent to one of u, w, at least one of u, w is on C. If the other vertex is also on C, then there is a vertex on C which is not v but is adjacent to exactly one of u, w, a contradiction. Therefore exactly one of u, w is a neighbour of v on C and the other is a leaf. □

Theorem 3.19. *Let X be an obstruction that has no cut-vertices. Suppose $\overline{U(X)}$ contains an induced C_5. Then $\overline{U(X)}$ is one of the graphs in Figure 3.7 and X or its dual contains the dotted arcs.*

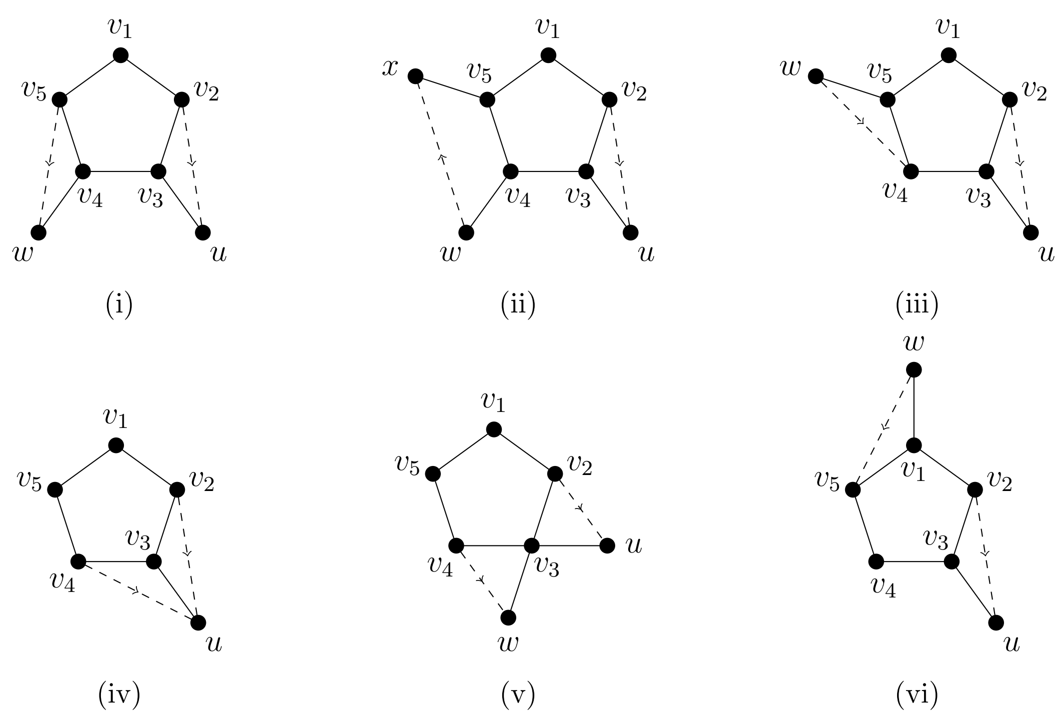


Figure 3.7: Obstructions X for which $\overline{U(X)}$ contains an induced C_5.

Proof: Let $C : v_1v_2v_3v_4v_5$ be an induced C_5 in $\overline{U(X)}$. By Lemma 3.18, C is the unique cycle in $\overline{U(X)}$ and has a vertex which is an arc-balancing but not a cut-vertex of $\overline{U(X)}$. Without loss of generality assume v_1 is such a vertex. By Lemma 3.18, v_1 balances an arc between an adjacent vertex on C and a leaf. Without loss of generality, assume v_1 balances an arc between v_2 and u and (v_2, u) is the arc. Since any vertex except v_1 that is adjacent to u is also adjacent to v_2, u is adjacent to v_3. Suppose that neither v_4 nor v_5 is an arc-balancing vertex. Then by Proposition 1.10, each of v_4, v_5 is a cut-vertex of $\overline{U(X)}$ or incident with an arc. Since v_4, v_5 are adjacent in $\overline{U(X)}$, there is no arc between them. So one of v_4, v_5 is not incident with an arc and hence must be a cut-vertex of $\overline{U(X)}$.

Suppose v_4 is a cut-vertex and v_5 is incident with an arc. By Lemma 3.18(b), there is a leaf w adjacent to v_4 and incident with an arc. Hence there is an arc between v_5 and w. Since we have $(v_2, u)\Gamma(u, v_1)\Gamma(v_1, v_3)\Gamma(v_4, v_1)\Gamma(v_1, w)\Gamma(w, v_5)$ and the two arcs are opposing in X, the second arc is (v_5, w). Thus $\overline{U(X)}$ is Figure 3.7(i). Suppose instead that v_4 is incident with an arc and v_5 is a cut-vertex. By Lemma 3.18, there is a leaf w adjacent to v_5 and incident with an arc. Hence there is an arc between v_4 and w. Since $(v_2, u)\Gamma(u, v_1)\Gamma(v_1, v_3)\Gamma(v_3, v_5)\Gamma(w, v_3)\Gamma(v_4, w)$, the second arc is (w, v_4) and $\overline{U(X)}$ is Figure 3.7(iii). Finally, suppose both v_4 and v_5 are cut-vertices. By Lemma 3.12, there are leaves w, x adjacent to v_4, v_5, respectively, and incident with arcs. Hence there is an arc between w and x. Since we have $(v_2, u)\Gamma(u, v_1)\Gamma(v_5, u)\Gamma(u, x)\Gamma(x, v_3)\Gamma(v_4, x)\Gamma(x, w)$, the second arc is (w, x). Thus $\overline{U(X)}$ is Figure 3.7(ii).

Suppose exactly one of v_4, v_5 is an arc-balancing vertex. Clearly they cannot both be arc-balancing vertices because v_1 is such a vertex and there are at most two arc-balancing vertices. Consider first the case when v_5 is an arc-balancing vertex. Then v_5 either balances an arc between v_1 and a leaf adjacent to v_2 or an arc between v_4 and a leaf adjacent to v_3. However, the former is not possible, as otherwise v_4 is not an arc-balancing vertex and not incident with an arc so it is a cut-vertex by Proposition 1.10. But then a leaf adjacent to it is not incident with an arc, a contradiction to Lemma 3.18. v_5 balances an arc between v_4 and a leaf w adjacent to v_3. Then we have $(v_2, u)\Gamma(u, v_1)\Gamma(v_1, v_3)\Gamma(w, v_1)\Gamma(v_5, w)\Gamma(w, v_4)$. The second arc is (v_4, w). When $w = u$, $\overline{U(X)}$ is Figure 3.7(iv); otherwise $\overline{U(X)}$ is Figure 3.7(v).

Consider now the case when v_4 is an arc-balancing vertex. Then v_4 either balances an arc between v_5 and a leaf adjacent to v_1 or an arc between v_3 and a leaf adjacent to v_2. If v_4 balances an arc between v_5 and a leaf w adjacent to v_1, then $(v_2, u)\Gamma(u, v_1)\Gamma(v_1, v_3)\Gamma(v_3, w)\Gamma(w, v_4)\Gamma(v_5, w)$. The second arc is (w, v_5) and $\overline{U(X)}$ is Figure 3.7(vi). On the other hand, if v_4 balances an arc between v_3 and a leaf adjacent to v_2, then v_5 is not an arc-balancing vertex and not incident with an arc so it is a cut-vertex by Proposition 1.10. But then a leaf adjacent to it is not incident with an arc, a contradiction to Lemma 3.18. Hence this is not possible. □

Chapter 4

Acyclic local tournament orientation completions

We now turn our attention to acyclic local tournament orientation completions. A partially oriented graph $X = (V, E \cup A)$ is called an *obstruction for acyclic local tournament orientation completions* if the following three properties hold:

1. X cannot be completed to an acyclic local tournament;

2. For each $v \in V$, $X - v$ can be completed to an acyclic local tournament;

3. For each $(u, v) \in A$, the partially oriented graph obtained from X by replacing (u, v) with the edge uv can be completed to an acyclic local tournament.

This definition mirrors the definition of obstructions for local tournament orientation completions, so the following proposition can be obtained in a similar way as Proposition 1.1.

Proposition 4.1. *A partially oriented graph H cannot be completed to an acyclic local tournament if and only if it critically contains an obstruction for acyclic local tournament orientation completions.* □

In the remainder of this chapter, we will prove the following:

Theorem 4.2. *Let X be an obstruction for acyclic local tournament orientations. Then X or its dual is a C_k ($k \geq 4$) or one of the graphs in Figures 4.1–4.3.*

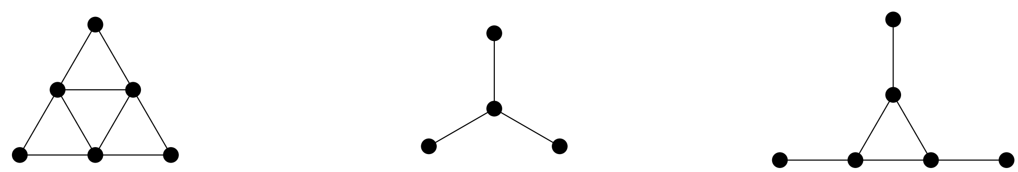
4.1 Proper interval graphs and Wegner's theorem

Obstructions for acyclic local tournament orientation completions which do not contain arcs can be derived from the next two theorems.

Theorem 4.3 ([10, 12]). *The following statements are equivalent for a graph G:*

- *G can be completed to an acyclic local tournament;*
- *G is a proper interval graph;*
- *G has a straight enumeration.* □

Theorem 4.4 (Wegner [25]). *A graph G is a proper interval graph if and only if it does not contain a C_k ($k \geq 4$), a tent, a claw, or a net as an induced subgraph. (See Figure 4.1.)* □



(i): tent. (ii): claw. (iii): net.

Figure 4.1: Forbidden induced subgraphs for proper interval graphs.

Corollary 4.5. *If X is an obstruction for acyclic local tournament orientation completions that does not contain an arc, then X is a C_k ($k \geq 4$) or a graph in Figure 4.1.*

Proof: This follows immediately from Theorems 4.3 and 4.4. □

4.2 Obstructions for acyclic local tournament orientation completions

It remains to determine the obstructions for acyclic local tournament orientation completions that contain arcs. Of these obstructions some cannot even be completed to local tournaments and the rest can be completed to local tournaments but not to acyclic local tournaments. We will distinguish two cases depending on whether they can be completed to local tournaments (See Theorems 4.8 and 4.9 below). Note that any obstruction for acyclic local tournament orientation completions that cannot be complete to a local tournament is an obstruction (for local tournament orientation completions) by definition.

Let X be an obstruction for acyclic local tournament orientation completions that contains arcs. Clearly, the dual of X is again an obstruction for acyclic local tournament orientation completions. Since $U(X)$ can be completed to an acyclic local tournament, it is a proper interval graph and has a straight enumeration by Theorem 4.3.

Let H be a partially oriented graph such that $U(X)$ is a proper interval graph. Observe that if (u, v) is a balanced arc in H then the partially oriented graph obtained from H by replacing (u, v) with (v, u) is isomorphic to H. Whether or not H can be completed to an acyclic local tournament merely depends on the unbalanced arcs in H. The following proposition is a reformulation of a result (Corollary 3.3) from [12].

Proposition 4.6 ([12])**.** *Let H be a partially oriented graph such that $U(H)$ is a proper interval graph and $\prec$ be a straight enumeration of $U(H)$. Suppose H does not contain a directed cycle. Then H can be completed to an acyclic local tournament if and only if it does not contain two unbalanced arcs, one positive and the other negative with respect to $\prec$.* □

Lemma 4.7. *Let X be an obstruction for acyclic local tournament orientation completions that contains arcs but no directed cycle, and let $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$. If X can be completed to a local tournament, then the following statements hold:*

(a) X contains exactly two unbalanced arcs, one positive and the other negative with respect to $\prec$;

(b) There exists a universal vertex incident with exactly one arc of X;

(c) Any vertex not incident with an arc is an arc-balancing vertex.

Proof: Since X cannot be completed to an acyclic local tournament, by Proposition 4.6 it contains two unbalanced arcs, one positive and the other negative with respect to $\prec$. The minimality of X ensures that X contains no other arcs. This proves (a).

For (b), let (v_i, v_j) and (v_s, v_t) be the two unbalanced arcs of X. Since X can be completed to a local tournament, v_iv_j and v_sv_t belong to different implication classes of $U(X)$. By Theorem 1.13, one of v_iv_j, v_sv_t is an edge of $U(X)$ between the unique non-trivial component of $\overline{U(X)}$ and a universal vertex of $U(X)$. That is, there exists a universal vertex incident with exactly one of $(v_i, v_j), (v_s, v_t)$.

Finally, for (c), suppose v is a vertex not incident with an arc. Since X is an obstruction for acyclic local tournament orientation completions, the subgraph $X - v$ can be completed to an acyclic local tournament, so at least one of (v_i, v_j) and (v_s, v_t) is balanced in $X - v$, which means v is an arc-balancing vertex. □

Theorem 4.8. *Let X be an obstruction for acyclic local tournament orientation completions that contains arcs. Suppose that X can be completed to a local tournament. Then X or its dual is one of the graphs in Figure 4.2.*

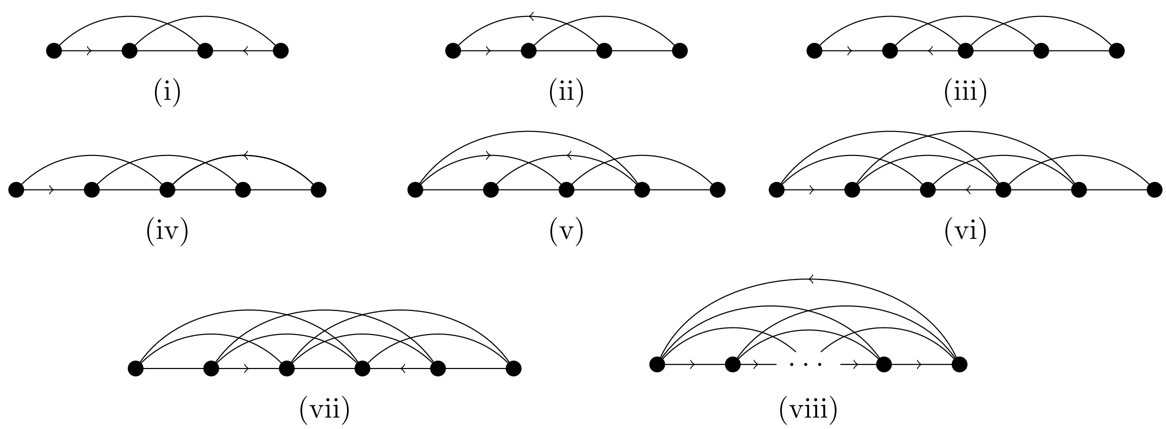


Figure 4.2: Obstructions for acyclic local tournament orientation completions containing arcs that can be completed to local tournaments.

Proof: It is easy to verify that each graph in Figures 4.2 is an obstruction for acyclic local tournament orientation completions and can be completed to a local tournament. Hence it suffices to show that X is one of them. Suppose X does not contain a directed cycle. Fix a straight enumeration $\prec$: $v_1, v_2, \ldots, v_n$ of $U(X)$. By Lemma 4.7(a) and (b), X contains exactly two unbalanced arcs, one positive and one negative, and there exists a universal vertex v_u incident with exactly one arc of X. Note that since X contains unbalanced arcs, $U(X)$ cannot be complete and so v_1v_n is not an edge of $U(X)$ by the umbrella property. In particular, $v_u \notin \{v_1, v_n\}$.

Clearly, $n \geq 3$. If $n = 3$, then X or its dual must contain the arcs (v_1, v_2) and (v_3, v_2) because the two arcs are unbalanced and opposing, contradicting the fact that X can be completed to a local tournament. Suppose $n = 4$. Without loss of generality, assume $v_u = v_2$. So, v_2v_4 is an edge of $U(X)$. If v_1v_3 is not an edge of $U(X)$, then the only unbalanced edges of $U(X)$ are those incident with v_2, so both unbalanced arcs of X are incident with v_2, contradicting the choice of v_u. Hence, v_1v_3 is an edge. Since both arcs of X are unbalanced, there is no arc between v_2 and v_3. It is now easy to see that X or its dual is one of Figure 4.2(i) or (ii).

Suppose instead $n = 5$. We claim that v_1v_3, v_2v_4, v_3v_5 are edges of $U(X)$. Since X contains a universal vertex, both of v_1v_3, v_3v_5 are edges of $U(X)$. If v_2v_4 is not an edge of $U(X)$, then $v_u = v_3$ and every unbalanced edge of $U(X)$ is incident with v_3, so both unbalanced arcs of X are incident with v_3, contradicting the choice of v_u. So, v_2v_4 is an edge of $U(X)$. Suppose neither v_1v_4 nor v_2v_5 are edges of $U(X)$. Then, v_3 is the unique universal vertex, so $v_u = v_3$. If the arc not incident with v_3 is between v_2 and v_4, then X or its dual critically contains Figure 4.2(i) or (ii), a contradiction to the minimality of X. Hence the arc not incident with v_3 is either between v_1 and v_2 or between v_4 and v_5. We may assume without loss of generality (v_1, v_2) is an arc. If v_5 is not incident with an arc, then it is an arc-balancing vertex by Lemma 4.7(c). Clearly, v_5 must balance the arc

(v_3, v_2), so X is Figure 4.2(iii). Otherwise if v_5 is incident with an arc, then X is (iv). Suppose instead that v_1v_4 or v_2v_5 is an edge of $U(X)$. Without loss of generality, assume v_1v_4 is an edge. If v_2v_5 is also an edge, then each of v_2, v_3, v_4 is a universal vertex and hence is not an arc-balancing vertex. By Lemma 4.7(c), each of v_2, v_3, v_4 is incident with an arc, so there is an arc with both endvertices among v_2, v_3, v_4, contradicting the fact that both arcs are unbalanced. So, v_2v_5 is not an edge. Any arc incident with v_5 does not have an arc-balancing vertex because there are two vertices adjacent to exactly one endvertex of such an arc. If v_1 or v_2 is an arc-balancing vertex, then it balances an arc incident with v_5, so neither v_1 nor v_2 is an arc-balancing vertex. By Lemma 4.7(c), both v_1 and v_2 are incident with arcs. Similarly, neither v_3 nor v_4 are arc-balancing vertices because they are universal, so they are both incident with arcs. Since both arcs of X are unbalanced, X or its dual must be Figure 4.2(v).

Suppose instead that $n \geq 6$. Since X contains exactly two arcs, it contains at most two arc-balancing vertices. Since any vertex not incident with an arc is an arc-balancing vertex by Lemma 4.7(c), X contains at most two vertices not incident with arcs. In particular, $n = 6$ and X contains two disjoint arc-balancing triples. We show that neither v_2 nor v_5 is universal. Assume v_2 is universal. Since X contains two disjoint arc-balancing triples, one of them contains only vertices succeeding v_1. Since v_2v_6 is an edge of $U(X)$, this arc-balancing triple induces a clique in $U(X)$ by the umbrella property, a contradiction. Hence, neither v_2 nor v_5 is universal by symmetry. So, $v_u \in \{v_3, v_4\}$. Assume $v_u = v_4$ without loss of generality. Let v_k be the arc-balancing vertex for the arc incident with v_4 and v_j be the other endvertex. Then, v_k is the unique vertex adjacent to v_4 and not v_j, so v_j is adjacent to every vertex except for v_k. It follows from the straight enumeration that $v_k \in \{v_1, v_6\}$.

Suppose $v_k = v_6$. If $v_j = v_1$, then v_1 is adjacent to v_5, contradicting the fact that v_5 is not a universal vertex, so $v_j \neq v_1$. Since v_j is not adjacent to v_6, we have either $v_j = v_3$ or $v_j = v_2$. First suppose $v_j = v_3$. Without loss of generality, assume v_6 is a (v_4, v_3)-balancing vertex. Since X contains two disjoint arc-balancing triples, $\{v_1, v_2, v_5\}$ is an arc-balancing triple. If v_1 balances an arc between v_2 and v_5, then v_6 is adjacent to both v_2 and v_5, contradicting the fact that v_2 is not a universal vertex. Clearly, v_2 cannot balance an arc between v_1 and v_5 by the straight enumeration. So, v_5 is a (v_1, v_2)-balancing vertex and thus X is Figure 4.2(vi). On the other hand, suppose $v_j = v_2$. Without loss of generality, assume v_6 is a (v_4, v_2)-balancing vertex. By a similar argument as above, $\{v_1, v_3, v_5\}$ is an arc-balancing triple. Clearly, v_3 cannot be the arc-balancing vertex by the straight enumeration. If v_1 is a (v_3, v_5)-balancing vertex, then the dual of X is Figure 4.2(vii). Otherwise v_5 is a (v_1, v_3)-balancing vertex and X is Figure 4.2(vi).

Otherwise, $v_k = v_1$. If $v_j = v_6$, then v_j is adjacent to v_2, so v_2 is a universal vertex, a contradiction. Hence, $v_j \neq v_6$. Since v_j is not adjacent to v_k, we have $v_j = v_5$. Without loss of generality, assume v_1 is a (v_5, v_4)-balancing vertex. By a similar argument as above,

$\{v_2, v_3, v_6\}$ is an arc-balancing triple. If v_2 balances an arc between v_3 and v_6, then v_6 must be adjacent to v_1, a contradiction. Clearly, v_3 cannot balance an arc between v_2 and v_6 by the straight enumeration. Hence, v_6 is a (v_2, v_3)-balancing vertex. It is now easy to see that X is Figure 4.2(vii).

On the other hand, suppose X contains a directed cycle. Let $C : v_1v_2 \ldots v_n$ denote a smallest directed cycle of X and assume $(v_1, v_2), (v_2, v_3), \ldots, (v_n, v_1)$ are arcs. By the choice of C, X does not contain arcs other than those of C. Clearly, every vertex of X is on C because if v is a vertex not on C, then $X - v$ still contains a directed cycle and therefore cannot be completed to an acyclic local tournament. We show that $U(X)$ is complete. Since X is an obstruction for acyclic local tournament orientation completions, the partially oriented graph X' obtained from X by replacing (v_n, v_1) with the edge v_nv_1 can be completed to an acyclic local tournament D. Clearly, D contains the arc (v_1, v_n) because it is acyclic, so v_{n-1} and v_1 are adjacent as they are both in the in-neighbourhood of v_n. Similarly, D contains the arc (v_1, v_{n-1}) because it is acyclic, so the same argument shows that v_{n-2} and v_1 are adjacent. By repeating this argument, we see that D contains the arcs (v_1, v_i) for each $i \neq 1$. Moreover, the out-neighbourhood of v_1 induces a clique, so it follows that $U(X)$ is complete. Thus, X is Figure 4.2(viii). $\square$

Theorem 4.9. *Let X be an obstruction for acyclic local tournament orientation completions that contains arcs. Suppose that X cannot be completed to a local tournament. Then X or its dual is one of the graphs in Figure 4.3.*

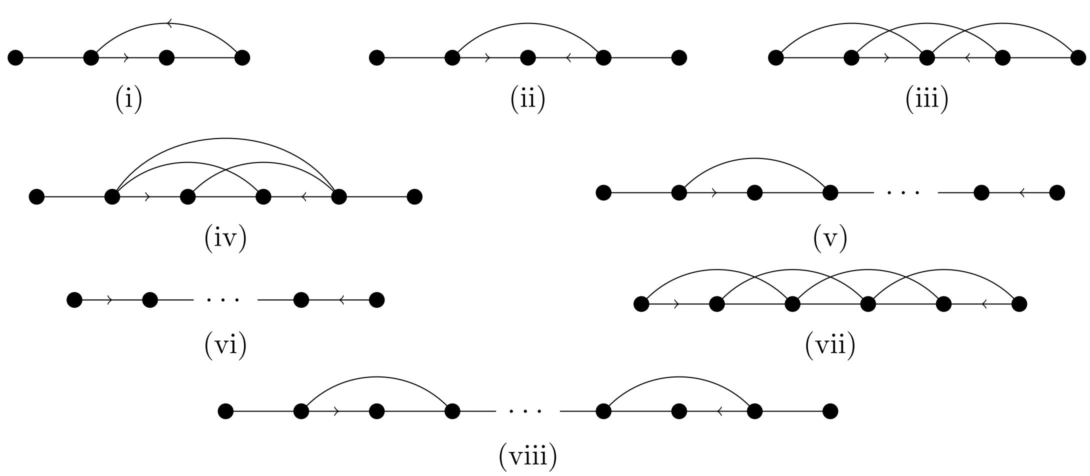


Figure 4.3: Obstructions for acyclic local tournament orientation completions containing arcs that cannot be completed to local tournaments.

Proof: It is easy to verify that each graph in Figures 4.3 is an obstruction for acyclic local tournament orientation completions and cannot be completed to a local tournament. Hence it suffices to show that X contains one of them as an induced subgraph. Since X an obstruction for acyclic local tournament orientation completions and cannot be completed to a local tournament, it is an obstruction for local tournament orientation completions. By Theorem 1.9 it has exactly two arcs which are opposing. Moreover, X does not contain any graph in Figure 4.2 as an induced subgraph.

Let $\prec$: $v_1, v_2, \ldots, v_n$ be a straight enumeration of $U(X)$ and let $(v_a, v_b), (v_c, v_d)$ be the two arcs in X. Then one of the two arcs is positive and the other is negative. Assume (v_a, v_b) is positive (i.e., $a < b$) and (v_c, v_d) is negative (i.e., $c > d$).

Consider first the case when the two arcs share an endvertex. Suppose that $v_a = v_d$ is the shared endvertex. By considering the dual of X if necessary we assume $b < c$. Then we have $a = d < b < c$ and the umbrella property of $\prec$ implies $v_b v_c$ is an edge of X. Since X does not contain Figure 4.2(ii), any vertex v_j with $j > c$ adjacent to v_b is adjacent to v_a. This together with the umbrella property of $\prec$ imply that any vertex adjacent to adjacent to v_b is adjacent to v_a. The arc (v_a, v_b) is unbalanced so there is a vertex v_i adjacent to v_a but not to v_b. Clearly, we must have $i < a$ and thus the subgraph of X induced by v_i, v_a, v_b, c_c is Figures 4.3(i). The case when $v_b = v_c$ is the shared endvertex can be treated analogously. Suppose that $v_b = v_d$ is the shared endvertex. If $v_a v_c$ is not an edge of X, then the subgraph of X induced by v_a, v_b, v_c is Figure 4.3(vi). So assume $v_a v_c$ is an edge. Since both arcs are unbalanced, for each of them there exists a vertex adjacent to exactly one of the two endvertices. Suppose there is a vertex v_i adjacent to v_a but not to v_b. Clearly, $i < a$. If there is a vertex v_j adjacent to v_c but not to $v_d = v_b$, then $j > c$ and the subgraph of X induced by v_i, v_a, v_b, v_c, v_j is Figure 4.3(ii). If there is a vertex v_k adjacent to $v_d = v_b$ but not to v_c, then $i < k < a$ and the subgraph of X induced by v_i, v_k, v_a, v_b, v_c is Figure 4.2(iii), a contradiction. Thus we may assume that any vertex adjacent to v_a except v_b is adjacent to v_b. Hence there is a vertex adjacent to v_b but not to v_a and let v_r be such a vertex. If there is a vertex v_ℓ adjacent to $v_d = v_b$ but not to v_c, then $\ell < a < b < c < r$ and the subgraph of X induced by $v_\ell, v_a, v_b, v_c, v_r$ is Figure 4.3(iii). If there is a vertex v_q adjacent to v_c but not to $v_d = v_b$, then $a < b < c < r < q$ and the subgraph of X induced by $v_\ell, v_a, v_b, v_c, v_r$ is Figure 4.2(iii), a contradiction. The proof for the case when $v_a = v_c$ is the same by considering the dual of X. Therefore we may further assume the endvertices of the two arcs are pairwise distinct.

Suppose that the endvertices of the two arcs are pairwise adjacent. Let v_i be a vertex adjacent to exactly one of v_a, v_b and v_j be a vertex adjacent to exactly one of v_c, v_d. Suppose first that v_i is adjacent to v_a but not to v_b and v_j is adjacent to v_d but not to v_c. Clearly, $\max\{i, j\} < \min\{a, d\}$. The umbrella property of $\prec$ implies $v_i v_j$ is an edge of X. Thus $v_i v_a v_d v_j$ is a C_4 in $U(X)$ which cannot be induced. So $v_i v_d$ or $v_j v_a$ is an edge of X. By symmetry assume $v_i v_d$ is an edge. If $v_i v_c$ is not an edge of X then the subgraph induced by v_i, v_a, v_b, v_c, v_d is Figure 4.2(v), a contradiction. So $v_i v_c$ is an edge, which implies $c < b$. Since v_j is not adjacent to v_c and $c < b$, v_j is not adjacent to v_b. If v_j is not adjacent to v_a, then the subgraph of X induced by $v_i, v_j, v_a, v_b, v_c, v_d$ is Figure 4.2(vi), a contradiction. If v_j is adjacent to v_a, then the subgraph of X induced by v_j, v_a, v_b, v_c, v_d

is Figure 4.2(v), a contradiction. Suppose now that v_i is adjacent to v_a but not to v_b and v_j is adjacent to v_c but not to v_d. (Note that the other two cases are symmetric.) If v_i is adjacent to neither of v_c, v_d and v_j is adjacent to neither of v_a, v_b, then the subgraph induced by $v_i, v_j, v_a, v_b, v_c, v_d$ is Figure 4.3(iv). If v_i is adjacent to exactly one of v_c, v_d, then it is adjacent to v_d, in which case the subgraph induced by v_i, v_a, v_b, v_c, v_d is Figure 4.2(v), a contradiction. So v_i is adjacent to both v_c, v_d. This implies $c < b$ because v_iv_b is not an edge of X. Thus v_jv_b is an edge following the umbrella property. If v_j is not adjacent to v_a then the subgraph of X induced by v_j, v_a, v_b, v_c, v_d is Figure 4.2(v), a contradiction. If v_j is adjacent to v_a, then the subgraph of X induced by $v_i, v_j, v_a, v_b, v_c, v_d$ is Figure 4.2(vii), a contradiction.

Suppose that the endvertices of the two arcs are not all pairwise adjacent. Without loss of generality assume $a < d$. Then we must have $b < c$ and in particular v_av_c is not an edge of X. Since X does not contain Figure 4.2(i) as an induced subgraph, we must have $b < d$ and at least one of v_av_d and v_bv_c is not an edge of X. By symmetry we assume v_av_d is not an edge of X. If $a < b-1$ then v_{a+1} is clearly not a cut-vertex of $U(X)$ and by Proposition 1.14 not a cut-vertex of $\overline{U(X)}$. Thus v_{a+1} can only be the (v_c, v_d)-balancing vertex by Proposition 1.10. Hence $v_{a+1}v_d$ is an edge of X, which implies v_bv_d is also an edge of X. Since v_{a+1} is the unique vertex adjacent to exactly one of the endvertices of (v_c, v_d), v_bv_c must be an edge of X. We see now that the subgraph of X induced by $v_a, v_{a+1}, v_b, v_c, v_d$ is Figure 4.2(iv), a contradiction. Hence v_a, v_b are consecutive in $\prec$. Similarly, v_c, v_d are consecutive in $\prec$. If v_bv_c is an edge of X, then any vertex adjacent to v_d except v_c is adjacent to v_c. So there must be a vertex v_j adjacent to v_c but not to v_d. The subgraph of X induced by v_a, v_b, v_c, v_d, v_j is a graph in Figure 4.3(v). So we may assume v_bv_c is not an edge of X. If v_bv_d is an edge of X, then the subgraph of X induced by v_a, v_b, v_c, v_d is a graph in Figure 4.3(vi). So we may further assume v_bv_d is not an edge of X.

Let v_k be the neighbour of v_b having the largest subscript k and let v_ℓ be the neighbour of v_d having the least subscript. Clearly, $b < k < d$ and $b < \ell < d$. Suppose neither v_av_k nor $v_\ell v_c$ is an edge of X. Consider first the case when $\ell < k$. If v_av_ℓ and v_kv_c are both edges of X, then the subgraph of X induced by $v_a, v_b, v_\ell, v_k, v_c, v_d$ is Figure 4.3(vii). If v_av_ℓ is not an edge of X, then the subgraph of X induced by $v_a, v_b, v_\ell, v_c, v_d$ is a graph in Figure 4.3(vi). Similarly, if v_kv_c is not an edge of X, then the subgraph of X induced by v_a, v_b, v_k, v_c, v_d is a graph in Figure 4.3(vi). When $k \leq \ell$, the subgraph of X induced by v_a, v_b, v_c, v_d together with the vertices in a shortest (v_k, v_ℓ)-path is a graph in Figure 4.3(vi). Suppose exactly one of v_av_k and $v_\ell v_c$ is an edge of X and by symmetry we assume it is v_av_k. Then any vertex adjacent to v_b except v_a is adjacent to v_a. So there must be a vertex v_i adjacent to v_a but not to v_b. Thus the subgraph of X induced by v_i, v_a, v_c, v_d and the vertices in a shortest (v_b, v_ℓ)-path is a graph in Figure 4.3(v). Finally suppose v_av_k and $v_\ell v_c$ are both edges of X. Then there must be a vertex v_i adjacent to v_a but not to v_b

and a vertex v_j adjacent to v_c but not to v_d. The subgraph induced by $v_i, v_a, v_b, v_c, v_d, v_j$ and the vertices in a shortest (v_b, v_d)-path is a graph in Figure 4.3(viii). This completes the proof. $\square$

Chapter 5
Conclusion and future work

The main results of the thesis now follow. Theorems 1.4, 2.2, 2.4, 2.6, 2.8, 2.10–2.11, 3.8, 3.11, 3.13–3.15, 3.17, and 3.19 provide a complete list of obstructions for orientation completions for local tournaments. Since each graph in the associated figures corresponds to a particular case considered, every graph in the associated figures is an obstruction. Thus, Theorem 1.2 follows. On the other hand, Theorem 1.3 follows from Corollary 4.5 and Theorems 4.8–4.9 in the same way.

We turn our attention to the computational aspects of obstructions for local tournament orientation completions. Naturally, we are interested in a recognition algorithm for obstructions. On the other hand, Proposition 1.1 implies that a partially oriented graph cannot be completed to a local tournament if and only if it critically contains an obstruction. We are also interested in an algorithm that finds an obstruction that is critically contained in a given partially oriented graph (if it exists).

According to [3], the orientation completion problem for both local tournaments and acyclic local tournaments is polynomial-time solvable. Thus the recognition problem can also be solved in polynomial-time by directly verifying the definition of obstructions. On the other hand, to find an obstruction that is critically contained in a given partially oriented graph (if it exists), it suffices verify that the given graph cannot be completed to a local tournament, and then to delete vertices or replace arcs with edges as long as the resulting graph still cannot be completed to a local tournament. When the deletion of any vertex or the replacement of any arc with an edge results in a graph that can be completed to a local tournament, then the graph is an obstruction. Similar algorithms work for the case of obstructions for acyclic local tournament orientation completions.

We conclude the thesis by listing a selection of relevant open problems for future research. As previously discussed, the orientation completion problem for transitive oriented graphs with the input restricted to undirected graphs is exactly the recognition problem for comparability graphs.

Problem 1. *What are the obstructions for transitive orientation completions?*

A digraph $D = (V, A)$ is said to be *quasi-transitive* if for any three vertices u, v, w, $(u, v), (v, w) \in A$ implies $(u, w) \in A$, $(w, u) \in A$, or both. Quasi-transitive orientations are of interest because although this is a weaker condition than transitive orientations, it is still true that a graph is a comparability graph if and only if it admits a quasi-transitive orientation, cf. [2]. Thus, similarly as above, the orientation completion problem for quasi-transitive oriented graphs with the input restricted to undirected graphs is exactly the recognition problem for comparability graphs.

Problem 2. *What are the obstructions for quasi-transitive orientation completions?*

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