Constraints Effecting Stability and Causality of Charged Relativistic Hydrodynamics

by

Raphael E. Hoult
B.Sc., University of Winnipeg, 2018

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ABSTRACT

First-order viscous relativistic hydrodynamics has long been thought to be unstable and acausal. This is not true; it is only with certain definitions of the hydrodynamic variables that the equations of motion display these properties. It is possible to define the hydrodynamic variables such that a fluid is both stable and causal at first order. This thesis does so for both uncharged and charged fluids, mostly for fluids at rest. Work has also been done in limited cases on fluids in motion. A class of stable and causal theories is identified via constraints on transport coefficients derived from linearized perturbations of the equilibrium state. Causality conditions are also derived for the full non-linear hydrodynamic equations.

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לא עליך המלאכה לגמור, ולא אתה בן חורין לבטל ממנה

-- רבי *טרפון* (פרקי אבות ב`:ט"ז)

Dedication

To Kasey Inkster;

Her love and support during the writing of this document and the months of research beforehand made this thesis possible.

Chapter 1

Introduction

This Masters thesis will study relativistic viscous hydrodynamics. The relativistic formulation of viscous hydrodynamics was pioneered by Eckart [1], as well as Landau and Lifshitz [2]. Each group developed a formalism, but their two formulations are inequivalent both conceptually and mathematically. Unfortunately, both theories share the same fatal property: they predict that the equilibrium state of a uniformly moving fluid is unstable, and that perturbations to the equilibrium state will propagate superluminally (i.e. “acausally”). These predictions are non-physical, and so both formulations have been abandoned as physical theories. The theories of Landau, Lifshitz, and Eckart are both examples of so-called “first-order hydrodynamics”, so named because only terms of first-derivative order appear in the entropy current.

The main approach to rectifying the issues of the Eckart and Landau formulations is a formalism developed by Müller [3], as well as by Israel and Stewart [4][5]. This approach treats first-order dissipative corrections to ideal fluid mechanics as dynamical variables unto themselves in addition to the degrees of freedom present in the theories of Landau and Eckart. Doing so also introduces five new parameters that must be kept track of – three relaxation times, and two coefficients accounting for possible viscous-heat flux couplings. The so-called “MIS theory” has been shown to predict a stable equilibrium state, as well as predicting that perturbations should propagate subluminally both near equilibrium [6] and far from equilibrium [7] when the parameters are subject to certaint conditions.

Developing a stable and causal theory of relativistic hydrodynamics is important to high-energy physics for myriad reasons. Relativistic hydrodynamics is used in numerous fields ranging from the study of quark-gluon plasmas in colliders [8](Section 11.10)[9][10] to various astrophysical processes, including (but not limited to) the study of plasmas with strong magnetic fields but shielded electric fields – this is magnetohydrodynamics, or MHD [11][12][13].

Despite the success of the Müller-Israel-Stewart (MIS) theory at rectifying the pathologies of Eckart and Landau, the accompanying complexity of the theory is challenging. Developing a stable first-order theory without additional degrees of freedom would make for a simpler framework.

Such a first-order theory *has* been developed. Works by Bemfica, Disconzi, Noronha, and Kovtun (BDNK) in [14], [15], and [16] show that, for an uncharged (in the sense of Noether charges) fluid, revisiting how hydrodynamic variables are defined out of equilibrium can lead to stable and causal fluid dynamics – all at first order.

This approach is distinct from the approach of MIS in that it introduces no new dynamical quantities. Rather, it takes advantage of a fundamental ambiguity in the formulation of relativistic hydrodynamics. In forming the effective macroscopic theory, one must provide a definition for the effective degrees of freedom. While these degrees of freedom have unambiguous definitions in equilibrium, once a system has departed from equilibrium each degree of freedom may be arbitrarily re-defined, so long as the various definitions agree in equilibrium.

Different definitions of the effective degrees of freedom will lead to mathematically inequivalent effective theories. Certain definitions are more advantageous than others, leading to theories that have stable equilibria and describe causal propagation.

In this thesis, I outline the work that I did to extend the work of BDNK to theories describing charged fluids, a much larger class of theory. The introduction of Noether charge adds a new degree of freedom as well as a new conserved current.

The approach used to find these stable and causal frames is ultimately that of effective field theory; one analyzes the structure of the equations and creates effective theories from allowed symmetries and effective degrees of freedom, rather than deriving quantities from underlying first principles[1]. We choose to be agnostic about the definitions of the effective degrees of freedom, and so when writing out the equations of motion for hydrodynamics in terms of these effective degrees of freedom (the so-called "constitutive relations"), all terms that are allowed by Lorentz symmetry and parity are written.To each of these terms is then attached some arbitrary coefficient, themselves functions of the degrees of freedom. These coefficients are called "transport coefficients", and it can be shown that a particular set of values for the transport coefficients corresponds directly to a choice for the definitions of the degrees of freedom.

[1] As opposed to, say, a kinetic theory derivation.

In other words, determining which values of the transport coefficients yield stable and causal equations is equivalent to determining the class of stable and causal first-order hydrodynamic theories. In this thesis, I will derive the constraints on the transport coefficients that, if satisfied, will yield a stable and causal linearized hydrodynamic theory for a fluid at rest, as well as the condition for long- and short-wavelength perturbations to be stable for a uniformly moving fluid. Additionally, I will derive constraints to ensure the full, non-linear equations propagate perturbations causally, including when the fluid is coupled to dynamical gravity.

The results of this thesis will primarily follow [17]. The outline of the rest of the thesis will be as follow:

- **Chapter 2** will introduce the Landau and Eckart "frames", and elucidate how and why they go wrong. In this chapter, the basic analysis techniques used for charged BDNK are introduced.

- **Chapter 3** analyzes the fundamentals of BDNK Hydrodynamics, and look at the methods used to constrain the class of stable and causal "frames", both in the charged and uncharged cases.

- **Chapter 4** contrasts BDNK hydrodynamics with the MIS theory, and puts the work in context. It then concludes and summarizes the thesis, and look to future works.

- There are four appendices: **Appendix A** covers the use of the ideal-order hydrodynamic equation to effectively perform a "frame-change"; **Appendix B** describes the so-called "Routh-Hurwitz criteria"; **Appendix C** details the derivation of the entropy current; and finally **Appendix D** serves as a compendium of lengthy equations, so that they need not go in the main body of the thesis.

Conventions Standard tensorial notation will be used (i.e. Einstein summation notation) throughout, where Greek indices run over all spacetime coordinates, and Latin indices run over purely spatial coordinates. The $(-+++)$ convention will be used for the Minkowski metric, $\eta^{\mu\nu}$. Natural units are used where $c = k_B = 1$. Where appearing inside a function, x represents all four spacetime dimensions – the index is supressed.

Chapter 2

Theory

2.1 Definition of a Fluid

To begin our discussion, we first must define what a fluid actually *is*. The classical way of defining a fluid is as follows[1] [8]:

Consider a system comprised of $\mathcal{N}$ particles, where $\mathcal{N} \ggg 1$. Let L be some length scale characterizing the system as a whole, and let ℓ be some length scale characterizing the constituent separation, usually taken to be the mean free path. One can then define a dimensionless number $\mathfrak{K}$ such that

$$\mathfrak{K} \equiv \frac{\ell}{L}. \tag{2.1}$$

This number $\mathfrak{K}$ is called the *Knudsen number*, and it characterizes the *separation of scales*; how much bigger L is than ℓ. The Knudsen number can also be related to the more well-known *Mach number* (the ratio of the fluid velocity to the speed of sound in the fluid) and the *Reynolds numbers* (a number characterizing the turbulence of the flow).

Suppose now that $L \gg \ell$ (i.e. $\mathfrak{K} \ll 1$). Then there may be some intermediate length scale dx, such that $L \gg dx \gg \ell$. If the volume element $dV \equiv dx^3$ contains a number of constituents N such that $\mathcal{N} \ggg N \ggg 1$, *and* if the system has rotational invariance at rest, then the volume element dV is called a *fluid element*, and the system as a whole is called a *fluid.*

Fluids are often defined by their macroscopic properties, namely the property that they "take the shape of their containers". There are numerous quotidien examples of fluids: water, air, gasoline, honey, and maple syrup are all examples of fluids of

[1] For a more modern approach, see [9].

varying viscosities that readily leap to mind. However, many high-energy phenomena may be characterized as fluids as well. A quark-gluon plasma may be considered a relativistic fluid. As well, plasmas in large stars should be described by relativistic hydrodynamics, as they are subject to a strong gravitational field.

Many different microscopic theories may all be modeled macroscopically as fluids; put differently, many theories have hydrodynamics as their low-energy/long distance ("IR") limit. The underlying microscopic theory is considered (in equilibrium) in the language of the grand canonical ensemble, where the fluid is in thermal and chemical equilibrium with some external bath. When a system is in (local) equilibrium[2], the equations that characterize the motion of the fluid elements are the equations of "ideal hydrodynamics". When the system departs from local equilibrium, the dynamics become more complicated, and the equations are those of *viscous* hydrodynamics. This thesis will investigate the latter case.

In general, one can characterize the equilibrium state of a system with a density operator $\hat{\varrho}$, which is given by [18][19]

$$\hat{\varrho} = \frac{1}{Z} e^{\beta_\mu P^\mu + \psi N}, \tag{2.2}$$

where $Z = \mathrm{Tr}\ e^{\beta_\mu P^\mu + \psi N}$ is the partition function. The vector β^μ is a timelike 4-vector, and ψ is a scalar. The operators P^μ and N are the momentum and $U(1)$ charge operators respectively. The equilibrium state is entirely specified by the choice of β^μ and ψ. The vector β^μ is the thermal vector, while the quantity ψ is related to the so-called *thermal potential.* These quantities may be written in terms of more familiar quantities from thermodynamics: T, the temperature, μ, the chemical potential, and u^α, the fluid element 4-velocity. In these terms, we can write that $\beta^\mu = \beta u^\mu$ and $\psi = \beta\mu$, where $\beta = 1/T$ is the inverse temperature. If the system is to truly be in equilibrium, β^μ must satisfy the Killing equation, i.e. $\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$, and ψ must be a constant.

If the partition function is known, we can derive the pressure p, energy density ϵ and so on. An "equation of state" defines how the pressure depends on the temperature and chemical potential, $p = p(T, \mu)$. Combining the equation of state with the

[2] In global equilibrium nothing moves, everything is constant, and the system is in hydrostatic equilibrium.

so-called "Gibbs-Duhem" equation

$$dp = sdT + nd\mu, \tag{2.3}$$

yields that the entropy density $s = \left(\frac{\partial p}{\partial T}\right)_\mu$ and the charge density $n = \left(\frac{\partial p}{\partial \mu}\right)_T$, where subscripts on derivatives details the dependent variables kept constant. These equalities for s and n, as well as the integrability of dp, lead immediately to the Maxwell relation

$$\left(\frac{\partial s}{\partial \mu}\right)_T = \left(\frac{\partial n}{\partial T}\right)_\mu. \tag{2.4}$$

Note that there is nothing special about p; the equation of state can relate any of the thermodynamic quantities listed above to T and μ. Therefore (assuming that the functions of T and μ can be inverted) one may write any of the thermodynamic quantities p, ϵ, and n as functions of one another[3], e.g. $p(\epsilon, n)$. We may also derive the following relation from the first law of thermodynamics, which expresses the energy density in terms of T, μ, given the equation of state:

$$\epsilon(T, \mu) = -p + sT + n\mu. \tag{2.5}$$

The Maxwell relation (2.4), via equation (2.5), reads

$$T\frac{\partial n}{\partial T} + \mu\frac{\partial n}{\partial \mu} = \frac{\partial \epsilon}{\partial \mu}. \tag{2.6}$$

We can also derive the following inequalities [18]:

$$\frac{\partial n}{\partial \mu} \geq 0, \qquad T\frac{\partial \epsilon}{\partial T} + \mu\frac{\partial \epsilon}{\partial \mu} \geq 0, \qquad \frac{\partial \epsilon}{\partial T}\frac{\partial n}{\partial \mu} - \frac{\partial n}{\partial T}\frac{\partial \epsilon}{\partial \mu} \geq 0. \tag{2.7}$$

The equations of motion for hydrodynamics have long puzzled mathematicians and physicists alike. The primary equations of hydrodynamics are the so-called "Navier-Stokes equations", equations which have befuddled those who attempt to analyze their long-term behaviour. It is still unknown whether, given reasonable initial data, these equations have a unique solution for all times (i.e. existence and uniqueness of a solution). While the traditional Navier-Stokes equations are equations of non-relativistic hydrodynamics, it is also unknown if the relativistic Navier-Stokes equations have

[3]We could also use s as well in place of either p, ϵ, and n. In systems with conserved mass, the mass density ρ is also involved.

unique solutions for all time, given reasonable initial data. This "Cauchy problem" for the Navier-Stokes equations is one of the seven "Millenium Problems" offered by the Clay Mathematics Institute in 2000[4] [20]. Work has been done [14] to determine that the relativistic hydrodynamics equations have *local* existence and uniqueness (i.e. on some finite (i.e. non-infinite) interval $t \in [0, T)$). Whether or not the Cauchy problem is solvable as $T \to \infty$ ("global existance/uniqueness") is at present unknown.

Relativistic hydrodynamics is the relevant formulation in three scenarios [8]: when the fluid element velocity is large, i.e. the fluid element moves at relativistic speeds; when the internal fluid constituents move at relativistic speeds (in this situation, the fluid element is said to be "hot"); or the fluid is in a strong external gravitational field, in which case the hydrodynamic equations described below will be coupled to Einstein's equations of general relativity.

This report will be primarily investigating relativistic fluids on a Minkowski background, and as such the third case will not apply, though we will make brief mention of such a coupling at the end of chapter three. The discussion in the remainder of chapter two primarily will follow [15] and [18].

2.2 Conservation Equations and Constitutive Relations

Relativistic hydrodynamics has two quantities that are conserved; the expectation value of the stress-energy tensor operator, $\langle T^{\mu\nu} \rangle$, of the microscopic theory, and the expectation value of the charge current operator, $\langle J^\mu \rangle$, associated with a possible $U(1)$ symmetry of the underlying microscopic theory. These two quantities obey conservation equations:

$$\nabla_\mu \langle T^{\mu\nu} \rangle = 0, \tag{2.8a}$$

$$\nabla_\mu \langle J^\mu \rangle = 0. \tag{2.8b}$$

where ∇_μ is the covariant derivative. Hereafter, we will drop the brakets $\langle \cdot \rangle$ – it is expected that we are discussing expectation values, and not the microscopic operators themselves. It is worth noting that the charge in question is not electrical charge[5], but rather a *Noether* charge such as Baryon number. The conservation equations for these two quantities may be considered the equations of motions for relativistic

[4] Only the non-relativistic equations are discussed in the problem.

[5] Electric charge necessitates the presence of a gauge field A^μ which would modify the equations of motion.

hydrodynamics.

If one had exact knowledge of how to solve the microscopic theory, there would be no need for any type of hydrodynamics. With complete knowledge of the state and the microscopic theory, the evolution of all the fundamental fields could be exactly determined. However, one does not always know how to solve the theory. It is in situations such as these that *effective theories*, such as hydrodynamics, become relevant.

The degrees of freedom of hydrodynamics are two scalars T and μ, and a timelike vector u^μ normalized such that $u_\mu u^\mu = -1$. We can identify these scalars in equilibrium with the temperature and the chemical potential respectively, and the vector with the fluid-element 4-velocity. Given these degrees of freedom, the next order of business is to find a way to express $T^{\mu\nu}$ and J^μ in terms of T, μ, and u^μ.

In equilibrium, absent an external field, these variables are constants. However, out of equilibrium, they may be promoted to continuous fields over the fluid elements. It is important to note that these *hydrodynamic variables* have no unique definition out of equilibrium. They may be arbitrarily re-defined at will, so long as they reduce to the correct values upon returning to equilibrium.[6] We assume that the variables are "slowly" varying, such that each order of the derivatives of the variables are smaller than the previous.

Given *any* timelike vector (denoted here by u^μ), it is possible to decompose any symmetric rank two tensor and rank one tensor into the following forms:

$$T^{\mu\nu} = \mathcal{E}u^\mu u^\nu + \mathcal{P}\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \tau^{\mu\nu}, \tag{2.9a}$$

$$J^\mu = \mathcal{N}u^\mu + \mathcal{J}^\mu, \tag{2.9b}$$

where $\Delta^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ is the *projection tensor*, which projects quantities with which it contracts to be orthogonal to u^μ (in the sense that $u_\mu \Delta^{\mu\nu} = 0$), and $g^{\mu\nu}$ is

[6] For temperature, one may think of this as a direct consequence of the zeroth law of thermodynamics; temperature is only defined by coming into equilibrium with reference systems. Out of equilibrium, "temperature" is whatever the thermometer making the measurement reads. Different thermometers will have different readings for the *same* fluid out of equilibrium. For fluid velocity, there is an ambiguity in *what* exactly "flows".

the inverse metric. The other quantities in the decomposition above are defined by

$$\begin{aligned}
\mathcal{E} &\equiv T^{\mu\nu} u_\mu u_\nu, \\
\mathcal{P} &\equiv \frac{1}{d} T^{\mu\nu} \Delta_{\mu\nu}, \\
\mathcal{Q}^\mu &\equiv -\Delta^{\mu\alpha} u^\nu T_{\alpha\nu}, \\
\tau^{\mu\nu} &\equiv \frac{1}{2}\left(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{d}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right) T^{\alpha\beta}, \\
\mathcal{N} &\equiv -J^\mu u_\mu, \\
\mathcal{J}^\mu &\equiv \Delta^{\mu\nu} J_\nu,
\end{aligned}$$

where d is the spatial dimensionality. The variables $\mathcal{E}$, $\mathcal{P}$, and $\mathcal{N}$ are all scalars, $\mathcal{Q}^\mu$ and $\mathcal{J}^\mu$ are transverse (relative to u^μ) vectors, and $\tau^{\mu\nu}$ is a transverse traceless symmetric two-tensor.

These quantities alone are not particularly useful, as they are merely definitions. It is the process of identifying these quantities with thermodynamic and hydrodynamic quantities that develops the *constitutive relations*. Here, the process about to be outlined diverges from the traditional approaches pioneered by Eckart, Landau, and Lifshitz.

Landau and Lifshitz made the association that

$$\begin{aligned}
T^{\mu\nu} &= \epsilon u^\mu u^\nu + (p+\tau)\,\Delta^{\mu\nu} + \tau^{\mu\nu}, \\
J^\mu &= n u^\mu + \mathcal{J}^\mu,
\end{aligned}$$

while Eckart wrote that

$$\begin{aligned}
T^{\mu\nu} &= \epsilon u^\mu u^\nu + (p+\tau)\,\Delta^{\mu\nu} + \mathcal{Q}^\mu u^\nu + \mathcal{Q}^\nu u^\mu + \tau^{\mu\nu}, \\
J^\mu &= n u^\mu,
\end{aligned}$$

where ϵ is the equilibrium energy density, p is the equilibrium pressure, and n is the equilibrium charge density. In both cases, τ, $\mathcal{Q}^\mu$, $\mathcal{J}^\mu$, and $\tau^{\mu\nu}$ still have not been assigned definitions in terms of the hydrodynamic variables. However, two *key* assumptions have already been made. In both theories the quantity T^{00} found by taking $T^{\mu\nu}u_\mu u_\nu$ in a locally co-moving reference frame is associated with the *equilibrium* energy density. Additionally, in both theories, J^0 (i.e. $-J^\mu u_\mu$ in a locally co-moving reference frame) is associated with the *equilibrium* charge density. In doing so, both have already made the assumption that charge density and energy density do not receive any corrections out of equilibrium based on our definitions of T, μ, and u^μ.

However, suppose we drop that supposition, and make no assumptions about the specific forms of the stress-energy tensor and charge current decompositions in terms of such things as the energy density. Let us rest solely on the fact that we wish to express the stress-energy tensor and the charge current in terms of the hydrodynamic variables. We will then assume that each of the quantities in the list above may be written as functions of the hydrodynamic variables. Further, we assume that the variables are slowly varying; if they were changing quickly, the system would not be near equilibrium, which is the regime of validity for hydrodynamics[7]. Given these assumptions, each of the quantities above may be written as a *derivative expansion* in the hydrodynamic variables [15]:

$$\begin{aligned}
\mathcal{E} &= \mathcal{E}_0(T,\mu,u) + \mathcal{E}_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right), \\
\mathcal{P} &= \mathcal{P}_0(T,\mu,u) + \mathcal{P}_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right), \\
\mathcal{N} &= \mathcal{N}_0(T,\mu,u) + \mathcal{N}_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right), \\
\mathcal{Q}^\mu &= \mathcal{Q}^\mu_0(T,\mu,u) + \mathcal{Q}^\mu_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right), \\
\mathcal{J}^\mu &= \mathcal{Q}^\mu_0(T,\mu,u) + \mathcal{J}^\mu_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right), \\
\tau^{\mu\nu} &= \tau^{\mu\nu}_0(T,\mu,u) + \tau^{\mu\nu}_1(\nabla T,\nabla\mu,\nabla u) + \mathcal{O}\left(\partial^2\right).
\end{aligned} \tag{2.10}$$

With these assumption made, the stress-energy tensor and the charge current are now functions of the hydrodynamic variables, and the conservation equations are now equations of motion for the hydrodynamic variables themselves. However, we have no way *a priori* of knowing what form the dependence of the stress-energy tensor and charge current on the hydrodynamic variables takes. As such, we must write all of the terms that are allowed by symmetry.

If we cut off the derivative expansion at first order in the derivative (as has been done above), the resulting system of equations is known as first-order hydrodynamics, or alternatively “viscous hydrodynamics”.

Let us first determine the zeroth-order dependence of the decomposition quantities. Let us analyze the equilibrium state where u^α points along the time axis. As well, for this analysis, assume a Minkowski background. As the system goes to equilibrium,

[7] Work has been done on hydrodynamics far from equilibrium; see for example [7][9].

all orders of the derivative expansion save for the zeroth order will vanish.

The quantity $\mathcal{E}_0(T,\mu,u)$ will clearly be, in equilibrium, T^{00}, which is the ideal fluid energy density $\epsilon(T,\mu)$. The quantity $\mathcal{P}_0(T,\mu,u)$ will be the coefficient of the spatial diagonal elements of the stress-energy tensor T^{ii}, or the pressure $p(T,\mu)$. The quantity $\mathcal{N}_0(T,\mu,u)$ is the zeroth component of the charge current; i.e. the ideal fluid charge density, $n(T,\mu)$. Since there is no way to form a transverse (to u^μ) vector out of just T, μ, u^μ, both $\mathcal{J}_0^\mu(T,\mu,u)$ and $\mathcal{Q}_0^\mu(T,\mu,u)$ must be zero. Similarly, $\tau_0^{\mu\nu}(T,\mu,u)=0$ because one cannot form a traceless, symmetric, transverse two-tensor out of just those three quantities.

At first order far more terms are allowed by Lorentz symmetry and parity. There are three scalars, three vectors, and one traceless, symmetric, transverse two-tensor. These building blocks are as follow:

$$\frac{u^\lambda\nabla_\lambda T}{T},\quad \nabla_\lambda u^\lambda,\quad u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right) \tag{2.11a}$$

$$u^\lambda\nabla_\lambda u^\mu,\quad \frac{\Delta^{\mu\lambda}\nabla_\lambda T}{T},\quad \Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right) \tag{2.11b}$$

$$\sigma^{\mu\nu}=\Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\nabla_\alpha u_\beta+\nabla_\beta u_\alpha-\frac{2}{d}\eta_{\alpha\beta}\nabla_\lambda u^\lambda\right) \tag{2.11c}$$

Each first-order term of the derivative expansion can be written as a linear combination of these building blocks, yielding

$$\mathcal{E}=\epsilon(T,\mu)+\varepsilon_1\frac{u^\lambda\nabla_\lambda T}{T}+\varepsilon_2\nabla_\lambda u^\lambda+\varepsilon_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right), \tag{2.12a}$$

$$\mathcal{P}=p(T,\mu)+\pi_1\frac{u^\lambda\nabla_\lambda T}{T}+\pi_2\nabla_\lambda u^\lambda+\pi_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right), \tag{2.12b}$$

$$\mathcal{N}=n(T,\mu)+\nu_1\frac{u^\lambda\nabla_\lambda T}{T}+\nu_2\nabla_\lambda u^\lambda+\nu_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right), \tag{2.12c}$$

$$\mathcal{Q}^\mu=\theta_1 u^\lambda\nabla_\lambda u^\mu+\theta_2\frac{\Delta^{\mu\lambda}\nabla_\lambda T}{T}+\theta_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right), \tag{2.12d}$$

$$\mathcal{J}^\mu=\gamma_1 u^\lambda\nabla_\lambda u^\mu+\gamma_2\frac{\Delta^{\mu\lambda}\nabla_\lambda T}{T}+\gamma_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right), \tag{2.12e}$$

$$\tau^{\mu\nu}=-\eta\sigma^{\mu\nu}=-\eta\Delta^{\mu\alpha}\Delta^{\nu\beta}\left\{\nabla_\alpha u_\beta+\nabla_\beta u_\alpha-\frac{2}{d}\eta_{\alpha\beta}\nabla_\lambda u^\lambda\right\}. \tag{2.12f}$$

These are all of the possible combinations of T,μ,u^μ that are allowed to zeroth and first order by Lorentz symmetry and parity. The set of coefficients that are present ($\{\varepsilon_i,\pi_i,\nu_i,\theta_i,\gamma_i,\eta\}$, $i\in\{1,2,3\}$) are called *transport coefficients*, as they relate the response of the stress-energy tensor to changes in the sources T, μ, and u^μ. They also describe derivative corrections to the ideal energy density, charge conductivity, heat flow, transverse charge flow, pressure, and shear. The quantity η is called the *shear viscosity*, and describes the resistance of the fluid to shearing. All of the transport coefficients are themselves functions of the hydrodynamic variables.

2.3 Ideal Charged Fluids

An "ideal fluid" is one where all of the transport coefficients are set to zero; i.e. where the derivative expansion is terminated at zeroth order. In this case, the constitutive relations take on the following simple form:

$$T^{\mu\nu} = \epsilon(T,\mu)u^\mu u^\nu + p(T,\mu)\Delta^{\mu\nu}, \tag{2.13a}$$

$$J^\mu = n(T,\mu)u^\mu. \tag{2.13b}$$

These equations are the equations of equilibrium as well; that is to say

$$T_0^{\mu\nu} = \epsilon(T_0,\mu_0)u_0^\mu u_0^\nu + p(T_0,\mu_0)\Delta_0^{\mu\nu},$$
$$J_0^\mu = n(T_0,\mu_0)u_0^\mu,$$

where $\Delta_0^{\mu\nu} = u_0^\mu u_0^\nu + g^{\mu\nu}$ and T_0, μ_0, and u_0^μ are the (constant) equilibrium values of those fields.[8] Since the manifold of equilibrium states is entirely characterized by β^μ and ψ, we can specify an equilibrium state by assigning equilibrium values to the hydrodynamic variables. We will refrain from assigning any value to T_0 and μ_0, but we *will* make the assumption that, in equilibrium, the fluid is at rest (i.e. $u_0^\mu = (1,\vec{0})$). This assumption does not fundamentally represent a loss of generality; a return to a more general equilibrium state can be achieved with the performance of a Lorentz boost.

Given some equilibrium state, we can look at the effect of adding a small, local

[8] We assume that the fluid is free from external fields. In the presence of some external fields, the equilibrium values may have a gradient, see for example the density of water in the presence of an external gravitational field in non-relativistic hydrodynamics [2].

perturbation to the fields in the form

$$T(x) = T_0 + \delta T(x), \qquad \mu(x) = \mu_0 + \delta\mu(x), \qquad u^\mu(x) = u_0^\mu + \delta u^\mu = \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix} + \begin{pmatrix} 0 \\ \vec{v}(x) \end{pmatrix}. \tag{2.14}$$

Keeping to only linear order in the perturbations, we get the perturbed, linearized stress-energy tensor and charge current:

$$\begin{aligned} T_0^{\mu\nu} + \delta T^{\mu\nu} &= \epsilon_0 \left(u_0^\mu u_0^\nu\right) + p_0 \Delta_0^{\mu\nu} + (\epsilon_0 + p_0)\left(\delta u^\mu u_0^\nu + u_0^\mu \delta u^\nu\right) + \left(\frac{\partial \epsilon}{\partial T}\delta T + \frac{\partial \epsilon}{\partial \mu}\delta\mu\right) u_0^\mu u_0^\nu \\ &\quad + \left(\frac{\partial p}{\partial T}\delta T + \frac{\partial p}{\partial \mu}\delta\mu\right) u_0^\mu u_0^\nu + \left(\frac{\partial p}{\partial T}\delta T + \frac{\partial p}{\partial \mu}\delta\mu\right) g^{\mu\nu}, \\ J_0^\mu + \delta J^\mu &= n_0 u_0^\mu + \left(\frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial \mu}\delta\mu\right) u_0^\mu + n_0 \delta u^\mu, \end{aligned}$$

where $\epsilon_0 \equiv \epsilon(T,\mu_0)$, $p_0 \equiv p(T_0, \mu_0)$, and $n_0 \equiv n(T_0, \mu_0)$. Due to the fact that the ideal equations and the equilibrium equations take the same form, we can note that $J_0^\mu = n_0 u_0^\mu$ and $T_0^{\mu\nu} = \epsilon_0 u_0^\mu u_0^\nu + p_0 \Delta_0^{\mu\nu}$. As such, we find (defining $w_0 \equiv \epsilon_0 + p_0$ to be the equilibrium enthalpy density) that

$$\begin{aligned} \delta T^{\mu\nu} &= w_0 \left(u_0^\mu \delta u^\nu + \delta u^\mu u_0^\nu\right) + \left(\left(\frac{\partial \epsilon}{\partial T} + \frac{\partial p}{\partial T}\right)\delta T + \left(\frac{\partial \epsilon}{\partial \mu} + \frac{\partial p}{\partial \mu}\right)\delta\mu\right) u_0^\mu u_0^\nu \\ &\quad + \left(\frac{\partial p}{\partial T}\delta T + \frac{\partial p}{\partial \mu}\delta\mu\right) g^{\mu\nu}, \end{aligned} \tag{2.15a}$$

$$\delta J^\mu = n_0 \delta u^\mu + \left(\frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial \mu}\delta\mu\right) u_0^\mu. \tag{2.15b}$$

Inserting the definitions for $\delta T^{\mu\nu}$ and δJ^μ in equations (2.15) into the conservation equations (2.8) yields the following conservation equations in Minkowski space:

$$\begin{aligned} \partial_\mu \delta T^{\mu\nu} &= \partial_\mu \Bigg[w_0 \left(\delta_t^\mu \delta_i^\nu + \delta_t^\nu \delta_i^\mu\right) v^i + \left(\left(\frac{\partial \epsilon}{\partial T} + s_0\right)\delta T + \left(\frac{\partial \epsilon}{\partial \mu} + n_0\right)\delta\mu\right) \delta_t^\mu \delta_t^\nu \\ &\quad + \left(s_0 \delta T + n_0 \delta\mu\right) \eta^{\mu\nu} \Bigg] = 0, \end{aligned} \tag{2.16a}$$

$$\partial_\mu \delta J^\mu = \partial_\mu \left[n_0 \delta_i^\mu v^i + \left(\frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial \mu}\delta\mu\right) \delta_t^\mu \right] = 0. \tag{2.16b}$$

While we could attempt to solve this problem in real-space, it is significantly easier to do the analysis in momentum space. Let us Fourier transform all of the perturbations into momentum space, and let us additionally without loss of generality align the wavevector k with the x-axis; by rotational invariance, this should not affect any of our results[9]. This transformation is functionally the same as assuming that the perturbations take the form of plane waves with wavevector pointing in the x direction, i.e. $\delta T = B_T e^{-i\omega t+ikx}$, $\delta\mu = B_\mu e^{-i\omega t+ikx}$, and $v^i = A^i e^{-i\omega t+ikx}$. If one makes this substitution in the above equations, one arrives at the following system of equations (writing only four dimenions for brevity; the transverse terms continue diagonally):

$$\begin{pmatrix} \frac{\partial\epsilon}{\partial T}(-i\omega) & \frac{\partial\epsilon}{\partial\mu}(-i\omega) & w_0(ik) & 0 & 0 \\ s_0(ik) & n_0(ik) & w_0(-i\omega) & 0 & 0 \\ \frac{\partial n}{\partial T}(-i\omega) & \frac{\partial n}{\partial\mu}(-i\omega) & n_0(ik) & 0 & 0 \\ 0 & 0 & 0 & w_0(-i\omega) & 0 \\ 0 & 0 & 0 & 0 & w_0(-i\omega) \end{pmatrix} \begin{bmatrix} B_T \\ B_\mu \\ A^x \\ A^y \\ A^z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2.17}$$

In order for the fluctuations to even exist, i.e. for the left-hand column vector be non-zero, the matrix must be singular. Taking the determinant of the matrix above and setting it equal to zero leads to the equation

$$\omega^3\left(\omega^2 w_0\left(\frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu} - \frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T}\right) + \left(n_0^2\frac{\partial\epsilon}{\partial T} + \frac{\partial n}{\partial\mu}s_0 w_0 - \frac{\partial\epsilon}{\partial\mu}n_0 s_0 - \frac{\partial n}{\partial T}n_0 w_0\right)k^2\right) = 0.$$

This equation gives the following five "dispersion relations"; relations that give the dependence of the angular frequency ω on the wavevector k. They are given by

$$\omega = 0, \qquad \omega = 0, \qquad \omega = 0,$$
$$\omega = \pm v_s k,$$

where

$$v_s^2 = \frac{n_0\left(n_0\frac{\partial\epsilon}{\partial T} - w_0\frac{\partial n}{\partial T}\right) - s_0\left(n_0\frac{\partial\epsilon}{\partial\mu} - w_0\frac{\partial n}{\partial\mu}\right)}{w_0\left(\frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu} - \frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T}\right)}.$$

We have no notion yet of what v_s is. While it looks complicated, it can actually be simplified quite nicely. We can use the following basic identities, two from the dependence of p on ϵ and n, and one from the Maxwell relation $(\partial s/\partial\mu)_T = (\partial n/\partial T)_\mu$

[9]Note that, should we boost to a moving Lorentz frame, this rotational invariance would be lost.

$$\frac{\partial p}{\partial T} = s_0 = p_\epsilon \frac{\partial \epsilon}{\partial T} + p_n \frac{\partial n}{\partial T}, \tag{2.18a}$$
$$\frac{\partial p}{\partial \mu} = n_0 = p_\epsilon \frac{\partial \epsilon}{\partial \mu} + p_n \frac{\partial n}{\partial \mu}, \tag{2.18b}$$
$$\frac{\partial \epsilon}{\partial \mu} = T \frac{\partial n}{\partial T} + \mu \frac{\partial n}{\partial \mu}. \tag{2.18c}$$

We can use these relations to eliminate $\frac{\partial \epsilon}{\partial T}$, $\frac{\partial \epsilon}{\partial \mu}$, and $\frac{\partial n}{\partial T}$ in favour of $p_\epsilon \equiv \left(\frac{\partial p}{\partial \epsilon}\right)_n$ and $p_n \equiv \left(\frac{\partial p}{\partial n}\right)_\epsilon$. Doing so will yield the far simpler form

$$v_s^2 = p_\epsilon + \frac{n_0}{w_0} p_n. \tag{2.19}$$

In a state with no charge in equilibrium, or a charged equilibrium state with an underlying conformal symmetry, equation (2.19) reduces to $v_s = \sqrt{p_\epsilon}$.

The dispersion relation $\omega = \pm v_s k$ may be re-written as $\omega^2 = v_s^2 k^2$. Since the perturbations are in the form of plane waves, this dispersion relation is clearly the requirement for a solution $\phi \propto e^{-i\omega t + i\vec{k}\cdot\vec{x}}$ to satisfy a differential equation of the form

$$\partial_t^2 \phi = v_s^2 \partial_i \partial^i \phi.$$

This is simply the wave equation, and so we can associate v_s with the **speed of sound** in the fluid.

2.4 First-Order Hydrodynamic Frames

As previously discussed, the hydrodynamic variables are not unique. One may re-define them at will, i.e. $T = T' + \delta T$. Such a change is known as a *change of frame* or a *frame redefinition*, and any particular choice of definition of T, μ, and u^μ is called a *hydrodynamic frame*[10]. The stress-energy tensor and charge current are independent of the frame choice, and so the transport coefficients must also transform during a change of frame in such a way as to keep the equations invariant. A redefinition of hydrodynamic variables is therefore accompanied by a change in transport coefficients.

[10] It is critical that these hydrodynamic frames not be confused with the Lorentz frames used in relativity. Any time a change of *Lorentz* frame occurs, it will be referred to explicitly as a "Lorentz boost", and Lorentz frames will be always explicitly referred to as either Lorentz frames or reference frames. The term "frame" on its own will be reserved solely for hydrodynamic frames.

There is, in fact, a one-to-one correspondence between a choice of values for the transport coefficients and a particular choice of frame. As such, from now on a particular choice of values for the transport coefficients will be called a hydrodynamic frame.

The transformation rules for each transport coefficient may be derived as follows. Consider the most general first-order frame redefinitions [15]:

$$\begin{aligned} T &= T' + \delta T = T' + a_1 \frac{u^\lambda \nabla_\lambda T}{T} + a_2 \nabla_\lambda u^\lambda + a_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right), \\ u^\mu &= (u')^\mu + \delta u^\mu = (u')^\mu + b_1 u^\lambda \nabla_\lambda u^\mu + b_2 \frac{\Delta^{\mu\lambda} \nabla_\lambda T}{T} + b_3 \Delta^{\mu\lambda} \nabla_\lambda \left(\frac{\mu}{T}\right), \\ \mu &= \mu' + \delta\mu = \mu' + c_1 \frac{u^\lambda \nabla_\lambda T}{T} + c_2 \nabla_\lambda u^\lambda + c_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right). \end{aligned}$$

By directly substituting these redefinitions into the constitutive relations, one can derive the transformation laws for the various transport coefficients. The vast majority of them will transform; only one, the shear viscosity, will be invariant under frame changes.

Keeping only to first order in derivatives of the hydrodynamic variables yields the rather nasty expressions for the stress-energy tensor

$$\begin{aligned} T^{\mu\nu} = &\Bigg[\epsilon(T', \mu') + \frac{\partial \epsilon}{\partial T}\left\{a_1 \frac{u^\lambda \nabla_\lambda T}{T} + a_2 \nabla_\lambda u^\lambda + a_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\right\} \\ &+ \frac{\partial \epsilon}{\partial \mu}\left\{c_1 \frac{u^\lambda \nabla_\lambda T}{T} + c_2 \nabla_\lambda u^\lambda + c_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\right\} \\ &+ \varepsilon_1 \frac{u^\lambda \nabla_\lambda T}{T} + \varepsilon_2 \nabla_\lambda u^\lambda + \varepsilon_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\Bigg] (u')^\mu (u')^\nu \\ &+ \Bigg[p(T', \mu') + \frac{\partial p}{\partial T}\left\{a_1 \frac{u^\lambda \nabla_\lambda T}{T} + a_2 \nabla_\lambda u^\lambda + a_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\right\} \\ &+ \frac{\partial p}{\partial \mu}\left\{c_1 \frac{u^\lambda \nabla_\lambda T}{T} + c_2 \nabla_\lambda u^\lambda + c_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\right\} \\ &+ \pi_1 \frac{u^\lambda \nabla_\lambda T}{T} + \pi_2 \nabla_\lambda u^\lambda + \pi_3 u^\lambda \nabla_\lambda \left(\frac{\mu}{T}\right)\Bigg] \left((u')^\mu (u')^\nu + \eta^{\mu\nu}\right) \\ &+ 2\left(\epsilon(T', \mu') + p(T', \mu')\right)\left(b_1 u^\lambda \nabla_\lambda u^{(\mu} + b_2 \frac{\Delta^{(\mu\lambda} \nabla_\lambda T}{T} + b_3 \Delta^{(\mu\lambda} \nabla_\lambda \left(\frac{\mu}{T}\right)\right) \\ &\times (u')^{\nu)} + 2\left(\theta_1 u^\lambda \nabla_\lambda u^{(\mu} + \theta_2 \frac{\Delta^{(\mu\lambda} \nabla_\lambda T}{T} + \theta_3 \Delta^{(\mu\lambda} \nabla_\lambda \left(\frac{\mu}{T}\right)\right)(u')^{\nu)} - \eta \sigma^{\mu\nu}, \end{aligned}$$

and the charge current

$$\begin{aligned}
J^\mu = &\Bigg[n(T',\mu') + \frac{\partial n}{\partial T}\Bigg\{a_1\frac{u^\lambda\nabla_\lambda T}{T} + a_2\nabla_\lambda u^\lambda + a_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right)\Bigg\} \\
&+ \frac{\partial n}{\partial \mu}\Bigg\{c_1\frac{u^\lambda\nabla_\lambda T}{T} + c_2\nabla_\lambda u^\lambda + c_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right)\Bigg\} \\
&+ \nu_1\frac{u^\lambda\nabla_\lambda T}{T} + \nu_2\nabla_\lambda u^\lambda + \nu_3 u^\lambda\nabla_\lambda\left(\frac{\mu}{T}\right)\Bigg](u')^\mu \\
&+ n(T',\mu')\left(b_1 u^\lambda\nabla_\lambda u^{(\mu} + b_2\frac{\Delta^{\mu\lambda}\nabla_\lambda T}{T} + b_3\Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right)\right) \\
&\gamma_1 u^\lambda\nabla_\lambda u^\mu + \gamma_2\frac{\Delta^{\mu\lambda}\nabla_\lambda T}{T} + \gamma_3\Delta^{\mu\lambda}\nabla_\lambda\left(\frac{\mu}{T}\right),
\end{aligned}$$

where $X^{(\mu\nu)} = \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right)$. The quantity $\sigma^{\mu\nu}$ is invariant under a first-order frame-redefinition up to second order, and so it has not been written out in full. In order for the form of the equations to remain the same the transport coefficients must obey the following transformation laws:

$$\begin{aligned}
\varepsilon_i &\to \varepsilon_i' - \left(\frac{\partial\epsilon}{\partial T}\right)_\mu a_i - \left(\frac{\partial\epsilon}{\partial\mu}\right)_T c_i, \\
\pi_i &\to \pi_i' - \left(\frac{\partial p}{\partial T}\right)_\mu a_i - \left(\frac{\partial p}{\partial\mu}\right)_T c_i, \\
\nu_i &\to \nu_i' - \left(\frac{\partial n}{\partial T}\right)_\mu a_i - \left(\frac{\partial n}{\partial\mu}\right)_T c_i, \\
\theta_i &\to \theta_i' - (\epsilon + p)\, b_i, \\
\gamma_i &\to \gamma_i' - n b_i, \\
\eta &\to \eta'.
\end{aligned}$$

As previously stated, the quantity η is invariant under field redefinitions. There are six other frame-invariant quantities that can be formed out of the transport coefficients. They are given by [15]

$$f_i \equiv \pi_i - p_\epsilon\varepsilon_i - p_n\nu_i, \tag{2.21a}$$

$$\ell_i \equiv \gamma_i - \frac{n}{w}\theta_i, \tag{2.21b}$$

where $\{i \in 1,2,3\}$ and $w = \epsilon + p$. These six quantities, along with the shear viscosity, are the only frame-invariants that we can form at first-order. Only five of these invariants are actually independent; the second law of thermodynamics imposes the "thermodynamic condition" that

$$\gamma_1 = \gamma_2, \qquad \theta_1 = \theta_2, \qquad \Longrightarrow \qquad \ell_1 = \ell_2. \tag{2.22}$$

The shear viscosity differs from the other invariants in one important way; it is an example of what is called a "physical transport coefficient". It is so called because its positivity leads directly to positive entropy production. Any transport coefficient that directly leads to entropy production is a physical transport coefficient.

There are three such physical transport coefficients in charged hydrodynamics at first order: the shear viscosity η, the bulk viscosity ζ, and the charge conductivity σ. These latter two quantities are defined in terms of the invariants by [15]

$$\zeta \equiv -f_2 + \frac{\left(w\frac{\partial n}{\partial \mu} - n\frac{\partial \epsilon}{\partial \mu}\right) f_1 + \left(n\left(\frac{\partial \epsilon}{\partial T} + \frac{\mu}{T}\frac{\partial \epsilon}{\partial \mu}\right) - w\left(\frac{\partial n}{\partial T} + \frac{\mu}{T}\frac{\partial n}{\partial \mu}\right)\right) f_3}{T\left(\frac{\partial \epsilon}{\partial T}\frac{\partial n}{\partial \mu} - \frac{\partial \epsilon}{\partial \mu}\frac{\partial n}{\partial T}\right)}, \tag{2.23a}$$

$$\sigma \equiv \frac{n}{\epsilon + p}\ell_1 - \frac{1}{T}\ell_3. \tag{2.23b}$$

There is also one more invariant, given by

$$\chi_T \equiv \frac{1}{T}\left(\ell_2 - \ell_1\right).$$

However, the condition $\ell_2 - \ell_1 = 0$ will force this transport parameter to be identically zero.

The three physical first-order transport coefficients characterize respectively the resistance of the fluid to shearing, resistance to bulk deformations, and ease of charge flow. In the case where the underlying theory obeys conformal symmetry, the bulk viscosity must be zero.

The bulk viscosity, similarly to the speed of sound, can be greatly simplified if we express it in terms of p_ϵ, p_n. Using equations (2.18) again, ζ becomes

$$\begin{aligned}\zeta &= \left(p_\epsilon \pi_1 - \pi_2\right) + p_\epsilon\left(\varepsilon_2 - p_\epsilon \varepsilon_1\right) \\ &\quad + \frac{1}{T}p_n\left(\pi_3 - p_\epsilon \varepsilon_3\right) + p_n\left(\nu_2 - p_\epsilon \nu_1\right) - \frac{1}{T}\left(p_n\right)^2 \nu_3.\end{aligned} \tag{2.24}$$

These physical transport coefficients may be brought into the equations of motion by substituting for one of the transport coefficients in their definitions. Typically, it is easiest to substitute ζ in for π_2 and σ in for γ_3.

2.5 The Landau-Lifshitz and Eckart Frames

There are two hydrodynamic frames (really *classes* of hydrodynamic frames) that are especially convenient and have been used frequently in the literature. These two frames are the Landau-Lifshitz frame (often just called the Landau frame), named for the authors of the famous textbook series who introduced it in their book on fluid dynamics [2], and the Eckart frame, named for theorist and geophysicist Carl Eckart[11] [1] who first used it in his papers on irreversible thermodynamic processes.

The primary difference between the two is the alignment of the fluid velocity u^μ. In the Landau frame, the fluid velocity is aligned with the heat flow, and as such $\mathcal{Q}^\mu = 0$. In the Eckart frame, the fluid velocity is aligned with the flow of charge, and as such $\mathcal{J}^\mu = 0$. Both of these frames are otherwise very simple, and only have a few transport coefficients.

We shall first investigate the Landau frame and identify its major properties and shortcomings, and then show that the Eckart frame, though it gives rise to different equations of motion, leads to the same shortcomings. Namely, both frames actually predict that any thermal equilibrium should be unstable if the fluid is moving uniformly, and also disturbances should propagate superluminally!

This is an obviously ridiculous prediction, and Chapter 3 will address one method to resolve these shortcomings.

2.5.1 The Landau Frame

One may arrive at the Landau frame from the previously outlined approach by setting $\varepsilon_i = \theta_i = \nu_i = 0$, $\{i \in 1, 2, 3\}$. Strictly speaking, since there are transport coefficients that do not have specified values (π_i, γ_i), the Landau "frame" is actually a *class* of hydrodynamic frames. However, all of these frames have the same functional form, since all of the remaining transport coefficients can be written in terms of the frame-invariants f_i and ℓ_i.

[11] Eckart is better known for the development of the Wigner-Eckart theorem in quantum mechanics. While Eckart worked in theoretical physics for the first half of his career, in the second half he served as a professor of geophysics as the University of California San Diego, and it was in this capacity that he derived his formulation of relativistic hydrodynamics.

In this class of frames, the stress-energy tensor and charge current are given (bearing in mind that, because $\varepsilon_i = \nu_i = 0$, it is the case that $\pi_i = f_i$) by

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p + f_1 \frac{u^\lambda \partial_\lambda T}{T} + f_2 \partial_\lambda u^\lambda + f_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) \Delta^{\mu\nu} - \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right) + \mathcal{O}(\partial^2), \tag{2.25a}$$

$$J^\mu = n u^\mu + \ell_1 \left(u^\lambda \partial_\lambda u^\mu + \frac{1}{T}\Delta^{\mu\lambda}\partial_\lambda T\right) + \ell_3 \Delta^{\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right) + \mathcal{O}(\partial^2). \tag{2.25b}$$

The ideal-order conservation equations (2.16) for charge current and entropy current can be used to re-write the terms proportional to f_i solely in terms of ζ – this is valid since any errors introduced by using the ideal equations will only enter at second order. For details on how this substitution works, as well as derivations of the conservation equations for ideal charge current and entropy current, see appendix A.

We can do something similar with the ℓ_i's; however, given there is only one transverse ideal equation, only one term can be removed in this way. In this particular instance we choose to eliminate the term proportional to $u^\lambda \partial_\lambda u^\mu$, leaving only terms proportional to $\frac{1}{T}\Delta^{\mu\lambda}\partial_\lambda T$ and $\frac{1}{T}\Delta^{\mu\lambda}\partial_\lambda \mu$.

The thermodynamic consistency condition (2.22) allows us to combine the two remaining terms into one term proportional to $\Delta^{\mu\lambda}\partial_\lambda\left(\frac{\mu}{T}\right)$. We can also use the definition of σ (2.23b) to remove ℓ_3:

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p - \zeta \partial_\lambda u^\lambda\right)\Delta^{\mu\nu} - \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right) + \mathcal{O}(\partial^2), \tag{2.26a}$$

$$J^\mu = n u^\mu - \sigma \Delta^{\mu\lambda}\partial_\lambda\left(\frac{\mu}{T}\right) + \mathcal{O}(\partial^2). \tag{2.26b}$$

Having made these substitutions and significantly cleaned up the equations, let us again create a small perturbation away from equilibrium as in (2.14). We then arrive at the following equations for the perturbations of the stress-energy tensor and the charge current:

$$\delta T^{\mu\nu} = w_0 \left(\delta u^\mu u_0^\nu + u_0^\mu \delta u^\nu\right) + \left(\frac{\partial \epsilon}{\partial T}\delta T + \frac{\partial \epsilon}{\partial \mu}\delta\mu\right) u_0^\mu u_0^\nu$$
$$+ \left(s_0 \delta T + n_0 \delta\mu - \zeta \partial_\lambda \delta u^\lambda\right) \Delta_0^{\mu\nu}$$
$$- \eta \Delta_0^{\mu\alpha} \Delta_0^{\nu\beta} \left(\partial_\alpha \delta u_\beta + \partial_\beta \delta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda \delta u^\lambda\right) + \mathcal{O}\left(\partial^2\right), \tag{2.27a}$$

$$\delta J^\mu = \left(\frac{\partial n}{\partial T}\delta T + \frac{\partial n}{\partial \mu}\delta\mu\right) u_0^\mu - \frac{\sigma}{T}\Delta_0^{\mu\lambda}\left(\partial_\lambda \delta\mu - \frac{\mu}{T}\partial_\lambda \delta T\right) + \mathcal{O}\left(\partial^2\right). \tag{2.27b}$$

Assuming the perturbations are once again in the form of plane waves with wavevector pointing in the $\hat{x}$ direction, the conservation equations for (2.27) may be written in matrix form (with the conserved charge current equation being shifted to the second row, and with four spacetime dimensions for compactness) as in equation (2.17). The matrix equation is given by

$$\begin{pmatrix} -i\frac{\partial\epsilon}{\partial T}\omega & -i\frac{\partial\epsilon}{\partial\mu}\omega & iw_0 k & 0 & 0 \\ -i\frac{\partial n}{\partial T}\omega - \frac{\mu}{T^2}\sigma k^2 & -i\frac{\partial n}{\partial\mu}\omega + \frac{\sigma}{T}k^2 & 0 & 0 & 0 \\ is_0 k & in_0 k & -iw_0\omega + \zeta k^2 + \eta\frac{2(d-1)}{d}k^2 & 0 & 0 \\ 0 & 0 & 0 & -iw_0\omega + \eta k^2 & 0 \\ 0 & 0 & 0 & 0 & -iw_0\omega + \eta k^2 \end{pmatrix}$$
$$\times \begin{bmatrix} B_T \\ B_\mu \\ A^x \\ A^y \\ A^z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In order for these equations to be soluble, the matrix must be singular, and so the determinant may be set to zero. This yields the following controlling equation:

$$\left(-iw_0\omega + \eta k^2\right)^{d-1} F(\vec{v}_0 = 0, \omega, k) = 0, \tag{2.28}$$

where

$$
\begin{aligned}
F(\vec{v}_0 = 0, \omega, k) = &\left(i \left(\frac{\partial \epsilon}{\partial \mu} \frac{\partial n}{\partial T} - \frac{\partial \epsilon}{\partial T} \frac{\partial n}{\partial \mu} \right) w_0 \right) \omega^3 \\
&+ \left(\frac{w_0}{T^2} \left(\frac{\partial \epsilon}{\partial T} T + \frac{\partial \epsilon}{\partial \mu} \mu \right) \sigma - \left(\frac{\partial \epsilon}{\partial \mu} \frac{\partial n}{\partial T} - \frac{\partial \epsilon}{\partial T} \frac{\partial n}{\partial \mu} \right) \gamma_s \right) k^2 \omega^2 \\
&+ \frac{i\omega}{T^2} \left[\gamma_s \left(\frac{\partial \epsilon}{\partial T} T + \frac{\partial \epsilon}{\partial \mu} \mu \right) \sigma k^4 \right. \\
&+ \left. T^2 \left(n \left(\frac{\partial \epsilon}{\partial T} n - \frac{\partial \epsilon}{\partial \mu} s \right) - w \left(\frac{\partial n}{\partial T} n - \frac{\partial n}{\partial \mu} s \right) \right) k^2 \right] \\
&- \frac{w_0^2}{T^2} \sigma k^4,
\end{aligned} \tag{2.29}
$$

with $\gamma_s = \left(\zeta + \frac{2(d-1)}{d} \eta \right)$. There are $d-1$ copies of the same expression that factor out from the main set; these are "modes of propagation" of quantities transverse to the propagation direction of the perturbations; they represent the diffusion of transverse momentum. In $d = 3$, there are two "transverse modes" representing the diffusion of the y and z components of linear momentum.

There are two main questions that concern us: whether the equilibrium state is stable against perturbations, and whether said perturbations propagate causally. These questions were discussed extensively by Lindblom and Hiscock in [21] and [22]. Mathematically, these two questions may be posed as the following constraints on the roots of the equation (2.28):

$$
\textbf{Stability:} \qquad \mathrm{Im}\,(\omega(k)) \leq 0, \tag{2.30}
$$

$$
\textbf{Causality:} \qquad 0 < \left(\lim_{k \to \infty} \frac{\mathrm{Re}\,(\omega(k))}{k} \right)^2 < 1. \tag{2.31}
$$

The stability constraint must be true at all k, but the causality constraint only need hold true at large k. There is another point that should be made – the causality condition supposes that the dispersion relation at large k is linear in k. If it is not, as we shall show presently, the differential equations representing these large-k modes are either *parabolic*, and as such propagate information instantaneously, or non-propagating.

In order to answer these questions, two different limits of equation (2.28) must be investigated: the small-k limit, i.e. the long-range limit in real-space, and the large-k limit, i.e. the local limit. The long-range limit is the range of applicability of hydrodynamics, since hydrodynamics is a framework for long-distance phenomena. The short-range limit is where we can look at qualities such as causality and propagation.

The $(d-1)$ transverse, or "shear", modes each have a dispersion relation given by

$$\omega = -\frac{i\eta k^2}{w_0}. \tag{2.32}$$

Immediately, there is a problem. This mode is stable, as shall be shown presently. However, since this solution is exact, it should be true for all values of k, including when $k \to \infty$. This dispersion relation is not linear in k. If this mode is to causally propagate, its dispersion relation must satisfy a **hyperbolic** differential equation, but this dispersion relation satisfies a parabolic differential equation.

Given a plane wave solution representing transverse momentum $\pi_\perp(t,x) \propto e^{-i\omega t + ikx}$, this dispersion relation is the requirement for the existance of a solution to the differential equation

$$\frac{\partial \pi_\perp}{\partial t} - \frac{\eta}{w_0}\partial_x^2 \pi_\perp = 0.$$

The discriminant of this equation is zero, and as such the equation is parabolic. While this is what one might expect for a mode describing diffusion, because its regime of validity extends up to large k, this mode will have instantaneous propagation, and as such will be acausal.

Despite this setback, let us press on.

The quantity $F(\vec{v}_0 = 0, \omega, k)$ is a cubic polynomial in ω and will, in general, have quite complicated roots. To simplify matters, we will investigate its solutions in both the large-k and small-k limits, rather than trying to derive its exact solutions.

In the limit that $k \to 0$, we get the following dispersion relations:

$$\omega = -iD_n k^2 + \mathcal{O}\left(k^3\right), \tag{2.33a}$$

$$\omega = \pm v_s k - i\Gamma k^2 + \mathcal{O}\left(k^3\right), \tag{2.33b}$$

where

$$D_n = \left(\frac{p_\epsilon^2}{v_s^4\left(\frac{\partial n}{\partial \mu} - \frac{n^2}{w^2 v_s^2}\right)}\right)\frac{\sigma}{T}, \tag{2.34}$$

is the diffusion constant for the diffusion of charge, and

$$\Gamma = \frac{1}{2w_0}\left(\gamma_s + \frac{p_n^2}{T v_s^2}\sigma\right), \tag{2.35}$$

is the diffusion constant for the diffusion of the longitudinal momentum. The controlling equation for the first mode is clearly (for some $\phi \propto e^{-i\omega t + ikx}$)

$$\frac{\partial \phi}{\partial t} = D_n \nabla^2 \phi,$$

which is the heat equation, and represents pure diffusion. The controlling equation for the second mode is

$$\frac{\partial \phi}{\partial t} = \mp v_s \nabla \phi + \Gamma \nabla^2 \phi.$$

This is a modified version of the wave equation that contains diffusion as well as linear propagation. Note that, because these dispersion relations are only expected to hold for small k, their parabolic nature does not pose a threat to causality.

Returning to the question of stability and causality, we begin with the transverse mode's dispersion relation (2.32). If $\eta > 0$ and $w_0 > 0$, then this mode is stable for all k. The condition $w_0 > 0$ is enforced by thermodynamics, and $\eta > 0$ is a direct result of the second law of thermodynamics. As such, the transverse modes are stable; and since the modes are exact, they are stable for all k. As was already demonstrated, they are not causal.

For the longitudinal modes, let us first look at the charge diffusion mode. We require that $D_n > 0$. This is equivalent to $\sigma > 0$ and $\frac{\partial n}{\partial \mu} > n_0^2 / w_0^2 v_s^2$, which are required to be true by the thermodynamics and the positivity of entropy production. Therefore, the charge diffusion mode is stable as $k \to 0$.

For the sound mode, the stability is solely dependent upon the positivity of Γ. This corresponds to demanding that $\eta > 0$, $\zeta > 0$, and $\sigma > 0$. These are all true by the second law of thermodynamics, and as such, the sound mode is also stable.

Finally, let us investigate the causality of the longitudinal modes. At large k, the dispersion relations are again non-linear. Neglecting the actual values of the coefficients, the modes are schematically

$$\begin{aligned}
\omega &= -i\alpha_1 + \mathcal{O}\left(\frac{1}{k^2}\right), \\
\omega &= -i\frac{\gamma_s}{w_0}k^2 + i\alpha_2 + \mathcal{O}\left(\frac{1}{k^2}\right), \\
\omega &= -i\alpha_3 k^2 - i\alpha_4 + \mathcal{O}\left(\frac{1}{k^2}\right).
\end{aligned}$$

All of the α_i are positive. The first mode is controlled by the equation $\frac{\partial \phi}{\partial t} + \alpha_1 \phi = 0$;

The x-dependence of ϕ will separate, and there will be no propagation at all. This is therefore a non-propagating mode.

For the modes that have a term proportional to k^2, the equation governing them is

$$\frac{\partial \phi}{\partial t} + a\frac{\partial^2 \phi}{\partial x^2} + b\phi = 0.$$

This equation is also parabolic, and the same issues will arise as before.

We have found a troubling fact – None of the large-k modes in the Landau frame are hyperbolic. They are therefore acausal (or non-propagating). Already, one might be inclined to look for a different theory that does not suffer from these shortcomings. However, let us drive the final nail into the coffin. While the theory was acausal, it at least predicted stable equilibrium states. This is only true for $\vec{v}_0 = 0$. As soon as we perform a Lorentz boost to a uniformly moving description, we lose stability as well.

2.5.2 Moving Frames

To investigate a uniformly moving fluid we will take the wave 4-vector $\tilde{p}^\mu = (\omega, \vec{k})$ and boost it in the $\hat{x}$-direction. This will not represent a fundamental loss of generality, since rotational invariance of the locally co-moving Lorentz frame implies that the only thing that matters is the angle between the boosted wavevector $\vec{k}$ and the boosted velocity $\vec{v}_0$.

In order to see that stability immediately falls by the wayside, one need merely investigate the shear modes; they will illustrate the point well enough.

For a boost in the $\hat{x}$-direction, the old components of the momentum are given in terms of the boosted components by (writing only 4 dimensions for brevity)

$$(\omega = \frac{\omega' - k' \cdot v_0}{\sqrt{1 - v_0^2}}, k_x = \frac{k'_x - \omega' v_0}{\sqrt{1 - v_0^2}}, k_y = k'_y, k_z = k'_z).$$

where v_0 is the uniform velocity associated with the movement of the fluid as a whole. These relations between $(\omega, \vec{k})$ and $(\omega', \vec{k}')$ can be used to express k^2 in terms of k' and ω', since k^2 is what actually appears in the transverse mode (and in $F(\omega, k)$ for that matter)

$$
\begin{aligned}
k^2 = k_x^2 + k_y^2 + k_z^2 &= \frac{1}{1-v_0^2}\left(k_x' - \omega' v_0\right)^2 + (k_y')^2 + (k_z')^2 \\
&= \frac{1}{1-v_0^2}\left((k_x')^2 - 2k_x'\omega' v_0 + (\omega')^2 v_0^2 + (k_y')^2 + (k_z')^2 - v_0^2(k_y')^2 - v_0^2(k_z')^2\right) \\
&= \frac{1}{1-v_0^2}\left((k')^2 + v_0^2(k_x')^2 - v_0^2(k_x')^2 + (\omega')^2 v_0^2 - 2\omega'\left(k'\cdot v_0\right) - v_0^2(k_y')^2 - v_0^2(k_z')^2\right) \\
&= \frac{1}{1-v_0^2}\left(\left(1-v_0^2\right)\left(k'\right)^2 + v_0^2(\omega')^2 - 2\omega'\left(k'\cdot v_0\right) + \left(k'\cdot v_0\right)^2\right).
\end{aligned}
$$

So, ultimately, we make the following transformations (dropping the primes):

$$
\omega \to \gamma_P\left(\omega - k\cdot v_0\right), \quad k^2 \to k^2 + \gamma_P^2\left(v_0^2\omega^2 - 2\omega\left(k\cdot v_0\right) + \left(k\cdot v_0\right)^2\right), \tag{2.36}
$$

where $\gamma_P = (1-v_0^2)^{-1/2}$. Performing these transformations on the transverse mode's dispersion relation (2.32) yields

$$
\omega = -\frac{i\eta k^2}{w_0} \to \gamma_P\left(\omega - k\cdot v_0\right) = -\frac{i\eta}{w_0}\left(k^2 + \gamma_P^2\left(v_0^2\omega^2 - 2\omega\left(k\cdot v_0\right) + \left(k\cdot v_0\right)^2\right)\right).
$$

This is now a quadratic polynomial in ω. Investigating the small-k limit is sufficient to show the shear mode's newfound instability. The small-k limit now yields two dispersion relations instead of one:

$$
\omega = \left(k\cdot v_0\right) - i\frac{\sqrt{1-v_0^2}\,\eta}{w_0}\left(k^2 - \left(k\cdot v_0\right)^2\right) + \mathcal{O}\left(k^3\right), \tag{2.37a}
$$

$$
\omega = i\frac{\sqrt{1-v_0^2}}{v_0^2\eta}w_0 + \frac{\left(2-v_0^2\right)}{v_0^2}\left(k\cdot v_0\right) + \mathcal{O}\left(k^2\right). \tag{2.37b}
$$

It is clear that (2.37a) remains stable as $k \to 0$; however, the other dispersion relation given by (2.37b) approaches a constant, positive imaginary value as $k \to 0$. The very condition that imposed stability of the shear mode when $\vec{v}_0 = 0$ now in turn dooms this new mode – if $\eta/w_0 > 0$, then the mode is unstable at small k, and any equilibrium state is unstable.

This result is clearly ridiculous – water does not explode if set in motion. The time for the mode to diverge is non-trivial as well – Lindblom and Hiscock found [21] that in Landau-Eckart type theories, perturbations away from equilibrium in a glass of water at room temperature and pressure would diverge with a characteristic time of 10^{-34} seconds or less[12]. This is clearly fatal to the theory. This issue, along with the causality problem, will be resolved in the next chapter.

For the purposes of comparison, let us also briefly analyze the Eckart frame. Both the Landau and Eckart frames belong to a class of frames found by Lindblom and Hiscock in [21] to be generically unstable to perturbations.

2.6 The Eckart Frame

This section does not aim to give a comprehensive overview of the Eckart frame, but rather aims simply to show that the same flaws that victimize the Landau frame are also present in the Eckart case. In most regards the Eckart frame is similar to the Landau frame, save for the alignment of the fluid velocity. While the Landau frame fluid velocity aligns with the heat flow, thereby rendering $\mathcal{Q}^\mu = 0$, the Eckart frame aligns the fluid velocity with the flow of charge, thereby rendering $\mathcal{J}^\mu = 0$. This is accomplished by setting $\varepsilon_i = \nu_i = \gamma_i = 0$. With this frame choice the charge current becomes extremely simple, but the stress-energy tensor becomes somewhat more complicated. The stress-energy tensor and the charge current are given by

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p + f_1 u^\lambda \partial_\lambda T + f_2 \partial_\lambda u^\lambda + f_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) \Delta^{\mu\nu} \tag{2.38a}$$

$$- 2\frac{w}{n}\ell_1 \left(u^\lambda \partial_\lambda u^{(\mu} + \frac{1}{T}\Delta^{(\mu\lambda}\partial_\lambda T\right) u^{\nu)} - 2\frac{w}{n}\ell_3 \Delta^{(\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right) u^{\nu)} - \eta\sigma^{\mu\nu}, \tag{2.38b}$$

$$J^\mu = n u^\mu. \tag{2.38c}$$

Making the same substitutions using the ideal-order equations (2.16) as in the Landau frame yields

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p - \zeta \partial_\lambda u^\lambda\right) \Delta^{\mu\nu} + 2\frac{w_0}{n_0} T\sigma \Delta^{(\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right) u^{\nu)} - \eta\sigma^{\mu\nu}, \tag{2.39a}$$

$$J^\mu = n u^\mu. \tag{2.39b}$$

Perturbing the equilibrium state and then passing to momentum space once again yields a linear system of equations in $B_T, B_\mu, A^x, A^y, A^z, ...$, as in the Landau frame. Ensuring that the coefficient matrix of that system of equations is singular yields a

[12] In all Landau-Eckart type theories *except* the Landau theory, these instabilities appear for a fluid at rest. The Landau theory is the singular limit of these types of theories; Lindblom and Hiscock do not explicitly give a value for the upper bound of the characteristic time, but claim it is also extremely short.

polynomial in angular frequency ω, with coefficients that are functions of wavevector norm k^2 as well as various thermodynamic quantities. The roots of that polynomial give the dispersion relations for the modes of propagation for the charged fluid. The transverse modes obey the same dispersion relation as in the Landau frame.

$$\omega = -\frac{i\eta k^2}{w_0}. \tag{2.40}$$

Had we made a different choice regarding which terms were eliminated via the ideal-order equations, this equation would look more complicated, and have an inequivalent controlling equation.

Turning now to the longitudinal modes, we can again look at the small-k limit and the large-k limit for $\vec{v}_0 = 0$. The small-k limit gives the following longitudinal modes:

$$\begin{aligned} \omega &= -iD_n k^2 + \mathcal{O}\left(k^3\right), \\ \omega &= \pm v_s k - i\frac{\Gamma}{2w_0}k^2 + \mathcal{O}\left(k^3\right), \end{aligned}$$

where v_s is the usual speed of sound (c.f. equation (2.19)), and

$$\begin{aligned} D_n &= \frac{\left(p_\epsilon\right)^2 \sigma}{v_s^2\left(\frac{\partial n}{\partial \mu}v_s^2 - \frac{n_0^2}{w_0}\right)}, \\ \Gamma &= \gamma_s + \left(\frac{p_n}{v_s}\right)^2 \sigma. \end{aligned}$$

The large k mode again has a non-linear dispersion relation, which is still problematic for the reasons outlined in the previous section. The large-k dispersion relations are quite complicated, and so will not be repeated here, as a detailed analysis of their structure would yield little useful information.

The same issues as in the Landau frame obviously persist in the Eckart frame, since the transverse modes are the same. What if we make a different choice for which terms to eliminate with the ideal-order equations? Eliminating $\Delta^{\mu\lambda}\partial_\lambda\left(\frac{\mu}{T}\right)$

instead yields

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + \left(p - \zeta \partial_\lambda u^\lambda\right) \Delta^{\mu\nu} - 2\frac{w_0^2}{n_0^2}\sigma \left(u^\lambda \partial_\lambda u^{(\mu} + \frac{\Delta^{(\mu\lambda}\partial_\lambda T}{T}\right) u^{\nu)} - \eta\sigma^{\mu\nu}, \tag{2.41a}$$

$$J^\mu = n u^\mu. \tag{2.41b}$$

For these constitutive relations perturbing the equilibrium state yields the following controlling equation for the transverse dispersion relations:

$$\sigma\omega^2 - i\frac{n^2}{w}\omega + \frac{n^2}{w^2}\eta k^2 = 0.$$

This is obviously not the same as equation (2.40). There are two modes with small-k dispersion relations given by

$$\omega = i\frac{n_0^2}{w_0\sigma} + \mathcal{O}\left(k^2\right),$$
$$\omega = -i\frac{\eta}{w_0}k^2 + \mathcal{O}\left(k^3\right).$$

The first mode is unstable, since $\sigma > 0$. This analysis makes clear an important fact: differing frames yield inequivalent dispersion relations, and therefore inequivalent controlling equations.

2.7 Gapped and Gapless Modes

Refering momentarily back to the Landau frame, we can note that in the case where $\vec{v}_0 = 0$, all of the small-k modes have the property that $\omega(k \to 0) = 0$, i.e. the mode is **gapless**.

However, after performing a Lorentz boost a new mode appeared that did not have this property. Instead, it had the property that $\omega(k \to 0) = i\Omega$ where Ω is some real constant. The angular frequency is non-zero even when the wave is not propagating, and dispersion happens even without movement; the mode is then said to be **gapped**.

While these gapped modes only appeared upon performing a Lorentz boost in the Landau frame, they appeared in the rest frame for the Eckart frame in the second case investigated. Additionally, they are ubiquitous in the general frame that is the central topic of the next chapter. As such it is worthwhile to briefly discuss the place of these gapped modes in hydrodynamics as a whole.

Hydrodynamics is a theory regarding conserved densities. Any such conserved density can only change its value in one manner – through fluxes to a different location. Consider a given equilibrium state with uniform energy density ϵ. If we shift the whole system **uniformly** to a new energy density $\epsilon' = \epsilon + \epsilon_0$, where ϵ_0 is a constant, the system can never return to the old equilibrium state, because the density can only change via fluxes, and the distribution is already uniform. The relaxation time is infinite.

Consider now a perturbation of the form $\epsilon' = \epsilon + \delta\epsilon = \epsilon + \epsilon_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}}$, i.e. the plane wave that arose from transforming to momentum space. If the perturbation is uniformly distributed, then it has no dependence on $\vec{x}$: this is equivalent to demanding that $\vec{k} = 0$. For a gapless mode, if $\vec{k} = 0$, then $\omega = 0$ as well, and the exponential $e^{-i\omega t + ikx} = 1$, leaving the perturbation as a constant shift $\epsilon' = \epsilon + \epsilon_0$. The relaxation time is simply $\tau = 1/\omega$, and so the relaxation time is infinite: exactly the behaviour we would expect for the types of conserved quantities that are considered in hydrodynamics.

Conversely, consider the same perturbation with a **gapped** mode. In this case, a uniformly distributed density (i.e. $\vec{k} = 0$) does *not* lead to $\omega = 0$ – rather, $\omega = i\Omega$, where $\Omega < 0$ implies stability, and $\Omega > 0$ diverges. The value Ω is the size of the gap, and the characteristic time (either for relaxation or divergence) for such a system is $\tau = 1/\Omega$ – decidedly non-infinite for non-zero Ω. If Ω is negative, the uniform distribution will uniformly decay until it returns to the original equilibrium state.

This is not possible with conserved densities, and so hydrodynamics has nothing to say about these modes – they are "non-hydrodynamic modes". Regardless of the physics of these modes, it is still important to ensure their stability (i.e. that $\Omega < 0$); when the full, non-linear hydrodynamic equations are solved numerically, these modes will still arise and will diverge if not properly treated, ruining the numerics and making the equations unsolvable. Conversely, if the gapped modes are stable, the equations are soluble, and the numerics will be reliable.

Chapter 3

BDNK Hydrodynamics and the Useful Frames

3.1 The General Frame

Now that the problematic nature of the most common frames has been established, we must search for a solution. The stress-energy tensor and charge current, in the most general frame possible[1] and in Minkowski space, are given by (c.f. equations (2.11))

$$\begin{aligned}
T^{\mu\nu} = & \left(\epsilon + \varepsilon_1 \frac{u^\lambda \partial_\lambda T}{T} + \varepsilon_2 \partial_\lambda u^\lambda + \varepsilon_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) u^\mu u^\nu \\
& + \left(p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \pi_2 \partial_\lambda u^\lambda + \pi_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) \Delta^{\mu\nu} \\
& + 2\theta_1 \left(u^\lambda \partial_\lambda u^{(\mu} + \frac{\Delta^{(\mu\lambda} \partial_\lambda T}{T}\right) u^{\nu)} + 2\theta_3 \Delta^{(\mu\lambda} \partial_\lambda \left(\frac{\mu}{T}\right) u^{\nu)} \\
& - \eta \Delta^{\mu\alpha} \Delta^{\nu\beta} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d} \eta_{\alpha\beta} \partial_\lambda u^\lambda\right), \qquad (3.1a)
\end{aligned}$$

$$\begin{aligned}
J^\mu = & \left(n + \nu_1 \frac{u^\lambda \partial_\lambda T}{T} + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) u^\mu \\
& + \gamma_1 \left(u^\lambda \partial_\lambda u^\mu + \frac{\Delta^{\mu\lambda} \partial_\lambda T}{T}\right) + \gamma_3 \Delta^{\mu\lambda} \partial_\lambda \left(\frac{\mu}{T}\right), \qquad (3.1b)
\end{aligned}$$

where as before $X^{(\mu\nu)} = \frac{1}{2}\left(X^{\mu\nu} + X^{\nu\mu}\right)$. Note that $\theta_1 = \theta_2$ and $\gamma_1 = \gamma_2$, as required by the thermodynamic consistency condition (2.22). The goal of this chapter is the following: find definitions for T, μ, and u^μ such that the resulting hydrodynamic theory is stable and causal. Due to the correspondence between variable definition and transport coefficient definition, this is equivalent to determining the subspace of

[1] A frame choice is, as previously stated, equivalent to defining T, μ, and u^μ, so *some* choice has to be made; one cannot "not pick a frame".

the parameter space swept out by the transport coefficients that is both stable and causal. These stable and causal frames will be dubbed the "useful" frames.

In order to familiarize the reader with some of the techniques used, and also to explore some of the fundamental properties of the set of useful frames, we will first examine the uncharged case. The term "uncharged" here is not quite the same as the use of the term in other areas of physics – the charge is not electric charge. Rather the charge in question is any conserved quantity associated with a global internal symmetry of the system, e.g. a $U(1)$ symmetry such as Baryon number. Electric charge may be added to the fluid by adding a gauge field, which modifies the equations of motion. An uncharged system has no such symmetries. An example of an uncharged system would be $SU(N)$ Yang-Mills theory. A charged system could be any number of things; a Quark-Gluon plasma (QGP) is the best relativistic system to think of, though a simple quotidien example of a non-relativistic charged system is a glass of water: in non-relativistic hydrodynamics, particle number is a conserved charge[2].

3.2 General Uncharged Fluids

The theory for the viscous hydrodynamics of an uncharged fluid was initially developed by Bemfica, Disconzi, Noronha, and Kovtun in [14], [15], and [16]. The theory has therefore come to be called BDNK hydrodynamics.

If a fluid is uncharged, then both n and μ do not appear in the equations of motion, and there is no conserved charge current. Therefore, the only relevant equation is the conservation of the stress-energy tensor. The stress-energy tensor in the general frame is given by

$$\begin{aligned} T^{\mu\nu} = &\left(\epsilon + \varepsilon_1 \frac{u^\lambda \partial_\lambda T}{T} + \varepsilon_2 \partial_\lambda u^\lambda\right) u^\mu u^\nu + \left(p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \pi_2 \partial_\lambda u^\lambda\right) \Delta^{\mu\nu} \\ &+ 2\theta_1 \left(u^\lambda \partial_\lambda u^{(\mu} + \frac{\Delta^{(\mu\lambda}\partial_\lambda T}{T}\right) u^{\nu)} \\ &- \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\lambda\right). \end{aligned} \tag{3.2}$$

There is now a significant reduction in complexity. Before, there were fourteen

[2]Non-relativistic hydrodynamics *always* has conserved particle number, and so the concept of "uncharged" does not exist; we will ignore this fact, as non-relativistic hydrodynamics is outside the scope of this thesis.

unique transport coefficients: now there are only six, which greatly simplifies matters.

Let us again perturb the equilibrium state where u_0^μ is in the time-direction, such that $T = T_0 + \delta T$, and $u^\mu = u_0^\mu + \delta u^\mu$. The conservation equation takes the form (taking into account that $u_0^\mu = \delta_t^\mu$ and $\delta u^\mu = \delta_i^\mu v^i$)

$$\begin{aligned}\partial_\mu T^{\mu\nu} = w_0 \left(\delta_t^\nu \partial_i v^i + \delta_j^\nu \partial_t v^j\right) + \left(\frac{\partial \epsilon}{\partial T}\partial_t \delta T + \frac{\varepsilon_1}{T_0}\partial_t^2 \delta T + \varepsilon_2 \partial_t \partial_i v^i\right)\delta_t^\nu \\ + \left(\frac{\partial p}{\partial T}\partial_j \delta T + \frac{\pi_1}{T_0}\partial_j \partial_t \delta T + \pi_2 \partial_j \partial_i v^i\right)\delta^{\nu j} \\ + \theta_1 \left(\partial_i \partial_t v^i + \frac{\partial_i \partial^i \delta T}{T}\right)\delta_t^\nu + \theta_1 \left(\partial_t \partial_t v^j + \frac{\partial_t \partial^j \delta T}{T}\right)\delta_j^\nu \\ - \eta \left(\partial_i \partial^i v^j + \frac{(d-2)}{d}\partial_i \partial^j v^i\right)\delta_j^\nu = 0.\end{aligned}$$

It is once again prudent to transition to momentum space. We align the wave-vector with the $\hat{x}$-axis as before, i.e. letting $\delta T = B_T e^{-i\omega t + ikx}$, $v^i = A^i e^{-i\omega t + ikx}$. For notational simplicity we assume four spacetime dimensions when writing matrices: since the transverse modes decouple, the matrix can be easily extended to arbitrary dimensionality. Finally, recall that with no charge, the speed of sound v_s is given by $v_s^2 = p_\epsilon \equiv \frac{\partial p}{\partial \epsilon}$, and the equilibrium enthalpy density is $w_0 \equiv (\epsilon_0 + p_0) = s_0 T_0$. We can therefore write that $\frac{\partial \epsilon}{\partial T} = v_s^{-2}\frac{w_0}{T_0}$. Bearing all of this in mind, the equations of motion take on the following matrix form:

$$\begin{bmatrix} -i\frac{w_0}{v_s^2 T}\omega - \frac{\varepsilon_1}{T_0}\omega^2 - \frac{\theta_1}{T_0}k^2 & iw_0 k + (\varepsilon_2 + \theta_1)\,\omega k & 0 & 0 \\ i\frac{w_0}{T_0}k + (\pi_1 + \theta_1)\frac{\omega k}{T_0} & -iw_0\omega - \theta_1\omega^2 + \left(-\pi_2 + \frac{2(d-1)}{d}\eta\right)k^2 & 0 & 0 \\ 0 & 0 & -iw_0\omega - \theta_1\omega^2 + \eta k^2 & 0 \\ 0 & 0 & 0 & -iw_0\omega - \theta_1\omega^2 + \eta k^2 \end{bmatrix}$$
$$\times \begin{pmatrix} B_T \\ A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Taking the determinant of this matrix yields

$$\left(\theta_1 \omega^2 + iw_0\omega - \eta k^2\right)^{d-1} F(\vec{v}_0 = 0, \omega, k) = 0,$$

where the longitudinal part of this equation, for zero global uniform velocity, is given by

$$F(\vec{v}_0 = 0, \omega, k) = v_s^2 \varepsilon_1 \theta_1 \omega^4 + i w_0 \left(v_s^2 \varepsilon_1 + \theta_1\right) \omega^3 \\ - \left(w_0^2 + v_s^2 k^2 \left(\theta_1 \pi_1 + \varepsilon_2 \left(\theta_1 + \pi_1\right) - \varepsilon_1 \pi_2 + 2\frac{(d-1)}{d} \varepsilon_1 \eta\right)\right) \omega^2 \\ - i w_0 k^2 \left(v_s^2 \left(\varepsilon_2 + \theta_1 + \pi_1\right) - \pi_2 + \frac{2\left(d-1\right)}{d}\eta\right)\omega \\ + k^2 v_s^2 w_0^2 - k^4 v_s^2 \theta_1 \left(-\pi_2 + \frac{2\left(d-1\right)}{d}\eta\right). \tag{3.3}$$

In the uncharged case, the definition of the bulk viscosity ζ is (c.f. Equation (2.24))

$$\zeta = -\pi_2 + v_s^2 \varepsilon_2 + v_s^2 \left(\pi_1 - v_s^2 \varepsilon_1\right). \tag{3.4}$$

Utilizing this definition to eliminate $-\pi_2$ in favour of ζ in (3.3), and defining $\gamma_s = \zeta + \frac{2(d-1)}{d}\eta$, yields

$$F(\vec{v}_0 = 0, \omega, k) = v_s^2 \varepsilon_1 \theta_1 \omega^4 + i w_0 \left(v_s^2 \varepsilon_1 + \theta_1\right) \omega^3 \\ - \left(w_0^2 + k^2 v_s^2 \left(v_s^4 \varepsilon_1^2 - v_s^2 \varepsilon_1 \left(\varepsilon_2 + \pi_1\right) + \gamma_s \varepsilon_1 + \theta_1 \pi_1 + \varepsilon_2 \left(\theta_1 + \pi_1\right)\right)\right) \omega^2 \\ - i w_0 k^2 \left(\gamma_s + v_s^4 \varepsilon_1 + v_s^2 \theta_1\right) \omega \\ + k^2 v_s^2 w_0^2 - k^4 v_s^2 \theta_1 \left(\gamma_s + v_s^2 \left(v_s^2 \varepsilon_1 - \varepsilon_2 - \pi_1\right)\right), \tag{3.5}$$

This is clearly more complicated than in the Landau frame. We begin by analyzing the transverse modes due to their relative simplicity.

3.2.1 Transverse Modes

The controlling equation for the transverse modes is

$$\theta_1 \omega^2 + i w_0 \omega - \eta k^2 = 0. \tag{3.6}$$

At small k, the modes are

$$\omega = -i\frac{w_0}{\theta_1} + \mathcal{O}\left(k^2\right), \tag{3.7a}$$

$$\omega = -i\frac{\eta}{w_0} k^2 + \mathcal{O}\left(k^3\right). \tag{3.7b}$$

The gapless mode (3.7b) is automatically stable (c.f. condition (2.30)) by the thermodynamics ($w_0 > 0$) and the positivity of entropy production ($\eta > 0$). In order for the gapped mode (3.7a) to be stable as $k \to 0$, we require that $\theta_1 > 0$.

An analysis of the causality of the transverse modes may be done by asymptotically expanding ω about $k = \infty$ in (3.6). Doing so, we find that all orders in k greater than or equal to second order are all zero; that is, ω obeys a dispersion relation at large k of the form $\omega = c_1 k + c_0 + \mathcal{O}\left(k^{-1}\right)$.

Since k is large, the most important term will be the largest power of k. Therefore, let us take $\omega = c\,k$. Substituting this expression for ω in to the controlling equation (3.6) yields that

$$c^2 = \frac{\eta}{\theta_1}.$$

The condition for causality (2.31) is that $0 < c^2 < 1$. We know $\eta > 0$, $\theta_1 > 0$, so the condition that $c^2 > 0$ is automatically satisfied. However, the condition $c^2 < 1$ gives an additional constraint:

$$\theta_1 > \eta > 0. \tag{3.8}$$

Finally, the equilibrium state ought to be stable for *all* k, not just as $k \to 0$. In order to ensure that this is the case, we will borrow a technique from control theory, the branch of mathematics and engineering dealing with the behaviour of dynamical systems: the so-called "Routh-Hurwitz criterion", hereafter referred to as the "RH criterion". For more information on the application of the criterion, as well as a list of stability conditions at second, third, fourth, and sixth orders, see Appendix B.

The RH criterion is a criterion that, if satisfied, ensures that all of the roots of a polynomial are in the left complex half-plane [23][24]. In order to make the connection between the RH criterion and the stability constraint, we make the substitution $\omega = i\Delta$. Making this substitution in (3.6) yields

$$a_2\Delta^2 + a_1\Delta + a_0 = 0, \tag{3.9}$$

where

$$\begin{aligned} a_2 &= 1, \\ a_1 &= \frac{w_0}{\theta_1}, \\ a_0 &= \frac{\eta}{\theta_1}. \end{aligned}$$

For a quadratic equation of the form above, the RH criteria are simply that $a_1 > 0$, $a_0 > 0$. The first condition is true by the small-k stability analysis. The second condition is true both by the small-k analysis, and also by the causality analysis. It is then clear that, in the case where $\vec{v}_0 = 0$, the transverse mode is stable and causal as long as $\theta_1 > \eta > 0$.

3.2.2 Longitudinal Modes

Turning now to the longitudinal modes, which are more complicated, the exact same procedure may be applied: examine stability requirements for the small-k modes, examine stability and causality constraints from the large-k modes, and then examine the Routh-Hurwitz criteria.

The first order of business is to look at the small-k modes. There are four of these modes: two gapped modes and two gapless ones. Their dispersion relations are given by

$$\omega = \pm v_s k - i\frac{\gamma_s}{2w_0}k^2, \tag{3.10a}$$

$$\omega = -i\frac{w_0}{v_s^2\varepsilon_1} + \mathcal{O}\left(k^2\right), \tag{3.10b}$$

$$\omega = -i\frac{w_0}{\theta_1} + \mathcal{O}\left(k^2\right). \tag{3.10c}$$

The constraints gained by demanding stability are that $\gamma_s > 0$, $\varepsilon_1 > 0$, and $\theta_1 > 0$. The first of these is required by the positivity of entropy production (see Appendix C for more details). The third constraint is already satisfied by demanding the stability and causality of the transverse modes. The second, $\varepsilon_1 > 0$, is a new constraint.

Turning to the large-k modes, if we again perform an asymptotic expansion of ω about $k = \infty$ in (3.5), the highest-order term in the expansion is again linear. Setting $\omega = ck$ yields the following controlling equation for the phase velocity c.

$$a_2 c^4 + a_1 c^2 + a_0 = 0, \tag{3.11}$$

where

$$a_2 = \varepsilon_1\theta_1, \tag{3.12a}$$

$$a_1 = -\left(\gamma_s\varepsilon_1 + v_s^4\varepsilon_1^2 + \varepsilon_2\theta_1 + (\varepsilon_2 + \theta_1)\,\pi_1 - v_s^2\varepsilon_1\,(\varepsilon_2 + \pi_1)\right), \tag{3.12b}$$

$$a_0 = -\theta_1\left(\gamma_s + v_s^2\left(v_s^2\varepsilon_1 - \varepsilon_2 - \pi_1\right)\right). \tag{3.12c}$$

In order for the roots of this equation for c^2 to be real and lie between 0 and 1, the following conditions must be obeyed:

$$a_1^2 - 4a_2a_0 > 0, \quad a_1 < 0, \quad 0 < a_0 < a_2, \quad a_2 + a_1 + a_0 > 0. \tag{3.13}$$

These constraints can be directly derived from the conditions given. Inserting the coefficients (3.12) into (3.13) yields the following large-k stability and causality constraints:

$$4\varepsilon_1\theta_1^2\tilde{\aleph} + \left(\varepsilon_1\tilde{\aleph} + \varepsilon_2\theta_1 + (\varepsilon_2 + \theta_1)\,\pi_1\right)^2 > 0, \tag{3.14a}$$

$$\varepsilon_1\tilde{\aleph} + \theta_1\pi_1 + \varepsilon_2\,(\theta_1 + \pi_1) > 0, \tag{3.14b}$$

$$\theta_1\tilde{\aleph} < 0, \tag{3.14c}$$

$$\varepsilon_1\theta_1 > 0, \tag{3.14d}$$

$$\theta_1\left(\tilde{\aleph} + \varepsilon_1\right) > 0, \tag{3.14e}$$

$$(\varepsilon_1 - \varepsilon_2)\,\theta_1 - \left(\tilde{\aleph} + \pi_1\right)(\varepsilon_1 + \theta_1) > 0, \tag{3.14f}$$

where $\tilde{\aleph} \equiv \gamma_s + v_s^2\,(v_s^2\varepsilon_1 - \varepsilon_2 - \pi_1)$ has been introduced as shorthand. These constraints can be simultaneously satisfied.

The final step is analysis of the RH criterion. The equation $F(\vec{v}_0 = 0, \omega, k) = 0$ can be written, following the substitution $\omega = i\Delta$, as a fourth-order polynomial of the form

$$a_4\Delta^4 + a_3\Delta^3 + a_2\Delta^2 + a_1\Delta + a_0 = 0,$$

with coefficients given by

$$a_4 = v_s^2\varepsilon_1\theta_1, \tag{3.15a}$$

$$a_3 = w_0\chi_1, \tag{3.15b}$$

$$a_2 = \left(w_0^2 + k^2v_s^2\left(\tilde{\aleph}\varepsilon_1 + \theta_1\pi_1 + \varepsilon_2\,(\theta_1 + \pi_1)\right)\right), \tag{3.15c}$$

$$a_1 = w_0k^2\chi_2, \tag{3.15d}$$

$$a_0 = k^2v_s^2w_0^2 - k^4v_s^2\theta_1\tilde{\aleph}. \tag{3.15e}$$

where $\chi_1 \equiv v_s^2\varepsilon_1 + \theta_1$, $\chi_2 \equiv \gamma_s + v_s^4\varepsilon_1 + v_s^2\theta_1$ have been introduced as shorthand. The RH criterion for a fourth-order polynomial gives constraints listed in Appendix C.

The constraints are:

$$a_4 > 0, \quad a_3 > 0, \quad a_0 > 0, \quad a_3 a_2 - a_1 a_4 > 0, \quad a_1 > \frac{a_3^2 a_0}{a_3 a_2 - a_1 a_4}. \tag{3.16}$$

These constraints imply that all of the coefficients a_n must be positive. Inserting the values given in (3.15) into (3.16) yields

$$v_s^2 \varepsilon_1 \theta_1 > 0, \tag{3.17a}$$

$$w_0 \chi_1 > 0, \tag{3.17b}$$

$$w_0^2 - k^2 \theta_1 \tilde{\aleph} > 0, \tag{3.17c}$$

$$\frac{w_0^2}{v_s^2}\chi_1 + k^2 \left[-\varepsilon_1 \theta_1 \chi_2 + \chi_1 \left(\tilde{\aleph}\varepsilon_1 + \theta_1 \pi_1 + \varepsilon_2 \left(\theta_1 + \pi_1 \right) \right) \right] > 0, \tag{3.17d}$$

$$\frac{w_0^2}{v_s^2}\chi_1 \left(\chi_2 - v_s^2 \chi_1 \right) + k^2 \left[-\varepsilon_1 \theta_1 \chi_2^2 + \chi_1 \chi_2 \left(\tilde{\aleph}\varepsilon_1 + \varepsilon_2 \theta_1 + \left(\varepsilon_2 + \theta_1 \right) \pi_1 \right) + \theta_1 \tilde{\aleph} \chi_1^2 \right] > 0. \tag{3.17e}$$

The first two constraints simply re-enforce that $\varepsilon_1 > 0$ and $\theta_1 > 0$. The third constraint needs to be true for all values of k. This implies that $\theta_1 \tilde{\aleph} < 0$, which is the same as the condition (3.14a) in the causality constraints. This condition is given, for positive θ_1, by

$$1 + \bar{\varepsilon}_1 < \bar{\varepsilon}_2 + \bar{\pi}_1, \tag{3.18}$$

where $\bar{\varepsilon}_1 = \frac{v_s^2 \varepsilon_1}{\gamma_s}$, $\bar{\varepsilon}_2 = \frac{\varepsilon_2}{\gamma_s}$, and $\bar{\pi}_1 = \frac{\pi_1}{\gamma_s}$. The constraint (3.17d) yields the non-linear relation

$$\bar{\varepsilon}_1^3 + \bar{\varepsilon}_1 \left(\frac{\bar{\varepsilon}_1}{v_s^2} - \bar{\theta}_1^2 + \bar{\varepsilon}_2 \bar{\pi}_1 \right) + \bar{\theta}_1 \left(\bar{\theta}_1 \bar{\pi}_1 + \bar{\varepsilon}_2 \left(\bar{\theta}_1 + \bar{\pi}_1 \right) \right) > \bar{\varepsilon}_1^2 \left(\bar{\varepsilon}_2 + \bar{\pi}_1 \right), \tag{3.19}$$

where $\bar{\theta}_1 = \frac{\theta_1}{\gamma_s}$. The final condition, (3.17e), gives another non-linear relation between the transport coefficients, namely

$$\frac{\bar{\varepsilon}_1^2}{v_s^2} + v_s^2 \left(\bar{\varepsilon}_1 - \bar{\varepsilon}_2 \right) \left(\bar{\varepsilon}_1 + \bar{\theta}_1 \right)^2 \left(\bar{\varepsilon}_1 - \bar{\pi}_1 \right) + \left(\bar{\varepsilon}_1 + \bar{\theta}_1 \right) \left(2\bar{\varepsilon}_1^2 - \bar{\varepsilon}_1 \left(\bar{\varepsilon}_2 + \bar{\pi}_1 \right) + \left(\bar{\theta}_1 + \bar{\theta}_1 \right) \left(\bar{\theta}_1 + \bar{\pi}_1 \right) \right) > 0. \tag{3.20}$$

These three conditions, along with the requirements that $\varepsilon_1, \theta_1 > 0$, are enough to define a class of stable frames. The parameter space is 5-dimensional, so only a slice of it is presented in Figures 3.1 and 3.2. This slice is the same slice as was used in [15]. In the second figure, the causality constraints (3.14) have been merged in – we can see that, while causality reduces the space of stable frames, it by no means eliminates it.

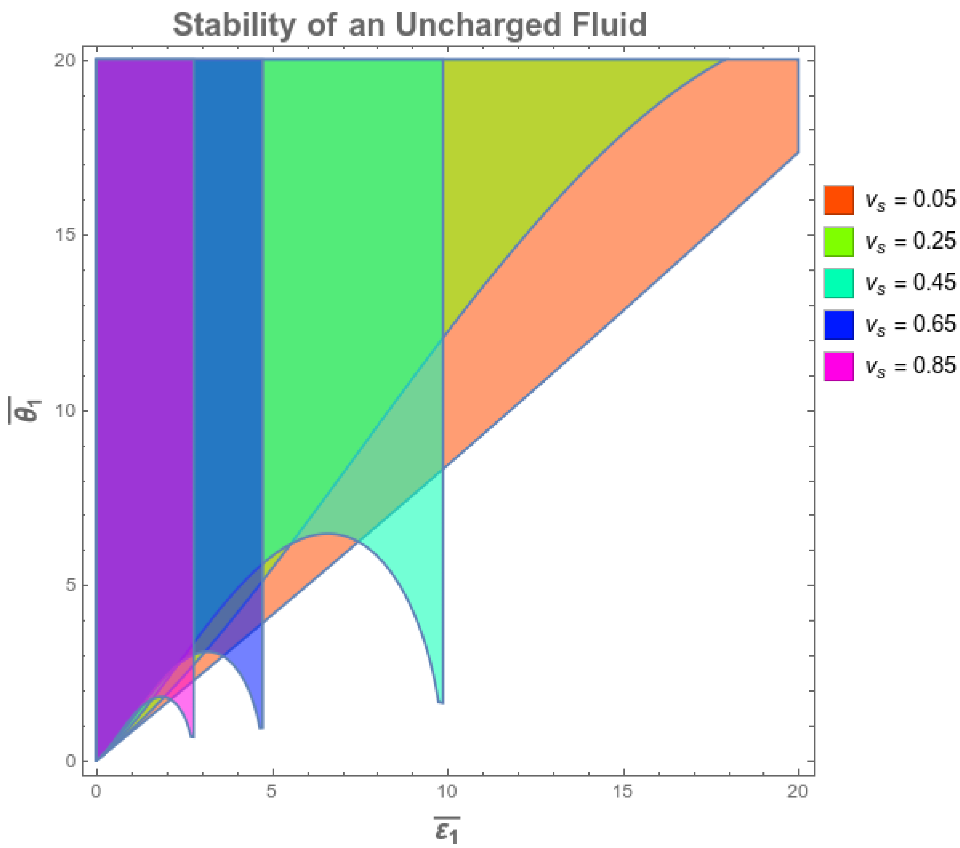

Figure 3.1: A slice of the parameter space constrained by the RH criteria (3.17), expressed in terms of dimensionless parameters $\bar{\varepsilon}_1$ and $\bar{\theta}_1$. This is specifically the slice where $\varepsilon_2 = 0$ and $\pi_1 = \frac{3}{v_s^2}\gamma_s$. I have plotted the constraints for 5 different values of the speed of sound, ranging from $v_s = 0.05$ to $v_s = 0.85$. All figures except Figure 3.5 made with [25].

A class of stable and causal frames has therefore been found. We have, however, only analyzed the stability and causality of the theory when the uniform global velocity $\vec{v}_0 = 0$. The final step is to perform the analysis for the case where $\vec{v}_0 \neq 0$, though we need not be quite as exhaustive – some results may be generalized in a straightforward way.

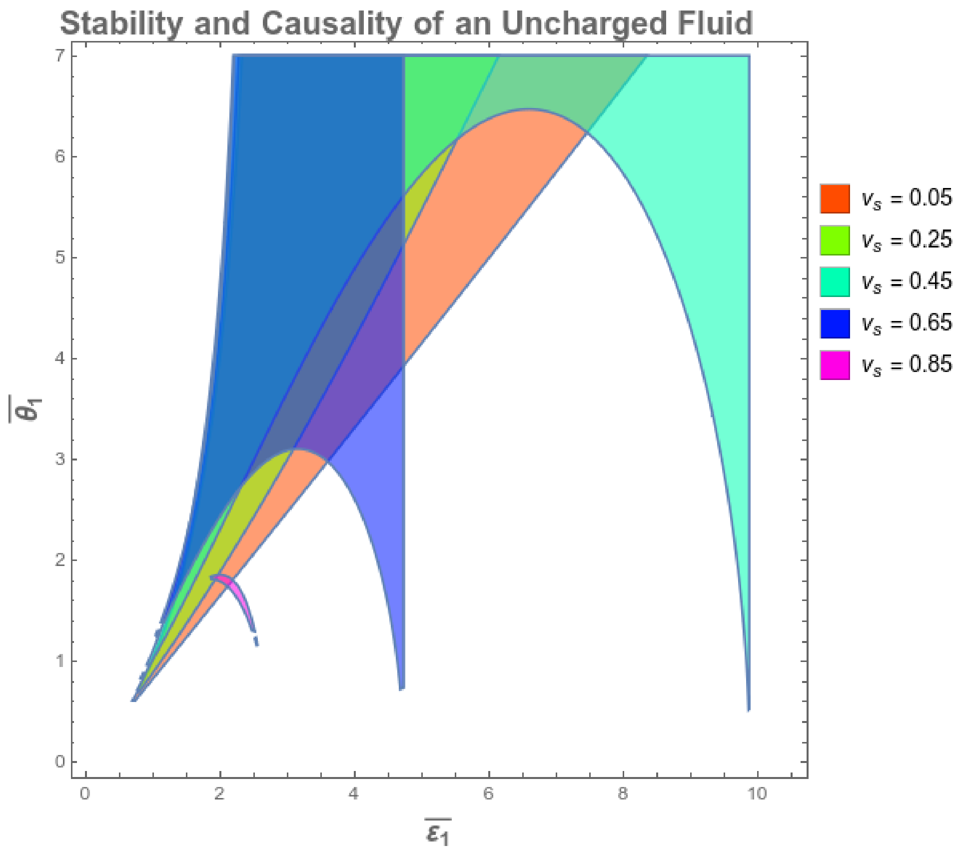

Figure 3.2: A slice of the parameter space constrained by the RH criteria (3.17) **and** the causality constraints (3.14), expressed in terms of dimensionless parameters $\bar{\varepsilon}_1$ and $\bar{\theta}_1$. This is specifically the slice where $\varepsilon_2 = 0$ and $\pi_1 = \frac{3}{v_s^2}\gamma_s$. I have plotted the constraints for 5 different values of the speed of sound, ranging from $v_s = 0.05$ to $v_s = 0.85$. Notice that the origin is excluded, eliminating the Landau frame.

Firstly, let us analyze the large-k modes. The large-k modes had linear dispersion relations, so let us examine the effect of performing a Lorentz boost on a linear dispersion relation of the form $\omega = c_0 k$. Performing the boost yields the following equation in the boosted variables ω' and k':

$$\begin{aligned}&\left(1-|v_0|^2c_0^2\right)(\omega')^2 - 2\left(1-c_0^2\right)|k'||v_0|\cos(\phi)\omega'\\&+\left[\left(1-c_0^2\right)|v_0|^2\cos(\phi)^2 - c_0^2\left(1-v_0^2\right)\right]|k'|^2 = 0,\end{aligned}$$

where ϕ is the angle between $\vec{v}_0$ and $\vec{k}'$. This controlling equation yields two new, *linear* modes of the form $\omega = c_v(\phi)|k|$, where $c_v(\phi)$ is given by

$$c_v(\phi) = \frac{(1-c_0^2)\,|v_0|}{1-|v_0|^2c_0^2}\cos(\phi) \pm \frac{c_0}{1-|v_0|^2c_0^2}\sqrt{(1-v_0^2)\,(1-c_0^2|v_0|^2-(1-c_0^2)\,|v_0|^2\cos(\phi)^2)}. \tag{3.21}$$

Note that this is a *generic* feature of linear dispersion relations – and so it applies equally to the sound mode in the small-k limit. There are a couple things to note here:

1. If $c_0^2 < 0$, i.e. if the mode is **unstable** when $v_0 = 0$, then the mode will **remain** unstable. The terms under the square root will be generically positive, and the c_0 before the root will be imaginary, leading to instability.

2. If $c_0^2 > 1$, i.e. if the mode is **acausal**, it will lead to the existance of an unstable Lorentz frame. If $c_0^2 > 1$, there exists a $v_0^2 < 1$ such that the terms under the square root are negative – this leads to a positive imaginary component for $c_v(\phi)$, leading to instability.

The converse of these is also true: if $0 < c_0^2 < 1$, then $0 < c_v^2 < 1$. Therefore, since our modes were causal for a non-moving fluid, they will remain causal for a moving fluid. [15]

Next is the small-k modes. The dispersion relations for the boosted transverse modes at small k are given by

$$\omega = -i\frac{w_0\sqrt{1-v_0^2}}{\theta - v_0^2\eta} + \left(\frac{v_0}{\theta - v_0^2\eta}\left(\theta - \left(2-v_0^2\right)\eta\right)\cos(\phi)\right)k + \mathcal{O}\left(k^2\right), \tag{3.22a}$$

$$\omega = v_0\cos(\phi)k - i\frac{\sqrt{1-v_0^2}\,(1-v_0^2\cos(\phi)^2)\,\eta}{w_0}k^2 + \mathcal{O}\left(k^3\right). \tag{3.22b}$$

The only requirement for the gap to be stable as $k \to 0$ is that $\theta > \eta$. The requirement for the gapless mode to be stable is simply that $\eta > 0$, which is of course satisfied. So, the transverse modes are stable at small k for $\vec{v}_0 \neq 0$.

The dispersion relations for the boosted small-k longitudinal modes are quite complicated. The dispersion relations for the gapless modes are given by

$$\omega = \pm c_v(\phi)k - i\Gamma(\phi)k^2 + \mathcal{O}\left(k^3\right), \tag{3.23}$$

where $c_v(\phi)$ is given by 3.21, and $\Gamma(\phi)$ is given in D.1. As long as $\gamma_s > 0$, this mode will be stable. The dispersion relations for the gapped modes are given by (recalling the shorthands $\tilde{\aleph} = \gamma_s + v_s^2\left(v_s^2\varepsilon_1 - \varepsilon_2 - \pi_1\right)$, $\chi_1 = v_s^2\varepsilon_1 + \theta_1$, and $\chi_2 = \gamma_s + v_s^2\chi_1$)

$$
\begin{aligned}
\omega = -i\Bigg[&\sqrt{1-v_0^2}w_0\left(-\chi_1 + v_0^2\chi_2\right) \pm \Bigg(\left(1-v_0^2\right)w_0^2\left(-\chi_1+v_0^2\chi_2\right)^2 \\
&+ 4\left(1-v_0^2\right)v_s^2\left(1-v_0^2v_s^2\right)w_0^2\left(-\varepsilon_1\theta_1 + v_0^4\theta_1\tilde{\aleph} + v_0^2\left(\tilde{\aleph}\varepsilon_1 + \varepsilon_2\theta + \left(\varepsilon_2+\theta\right)\pi_1\right)\right)\Bigg)^{1/2}\Bigg] \\
&\times\left[2v_s^2\left(-\varepsilon_1\theta_1 + v_0^4\theta_1\tilde{\aleph} + v_0^2\left(\tilde{\aleph}\varepsilon_1 + \varepsilon_2\theta + \left(\varepsilon_2+\theta\right)\pi_1\right)\right)\right]^{-1} + \mathcal{O}\left(k\right).
\end{aligned}
$$

We wish to show that the constraints (3.17) and (3.13) ensure stability. This can be done via the RH criterion. The controlling equation for the gaps $\omega = \Omega$ is given by

$$
\begin{aligned}
&-v_s^2\left(\varepsilon_1\theta - v_0^4\theta\tilde{\aleph} - v_0^2\left(\varepsilon_1\tilde{\aleph} + \varepsilon_2\theta + \left(\varepsilon_2+\theta\right)\pi_1\right)\right)\Omega^2 \\
&-i\sqrt{1-v_0^2}w_0\left(v_s^2\varepsilon_1 + \theta_1 - v_0^2\chi_2\right)\Omega + \left(1-v_0^2\right)\left(1-v_0^2v_s^2\right)w_0^2 = 0.
\end{aligned}
\tag{3.24}
$$

Let $\Omega = i\Delta$, as generally done when using the RH criteria. The controlling equation (3.24) becomes

$$
a_2\Delta^2 + a_1\Delta + a_0 = 0,
$$

with

$$
\begin{aligned}
a_2 &= v_s^2\left(\varepsilon_1\theta - v_0^4\theta\tilde{\aleph} - v_0^2\left(\varepsilon_1\tilde{\aleph} + \varepsilon_2\theta + \left(\varepsilon_2+\theta\right)\pi_1\right)\right), \\
a_1 &= \sqrt{1-v_0^2}w_0\left(\chi_1 - v_0^2\chi_2\right), \\
a_0 &= \left(1-v_0^2\right)\left(1-v_0^2v_s^2\right)w_0^2.
\end{aligned}
$$

For a quadratic, the stability requirement is simply that all of the coefficients must have the same sign. We can see clearly that a_0 is positive, so this amount to requiring that $a_1 > 0$, $a_2 > 0$. The first of these conditions amounts to $\chi_1 > v_0^2\chi_2$. To ensure this is true for all v_0, it must be that $\chi_1 > \chi_2$. In order for this to be true, we must satisfy the constraint that

$$
v_s^2\varepsilon_1 + \theta_1 > \frac{\gamma_s}{1-v_s^2}.
$$

For the second to be true, we require the modes to satisfy the following non-linear equation in the transport coefficients

$$
\left(\varepsilon_1 + \theta_1\right)\left(-\tilde{\aleph}\right) + \left(\varepsilon_1 - \pi_1\right)\theta_1 > \varepsilon_2\left(\theta_1 + \pi_1\right).
$$

Both of these restrictions lie squarely within the stable and causal region, as is demonstrated in figure 3.3.

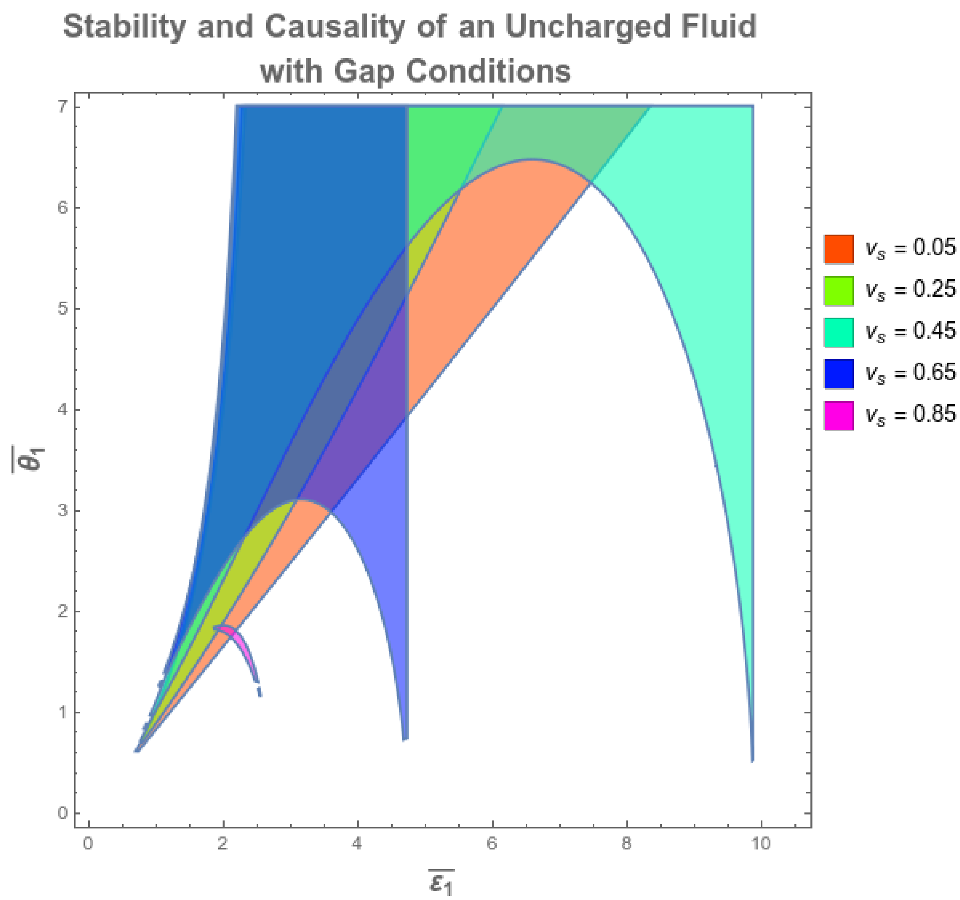

Figure 3.3: A slice of the parameter space constrained by the RH criteria **and** the causality constraints **and** the stability conditions for the gaps when $v_0 \neq 0$. The same slice is pictured as in figure 3.2. Note that the two figures are identical; the stable and causal region is still stable after boosting (absent RH criteria analysis).

While work has not been done to analyze the RH criterion for the longitudinal modes of a moving fluid, in [15] (c.f. Figure 5) numerical work was done to show there exists a frame which has stable and causal propagation for a moving fluid at all k. A full analysis of the RH criterion is still outstanding.

The discussion above was for a generic uncharged fluid. However, it is illustrative to also briefly examine the specific case of a *conformal* uncharged fluid, as they are in general more tractable.

3.3 Uncharged Conformal Fluids

A "conformal" symmetry is one which leaves *angles* invariant. Quantum field theories that are invariant under these conformal transformations are known as "conformal field theories", or CFTs [26]. A fluid that has an underlying conformal symmetry is called a "conformal fluid".

The additional assumption of conformality leads to significant simplification. For one thing, in the uncharged case, the conformal symmetry specifies the equation of state: $p(T) = \alpha T^{d+1}$, where α is a constant. This equation of state defines all the other thermodynamic quantities:

$$\begin{aligned}
\epsilon &= -p + \frac{\partial p}{\partial T} T = -\alpha T^{d+1} + \alpha\left(d+1\right) T^{d+1} = d\alpha T^{d+1} = dp; \\
s &= \frac{\partial p}{\partial T} = \left(d+1\right)\alpha T^{d} = \frac{p}{\left(d+1\right)T}; \\
w &= \epsilon + p = \left(d+1\right)\alpha T^{d+1}; \\
v_s^2 &= p_\epsilon = \frac{\frac{\partial p}{\partial T}}{\frac{\partial \epsilon}{\partial T}} = \frac{\left(d+1\right)\alpha T^d}{d\left(d+1\right)\alpha T^d} = \frac{1}{d}.
\end{aligned}$$

The main takeaway from the thermodynamics is that $v_s = \frac{1}{\sqrt{d}}$. Another consequence of conformal symmetry [15] is that

$$\varepsilon_i = d\pi_i, \quad \pi_1 = d\pi_2. \tag{3.25}$$

These relations reduce the set of $\{\varepsilon_1, \varepsilon_2, \pi_1, \pi_2\}$ to just π_1. This along with the thermodynamic consistency condition (2.22) means that there are only three transport coefficients: π_1, θ_1, η. In a conformal fluid, the bulk viscosity ζ is identically zero; this is obvious intuitively, as scale invariance would forbid any type of bulk deformation. The stress-energy tensor is then simply [14]

$$\begin{aligned}
T^{\mu\nu}_{CFT} = &\left(p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \frac{\pi_1}{d}\partial_\lambda u^\lambda\right)\left(\left(d+1\right)\ u^\mu u^\nu + \eta^{\mu\nu}\right) + 2\theta_1 \left(u^\lambda \partial_\lambda u^{(\mu} + \frac{\Delta^{(\mu\lambda}\partial_\lambda T}{T}\right) u^{\nu)} \\
&- \eta \Delta^{\mu\alpha}\Delta^{\nu\beta}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha - \frac{2}{d}\eta_{\alpha\beta}\partial_\lambda u^\beta\right) + \mathcal{O}\left(\partial^2\right).
\end{aligned}$$

There are only three relevant parameters that control the equations: π_1, θ, η. Since the transverse modes only involve θ_1 and η, they will not be changed by the conformal symmetry. We will therefore only analyze the longitudinal modes. Perturbing the equilibrium state and transferring to momentum space yields the following longitudinal equation:

$$
\begin{aligned}
&d\theta_1\pi_1\omega^4 + idw_0\left(\theta_1+\pi_1\right)\omega^3 - \left(dw_0^2 + 2\pi_1\left(\left(d-1\right)\eta+\pi_1\right)k^2\right)\omega^2 \\
&-iw_0k^2\left(2\left(d-1\right)\eta+\theta_1+\pi_1\right)\omega + k^2w_0^2 + \frac{\theta_1}{d}\left(\pi_1 - 2\left(d-1\right)\eta\right)k^4 = 0.
\end{aligned} \tag{3.26}
$$

The small-k modes are

$$
\begin{aligned}
&\omega = \pm\frac{1}{\sqrt{d}}k - i\frac{\gamma_s}{2w_0}k^2 + \mathcal{O}\left(k^3\right), \\
&\omega = -i\frac{w_0}{\theta_1} + \mathcal{O}\left(k\right), \quad \omega = -i\frac{w}{\pi_1} + \mathcal{O}\left(k\right),
\end{aligned}
$$

where $\gamma_s = \frac{2(d-1)}{d}\eta$, since $\zeta = 0$. The constraints are clearly the same as in the non-conformal case: $\theta_1 > 0$, $\pi_1 > 0$, $\eta > 0$.

Next is the causality analysis. Performing an asymptotic expansion of ω about $k = \infty$ in (3.26) yields the following controlling equation for the coefficient c in the linear dispersion relation $\omega = ck$:

$$
d\theta_1\pi_1c^4 - 2\pi_1\left(\left(d-1\right)\eta+\theta_1\right)c^2 + \frac{\theta_1}{d}\left(\pi_1 - 2\left(d-1\right)\eta\right) = 0. \tag{3.27}
$$

The requirements for the roots of this equation to be real and positive, as well as less than one, are as in (3.13). The constraints required to ensure $1 > c^2 > 0$ are therefore

$$
\begin{aligned}
4\pi_1^2\left(\left(d-1\right)\eta+\theta_1\right)^2 - 4\theta_1^2\pi_1\left(\pi_1 - 2\left(d-1\right)\eta\right) &> 0, \\
-2\pi_1\left(\left(d-1\right)\eta+\theta_1\right) &< 0, \\
\frac{\theta_1}{d}\left(\pi_1 - 2\left(d-1\right)\eta\right) &< d\theta_1\pi_1, \\
d\theta_1\pi_1 &> 0, \\
\frac{\theta_1}{d}\left(\pi_1 - 2\left(2\left(d-1\right)\right)\eta\right) &> 0, \\
d\theta_1\pi_1 - 2\pi_1\left(\left(d-1\right)\eta+\theta_1\right) + \frac{\theta_1}{d}\left(\pi_1 - 2\left(d-1\right)\eta\right) &> 0.
\end{aligned}
$$

Most of these constraints are redundant. Utilizing the constraints derived from the small-k modes (specifically $\pi_1 > 0$ and $\theta_1 > 0$), these causality constraints can reduce this down to two constraints [14][15][17]:

$$2\left(d-1\right)\eta < \pi_1, \tag{3.28a}$$

$$1-2\left(\frac{1}{\left(d-1\right)}\frac{\eta}{\pi_1}+\frac{d}{d-1}\frac{\eta}{\theta_1}\right) > 0. \tag{3.28b}$$

These constraints define the causal parameter space for an uncharged conformal fluid. Note that these constraints are significantly simpler than for a non-conformal fluid. Additionally, they have one other feature – satisfying these causality constraints as well as the constraints needed for the stability of the gaps is sufficient to ensure stability for all momenta. This can be demonstrated by once more analyzing the RH criterion.

Looking at the constraints in (3.17), it is straightforward to find the CFT equivalents (simplifying somewhat using the positivity of $\pi_1,\ \theta_1$ implied by the first two constraints):

$$\begin{aligned}
\pi_1\theta_1 &> 0,\\
\left(\theta_1+\pi_1\right) &> 0,\\
\frac{dw_0^2}{\theta_1}+k^2\left(\pi_1-2\left(d-1\right)\eta\right) &> 0,\\
dw_0^2\left(\theta_1+\pi_1\right)+k^2\left(2\left(d-1\right)\eta\pi_1^2+\theta_1\pi_1\left(\theta_1+\pi_1\right)\right) &> 0,\\
dw_0^2\left(\theta_1+\pi_1\right)+k^2\left(\left(\pi_1+\theta_1\right)^3+2\left(d-1\right)\eta\pi_1^2\right) &> 0.
\end{aligned}$$

These conditions are satisfied for all values of k as long as $\theta_1 > \eta$ and $\pi_1 > 2\left(d-1\right)\eta$, which are both required by the causality constraints. Therefore causality (in addition to the small-k constraint of positivity for π_1, θ_1) for a conformal fluid at rests implies stability.

Stable and causal frames therefore exist in an uncharged conformal fluid at rest. While there has not been an analysis of the boosted fluid, it does not provide any particular insight relative to the generic uncharged fluid. The techniques used to analyze the uncharged fluid can now be applied to the charged fluid in the following sections.

3.4 Useful Charged Fluids

The general charged stress-energy tensor and charge current were given in (3.1). In order to simplify the discussion, and make the connection to the uncharged case more easily, we restrict our analysis to the following class of frames: those where $\varepsilon_3 = \pi_3 = \theta_3 = 0$. The reasoning behind this choice of frame is as follows:

The causality of a system of partial differential equations is determined by the principal part of the system, i.e. the terms with the highest order derivatives. The only dependence on gradients of μ in the principal part of the conservation equations for the stress-energy tensor is in the terms proportional to ε_3, π_3, and θ_3. Therefore, if those transport coefficients are set to zero, the principal part of the stress-energy conservation equation will contain no gradients of μ, and the causality controlling equation will neatly factorize, making constraints far easier to identify.

In contrast to the previous section, we first perform an analysis in a CFT, and then move on to the general case; this is done due to the fact that a CFT is generally more tractable than a generic fluid.

3.4.1 Charged Conformal Fluids

In addition to the CFT constraints (3.25), the following condition is also true [17]:

$$\nu_1 = d\, \nu_2.$$

Upon specifying to a class of frames with ε_3, π_3, and θ_3 equal to zero, we are left with only a few relevant parameters:

Transport Coefficients: π_1, θ_1, ν_1, ν_3, γ_1, η, σ.

Note that, recalling (2.23b), in this frame $\sigma \equiv \frac{n}{w}\left(\gamma_1 - \frac{n}{w}\theta_1\right) - \frac{1}{T}\gamma_3$. This definition has been used to replace γ_3 with σ. Further specifying a frame choice that we will only use for this CFT case, we choose to set $\nu_1 = \gamma_1 = 0$. Wit this choice, there are only **five** relevant transport coefficients: π_1, θ_1, ν_3, η, and σ. The quantities η and σ are of course the shear viscosity and the charge conductivity. The other three also have interpretations:

- π_1 may be thought of as the relaxation time for the longitudinal momentum.
- θ_1 may be thought of as the relaxation time for the transverse momentum.

- ν_3 may be thought of as the relaxation time for the charge density.

Given only these five transport coefficients, along with two unitless thermodynamic quantities $\kappa \equiv \frac{nT}{w}$ and $\lambda \equiv \frac{T^2}{w}\frac{\partial n}{\partial \mu}$, the conformal fluid can be entirely characterized. Let us examine the stability and causality constraints. Recall that, as mentioned when defining the speed of sound in section 2.3, in a conformal fluid $p_n = 0$.

With this choice of frame, the stress-energy tensor and the charge current become

$$\begin{aligned} T^{\mu\nu} &= \left(p + \pi_1 \frac{u^\lambda \partial_\lambda T}{T} + \frac{\pi_1}{d}\partial_\lambda u^\lambda\right)\left(\left(d+1\right) u^\mu u^\nu + \eta^{\mu\nu}\right) \\ &\quad + \theta_1 \left(u^\lambda \partial_\lambda u^\mu + \frac{\Delta^{\mu\lambda}\partial_\lambda T}{T}\right) u^\nu + \theta_1 \left(u^\lambda \partial_\lambda u^\nu + \frac{\Delta^{\nu\lambda}\partial_\lambda T}{T}\right) u^\mu \\ &\quad - \eta\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right), \\ J^\mu &= nu^\mu + \nu_3 u^\mu u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right) + \gamma_3 \Delta^{\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right) + \mathcal{O}\left(\partial^2\right). \end{aligned}$$

Once more perturbing the equilibrium state, we can repeat the procedure by now performed numerous times. After transferring to momentum space, and ensuring the singularity of the coefficient matrix, we are left with the following determinant

$$\left(\theta_1\omega^2 + w_0\omega - \eta k^2\right)^{d-1} F(\vec{v}_0 = 0, \omega, k) = 0, \tag{3.29}$$

where $F(\vec{v}_0, \omega, k)$ is a sixth-order polynomial in ω of the form $\sum_{n=0}^{6} a_n \omega^n$. The conformal symmetry can be used to attempt to simplify the polynomial, but it is still quite long. As such, for the sake of brevity, it has been printed in full in Appendix D.2.

The transverse modes are identical to the uncharged case, and so will be neglected. Looking first at the small-k limit, there are 6 longitudinal modes: three gapped modes, and three gapless modes. Their dispersion relations are given by

$$\omega = -i\frac{T^2\sigma}{w_0\left(\lambda - d\kappa^2\right)}k^2 + \mathcal{O}\left(k^3\right), \tag{3.30a}$$

$$\omega = \pm\frac{1}{\sqrt{d}}k - i\frac{1}{2w_0}\gamma_s k^2 + \mathcal{O}\left(k^3\right), \tag{3.30b}$$

$$\omega = -\frac{iw_0}{\theta_1} + \mathcal{O}\left(k^2\right), \tag{3.30c}$$

$$\omega = -\frac{iw_0}{2T\nu_3\pi_1}\left[T\nu_3 + \lambda\pi_1 \pm \sqrt{T^2\nu_3^2 - 2T\left(\lambda - 2d\kappa^2\right)\nu_3\pi_1 + \lambda^2\pi_1^2}\right] + \mathcal{O}\left(k^2\right). \tag{3.30d}$$

The stability of the gapless modes is ensured by the thermodynamics and the positivity of entropy production. For the gapped modes, the stability of these modes implies that $\nu_3 > 0, \pi_1 > 0,\ \ \theta_1 > 0$. This condition can be found by applying the RH criterion to the controlling equation of (3.30d).

For the large-k limit, if we asymptotically expand ω about $k = \infty$ in $F(\vec{v} = 0, \omega, k)$, the highest-order term is linear as in the uncharged case, i.e. $\omega = ck + \mathcal{O}\left(k^0\right)$. The controlling equation for the coefficient c is given by:

$$\left(\theta_1\kappa^2 + T^2\sigma - c^2T\nu_3\right)\left(\left(c^2d - 1\right)^2\theta_1\pi_1 - 2\left(d-1\right)\eta\left(\theta_1 + c^2d\pi_1\right)\right) = 0. \tag{3.31}$$

There is a neat de-coupling between the charge and momentum/energy propagations due to our frame choice. Demanding that c^2 be real and lie between 0 and 1 (i.e. causality constraint (2.31)) means that the first bracket gives a simple causality constraint on ν_3, namely that

$$\nu_3 > T\sigma + \frac{\kappa^2}{T}\theta_1. \tag{3.32}$$

The other bracket in equation (3.31) is **precisely** the same controlling equation as the uncharged case, i.e. equation (3.27). As such, the constraints it imposes are identical:

$$\pi_1 > 2\left(d-1\right)\eta, \qquad 1 - 2\left(\frac{d}{d-1}\frac{\eta}{\theta_1} - \frac{1}{d-1}\frac{\eta}{\pi_1}\right) > 0.$$

The sets of constraints (3.32) and (3.28) can be simultaneously satisfied.

Having analyzed both the small-k and large-k modes, the next logical step is to look at the RH criterion. This is a significantly more difficult task for a charged fluid.

For a sextic polynomial of the form

$$a_6\Delta^6 + a_5\Delta^5 + a_4\Delta^4 + a_3\Delta^3 + a_2\Delta^2 + a_1\Delta + a_0 = 0,$$

the constraints derived from the RH criterion are given in Appendix B; they are

$$a_6 > 0 \tag{3.33a}$$

$$a_5 > 0 \tag{3.33b}$$

$$a_5 a_4 > a_6 a_3 \tag{3.33c}$$

$$a_3\left(a_4 a_5 - a_3 a_6\right) - a_2 a_5^2 + a_1 a_5 a_6 > 0 \tag{3.33d}$$

$$\begin{aligned} &a_2^2 a_5^2 - a_0 a_4 a_5^2 + a_0 a_3 a_6 a_5 + a_2\left(\left(a_3^2 - 2a_1 a_5\right)a_6 - a_3 a_4 a_5\right) \\ &+ a_1\left(a_5 a_4^2 - a_3 a_6 a_4 + a_1 a_6^2\right) < 0 \end{aligned} \tag{3.33e}$$

$$\begin{aligned} &a_1\Bigg(a_2^2 a_5^2 - a_0 a_4 a_5^2 + a_0 a_3 a_6 a_5 \\ &+ a_2\left(\left(a_3^2 - 2a_1 a_5\right)a_6 - a_3 a_4 a_5\right) + a_1\left(a_5 a_4^2 - a_3 a_6 a_4 + a_1 a_6^2\right)\Bigg) \\ &+ a_0\left(a_5\left(a_4 a_3^2 + a_0 a_5^2 - \left(a_2 a_3 + a_1 a_4\right)a_5\right) - a_3\left(a_3^2 - 2a_1 a_5\right)a_6\right) < 0 \end{aligned} \tag{3.33f}$$

$$a_0 > 0 \tag{3.33g}$$

Plugging the coefficients of $F(\vec{v}_0 = 0, \omega, k)$ (after making the substitution $\omega = i\Delta$) into these conditions leads to an extremely long (i.e. thousands of terms) set of conditions, and so I will refrain from printing the complete set of constraints in this thesis. It does appear, however, as though the causality constraints along with the small-k constraints remain sufficient to ensure stability.

Now that this charged conformal fluid has been shown to be stable (at least for small k) and causal for $\vec{v}_0 = 0$, let us finally show it for $\vec{v}_0 \neq 0$. Performing a Lorentz boost in the same manner as the previous sections, we get new small-k modes. The gapless modes are given by

$$\begin{aligned} \omega &= v_0\cos(\phi)k - i\frac{T^2\sqrt{1-v_0^2}\left(1 - v_0^2\cos(\phi)^2\right)}{w_0\left(\lambda - d\kappa^2\right)}\sigma k^2 + \mathcal{O}\left(k^3\right), \\ \omega &= -\frac{1}{2\left(d - v_0^2\right)}\Bigg[2\left(d-1\right)v_0\cos(\phi) \\ &\pm\sqrt{2}\Big\{\left(1 - v_0^2\right)\left(2d - \left(d+1\right)v_0^2 + \left(d-1\right)v_0^2\cos(2\phi)\right)\Big\}^{1/2}\Bigg]k + \mathcal{O}\left(k^2\right). \end{aligned}$$

These modes are ensured to be stable by Lorentz symmetry ($1 > v_0^2 > 0$), the thermodynamics ($\lambda > d\kappa^2$), and the positivity of entropy production ($\sigma > 0$).

The gapped modes are given by the roots of a cubic polynomial

$$a_3\Delta^3 + a_2\Delta^2 + a_1\Delta + a_0 \tag{3.34}$$

with gaps given by $\omega = i\Delta$. The coefficients are somewhat lengthy, and their exact form is not particularly illuminating, so they are given in appendix D.3.

Showing that the gapped modes remain stable is in principle simple; show that the stability and causality conditions for $\vec{v}_0 = 0$ satisfy the RH criteria for this equation. The RH criteria for a cubic polynomial are given in B; they are given by

$$a_0 > 0, \quad a_3 > 0, \quad a_1 > 0, \quad a_1 a_2 > a_0 a_3.$$

These conditions can be simultaneously satisfied, see e.g. figure 3.4. They can be cast into a unitless form by introducing $\bar{\nu}_3 = \frac{T\nu_3}{\kappa^2\gamma_s}$, $\bar{\theta}_1 = \frac{\theta_1}{\gamma_s}$, $\bar{\pi}_1 = \frac{\pi_1}{\gamma_s}$, and $\bar{\sigma} = \frac{T^2\sigma}{\gamma_s\kappa^2}$.

As was previously demonstrated, modes that propagate causally when the fluid is at rest will propagate causally when the fluid is moving. Analyzing the RH criterion of a fluid in motion is prohibitively complicated, even for a conformal fluid, and so is left to future work.

3.4.2 Generic Charged Fluids

For a general charged fluid, there is no assumption of conformal symmetry. We will, however, make the same basic de-coupling frame choice ($\pi_3 = \varepsilon_3 = \theta_3 = 0$), as well as two additional further de-coupling/simplifying choices. Recall the definitions of the bulk viscosity and the charge conductivity in terms of the transport coefficients, given the frame choice above:

$$\zeta = p_\epsilon\pi_1 - \pi_2 + p_\epsilon\left(\varepsilon_2 - p_\epsilon\varepsilon_1\right) + p_n\left(\nu_2 - p_\epsilon\nu_1\right) - \frac{1}{T}p_n^2\nu_3, \tag{3.35a}$$

$$\sigma = -\frac{\gamma_3}{T} + \frac{n\gamma_1}{w} - \frac{n^2\theta_1}{w^2}. \tag{3.35b}$$

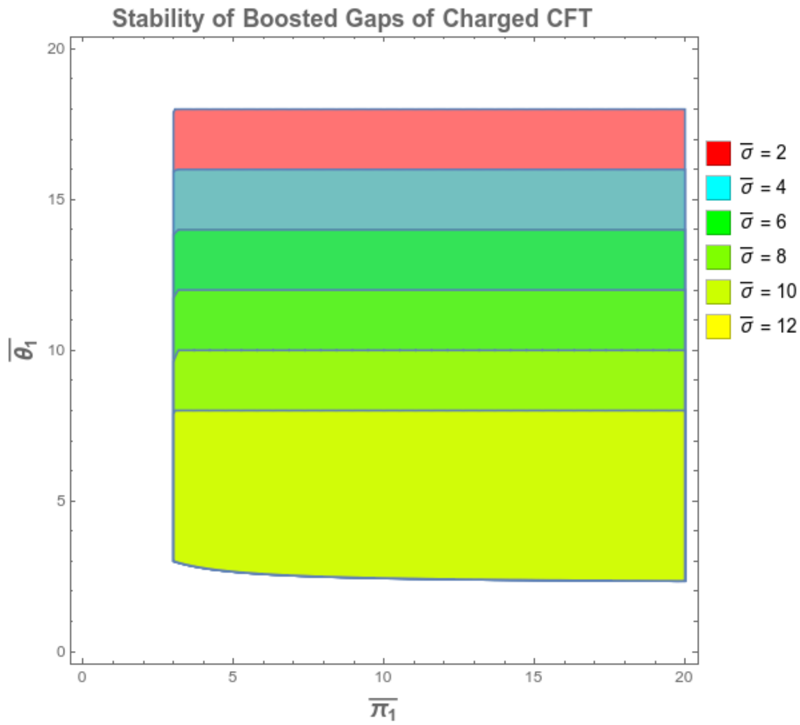

Figure 3.4: A slice of the parameter space for $d = 3$ with $\lambda = 4$, $\kappa = 1$, and $\bar{\nu}_3 = 20$. Plotted for various values of $\bar{\sigma}$, this plot shows the regions of the parameter space where the fluid at rest is stable and causal, and the boosted gaps are stable. The regions are overlayed, and all have the same lower boundary.

Further specializing to the class of frames where $\nu_2 = p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3$ and $\gamma_1 = \frac{n\theta_1}{w}$ leads to extremely simple expressions for σ and ζ:

$$\zeta = p_\epsilon \pi_1 - \pi_2 + p_\epsilon \left(\varepsilon_2 - p_\epsilon \varepsilon_1\right), \tag{3.36a}$$

$$\sigma = -\frac{\gamma_3}{T}. \tag{3.36b}$$

This frame choice has two benefits: it makes the charge conductivity exceedingly simple, and also casts the bulk viscosity into exactly the same form as in the uncharged case (though of course, when the charged fluid is not conformal, $p_\epsilon \neq v_s^2$).

In this “decoupled frame”, the stress-energy tensor and charge current are given by

$$T^{\mu\nu} = \left(\epsilon + \varepsilon_1 u^\lambda \partial_\lambda T + \varepsilon_2 \partial_\lambda u^\lambda\right) u^\mu u^\nu + \left(p + \pi_1 u^\lambda \partial_\lambda T + \pi_2 \partial_\lambda u^\lambda\right) \Delta^{\mu\nu} \tag{3.37a}$$

$$+ \theta_1 \left(u^\lambda \partial_\lambda u^\mu + \frac{\Delta^{\mu\lambda}\partial_\lambda T}{T}\right) u^\nu + \theta_1 \left(u^\lambda \partial_\lambda u^\nu + \frac{\Delta^{\nu\lambda}\partial_\lambda T}{T}\right) u^\mu - \eta\sigma^{\mu\nu}, \tag{3.37b}$$

$$J^\mu = \left(n + \nu_1 u^\lambda \partial_\lambda T + \left(p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3\right) \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) u^\mu \tag{3.37c}$$

$$+ \frac{n\theta_1}{w}\left(u^\lambda \partial_\lambda u^\mu + \frac{\Delta^{\mu\lambda}\partial_\lambda T}{T}\right) - T\sigma\Delta^{\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right). \tag{3.37d}$$

Perturbing the equilibrium state and passing to momentum space once more, ensuring the singularity of the coefficient matrix yields the exact same transverse shear modes as in the uncharged case (i.e. (3.7)). The longitudinal equation (after substituting ζ for π_2) is once more a sextic polynomial of the form

$$F(\vec{v}_0 = 0, \omega, k) = a_6\omega^6 + a_5\omega^5 + a_4\omega^4 + a_3\omega^3 + a_2\omega^2 + a_1\omega + a_0 \tag{3.38}$$

The form of the coefficients is again horrible, and so can be found in appendix D.4.

The small-k modes at $\vec{v}_0 = 0$ are given by

$$\omega = -i\left(\frac{p_\epsilon^2 T^2 \sigma}{w_0 v_s^2 \left(v_s^2 \lambda - T\kappa\right)}\right) k^2 + \mathcal{O}\left(k^3\right), \tag{3.39a}$$

$$\omega = \pm v_s k - i\frac{1}{2w_0}\left(\gamma_s + \frac{p_n^2\sigma}{v_s^2}\right) k^2 + \mathcal{O}\left(k^3\right), \tag{3.39b}$$

$$\omega = -i\frac{w_0}{\theta_1} + \mathcal{O}\left(k\right), \tag{3.39c}$$

$$\begin{aligned}\omega = -\frac{i}{2p_\epsilon^2 T^2 \varepsilon_1 \nu_3}\Bigg[& p_\epsilon^2 T w_0 \varepsilon_1 \lambda - n p_\epsilon T^3 \nu_1 + p_n p_\epsilon T w_0 \lambda \nu_1 - n p_n T^2 \nu_3 + p_\epsilon T^2 w_0 \nu_3 + p_n^2 w \lambda \nu_3 \\ & \pm \Bigg(4 p_\epsilon^2 T^3 w_0^2 \varepsilon_1 \left(T\kappa - v_s^2\lambda\right)\nu_3 - \bigg(p_\epsilon^2 T w_0 \varepsilon_1 \lambda + p_n\left(p_n w_0 \lambda - nT^2\right)\nu_3 \\ & + p_\epsilon T\left(T w_0 \nu_3 + p_n w_0 \lambda \nu_1 - nT^2\nu_1\right)\bigg)^2\Bigg)^{1/2}\Bigg] + \mathcal{O}\left(k\right).\end{aligned} \tag{3.39d}$$

In order for the gapless modes to be stable, the only conditions are the constraints of thermodynamics and the positivity of γ_s and σ, which are guaranteed by the positivity of the entropy current. The positivity of θ_1 obviously makes gap (3.39c) stable. The gaps in (3.39d) have the following controlling equation:

$$-p_\epsilon^2 T^2 \varepsilon_1 \nu_3 \omega^2 - i w_0 \Bigg(p_\epsilon^2 T_0 w_0 \varepsilon_1 \lambda + w_0 p_\epsilon T \left(p_n \lambda - T\kappa\right) \nu_1 + w \left(p_\epsilon T^2 + p_n^2 \lambda - T p_n \kappa\right) \nu_3 \Bigg) \omega + w_0^2 T v_s^2 \left(\lambda - \frac{1}{v_s^2}\kappa^2\right) = 0. \tag{3.40}$$

The RH criterion can be used to ensure stability. For a quadratic polynomial, the criterion is simply that all of the coefficients of the polynomial have the same sign. Making the substitution that $\omega = i\Delta$, the controlling equation becomes

$$p_\epsilon^2 T^2 \varepsilon_1 \nu_3 \omega^2 + w_0 \Bigg(p_\epsilon^2 T_0 w_0 \varepsilon_1 \lambda + w_0 p_\epsilon T \left(p_n \lambda - T\kappa\right) \nu_1 + w \left(p_\epsilon T^2 + p_n^2 \lambda - T p_n \kappa\right) \nu_3 \Bigg) \omega + w_0^2 T v_s^2 \left(\lambda - \frac{1}{v_s^2}\kappa^2\right) = 0. \tag{3.41}$$

The second-order term of (3.41) will obviously be positive if $\varepsilon_1 > 0, \nu_3 > 0$. The zeroth-order term is positive by the thermodynamics ($\lambda - \kappa^2/v_s^2 > 0$). The first-order term actually presents a new constraint on the transport coefficients that must be satisfied.

The condition that the first-order coefficient be positive yields the constraint

$$\frac{p_\epsilon^2 \lambda}{T}\varepsilon_1 + \varrho \nu_3 > -\frac{p_\epsilon v_s^2}{\kappa}\left(\left(\lambda - \frac{\kappa^2}{v_s^2}\right) - p_\epsilon \lambda\right)\nu_1,$$

where $\varrho = p_\epsilon + \frac{p_n^2}{T^2}\lambda - \frac{\kappa}{T} p_n \geq 0$ by the thermodynamics (c.f. conditions 2.6 of [17]). In the conformal fluid analysis, the positivity of the transport coefficients was enough to confirm stability of the gaps, because of the choice $\nu_1 = 0$. Making the same choice here would lead to this gap being generically stable.

For the large-k modes, the highest-order terms of the dispersion relations are once again linear in k, i.e. $\omega = ck + \mathcal{O}(k^0)$. The controlling equation for c is

$$\left(c^2 - \frac{\sigma T}{\nu_3}\right)\Bigg(\varepsilon_1 \theta_1 c^4 - \Big(\gamma_s \varepsilon_1 + p_\epsilon^2 \varepsilon_1^2 + \varepsilon_2 \theta_1 + \left(\varepsilon_2 + \theta_1\right)\pi_1 - p_\epsilon \varepsilon_1 \left(\varepsilon_2 + \pi_1\right)\Big) c^2 - \gamma_s \theta_1 - p_\epsilon \theta_1 \left(p_\epsilon \varepsilon_1 - \varepsilon_2 - \pi_1\right)\Bigg) = 0. \tag{3.42}$$

There are six solutions to this equation for c, corresponding to the six large-k longitudinal modes. The modes described by the first factor of (3.42) are causal so long as

$$\nu_3 > \sigma T. \tag{3.43}$$

The four modes corresponding to the second factor of (3.42) obey the same dispersion relations as the longitudinal modes of an uncharged fluid. This is because the controlling equation for these modes is identical to equation (3.11), and as such, the conditions are identical to conditions (3.14).

We now know the conditions for stability of the small-k modes, as well as the conditions for stability and causality of the large-k modes. The only thing left to do for the $\vec{v}_0 = 0$ case is examine the RH criterion. Unfortunately, the conditions that arise from the RH criterion are immensely complicated, and at present it is not known if simultaneously satisfying them all is possible. Given that the conditions are thousands of terms long, analysis by hand in impractical, and as such I have chosen not to put the criteria explicitly in this thesis – they may be derived by putting the coefficients in appendix D.4 into the conditions described in (3.33).

Given the lack of tractability of the RH criteria even for a fluid at rest, there is not much reason to analyze the boosted case. Hopefully this will encourage future work on the subject; the criteria being solved for $v_0 \neq 0$ would confirm that the fluid is generically stable, given the constraints outlined above. The small-k dispersion relations for the boosted sound modes can be found by analogy with (3.23), and the dispersion relation of the charge mode can all be found by analogy with (3.22b). The shear modes have boosted dispersion relations equal to (3.22).

3.5 Non-Linear, Real-Space Causality

All of the previous sections have analyzed the **linearized** equations, where the transport coefficients themselves have no dependence on the thermodynamic variables, and non-linear terms in the hydrodynamic equations were neglected. However, it is possible to show that the derived causality constraints are true not only at the linear level, but also for the full, non-linear equations. This can be done by analyzing the structure of the real-space differential equations. To proceed, some of the theory of systems of partial differential equations must be introduced.

The following employs the methods utilized by BDN in [14] and [16]. For more information on the theory of partial differential equations, the interested reader may refer to [27], especially Chapter VI.

3.5.1 General Theory

A second-order quasi-linear differential equation in 4 independent spacetime variables x_0, x_1, x_2, x_3 may be written in the form

$$L[u] = a^{\mu\nu}\partial_\mu\partial_\nu u + d = 0, \tag{3.44}$$

where $a^{\mu\nu} = a^{\nu\mu}$ and d are functions of the independent variables, the field u, and the first order derivatives $\partial_\mu u$, where μ runs over the four spacetime variables. The second-order terms are called the *principal part* of the differential equation, and are the only part relevant to this discussion.

Consider some spacetime hypersurface C described by the equation $\phi(x) = 0$, which is assumed to separate a region $\phi > 0$ and $\phi < 0$. On this hypersurface, one is given initial conditions, or "Cauchy data", comprising the value of the field u on the surface, as well as the value of the derivative of the field in the direction of the gradient of the scalar field ϕ, which is denoted $\partial_\phi u$. As an example, for a purely spacelike hypersurface at $t = 0$, one could simply set $\phi(x) = t$, and the derivative would be the time-derivative of u.

One can do a coordinate transformation from the set of coordinates (x_0, x_1, x_2, x_3) to a different coordinate system $(\phi, \eta_1, \eta_2, \eta_3)$ where η_i are "internal" variables that point along the surface, and ϕ is the scalar field that controls the surface – naturally, it departs from the surface, and so is called "external". In this coordinate system, the Cauchy data will be given solely in terms of the internal variables, η_i.

After this transformation, the differential equation takes on the form

$$Q\partial_\phi^2 u = \tilde{J}, \tag{3.45}$$

where $\tilde{J}$ contains all of the quantities that may be derived from the Cauchy data, i.e. u, all derivatives with respect to internal variables η up to second order, and the derivative of u with respect to external variable ϕ. The quantity Q is called the *characteristic element*. If $Q \neq 0$, then the external derivative $\partial_\phi^2 u$ may be determined on the hypersurface C, as may all higher-order derivatives. If this is the case, the hypersurface C is said to be "free". On the other hand, if $Q = 0$, then $\partial_\phi^2 u$ is not uniquely determined by the Cauchy data, the surface is called a *characteristic surface*, and the differential equation (3.45) represents a constraint on the initial data.

Cauchy's Theorem states the following:

For an initial surface C, if it is free, there is one, and only one, unique solution to the differential equation, given sufficient initial data. If the initial surface C is characteristic, then the differential equation imposes a constraint on the Cauchy data. If this constraint is satisfied, there are infinitely many solutions. If it is not, there are no solutions.

This characteristic element can be shown to be the following [27]:

$$Q \equiv a^{\mu\nu} \left(\partial_\mu \phi\right) \left(\partial_\nu \phi\right) . \tag{3.46}$$

The characteristic equation $Q = 0$ is a first-order differential equation for the scalar ϕ, and determines where the characteristic surfaces of the equations are (if they exist). The functions ϕ must be real. If there are **no** real solutions $\phi(x)$ to the equation $Q = 0$, the differential equation (3.44) is said to by elliptic. If there exist two real solutions (as the differential equation is of second order), then (3.44) is said to be **hyperbolic**. If the characteristic equation $Q = 0$ can be reduced to an equation in fewer variables $\xi_\mu = \partial_\mu \phi$, then (3.44) is called "parabolic".

Hyperbolicity is an absolute requirement if the equation is intended to model finite speed propagation, e.g. subluminal propagation. Hyperbolicity is defined (for a single second-order differential equation) by the existance of two characteristic surfaces passing through every point.

The requirement of hyperbolicity for causality is implied by the following. Consider some initial, free hypersurface ϕ (i.e. one on which initial data may be freely prescribed at some initial time t_0). Then consider the solution to the differential equation (3.44), denoted by u, at some point $P = P(t, x, y, z)$, where $t > t_0$. The characteristic surfaces passing through the point P will intersect the initial hypersurface ϕ, creating some closed region Ω.

The closed region of ϕ defined by the outer reaches of the characteristic surfaces is called the *domain of dependence* of the solution at point P. The solution u will formally **only** depend on the initial data on this region Ω, and will be totally immune to changes in initial data outside of this region. In physics parlance, if the outer characteristic surface is the lightcone we would say that the solution is *causally disconnected* from the region outside of Ω. The characteristic surfaces define the *past* J^- of the solution at the point P. For an illustration, refer to Figure 3.5.

With characteristic surfaces defining the domain of dependence, they may be

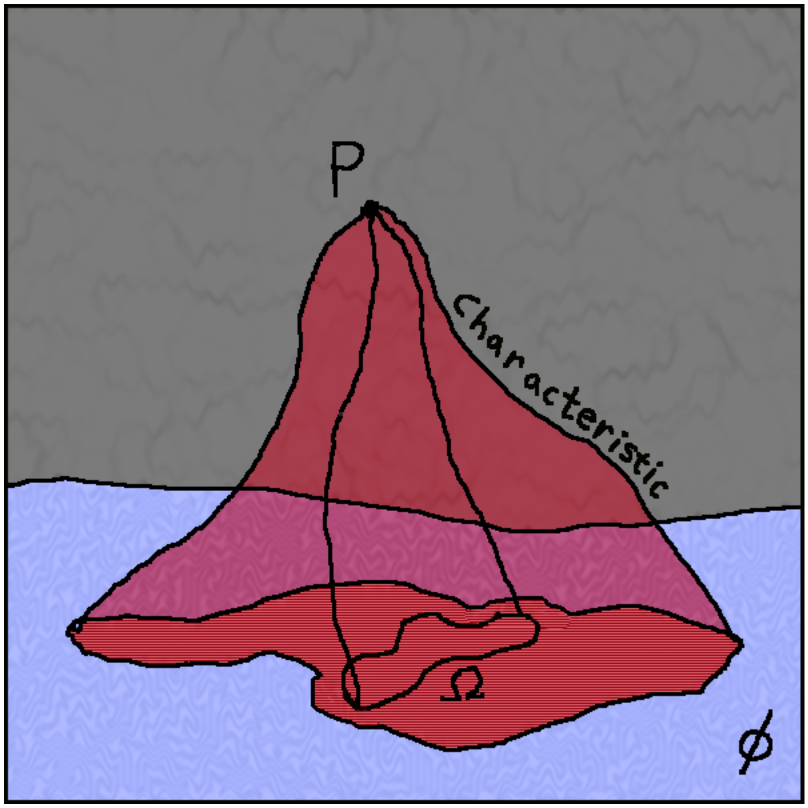

Figure 3.5: An illustration of how the characteristic surfaces through a point P intersect an initial hypersurface ϕ to define the **domain of dependence** Ω. The conoid defined by the characteristic surfaces encloses the past of the solution at P. Note that it is only the *outmost* characteristic surface that defined the domain of dependence; the internal characteristic surface has no effect.

thought of as *wave fronts* at any given time slice. They determine the furthest extent to which the effect of a disturbance may have reached. If the characteristic surfaces form layers within the lightcone, each layer may be thought of as a *mode of propagation*, each moving at a different speed.

Hyperbolicity is required for finite propagation speeds, but it is insufficient for *causality*. In order to avoid acausal propagation, the characteristic surfaces passing some point P must all reside *within* or *on* the lightcone at that point P.

So, we see that there are two conditions that must be met to ensure causality:

1. There must exist the correct number of characteristic surfaces, and

2. These characteristic surfaces must all remain within or on the lightcone.

The theory is modified only slightly for the case of a system of second-order quasi-linear differential equations in some k fields. We will assume there are *five* fields, and as such five equations. Previously, the definition of hyperbolicity was that there were two real solutions, as the differential equation is of second order. With five differential equations, there must now be *ten* real solutions.

Let all of the fields be collected in a 5-dimensional vector, U. Then the system of differential equations may be written as

$$L_a[u] = (A^{\mu\nu})_{ab}\,\partial_\mu\partial_\nu U^b + d_a = 0, \tag{3.47}$$

where μ, ν are labels on the matrix A, while $a, b \in \{1, ..., 5\}$ are matrix indices. Summation notation applies to a, b as well as μ, ν. With this system in place, the characteristic equation may be written as

$$Q = \det\ \left((A^{\mu\nu})_{ab}\,(\partial_\mu\phi)\,(\partial_\nu\phi)\right) = 0. \tag{3.48}$$

We define here the co-vector $\xi_\mu = \partial_\mu\phi$, which is the *normal* to the hypersurface defined by $\phi = 0$. Then the characteristic condition may be written as

$$Q = \det\ \left((A^{\mu\nu})_{ab}\,\xi_\mu\xi_\nu\right) = 0.$$

In order for the characteristic surfaces to be entirely *inside* the lightcone at a given point, these *normals* must point entirely *out* of the lightcone; these normals will themselves form a cone called the normal cone. This is illustrated in Figure 3.6. The condition for causality of a system of five second-order partial differential equations then becomes the following:

1. There must be *ten* real solutions to the algebraic equation $Q = 0$ of the form $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$, and
2. All of these solutions must lie *outside* the lightcone.

With these conditions in hand, constraints may be derived on the transport coefficients in order to ensure causality.

3.5.2 Causality of a Charged Conformal Fluid

Inspection of the conservation equations for both $T^{\mu\nu}$ and J^μ show that there are five equations in five degrees of freedom. However, working with the physical fluid

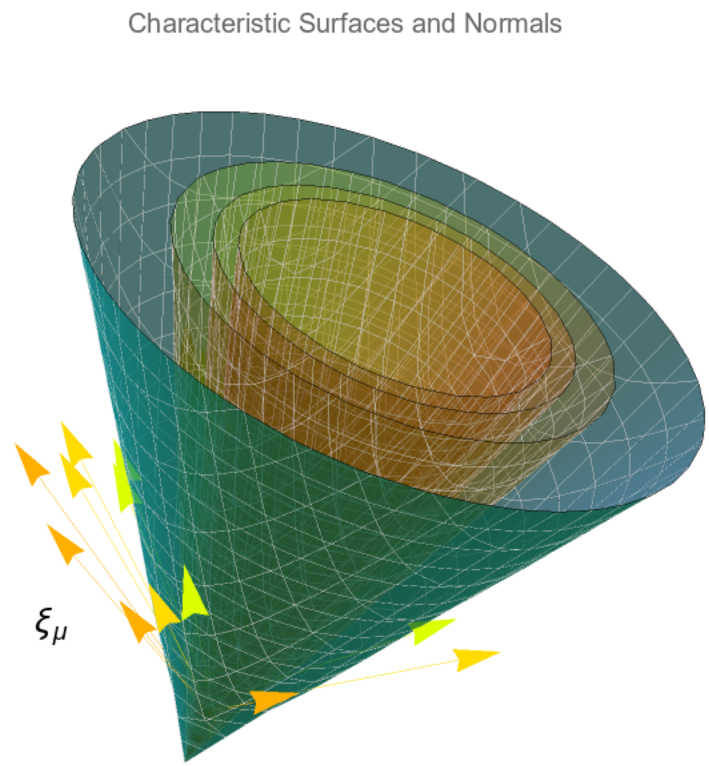

Figure 3.6: A graphical representation of the light cone with characteristic surfaces entirely inside. The normal cone has been represented by arrows, with different colours corresponding to different modes of propagation. All of the normal-cone arrows point out of the light-cone.

velocity $\vec{v}$ in the non-linear regime is troublesome due to the factors of $(1-v^2)^{-1/2}$. If possible, it is preferable to continue to work with the fluid 4-vector. Let us refer back to the equilibrium density operator of the previous chapter, $\hat{\varrho}$. If we examine its definition in Equation (2.2), we see two thermodynamic quantities that appear: β^μ and ψ. The thermal vector β^μ is a Killing vector of the background metric. Choosing to work with these quantities instead of the temperature, spatial fluid velocity, and chemical potential means there will still be five equations for five degrees of freedom, but we will not have to explicitly deal with any nasty square roots.

The quantity $\psi = \frac{\mu}{T}$ appears explicitly in the derivative expansion, and so no further work needs to be done to transform our hydrodynamic variables as written to the correct form. The thermal vector β^μ on the other hand requires some more work. Recall its definition: $\beta^\mu = \frac{u^\mu}{T}$. Then obviously $u^\mu = T\beta^\mu$. This fact along with

$\beta^2 = -1/T^2$ can be used to derive that

$$\partial_\alpha T = T^2 u_\rho \partial_\alpha \beta^\rho, \qquad \partial_\alpha u^\gamma = T\Delta^\gamma_\rho \partial_\alpha \beta^\rho.$$

The **principal part** of the divergences of the stress-energy tensor and charge current (with $\theta_3 = \pi_3 = \varepsilon_3 = 0$) can then be written:

$$\begin{aligned}
\partial_\mu T^{\mu\nu}|_{\mathcal{P}} = T\Bigg\{ &\varepsilon_1 u^\lambda u_\rho u^\mu u^\nu + \varepsilon_2 u^\mu u^\nu \Delta^\lambda_\rho + \pi_1 \Delta^{\mu\nu} u^\lambda u_\rho + \pi_2 \Delta^{\mu\nu}\Delta^\lambda_\rho \\
&+ \theta_1 \left(\Delta^{\mu\lambda} u^\nu u_\rho + u^\lambda u^\nu \Delta^\mu_\rho + \Delta^{\nu\lambda} u^\mu u_\rho + u^\lambda u^\mu \Delta^\nu_\rho\right) \\
&- \eta \left(\Delta^{\mu\lambda}\Delta^\nu_\rho + \Delta^\mu_\rho \Delta^{\nu\lambda} - \frac{2}{d}\Delta^{\mu\nu}\Delta^\lambda_\rho\right)\Bigg\}\partial_\mu\partial_\lambda\beta^\rho + 0^{\mu\lambda\nu}\partial_\mu\partial_\nu\psi, \qquad (3.49a)\\
\partial_\mu J^\mu|_{\mathcal{P}} = T\Bigg[&\nu_1 u^\mu u^\lambda u_\rho + \nu_2 u^\mu \Delta^\lambda_\rho + \gamma_1\left(\Delta^{\mu\lambda}u_\rho + u^\lambda \Delta^\mu_\rho\right)\Bigg]\partial_\mu\partial_\lambda\beta^\rho \\
&+ \Bigg[\nu_3 u^\mu u^\lambda + \gamma_3 \Delta^{\mu\lambda}\Bigg]\partial_\lambda\partial_\mu\psi, \qquad (3.49b)
\end{aligned}$$

where $|_{\mathcal{P}}$ means the principal part of the equation, and $0^{\mu\lambda\nu}$ is the zero tensor. If we choose to specify that the fluid is conformal, then conformality imposes that $\varepsilon_i = d\pi_i, \pi_1 = d\pi_2, \nu_1 = d\nu_2$. To match the results of section 3.4.1, we further specify that $\nu_1 = 0, \gamma = 0$. Then the principal part of the system of differential equations is:

$$\begin{aligned}
\partial_\mu T^{\mu\nu}|_{\mathcal{P}} = T\Bigg\{ &d\pi_1 u^\lambda u_\rho u^\mu u^\nu + \pi_1 u^\mu u^\nu \Delta^\lambda_\rho + \pi_1 \Delta^{\mu\nu} u^\lambda u_\rho + \frac{1}{d}\pi_1 \Delta^{\mu\nu}\Delta^\lambda_\rho \\
&+ \theta_1 \left(\Delta^{\mu\lambda} u^\nu u_\rho + u^\lambda u^\nu \Delta^\mu_\rho + \Delta^{\nu\lambda} u^\mu u_\rho + u^\lambda u^\mu \Delta^\nu_\rho\right) \\
&- \eta \left(\Delta^{\mu\lambda}\Delta^\nu_\rho + \Delta^\mu_\rho \Delta^{\nu\lambda} - \frac{2}{d}\Delta^{\mu\nu}\Delta^\lambda_\rho\right)\Bigg\}\partial_\mu\partial_\lambda\beta^\rho + 0^{\mu\lambda\nu}\partial_\mu\partial_\nu\psi,\\
\partial_\mu J^\mu|_{\mathcal{P}} = T\Bigg[&0^{\mu\lambda}_\rho\Bigg]\partial_\mu\partial_\lambda\beta^\rho \\
&+ \Bigg[\nu_3 u^\mu u^\lambda + \gamma_3 \Delta^{\mu\lambda}\Bigg]\partial_\lambda\partial_\mu\psi.
\end{aligned}$$

These equations can be cast in matrix form by creating a column vector U such that $U = \{\beta^\mu, \psi\}$. The equations then become

$$(A_{ij})^{\mu\lambda}\,\partial_\mu\partial_\lambda U^j + \delta_i = 0, \qquad (3.50)$$

where δ contains all the lower-order terms. Here, μ, λ may be thought of as being labels for $(d+1)^2$ matrices, and $i, j \in [0, d+1]$ are matrix indices. In the range $[0, d]$, they correspond to ν and ρ in the equations above. The fact that the principal part of $\partial_\mu T^{\mu\nu}$ does not depend on ψ at all, and the principal part of $\partial_\mu J^\mu$ does not depend on β^μ leads to the matriices $(A_{ij})^{\mu\lambda}$ above decomposing into two smaller block matrices, one of size $(d+1) \times (d+1)$, and the other of size one. Schematically, this is

$$(A_{ij})^{\mu\lambda} = \begin{bmatrix} (\mathcal{A}_{\text{stress energy}})_\rho^{\mu\nu\lambda} & 0 \\ 0 & \mathcal{A}_{\text{charge current}} \end{bmatrix}.$$

It is clear that the determinant of such a matrix will factorize. Now, to determine if this system has causal propagation, there are two conditions that must be met:

1. The system must be **hyperbolic**
2. The normal cone must be entirely **outside** the light cone.

To determine both of these qualities, we investigate the characteristic element, and look for solutions to the characteristic equation:

$$Q \equiv \det \left((A_{ij})^{\mu\lambda} \xi_\mu \xi_\lambda \right) = 0. \tag{3.51}$$

These ξ define the normal cone. As is clear from the schematic matrix above, the characteristic element Q will factorize into two parts:

$$\mathcal{M}_{\text{charge}} \times \mathcal{M}_{\text{stress energy}} = 0.$$

The first part, $\mathcal{M}_{\text{charge}}$, is given by

$$\nu_3 \left(u \cdot \xi \right) + \gamma_3 \left(\xi \cdot \Delta \cdot \xi \right) \quad = \nu_3 \left(\quad \left(u \cdot \xi \right) - \tau_\nu \left(\xi \cdot \Delta \cdot \xi \right) \right),$$

where $\tau_\nu = -\gamma_3/\nu_3$. At a **specific** given point, let us now pass to a locally co-moving reference frame in which $u^\mu = \left(1, \vec{0}\right)$. Then the equation $\mathcal{M}_{\text{charge}} = 0$ becomes

$$\xi_0^2 = \tau_\nu |\vec{\xi}|^2.$$

If ξ_0 is to be real, then we require that $\tau_\nu > 0$. If ξ as a whole is to lie **outside** the lightcone, we require that $\tau_\nu < 1$. Therefore, the causality condition is given by

$$0 < \frac{-\gamma_3}{\nu_3} < 1. \tag{3.52}$$

If we substitute $T\sigma \equiv -\kappa^2\theta_1 - \gamma_3$ into condition (3.52) for γ_3, then condition (3.52) becomes

$$\nu_3 > T\sigma + \frac{n^2 T}{w^2}\theta_1. \tag{3.53}$$

This is identical to condition (3.32) of section 3.4.1. This shows that the charge causality condition is not merely true at **linear** order, but for the full non-linear equations as well. Let us now address the other half of the characteristic equation, $\mathcal{M}_{\text{stress energy}}$. The stress-energy part of the characteristic equation is given by

$$\begin{aligned}
\left(\mathcal{A}_{\text{stress energy}}\right)^{\mu\nu\lambda}_{\rho}\xi_\mu\xi_\lambda &= d\pi_1\left(u\cdot\xi\right)^2 u^\nu u_\rho + \pi_1\left(u\cdot\xi\right)u^\nu\left(\left(u\cdot\xi\right)u_\rho + \xi_\rho\right) \\
&+ \pi_1\left(u\cdot\xi\right)\left(\left(u\cdot\xi\right)u^\nu + \xi^\nu\right)u_\rho + \frac{1}{d}\pi_1\left(\left(u\cdot\xi\right)u^\nu + \xi^\nu\right)\left(\left(u\cdot\xi\right)u_\rho + \xi_\rho\right) \\
&+ \theta_1\Bigg(\left(\xi\cdot\Delta\cdot\xi\right)u^\nu u_\rho + \left(u\cdot\xi\right)u^\nu\left(\left(u\cdot\xi\right)u_\rho + \xi_\rho\right) \\
&+ \left(\left(u\cdot\xi\right)u^\nu + \xi^\nu\right)\left(u\cdot\xi\right)u_\rho + \left(u\cdot\xi\right)^2\Delta^\nu_\rho\Bigg) \\
&- \eta\left(\left(\xi\cdot\Delta\cdot\xi\right)\Delta^\nu_\rho + \left(\frac{d-2}{d}\right)\left(\left(u\cdot\xi\right)u_\rho + \xi_\rho\right)\left(\left(u\cdot\xi\right)u^\nu + \xi^\nu\right)\right).
\end{aligned}$$

Consider an expression of the form

$$Au^\alpha u_\beta + B\Delta^\alpha_\beta + Cu^\alpha\xi_\beta + D\xi^\alpha u_\beta + E\xi^\alpha\xi_\beta.$$

We would like to know, before determining $\{A, B, C, D, E\}$, what the determinant of such an expression would be. Explict calculation reveals [17]

$$\begin{aligned}
&\det\left(Au^\alpha u_\beta + B\Delta^\alpha_\beta + Cu^\alpha\xi_\beta + D\xi^\alpha u_\beta + E\xi^\alpha\xi_\beta\right) \\
&= B^{d-1}\left(-AB + B\left(C+D\right)\left(u\cdot\xi\right) - BE\left(u\cdot\xi\right)^2 + \left(CD - AE\right)\left(\xi\cdot\Delta\cdot\xi\right)\right).
\end{aligned}$$

Casting the stress-energy part of the characteristic equation into a form where A, B,

C, D, and E can be read off yields

$$\begin{aligned}
(\mathcal{A}_{\text{stress energy}})^{\mu\nu\lambda}_{\rho}\,\xi_\mu\xi_\lambda = &\left\{\left(\frac{(d+1)^2}{d}\pi_1 + 2\theta_1 - \left(\frac{d-2}{d}\right)\eta\right)(u\cdot\xi)^2 + (\xi\cdot\Delta\cdot\xi)\,\theta_1\right\}u^\nu u_\rho \\
&+\left\{\theta_1\,(u\cdot\xi)^2 - \eta\,(\xi\cdot\Delta\cdot\xi)\right\}\Delta^\nu_\rho \\
&+\left\{\left(\frac{d+1}{d}\right)\pi_1 + \theta_1 - \left(\frac{d-2}{d}\right)\eta\right\}(u\cdot\xi)\,u^\nu\xi_\rho \\
&+\left\{\left(\frac{d+1}{d}\right)\pi_1 + \theta_1 - \left(\frac{d-2}{d}\right)\eta\right\}(u\cdot\xi)\,\xi^\nu u_\rho \\
&+\left\{\frac{1}{d}\pi_1 - \left(\frac{d-2}{d}\right)\eta\right\}\xi^\nu\xi_\rho.
\end{aligned}$$

Reading off B, we get $d-1$ copies of $\left((u\cdot\xi)^2\,\theta_1 - \eta\,(\xi\cdot\Delta\cdot\xi)\right)$, or equivalently,

$$(u\cdot\xi)^2 - \tau_\theta\,(\xi\cdot\Delta\cdot\xi)\,,$$

where $\tau_\theta = \eta/\theta_1$. By again passing to a locally co-moving reference frame (i.e. one where $u^\alpha = (1,\vec{0})$), we can identify the constraint

$$\theta > \eta > 0. \tag{3.54}$$

This matches perfectly the constraints we already found in the small-k limit in (3.8). The positivity of η does not come directly from this constraint, but rather from the positivity of the entropy current production (see Appendix C). As in the linearized theory, there are $d-1$ copies of the transverse mode.

Plugging the values of A, B, C, D, and E in to

$$\left(-AB + B\,(C+D)\,(u\cdot\xi) - BE\,(u\cdot\xi)^2 + (CD - AE)\,(\xi\cdot\Delta\cdot\xi)\right)$$

yields the equation

$$\begin{aligned}
&d\theta_1\pi_1\,(u\cdot\xi)^4 - (2\,(d-1)\,\eta + 2\theta_1)\,\pi_1\,(u\cdot\xi)^2\,(\xi\cdot\Delta\cdot\xi) \\
&-\left(\frac{2\,(d-1)}{d}\eta - \pi_1\right)(\xi\cdot\Delta\cdot\xi)^2 = 0.
\end{aligned} \tag{3.55}$$

Boosting to a locally co-moving reference frame, the equation (3.55) can be solved for ξ_0, giving a solution of the form $\xi_0 = c|\vec{\xi}|$. If we anticipate this final result of

$\xi_0 = c|\vec{\xi}|$ and plug this form into (3.55), then this is **exactly** the controlling equation for the propagation speed in the large-k analysis, (3.27). The causality constraints will therefore be identical as in the linearized analysis:

$$\pi_1 > 2\,(d-1)\,\eta, \quad 1 - \frac{2}{d-1}\left(\frac{d\eta}{\theta_1} + \frac{\eta}{\pi_1}\right) > 0 \tag{3.56}$$

The causality constraints (3.56) derived from the linearized analysis in momentum space therefore also hold true for the full non-linear hydrodynamic equations in real-space.

3.5.3 Generic Charged Fluid

Returning to the theory for a generic charged fluid, which may or may not have a conformal symmetry, the analysis will essentially be identical – because $\varepsilon_3 = \pi_3 = \theta_3 = 0$, the charged and uncharged parts of the characteristic equation will decouple. If we make the further specification (as in section 3.4.2) that $\nu_2 = p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3$ and $\gamma_1 = \frac{n}{w}\theta_1$, then the conductivity becomes extremely simple ($\sigma = -\frac{\gamma_3}{T}$, c.f. the version of σ substituted into (3.53)), and the bulk viscosity takes on its uncharged form (3.4). This "decoupled frame" will yield simple causality constraints. In this frame ($\{\varepsilon_3 = \theta_3 = \pi_3 = 0,\ \nu_2 = p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3,\ \gamma_1 = \frac{n}{w}\theta_1\}$), the principal part of the conservation equations will be

$$\begin{aligned}
\partial_\mu T^{\mu\nu}|_{\mathcal{P}} = T\Big\{ & \varepsilon_1 u^\lambda u_\rho u^\mu u^\nu + \varepsilon_2 u^\mu u^\nu \Delta^\lambda_\rho + \pi_1 \Delta^{\mu\nu} u^\lambda u_\rho + \pi_2 \Delta^{\mu\nu}\Delta^\lambda_\rho \\
& + \theta_1 \left(\Delta^{\mu\lambda} u^\nu u_\rho + u^\lambda u^\nu \Delta^\mu_\rho + \Delta^{\nu\lambda} u^\mu u_\rho + u^\lambda u^\mu \Delta^\nu_\rho\right) \\
& - \eta \left(\Delta^{\mu\lambda}\Delta^\nu_\rho + \Delta^\mu_\rho \Delta^{\nu\lambda} - \frac{2}{d}\Delta^{\mu\nu}\Delta^\lambda_\rho\right)\Big\}\partial_\mu\partial_\lambda \beta^\rho + 0^{\mu\lambda\nu}\partial_\mu\partial_\nu\psi,
\end{aligned} \tag{3.57a}$$

$$\begin{aligned}
\partial_\mu J^\mu|_{\mathcal{P}} = T\Big[& \nu_1 u^\mu u^\lambda u_\rho + \left(p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3\right) u^\mu \Delta^\lambda_\rho + \frac{n_0}{w_0}\theta_1 \left(\Delta^{\mu\lambda} u_\rho + u^\lambda \Delta^\mu_\rho\right)\Big]\partial_\mu\partial_\lambda\beta^\rho \\
& + \Big[\nu_3 u^\mu u^\lambda - T\sigma\Delta^{\mu\lambda}\Big]\partial_\lambda\partial_\mu\psi.
\end{aligned} \tag{3.57b}$$

This can once more be written in matrix form as

$$(A_{ij})^{\mu\lambda}\,\partial_\mu\partial_\lambda U^j + \delta_i = 0, \tag{3.58}$$

where $U^j = \{\beta^\alpha, \psi\}$. The characteristic equation $Q = 0$ will again factorize. The charged part will yield $\xi^0 = \pm\sqrt{\frac{T\sigma}{\nu_3}}|\vec{\xi}|$, which gives the very simple constraint

$$\nu_3 > T\sigma. \tag{3.59}$$

The stress-energy characteristic equation will read, after substituting ζ for π_2 in (3.57a) and writing in terms of $\gamma_s \equiv \zeta + \frac{2(d-1)}{d}\eta$,

$$\begin{aligned}&\Big(\theta_1 (u\cdot\xi)^2 - \eta\,(\xi\cdot\Delta\cdot\xi)\Big)^{d-1}\Big\{\varepsilon_1\theta_1 (u\cdot\xi)^4 \\ &-\big(\big(\gamma_s + p_\epsilon^2\varepsilon_1 - p_\epsilon(\varepsilon_2+\pi_1)\big)\varepsilon_1 + \varepsilon_2\theta_1 + (\varepsilon_2+\theta_1)\pi_1\big)(u\cdot\xi)^2(\xi\cdot\Delta\cdot\xi) \\ &-\theta_1\left(\gamma_s + p_\epsilon(p_\epsilon\varepsilon_1 - \varepsilon_2 - \pi_1)\right)(\xi\cdot\Delta\cdot\xi)^2\Big\} = 0.\end{aligned} \tag{3.60}$$

There are again $d-1$ modes propagating at speed $c^2 = \eta/\theta_1$, yielding the same constraints as (3.54). The second factor in (3.60) can by boosted to a locally co-moving reference frame; we then find $\xi_0 = c|\vec{\xi}|$. If we plug this solution for ξ_0 into the second factor of (3.60) in a locally co-moving reference frame, it yields an equation for c:

$$\begin{aligned}&\varepsilon_1\theta_1 c^4 \\ &-\big(\big(\gamma_s + p_\epsilon^2\varepsilon_1 - p_\epsilon(\varepsilon_2+\pi_1)\big)\varepsilon_1 + \varepsilon_2\theta_1 + (\varepsilon_2+\theta_1)\pi_1\big)c^2 \\ &-\theta_1\left(\gamma_s + p_\epsilon(p_\epsilon\varepsilon_1 - \varepsilon_2 - \pi_1)\right) = 0\end{aligned} \tag{3.61}$$

Equation (3.61) is identical to (3.11), and so the constraints on the transport coefficients will be identical to (3.14).

3.5.4 Coupling to Gravity

A surprising result that emerges from this non-linear analysis is that the causality results are **generically true** even when coupled to dynamical gravity. The proof is briefly sketched out hereafter.

If the background is curved, then all partial derivatives ought to have been upgraded to covariant derivatives, $\Delta_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_{\mu\lambda}A^\lambda$. The connection Γ is of order one in derivatives of the metric. With a curved background the principal part of the equations of motion (i.e. the conservation equations and the Einstein equations) will be [16][17]

$$\begin{aligned}
\partial_\mu T^{\mu\nu}|_{\mathcal{P}} = T\Big\{ & \varepsilon_1 u^\lambda u_\rho u^\mu u^\nu + \varepsilon_2 u^\mu u^\nu \Delta^\lambda_\rho + \pi_1 \Delta^{\mu\nu} u^\lambda u_\rho + \pi_2 \Delta^{\mu\nu}\Delta^\lambda_\rho \\
& + \theta_1 \left(\Delta^{\mu\lambda} u^\nu u_\rho + u^\lambda u^\nu \Delta^\mu_\rho + \Delta^{\nu\lambda} u^\mu u_\rho + u^\lambda u^\mu \Delta^\nu_\rho\right) \\
& - \eta\left(\Delta^{\mu\lambda}\Delta^\nu_\rho + \Delta^\mu_\rho \Delta^{\nu\lambda} - \frac{2}{d}\Delta^{\mu\nu}\Delta^\lambda_\rho\right)\Big\}\partial_\mu\partial_\lambda\beta^\rho + 0^{\mu\lambda\nu}\partial_\mu\partial_\nu\psi \\
& + C^{\mu\lambda\nu}_{\alpha\beta}\partial_\mu\partial_\lambda g^{\alpha\beta}, \qquad (3.62a) \\
\partial_\mu J^\mu|_{\mathcal{P}} = T\Big[& \nu_1 u^\mu u^\lambda u_\rho + \left(p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3\right) u^\mu \Delta^\lambda_\rho + \frac{n}{w}\theta_1\left(\Delta^{\mu\lambda}u_\rho + u^\lambda \Delta^\mu_\rho\right)\Big]\partial_\mu\partial_\lambda\beta^\rho \\
& + \Big[\nu_3 u^\mu u^\lambda + \gamma_3 \Delta^{\mu\lambda}\Big]\partial_\lambda\partial_\mu\psi + \tilde{C}^{\mu\lambda}_{\alpha\beta}\partial_\mu\partial_\beta g^{\alpha\beta}, \qquad (3.62b) \\
G^{\mu\nu}|_{\mathcal{P}} = \; & g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu}, \qquad (3.62c)
\end{aligned}$$

where C contains whatever terms come before the second derivative of the metric in the stress-energy, $\tilde{C}$ does the same thing in the charge current, and $g^{\alpha\beta}\partial_\alpha\partial_\beta g^{\mu\nu}$ is the principal part of the Einstein equations in wave gauge, where $G^{\mu\nu}$ is the Einstein tensor. These equations may be cast in matrix form as

$$\mathfrak{M}^{\alpha\beta}\partial_\alpha\partial_\beta U = B,$$

where $U = \{\beta^\mu, \psi, g^{\alpha\beta}\}^T$ contains all the dynamical variables and B contains the non-principal part of the system. The indices α, β act as labels on the matrices $\mathfrak{M}$, which may be written in block-matrix form as

$$\mathfrak{M}^{\alpha\beta} = \begin{bmatrix} \mathfrak{m}^{\alpha\beta}_{(d+2)\times(d+2)} & (C\cup\tilde{C})_{(d+2)\times 10} \\ 0_{10\times(d+2)} & g^{\alpha\beta}\mathbb{I}_{10\times 10} \end{bmatrix}.$$

The matrix $\mathfrak{m}$ is the matrix for the coefficients that come before the hydrodynamic variables in (3.62). The matrix $\mathfrak{m}^{\mu\nu} = (A_{ij})^{\mu\nu}$ from (3.58). The matrix $C\cup\tilde{C}$ contains the prefactors C and $\tilde{C}$ arranged in such a way that $\tilde{C}$ is in the same line as the line of $\mathfrak{m}$ that came from the charge current. The symbol $\mathbb{I}$ is just the identity matrix.

The characteristic equation of such a system of equations will be merely

$$\left(g^{\alpha\beta}\xi_\alpha\xi_\beta\right)^{10}\det\left(\mathfrak{m}^{\alpha\beta}\xi_\alpha\xi_\beta\right) = 0.$$

The first term defines the lightcone; the second gives the flat-space characteristic equation. It is clear then that any effect of the curvature decouples from the matter sector, and all of the causality constraints previously derived in flat space are equally valid in curved space.

Chapter 4

Discussion and Conclusion

In the preceding chapters, we investigated the failings of the Eckart and Landau classes of frames – specifically the fact that they predict that a uniformly moving equilibrium state will decay, and that they predict superluminal propagation. This issue can be resolved by introducing the BDNK formulation of first-order viscous hydrodynamics [14][16][15][17], and analyzing the stability and causality of that formulation. First we reviewed the stability and causality of an uncharged fluid, and then extended the formulation to that of a fluid with a U(1) charge.

We found that a hydrodynamic theory describing a charged fluid, in a frame dubbed the "de-coupled frame" with $\theta_3 = \varepsilon_3 = \pi_3 = 0$, had the exact same causality constraints as in the uncharged case. We established the formulation of the stability criteria, though the exact constraints they provide have not yet been determined, nor has it been determined if they can be satisfied for all k. We also showed that causality is sufficient to determine the stability of the gaps in the theory for a conformal charged fluid.

Having established that there exists a stable and causal first-order hydrodynamics, let us now compare it with the current prevailing formulation, the Müller-Israel-Stewart theory, or MIS theory. Both formulations have advantages and drawbacks.

4.1 Why Landau Frame At All?

A question that may have been percolating in the reader's mind over the course of this thesis is "Given the known problems with the Landau frame, why would anyone continue to use the Landau frame at all?".

In most cases of physical interest in high-energy physics, we do *not* initially know the hydrodynamic variables. Instead, we may know the expectation values of the stress-energy tensor and the charge current. In Landau frame, the fluid velocity is aligned with the flow of the energy current, and is an eigenvector of the stress-energy tensor. It then becomes simple in principle to find the form of u^μ:

$$T^{\alpha\beta}u_\beta = -\epsilon u^\alpha.$$

The eigenvalue associated with u^α is the energy density ϵ. If we then take this fluid velocity and contract it with the charge current $J_\alpha u^\alpha$, we get the form of n. If we know the form of ϵ and n, then given an equation of state of the form $p = p(T, \mu)$, we should be able to find definitions for T and μ. The Landau frame therefore lends itself to a simple definition of the hydrodynamic variables directly from the stress-energy tensor and the charge current.

For many situations requiring numerical simulation, initial conditions are often given in terms of the stress-energy tensor and the charge current, not the hydrodynamic variables. Having a simple dictionary from those microscopic quantities to the hydro variables is valuable, and so motivates the use of the Ladau frame. Instead of discarding the definitions of the hydrodynamic variables that the Landau frame gives, the MIS theory introduces new dynamical variables that act to remove the pathologies of the Landau class of frames. One is therefore able to retain the simple dictionary from the stress-energy tensor and charge current at the expense of making the equations significantly more complicated.

4.2 Müller-Israel-Stewart Theory

The MIS theory was first proposed in the non-relativistic case by Müller [3], and then in the relativistic case by Israel [4], with further work done by Israel and Stewart [5]. It promotes the bulk stress Π, heat current $\mathcal{Q}^\alpha$, and shear stress $\pi^{\mu\nu}$ to dynamical quantities, and introduces five new transport coefficients; three relaxation times β_0, β_1, β_2, and two coefficients detailing the interaction between the heat flux and the two stresses, α_0 and α_1. Later formulations added other coefficients to handle gradients of these five terms. With these additions, it is possible to construct a stable theory in MIS.

Lindblom and Hiscock developed a linearized analysis of the stability and causality of the MIS theory in [6]. In that work they showed that the equilibrium state is stable in MIS theory, given sufficiently large relaxation times $\beta_i \gtrsim (\epsilon + p)^{-1}$, where ϵ is the equilibrium energy density and p is the equilibrium pressure. They also showed that, for the linearized theory, stability of the equilibrium state implies that the linearized equations are hyperbolic, and that perturbations will propagate causally. Finally, they showed the reverse to be true as well.

Bemfica, Disconzi, Hoang, Noronha, and Radosz found in [7] conditions that are necessary for MIS-like theories to be causal far from equilibrium. They also found conditions that are sufficient (though possibly not all necessary) for causality, local exitence, and uniqueness of solutions in these MIS-like theories.

These features identified in MIS are *not* generically true in BDNK. The conditions in the linearized theory that ensure stability in BDNK do not imply the constraints that lead to perturbations propagating causally, nor vice-versa. This is demonstrated in Figures 3.1 and 3.2. Figure 3.1 illustrates the region of the parameter space where the equilibrium state is stable for all values of the wavevector, while Figure 3.2 illustrates the region of the parameter space where the equilibrium state is stable for all values of the wavevector *and* where perturbations propagate subluminally. The two regions are not the same.

The advantages of MIS are then essentially as follows: causality and stability have a one-to-one correspondence, and so one need only ensure the one in order to ensure the other; the procedure used by MIS can be done on the Landau or Eckart frames, leading to a straightforward derivation of the hydrodynamic variables from the stress-energy tensor and the charge current[1]; finally, MIS has been in use for about forty years at the time of writing this thesis, researchers are very familiar with it, and know how to implement it numerically [28][29].

There also exist some clear drawbacks to MIS. It increases the complexity of the equations considerably, including more dynamical degrees of a freedom. Additionally, since MIS does not just use the hydrodynamic variables T, μ, and u^μ, its status as a true "hydrodynamic" theory is questionable.

Let us now contrast BDNK with MIS. While not perfect, BDNK has some clear advantages over MIS. BDNK uses only the five ideal-order hydrodynamic degrees of freedom: the temperature T, the chemical potential μ, and the fluid four-velocity u^μ. This represents a large reduction in complexity; MIS has an additional scalar, vector, and tensor to evolve. It is also not significantly more difficult to derive the fluid velocity, temperature, or charge current in BDNK than in MIS. One can simply derive the hydrodynamic variables in the Landau frame, perform a frame-redefinition

[1]Technically, one could perform the MIS procedure on BDNK hydrodynamics as well, but there is not much point given that BDNK theory is already stable and causal in the right frame.

to one's frame of choice, and then evolve the variables according to the hydrodynamic equations.

One disadvantage is that in BDNK causality and stability do not imply one another – there are separate constraints for each, though they can be simultaneously satisfied. We have shown, however, that in the linearized regime a violation of causality *does* imply instability in a boosted reference frame (c.f. equation (3.21)). Another disadvantage is that, given the relative novelty of this formulation, approaches to implementing BDNK numerically have not yet been explored (to my knowledge).

While BDNK is still young, it has the potential to replace MIS as the predominant hydrodynamic framework. Its simplicity is a major draw over the complexity of MIS. There are also many interesting systems that could be analyzed with BDNK that may yield interesting new qualitative results such as the evolution of the quark-gluon plasma, viscous cosmology, and relativistic magnetohydrodynamics.

4.3 Conclusion

This thesis studied the constraints that effect both stability and causality in a first-order theory of relativistic hydrodynamics. For a conformal charged fluid, the causality constraints are especially simple: in order to ensure causality, in a frame where $\theta_3 = \pi_3 = \nu_1 = \gamma_1 = 0$, the transport coefficients must satisfy the constraints

$$\pi_1 > 2\,(d-1)\,\eta, \tag{4.1}$$

$$1 - 2\left(\frac{d}{d-1}\frac{\eta}{\theta_1} + \frac{1}{d-1}\frac{\eta}{\pi_1}\right) > 0, \tag{4.2}$$

$$\nu_3 > T\sigma + \frac{nT}{w}\theta_1. \tag{4.3}$$

In a generic charged fluid, for both the linearized and the full non-linear equations, the causality constraints for a frame where $\varepsilon_3 = \theta_3 = \pi_3 = 0$, $\nu_2 = p_\epsilon \nu_1 + \frac{p_n}{T}\nu_3$, and

$\gamma_1 = \frac{n}{w}\theta_1$ (the so-called de-coupled frame) are given by (3.59), (3.54), and (3.61):

$$\nu_3 > T\sigma, \tag{4.4a}$$
$$\theta_1 > \eta, \tag{4.4b}$$
$$4\varepsilon_1\theta_1^2\tilde{\aleph} + \left(\varepsilon_1\tilde{\aleph} + \varepsilon_2\theta_1 + (\varepsilon_2 + \theta_1)\,\pi_1\right)^2 > 0, \tag{4.4c}$$
$$\varepsilon_1\tilde{\aleph} + \theta_1\pi_1 + \varepsilon_2\,(\theta_1 + \pi_1) > 0, \tag{4.4d}$$
$$\theta_1\tilde{\aleph} < 0, \tag{4.4e}$$
$$\varepsilon_1\theta_1 > 0, \tag{4.4f}$$
$$\theta_1\left(\tilde{\aleph} + \varepsilon_1\right) > 0 \tag{4.4g}$$
$$(\varepsilon_1 - \varepsilon_2)\,\theta_1 - \left(\tilde{\aleph} + \pi_1\right)(\varepsilon_1 + \theta_1) > 0, \tag{4.4h}$$

where $\tilde{\aleph} = \gamma_s + p_\epsilon\,(p_\epsilon\varepsilon_1 - \varepsilon_2 - \pi_1)$. Finally, while the Routh-Hurwitz criterion for stability has not been fully analyzed, the small-k modes have been. For the theory describing a generic charged fluid, there exists a constraint that must be satisfied for the gaps to be stable: the positivity of the transport coefficients, and

$$\frac{p_\epsilon^2\lambda}{T}\varepsilon_1 + \varrho\nu_3 > -\frac{p_\epsilon(1-p_\epsilon)v_s^2}{\kappa}\left(\lambda - \frac{\kappa^2}{(1-p_\epsilon)v_s^2}\right)\nu_1. \tag{4.5}$$

where $\kappa = \frac{nT}{\epsilon+p}$, $\lambda = \frac{\partial n}{\partial\mu}\frac{T^2}{\epsilon+p}$, and $\varrho = p_\epsilon + \frac{p_n^2}{T^2}\lambda - \frac{\kappa}{T}p_n > 0$. In the analysis of the theory describing a conformal fluid, we set $\nu_1 = 0$, and so this constraint was automatically satisfied for $\pi_1 > 0$, $\nu_3 > 0$. Doing so in the generic case may lead to simplified stability conditions.

In order to calculate the causality constraints, we utilized two methods, one linear and one non-linear. They yielded identical results. In the linear case, we derived the controlling equation for the dispersion relations of the modes, and performed an asymptotic expansion of ω in k. This gave a linear dispersion relation for ω of the form $\omega = ck$. The velocity c had the following controlling equation (3.42):

$$\Big\{\varepsilon_1\theta_1c^4 - \big(\gamma_s\varepsilon_1 + p_\epsilon^2\varepsilon_1^2 + \varepsilon_2\theta_1 + (\varepsilon_2+\theta_1)\,\pi_1 - p_\epsilon\varepsilon_1\,(\varepsilon_2+\pi_1)\big)\,c^2$$
$$- \gamma_s\theta_1 - p_\epsilon\theta_1\,(p_\epsilon\varepsilon_1 - \varepsilon_2 - \pi_1)\Big\} \times \left(c^2 - \frac{\sigma T}{\nu_3}\right) = 0.$$

Demanding that $0 < c < 1$ gave the constraints (4.4).

The non-linear method was slightly more involved. Using the general theory of partial differential equations, we derived the so-called “characteristic equation” for the hydrodynamic equations, using the equilibrium variables β^μ and ψ instead of T, u^α, and μ. The characteristic equation involved normals to the surfaces denoting the wavefronts for the modes of propagation; these were denoted by ξ. We found the set of $\xi_0 = \xi_0(\xi_1, \xi_2, \xi_3)$ that swept out these “characteristic surfaces”. In order for propagation to be subluminal, all of the surfaces were required to be inside the lightcone; which mean the covectors ξ needed to point out of the lightcone. Demanding this condition, and also demanding that none of the ξ_0 be equal to zero or be imaginary (the condition for hyperbolicity), led to the exact same constraints as in the linear case.

For the stability constraints, we used a technique from control theory, the branch of engineering and mathematics investigating the behaviour of dynamical systems. This technique is the so-called ”Routh-Hurwitz criterion”. This criterion was algorithmically generated for a polynomial of a given order; the constraints created by the criterion, if satisfied, would ensure stability. However, given that the longitudinal equations of hydrodynamics take the form of a sixth order polynomial, the RH criterion created intractable conditions. Hopefully, further work will yield the parameter space leading to stability for a non-CFT charged fluid.

Altogether, we have presented conditions for stability of the equilibrium state of both conformal and non-conformal charged fluid in the long-wavelength and short-wavelength limits. We also derived conditions for causal propagation of perturbations for both the linearized equations and the full, non-linear first-order hydrodynamic equations. We also showed these causality constraints hold true even if the fluid is coupled to dynamical gravity.

Further Works

Given this new theory for charged relativistic hydrodynamics, there are a number areas where the theory could be applied and pushed further.

Firstly, the conditions for generic stability still remain unsolved. If the constraints generated by the Routh-Hurwitz criterion for the generic charged fluid could be satisfied at all scales, then the stability of the theory would be confirmed.

Secondly, a derivation of charged BDNK hydrodynamics from a kinetic theory basis would provide a “first principles” motivation for the theory, as well as possibly giving insight as to whether there exists a stable, causal "kinetic frame".

As well, a comparison of BDNK could be made to holography. One could evolve some initial stress-energy tensor according to holography, then take that same initial stress energy tensor, write it in terms of the hydrodynamic variables of BDNK, evolve those hydrodynamic variables according to BDNK, and then compare the results. Since holography is supposed to describe the exact evolution of the stress-energy tensor, this could be an interesting test of BDNK.

Additionally, work could be done to investigate the well-posedness of *charged* BDNK hydro, á la [14]. Given the local existence and uniqueness of solutions for BDNK in the uncharged case, the charged case ought to be locally well-posed as well.

Finally, it could be interesting to apply BDNK hydrodynamics to the area of relativistic magnetohydrodynamics, especially to neutron star mergers. If viscous effects contribute to the behaviour of the stars during the merger, it would be enlightening to analyze these collisions using BDNK.

Appendix A

Frame Changes via Ideal Equations

If one chooses to truncate the derivative expansion (2.10) at zeroth order, the stress-energy tensor and charge current are given by

$$\begin{aligned} T^{\mu\nu} &= \epsilon u^\mu u^\nu + p\Delta^{\mu\nu}, \\ J^\mu &= n u^\mu. \end{aligned}$$

The conservation laws are therefore

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= \left(\partial_\mu \epsilon\right) u^\mu u^\nu + \epsilon \left(\partial_\mu u^\mu\right) u^\nu + \epsilon u^\mu \left(\partial_\mu u^\nu\right) + \left(\partial_\mu p\right) \Delta^{\mu\nu} + p \left(\partial_\mu \Delta^{\mu\nu}\right) = 0, \\ \partial_\mu J^\mu &= \partial_\mu \left(n u^\mu\right) = 0. \end{aligned}$$

We can break the conservation of the stress-energy tensor into longitudinal and transverse (relative to u^μ) directions:

$$\begin{aligned} u_\nu \partial_\mu T^{\mu\nu} &= -u^\mu \partial_\mu \epsilon - \epsilon \left(\partial_\mu u^\mu\right) - p \partial_\mu u^\mu = -u^\mu \partial_\mu \epsilon - w \partial_\mu u^\mu = 0 \\ &\implies u^\mu \left(\frac{\partial \epsilon}{\partial T} \partial_\mu T + \frac{\partial \epsilon}{\partial \mu} \partial_\mu \mu\right) = -w \partial_\mu u^\mu \qquad \text{(A.1)} \\ \Delta^\alpha_\nu \partial_\mu T^{\mu\nu} &= \epsilon \Delta^\alpha_\nu u^\mu \partial_\mu u^\nu + \left(\partial_\mu p\right) \Delta^{\mu\alpha} + p \Delta^\alpha_\nu \partial_\mu \Delta^{\mu\nu} = 0 \\ &\implies w u^\mu \partial_\mu u^\alpha + \left(s \partial_\mu T + n \partial_\mu \mu\right) \Delta^{\alpha\mu} = 0^\alpha. \qquad \text{(A.2)} \end{aligned}$$

The charge current conservation can also be written in the form

$$n \partial_\mu u^\mu + \frac{\partial n}{\partial T} u^\mu \partial_\mu T + \frac{\partial n}{\partial \mu} u^\mu \partial_\mu \mu = 0. \qquad \text{(A.3)}$$

From these equations, it is simple to show that the entropy current is conserved for ideal fluids. Writing (A.1) and (A.3) in the simpler forms $\partial_\mu (nu^\mu) = 0$ and $\partial_\mu (\epsilon u^\mu) = -p\partial_\mu u^\mu$, one can make use of the relations $\epsilon + p = sT + n\mu$ and $\partial_\mu p = s\partial_\mu T + n\partial_\mu \mu$ to find that

$$\partial_\mu S^\mu = \partial_\mu (su^\mu) = 0 \tag{A.4}$$

There are two equations (A.1), (A.3), relating the three one-derivative scalars $u^\mu\partial_\mu T$, $\partial_\mu u^\mu$, and $u^\mu\partial_\mu\mu$. These ideal-order equations can be used to write two of these scalars in terms of a third. Note that this will introduce error into the equations: these relations only hold true at ideal order. However, because we are using the conservation equations to derive the relations, any first-order corrections to the ideal variables become second order after taking the derivative, making them negligible at first order, and so we can happily accept this error.

There is, however, one major consequence to our choices. While the error will not appear until second order, the equations that these substitutions yield are not equivalent, and essentially constitute a change in frame; referring to the work done in Chapter 2, a different choice of which scalar to eliminate will change where the bulk viscosity ζ appears in the equations.

The following demonstrates exactly how these ideal-equation frame changes are done. Consider the Landau frame, where the bulk term is given by

$$T^{\mu\nu} \supset \left(p + f_1 \frac{1}{T} u^\lambda \partial_\lambda T + f_2 \partial_\lambda u^\lambda + f_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) \Delta^{\mu\nu}.$$

Let us now follow in the footsteps of the main body of the thesis and eliminate $u^\lambda\partial_\lambda T$ and $u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)$. This is done in the following manner:

$$\begin{aligned}
&u^\mu \left(\frac{\partial\epsilon}{\partial T}\partial_\mu T + \frac{\partial\epsilon}{\partial\mu}\partial_\mu\mu\right) = -w\partial_\mu u^\mu + \mathcal{O}\left(\partial^2\right), \\
&\frac{\partial\epsilon}{\partial T}\frac{1}{T}u^\mu\partial_\mu T + \frac{\partial\epsilon}{\partial\mu}\frac{1}{T}u^\lambda\partial_\lambda\mu - \frac{\partial\epsilon}{\partial\mu}\frac{\mu}{T^2}u^\lambda\partial_\lambda T + \frac{\partial\epsilon}{\partial\mu}\frac{\mu}{T^2}u^\lambda\partial_\lambda T = -\frac{w}{T}\partial_\mu u^\mu + \mathcal{O}\left(\partial^2\right), \\
&(\mathcal{D}\epsilon)\frac{1}{T}u^\mu\partial_\mu T + \frac{\partial\epsilon}{\partial\mu}u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right) = -\frac{w}{T}\partial_\mu u^\mu + \mathcal{O}\left(\partial^2\right),
\end{aligned}$$

where $\mathcal{D} = \frac{\partial}{\partial T} + \frac{\mu}{T}\frac{\partial}{\partial\mu}$. Doing a similar analysis for the charge conservation equation yields

$$(\mathcal{D}n)\frac{1}{T}u^\mu\partial_\mu T + \frac{\partial n}{\partial\mu}u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right) = -\frac{n}{T}\partial_\lambda u^\lambda + \mathcal{O}\left(\partial^2\right).$$

Solving these equations yields the following expressions for $u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)$ and $\frac{1}{T}u^\lambda\partial_\lambda T$:

$$u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right) = -\frac{\tilde{\alpha}}{\tilde{\theta}T}\partial_\lambda u^\lambda + \mathcal{O}\left(\partial^2\right), \tag{A.5}$$

$$\frac{1}{T}u^\lambda\partial_\lambda T = \frac{1}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)\partial_\lambda u^\lambda + \mathcal{O}\left(\partial^2\right), \tag{A.6}$$

where $\tilde{\theta} = \frac{\partial n}{\partial\mu}\frac{\partial\epsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial\epsilon}{\partial\mu}$ and $\tilde{\alpha} = n\left(\mathcal{D}\epsilon\right) - w\left(\mathcal{D}n\right)$. It is straightforward to show, using the Maxwell relation $\frac{\partial\epsilon}{\partial\mu} = T\frac{\partial n}{\partial T} + \mu\frac{\partial n}{\partial\mu}$, that $\tilde{\theta} = \frac{1}{p_\epsilon T}\left(w\frac{\partial n}{\partial\mu} - n\frac{\partial\epsilon}{\partial\mu}\right)$ and $\tilde{\alpha} = \frac{p_n}{p_\epsilon T}\left(w\frac{\partial n}{\partial\mu} - n\frac{\partial\epsilon}{\partial\mu}\right) = p_n\tilde{\theta}$. Plugging the relations (A.5) and (A.6) in to the first order stress-energy tensor yields

$$T^{\mu\nu} \supset \left(p - \left(-f_2 + f_1\frac{\left(\frac{\partial n}{\partial\mu}w - n\frac{\partial\epsilon}{\partial\mu}\right)}{T\tilde{\theta}} + f_3\frac{\tilde{\alpha}}{\tilde{\theta}T}\right)\partial_\lambda u^\lambda\right)\Delta^{\mu\nu}.$$

The term in brackets before $\partial_\lambda u^\lambda$ is exactly the definition of ζ, and so we can write

$$T^{\mu\nu} \supset \left(p - \zeta\partial_\lambda u^\lambda\right)\Delta^{\mu\nu},$$

which is exactly the form that was used previously. However, what if we had made a different choice? The decision to write everything in terms of $\partial_\lambda u^\lambda$ was totally arbitrary. Suppose we instead write everything in terms of $u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)$. Then the other two terms will be given by

$$\frac{1}{T}u^\lambda\partial_\lambda T = -\frac{\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)}{\tilde{\alpha}}u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right) + \mathcal{O}\left(\partial^2\right), \qquad \partial_\lambda u^\lambda = -\frac{\tilde{\theta}T}{\tilde{\alpha}}u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right) + \mathcal{O}\left(\partial^2\right).$$

Plugging these into the $\Delta^{\mu\nu}$ term of the stress-energy tensor and rearranging slightly gives

$$T^{\mu\nu} \supset \left(p + \frac{\tilde{\alpha}}{\tilde{\theta}T}\left(\frac{\left(\frac{\partial n}{\partial\mu}w - \frac{\partial\epsilon}{\partial\mu}n\right)}{\tilde{\theta}T}f_1 - f_2 + \frac{\tilde{\alpha}}{\tilde{\theta}T}f_3\right)u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)\right)\Delta^{\mu\nu}$$

$$T^{\mu\nu} \supset \left(p + \frac{\tilde{\alpha}}{\tilde{\theta}T}\zeta u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)\right)\Delta^{\mu\nu}.$$

This is clearly a different form than the previous case, and will yield **mathematically inequivalent** equations of motion when we truncate the derivative expansion at first order. Therefore, we can see that the term "Landau Frame" is actually a bit of a misnomer. The "Landau Frame" is actually a *class* of inequivalent frames that all share a few defining properties.

We can also maneuver around the σ that appears in the charge current in the Landau frame, and in the stress-energy tensor in the Eckart frame via the transverse ideal equations. Doing so will yield different modes (in particular, choosing to eliminate $\Delta^{\mu\lambda}\partial_\lambda\left(\frac{\mu}{T}\right)$ will yield a massive shear mode).

To finish off this appendix, we will write the transverse equation (A.2) in a form that will be more useful:

$$u^\mu \partial_\mu u^\alpha = -\frac{\Delta^{\mu\alpha}\partial_\mu T}{T} - \kappa \Delta^{\mu\alpha}\partial_\mu \left(\frac{\mu}{T}\right). \tag{A.7}$$

As in the body of the thesis, $\kappa \equiv nT/w$. In particular, we will make use of this relation in Appendix C.

Appendix B

The Routh-Hurwitz Criteria

The Routh-Hurwitz (RH) criterion is a criterion that generates a set of programatically generated conditions that come from control theory; it describes how many roots of a polynomial are in the left complex half-plane. In the main body of the text, in order to use the RH criterion, the substitution $\omega = i\Delta$ was made to transfer all roots in the lower half-plane to the left half-plane. In order to ensure stability, we require that *all* roots be in the left half-plane, so that their real part is negative – thereby ensuring that the imaginary part of ω will be negative. Conditions on the transport parameters come from insisting on stability.

The RH criterion was first derived by E. J. Routh in 1877 [30]. Its application to stability and its use for identifying poles in the left half-plane was discovered by Adolf Hurwitz in 1895 [31]. For more information, see [23].

While the fundamental quantity involved with the RH criterion is the so-called "Routh array", we will not need to make reference to it. Consider a polynomial of order n, given by

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_1 x + a_0 \tag{B.1}$$

where all of the coefficients a_n are real. Let us split this polynomial into two smaller polynomials, one containing the even powers of x and one containing the odd; i.e.

$$P_0(x) = a_n x^n + a_{n-2} x^{n-2} + ...$$
$$P_1(x) = a_{n-1} x^{n-1} + a_{n-3} x^{n-3} + ...$$

We will now programatically generate more polynomials. We create a new polynomial P_2 from the remainder of dividing P_0 by P_1. We then create another new polynomial

P_3 from the remainder of dividing P_1 by P_2. We continue in this fashion until we get a zeroth-order polynomial, at which point the algorithm stops. Since the polynomials we deal with in this thesis are of a type referred to as "regular", this point will be at polynomial P_n.

The part of each sub-polynomial that we care about is the **leading order term**. We could then put each of these leading order terms (or rather, their coefficients) into a chain, in the form

$$G_S = \{P_0, P_1, P_2, ..., P_n\}$$

In order to ensure stability (i.e. to ensure all poles of the original polynomial fall in the left-hand plane), *all of the entries in the chain must be the same sign*. This criterion is the Routh-Hurwitz criterion. If it is satisfied, then all of the poles of the polynomial (B.1) are in the left half-plane.

Let us look at the application of the RH criterion to polynomials of second, third, fourth, and sixth order.

B.1 Second-Order Polynomial

Consider a polynomial of the form

$$F[x] = a_2x^2 + a_1x + a_0$$

We can take the even and odd terms and create the sub-polynomials

$$\begin{aligned} P_0 &= a_2x^2 + a_0, \\ P_1 &= a_1x, \\ P_2 &= a_0. \end{aligned}$$

The chain is then

$$G_2 = \{a_2, a_1, a_0\}.$$

For a second-order polynomial, the Routh-Hurwitz criterion tells us that all of the coefficients must be the same sign.

B.2 Third-Order Polynomial

Consider a polynomial of the form

$$F[x] = a_3x^3 + a_2x^2 + a_1x + a_0$$

We can create the sub-polynomials

$$\begin{aligned} P_0 &= a_3x^3 + a_1x \\ P_1 &= a_2x^2 + a_0 \end{aligned}$$

We can now programatically create the remaining polynomials:

$$\begin{aligned} P_2 &= Rem(P_0, P_1) = \frac{a_1a_2 - a_3a_0}{a_2} \\ P_3 &= Rem(P_1, P_2) = a_0 \end{aligned}$$

With these four sub-polynomials, the chain is given by

$$G_s = \{a_3, a_2, \frac{a_1a_2 - a_3a_0}{a_2}, a_0\}$$

This yields the constraints

$$a_3 > 0, \quad a_2 > 0, \quad , a_1a_2 > a_3a_0, \quad a_0 > 0$$

We could just as well have required that all of the coefficients be negative. When actually applying these criteria to the transport coefficients, there are certain positivity constraints imposed by small-k stability, as well as the positivity of the divergence of the entropy current (see C). These will tell us whether to enforce positivity or negativity.

B.3 Fourth-Order Polynomials

Let

$$F[x] = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

The sub-polynomials are given by

$$\begin{aligned} P_0 &= a_4x^4 + a_2x^2 + a_0 \\ P_1 &= a_3x^3 + a_1x \\ P_2 &= \left(\frac{a_3a_2 - a_1a_4}{a_3}\right)x^2 + a_0 \\ P_3 &= \left(a_1 - \frac{a_0a_3^2}{a_2a_3 - a_1a_4}\right)x \\ P_4 &= a_0 \end{aligned}$$

Therefore the chain is

$$G_s = \{a_4, a_3, \left(\frac{a_3a_2 - a_1a_4}{a_3}\right), \left(a_1 - \frac{a_0a_3^2}{a_2a_3 - a_1a_4}\right), a_0\}$$

and the conditions for stability are

$$a_4 > 0, \quad a_3 > 0, \quad a_3a_2 > a_1a_4, \quad a_1\left(a_2a_3 - a_1a_4\right) > a_0a_3^2, \quad a_0 > 0$$

B.4 Sixth-Order Polynomials

Let

$$F[x] = a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

The sub-polynomials get quite lengthy, so I will relate here only the chain:

$$\begin{aligned} G_s = \{&a_6, a_5, \frac{a_5a_4 - a_3a_6}{a_5}, \left(\frac{a_3\left(a_4a_5 - a_3a_6\right) - a_2a_5^2 + a_1a_5a_6}{a_4a_5 - a_3a_6}\right), \\ &\frac{a_2^2a_5^2 - a_0a_4a_5^2 + a_0a_3a_6a_5 + a_2\left(\left(a_3^2 - 2a_1a_5\right)a_6 - a_3a_4a_5\right) + a_1\left(a_5a_4^2 - a_3a_6a_4 + a_1a_6^2\right)}{a_6a_3^2 - a_4a_5a_3 + a_5\left(a_2a_5 - a_1a_6\right)}, \\ &a_1 + \frac{a_0\left(a_5\left(a_4a_3^2 + a_0a_5^2 - \left(a_2a_3 + a_1a_4\right)a_5\right) - a_3\left(a_3^2 - 2a_1a_5\right)a_6\right)}{a_1^2a_6^2 + \left(a_3\left(a_0a_5 - a_1a_4\right) + a_2\left(a_3^2 - 2a_1a_5\right)\right)a_6 + a_5\left(a_5a_2^2 - a_3a_4a_2 + a_4\left(a_1a_4 - a_0a_5\right)\right)}, \\ &a_0\}, \end{aligned}$$

leading to the set of constraints

$$a_6 > 0$$
$$a_5 > 0$$
$$a_5 a_4 > a_6 a_3$$
$$a_3\left(a_4 a_5 - a_3 a_6\right) - a_2 a_5^2 + a_1 a_5 a_6 > 0$$
$$a_2^2 a_5^2 - a_0 a_4 a_5^2 + a_0 a_3 a_6 a_5 + a_2\left(\left(a_3^2 - 2a_1 a_5\right) a_6 - a_3 a_4 a_5\right) + a_1\left(a_5 a_4^2 - a_3 a_6 a_4 + a_1 a_6^2\right) < 0$$
$$a_1\left(a_2^2 a_5^2 - a_0 a_4 a_5^2 + a_0 a_3 a_6 a_5 + a_2\left(\left(a_3^2 - 2a_1 a_5\right) a_6 - a_3 a_4 a_5\right) + a_1\left(a_5 a_4^2 - a_3 a_6 a_4 + a_1 a_6^2\right)\right)$$
$$+ a_0\left(a_5\left(a_4 a_3^2 + a_0 a_5^2 - \left(a_2 a_3 + a_1 a_4\right) a_5\right) - a_3\left(a_3^2 - 2a_1 a_5\right) a_6\right) < 0$$
$$a_0 > 0$$

While these constraints are not simple, if satisfied, stability is ensured.

Appendix C

Entropy Current

In Appendix A, it was shown (A.4) that in the ideal case the entropy current $S^{\mu}_{\text{ideal}} = su^{\mu}$ is conserved. At first order in the derivative expansion, the entropy current gets corrections to it. In this appendix, we compute those corrections, and see what constraints they impose on the transport coefficients. This chapter largely follows and expands on [2] and [15].

We can define the covariant form of the entropy current as

$$TS^{\mu} = pu^{\mu} - T^{\mu\nu}u_{\nu} - \mu J^{\mu}.$$

where at ideal order this will reduce to TS^{μ}_{ideal}. We split both $T^{\mu\nu}$ and J^{μ} into their ideal parts and their corrections (*i.e.* $T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} + T^{\mu\nu}_{(1)}$), which yields

$$S^{\mu} = su^{\mu} - \frac{u_{\nu}}{T}T^{\mu\nu}_{(1)} - \frac{\mu}{T}J^{\mu}_{(1)}.$$

Note here that

$$\begin{aligned} T^{\mu\nu}_{(1)} &= (\mathcal{E} - \epsilon)\, u^{\mu}u^{\nu} + (\mathcal{P} - p)\, \Delta^{\mu\nu} + \mathcal{Q}^{\mu}u^{\nu} + \mathcal{Q}^{\nu}u^{\mu} + \tau^{\mu\nu}, \\ J^{\mu}_{(1)} &= (\mathcal{N} - n)\, u^{\mu} + \mathcal{J}^{\mu}. \end{aligned}$$

The divergence of the entropy current is given by

$$\partial_{\mu}S^{\mu} = \partial_{\mu}(su^{\mu}) - \partial_{\mu}\left(\frac{u^{\nu}}{T}T^{\mu\nu}_{(1)}\right) - \partial_{\mu}\left(\frac{\mu}{T}J^{\mu}_{(1)}\right). \tag{C.1}$$

We want to find a way to express the divergence of the entropy current solely in terms of the hydrodynamic variables T, u^{μ}, μ, as well as the transport coefficients. Our first step will be to eliminate $\partial_\mu (su^\mu)$ from the divergence of the entropy current. To begin this process, we examine the longitudinal equation for $T^{\mu\nu}$:

$$\begin{aligned}
u_\nu \partial_\mu T^{\mu\nu} = u_\nu \partial_\mu \left(\epsilon u^\mu u^\nu + p\Delta^{\mu\nu} + T^{\mu\nu}_{(1)} \right) &= 0, \\
\implies -u^\mu \partial_\mu \epsilon - w\partial_\mu u^\mu + u_\nu \partial_\mu T^{\mu\nu}_{(1)} &= 0, \\
u^\mu \partial_\mu (-p + sT + n\mu) + (sT + n\mu)\partial_\mu u^\mu &= u_\nu \partial_\mu T^{\mu\nu}_{(1)}, \\
u^\mu (-s\partial_\mu T - n\partial_\mu \mu + \partial_\mu (sT) + \partial_\mu (n\mu)) & \\
+ (sT + n\mu)\partial_\mu u^\mu &= u_\nu \partial_\mu T^{\mu\nu}_{(1)}, \\
u^\mu (T\partial_\mu s + \mu \partial_\mu n) + (sT + n\mu)\partial_\mu u^\mu &= u_\nu \partial_\mu T^{\mu\nu}_{(1)}, \\
T\partial_\mu (su^\mu) + \mu\partial_\mu (nu^\mu) &= u_\nu \partial_\mu T^{\mu\nu}_{(1)}.
\end{aligned}$$

In order to use this equation to remove $\partial_\mu (su^\mu)$ from (C.1), we can employ the divergence of the charge current:

$$\partial_\mu J^\mu = \partial_\mu (nu^\mu) + \partial_\mu J^\mu_{(1)} = 0 \implies \partial_\mu (nu^\mu) = -\partial_\mu J^\mu_{(1)}.$$

So,

$$\partial_\mu (su^\mu) = \frac{u_\nu}{T}\partial_\mu T^{\mu\nu}_{(1)} + \frac{\mu}{T}\partial_\mu J^\mu_{(1)}.$$

Inserting this equation into the divergence of the entropy current (C.1) yields

$$\partial_\mu S^\mu = -T^{\mu\nu}_{(1)}\partial_\mu \left(\frac{u_\nu}{T}\right) - J^\mu_{(1)}\partial_\mu \left(\frac{\mu}{T}\right). \tag{C.2}$$

Before we proceed, we note the following facts:

Consider **any** symmetric tensor $X^{\mu\nu}$ and decompose it as we do for $T^{\mu\nu}$:

$$X^{\mu\nu} = \mathcal{E}_X u^\mu u^\nu + \mathcal{P}_X \Delta^{\mu\nu} + (\mathcal{Q}^\mu_X u^\nu + \mathcal{Q}^\nu_X u^\mu) + \tau^{\mu\nu}_X,$$

where the definitions are the same as in the main body of the thesis:

$$\begin{aligned}
\mathcal{E}_X \equiv u_\mu u_\nu X^{\mu\nu},\ \mathcal{P}_X \equiv \frac{1}{d}\Delta_{\mu\nu}X^{\mu\nu},\ \mathcal{Q}_{X,\mu} \equiv -\Delta_{\mu\alpha}u_\beta X^{\alpha\beta}, \\
\tau_{X,\mu\nu} \equiv \frac{1}{2}\left(\Delta_{\mu\alpha}\Delta_{\nu\beta} + \Delta_{\nu\alpha}\Delta_{\mu\beta} - \frac{2}{d}\Delta_{\mu\nu}\Delta_{\alpha\beta}\right) X^{\alpha\beta}.
\end{aligned}$$

If we take this symmetric tensor $X^{\mu\nu}$ and then contract it with some *other* symmetric tensor $Y_{\mu\nu}$, we get

$$X_{\mu\nu}Y^{\mu\nu} = \mathcal{E}_X\mathcal{E}_Y + d\mathcal{P}_X\mathcal{P}_Y - 2\mathcal{Q}_{X,\mu}\mathcal{Q}_Y^{\mu} + \tau_{X,\mu\nu}\tau_Y^{\mu\nu}.$$

Now that we have established these baseline facts, let us define the following tensor:

$$X_{\mu\nu} = \frac{1}{2}\left(\partial_\mu\left(u_\nu/T\right) + \partial_\nu\left(u_\mu/T\right)\right),$$

With this definition, the first term of the divergence of the entropy current becomes $\partial_\mu S^\mu \supset -T^{\mu\nu}_{(1)}X_{\mu\nu}$, taking advantage of the symmetry of $T^{\mu\nu}$. We can easily identify via the definitions of the decompositions that

$$\mathcal{E}_X = \frac{u^\lambda\partial_\lambda T}{T^2},\ \mathcal{P}_X = \frac{1}{d}\frac{\partial_\lambda u^\lambda}{T},\ \mathcal{Q}_X^\mu = -\frac{1}{2T}\left(\frac{\Delta^{\mu\lambda}\partial_\lambda T}{T} + u^\lambda\partial_\lambda u^\mu\right),\ \tau_X^{\mu\nu} = \frac{1}{2T}\sigma^{\mu\nu}.$$

This gives the first term of (C.2) as

$$\begin{aligned}
T^{\mu\nu}_{(1)}X_{\mu\nu} &= \mathcal{E}_X\mathcal{E}_T + d\mathcal{P}_X\mathcal{P}_T - 2\mathcal{Q}_{X,\mu}\mathcal{Q}_T^\mu + \tau_{X,\mu\nu}\tau_T^{\mu\nu} \\
&= \frac{1}{T}\Bigg[\left(\frac{u^\alpha\partial_\alpha T}{T}\right)\left(\varepsilon_1\frac{u^\lambda\partial_\lambda T}{T} + \varepsilon_2\partial_\lambda u^\lambda + \varepsilon_3 u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)\right) \\
&+ \left(\partial_\alpha u^\alpha\right)\left(\pi_1\frac{u^\lambda\partial_\lambda T}{T} + \pi_2\partial_\lambda u^\lambda + \pi_3 u^\lambda\partial_\lambda\left(\frac{\mu}{T}\right)\right) \\
&+ \left(\frac{\Delta_\mu^\alpha\partial_\alpha T}{T} + u^\alpha\partial_\alpha u_\mu\right)\left(\theta_1 u^\lambda\partial_\lambda u^\mu + \theta_2\frac{1}{T}\Delta^{\mu\lambda}\partial_\lambda T + \theta_3\Delta^{\mu\lambda}\partial_\lambda\left(\frac{\mu}{T}\right)\right) \\
&- \frac{\eta}{2}\sigma_{\mu\nu}\sigma^{\mu\nu}\Bigg] + \mathcal{O}\left(\partial^2\right).
\end{aligned}$$

For the second term of the divergence of the entropy current (C.2), we can follow a similar process for J^μ. Note that we can decompose any vector P^μ along some timelike vector u^μ

$$P^\mu = \mathcal{N}_P u^\mu + j_P^\mu,$$

where

$$\mathcal{N}_p = -P^\mu u_\mu, \qquad j_P^\mu = \Delta^{\mu\nu}P_\nu.$$

If we take the vector P^μ and contract it with some *other* vector R_μ, we get

$$P_\mu R^\mu = -\mathcal{N}_p\mathcal{N}_R + j_{P,\mu}j_R^\mu.$$

Define $P_\mu = \partial_\mu \left(\frac{\mu}{T}\right)$. Then $\mathcal{N}_p = -u^\mu \partial_\mu \left(\frac{\mu}{T}\right)$ and $j^\mu_P = \Delta^{\mu\nu}\partial_\nu \left(\frac{\mu}{T}\right)$ and we have

$$J^\mu_{(1)}\partial_\mu \left(\frac{\mu}{T}\right) = \left(\nu_1 \frac{u^\lambda \partial_\lambda T}{T} + \nu_2 \partial_\lambda u^\lambda + \nu_3 u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right)\right) u^\mu \partial_\mu \left(\frac{\mu}{T}\right)$$
$$+ \left(\gamma_1 u^\lambda \partial_\lambda u^\mu + \frac{\gamma_2}{T}\Delta^{\mu\nu}\partial_\nu T + \gamma_3 \Delta^{\mu\nu}\partial_\nu \left(\frac{\mu}{T}\right)\right) \Delta_{\mu\lambda}\partial^\lambda \left(\frac{\mu}{T}\right) + \mathcal{O}\left(\partial^2\right).$$

Let us use the following shorthand:

$$\chi_1 = \frac{u^\lambda \partial_\lambda T}{T}, \qquad \chi_2 = \partial_\lambda u^\lambda, \qquad \chi_3 = u^\lambda \partial_\lambda \left(\frac{\mu}{T}\right),$$
$$\xi^\mu_1 = u^\lambda \partial_\lambda u^\mu, \qquad \xi^\mu_2 = \frac{\Delta^{\mu\lambda}\partial_\lambda T}{T}, \qquad \xi^\mu_3 = \Delta^{\mu\lambda}\partial_\lambda \left(\frac{\mu}{T}\right).$$

Then the divergence of the entropy current, up to first order, is given by

$$T\partial_\mu S^\mu = -\frac{\varepsilon_1}{T}\chi_1^2 - \frac{1}{T}\left(\varepsilon_2 + \pi_1\right)\chi_1\chi_2 - \left(\frac{\varepsilon_3}{T} + \nu_1\right)\chi_1\chi_3 - \frac{\pi_2}{T}\chi_2^2 - \left(\frac{\pi_3}{T} + \nu_2\right)\chi_2\chi_3 - \nu_3\chi_3^2$$
$$- \frac{\theta_1}{T}\xi^\mu_1\xi_{1,\mu} - \frac{(\theta_1+\theta_2)}{T}\xi^\mu_1\xi_{2,\mu} - \frac{\theta_2}{T}\xi^\mu_2\xi_{2,\mu} - \left(\frac{\theta_3}{T} + \gamma_1\right)\xi^\mu_3\xi_{1,\mu} - \left(\frac{\theta_3}{T} + \gamma_2\right)\xi^\mu_3\xi_{2,\mu}$$
$$- \gamma_3\xi^\mu\xi_{3,\mu} + \frac{1}{2T}\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right).$$

We would like the right-hand side of this equation to be positive semi-definite. However, we only need it to be so up to second order; so, we can shift some terms up to second order without issue. Let us use the ideal order equations to write χ_1 and χ_3 in terms of χ_2, and ξ^μ_2 in terms of $\xi^\mu_{1,3}$.

Let us define the following two quantities as shorthand; these quantities are also introduced in Appendix A.

$$\tilde{\theta} = \frac{\partial n}{\partial \mu}\frac{\partial \epsilon}{\partial T} - \frac{\partial n}{\partial T}\frac{\partial \epsilon}{\partial \mu}$$
$$\tilde{\alpha} = n\left(\frac{\partial \epsilon}{\partial T} + \frac{\mu}{T}\frac{\partial \epsilon}{\partial \mu}\right) - w\left(\frac{\partial n}{\partial T} + \frac{\mu}{T}\frac{\partial n}{\partial \mu}\right)$$

With these shorthands in place, we can make use of relations (A.5), (A.6), and (A.7):

$$\begin{aligned}
\chi_3 &= -\frac{\tilde{\alpha}}{\tilde{\theta}T}\chi_2 + \mathcal{O}\left(\partial^2\right), \\
\chi_1 &= \frac{1}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)\chi_2 + \mathcal{O}\left(\partial^2\right), \\
\xi_2^\mu &= -\left(\xi_1^\mu + \frac{nT}{w}\xi_3^\mu\right) + \mathcal{O}\left(\partial^2\right).
\end{aligned}$$

Using these relations, the divergence of the entropy current (C.2) becomes

$$\begin{aligned}
T\partial_\mu S^\mu = &\left[-\frac{\varepsilon_1}{\tilde{\theta}^2T^3}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)^2 - (\varepsilon_2+\pi_1)\frac{1}{\tilde{\theta}T^2}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)\right. \\
&\left. + \left(\frac{\varepsilon_3}{T}+\nu_1\right)\frac{1}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)\frac{\tilde{\alpha}}{\tilde{\theta}T} - \frac{\pi_2}{T} + \left(\frac{\pi_3}{T}+\nu_2\right)\frac{\tilde{\alpha}}{\tilde{\theta}T} - \nu_3\frac{\tilde{\alpha}^2}{\tilde{\theta}^2T^2}\right]\chi_2^2 \\
& + \left[(\theta_1-\theta_2)\frac{n}{w} + (\gamma_2-\gamma_1)\right]\xi_1^\mu\xi_{3\mu} + \left[-\theta_2\frac{n^2T}{w^2} + \left(\frac{\theta_3}{T}+\gamma_2\right)\frac{nT}{w} - \gamma_3\right]\xi_{3\mu}\xi_3^\mu \\
& + \frac{1}{2}\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right).
\end{aligned} \tag{C.3}$$

Let us now re-state the definitions of the frame-invariant transport coefficients:

$$\begin{aligned}
f_i &\equiv \pi_i - p_\epsilon\varepsilon_i - p_n\nu_i, \\
\ell_i &\equiv \gamma_i - \frac{n}{w}\theta_i.
\end{aligned}$$

We can write p_ϵ and p_n in terms of derivatives of ϵ and n with respect to T and μ:

$$p_\epsilon = \frac{1}{\tilde{\theta}}\left(s\frac{\partial n}{\partial\mu} - n\frac{\partial n}{\partial T}\right), \qquad p_n = \frac{1}{\tilde{\theta}}\left(n\frac{\partial\epsilon}{\partial T} - s\frac{\partial\epsilon}{\partial\mu}\right).$$

The terms that are proportional to χ_2^2 can be grouped together in the form

$$\begin{aligned}
&\left\{\left[\frac{1}{T^2}\left(-\pi_1 - \frac{\varepsilon_1}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - w\frac{\partial n}{\partial\mu}\right) + \nu_1\frac{\tilde{\alpha}}{\tilde{\theta}}\right)\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right)\right]\right. \\
&+ \frac{\tilde{\theta}}{T}\left[-\pi_2 - \frac{\varepsilon_2}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right) + \nu_2\frac{\tilde{\alpha}}{\tilde{\theta}}\right] \\
&\left. - \frac{\tilde{\alpha}}{T^2}\left[-\pi_3 - \frac{\varepsilon_3}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - \frac{\partial n}{\partial\mu}w\right) + \nu_3\frac{\tilde{\alpha}}{\tilde{\theta}}\right]\right\}\chi_2^2.
\end{aligned}$$

Using the relation $\frac{\partial\epsilon}{\partial\mu} = T\frac{\partial n}{\partial T} + \mu\frac{\partial n}{\partial\mu}$ makes it straightforward[1] to show that $\frac{1}{\tilde{\theta}T}\left(\frac{\partial\epsilon}{\partial\mu}n - w\frac{\partial n}{\partial\mu}\right) = -p_\epsilon$ and $\tilde{\alpha}/\tilde{\theta} = p_n$. We can then re-write the entropy current divergence in terms of the ℓ_i and f_i:

$$\begin{aligned}T\partial_\mu S^\mu = \frac{\tilde{\theta}}{T}\Big\{-f_2 + \frac{1}{\tilde{\theta}T}f_1\left(\frac{\partial n}{\partial\mu}w - \frac{\partial\epsilon}{\partial\mu}n\right) + \frac{\tilde{\alpha}}{\tilde{\theta}T}f_3\Big\}\chi_2^2 \\ + \Big[\ell_2 - \ell_1\Big]\xi_1^\mu\xi_{3\mu} + \Big[\frac{nT}{w}\ell_2 - \frac{nT}{w}\ell_1 + \frac{nT}{w}\ell_1 - \ell_3\Big]\xi_{3\mu}\xi_3^\mu \\ + \frac{1}{2}\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right).\end{aligned}$$

Recall that the definitions of ζ, σ, and χ_T from (2.23)

$$\begin{aligned}\zeta &= -f_2 + \frac{1}{\tilde{\theta}T}\left(f_1\left(w\frac{\partial n}{\partial\mu} - n\frac{\partial\epsilon}{\partial\mu}\right) + \tilde{\alpha}f_3\right), \\ \sigma &= \frac{n}{w}\ell_1 - \frac{1}{T}\ell_3, \\ \chi_T &= \frac{1}{T}\left(\ell_2 - \ell_1\right).\end{aligned}$$

We can use these definitions to write the entropy current solely in terms of the these transport coefficients

$$\partial_\mu S^\mu = \left(\frac{\tilde{\theta}}{T^2}\zeta\right)\chi_2^2 + \left(\chi_T\right)\xi_1^\mu\xi_{3\mu} + \left(\kappa\chi_T + \sigma\right)\xi_{3\mu}\xi_3^\mu + \left(\frac{\eta}{2T}\right)\sigma_{\mu\nu}\sigma^{\mu\nu} + \mathcal{O}\left(\partial^2\right),$$

where $\kappa = \frac{nT}{w}$. If we want the divergence of the entropy current to be positive semi-definite for **all** values of the hydrodynamic variables, then we require the cross-terms to vanish. This is equivalent to demanding that $\chi_T = 0$, or rather that $\ell_2 = \ell_1$. The divergence of the entropy current is, truncating to first order, therefore[2]

$$\partial_\mu S^\mu = \left(\frac{\tilde{\theta}}{T^2}\zeta\right)\chi_2^2 + \left(\sigma\right)\xi_{3\mu}\xi_3^\mu + \left(\frac{\eta}{2T}\right)\sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0. \tag{C.4}$$

We can see then why ζ, η, and σ are referred to as the "physical" transport coefficients – they are the quantities that, when non-zero, contribute to entropy production. Since $\chi_2^2 > 0$, $\xi_{3,\mu}\xi_3^\mu > 0$, and $\sigma_{\mu\nu}\sigma^{\mu\nu} > 0$, each coefficient must be independently positive

[1] This was shown in Appendix A immediately after equation (A.6).

[2] Note that $\tilde{\theta} > 0$ (c.f. (2.7)).

to ensure that the entropy current in positive semi-definite. The positivity of the divergence of the entropy current therefore guarantees that

$$\zeta > 0, \qquad \sigma > 0, \qquad \eta > 0.$$

Appendix D

Repository of Long Equations and Conditions

D.1 Uncharged Generic Fluid: Boosted Longitudinal Mode

The term $\Gamma(\phi)$ in equation (3.23) is given by

$$\Gamma_v(\phi) = (v_0\cos(\phi) - c_v)\Big[2\,(d-1)\left(2-\left(1-2c_v^2\right)v_0^2\right)\eta + \left(2-(d+1)\,v_0^2 - 2c_v^2\left(d-v_0^2\right)\right)(\theta_1+\pi_1)$$
$$+ (d-1)\,v_0\,(2\eta-\theta_1-\pi_1)\,(-4c_v\cos(\phi)+v_0\cos(2\phi))\Big]$$
$$\times\left[4\sqrt{1-v_0^2}\,w_0\left((d-1)\,v_0\cos(\phi) - c_v\left(d-v_0^2\right)\right)\right]^{-1}.$$

D.2 Charged CFT: Longitudinal Equation

For the charged conformal fluid, the longitudinal function $F(\omega,k)$ is a sextic polynomial of the form $\sum_n a_n\omega^n$ (c.f. equation (D.2)). The coefficients are given by

$$a_6 = d\pi_1\theta_1\nu_3,$$
$$a_5 = \frac{id\Big(\nu_3T_0\Big(\Big(\theta_1+\pi_1\Big)w_0 - \theta_1\mu_0n_0\Big) + \theta_1w_0\Big(\kappa\mu_0\nu_3+\pi_1\lambda\Big)\Big)}{T_0},$$
$$a_4 = \frac{d^2\theta_1\kappa^2w_0^2}{T_0} + k^2\left(-2\pi_1d\eta\nu_3 - \frac{\pi_1d\theta_1^2\kappa^2}{T_0} - \pi_1d\theta_1\sigma T_0 + 2\pi_1\nu_3\Big(\eta-\theta_1\Big)\right) + \frac{d\theta_1\lambda\mu_0n_0w_0}{T_0}$$
$$+ d\mu_0\nu_3n_0w_0 - \frac{dw_0^2\Big(\Big(\theta_1+\pi_1\Big)\lambda + \kappa\mu_0\nu_3\Big)}{T_0} - \frac{d\theta_1\kappa\lambda\mu_0w_0^2}{T_0^2} - d\nu_3w_0^2,$$
$$a_3 = -\frac{i\Big(d\lambda w_0^2\Big(T_0\Big(w_0-\mu_0n_0\Big)+\kappa\mu_0w_0\Big) - d^2\kappa^2T_0w_0^3\Big)}{T_0^2}$$

$$
\begin{aligned}
&-\frac{ik^2}{T_0^2}\Bigg(d\bigg(T_0^2\Big(2\eta\nu_3\Big(w_0-\mu_0 n_0\Big)+\theta_1\kappa\mu_0\sigma w_0\Big)+\kappa T_0\Big(\theta_1^2\kappa\Big(w_0-\mu_0 n_0\Big)+2\eta\mu_0\nu_3 w_0\Big)\\
&+\sigma T_0^3\Big(\Big(\theta_1+\pi_1\Big)w_0-\theta_1\mu_0 n_0\Big)+\pi_1 T_0 w_0\Big(2\eta\lambda+\theta_1\kappa^2\Big)+\theta_1^2\kappa^3\mu_0 w_0\Big)\\
&+T_0\Big(\pi_1 w_0\Big(-2\eta\lambda+2\theta_1\lambda+\nu_3 T_0\Big)-\nu_3\Big(2\eta-\theta_1\Big)\Big(T_0\Big(w_0-\mu_0 n_0\Big)+\kappa\mu_0 w_0\Big)\Big)\bigg)\Bigg),\\
a_2&=\frac{k^4}{dT_0^2}\Bigg(2\pi_1 d^2\eta\theta_1\kappa^2 T_0+2\pi_1 d^2\eta\sigma T_0^3+2\pi_1 d\theta_1\kappa^2 T_0\Big(\theta_1-\eta\Big)-2d\eta\theta_1\nu_3 T_0^2\\
&+2\pi_1 d\sigma T_0^3\Big(\theta_1-\eta\Big)+\Big(2\eta+\pi_1\Big)\theta_1\nu_3 T_0^2\Bigg)+\frac{k^2}{dT_0^2}\Bigg(-2d^3\eta\kappa^2 T_0 w_0^2\\
&+d^2 T_0 w_0\Big(-\mu_0 n_0\Big(2\eta\lambda+\theta_1\kappa^2\Big)+\kappa^2 w_0\Big(2\eta-\theta_1\Big)+2\eta\lambda w_0\Big)-d^2\mu_0 n_0\sigma T_0^3 w_0+d^2\kappa\mu_0\sigma T_0^2 w_0^2\\
&+d^2\sigma T_0^3 w_0^2+d^2\kappa\mu_0 w_0^2\Big(2\eta\lambda+\theta_1\kappa^2\Big)+dT_0 w_0\Big(\kappa\mu_0\nu_3 w_0-\lambda\Big(2\eta-\theta_1\Big)\Big(w_0-\mu_0 n_0\Big)\Big)\\
&-d\mu_0\nu_3 n_0 T_0^2 w_0+\pi_1 d\lambda T_0 w_0^2+d\nu_3 T_0^2 w_0^2+d\kappa\lambda\mu_0 w_0^2\Big(\theta_1-2\eta\Big)\Bigg),\\
a_1&=-\frac{ik^4}{dT_0^2}\Bigg(2d^2\eta\theta_1\kappa^2\mu_0\Big(n_0 T_0-\kappa w_0\Big)-2d^2\eta\sigma T_0^2\Big(T_0\Big(w_0-\mu_0 n_0\Big)+\kappa\mu_0 w_0\Big)\\
&-2d^2\eta\theta_1\kappa^2 T_0 w_0+d\bigg(-\theta_1\Big(T_0\Big(\kappa^2\mu_0 n_0\Big(2\eta-\theta_1\Big)-2\eta w_0\Big(\kappa^2+\lambda\Big)+\Big(\theta_1+\pi_1\Big)\kappa^2 w_0\Big)\\
&+\kappa^3\mu_0 w_0\Big(\theta_1-2\eta\Big)\Big)+\sigma T_0^2\Big(2\eta-\theta_1\Big)\Big(T_0\Big(w_0-\mu_0 n_0\Big)+\kappa\mu_0 w_0\Big)+\pi_1(-\sigma)T_0^3 w_0\Big)\\
&-\Big(2\eta+\pi_1\Big)\theta_1\lambda T_0 w_0\bigg)\Bigg)-\frac{ik^2}{dT_0^2}\Big(d^2\kappa^2 T_0 w_0^3-d\lambda w_0^2\Big(T_0\Big(w_0-\mu_0 n_0\Big)+\kappa\mu_0 w_0\Big)\Big),\\
a_0&=\frac{\theta_1 k^6\Big(2(d-1)\eta-\pi_1\Big)\Big(\theta_1\kappa^2+\sigma T_0^2\Big)}{dT_0}\\
&-\frac{k^4 w_0\Big(\theta_1\kappa^2\mu_0\Big(\kappa w_0-n_0 T_0\Big)+\sigma T_0^2\Big(T_0\Big(w_0-\mu_0 n_0\Big)+\kappa\mu_0 w_0\Big)\Big)}{T_0^2}.
\end{aligned}
$$

D.3 Conformal Charged Fluid: Boosted Gaps

The coefficients of the gap-controlling equation (3.34) once the substitution $\omega = i\Delta$ has been made are

$$
a_3 = T\sqrt{1-v_0^2}\Big(\pi_1 d^2\theta_1+\theta_1 v_0^4\Big(-2(d-1)\eta+\pi_1\Big)-2\pi_1 d v_0^2\Big((d-1)\eta+\theta_1\Big)\Big)
$$

$$\times\left(-v_0^2\left(\theta_1\kappa^2+\sigma T^2\right)+\nu_3 T\right),$$

$$a_2 = T\left(v_0^2-1\right)w\left(v_0^4\left(-\theta_1\left(2d^2\eta\kappa^2+d\theta_1\kappa^2+d\left(-2\eta\left(\kappa^2+\lambda\right)+\pi_1\kappa^2+\sigma T^2\right)+\left(2\eta+\pi_1\right)\lambda\right)\right.\right.$$
$$\left.-\,d\sigma T^2\left(2(d-1)\eta+\pi_1\right)\right)-d^2\left(\pi_1\theta_1\lambda+\left(\theta_1+\pi_1\right)\nu_3 T\right)+dv_0^2\left(\pi_1\left(2(d-1)\eta\lambda+d\sigma T^2\right)\right.$$
$$\left.\left.+\,\theta_1\left(d\theta_1\kappa^2+\pi_1\left(d\kappa^2+2\lambda\right)+d\sigma T^2\right)+\nu_3 T\left(2(d-1)\eta+\theta_1+\pi_1\right)\right)\right),$$

$$a_1 = dT\left(1-v_0^2\right)^{3/2}w^2\left(d\left(\theta_1+\pi_1\right)\lambda+\sigma T^2 v_0^2\left(v_0^2-d\right)+\nu_3 T\left(d-v_0^2\right)\right.$$
$$\left.+\,d\kappa^2\left(v_0^2\left(2(d-1)\eta+\theta_1\right)-d\theta_1\right)-\lambda v_0^2\left(2(d-1)\eta+\theta_1+\pi_1\right)\right),$$

$$a_0 = dT\left(v_0^2-1\right)^2 w^3\left(d-v_0^2\right)\left(\lambda-d\kappa^2\right).$$

D.4 Charged Generic Fluid: Longitudinal equation

The longitudinal modes of the charged generic fluid once more have dispersion relations controlled by a sextic polynomial of the form $\sum a_n\omega^n$, (3.38). The coefficients are given by

$$a_6 = \varepsilon_1\theta_1\nu_3,$$

$$a_5 = \frac{iw_0}{T_0^2p_\epsilon^2}\left(\theta_1\nu_3 p_n\left(\lambda p_n-\kappa T_0\right)+\theta_1 T_0 p_\epsilon\left(\lambda\nu_1 p_n+T_0\left(\nu_3-\kappa\nu_1\right)\right)+\varepsilon_1 T_0 p_\epsilon^2\left(\theta_1\lambda+\nu_3 T_0\right)\right),$$

$$a_4 = -\frac{k^2}{T_0^2p_\epsilon^2}\left(\nu_3 T_0^2 p_\epsilon^2\left(\varepsilon_2\theta_1+\pi_1\left(\varepsilon_2+\theta_1\right)+\varepsilon_1^2p_\epsilon^2-\left(\varepsilon_2+\pi_1\right)\varepsilon_1 p_\epsilon+\varepsilon_1\gamma_s\right)+\varepsilon_1\theta_1\sigma T_0^3p_\epsilon^2\right)$$
$$-\frac{1}{T_0^2p_\epsilon^2}\left[T_0w_0^2\left(-\kappa\nu_3 p_n+\lambda p_\epsilon\left(\nu_1 p_n+\varepsilon_1 p_\epsilon\right)+\theta_1\left(\lambda p_\epsilon-\kappa^2\right)\right)+\lambda w_0^2 p_n\left(\theta_1\kappa+\nu_3 p_n\right)\right.$$
$$\left.+\,T_0^2w_0^2\left(\nu_3-\kappa\nu_1\right)p_\epsilon\right],$$

$$a_3 = -\frac{1}{T_0^2p_\epsilon^2}\left[ik^2\left(T_0w_0\left(p_n\left(\theta_1\lambda\sigma p_n-\kappa\nu_3\gamma_s\right)+p_\epsilon^2\left(\varepsilon_2\kappa\mu_0\nu_3+\varepsilon_2\lambda\left(\theta_1-\nu_1 p_n+\pi_1\right)\right.\right.\right.\right.$$
$$\left.\left.+\,\theta_1\left(\kappa\mu_0\nu_3+\pi_1\lambda+\lambda\nu_1 p_n\right)+\varepsilon_1\left(\theta_1\kappa^2+\lambda\gamma_s\right)\right)+p_\epsilon\left(-\pi_1\theta_1\kappa^2+\kappa\nu_3\left(\varepsilon_2-\theta_1\right)p_n+\lambda\nu_1 p_n\gamma_s\right)$$
$$\left.-\,\varepsilon_1\lambda p_\epsilon^3\left(\varepsilon_2-\nu_1 p_n+\pi_1\right)+\varepsilon_1^2\lambda p_\epsilon^4\right)+T_0^2w_0\left(\theta_1(-\kappa)\sigma p_n+\nu_3 p_\epsilon\left(p_\epsilon\left(\varepsilon_1 p_\epsilon-\varepsilon_2\right)+\gamma_s\right)\right.$$
$$\left.\left.-\,\kappa\nu_1 p_\epsilon\left(2\theta_1 p_\epsilon+\gamma_s\right)\right)\right)+\lambda w_0 p_n\left(\nu_3 p_n\left(p_\epsilon\left(-\varepsilon_2+\theta_1+\varepsilon_1 p_\epsilon\right)+\gamma_s\right)+\pi_1\theta_1\kappa p_\epsilon\right)$$

$$+ T_0^3 p_\epsilon\bigg(\nu_3 s_0\bigg(\varepsilon_2+\theta_1\bigg)p_\epsilon+\sigma w_0\bigg(\theta_1+\varepsilon_1 p_\epsilon\bigg)\bigg)\bigg)\bigg)\bigg] - \frac{i\bigg(\kappa\lambda w_0^3 p_n + T_0 w_0^3\bigg(\lambda p_\epsilon-\kappa^2\bigg)\bigg)}{T_0^2 p_\epsilon^2},$$

$$a_2 = \frac{k^4}{T_0^3 p_\epsilon^2}\bigg(\theta_1\nu_3 T_0^3 p_\epsilon^2\bigg(p_\epsilon\bigg(\varepsilon_2-\varepsilon_1 p_\epsilon+\pi_1\bigg)-\gamma_s\bigg)$$

$$+ \sigma T_0^4 p_\epsilon^2\bigg(\varepsilon_2\theta_1+\pi_1\bigg(\varepsilon_2+\theta_1\bigg)+\varepsilon_1^2 p_\epsilon^2-\bigg(\varepsilon_2+\pi_1\bigg)\varepsilon_1 p_\epsilon+\varepsilon_1\gamma_s\bigg)\bigg)$$

$$+ \frac{k^2}{T_0^3 p_\epsilon^2}\bigg(T_0^2 w_0^2\bigg(\lambda\sigma p_n^2+p_\epsilon^2\bigg(\varepsilon_2(-\lambda)+\kappa\mu_0\nu_3+\lambda\nu_1 p_n\bigg)-\kappa^2 p_\epsilon\bigg(\theta_1+\nu_1 p_n\bigg)+\gamma_s\bigg(\lambda p_\epsilon-\kappa^2\bigg)$$

$$+ \varepsilon_1\lambda p_\epsilon^3\bigg)+T_0 w_0^2\bigg(\theta_1\kappa^3\bigg(-p_n\bigg)+\kappa\lambda p_n\bigg(p_\epsilon\bigg(2\theta_1+\varepsilon_1 p_\epsilon\bigg)+\gamma_s\bigg)+p_n^2\bigg(\lambda\bigg(\kappa\nu_1+\nu_3\bigg)p_\epsilon-\kappa^2\nu_3\bigg)$$

$$+ \kappa\lambda\mu_0\bigg(\varepsilon_2+\theta_1\bigg)p_\epsilon^2\bigg)-\kappa\sigma T_0^3 w_0^2 p_n+\kappa\lambda w_0^2 p_n^2\bigg(\theta_1\kappa+\nu_3 p_n\bigg)+\lambda s_0 T_0^3 w_0\bigg(\varepsilon_2+\theta_1\bigg)p_\epsilon^2$$

$$+ \nu_3 s_0 T_0^4 w_0 p_\epsilon^2-\kappa\nu_1 T_0^3 w_0^2 p_\epsilon^2+\sigma T_0^4 w_0^2 p_\epsilon\bigg),$$

$$a_1 = \frac{ik^4}{T_0^3 p_\epsilon^2}\bigg(\sigma T_0^2 w_0 p_n\gamma_s\bigg(\lambda p_n-\kappa T_0\bigg)+T_0 w_0 p_\epsilon^2\bigg(\bigg(\varepsilon_2+\pi_1\bigg)\theta_1\kappa\lambda p_n-\theta_1 T_0\bigg(\kappa\bigg(\pi_1\kappa+\nu_3 p_n\bigg)+\lambda\gamma_s\bigg)$$

$$+ \sigma T_0\bigg(\varepsilon_1\lambda p_n^2+\kappa T_0\bigg(\mu_0\bigg(\varepsilon_2+\theta_1\bigg)-\varepsilon_1 p_n\bigg)-\varepsilon_2 T_0^2\bigg)\bigg)+\sigma T_0^2 w_0 p_\epsilon\bigg(\bigg(\varepsilon_2+\pi_1\bigg)(-\lambda)p_n^2$$

$$+ \bigg(\varepsilon_2+\pi_1\bigg)\kappa T_0 p_n+T_0^2\gamma_s\bigg)-\theta_1\kappa\lambda T_0 w_0 p_n p_\epsilon\gamma_s+T_0 w_0 p_\epsilon^3\bigg(-\varepsilon_1\theta_1\kappa\lambda p_n$$

$$+ \theta_1 T_0\bigg(\varepsilon_1\kappa^2+\bigg(\varepsilon_2+\pi_1\bigg)\lambda\bigg)+\varepsilon_1\sigma T_0^3-\theta_1\kappa\nu_1 T_0^2\bigg)+\theta_1\kappa^2 T_0^2 w_0 p_\epsilon\gamma_s+\sigma s_0 T_0^5\bigg(\varepsilon_2+\theta_1\bigg)p_\epsilon^2$$

$$- \varepsilon_1\theta_1\lambda T_0^2 w_0 p_\epsilon^4\bigg)+\frac{ik^2}{T_0^3 p_\epsilon^2}\bigg(\kappa^2 w_0^3 p_n\bigg(\lambda p_n-\kappa T_0\bigg)+2\kappa\lambda T_0 w_0^3 p_n p_\epsilon$$

$$+ \lambda s_0 T_0^3 w_0^2 p_\epsilon^2-\kappa^2 T_0^2 w_0^3 p_\epsilon+\kappa\lambda\mu_0 T_0 w_0^3 p_\epsilon^2\bigg),$$

$$a_0 = \theta_1 k^6\sigma T_0\bigg(p_\epsilon\bigg(-\varepsilon_2+\varepsilon_1 p_\epsilon-\pi_1\bigg)+\gamma_s\bigg)+k^4\sigma\bigg(-s_0 T_0^2 w_0-\kappa\mu_0 w_0^2\bigg).$$

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