Enumerating Digitally Convex Sets in Graphs

by

MacKenzie Carr
B.Sc.(Hons), Acadia University, 2018

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ABSTRACT

Given a finite set V, a *convexity*, $\mathscr{C}$, is a collection of subsets of V that contains both the empty set and the set V and is closed under intersections. The elements of $\mathscr{C}$ are called *convex sets*. We can define several different convexities on the vertex set of a graph. In particular, the digital convexity, originally proposed as a tool for processing digital images, is defined as follows: a subset $S \subseteq V(G)$ is *digitally convex* if, for every $v \in V(G)$, we have $N[v] \subseteq N[S]$ implies $v \in S$. Or, in other words, each vertex v that is not in the digitally convex set S needs to have a private neighbour in the graph with respect to S. In this thesis, we focus on the generation and enumeration of digitally convex sets in several classes of graphs. We establish upper bounds on the number of digitally convex sets of 2-trees, k-trees and simple clique 2-trees, as well as conjecturing a lower bound on the number of digitally convex sets of 2-trees and a generalization to k-trees. For other classes of graphs, including powers of cycles and paths, and Cartesian products of complete graphs and of paths, we enumerate the digitally convex sets using recurrence relations. Finally, we enumerate the digitally convex sets of block graphs in terms of the number of blocks in the graph, rather than in terms of the order of the graph.

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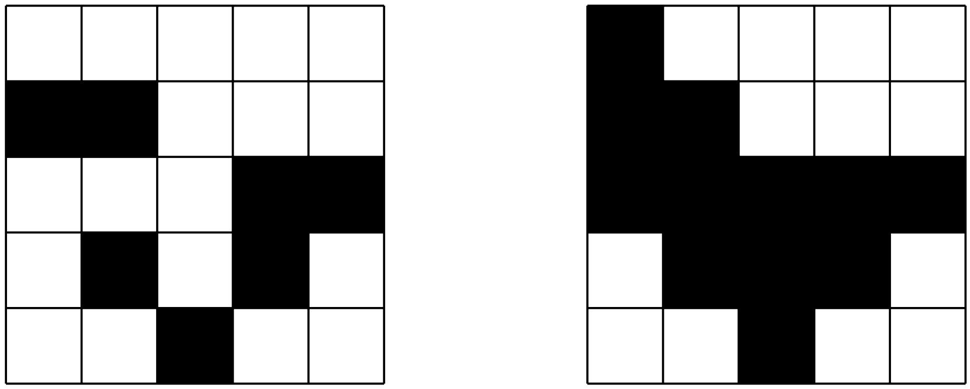
Chapter 1

Introduction

Digital convexity was introduced initially as a tool for processing and smoothing digital images [22]. In a black and white digital image, taking a smallest digitally convex set of black pixels containing the black pixels in the original image is a method of smoothing the digital image. Smoothing an image is sometimes required for processing or storing the image, as a smoothed image often requires less memory space to store. As an example, Figure 1.1 shows a black and white digital image in Figure 1.1(a) and its corresponding smoothed image using digital convexity in Figure 1.1(b). The black pixels in the smoothed image form a smallest digitally convex set containing all of the black pixels in the original image. Digital convexity has been extended to graphs as a way of generalizing the digital image structure, which is the focus of this thesis.

In the following chapters, we examine the generation and enumeration of the digitally convex sets of a variety of graph classes. Enumerating the digitally convex sets of a class of graphs corresponds to determining the number of "smoothed images" that a given graph structure can have. In Chapter 2, we review relevant notation and background, including the definition of digital convexity, other convexities defined on graphs and problems that have been explored using these convexities. In Chapter 3,



(a) A black and white digital image (b) The corresponding smoothed image

Figure 1.1: Smoothing of a black and white digital image using digital convexity

we extend previous results on the enumeration of the digitally convex sets of trees to 2-trees and to the more general k-trees. In Chapter 4, we show how other mathematical objects, such as binary strings and arrays, can be used to enumerate the digitally convex sets of classes of graphs such as cycles and Cartesian products of paths. In Chapter 5, we show how the digitally convex sets of block graphs can be enumerated in terms of the number of blocks in the graph and how this gives more information about the structure of the sets than enumerating them in terms of the order of the graph. Finally, in Chapter 6, we summarize our results and suggest some directions for future research.

For all graph theory terms and concepts not defined in this thesis, refer to [1].

Chapter 2

Notation and Background

First, we note that throughout this thesis, we use $A \subseteq B$ to denote that the set A is a subset of the set B. We use $A \subsetneq B$ to denote that A is a proper subset of B.

Given a finite set V, a collection, $\mathscr{C}$, of subsets of V is called a *convexity* or *alignment* if it contains $\emptyset$ and V and is closed under intersections. The elements of a convexity $\mathscr{C}$ are called *convex sets* and the ordered pair $(V, \mathscr{C})$ is an *aligned space*. For any subset $S \subseteq V$, the *convex hull* of S, denoted by $CH_{\mathscr{C}}(S)$, is the smallest convex set that contains S. For any $S \subseteq V$, if $CH_{\mathscr{C}}(S) = S$, then S is a convex set.

As an example, let $V = \{1, 2, 3, 4, 5\}$. The collection $\mathscr{C} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{4, 5\}, \{1, 2, 3, 4, 5\}\}$ is a convexity. We have $CH_{\mathscr{C}}(\{4\}) = \{4, 5\}$ and $CH_{\mathscr{C}}(\{3, 4\}) = V$, since $\{4, 5\}$ is the smallest convex set containing $\{4\}$ and the only convex set containing $\{3, 4\}$ is the entire set V. Since $\{1\}$ is a convex set, we have $CH_{\mathscr{C}}(\{1\}) = \{1\}$.

Van de Vel provides an in-depth study of abstract convex structures in [26].

2.1 Convexity in graphs

There are many convexities defined on the vertex set of a graph, many of which use an interval notion, as does the definition of Euclidean convexity. Several of these convexities were studied by Farber and Jamison in [12]. A set $S \subseteq V(G)$ is *g-convex* if, for every $a, b \in S$, every vertex on some *a*-*b* *geodesic*, or shortest *a*-*b* path, belongs to S. The collection of vertices that are on some *a*-*b* geodesic forms the *geodesic interval* between a and b. So the definition of a g-convex set can be restated in terms of geodesic intervals; a set $S \subseteq V(G)$ is g-convex if it contains the geodesic interval between a and b, for every $a, b \in S$. The collection of all g-convex sets in a graph G forms the *geodesic convexity* of G.

Similar to the geodesic convexity of a graph is the monophonic convexity. A set $S \subseteq V(G)$ is *m-convex* if it contains every vertex that lies on some induced *a*-*b* path, for every $a, b \in S$. The set of vertices that are on some induced *a*-*b* path is called the *monophonic interval* between a and b. So the definition of an m-convex set can, as with the definition of a g-convex set, be stated in terms of intervals. The collection of all m-convex sets in a graph G forms the *monophonic convexity* of G.

Several other convexities have been similarly defined using paths between pairs of vertices, including the *simple path convexity* [12] and the *triangle path convexity* [8].

Cáceres and Oellermann [6] introduced a graph convexity that uses Steiner trees in the graph. For a connected graph G and a set X of at least two vertices of G, a *Steiner tree* for X is a connected subgraph of smallest size that contains every vertex in X. The *Steiner interval* for X is the set of all vertices that belong to some Steiner tree for X. Then, for any integer $k \geq 2$, a set $S \subseteq V(G)$ is *k-Steiner convex*, or *g_k-convex*, if S contains the Steiner interval for every subset of k vertices of S. Note that when $k = 2$, the Steiner interval for a pair of vertices $a, b \in S$ is equivalent to the geodesic interval between a and b, because a connected subgraph of smallest size containing a and b must be a shortest *a*-*b* path. Therefore, a set S is g_2-convex (or 2-Steiner convex) if and only if it is g-convex.

Other graph convexities defined in terms of intervals have been studied in [7, 10] and [19].

A graph convexity that is not defined in terms of intervals is the digital convexity, introduced by Rosenfeld and Pfaltz in [22]. Rather than using an interval definition, the digital convexity is instead defined in terms of neighbourhoods. The *open neighbourhood* of a vertex $v \in V(G)$, denoted by $N_G(v)$ or $N(v)$ when the graph G is obvious, is defined as $N_G(v) = \{x \in V(G) \mid xv \in E(G)\}$. Similarly, the *closed neighbourhood* of v, denoted by $N_G[v]$ or $N[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the closed neighbourhood of S, denoted by $N_G[S]$ or $N[S]$, is defined as $N_G[S] = \bigcup_{v \in S} N_G[v]$.

A set $S \subseteq V(G)$ is *digitally convex* if $N_G[v] \subseteq N_G[S]$ implies $v \in S$ for every $v \in V(G)$. For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, if $N_G[v] - N_G[S - \{v\}] \neq \emptyset$, then we say that v has a *private neighbour with respect to* S in G. Thus, S is digitally convex if and only if, for every $v \notin S$, v has a private neighbour with respect to S. In other words, either $v \notin N[S]$ or there is some $x \in N(v)$ with $x \notin N[S]$. Note that private neighbours are not necessarily unique and a vertex v can be a private neighbour for multiple vertices. For a graph G, the collection of all digitally convex sets in G is the *digital convexity* of G, denoted by $\mathscr{D}(G)$. The number of digitally convex sets in G is denoted by $n_{\mathscr{D}}(G)$.

As an example of the digital convexity in graphs, consider the complete graph, K_n. For any $n \geq 1$, the only digitally convex sets in this graph are $\emptyset$ and $V(K_n)$. As each vertex is a universal vertex, the closed neighbourhood of any nonempty subset of $V(K_n)$ is the entire vertex set.

Consider instead the graph G in Figure 2.1. The collection of digitally convex sets in this graph is $\mathscr{D}(G) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,3\}, \{3,5\}, \{2,6\}, \{4,6\}, \{2,3,4\}, \{1,5,6\}, \{1,2,3,4,5,6\}\}$. The set $S = \{1,3\}$, for example, is digitally convex because the vertex 5 is not in the neighbourhood of S, so it is a private

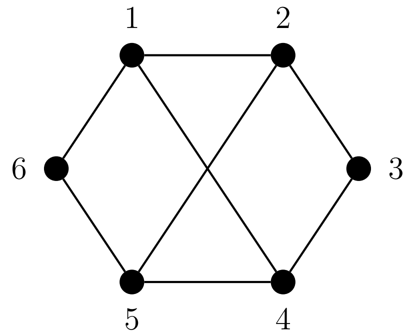


Figure 2.1: A graph G with $n_{\mathscr{D}}(G) = 14$

neighbour for each of the vertices 2, 4, 5 and 6. The set $\{2,4\}$ is not digitally convex in G, because $N[3] = \{2,3,4\} \subseteq N[\{2,4\}] = \{1,2,3,4,5\}$. Adding the vertex 3 to this set gives the convex hull of $\{2,4\}$, i.e. $CH_{\mathscr{D}}(\{2,4\}) = \{2,3,4\}$.

There are several problems related to graph convexity that have been explored using the various convexities described above. For example, Pfaltz and Jamison [21] studied digital convexity in the context of closure systems. Buzatu and Cataranciuc [4] and Gonzáles, Grippo, Safe and Santos [14] examined the problem of covering graphs with convex sets, using geodesic convexity and monophonic convexity, respectively. Dourado, Gimbel, Kratochvìl, Protti and Szwarcfiter [9] examined the problem of determining the hull number, or size of a largest proper g-convex subset of $V(G)$, and characterized the graphs with a given hull number. Brown and Oellermann [2] studied the graphs with a smallest possible set of g-convex or m-convex sets. The graphs whose g-convex (m-convex) sets are exactly $\emptyset$, all singletons, all edges and $V(G)$ are *g-minimal* (resp. *m-minimal*). Brown and Oellermann characterized these graphs and examined the properties of g-minimal and m-minimal graphs. More general is the problem of enumerating the convex sets of a given graph, or class of graphs. In the case of geodesic convexity, it can be shown that the number of g-convex sets of a tree is equal to the number of its subtrees, a problem which is explored in [24, 25] and [27]. Brown and Oellermann [3] determined that the problem of enumerating the g-convex sets of a cograph can be performed in linear time but, for an arbitrary graph, the problem is #P-complete. Lafrance, Oellermann and Pressey examined the reconstruction of a tree from its digitally convex sets in [16] and the enumeration of the digitally convex sets of trees and cographs in [15].

In this thesis, we extend many of the results in [15] and enumerate the digitally convex sets of several other classes of graphs. In the remainder of this chapter, we state relevant properties of digitally convex sets, as well as results from [15] that will be used in later chapters.

2.2 Properties of digitally convex sets

Digital convexity is closely related to domination in a graph. For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, if $N[v] \subseteq N[S]$, then S is said to be a *local dominating set* for v. Thus, a digitally convex set contains every vertex for which it is a local dominating set. Cáceres, Márquez, Morales and Puertas [5] and Oellermann [20] examine the relationship between digital convexity and other domination parameters. In particular, in [5], the following result is given.

Theorem 2.1 (Cáceres, Márquez, Morales and Puertas [5])**.** *Let G be a graph, let $\delta(G)$ denote the minimum degree of G and let con(G) denote the cardinality of a largest proper digitally convex set of $V(G)$. Then*

(a) for any $v \in V(G)$, the set $V(G) - N_G[v]$ is digitally convex in G,

(b) $con(G) \geq n - k - 1$ if and only if $\delta(G) \leq k$, and

(c) $con(G) = n - \delta(G) - 1$.

Note that parts (b) and (c) of Theorem 2.1 follow directly from part (a) and solve the digital convexity equivalent of the hull number problem explored in [9]. Lafrance, Oellermann and Pressey [15] give the following properties that aid in generating the digitally convex sets of a graph.

Theorem 2.2 (Lafrance, Oellermann and Pressey [15])**.**

(a) *If S is digitally convex in the graph G, the set $\varphi(S) = V(G) - N[S]$ is also digitally convex in G. Furthermore, φ defines a bijection from $\mathscr{D}(G)$ to itself.*

(b) *The graph G has an even number of digitally convex sets.*

(c) *A vertex $v \in V(G)$ appears in at most half of the digitally convex sets of G.*

(d) *A vertex $v \in V(G)$ appears in exactly half of the digitally convex sets of G if and only if v is a simplicial vertex.*

2.3 Digital convexity in trees

Lafrance, Oellermann and Pressey [15] developed an algorithm for generating the digitally convex sets of a tree. This algorithm follows the construction of the tree, beginning with a K_2, whose digitally convex sets are known to be $\emptyset$ and $V(K_2)$. At each step, a leaf is added to the tree and the digitally convex sets of the new tree are generated using the digitally convex sets of the tree generated in the previous step. Lafrance, Oellermann and Pressey then prove that, for any tree T, the algorithm generates the entire collection $\mathscr{D}(T)$ of digitally convex sets. The algorithm is stated below.

Algorithm 1 (Lafrance, Oellermann and Pressey [15])**.** *Generating the collection $\mathscr{S}_T$ of digitally convex sets of a tree T of order $n \geq 2$.*

1. *If $n = 2$, then $\mathscr{S}_T = \{\emptyset, V(T)\}$.*

2. *Suppose $n \geq 3$. Then, let v be a leaf of T and let u be its neighbour. Use this algorithm to find the collection $\mathscr{S}_{T-v}$ of all digitally convex sets of the tree $T-v$. Then, for each $S \in \mathscr{S}_{T-v}$, generate the sets in $\mathscr{S}_T$ as follows: Let $\mathscr{S}_T = \emptyset$.*

   (a) *If $u \notin S$, add S to $\mathscr{S}_T$.*

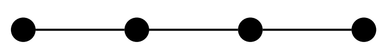
   (b) *If $u \notin S$ and for every $a \in N_{T-v}[u] - S$, we have $N_T[a] \nsubseteq N_T[S \cup \{v\}]$, add $S \cup \{v\}$ to $\mathscr{S}_T$.*

   (c) *If $u \in S$, add $S \cup \{v\}$ to $\mathscr{S}_T$.*

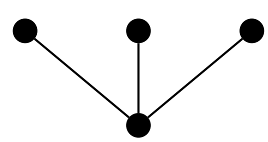
   (d) *If $u \in S$ and $N_{T-v}[u] \subseteq N_{T-v}[S - \{u\}]$, add $S - \{u\}$ to $\mathscr{S}_T$.*

Theorem 2.3 (Lafrance, Oellermann and Pressey [15]). *Let T be a tree of order $n \geq 2$. Then the collection $\mathscr{S}_T$ generated by Algorithm 1 is $\mathscr{D}(T)$.*

It is not obvious from Algorithm 1 how many digitally convex sets are generated at each step. The number of digitally convex sets constructed from each case depends mainly on the neighbour, u, of the new leaf being added. There are many choices for this vertex u, many of which lead to non-isomorphic trees with different numbers of digitally convex sets. Thus, other methods must be used in enumerating the digitally convex sets of trees.



(a) $n_{\mathscr{D}}(T_1) = 6$



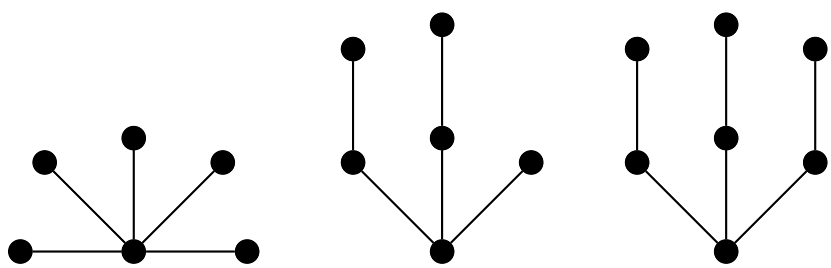
(b) $n_{\mathscr{D}}(T_2) = 8$

Figure 2.2: Two non-isomorphic trees of order four with different numbers of digitally convex sets.

In the case of paths, there is a unique graph for a given order. Lafrance, Oellermann and Pressey show that the number of digitally convex sets of a path can be expressed in terms of the Fibonacci numbers. Recall that the Fibonacci sequence $f_1, f_2, \ldots$ is defined recursively as follows: $f_1 = f_2 = 1$ and, for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$.

Proposition 2.4 (Lafrance, Oellermann and Pressey [15]). *If P_n is the path of order n, then $n_{\mathscr{D}}(P_n) = 2f_n$.*

Since non-isomorphic trees of the same order can have a different number of digitally convex sets, only upper and lower bounds on the number of digitally convex sets of trees of a given order can be constructed. Lafrance, Oellermann and Pressey show that these bounds are attained by the stars and the spiderstars, respectively. The star of order n is the graph $K_{1,n-1}$. The spiderstar S_n of order $n = 2k+1$ is obtained from the star $K_{1,k}$ by subdividing each edge exactly once, and that of order $n = 2k$ is obtained by subdividing all but one edge exactly once. The star of order six and the spiderstars of orders six and seven are shown in Figure 2.3.



(a) $K_{1,5}$ (b) S_6 (c) S_7

Figure 2.3: The star of order six and the spiderstars of orders six and seven

Theorem 2.5 (Lafrance, Oellermann and Pressey [15]). *Let T be a tree of order n. Then,*

$$\left.\begin{array}{ll} for\ n\ even, & 2 \cdot 2^{\frac{n}{2}} - 2 \\ for\ n\ odd, & 3 \cdot 2^{\frac{n-1}{2}} - 2 \end{array}\right\} \leq n_{\mathscr{D}}(T) \leq 2^{n-1}.$$

The lower bound is attained by the spiderstar, S_n, and the upper bound is attained by the star $K_{1,n-1}$.

To prove the upper bound, Lafrance, Oellermann and Pressey showed that the removal of an edge incident with a leaf does not decrease the number of digitally convex sets in the graph. Thus, the number of digitally convex sets in a tree of order n is bounded above by the number of digitally convex sets in the disjoint union of K_2 and $\overline{K}_{n-2}$. The proof of the lower bound, however, required the use of the following lemmas, both proven in [15].

Lemma 2.6. *Let T be a tree of order $n \geq 2$ and $v \in V(T)$. Let T' be the tree formed by adding two new vertices v_1 and v_2 to T and edges vv_1 and v_1v_2. Then $n_{\mathscr{D}}(T') \geq 2n_{\mathscr{D}}(T) + 2$.*

Lemma 2.7. *Let T be a tree of order $n \geq 4$ containing two leaves v_1 and v_2, both adjacent to the same vertex, v. Let T_1 be the tree formed by deleting the edge vv_2 from T and adding the edge v_1v_2. Then, $n_{\mathscr{D}}(T_1) \leq n_{\mathscr{D}}(T)$.*

We state these results on the digitally convex sets of trees here because we show, in the following chapter, that there are analogous results on the number of digitally convex sets of 2-trees.

Chapter 3

Digital Convexity in k-trees

For $k \geq 1$, a *k-tree* is a graph defined as follows: a $k+1$-clique, K_{k+1}, is a k-tree, and a k-tree of order $n > k+1$ is constructed by adding a vertex v adjacent to k pairwise adjacent vertices (i.e. the vertices of a k-clique) in a k-tree of order $n-1$. Note that the 1-trees are exactly the trees. Figure 3.1 shows a 3-tree of order eight.

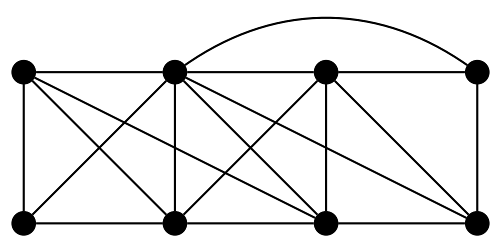


Figure 3.1: A 3-tree of order eight

In this chapter, we extend the results of Lafrance, Oellermann and Pressey [15] from trees to k-trees, generalizing both their algorithm for generating the digitally convex sets of a tree, and the upper bound that they gave for the number of digitally convex sets of a tree. We conjecture a lower bound on the number of digitally convex sets of a 2-tree and give a class of 2-trees that achieve the bound, later conjecturing a generalization to k-trees. Finally, we examine the digitally convex sets of a specific subclass of 2-trees, the simple clique 2-trees. The upper bound on the number of digitally convex sets of a 2-tree is no longer sharp when restricted to simple clique

2-trees, so we establish a sharp upper bound for the number of digitally convex sets of a simple clique 2-tree. In addition, we prove that the conjectured lower bound on the number of digitally convex sets of a 2-tree holds for 2-paths, a subclass of simple clique 2-trees.

3.1 Generating and enumerating digitally convex sets in 2-trees

In Section 2.3, we stated the algorithm developed by Lafrance, Oellermann and Pressey [15] to generate the digitally convex sets of a tree. This algorithm follows the construction of a tree, and the digitally convex sets generated at each step depend on the support vertex of the new leaf that is added at each step. In the construction of a 2-tree, we add a new vertex v adjacent to two adjacent vertices u and w. So in generating the digitally convex sets of a 2-tree, the digitally convex sets that are constructed at each step will depend on both u and w.

Algorithm 2. *Generating the collection $\mathscr{S}_G$ of digitally convex sets of a 2-tree G of order $n \geq 3$.*

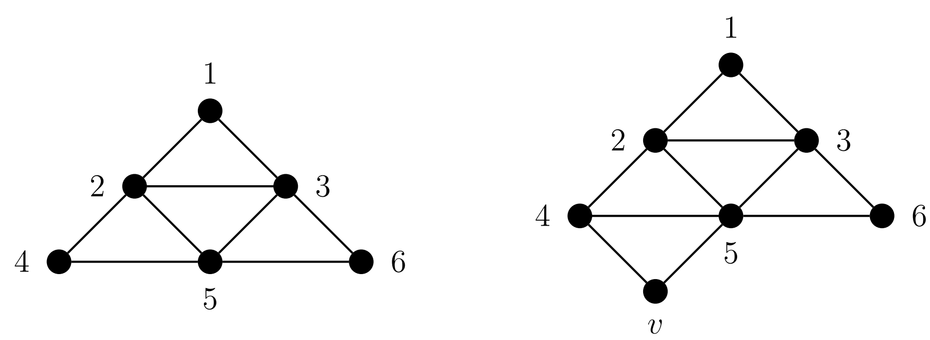
1. *If $n = 3$, then $\mathscr{S}_G = \{\emptyset, V(G)\}$.*
2. *Suppose $n > 3$ and let v be a vertex of degree 2, with neighbours u and w. Use the algorithm to generate $\mathscr{S}_{G-v}$. Obtain $\mathscr{S}_G$ from $\mathscr{S}_{G-v}$ as follows: Let $\mathscr{S}_G = \emptyset$. For each $S \in \mathscr{S}_{G-v}$, proceed as follows.*
   - (a) *If $u, w \notin S$, then add S to $\mathscr{S}_G$.*
   - (b) *If $u, w \notin S$ and for every $a \in (N_{G-v}[u] \cup N_{G-v}[w]) - S$, we have $N_G[a] \nsubseteq N_G[S \cup \{v\}]$, then add $S \cup \{v\}$ to $\mathscr{S}_G$.*

(c) If $u \in S$ *or* $w \in S$*, then add* $S \cup \{v\}$ *to* $\mathscr{S}_G$*.*

(d) If $u \in S$*,* $w \notin S$ *and* $N_{G-v}[u] \subseteq N_{G-v}[S - \{u\}]$*, then add* $S - \{u\}$ *to* $\mathscr{S}_G$*.*

(e) If $u \notin S$*,* $w \in S$ *and* $N_{G-v}[w] \subseteq N_{G-v}[S - \{w\}]$*, then add* $S - \{w\}$ *to* $\mathscr{S}_G$*.*

(f) If $u, w \in S$ *and* $N_{G-v}[\{u, w\}] \subseteq N_{G-v}[S - \{u, w\}]$*, then add* $S - \{u, w\}$ *to* $\mathscr{S}_G$*.*



(a) $G - v$ (b) G

Figure 3.2: Algorithm 2 generates the digitally convex sets of G using those of $G - v$.

As an example of step 2 in Algorithm 2, refer to the 2-tree in Figure 3.2(a), to which we add the vertex v to obtain the 2-tree in Figure 3.2(b). The digitally convex sets of $G - v$ are $\mathscr{D}(G - v) = \{\emptyset, \{1\}, \{4\}, \{6\}, \{1, 2, 4\}, \{1, 3, 6\}, \{4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$. In this case, the vertices u and w in the algorithm are the vertices 4 and 5, respectively.

The sets $\emptyset$, $\{1\}$, $\{6\}$ and $\{1, 3, 6\}$ all satisfy case 2(a) of Algorithm 2, so each of these sets is added to $\mathscr{S}_G$.

The sets $\emptyset$ and $\{6\}$ both satisfy case 2(b) of Algorithm 2, so $\{v\}$ and $\{6, v\}$ are added to $\mathscr{S}_G$. Neither $\{1\}$ nor $\{1, 3, 6\}$ satisfies case 2(b) because $2 \in (N_{G-v}[4] \cup N_{G-v}[5]) - S$ and $N_G[2] \subseteq N_G[S \cup \{v\}]$ for both $S = \{1\}$ and $S = \{1, 3, 6\}$.

The sets $\{4\}$, $\{1, 2, 4\}$, $\{4, 5, 6\}$ and $\{1, 2, 3, 4, 5, 6\}$ satisfy case 2(c) of Algorithm 2, so $\{4, v\}$, $\{1, 2, 4, v\}$, $\{4, 5, 6, v\}$ and $\{1, 2, 3, 4, 5, 6, v\}$ are all added to $\mathscr{S}_G$.

The set $\{1,2,4\}$ satisfies case 2(d) of Algorithm 2, as $N_{G-v}[4] = \{2,4,5\} \subseteq N_{G-v}[\{1,2\}] = \{1,2,3,4,5\}$. So $\{1,2\}$ is added to $\mathscr{S}_G$. The set $\{4\}$ does not satisfy case 2(d), as $N_{G-v}[4] \not\subseteq \emptyset$. There are no sets satisfying case 2(e).

Finally, $\{1,2,3,4,5,6\}$ satisfies case 2(f) of Algorithm 2, as $N_{G-v}[\{4,5\}] = \{2,3,4,5,6\} \subseteq N_{G-v}[\{1,2,3,6\}] = V(G-v)$. So $\{1,2,3,6\}$ is added to $\mathscr{S}_G$. The set $\{4,5,6\}$ does not satisfy case 2(f) because $\{2,3,4,5,6\} \not\subseteq N_{G-v}[\{6\}] = \{3,5,6\}$.

Now we have $\mathscr{S}_G = \{\emptyset, \{1\}, \{6\}, \{v\}, \{1,2\}, \{6,v\}, \{4,v\}, \{1,3,6\}, \{1,2,3,6\}, \{1,2,4,v\}, \{4,5,6,v\}, \{1,2,3,4,5,6,v\}\}$. The following result proves that this collection of digitally convex sets is exactly $\mathscr{D}(G)$.

Theorem 3.1. *Let G be a 2-tree of order $n \geq 3$. Then the collection $\mathscr{S}_G$ generated by Algorithm 2 is $\mathscr{D}(G)$.*

Proof. We use induction on n. First, let $n = 3$. Then $G \cong K_3$ and it is known that $\mathscr{D}(K_3) = \{\emptyset, V(K_3)\}$. So the algorithm correctly generates the collection of digitally convex sets for $n = 3$.

Now suppose $n > 3$. First, we show that, for each set $S \in \mathscr{S}_{G-v}$, the sets added to $\mathscr{S}_G$ by Algorithm 2 are digitally convex in G.

(a) Suppose $u, w \notin S$. Then, $v \notin N_G[S]$, so v is its own private neighbour with respect to S in G. Thus, S is digitally convex in G.

(b) Suppose $u, w \notin S$ and for every $a \in (N_{G-v}[u] \cup N_{G-v}[w]) - S$, we have $N_G[a] \not\subseteq N_G[S \cup \{v\}]$. Then, each such vertex a has a private neighbour with respect to $S \cup \{v\}$. So $S \cup \{v\}$ is digitally convex in G.

(c) Suppose $u \in S$. Then, because $N_G[v] \subseteq N_G[u]$, each vertex $x \in V(G)-(S\cup\{v\})$ has the same private neighbour with respect to $S \cup \{v\}$ in G as with respect to S in $G - v$. Thus, $S \cup \{v\}$ is digitally convex in G. Similarly, if $w \in S$, then $S \cup \{v\}$ is digitally convex in G.

(d) Suppose $u \in S$, $w \notin S$ and $N_{G-v}[u] \subseteq N_{G-v}[S - \{u\}]$. Then, $v \notin N_G[S - \{u\}]$ so v is a private neighbour for itself and for u with respect to $S - \{u\}$ in G. Thus, $S - \{u\}$ is digitally convex in G.

(e) Suppose $u \notin S$, $w \in S$ and $N_{G-v}[w] \subseteq N_{G-v}[S - \{w\}]$. Then, using the same argument as in case (d), $S - \{w\}$ is digitally convex in G.

(f) Suppose $u, w \in S$ and $N_{G-v}[\{u, w\}] \subseteq N_{G-v}[S - \{u, w\}]$. Then, using the same argument as in case (d), $S - \{u, w\}$ is digitally convex in G.

Thus, $\mathscr{S}_G \subseteq \mathscr{D}(G)$.

Now we show that each digitally convex set $S \in \mathscr{D}(G)$ is in $\mathscr{S}_G$. In other words, each digitally convex set in G is generated by Algorithm 2. Let $S \in \mathscr{D}(G)$.

If $v \in S$, then S satisfies one of the following two cases.

- If at least one of u or w is in S, then $S - \{v\}$ is digitally convex in $G - v$. Each vertex $x \in V(G) - S$ has the same private neighbour with respect to $S - \{v\}$ in $G - v$ as with respect to S in G. Thus, the set $S - \{v\}$ satisfies case 2(c), and S is added to $\mathscr{S}_G$ by Algorithm 2.

- If $u, w \notin S$ then, by definition of a digitally convex set, each vertex $a \in (N_G[u] \cup N_G[w]) - S$ has a private neighbour with respect to S. Since $v \in S$, the set of vertices $(N_G[u] \cup N_G[w]) - S$ is equal to $(N_{G-v}[u] \cup N_{G-v}[w]) - (S - \{v\})$ and each of these vertices must have a private neighbour with respect to $S - \{v\}$ in $G - v$. Thus, the set $S - \{v\}$ satisfies case 2(b), and S is added to $\mathscr{S}_G$ by Algorithm 2.

If $v \notin S$, then it must be the case that $u, w \notin S$ because both u and w dominate $N[v]$ in G. The set S satisfies one of the following cases.

- If v is the only private neighbour of both u and w with respect to S in G, then $N_{G-v}[\{u,w\}] \subseteq N_{G-v}[S]$. This means that the set $S \cup \{u,w\}$ is digitally convex in $G-v$. It satisfies case 2(f), and S is added to $\mathscr{S}_G$ by Algorithm 2.

- Similarly, if v is the only private neighbour of u with respect to S, but not of w, then the set $S \cup \{u\}$ is digitally convex in $G-v$. It satisfies case 2(d), and S is added to $\mathscr{S}_G$ by Algorithm 2.

- The same argument shows that if v is the only private neighbour of w with respect to S, but not of u, then $S \cup \{w\}$ satisfies case 2(e). So S is added to $\mathscr{S}_G$ by Algorithm 2.

- Finally, if both u and w have a private neighbour with respect to S that is not the vertex v, then they have this same private neighbour with respect to S in $G-v$. Thus, S is digitally convex in $G-v$ and satisfies case 2(a). So S is added to $\mathscr{S}_G$ by Algorithm 2.

Therefore $\mathscr{S}_G = \mathscr{D}(G)$. □

As was the case with Algorithm 1, it is not clear from Algorithm 2 how many digitally convex sets are generated for a given 2-tree. Thus, there is no closed formula for the number of digitally convex sets of a 2-tree of a given order. We show, however, that for a particular subclass, there is a nice recurrence.

Definition 3.2 (Bondy and Murty [1])**.** *Given a graph $G=(V,E)$ and a positive integer d, the d^{th} power of G is the graph $G^d=(V,E')$, such that two vertices are adjacent if and only if they are distance at most d apart in the graph G.*

The following result describes how to enumerate the digitally convex sets in the square of a path, P_n^2. We denote the vertices of this graph by $v_1, v_2, \ldots, v_n$, with

$v_iv_{i+1} \in E(P_n^2)$ and $v_jv_{j+2} \in E(P_n^2)$, for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n-2$. Figure 3.3 shows the graph P_n^2 with $n = 4, 5$ and 6. Note that P_m^2 is an induced subgraph of P_n^2 for any $m \leq n$. Markenzon, Justel and Paciornik [17] showed that these graphs are 2-trees and, in general, that the k^{th} power of a path P_n^k is a k-tree.

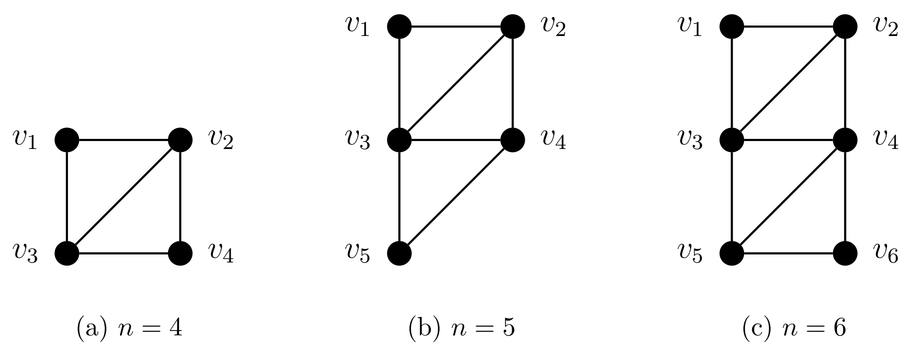


Figure 3.3: The square of a path, P_n^2

Theorem 3.3. *Let P_n^2 be the square of the path of order n. Then $n_{\mathscr{D}}(P_3^2) = 2$, $n_{\mathscr{D}}(P_4^2) = 4$, $n_{\mathscr{D}}(P_5^2) = 6$ and, for $n \geq 6$, $n_{\mathscr{D}}(P_n^2) = n_{\mathscr{D}}(P_{n-1}^2) + n_{\mathscr{D}}(P_{n-3}^2)$.*

Proof. First, we prove the initial conditions. For $n = 3$, $P_3^2 \cong K_3$ so $\mathscr{D}(P_3^2) = \{\emptyset, V(P_3^2)\}$. For $n = 4$, $\mathscr{D}(P_4^2) = \{\emptyset, \{v_1\}, \{v_4\}, V(P_4^2)\}$. For $n = 5$, $\mathscr{D}(P_5^2) = \{\emptyset, \{v_1\}, \{v_5\}, \{v_1, v_2\}, \{v_4, v_5\}, V(P_5^2)\}$. Thus, $n_{\mathscr{D}}(P_3^2) = 2$, $n_{\mathscr{D}}(P_4^2) = 4$, and $n_{\mathscr{D}}(P_5^2) = 6$.

Now suppose $n \geq 6$. We begin by showing that $n_{\mathscr{D}}(P_n^2) \geq n_{\mathscr{D}}(P_{n-1}^2) + n_{\mathscr{D}}(P_{n-3}^2)$. Let $S \in \mathscr{D}(P_{n-1}^2)$. If $v_{n-1} \in S$, then $N_{P_n^2}[v_n] \subseteq N_{P_n^2}[S]$. So $S \cup \{v_n\}$ is digitally convex in P_n^2.

If $v_{n-1} \notin S$ and $v_{n-3} \notin N_{P_{n-1}^2}[S]$, then $v_{n-3} \notin N_{P_n^2}[S \cup \{v_n\}]$. Then v_{n-3} is a private neighbour for itself, as well as for v_{n-2} and v_{n-1} with respect to $S \cup \{v_n\}$ in P_n^2. Thus, $S \cup \{v_n\}$ is digitally convex in P_n^2.

If $v_{n-1} \notin S$ and $v_{n-3} \in N_{P_{n-1}^2}[S]$, then it must be the case that $v_{n-3}, v_{n-2} \notin S$ because both vertices dominate $N[v_{n-1}]$ in P_{n-1}^2. Thus, in P_n^2, the vertex v_n is its own private neighbour with respect to S, so S is digitally convex in P_n^2.

Now, let $S \in \mathscr{D}(P^2_{n-3})$. In P^2_n, the vertex v_n is a private neighbour for itself, as well as for v_{n-1} and v_{n-2} with respect to S. Thus, S is digitally convex in P^2_n. Note that this case gives different digitally convex sets than the previous case. In P^2_{n-1}, if $v_{n-4} \in S$ or $v_{n-5} \in S$ and $v_{n-1}, v_{n-2}, v_{n-3} \notin S$, then $v_{n-3} \in N_{P^2_{n-1}}[S] - S$ with the vertex v_{n-1} as a private neighbour. These are the digitally convex sets counted in the previous case. However, it is impossible to have a digitally convex set S in P^2_{n-3} with $v_{n-3} \in N_{P^2_{n-3}}[S] - S$, because $N[v_{n-3}]$ is dominated by both neighbours of v_{n-3}. So the sets counted in the previous case are not counted again here.

Each set of $\mathscr{D}(P^2_{n-1}) \cup \mathscr{D}(P^2_{n-3})$ is associated in a one-to-one manner with a set in $\mathscr{D}(P^2_n)$. So $n_{\mathscr{D}}(P^2_n) \geq n_{\mathscr{D}}(P^2_{n-1}) + n_{\mathscr{D}}(P^2_{n-3})$. Now, we show the reverse inequality.

Let $S \in \mathscr{D}(P^2_n)$. If $v_n \in S$, then each vertex $v_i \in V(P^2_n) - S$ has a private neighbour with respect to S that is in $V(P^2_{n-1})$. Thus, $S - \{v_n\}$ is digitally convex in P^2_{n-1}.

If $v_n \notin S$ and $v_{n-3} \in N_{P^2_n}[S] - S$, then it must be the case that $v_{n-3}, v_{n-2}, v_{n-1} \notin S$, as the vertices v_{n-1} and v_{n-2} both dominate $N[v_n]$ in P^2_n. Thus, $v_{n-1} \notin N_{P^2_n}[S]$ and is a private neighbour with respect to S in P^2_{n-1} for all of the vertices in its closed neighbourhood. So S is digitally convex in P^2_{n-1}.

If $v_n \notin S$ and $v_{n-3} \notin N_{P^2_n}[S] - S$, then each vertex $v_i \in V(P^2_n) - (S \cup \{v_{n-2}, v_{n-1}, v_n\})$ has a private neighbour with respect to S that is in P^2_{n-3}. The vertices v_{n-1} and v_{n-2} each dominate $N[v_n]$ in P^2_n, so they are not in S. Thus, S is also digitally convex in P^2_{n-3}.

Since each set in $\mathscr{D}(P^2_n)$ has a corresponding set in either $\mathscr{D}(P^2_{n-1})$ or $\mathscr{D}(P^2_{n-3})$, we have $n_{\mathscr{D}}(P^2_n) = n_{\mathscr{D}}(P^2_{n-1}) + n_{\mathscr{D}}(P^2_{n-3})$. □

The proof of Theorem 3.3 also describes a method for generating the digitally convex sets of P^2_n from those of P^2_{n-1} and P^2_{n-3}, or vice versa. Here, we give an algorithm for generating the digitally convex sets of P^2_n, which follows the proof above.

Algorithm 3. *Generating the collection $\mathscr{D}(P_n^2)$ of all digitally convex sets of the square of the path of order $n \geq 3$.*

1. *If $n = 3$, then $\mathscr{D}(P_n^2) = \{\emptyset, V(P_n^2)\}$.*

2. *If $n = 4$, then $\mathscr{D}(P_n^2) = \{\emptyset,\ \{v_1\},\ \{v_4\},\ V(P_n^2)\}$.*

3. *If $n = 5$, then $\mathscr{D}(P_n^2) = \{\emptyset,\ \{v_1\},\ \{v_5\},\ \{v_1, v_2\},\ \{v_4, v_5\},\ V(P_n^2)\}$.*

4. *Suppose $n > 5$. Use the algorithm to generate $\mathscr{D}(P_{n-3}^2)$ and $\mathscr{D}(P_{n-1}^2)$. Obtain $\mathscr{D}(P_n^2)$ as follows: Set $\mathscr{S}_n = \emptyset$.*

   (a) For each $S \in \mathscr{D}(P_{n-1}^2)$

      (i) if $v_{n-1} \in S$, then add $S \cup \{v_n\}$ to $\mathscr{S}_n$.

      (ii) if $v_{n-1} \notin S$ and $v_{n-3} \notin N_{P_{n-1}^2}[S]$, then add $S \cup \{v_n\}$ to $\mathscr{S}_n$.

      (iii) if $v_{n-1} \notin S$ and $v_{n-3} \in N_{P_{n-1}^2}[S]$, then add S to $\mathscr{S}_n$.

   (b) For each $S \in \mathscr{D}(P_{n-3}^2)$, add S to $\mathscr{S}_n$.

   (c) Then, $\mathscr{D}(P_n^2) = \mathscr{S}_n$.

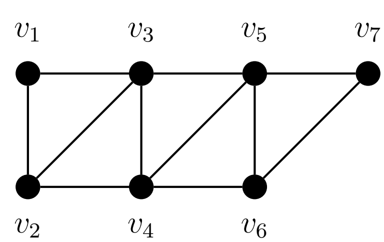


Figure 3.4: P_7^2

As an example of step 4 in Algorithm 3, consider P_7^2, shown in Figure 3.4. To generate $\mathscr{D}(P_7^2)$, we require the digitally convex sets of P_4^2 and P_6^2.

$$\mathscr{D}(P_4^2) = \{\emptyset, \{v_1\}, \{v_4\}, \{v_1, v_2, v_3, v_4\}\}$$

$\mathscr{D}(P_6^2) = \{\emptyset, \{v_1\}, \{v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_6\}, \{v_5, v_6\}, \{v_4, v_5, v_6\}, \{v_1, v_2, v_3, v_4, v_5, v_6\}\}$

The digitally convex sets in P_6^2 that satisfy case 4(a)(i) are $\{v_6\}$, $\{v_5, v_6\}$, $\{v_4, v_5, v_6\}$ and $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Thus, Algorithm 3 generates, relative to these sets, the digitally convex sets $\{v_6, v_7\}$, $\{v_5, v_6, v_7\}$, $\{v_4, v_5, v_6, v_7\}$, and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ for P_7^2.

The digitally convex sets in P_6^2 that satisfy case 4(a)(ii) are $\emptyset$ and $\{v_1\}$. Thus, Algorithm 3 generates, relative to these sets, the digitally convex sets $\{v_7\}$ and $\{v_1, v_7\}$ for P_7^2.

Finally, the digitally convex sets in P_6^2 that satisfy 4(a)(iii) and those of P_4^2 that satisfy 4(b) of the algorithm give rise to the digitally convex sets $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\emptyset$, $\{v_1\}$, $\{v_4\}$ and $\{v_1, v_2, v_3, v_4\}$ for P_7^2.

This gives $\mathscr{D}(P_7^2) = \{\emptyset$, $\{v_1\}$, $\{v_1, v_2\}$, $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_3, v_4\}$, $\{v_7\}$, $\{v_6, v_7\}$, $\{v_5, v_6, v_7\}$, $\{v_4, v_5, v_6, v_7\}$, $\{v_1, v_7\}$, $\{v_4\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}\}$.

3.1.1 Upper Bound

As with trees, it is not the case that all 2-trees of order n have the same number of digitally convex sets. For example, the 2-tree in Figure 3.5 has ten digitally convex sets, while P_6^2 has only eight digitally convex sets. This difference means we cannot construct a formula for the number of digitally convex sets in a 2-tree of order n, but we can construct upper and lower bounds on the number of digitally convex sets.

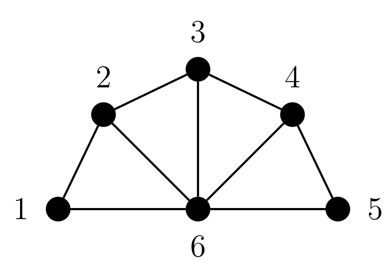


Figure 3.5: $\mathscr{D}(G) = \{\emptyset$, $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, $\{3\}$, $\{5\}$, $\{4, 5\}$, $\{3, 4, 5\}$, $\{1, 5\}$, $V(G)\}$

Theorem 3.4. *Let G be a 2-tree of order n. Then $n_{\mathscr{D}}(G) \leq 2^{n-2}$.*

Proof. Let v be a vertex of degree 2 in G, with neighbours u and w. Let $uv = e_1$ and $vw = e_2$, as shown in Figure 3.6.

Claim: $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{e_1, e_2\})$.

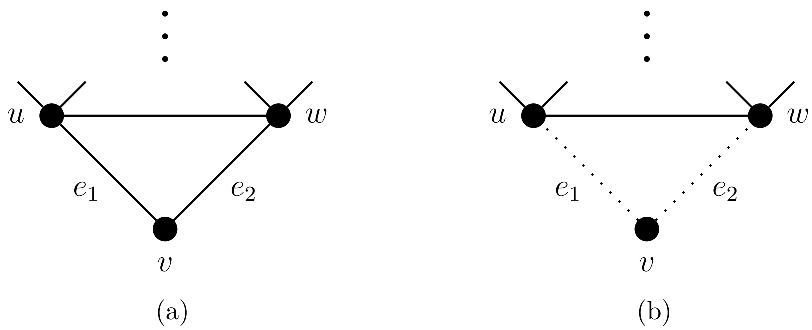


Figure 3.6: Remove edges e_1 and e_2 incident with v

We show that there is an injection from $\mathscr{D}(G)$ to $\mathscr{D}(G - \{e_1, e_2\})$. Let $S \in \mathscr{D}(G)$.

If $v \in S$, then each vertex $x \in V(G) - S$ has a private neighbour with respect to S in $G - \{e_1, e_2\}$. So S is digitally convex in $G - \{e_1, e_2\}$.

If $v \notin S$, then $u, w \notin S$ because each of these vertices dominates $N_G[v]$. If $(N_G[u] \cup N_G[w]) - \{v\} \subseteq N_G[S]$, then $u, w \in N_G[S] - S$ and v is the only private neighbour of u and of w with respect to S. Then the set $S \cup \{u, w\}$ is digitally convex in $G - \{e_1, e_2\}$. Similarly, if $N_G[u] - \{v\} \subseteq N_G[S]$ and $N_G[w] - \{v\} \nsubseteq N_G[S]$, then $S \cup \{u\}$ is digitally convex in $G - \{e_1, e_2\}$. If $N_G[w] - \{v\} \subseteq N_G[S]$ and $N_G[u] - \{v\} \nsubseteq N_G[S]$, then $S \cup \{w\}$ is digitally convex in $G - \{e_1, e_2\}$. If both u and w have a private neighbour in $V(G) - \{v\}$ with respect to S, then S is digitally convex in $G - \{e_1, e_2\}$. This completes the proof of the claim.

Thus, $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{e_1, e_2\})$. Repeat this process, removing the edges incident with a vertex of degree 2, until the remaining graph has $n - 2$ components: K_3 and $n - 3$ isolated vertices. Each component has two digitally convex sets, as each component is a clique. Overall, this gives 2^{n-2} digitally convex sets. Applying the above inequality each time a pair of edges is removed, we get $n_{\mathscr{D}}(G) \leq 2^{n-2}$. □

In Theorem 2.5, Lafrance, Oellermann and Pressey [15] showed that $n_{\mathscr{D}}(T) \leq 2^{n-1}$ for a tree T of order n, a bound that is very similar to the bound for 2-trees in Theorem 3.4. The subclass of 2-trees that attain the upper bound also have a structure that is very similar to that of the stars, $K_{1,n-1}$.

Proposition 3.5. *The upper bound given in Theorem 3.4 is attained by the graph* $K_2 + \overline{K}_{n-2}$.

Proof. Let x, y be the vertices of the K_2 and let $v_1, v_2, \ldots, v_{n-2}$ be the remaining vertices. See Figure 3.7 for an example with $n = 7$. Let $S \subsetneq \{v_1, v_2, \ldots, v_{n-2}\}$.

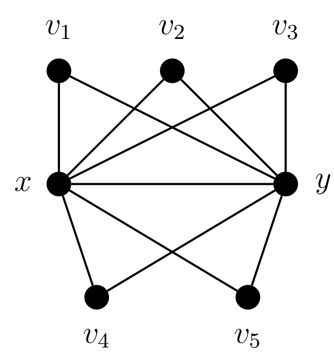


Figure 3.7: The graph $K_2 + \overline{K}_5$

Claim: S is digitally convex in $K_2 + \overline{K}_{n-2}$. There is some $v_i \notin S$ and since $v_iv_j \notin E(K_2 + \overline{K}_{n-2})$ for $i \neq j$, we have $v_i \notin N[S]$. This vertex v_i is a private neighbour for itself, as well as for both x and y with respect to S. Thus, S is a digitally convex set.

There are $2^{n-2} - 1$ such sets S. Since both x and y are universal vertices, the only digitally convex set containing either of these vertices is the set $V(K_2 + \overline{K}_{n-2})$. Similarly, $\{v_1, v_2, \ldots, v_{n-2}\}$ forms a dominating set in $K_2 + \overline{K}_{n-2}$. So the only digitally convex set containing all of these vertices is the entire vertex set. In total, this gives 2^{n-2} digitally convex sets in $K_2 + \overline{K}_{n-2}$. □

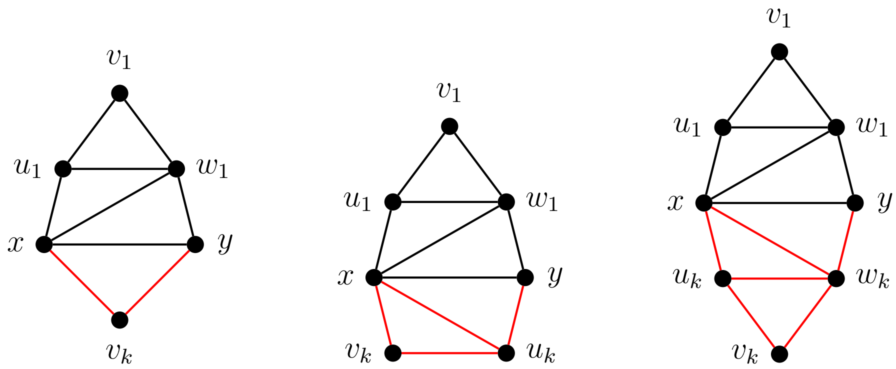
3.1.2 Lower Bound

In this section, we conjecture a lower bound on the number of digitally convex sets in 2-trees that is very similar to the lower bound on the number of digitally convex sets in trees, stated in Theorem 2.5. We then describe a method of proving this conjecture by dividing all possible 2-trees into several cases and, for each of these cases, showing a relationship between the number of digitally convex sets in a 2-tree of order n and the number in a 2-tree of order $n-3$. Several of these relationships are stated as lemmas, with the proofs of these lemmas given at the end of this section. However, the relationship for three of these cases remain conjectures. Proving these remaining cases would complete the proof of the lower bound on the number of digitally convex sets in a 2-tree.

Recall that a spiderstar of order $n = 2k+1$ is obtained from the star $K_{1,k}$ by subdividing each edge exactly once, and the spiderstar of order $n = 2k$ is obtained from $K_{1,k}$ by subdividing all but one edge exactly once. Before stating the main conjecture in this section, we define a subclass of 2-trees that is similar to the spiderstars, the 2-spiderstars, $S_{2,n}$. The 2-spiderstar of order n is constructed in the following way:

1. begin with a K_2 with vertices x, y.
2. for $i = 1, 2, \ldots, \lfloor \frac{n-2}{3} \rfloor$, add vertices w_i, u_i, v_i and edges $xw_i, yw_i, xu_i, w_iu_i, w_iv_i, u_iv_i$.
3. if $(n-2) \equiv 0 \pmod 3$, then let $k = \lceil \frac{n-2}{3} \rceil$.
4. if $(n-2) \equiv 1 \pmod 3$, then add a vertex v_k (where $k = \lceil \frac{n-2}{3} \rceil$) and edges xv_k, yv_k.
5. if $(n-2) \equiv 2 \pmod 3$, then add vertices u_k, v_k (where $k = \lceil \frac{n-2}{3} \rceil$) and edges xu_k, yu_k, xv_k, u_kv_k.

Figure 3.8 shows the 2-spiderstars of orders 6, 7 and 8.



(a) $n \equiv 0 \pmod 3$ (b) $n \equiv 1 \pmod 3$ (c) $n \equiv 2 \pmod 3$

Figure 3.8: Construction of the 2-spiderstars, with $k = \lceil \frac{n-2}{3} \rceil$ and differences depending on n indicated by red edges

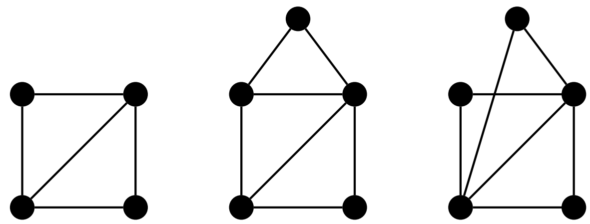
Conjecture 3.6. *Let G be a 2-tree of order $n \geq 3$. Then*

$$n_{\mathscr{D}}(G) \geq \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Moreover, this bound is attained by the 2-spiderstars.

We now show how this conjecture might be proven by induction on n, by using Lemmas 3.7 - 3.11, Lemma 3.15 and Conjectures 3.12 - 3.14. These lemmas and conjectures divide the class of 2-trees into nine subclasses. We later show, in Lemma 3.16, that the union of these subclasses gives the full class of 2-trees. If $n = 3$, then $G \cong K_3$, so $n_{\mathscr{D}}(G) = 2 = 3 \cdot 2^{\frac{3}{3}} - 4$. If $n = 4$, then G must be the 2-tree shown in Figure 3.9(a). So $n_{\mathscr{D}}(G) = 4 = 4 \cdot 2^{\frac{3}{3}} - 4$. If $n = 5$, then G must be either the 2-tree in Figure 3.9(b) or in Figure 3.9(c), which have six and eight digitally convex sets, respectively. So $n_{\mathscr{D}}(G) \geq 6 = 5 \cdot 2^{\frac{3}{3}} - 4$.

Now, suppose that there exists some $k \geq 6$ such that the result holds for 2-trees of order n, where $3 \leq n < k$. Let G be a 2-tree of order k. We now make use of the following lemmas to apply the induction hypothesis to a 2-tree of order $k - 3$. The



(a) $n = 4$ (b) $n = 5$ (c) $n = 5$

Figure 3.9: All 2-trees of order 4 and 5

proofs of these lemmas are given at the end of this section.

Lemma 3.7. *Let G be a 2-tree of order at least 3 and let $xy \in E(G)$. Construct the 2-tree G_1 by adding the vertices v_1, v_2, v_3 and edges xv_1, yv_1, xv_2, v_1v_2, v_1v_3, v_2v_3 to G (see Figure 3.10). Then $n_{\mathscr{D}}(G_1) \geq 2n_{\mathscr{D}}(G) + 4$.*

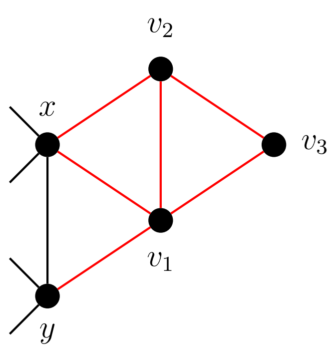


Figure 3.10: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_1

Lemma 3.8. *Let G be a 2-tree of order at least 3 and let $xy \in E(G)$. Construct the 2-tree G_2 by adding the vertices v_1, v_2, v_3 and edges v_1x, v_1y, v_1v_2, v_2x, v_2v_3, v_3x to G (see Figure 3.11). Then $n_{\mathscr{D}}(G_2) \geq 2n_{\mathscr{D}}(G) + 4$.*

Lemma 3.9. *Let G be a 2-tree of order at least 3 and let $xy \in E(G)$. Construct the 2-tree G_3 by adding the vertices v_1, v_2, v_3 and edges v_1x, v_1y, v_1v_2, v_1v_3, v_2x, v_3y to G (see Figure 3.12). Then $n_{\mathscr{D}}(G_3) \geq 2n_{\mathscr{D}}(G) + 4$.*

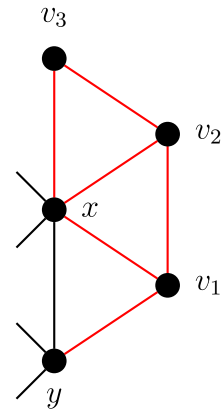


Figure 3.11: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_2

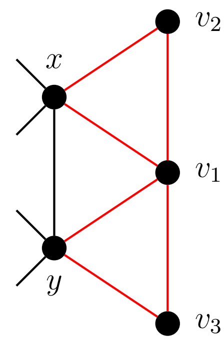


Figure 3.12: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_3

Lemma 3.10. *Let G be a 2-tree of order at least 3 and let $xy \in E(G)$. Construct the 2-tree G_4 by adding the vertices v_1, v_2, v_3 and edges v_1x, v_1y, v_2x, v_2y, v_2v_3, v_3x to G (see Figure 3.13). Then $n_{\mathscr{D}}(G_4) \geq 2n_{\mathscr{D}}(G) + 4$.*

Lemma 3.11. *Let G be a 2-tree of order at least 4, with x a vertex of degree 2 or 3 in G. In the first case, let $N_G(x) = \{y, z\}$ and, in the second case, let $N_G(x) = \{w, y, z\}$ with $yz \notin E(G)$. Construct the 2-tree G_5 by adding the vertices v_1, v_2, v_3 and edges v_1v_2, v_1x, v_2x, v_2y, v_3x, v_3z to G (see Figure 3.14). Then $n_{\mathscr{D}}(G_5) \geq 2n_{\mathscr{D}}(G) + 4$.*

Suppose G can be constructed from a 2-tree of order $k-3$ by the addition of vertices v_1, v_2, and v_3 using the process described in one of Lemma 3.7 - Lemma 3.11. Let $\mathscr{G}_1$ be the collection of 2-trees that can be constructed from a 2-tree of order $k-3$

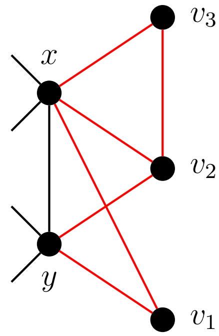


Figure 3.13: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_4

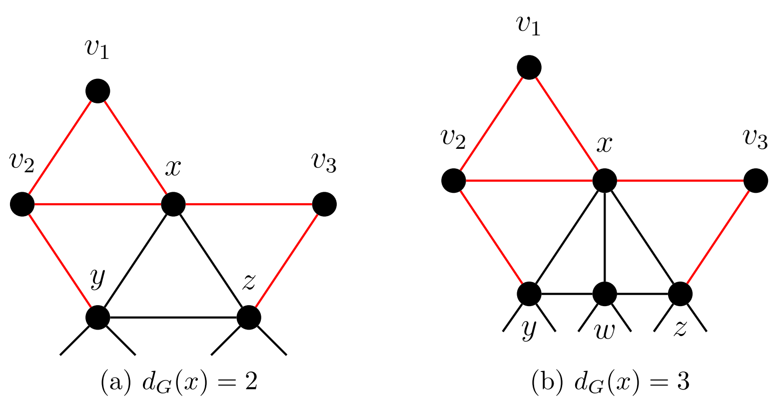


Figure 3.14: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_5

using the process in Lemma 3.7, let $\mathscr{G}_2$ be the collection of 2-trees constructed using the process in Lemma 3.8, and so on. Then $\mathscr{G}_5$ is the collection of 2-trees constructed using the process in Lemma 3.11.

Then, by the lemmas stated above, we have $2n_{\mathscr{D}}(G - \{v_1, v_2, v_3\}) + 4 \leq n_{\mathscr{D}}(G)$. By the induction hypothesis, we have

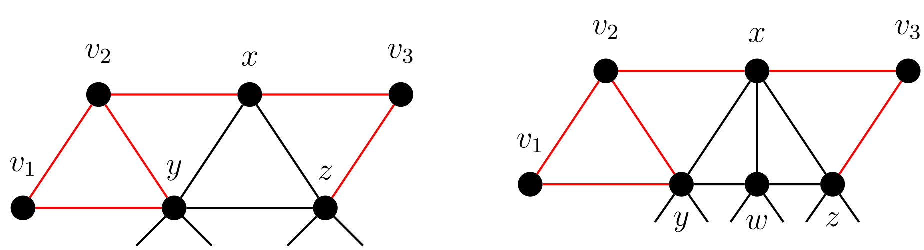
$$n_{\mathscr{D}}(G) \geq 2n_{\mathscr{D}}(G - \{v_1, v_2, v_3\}) + 4 \geq \begin{cases} 2(3 \cdot 2^{\frac{n}{3}-1} - 4) + 4, & \text{if } n \equiv 0 \pmod 3 \\ 2(4 \cdot 2^{\frac{n-1}{3}-1} - 4) + 4, & \text{if } n \equiv 1 \pmod 3 \\ 2(5 \cdot 2^{\frac{n-2}{3}-1} - 4) + 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$$= \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

as desired.

We now state three conjectures that, if true, would allow for the completion of the proof of Conjecture 3.6.

Conjecture 3.12. *Let G be a 2-tree of order at least 4, with x a vertex of degree 2 or 3 in G. In the first case, let $N_G(x) = \{y, z\}$ and, in the second case, let $N_G(x) = \{w, y, z\}$ with $yz \notin E(G)$. Construct the 2-tree G_6 by adding the vertices v_1, v_2, v_3 and edges v_1v_2, v_1y, v_2x, v_2y, v_3x, v_3z to G (see Figure 3.15). Then, $n_{\mathscr{D}}(G_6) \geq 2n_{\mathscr{D}}(G) + 4$.*



(a) $d_G(x) = 2$ (b) $d_G(x) = 3$

Figure 3.15: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_6

Conjecture 3.13. *Let G be a 2-tree of order at least 5, with w a vertex of degree 2, adjacent to a vertex x of degree at least 3 and a vertex z. Let y be another neighbour of x.*

Construct the 2-tree G_7 by adding the vertices v_1, v_2, v_3 and edges v_1v_2, v_1y, v_2y, v_2x, v_3z, v_3w to G (see Figure 3.16(a)).

Similarly, construct the 2-tree G_7' by adding the vertices v_1', v_2', v_3' and edges $v_1'v_2'$, $v_1'x$, $v_2'x$, $v_2'y$, $v_3'w$, $v_3'z$ to G (see Figure 3.16(b)).

Construct the 2-tree G_7'' by adding the vertices v_1'', v_2'', v_3'' and edges $v_1''v_2''$, $v_1''x$, $v_2''x$, $v_2''y$, $v_3''x$, $v_3''w$ to G (see Figure 3.16(c)).

Then $n_{\mathscr{D}}(G_7), n_{\mathscr{D}}(G_7'), n_{\mathscr{D}}(G_7'') \geq 2n_{\mathscr{D}}(G) + 4$.

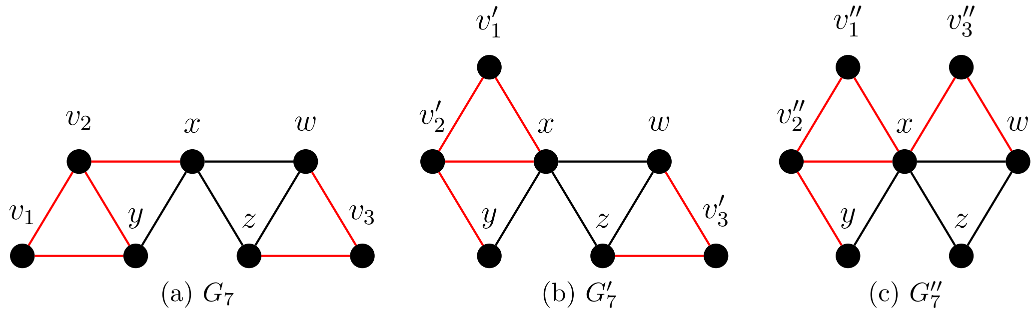


Figure 3.16: Vertices v_1, v_2, v_3 and the red edges are added to G to form G_7, G_7' and G_7''

Conjecture 3.14. *Let G be a 2-tree of order at least 5, with x a vertex of degree at least 3. Let w, y and z be neighbours of x.*

Construct G_8 by adding the vertices v_1, v_2, v_3 and edges v_1x, v_1y, v_2x, v_2z, v_3x, v_3w to G (see Figure 3.17(a)). Then $n_{\mathscr{D}}(G_8) \geq 2n_{\mathscr{D}}(G) + 4$.

Construct G_8' by adding the vertices v_1', v_2', v_2' and u and edges ux, uy, $v_1'u$, $v_1'y$, $v_2'x$, $v_2'z$, $v_3'x$, $v_3'z$ to G (see Figure 3.17(b)). Then, $n_{\mathscr{D}}(G_8') \geq 2n_{\mathscr{D}}(G_8' - \{v_1', v_2', v_3'\}) + 4$.

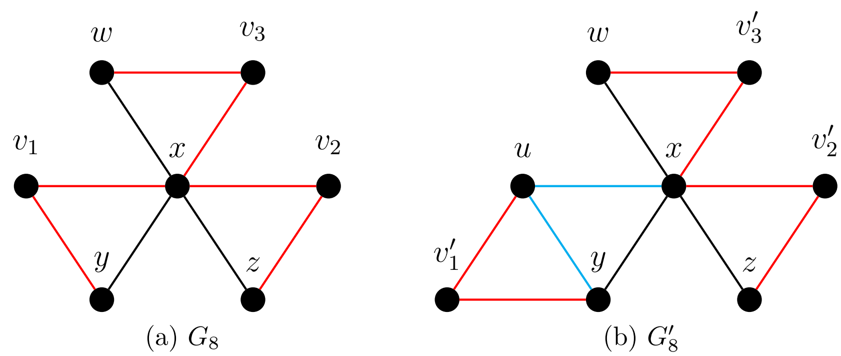


Figure 3.17: The vertices incident with the red edges are added to form G_8 and G_8'. The vertex u and blue edges are added to form $G_8' - \{v_1', v_2', v_3'\}$

Now, suppose G can be constructed from a 2-tree of order $k-3$ by the addition of vertices v_1, v_2, and v_3 using the process described in one of Conjecture 3.12 - Conjecture 3.14. As above, we let $\mathscr{G}_6$ be the collection of 2-trees that can be constructed from a 2-tree of order $k-3$ using the process described in Conjecture 3.12, $\mathscr{G}_7$ the collection of 2-trees constructed using one of the processes in Conjecture 3.13, and $\mathscr{G}_8$ the collection of 2-trees constructed using one of the processes in Conjecture 3.14.

Then, provided each of the conjectures above holds, we have $2n_{\mathscr{D}}(G-\{v_1,v_2,v_3\})+4 \leq n_{\mathscr{D}}(G)$. As above, by the induction hypothesis, we have

$$n_{\mathscr{D}}(G) \geq 2n_{\mathscr{D}}(G-\{v_1,v_2,v_3\})+4 \geq \begin{cases} 2(3\cdot 2^{\frac{n}{3}-1}-4)+4, & \text{if } n \equiv 0 \pmod 3 \\ 2(4\cdot 2^{\frac{n-1}{3}-1}-4)+4, & \text{if } n \equiv 1 \pmod 3 \\ 2(5\cdot 2^{\frac{n-2}{3}-1}-4)+4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

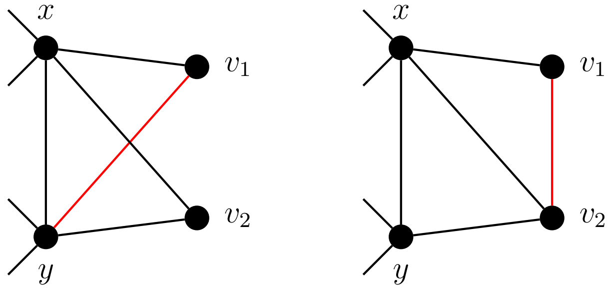
$$= \begin{cases} 3\cdot 2^{\frac{n}{3}}-4, & \text{if } n \equiv 0 \pmod 3 \\ 4\cdot 2^{\frac{n-1}{3}}-4, & \text{if } n \equiv 1 \pmod 3 \\ 5\cdot 2^{\frac{n-2}{3}}-4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

as desired.

Finally, suppose G has two vertices a and b, both of degree 2, with the same open neighbourhoods in G. Let $\mathscr{G}_9$ be the collection of 2-trees satisfying these conditions. Then, we can construct another 2-tree of order k with the same vertex set and at most $n_{\mathscr{D}}(G)$ digitally convex sets using the process described in the following lemma.

Lemma 3.15. *Let G be a 2-tree of order $n \geq 4$, containing two vertices v_1 and v_2 of degree 2, both adjacent to vertices x and y. Construct the 2-tree G^* by removing the edge yv_1 from G and adding the edge v_1v_2 (see Figure 3.18). Then, $n_{\mathscr{D}}(G^*) \leq n_{\mathscr{D}}(G)$.*

Once we apply the process in Lemma 3.15 to get the 2-tree G^*, either $G^* \notin \mathscr{G}_9$ or $G^* \in \mathscr{G}_9$ and Lemma 3.15 can be applied again until this is no longer the case.



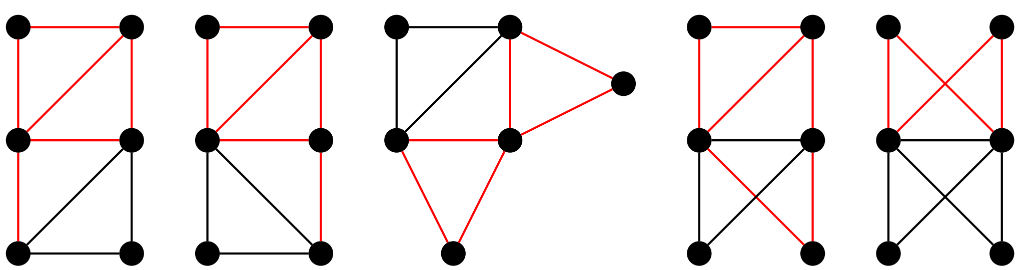
(a) G (b) G^*

Figure 3.18: The edges removed from G and added to form G^* are highlighted in red

Now, we show that if $G^* \notin \mathscr{G}_9$, then it must be the case that $G^* \in \mathscr{G}_i$ for $1 \leq i \leq 8$. Moreover, we show that any 2-tree of order k must be in one of the collections $\mathscr{G}_i$ with $i = 1, 2, \ldots, 9$.

Lemma 3.16. *Let G be a 2-tree of order $n \geq 6$. Then $G \in \mathscr{G}_i$ for some $i = 1, 2, \ldots, 9$.*

Proof. We prove this result by induction on n. First, suppose $n = 6$. Figure 3.19 shows all five non-isomorphic 2-trees of order 6. The edges highlighted in red in Figure 3.19(a)-(e) show that $G_1 \in \mathscr{G}_1$, $G_2 \in \mathscr{G}_2$, $G_3 \in \mathscr{G}_3$, $G_4 \in \mathscr{G}_4$ and $G^* \in \mathscr{G}_9$. Therefore, the result holds for $n = 6$.



(a) G_1 (b) G_2 (c) G_3 (d) G_4 (e) G^*

Figure 3.19: All non-isomorphic 2-trees of order 6

Now suppose that there exists an $\ell \geq 7$ such that the statement holds for all 2-trees of order $6 \leq n < \ell$. Consider a 2-tree G of order ℓ, with v a vertex of degree 2 in G. It is clear from the construction of a 2-tree that such a vertex v exists. By the induction hypothesis, the 2-tree $G - v$ is in one of the collections $\mathscr{G}_i$ for $i = 1, 2, \ldots, 9$. We now consider each of these cases separately and show that any possible addition of v to form the 2-tree G also results in a 2-tree that is in one of the collections $\mathscr{G}_i$, $1 \leq i \leq 9$.

Suppose $G - v \in \mathscr{G}_1$. Let the vertices v_1, v_2, v_3, x, y be as in Figure 3.10. Now, we examine each possible neighbourhood of v in G. If $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_9$, as both v and v_3 have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_2, v_3\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_1, v_3\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_3$. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_4$. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_5$. For any other neighbourhood of v, $G \in \mathscr{G}_1$, as the vertices v_1, v_2, v_3 have the same degree as in $G - v$.

Now, suppose $G - v \in \mathscr{G}_2$. Let the vertices v_1, v_2, v_3, x, y be as in Figure 3.11. Now, we examine each possible neighbourhood of v in G. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_9$, as the vertices v and v_3 are both of degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_3, v_2\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_3, x\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, v_1\}$, then $G \in \mathscr{G}_3$. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_4$. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_6$. For any other neighbourhood of v, $G \in \mathscr{G}_2$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Now, suppose $G - v \in \mathscr{G}_3$. Let the vertices v_1, v_2, v_3, x, y be as in Figure 3.12. Now, we examine each possible neighbourhood of v in G. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_9$, as the vertices v and v_2 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_9$, as the vertices v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_3, y\}$ or $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_6$. If $N_G(v) = \{v_1, v_3\}$ or $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_5$. For any other neighbourhood of v, $G \in \mathscr{G}_3$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Now, suppose $G - v \in \mathscr{G}_4$. Let the vertices v_1, v_2, v_3, x, y be as in Figure 3.13. We examine each possible neighbourhood of v in G. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_9$, as the vertices v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_2, v_3\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_3, x\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, y\}$, then $G \in \mathscr{G}_3$. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_5$. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_5$. For any other neighbourhood of v, $G \in \mathscr{G}_4$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Suppose $G - v \in \mathscr{G}_5$. Let the vertices $v_1, v_2, v_3, w, x, y, z$ be as in Figure 3.14. Now, we examine each possible neighbourhood of v in G. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_9$, as v and v_1 both have degree 2 and have the same open neighbourhood in G. Similarly, if $N_G(v) = \{x, z\}$, then $G \in \mathscr{G}_9$, as v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, y\}$, then $G \in \mathscr{G}_3$. If $N_G(v) = \{x, y\}$, then $G \in \mathscr{G}_4$. If $N_G(v) = \{v_3, x\}$, then $G \in \mathscr{G}_5$ in the case that $d(x) = 5$ and $G \in \mathscr{G}_7$ in the case that $d(x) = 6$. If $N_G(v) = \{v_3, z\}$, then $G \in \mathscr{G}_6$ in the case that $d(x) = 5$ and $G \in \mathscr{G}_7$ in the case that $d(x) = 6$. In the case that $d(x) = 6$, then x has another neighbour, w. If $N_G(v) = \{x, w\}$, then $G \in \mathscr{G}_8$. For any other neighbourhood of v, $G \in \mathscr{G}_5$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Suppose $G - v \in \mathscr{G}_6$. Let the vertices $v_1, v_2, v_3, w, x, y, z$ be as in Figure 3.15. Now, we examine each possible neighbourhood of v in G. If $N_G(v) = \{v_2, y\}$, then $G \in \mathscr{G}_9$, as v and v_1 both have degree 2 and have the same open neighbourhood in G. Similarly, if $N_G(v) = \{x, z\}$, then $G \in \mathscr{G}_9$, as v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_3$. If $N_G(v) = \{x, y\}$, then $G \in \mathscr{G}_4$. If $N_G(v) = \{v_3, x\}$, then $G \in \mathscr{G}_6$ in the case that $d(x) = 4$ and $G \in \mathscr{G}_7$ in the case that $d(x) = 5$. If $N_G(v) = \{v_3, z\}$, then $G \in \mathscr{G}_7$. In the case that $d(x) = 5$, then x has another neighbour, w. If $N_G(v) = \{x, w\}$, then $G \in \mathscr{G}_8$. For any other neighbourhood of v, $G \in \mathscr{G}_6$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Suppose $G - v \in \mathscr{G}_7$. Let the vertices v_2, w, x, y, z be as in Figure 3.16. There are two cases for the neighbourhoods of each of the vertices v_1 and v_3 of degree 2. In the first case for the neighbourhood of v_1, we have $v_1v_2, v_1x \in E(G - v)$. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_9$, as v and v_1 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_1, x\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, y\}$, then $G \in \mathscr{G}_3$.

In the second case, we have $v_1v_2, v_1y \in E(G - v)$. If $N_G(v) = \{v_2, y\}$, then $G \in \mathscr{G}_9$, as v and v_1 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_1, v_2\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_1, y\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{v_2, x\}$, then $G \in \mathscr{G}_3$.

Similarly, in the first case for the neighbourhood of v_3, we have $v_3w, v_3z \in E(G - v)$. If $N_G(v) = \{w, z\}$, then $G \in \mathscr{G}_9$, as v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_3, w\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_3, z\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{w, x\}$, then $G \in \mathscr{G}_3$.

In the second case, we have $v_3w, v_3x \in E(G - v)$. If $N_G(v) = \{w, x\}$, then $G \in \mathscr{G}_9$, as v and v_3 both have degree 2 and have the same open neighbourhood in G. If $N_G(v) = \{v_3, w\}$, then $G \in \mathscr{G}_1$. If $N_G(v) = \{v_3, x\}$, then $G \in \mathscr{G}_2$. If $N_G(v) = \{w, z\}$, then $G \in \mathscr{G}_3$.

For any of the above cases, if $N_G(v) = \{x, y\}$ or $N_G(v) = \{x, z\}$, then $G \in \mathscr{G}_4$. For any other neighbourhood of v, $G \in \mathscr{G}_7$, as the vertices v_1, v_2, v_3 each have the same degree as in $G - v$.

Now, suppose $G - v \in \mathscr{G}_8$. Let the vertices v_2, v_3, w, x, y, z be as in Figure 3.17. Now, we have two cases for the neighbourhood v_1. In the first case, we have $v_1x, v_1y \in$

$E(G-v)$. If $N_G(v)=\{w,x\}$, $N_G(v)=\{x,y\}$, or $N_G(v)=\{x,z\}$, then $G\in\mathscr{G}_9$, as v and one of v_1,v_2,v_3 have degree 2 and have the same open neighbourhood in G. If $N_G(v)=\{v_1,x\}$, $N_G(v)=\{v_2,x\}$, or $N_G(v)=\{v_3,x\}$, then $G\in\mathscr{G}_8$, as v and two of v_1,v_2,v_3 have degree 2 and have x as a neighbour. If $N_G(v)=\{v_1,y\}$, $N_G(v)=\{v_2,z\}$, or $N_G(v)=\{v_3,w\}$, then $G\in\mathscr{G}_8$.

In the second case, we have a vertex u of degree 3 in $G-v$ and edges $v_1u, v_1y, ux, uy\in E(G-v)$. If $N_G(v)=\{u,y\}$, $N_G(v)=\{w,x\}$, or $N_G(v)=\{x,z\}$, then $G\in\mathscr{G}_9$, as v and one of v_1,v_2,v_3 have degree 2 and the same open neighbourhood in G. If $N_G(v)=\{v_1,u\}$, then $G\in\mathscr{G}_1$. If $N_G(v)=\{v_1,y\}$, then $G\in\mathscr{G}_2$. If $N_G(v)=\{u,x\}$, then $G\in\mathscr{G}_4$. If $N_G(v)=\{v_3,w\}$, $N_G(v)=\{v_3,x\}$, $N_G(v)=\{v_2,x\}$, or $N_G(v)=\{v_2,z\}$, then $G\in\mathscr{G}_7$.

For any other neighbourhood of v, in either of the above cases, $G\in\mathscr{G}_8$, as the vertices v_1,v_2,v_3 each have the same degree as in $G-v$.

Finally, suppose $G-v\in\mathscr{G}_9$. Let v_1,v_2,x,y be as in Figure 3.18. Now, we examine each possible neighbourhood of v in G. If $N_G(v)=\{v_1,x\},\{v_1,y\},\{v_2,x\}$, or $\{v_2,y\}$, then $G\in\mathscr{G}_4$. For any other neighbourhood of v, $G\in\mathscr{G}_9$, as both v_1 and v_2 have degree 2 and the same open neighbourhood in G. $\square$

Thus, every 2-tree G of order $k\geq 6$ satisfies the conditions of one of the cases considered above and, provided Conjectures 3.12, 3.13 and 3.14 all hold, satisfies

$$n_{\mathscr{D}}(G)\geq\begin{cases}3\cdot 2^{\frac{n}{3}}-4, & \text{if } n\equiv 0 \pmod 3\\ 4\cdot 2^{\frac{n-1}{3}}-4, & \text{if } n\equiv 1 \pmod 3\\ 5\cdot 2^{\frac{n-2}{3}}-4, & \text{if } n\equiv 2 \pmod 3\end{cases}$$

Finally, we show that the 2-spiderstars attain this conjectured lower bound.

Proposition 3.17. *Let $S_{2,n}$ be the 2-spiderstar of order $n \geq 3$. Then,*

$$n_{\mathscr{D}}(S_{2,n}) = \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \textit{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \textit{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \textit{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. We prove this result by constructing $\mathscr{D}(S_{2,n})$ for each n. Let the vertices x, y, $v_1, v_2, \ldots, v_k$, $u_1, u_2, \ldots, u_k$, $w_1, w_2, \ldots, w_k$ be as in the definition of 2-spiderstars and as in Figure 3.8. First, we note that any digitally convex set containing a vertex u_i must also contain the vertex v_i and, similarly, any digitally convex set containing w_i must also contain u_i and v_i. Furthermore, if a digitally convex set contains some u_i, then for each $1 \leq j \leq k-1$, it either contains v_j and u_j or neither of these vertices. Similarly, if a digitally convex set contains some w_i, then for each $1 \leq j \leq k-1$, it either contains all of v_j, u_j and w_j or none of these vertices.

For simplicity, let $V_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$, $U_{k-1} = \{u_1, u_2, \ldots, u_{k-1}\}$, $W_{k-1} = \{w_1, w_2, \ldots, w_{k-1}\}$. Additionally, let R_V be any subset of V_{k-1}, R_U any subset of U_{k-1} and R_W any subset of W_{k-1}.

Claim 1: The following sets are digitally convex in $S_{2,n}$ for any value of n.

(a) $S_1^* = R_V$.

(b) $S_2^* = R_U \cup \{v_i \in V_{k-1} \mid u_i \in R_U\}$.

The vertex $x \notin N[V_{k-1}]$, so x is a private neighbour with respect to S_1^* for each u_i and w_i with $v_i \in S_1^*$. Every other vertex is a private neighbour for itself with respect to S_1^*. So S_1^* is digitally convex in $S_{2,n}$.

Similarly, the vertex $y \notin N[U_{k-1}]$, so y is a private neighbour with respect to S_2^* for x and for each w_i with $u_i \in S_2^*$. Every other vertex is a private neighbour for itself with respect to S_2^*. So S_2^* is digitally convex in $S_{2,n}$. This completes the proof of Claim 1.

There are 2^{k-1} possible subsets R_V and 2^{k-1} possible subsets of R_U, giving a total of $2 \cdot 2^{k-1} - 1$ distinct digitally convex sets of these types in $S_{2,n}$ for any $n \geq 3$. Note that we subtract one to avoid counting the empty set twice, as it is a subset of both R_V and R_U.

Now, suppose $n \equiv 0 \pmod 3$. Then, $S_{2,n}$ has a vertex v_k adjacent to both x and y.

Claim 2: The following sets are digitally convex in $S_{2,n}$.

(a) $S_3^* = R_W \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(b) $S_4^* = R_W \cup \{v_k, x, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(c) $S_5^* = R_W \cup \{v_k\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

(d) $S_6^* = R_W \cup \{v_k, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

The vertex v_k is a private neighbour for both x and y with respect to S_3^*. Every other vertex not in S_3^* is a private neighbour for itself with respect to S_3^*. Therefore, S_3^* is digitally convex in $S_{2,n}$.

For each $w_i \notin R_W$, the vertex v_i is a private neighbour with respect to S_4^* for itself, as well as for w_i and u_i. Thus, S_4^* is digitally convex in $S_{2,n}$.

There must be some $w_i \notin S_5^*$, since $R_W \neq W_{k-1}$, so w_i is a private neighbour with respect to S_5^* for x and y. Every other vertex is a private neighbour for itself with respect to S_5^*. So S_5^* is digitally convex in $S_{2,n}$.

Similarly, there is some $w_i \notin S_6^*$. Then, u_i is a private neighbour with respect to S_6^* for w_i and x. For each other $w_j \notin R_W$, the vertex u_j is a private neighbour with respect to S_6^* for w_j. Every other vertex is a private neighbour for itself with respect to S_6^*. So S_6^* is digitally convex in $S_{2,n}$. This completes the proof of Claim 2.

There are 2^{k-1} digitally convex sets S_3^*. However one of these is the empty set so we will count only $2^{k-1}-1$ of these sets. There are 2^{k-1} digitally convex sets S_4^*, as there are 2^{k-1} possible sets R_W. Similarly, there are $2^{k-1}-1$ digitally convex sets S_5^* and the same number of digitally convex sets S_6^*. Overall, we get a total of $n_{\mathscr{D}}(S_{2,n}) = 6 \cdot 2^{k-1} - 4 = 3 \cdot 2^k - 4 = 3 \cdot 2^{\frac{n}{3}} - 4$ digitally convex sets when $n \equiv 0 \pmod 3$.

Suppose $n \equiv 1 \pmod 3$. Then, $S_{2,n}$ has vertices u_k and v_k.

Claim 3: The following sets are digitally convex in $S_{2,n}$.

(a) $S_3^* = R_U \cup \{v_k\} \cup \{v_i \in V_{k-1} \mid u_i \in R_U\}$.

(b) $S_4^* = R_W \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(c) $S_5^* = R_W \cup \{u_k, v_k, x, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(d) $S_6^* = R_W \cup \{y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(e) $S_7^* = R_W \cup \{u_k, v_k\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

(f) $S_8^* = R_W \cup \{u_k, v_k, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

The vertex y is a private neighbour with respect to S_3^* for itself, as well as for x, for u_k, and for every vertex w_i. Every other vertex is a private neighbour for itself with respect to S_3^*. So S_3^* is a digitally convex set in $S_{2,n}$.

Similar to the proof of Claim 2(a), the vertex u_k is a private neighbour with respect to S_4^* for x and y, and every other vertex is its own private neighbour with respect to S_4^*. So S_4^* is digitally convex in $S_{2,n}$.

The proof that S_5^* is digitally convex in $S_{2,n}$ is identical to that of Claim 2(b).

The vertex v_k is a private neighbour with respect to S_6^* for x and for u_k. For each $w_i \notin R_W$, the vertex u_i is a private neighbour with respect to S_6^* for itself and for w_i.

Every other vertex not in S_6^* is a private neighbour for itself with respect to S_6^*. So S_6^* is digitally convex in $S_{2,n}$.

The proofs that S_7^* and S_8^* are digitally convex in $S_{2,n}$ are identical to those of Claim 2(c) and Claim 2(d), respectively. This completes the proof of Claim 3.

There are 2^{k-1} digitally convex sets S_3^*, as each one corresponds to a subset of R_U. Similar to the above argument, there are 2^{k-1} digitally convex sets S_4^*, but we will count only $2^{k-1}-1$ of these to avoiding double counting the empty set. There are 2^{k-1} digitally convex sets S_5^* and the same number of digitally convex sets S_6^*, as each corresponds to a possible subset of R_W. Similarly, there are $2^{k-1}-1$ digitally convex sets S_7^* and the same number of digitally convex sets S_8^*. Overall, we get a total of $n_{\mathscr{D}}(S_{2,n}) = 8 \cdot 2^{k-1} - 4 = 4 \cdot 2^k - 4 = 4 \cdot 2^{\frac{n-1}{3}} - 4$ digitally convex sets when $n \equiv 1 \pmod 3$.

Finally, suppose $n \equiv 2 \pmod 3$. Then, $S_{2,n}$ has vertices w_k, u_k and v_k.

Claim 4: The following sets are digitally convex in $S_{2,n}$.

(a) $S_3^* = R_V \cup \{v_k\}$.

(b) $S_4^* = R_U \cup \{u_k, v_k\} \cup \{v_i \in V_{k-1} \mid u_i \in R_U\}$.

(c) $S_5^* = R_W \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(d) $S_6^* = R_W \cup \{x, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(e) $S_7^* = R_W \cup \{v_k, u_k, w_k, x, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(f) $S_8^* = R_W \cup \{y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$.

(g) $S_9^* = R_W \cup \{v_k, u_k, w_k\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

(h) $S_{10}^* = R_W \cup \{v_k, u_k, w_k, y\} \cup \{v_i \in V_{k-1}, u_i \in U_{k-1} \mid w_i \in R_W\}$, if $R_W \neq W_{k-1}$.

The vertex x is a private neighbour with respect to S_3^* for each w_i and u_i, including w_k and u_k. Similarly, the vertex y is a private neighbour with respect to S_4^* for x and for each w_i. In addition, the vertex w_k is a private neighbour with respect to S_5^* for x and y. In all three cases, every other vertex is a private neighbour for itself. So S_3^*, S_4^* and S_5^* are all digitally convex in $S_{2,n}$.

The vertex v_k is a private neighbour with respect to S_6^* for itself, as well as for u_k and w_k. Similarly, for any $w_i \notin S_6^*$, the vertex v_i is a private neighbour for itself, for u_i and for w_i. Every other vertex is a private neighbour for itself with respect to S_6^*. So S_6^* is digitally convex in $S_{2,n}$.

The proof that S_7^* is digitally convex in $S_{2,n}$ is identical to that of Claim 2(b).

The vertex u_k is a private neighbour with respect to S_8^* for itself, as well as for x and for w_k. For each $w_i \notin S_8^*$, the vertex u_i is a private neighbour for itself and for w_i. Every other vertex is a private neighbour for itself with respect to S_8^*. So S_8^* is digitally convex in $S_{2,n}$.

Finally, the proofs that S_9^* and S_{10}^* are digitally convex in $S_{2,n}$ are identical to those of Claim 2(c) and Claim 2(d), respectively. This completes the proof of Claim 4.

Similar to the previous arguments, there are 2^{k-1} digitally convex sets S_3^*, as each one corresponds to a possible subset of R_V. There are the same number of digitally convex sets S_4^*. There are 2^{k-1} digitally convex sets S_5^*, but we count only $2^{k-1}-1$ of these to avoid double counting the empty set. There are 2^{k-1} digitally convex sets S_6^*, as each one corresponds to a possible set R_W, and the same number of sets S_7^* and of sets S_8^*. There are $2^{k-1}-1$ digitally convex sets S_9^* and the same number of sets S_{10}^*. Overall, we get a total of $n_{\mathscr{D}}(S_{2,n}) = 10 \cdot 2^{k-1} - 4 = 5 \cdot 2^k - 4 = 5 \cdot 2^{\frac{n-2}{3}} - 4$ digitally convex sets when $n \equiv 2 \pmod 3$. $\square$

Proofs of lemmas

We complete this section by giving the proofs of Lemmas 3.7 - 3.11 and Lemma 3.15.

Proof of Lemma 3.7. First, we show that each digitally convex set in $\mathscr{D}(G)$ corresponds to two distinct digitally convex sets in $\mathscr{D}(G_1)$. Moreover, we show that, for two different digitally convex sets S_1 and S_2 in G, their corresponding digitally convex sets in G_1 are all distinct.

Let $S \in \mathscr{D}(G)$. Then, in G_1, the vertex v_3 is a private neighbour for itself, as well as for v_1 and v_2 with respect to S. Thus, S is also digitally convex in G_1.

If $x \notin N_G[S]$, then $S \cup \{v_3\}$ is digitally convex in G_1. The vertices v_1 and v_2 both have x as a private neighbour, because $x \notin N_{G_1}[v_3]$. Note that, because each of these digitally convex sets contains v_3, they were not counted as digitally convex sets of G_1 above. If $x, y \in N_G[S]$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_1, as every vertex not in $S \cup \{v_1, v_2, v_3\}$ has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$ in G_1 as with respect to S in G. Note that because these digitally convex sets of G_1 contain all of v_1, v_2, v_3, they were not counted as digitally convex sets of G_1 above. If $x \in N_G[S]$ and $y \notin N_G[S]$, then $S \cup \{v_2, v_3\}$ is digitally convex in G_1. Neither v_2 nor v_3 is adjacent to y, so y is a private neighbour for itself and for v_1. Note that, because these digitally convex sets contain v_2 and v_3 but not v_1, they were not counted as digitally convex sets of G_1 above.

In addition to the $2\mathscr{D}(G)$ digitally convex sets of G_1 described above, the sets $S_1 = \{v_2, v_3\}$, $S_2 = \{v_1, v_2, v_3\}$, $S_3 = V(G) - \{x\}$ and $S_4 = V(G) - \{x, y\}$ are each digitally convex in G_1, as shown next. The vertex y is a private neighbour for both x and v_1 with respect to S_1. As $x \notin N_G[S_1 - \{v_2, v_3\}]$, the set S_1 was not counted above as a digitally convex set of G_1. Since G is assumed to have order at least 3, both x and y must have another neighbour that is a private neighbour with respect to S_2. Since $x, y \notin N_G[S_2 - \{v_1, v_2, v_3\}]$, the set S_2 was not counted above as a digitally convex set of G_1. The vertex v_2 is a private neighbour for both x and v_1 with respect to S_3. Finally, the vertex v_1 is a private neighbour for both x and y with respect to S_4. Both S_3 and S_4 have cardinality greater than $n - \delta(G) - 1$ so, by Theorem 2.1, neither of S_3 or S_4 is digitally convex in G. Thus, neither was counted above as a digitally convex set of G_1.

Therefore, $n_{\mathscr{D}}(G_1) \geq 2n_{\mathscr{D}}(G) + 4$. □

Proof of Lemma 3.8. First, we show that each digitally convex set in $\mathscr{D}(G)$ corresponds to two distinct digitally convex sets in $\mathscr{D}(G_2)$. Moreover, we show that, for two different digitally convex sets S_1 and S_2 in G, their corresponding digitally convex sets in G_2 are all distinct.

Let $S \in \mathscr{D}(G)$. If $x, y \notin S$, then S is digitally convex in G_2, as v_1, v_2, v_3 are all private neighbours for themselves. The set $S \cup \{v_3\}$ is also digitally convex in G_2, as the vertex v_1 is a private neighbour for v_2 and for x. Every other vertex has the same private neighbour in G_2 as in G. Note that none of these digitally convex sets contains x or y.

If $x \in S$ or $y \in S$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_2, as every vertex not in $S \cup \{v_1, v_2, v_3\}$ has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$ in G_2 as with respect to S in G. Note that each of these digitally convex sets contains v_1, v_2 and v_3, as well as one of x or y. If $x \notin S$ and $y \in S$, then the set S is also digitally convex in G_2, as the vertex v_2 is a private neighbour for itself, v_1 and x with respect to S. Note that because each of these sets contains y, they were not counted above as digitally convex sets of G_2. If $x \in S$, then the set $(S - \{x\}) \cup \{v_1\}$ is digitally convex in G_2. Note that S can contain y or not. As above, the vertex v_3 is a private neighbour for itself, v_2 and x with respect to $(S - \{x\}) \cup \{v_1\}$. The vertex y is either in S or has a private neighbour with respect to S that is in $V(G) - \{x\}$. So it has this same private neighbour with respect to $(S - \{x\}) \cup \{v_1\}$. Note that these sets contain v_1 but do not contain v_2 or v_3, so they were not counted above as digitally convex sets of G_2.

In addition to the $2n_{\mathscr{D}}(G)$ digitally convex sets described above, the sets $S_1 = \{v_2, v_3\}$, $S_2 = \{v_1, v_2, v_3\}$, $S_3 = V(G) - \{x\}$ and $S_4 = V(G) - \{x, y\}$ are each digitally convex in G_2, as shown next. The vertex y is a private neighbour for both v_1 and x with respect to S_1. As $v_1 \notin S_1$, it was not counted above as a digitally convex set of G_2. Since G is assumed to have order at least 3, both x and y must have another neighbour that is a private neighbour with respect to S_2. Moreover, because $x, y \notin N_G[S - \{v_1, v_2, v_3\}]$, the set S_2 was not counted above as a digitally convex set of G_2. The vertex v_2 is a private neighbour for itself, x, v_1 and v_3 with respect to S_3. Finally, the vertex v_1 is a private neighbour for both x and y with respect to S_4. Both S_3 and S_4 have cardinality greater than $n - \delta(G) - 1$ so, by Theorem 2.1, neither of S_3 or S_4 is digitally convex in G. Thus, neither was counted above as a digitally convex set of G_2.

Therefore, $n_{\mathscr{D}}(G_2) \geq 2n_{\mathscr{D}}(G) + 4$. □

Proof of Lemma 3.9. As in the proof of the previous lemma, we begin by showing that each digitally convex set in $\mathscr{D}(G)$ corresponds to two distinct digitally convex sets in $\mathscr{D}(G_3)$. Moreover, we show that, for two different digitally convex sets S_1 and S_2 in G, their corresponding digitally convex sets in G_3 are all distinct.

Let $S \in \mathscr{D}(G)$. If $x, y \notin S$, then S is digitally convex in G_3, as v_1, v_2, v_3 are all private neighbours for themselves. Note that none of these sets contain any of x, y, v_1, v_2, v_3. If $x \in S$ or $y \in S$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_3, as every vertex not in $S \cup \{v_1, v_2, v_3\}$ has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$ in G_3 as with respect to S in G. Note that these sets contain all of

v_1, v_2, v_3 and at least one of x or y.

We now describe a second digitally convex set in $\mathscr{D}(G_3)$ for each of the sets in $\mathscr{D}(G)$. If $x, y \notin S$, then we consider four possible cases. If, for every vertex $a \in V(G) - S$, we have $N_G[a] - \{x, y\} \nsubseteq N_G[S]$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_3. Every vertex not in $S \cup \{v_1, v_2, v_3\}$, including x and y, has a private neighbour with respect to S in G that is not x or y. Each of these vertices has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$. Note that these sets contain all of v_1, v_2, v_3 but neither of x or y, so they were not counted above as digitally convex sets of G_3.

If, for every $a \in V(G) - S$, we have $N_G[a] - \{y\} \nsubseteq N_G[S]$ and there exists at least one vertex $b \in V(G) - N_G[S]$ for which $N_G[b] - \{x\} \subseteq N_G[S]$, then $S \cup \{v_3\}$ is digitally convex in G_3. Since $N_G[a] - \{y\} \nsubseteq N_G[S]$ for every vertex $a \in V(G) - S$, each vertex a has a private neighbour with respect to S that is not y. Thus, a still has this same private neighbour with respect to $S \cup \{v_3\}$. Since $xv_3 \notin E(G_3)$, each vertex that has only x as a private neighbour with respect to S in G also has this private neighbour with respect to $S \cup \{v_3\}$ in G_3. The vertex v_2 is a private neighbour for itself and v_1 with respect to $S \cup \{v_3\}$. Note that these sets contain v_3 but not y.

If, for every $a \in V(G) - S$, we have $N_G[a] - \{x\} \nsubseteq N_G[S]$ and there exists at least one vertex $b \in V(G) - N_G[S]$ for which $N_G[b] - \{y\} \subseteq N_G[S]$, then $S \cup \{v_2\}$ is digitally convex in G_3. Since $N_G[a] - \{x\} \nsubseteq N_G[S]$ for every vertex $a \in V(G) - S$, each vertex a has a private neighbour with respect to S that is not x. Thus, a still has this same private neighbour with respect to $S \cup \{v_2\}$. Since $yv_2 \notin E(G_3)$, each vertex that has only y as a private neighbour with respect to S in G also has this private neighbour with respect to $S \cup \{v_2\}$ in G_3. The vertex v_3 is a private neighbour for itself and v_1 with respect to $S \cup \{v_2\}$. Note that these sets contain v_2 but not x.

Finally, if there exists vertices $a, b \in V(G) - S$ such that $N_G[a] - \{x\} \subseteq N_G[S]$

and $N_G[b]-\{y\} \subseteq N_G[S]$, then let A be the set of vertices in $V(G)-\{x,y\}$ satisfying $N_G[a]-\{x\} \subseteq N_G[S]$ and let B be the set of vertices in $V(G)-\{x,y\}$ satisfying $N_G[b]-\{y\} \subseteq N_G[S]$. Then, $S' = S \cup A \cup B$ is digitally convex in G_3, as v_2 is a private neighbour for itself, v_1 and x with respect to S' and v_3 is a private neighbour for itself and y with respect to S'. By definition of A and B, every other vertex $z \notin S'$ has a private neighbour z' with respect to S' that is neither x or y and also must satisfy $z' \notin N_G[a]$ for every $a \in A$ and $z' \notin N_G[b]$ for every $b \in B$.

If $x \in S$ and $y \notin S$, then $S \cup \{v_2\}$ is digitally convex in G_3, as every vertex in $V(G)-S$ has the same private neighbour with respect to $S \cup \{v_2\}$ in G_3 as with respect to S in G. The vertex v_3 is a private neighbour for itself and v_1. Note that these sets contain both v_2 and x, so they were not counted above as digitally convex sets of G_3. Similarly, if $x \notin S$ and $y \in S$, then $S \cup \{v_3\}$ is digitally convex in G_3. Since these sets all contain both y and v_3, they were not counted above as digitally convex sets of G_3.

Finally, if both $x, y \in S$, then we consider three cases. If $S-\{y\} \notin \mathscr{D}(G)$, then $(S-\{y\}) \cup \{v_2\}$ is digitally convex in G_3, as the vertex v_3 is a private neighbour for itself, v_1 and y with respect to $(S-\{y\}) \cup \{v_2\}$. Every other vertex has the same private neighbour with respect to $(S-\{y\}) \cup \{v_2\}$ in G_3 as with respect to S in G. Note that these sets contain x and v_2, but since $S-\{y\} \notin \mathscr{D}(G)$, these sets were not counted above as digitally convex sets of G_3.

If $S-\{y\} \in \mathscr{D}(G)$ and $S-\{x\} \notin \mathscr{D}(G)$, then $(S-\{x\}) \cup \{v_3\}$ is digitally convex in G_3. Similar to above, the vertex v_2 is a private neighbour for itself, v_1 and x with respect to $(S-\{x\}) \cup \{v_3\}$. Every other vertex has the same private neighbour with respect to $(S-\{x\}) \cup \{v_3\}$ in G_3 as with respect to S in G. Note that these sets contain y and v_3 but, since $S-\{x\} \notin \mathscr{D}(G)$, these sets were not counted above as digitally convex sets of G_3.

If both $S - \{x\}, S - \{y\} \in \mathscr{D}(G)$, then it must be the case that $S - \{x, y\} \in \mathscr{D}(G)$ since the digital convexity is closed under intersections. Then, $S - \{x, y\}$ must satisfy $x, y \notin S - \{x, y\}$ and, for every vertex $a \in V(G) - (S - \{x, y\})$, we have $N_G[a] - \{x, y\} \not\subseteq N_G[S - \{x, y\}]$. If these conditions are not true, then that contradicts S, $S - \{x\}$ and $S - \{y\}$ all being digitally convex in G. Then, $(S - \{x, y\}) \cup \{v_3\}$ is digitally convex in G_3, as $S - \{x, y\} \in \mathscr{D}(G)$ implies that every vertex not in $S - \{x, y\}$ has a private neighbour with respect to $S - \{x, y\}$ in $V(G)$. Furthermore, $N_G[a] - \{x, y\} \not\subseteq N_G[S - \{x, y\}]$ for every vertex $a \in V(G) - (S - \{x, y\})$ implies that every vertex not in $V(G) - (S - \{x, y\})$ has a private neighbour in $V(G)$ that is neither x nor y. Thus, each vertex has the same private neighbour with respect to $(S - \{x, y\}) \cup \{v_3\}$ in G_3 as with respect to $S - \{x, y\}$ in G. The vertex v_2 is a private neighbour for itself and v_1 with respect to $(S - \{x, y\}) \cup \{v_3\}$. Note that the sets $S - \{x, y\}$ and $(S - \{x, y\}) \cup \{v_1, v_2, v_3\}$ were counted above as digitally convex sets of G_3, but $(S - \{x, y\}) \cup \{v_3\}$ was not.

In addition to the $2n_{\mathscr{D}}(G)$ digitally convex sets of G_3 described above, the sets $S_1 = \{v_2\}$, $S_2 = V(G_3) - \{x, v_1, v_2\}$ and $S_3 = V(G) \cup \{v_3\} - N_G(w)$, where $w \in N_G(x) \cap N_G(y)$, are all digitally convex in G_3, as shown next. The vertex y is a private neighbour for itself, x and v_1 with respect to S_1. Every other vertex in $V(G_3) - S_1$ is a private neighbour for itself with respect to S_1. The vertex v_2 is a private neighbour for itself, v_1 and x with respect to S_2. Note that $V(G) - \{x\}$ is not digitally convex in G, so this set was not counted above as a digitally convex set of G_3. Finally, every vertex in $N_G(w)$ has w as a private neighbour with respect to S_3, and v_2 is a private neighbour for itself and v_1 with respect to S_3. Note that, in particular, both x and y have w as a private neighbour. Further note that such a vertex w must exist by the assumption that G has order at least 3. Thus, $n_{\mathscr{D}}(G_3) \geq 2n_{\mathscr{D}}(G) + 3$. Moreover, by Theorem 2.2, every graph has an even number of digitally convex sets, so it must be the case that $n_{\mathscr{D}}(G_3) \geq 2n_{\mathscr{D}}(G) + 4$. □

Proof of Lemma 3.10. As in the proof of the previous lemma, we begin by showing that each digitally convex set in $\mathscr{D}(G)$ corresponds to two distinct digitally convex sets in $\mathscr{D}(G_4)$. Moreover, we show that, for two different digitally convex sets S_1 and S_2 in G, their corresponding digitally convex sets in G_4 are all distinct.

Let $S \in \mathscr{D}(G)$. If $x, y \notin S$, then S is digitally convex in G_4, as v_1, v_2, v_3 are all private neighbours for themselves with respect to S. If $x \in S$ or $y \in S$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_4, as every vertex not in $S \cup \{v_1, v_2, v_3\}$ has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$ in G_4 as with respect to S in G.

We now describe a second digitally convex set in $\mathscr{D}(G_4)$ for each of the sets in $\mathscr{D}(G)$. If $x, y \notin S$, then we consider four cases. If, for every vertex $a \in V(G) - S$, we have $N_G[a] - \{x, y\} \not\subseteq N_G[S]$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_4. Every vertex not in $S \cup \{v_1, v_2, v_3\}$, including x and y, has a private neighbour with respect to S in G that is not x or y. Each of these vertices has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$. Note that these sets contain all of v_1, v_2, v_3 but neither of x or y, so they were not counted above as digitally convex sets of G_4.

If, for every $a \in V(G) - S$, we have $N_G[a] - \{x\} \not\subseteq N_G[S]$ and there exists at least one vertex $b \in V(G) - N_G[S]$ for which $N_G[b] - \{y\} \subseteq N_G[S]$, then $S \cup \{v_3\}$ is digitally convex in G_4. Since $N_G[a] - \{x\} \not\subseteq N_G[S]$ for every vertex $a \in V(G) - S$, each vertex a has a private neighbour with respect to S that is not x. Thus, a still has this same private neighbour with respect to $S \cup \{v_3\}$. Since $yv_3 \notin E(G_4)$, each vertex that has only y as a private neighbour with respect to S in G also has this private neighbour with respect to $S \cup \{v_3\}$ in G_4. The vertex y is a private neighbour for itself, v_1, v_2 and x with respect to $S \cup \{v_3\}$. Note that these sets contain v_3 but not x or v_2.

If, for every $a \in V(G)-(S\cup\{y\})$, we have $N_G[a]-\{x\} \nsubseteq N_G[S]$ and $N_G[y]-\{x\} \subseteq N_G[S]$, then $S \cup \{v_1\}$ is digitally convex in G_4. Since $N_G[a] - \{x\} \nsubseteq N_G[S]$ for every vertex $a \in V(G) - (S \cup \{y\})$, each vertex a has a private neighbour with respect to S that is not x. Thus, a still has this same private neighbour with respect to $S \cup \{v_1\}$. The vertex v_2 is a private neighbour with respect to $S \cup \{v_1\}$ for itself, x, y and v_3. Note that these sets contain v_1 but none of x, y, v_2 or v_3.

If there exists at least one $a \in V(G) - (S \cup \{y\})$ for which $N_G[a] - \{x\} \subseteq N_G[S]$, then let A be the set of vertices in $V(G) - \{x, y\}$ satisfying $N_G[a] - \{x\} \subseteq N_G[S]$. Then $S' = S \cup A$ is digitally convex in G_4, as v_1 is a private neighbour for itself, x and y and v_3 is a private neighbour for itself and v_2 with respect to S'. By definition of A, every other vertex $z \notin S'$ has a private neighbour z' with respect to S' that is not x and must satisfy $z' \notin N_G[a]$ for every $a \in A$.

If $x \in S$ and $y \notin S$, then $(S - \{x\}) \cup \{v_2, v_3\}$ is digitally convex in G_4, as the vertex v_1 is a private neighbour for itself, x and y with respect to $(S-\{x\})\cup\{v_2, v_3\}$. Every other vertex has the same private neighbour with respect to $(S-\{x\})\cup\{v_2, v_3\}$ in G_4 as with respect to S in G. Note that these sets contain v_2 and v_3 but neither of x or y.

If $x \notin S$ and $y \in S$, then $S \cup \{v_1\}$ is digitally convex in G_4, as v_3 is a private neighbour for itself, v_2 and x with respect to $S \cup \{v_1\}$. Every other vertex has the same private neighbour with respect to $S \cup \{v_1\}$ in G_4 as with respect to S in G. These sets each contain both v_1 and y so they were not counted above as digitally convex sets of G_4.

Now, if $x, y \in S$, then we consider three possible cases. If $S - \{x\} \notin \mathscr{D}(G)$, then $(S - \{x\}) \cup \{v_1\}$ is digitally convex in G_4, as v_3 is a private neighbour for itself, v_2 and x with respect to $(S - \{x\}) \cup \{v_1\}$. In addition, since $S - \{x\} \notin \mathscr{D}(G)$, these sets were not counted above as digitally convex sets of G_4.

If $S - \{x\} \in \mathscr{D}(G)$ and $S - \{y\} \notin \mathscr{D}(G)$, then $(S - \{x, y\}) \cup \{v_2, v_3\}$ is digitally convex in G_4, as v_1 is a private neighbour for itself, x and y with respect to $(S - \{x, y\}) \cup \{v_2, v_3\}$. Every other vertex has the same private neighbour with respect to $(S - \{x, y\}) \cup \{v_2, v_3\}$ in G_4 as with respect to S in G. In addition, since $S - \{y\} \notin \mathscr{D}(G)$, these sets were not counted above as digitally convex sets of G_4.

Finally, if $S - \{x\}, S - \{y\} \in \mathscr{D}(G)$, then it must be the case that $S - \{x, y\} \in \mathscr{D}(G)$, since the digital convexity is closed under intersections. As both S and $S - \{x, y\}$ are digitally convex, each vertex $a \in V(G) - (S - \{x, y\})$ must have a private neighbour with respect to $S - \{x, y\}$ that is in $V(G) - \{x, y\}$. This means that the set $(S - \{x, y\}) \cup \{v_1, v_2, v_3\}$ was counted above as a digitally convex set of G_4. In addition, the set $S' = (S - \{x, y\}) \cup \{v_1\}$ is digitally convex in G_4, as v_2 is a private neighbour for itself, v_3, x and y with respect to S'. This set does not contain y, so it was not counted above as a digitally convex set of G_4.

In addition to the $2n_{\mathscr{D}}(G)$ digitally convex sets described above, the sets $S_1 = \{v_3\}$, $S_2 = (V(G) - \{x, y\}) \cup \{v_1\}$ and $S_3 = (V(G) - \{x, y\}) \cup \{v_2, v_3\}$ are each digitally convex in G_4, as shown next. The vertex y is a private neighbour for itself, v_2 and x with respect to S_1. Every other vertex is a private neighbour for itself with respect to S_1. The only digitally convex sets above that contain v_3 and not v_2 also contain at least one vertex in $V(G)$, so S_1 was not counted above as a digitally convex set of G_4. The vertex v_2 is a private neighbour for itself, v_3, x and y with respect to S_2. Since $V(G) - \{x, y\}$ has cardinality $n - \delta(G) > n - \delta(G) - 1$, the set $V(G) - \{x, y\}$ is not digitally convex in G, by Theorem 2.1. In addition, neither of the sets $V(G) - \{x\}$ or $V(G) - \{y\}$ is digitally convex in G, so the set S_2 could only correspond to the digitally convex set $V(G)$ of G above. However, since $V(G) - \{x\}$ is not digitally convex in G, the set S_2 was not counted as a digitally convex set of G_4 above. The vertex v_1 is a private neighbour for itself, x and y with respect to S_3. Similar to S_2, since none of $V(G)-\{x,y\}$, $V(G)-\{x\}$ or $V(G)-\{y\}$ is digitally convex in G, the set S_3 was not counted as a digitally convex set of G_4 above.

Thus, $n_{\mathscr{D}}(G_4) \geq 2n_{\mathscr{D}}(G)+3$. However, by Theorem 2.2, every graph has an even number of digitally convex sets. Therefore, $n_{\mathscr{D}}(G_4) \geq 2n_{\mathscr{D}}(G)+4$. □

Proof of Lemma 3.11. As in the proof of the previous lemma, we begin by showing that each digitally convex set in $\mathscr{D}(G)$ corresponds to two distinct digitally convex sets in $\mathscr{D}(G_5)$. Moreover, we show that, for two different digitally convex sets S_1 and S_2 in G, their corresponding digitally convex sets in G_5 are all distinct.

Let $S \in \mathscr{D}(G)$. If $x,y,z \notin S$, then S is digitally convex in G_5, as v_1, v_2, v_3 are each private neighbours for themselves with respect to S. If $x,y,z \notin S$ and $N_G(y)-\{x\} \subseteq N_G[S]$, then $S \cup \{y, v_1, v_2\}$ is digitally convex in G_5, as both x and z have v_3 as a private neighbour in G_5. Any other vertices in $N_G(y)-S$ have the same private neighbour in G_5 as in G. If $x,y,z \notin S$ and $N_G(y)-\{x\} \not\subseteq N_G[S]$, then $S \cup \{v_1, v_2\}$ is digitally convex in G_5, as both x and z have v_3 as a private neighbour in G_5. The vertex y has a private neighbour with respect to S in $V(G)-\{x,y\}$, so it has this same private neighbour with respect to $S \cup \{v_1, v_2\}$ in G_5.

If $y \in S$ and $x, z \notin S$, then S is digitally convex in G_5, as v_1 is a private neighbour for itself and v_2 with respect to S in G_5. The vertex v_3 is also a private neighbour for itself with respect to S in G_5. Let A be the set of vertices a in $V(G)-\{x\}$ satisfying $N_G(a)-\{z\} \subseteq N_G[S]$. Then, the set $S' = S \cup \{z, v_3\} \cup A$ is digitally convex in G_5, as v_1 is a private neighbour for itself, v_2 and x with respect to S'. Each vertex that has only z as a private neighbour with respect to S in G has been added to A, and every other vertex has the same private neighbour with respect to S' in G_5 as with respect to S in G. Note that this case is only possible when $d_G(x)=3$.

If $z \in S$ and $x, y \notin S$, then $(S-\{z\}) \cup \{v_1\}$ is digitally convex in G_5, as v_3 is a private neighbour for itself, z and x. Since $x \in N_G[S] - S$, it must be the case that y is a private neighbour with respect to S for x in G, so $y \notin N_G[S]$. Thus, it must also be the case that $y \notin N_G[(S - \{z\}) \cup \{v_1\}]$. So y is a private neighbour for itself and v_2 with respect to $(S - \{z\}) \cup \{v_1\}$. In addition, the set $S \cup \{v_1, v_3\}$ is digitally convex in G_5, since y is a private neighbour for itself, x and v_2. Note that this case is only possible when $d_G(x) = 3$.

If $x \in S$, then $S \cup \{v_1, v_2, v_3\}$ is digitally convex in G_5. Every vertex not in $S \cup \{v_1, v_2, v_3\}$ has the same private neighbour with respect to $S \cup \{v_1, v_2, v_3\}$ in G_5 as with respect to S in G. If $x \in S$ and $y \notin S$ or $z \notin S$, then $(S - \{x\}) \cup \{v_3\}$ is digitally convex in G_5, as v_1 is a private neighbour for x and v_2. If $z \notin S$, then z has a private neighbour with respect to S in $V(G) - \{x, z\}$. So it has this same private neighbour with respect to $(S - \{x\}) \cup \{v_3\}$. If $x, y, z \in S$ and $S - \{y\}$ is not digitally convex, then $(S - \{x, y\}) \cup \{v_3\}$ is digitally convex in G_5, as v_2 is a private neighbour for itself, v_1, x and y. Note that we require that $S - \{y\}$ is not digitally convex so that the set $(S - \{x, y\}) \cup \{v_3\}$ is not counted twice. Finally, if $x, y, z \in S$ and $S - \{y\}$ is digitally convex, then $(S - \{x, z\}) \cup \{v_1, v_2\}$ is digitally convex in G_5, as v_3 is a private neighbour for itself, x and z. Note that, because $S - \{y\}$ is also digitally convex, the vertex y has a private neighbour with respect to $S - \{y\}$ in $V(G) - \{x\}$. Then, if $S - \{x, y, z\}$ is also digitally convex in G, it must be the case that $N_G(y) - \{x\} \not\subseteq N_G[S - \{x, y, z\}]$. So the digitally convex set $(S - \{x, z\}) \cup \{v_1, v_2\}$ contains y, v_1, v_2 but was not counted above.

In addition to the $2n_{\mathscr{D}}(G)$ digitally convex sets of G_5 described above, the sets $S_1 = \{v_1, v_3\}$, $S_2 = V(G) - \{x, y, z\}$ and $S_3 = V(G) - \{x, z\}$ are all digitally convex in G_5, as shown next. The vertex y is a private neighbour for itself, x and v_2 with respect to S_1. If $d_G(x) = 2$, then y is also a private neighbour for z. If $d_G(x) = 3$, then w, the other neighbour of x in G, is a private neighbour for z with respect to S_1.

The vertex v_3 is a private neighbour for x and z with respect to S_2 and v_2 is a private neighbour for itself and y with respect to S_2. Finally, v_3 is a private neighbour for itself, x and z with respect to S_3 and v_1 is a private neighbour for itself and for v_2. Since $y, w \in S_3$ and these vertices dominate $N_G[x]$, the set S_3 is not digitally convex in G and was not counted above.

If $d_G(x) = 2$, then $S_4 = \{v_1\}$ is digitally convex in G_5, as the vertex y is a private neighbour for v_2 and x with respect to S_1. Furthermore, the vertex z dominates $N_G[x]$ in G, so $\{z\}$ is not a digitally convex set in G. So S_4 was not counted above as a digitally convex set. If $d_G(x) = 3$, then $S_4 = \{v_1, v_2, v_3\}$ is digitally convex in G_5, as the vertex w is a private neighbour for itself, x, y and z with respect to S_4. Since the only sets above containing v_1, v_2, v_3 also contain x, the set S_4 was not counted in the $2n_{\mathscr{D}}(G)$ digitally convex sets above.

Therefore, $n_{\mathscr{D}}(G_5) \geq 2n_{\mathscr{D}}(G) + 4$. $\square$

Proof of Lemma 3.15. First, we use $\mathscr{D}(G)$ to generate the digitally convex sets of G^*. In this construction, we will have some digitally convex sets in $\mathscr{D}(G)$ that correspond to two digitally convex sets in $\mathscr{D}(G^*)$ and some that correspond to none of the digitally convex sets in $\mathscr{D}(G^*)$. We then show that, despite this, the number of digitally convex sets in G^* is at most $n_{\mathscr{D}}(G)$.

Let $\mathscr{S}_{G^*} = \emptyset$. Now, for each $S \in \mathscr{D}(G)$, proceed as follows.

1. (a) If $v_1, v_2 \in S$, then add S to $\mathscr{S}_{G^*}$. Every vertex has the same private neighbour with respect to S in G^* as in G.

   (b) If $v_1, v_2, x, y \in S$, then add $S - \{x, v_1, v_2\}$ to $\mathscr{S}_{G^*}$. The vertex v_1 is a private neighbour for itself, x and v_2 with respect to $S - \{x, v_1, v_2\}$ in G^*.

   (c) If $v_1, v_2, y \in S$ and $x \notin S$, then add $S - \{v_1, v_2\}$ to $\mathscr{S}_{G^*}$. The vertex v_1 is a private neighbour for itself and v_2 with respect to $S - \{v_1, v_2\}$ in G^*.

2. (a) If $v_1, v_2 \notin S$, then add S to $\mathscr{S}_{G^*}$. It must be the case that $x, y \notin S$. So v_2 is a private neighbour for itself, as well as for x, y and v_1 with respect to S in G^*.

   (b) If $v_1, v_2 \notin S$ and $y \notin N_G[S]$, then add $S \cup \{v_1\}$ to $\mathscr{S}_{G^*}$. In G^*, $yv_1 \notin E(G^*)$ so $y \notin N_{G^*}[S \cup \{v_1\}]$. Thus, y is a private neighbour for itself, x and v_2 with respect to $S \cup \{v_1\}$ in G^*.

3. If $v_1 \in S$ and $v_2 \notin S$, then add nothing to $\mathscr{S}_{G^*}$.

4. If $v_1 \notin S$ and $v_2 \in S$, then add nothing to $\mathscr{S}_{G^*}$.

Note that the digitally convex sets of G that satisfy the conditions of cases 1(b) and 1(c) also satisfy the conditions of case 1(a). So each of these corresponds to two digitally convex sets in G^*. Similarly, the digitally convex sets satisfying the conditions of case 2(b) also satisfy the conditions of case 2(a), so these sets correspond to two digitally convex sets in G^*.

We have shown that $\mathscr{S}_{G^*} \subseteq \mathscr{D}(G^*)$, so now we show that each set in $\mathscr{D}(G^*)$ is added to $\mathscr{S}_{G^*}$ exactly once. There are four cases for digitally convex sets S^* in G^*: both $v_1, v_2 \in S^*$, $y \in S^*$ but $x, v_1, v_2 \notin S^*$, $x, y, v_1, v_2 \notin S^*$, or $v_1 \in S^*$ but $v_2, x, y \notin S^*$.

In the first case with $v_1, v_2 \in S^*$, because $x, y \in N[\{v_1, v_2\}]$ in both G and G^*, this set S^* is also digitally convex in G. Thus, it is added to $\mathscr{S}_{G^*}$ in step 1(a). If $y \in S^*$ but $x, v_1, v_2 \notin S^*$ and x has a private neighbour in $V(G^*) - \{v_1\}$ with respect to S^*, then x has a private neighbour with respect to $S^* \cup \{v_1, v_2\}$ in G. Thus, $S^* \cup \{v_1, v_2\}$ is digitally convex in G and S^* is added to $\mathscr{S}_{G^*}$ in step 1(c). If x does not have a private neighbour in $V(G^*) - \{v_1\}$, then $S^* \cup \{x, v_1, v_2\}$ is digitally convex in G and S^* is added to $\mathscr{S}_{G^*}$ in step 1(b). If $x, y, v_1, v_2 \notin S^*$, then $v_1, v_2 \notin N[S]$ in both G^* and G. Thus, S^* is digitally convex in G and is added to $\mathscr{S}_{G^*}$ in step 2(a). Finally, if $v_1 \in S^*$ but $v_2, x, y \notin S^*$, then y must be a private neighbour for both x and v_2 with respect to S^*, i.e. $y \notin N_{G^*}[S^*]$. In G, the set $S^* - \{v_1\}$ is digitally convex and $y \notin N_G[S^* - \{v_1\}]$. Thus, S^* is added in step 2(b). Therefore, $\mathscr{S}_{G^*} = \mathscr{D}(G^*)$.

Next, we show that the number of digitally convex sets satisfying the conditions of step 1(b) or step 1(c) is equal to the number of digitally convex sets satisfying the conditions of step 2(b). To do this, we use a restriction of the bijection φ in Theorem 2.2. The sets S_1 satisfying cases 1(b) and 1(c) both contain v_1, v_2 and y. Thus, the sets $\varphi(S_1) = V(G) - N[S_1]$ must satisfy $v_1, v_2 \notin \varphi(S_1)$. They must also satisfy $y \notin N[\varphi(S_1)]$, because $y \notin \varphi(S_1)$ by definition and $y \notin N[S_1] - S_1 = N[\varphi(S_1)] - \varphi(S_1)$. This means that these sets satisfy the conditions of step 2(b). Now, suppose the digitally convex set S_2 satisfies the conditions of step 2(b). Then $v_1, v_2 \notin S_2$ and $y \notin N_G[S_2]$. Since $xy \in E(G)$, it must also be true that $x \notin S_2$ so $v_1, v_2 \notin N_G[S_2]$. Thus, $v_1, v_2, y \in \varphi(S_2)$. If $x \notin N_G[S_2]$, then $x \in \varphi(S_2)$, so $\varphi(S_2)$ satisfies the conditions of step 1(b). Otherwise, $x \notin \varphi(S_2)$, so $\varphi(S_2)$ satisfies the conditions of step 1(c). Thus, φ is a bijection between the digitally convex sets satisfying the conditions of step 1(b) or 1(c) and those satisfying step 2(b).

Now we show that the number of digitally convex sets satisfying the conditions of step 3 is at least the number that satisfy the conditions of step 2(b). Let S_3 be a digitally convex set satisfying step 2(b). Then $y \notin N_G[S]$ and so $x \notin S$, implying that $v_2 \notin N_G[S]$. Then $S_3 \cup \{v_1\}$ is a digitally convex set in G, since v_2 is a private neighbour for itself, x and y with respect to $S_3 \cup \{v_1\}$. The set $S_3 \cup \{v_1\}$ satisfies the conditions of step 3.

It is clear from the construction of $\mathscr{D}(G^*)$ that $n_{\mathscr{D}}(G)$ is the number of digitally convex sets satisfying steps 1(a), 2(a), 3 and 4. Furthermore, $n_{\mathscr{D}}(G^*)$ is the number of digitally convex sets produced from steps 1(a), 1(b), 1(c), 2(a) and 2(b). Evidently, the number of digitally convex sets satisfying the conditions of step 3 is equal to the number satisfying step 4. From above, each of these is at least the number of convex sets satisfying step 2(b), which is the same as the number satisfying step 1(b) or 1(c). Thus, $n_{\mathscr{D}}(G)$ is at least the number of digitally convex sets satisfying steps 1(a), 2(a), 1(b), 1(c) and 2(b), i.e. $n_{\mathscr{D}}(G^*)$. Therefore, $n_{\mathscr{D}}(G^*) \leq n_{\mathscr{D}}(G)$. □

3.2 Digital convexity in k-trees

Several of the results in Section 3.1 generalize to k-trees, linking them to the results of Lafrance, Oellermann and Pressey [15] involving trees, stated in Chapter 2. We begin by generalizing Algorithm 2, for generating the digitally convex sets of a 2-tree, to generate the digitally convex sets of a k-tree, and explain how this algorithm can be applied to chordal graphs. We then examine the digitally convex sets of the k^{th} power of a path, for any integer $k \geq 1$, and give an upper bound on the number of digitally convex sets in a k-tree of order n. In Theorem 3.20, we extend the recurrence relation for the number of digitally convex sets in k^{th} power of paths to an even more general type of structure. Finally, we conjecture a lower bound on the number of digitally convex sets in a k-tree, based on the lower bound for trees and the conjectured lower bound for 2-trees.

From Algorithm 2, we can rewrite the cases of step 2 to be in terms of the vertices in $N_G(v)$, instead of the individual vertices u and w. This will make the generalization to k-trees more natural. The conditions for each case in Algorithm 2 can be rewritten as follows:

(a) If $N_G(v) \cap S = \emptyset$, then add S to $\mathscr{S}_G$.

(b) If $N_G(v) \cap S = \emptyset$ and for every $a \in N_{G-v}[N_G(v)] - S$, we have $N_G[a] \not\subseteq N_G[S \cup \{v\}]$, then add $S \cup \{v\}$ to $\mathscr{S}_G$.

(c) If $N_G(v) \cap S \neq \emptyset$, then add $S \cup \{v\}$ to $\mathscr{S}_G$.

(d)-(f) If $N_G(v) \cap S \neq \emptyset$ and $N_{G-v}[N_G(v) \cap S] \subseteq N_{G-v}[S - N_G(v)]$ then add $S - N_G(v)$ to $\mathscr{S}$. In Algorithm 2, this case is split into three cases, corresponding to the three possible nonempty intersections of $N_G(v)$ and S: $\{u\}$, $\{w\}$ and $\{u, w\}$.

For k-trees, the conditions are the same, with only a difference in the neighbourhood of the new vertex v. This neighbourhood contains two pairwise adjacent vertices in Algorithm 2 and now it contains k pairwise adjacent vertices. We state Algorithm 4 in terms of the individual vertices $u_1, u_2, \ldots, u_k$, to keep with the formatting of Algorithm 2.

Algorithm 4. *Generating the collection $\mathscr{S}_G$ of digitally convex sets of a k-tree G of order $n \geq k+1$.*

1. *If $n = k+1$, then $\mathscr{S}_G = \{\emptyset, V(G)\}$.*

2. *Suppose $n > k+1$ and let v be a vertex of degree k, with neighbours $u_1, u_2, \ldots, u_k$. Use the algorithm to generate $\mathscr{S}_{G-v}$. Generate $\mathscr{S}_G$ from $\mathscr{S}_{G-v}$ as follows: For each $S \in \mathscr{S}_{G-v}$*

   (a) If $u_i \notin S$ for every $i = 1, 2, \ldots, k$, then add S to $\mathscr{S}_G$.

   (b) If $u_i \notin S$ for every $i = 1, 2, \ldots, k$ and for every $a \in (\cup_{i=1}^{k} N_{G-v}[u_i]) - S$, we have $N_G[a] \nsubseteq N_G[S \cup \{v\}]$, then add $S \cup \{v\}$ to $\mathscr{S}_G$.

   (c) If $u_i \in S$ for some $i = 1, 2, \ldots, k$, then add $S \cup \{v\}$ to $\mathscr{S}_G$.

   (d) For any $\emptyset \neq X \subseteq [k]$, if $u_i \in S$ for every $i \in X$ and $u_j \notin S$ for $j \notin X$, and $N_{G-v}[\{u_i \mid i \in X\}] \subseteq N_{G-v}[S - \{u_i \mid i \in X\}]$, then add $S - \{u_i \mid i \in X\}$ to $\mathscr{S}_G$.

Theorem 3.18. *Let G be a k-tree. Then the collection $\mathscr{S}_G$ generated by Algorithm 4 is $\mathscr{D}(G)$.*

We omit the full proof due to its similarity to that of Theorem 3.1, which can be seen by generalizing the cases in step 2, as above.

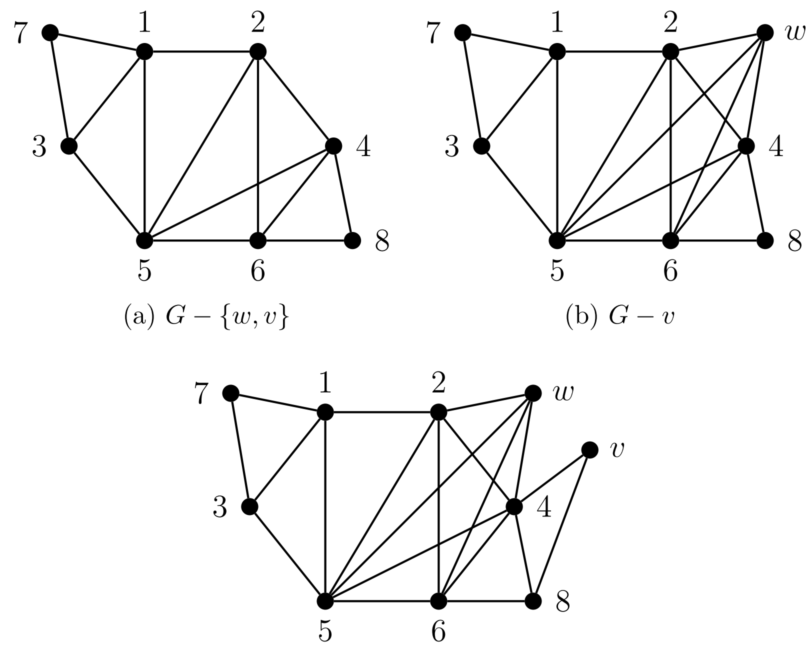
Chordal Graphs

Step 2 of Algorithm 4 does not require the degree, k, of the new vertex v in G to be the same as the vertex v' added at the previous step to form $G-v$. By restating this step to let v be a simplicial vertex of degree k with neighbours $u_1, u_2, \ldots, u_k$, the algorithm can be applied to graphs in which all simplicial vertices do not necessarily have degree k. In particular, this allows Algorithm 4 to be used to generate the digitally convex sets of a chordal graph, using a perfect elimination ordering. Several of the classes of graphs considered in this thesis are chordal graphs to which this algorithm can be applied, including the k-trees in the current chapter and block graphs in Chapter 5.

As an example, consider the graph shown in Figure 3.20. The simplicial vertex w is added to the chordal graph $G - \{w, v\}$ in Figure 3.20(a) to form the chordal graph $G - v$ in Figure 3.20(b), followed by adding the vertex v to form the chordal graph G in Figure 3.20(c). The digitally convex sets of $G - \{w, v\}$ are as follows:

$$\mathscr{D}(G - \{w, v\}) = \{\emptyset, \{2\}, \{2, 5\}, \{7\}, \{3, 7\}, \{1, 3, 7\}, \{1, 2, 3, 5, 7\}, \{8\}, \{4, 6, 8\},$$
$$\{2, 4, 6, 8\}, \{2, 4, 5, 6, 8\}, \{7, 8\}, \{3, 7, 8\}, \{1, 2, 3, 4, 5, 6, 7, 8\}\}.$$

Now, because w is a simplicial vertex of degree 4 and v a simplicial vertex of degree 2, applying step 2 of Algorithm 4 twice, using $\mathscr{D}(G - \{w, v\})$ above, will generate the digitally convex sets of $G - v$ and then of G. The neighbours of w in $G - v$ are the vertices 2, 4, 5 and 6, so when running step 2 of Algorithm 4, these will be the vertices u_i.



(a) $G - \{w, v\}$ (b) $G - v$

(c) G

Figure 3.20: Algorithm 4 uses the digitally convex sets of $G-\{w,v\}$ to generate those of $G - v$ and G

The sets $\emptyset$, $\{7\}$, $\{3,7\}$, $\{1,3,7\}$, $\{8\}$, $\{7,8\}$ and $\{3,7,8\}$ all satisfy the conditions of case (a), as they do not contain the vertices 2, 4, 5 or 6, and so they all get added to $\mathscr{S}_{G-v}$. The only set that satisfies case (b) is $\emptyset$, so the set $\{w\}$ gets added to $\mathscr{S}_{G-v}$. The sets $\{2\}$, $\{2,5\}$, $\{1,2,3,5,7\}$, $\{4,6,8\}$, $\{2,4,6,8\}$, $\{2,4,5,6,8\}$ and $\{1,2,3,4,5,6,7,8\}$ all satisfy the conditions of case (c), as they each contain one of the vertices 2, 4, 5, or 6. So for each of these sets S, the set $S \cup \{w\}$ is added to $\mathscr{S}_{G-v}$. Finally, the set $\{1,2,3,4,5,6,7,8\}$ satisfies the conditions of case (d). The vertices 2, 4, 5 and 6 all have w as a private neighbour in $G - v$ with respect to the set $S = \{1,3,7,8\}$ so S gets added to $\mathscr{S}_{G-v}$. Now

$$\mathscr{D}(G-v) = \{\emptyset, \{7\}, \{3,7\}, \{1,3,7\}, \{8\}, \{7,8\}, \{3,7,8\}, \{1,3,7,8\}, \{w\}, \{2,w\},$$
$$\{2,5,w\}, \{1,2,3,5,7,w\}, \{4,6,8,w\}, \{2,4,6,8,w\},$$
$$\{2,4,5,6,8,w\}, \{1,2,3,4,5,6,7,8,w\}\}.$$

Now, we repeat step 2 of Algorithm 4 using the digitally convex sets of $G-v$ to generate those of G, where the vertices 4 and 8 are the neighbours of v in G. The sets $\emptyset$, $\{7\}$, $\{3,7\}$, $\{1,3,7\}$, $\{w\}$, $\{2,w\}$, $\{2,5,w\}$ and $\{1,2,3,5,7,w\}$ satisfy case (a), as none of them contain the vertices 4 or 8. So they all get added to $\mathscr{S}_G$. The sets $\emptyset$, $\{7\}$, $\{3,7\}$ and $\{1,3,7\}$ all satisfy case (b), as the vertex 6 is not in the neighbourhood of any of these sets, or in the neighbourhood of the vertex v. Thus, for each of these sets S, the vertex 6 can be a private neighbour for the vertices in the closed neighbourhood of 4 and 8. So the set $S \cup \{v\}$ gets added to $\mathscr{S}_G$. The sets $\{8\}$, $\{7,8\}$, $\{3,7,8\}$, $\{4,6,8,w\}$, $\{2,4,6,8,w\}$, $\{2,4,5,6,8,w\}$ and $\{1,2,3,4,5,6,7,8,w\}$ each satisfy case (c), as they each contain one of the vertices 4 or 8. So for each set S, the set $S \cup \{v\}$ gets added to $\mathscr{S}_G$. Finally, $\{4,6,8,w\}$, $\{2,4,6,8,w\}$, $\{2,4,5,6,8,w\}$ and $\{1,2,3,4,5,6,7,8,w\}$ each satisfy case (d). The vertices 4 and 8 each have v as a private neighbour in G with respect to the sets $\{6,w\}$, $\{2,6,w\}$, $\{2,5,6,w\}$ and $\{1,2,3,5,6,7,w\}$, so these each get added to $\mathscr{S}_G$. Now

$$\mathscr{D}(G) = \mathscr{S}_G = \{\emptyset, \{7\}, \{3,7\}, \{1,3,7\}, \{w\}, \{2,w\}, \{2,5,w\}, \{6,w\}, \{2,6,w\},$$
$$\{2,5,6,w\}, \{1,2,3,5,7,w\}, \{1,2,3,5,6,7,w\}, \{v\}, \{7,v\}, \{3,7,v\}, \{1,3,7,v\}, \{8,v\},$$
$$\{7,8,v\}, \{3,7,8,v\}, \{1,3,7,8,v\}, \{4,6,8,w,v\}, \{2,4,6,8,w,v\}, \{2,3,5,6,8,w,v\},$$
$$\{1,2,3,4,5,6,7,8,w,v\}\}$$

Thus, Algorithm 4 can be used to generate the digitally convex sets of both k-trees and chordal graphs.

Another result that extends naturally from 2-trees to the more general k-trees is Theorem 3.3, enumerating the digitally convex sets of the square of the path of order n. Theorem 3.3 uses the digitally convex sets of P^2_{n-1} and P^2_{n-3} to enumerate those of P^2_n, as the vertex v_n is not adjacent to any of the vertices in P^2_{n-3}. For the k^{th} power of a path, P^k_n, the vertex v_n has degree k. So it is adjacent to the vertices $v_{n-1}, v_{n-2}, \ldots, v_{n-k}$ but not adjacent to the vertex v_{n-k-1}. So, instead, we use the digitally convex sets of P^k_{n-1} and P^k_{n-k-1} to enumerate those of P^k_n.

Theorem 3.19. *If P^k_n is the k^{th} power of the path of order n, then $n_{\mathscr{D}}(P^k_n) = n_{\mathscr{D}}(P^k_{n-1}) + n_{\mathscr{D}}(P^k_{n-k-1})$, for $n \geq 4 + k$.*

The proof of this theorem is omitted here, as it follows the same format as that of Theorem 3.3, with the role of the graph P^2_{n-3} and the vertex v_{n-3} being replaced with P^k_{n-k-1} and v_{n-k-1}, respectively. Base cases are not provided in Theorem 3.19, as their number and value both differ depending on k. Note that for $k = 1$, these graphs are simply the paths, and the recurrence given here matches that of Proposition 2.4.

This result can be further generalized, as the proof depends only on the vertices $v_{n-k-1}, v_{n-k}, \ldots, v_n$ and the structure of their neighbourhoods. The proof requires that the vertex v_{n-k-1} be adjacent to a k-clique in P^k_{n-k-1}, though if v_{n-k-1} has other neighbours, then they do not affect the proof. That is, v_{n-k-1} is not required to be simplicial. Denote the vertices of the k-clique by $u_1, u_2, \ldots, u_k$. Then, the proof requires that for each $j = 1, 2, \ldots, k$, the vertex v_{n-j} be adjacent to each u_i with $i < j$, to the vertex v_{n-k-1}, and to each v_{n-p} with $j < p \leq k$. Finally, the proof requires the vertex v_n to be adjacent to $v_{n-1}, v_{n-2}, \ldots, v_{n-k}$. The case with $k = 3$ is illustrated in Figure 3.21. This generalization is summarized in the following theorem.

Theorem 3.20. *Let G be a graph containing a $k{+}1$-clique with vertices $v, u_1, u_2, \ldots, u_k$, for some $k > 1$. For any ℓ with $1 \leq \ell \leq k$, let G_ℓ be the graph formed by adding to G*

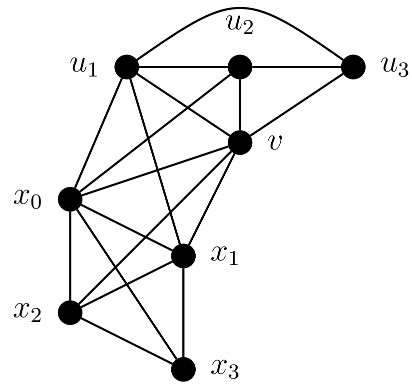


Figure 3.21: The structure described in Theorem 3.20 with $k = \ell = 3$

the pairwise adjacent vertices $x_0, x_1, \ldots, x_\ell$, forming an $\ell+1$-clique, and the edges $x_i v$, for $i = 0, 1, \ldots, \ell - 1$, and $x_i u_j$, for $i + j < \ell$. Then $n_{\mathscr{D}}(G_\ell) = n_{\mathscr{D}}(G) + n_{\mathscr{D}}(G_\ell - x_\ell)$.

Proof. First, we show that $n_{\mathscr{D}}(G_\ell) \geq n_{\mathscr{D}}(G) + n_{\mathscr{D}}(G_\ell - x_\ell)$. Let $S \in \mathscr{D}(G)$. Since $x_\ell \notin N_{G_\ell}[S]$, x_ℓ is a private neighbour for the vertices $x_0, x_1, x_2, \ldots, x_\ell$ with respect to S. Thus, S is digitally convex in G_ℓ.

Let $S \in \mathscr{D}(G_\ell - x_\ell)$.

- If $x_{\ell-1} \in S$ then $S \cup \{x_\ell\}$ is digitally convex in G_ℓ, because $N[x_\ell]$ is dominated by $x_{\ell-1}$ in G_ℓ.

- If $x_{\ell-1} \notin S$ and $v \notin N_{G_\ell - x_\ell}[S]$, then $v \notin N_{G_\ell}[S \cup \{x_\ell\}]$ because $vx_\ell \notin E(G_\ell)$. Thus, v is a private neighbour for itself as well as for $x_0, x_1, x_2, \ldots, x_{\ell-1}$ in G_ℓ with respect to $S \cup \{x_\ell\}$. So $S \cup \{x_\ell\}$ is digitally convex in G_ℓ.

- If $x_{\ell-1} \notin S$ and $v \in N_{G_\ell - x_\ell}[S]$, then it must be the case that $v \in N_{G_\ell - x_\ell}[S] - S$, as v dominates $N[x_{\ell-1}]$ in $G_\ell - x_\ell$. The vertices $x_0, x_1, x_2, \ldots, x_{\ell-2}$ also dominate $N[x_{\ell-1}]$ so none of these can be in S. In G_ℓ, $x_\ell \notin N_{G_\ell}[S]$ so x_ℓ is a private neighbour for itself with respect to S in G_ℓ, and the set S is digitally convex in G_ℓ.

Since each digitally convex set in $\mathscr{D}(G) \cup \mathscr{D}(G_\ell - x_\ell)$ has a corresponding set in $\mathscr{D}(G_\ell)$ and this correspondence is injective, we have $n_\mathscr{D}(G_\ell) \geq n_\mathscr{D}(G) + n_\mathscr{D}(G_\ell - x_\ell)$. Now, we show the reverse inequality.

Let $S \in \mathscr{D}(G_\ell)$.

- If $x_\ell \in S$ then each vertex in $V(G_\ell) - S$ has a private neighbour with respect to S that is in $V(G_\ell - x_\ell)$. Thus, $S - \{x_\ell\}$ is digitally convex in $G_\ell - x_\ell$.

- If $x_\ell \notin S$ and $v \in N_{G_\ell}[S] - S$, then it must be the case that $x_0, x_1, x_2, \ldots, x_{\ell-1} \notin S$, as each of these dominates $N_{G_\ell}[x_\ell]$. Thus, $x_{\ell-1} \notin N_{G_\ell}[S]$ and is a private neighbour for $x_0, x_1, x_2, \ldots, x_{\ell-1}, v$ with respect to S in $G_\ell - x_\ell$. So S is digitally convex in $G_\ell - x_\ell$.

- If $x_\ell \notin S$ and $v \notin N_{G_\ell}[S] - S$, then each vertex $y \in V(G_\ell) - (S \cup \{x_0, x_1, x_2, \ldots, x_\ell\})$ has a private neighbour with respect to S that is in G. Each $x_i \notin S$ because it dominates $N_{G_\ell}[x_\ell]$ for $i < \ell$. Thus, S is digitally convex in G.

Therefore, $n_\mathscr{D}(G_\ell) = n_\mathscr{D}(G) + n_\mathscr{D}(G_\ell - x_\ell)$. □

We turn now to an upper bound on the number of digitally convex sets in a k-tree. Recall that for a tree T and a 2-tree G each of order n, we have $n_\mathscr{D}(T) \leq 2^{n-1}$ and $n_\mathscr{D}(G) \leq 2^{n-2}$. Each of these has the form 2^{n-k}, as trees are equivalent to k-trees with $k = 1$. The following result shows that this pattern extends to $k > 2$.

Theorem 3.21. *If G is a k-tree, then $n_\mathscr{D}(G) \leq 2^{n-k}$.*

To prove this result, we use a technique identical to that of Theorem 3.4, removing the edges $e_1, e_2, \ldots, e_k$ incident with a vertex of degree k in G. The new graph then satisfies the inequality $n_\mathscr{D}(G) \leq n_\mathscr{D}(G - \{e_1, e_2, \ldots, e_k\})$. By continuing to remove the edges incident with a vertex of degree k until the remaining graph has the $n - k$

components consisting of K_{k+1} and $n-k-1$ isolated vertices, we obtain the desired inequality, as each component is a clique with exactly two digitally convex sets. We omit the full proof due to its similarity to that of Theorem 3.4. As for both trees and 2-trees, this upper bound is sharp.

Proposition 3.22. *For a given integer $k \geq 1$, the upper bound given in Theorem 3.21 is attained by the graph $K_k + \overline{K}_{n-k}$.*

The digitally convex sets are, as in the proof of Proposition 3.5, subsets $S \subsetneq V(\overline{K}_{n-k})$, along with the entire vertex set $V(K_k + \overline{K}_{n-k})$. So $K_k + \overline{K}_{n-k}$ has a total of 2^{n-k} digitally convex sets, for any integer $k \geq 1$ and $n > k$.

We examine now whether a lower bound on the number of digitally convex sets in a k-tree can be constructed using the lower bound given in Theorem 2.5 for trees and the conjectured lower bound given in Conjecture 3.6 for 2-trees. Provided Conjecture 3.6 holds, for a tree T and a 2-tree G, each of order n, we have

$$n_{\mathscr{D}}(T) \geq \begin{cases} 2 \cdot 2^{\frac{n}{2}} - 2, & \text{if } n \equiv 0 \pmod 2 \\ 3 \cdot 2^{\frac{n-1}{2}} - 2, & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

$$n_{\mathscr{D}}(G) \geq \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

By letting $n = (k+1)\ell + p$, with $p \in \{0, 1, \ldots, k\}$, each of these satisfies the more general formula

$$n_{\mathscr{D}}(H) \geq (k+1+p) \cdot 2^{\ell} - 2k$$

Similarly, we can generalize the graphs that attain this lower bound for trees and conjectured lower bound for 2-trees, the spiderstars and the 2-spiderstars, to form the k-spiderstars. The k-spiderstar of order n, $S_{k,n}$, is constructed in the following way:

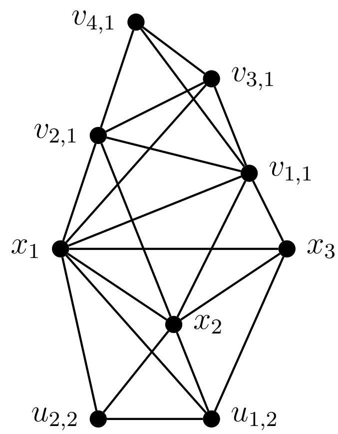


Figure 3.22: The 3-spiderstar with 9 vertices, $S_{3,9}$

1. Begin with K_k with vertices $x_1, x_2, \ldots, x_k$.

2. For $i = 1, 2, \ldots, \lfloor \frac{n-k}{k+1} \rfloor$, add vertices $v_{1,i}, v_{2,i}, \ldots, v_{k+1,i}$ and edges $x_j v_{h,i}$ with $j + h \leq k+1$ and $v_{g,i} v_{h,i}$ with $g < h$.

3. Let $p \equiv n-k \pmod{k+1}$ and $b = \lceil \frac{n-k}{k+1} \rceil$. If $p \not\equiv 0 \pmod{k+1}$, then add vertices $u_{1,b}, u_{2,b}, \ldots, u_{p,b}$ and edges $x_j u_{h,b}$ with $j + h \leq k+1$ and $u_{g,b} u_{h,b}$ with $g < h$.

An example with $n = 9$ and $k = 3$ is given in Figure 3.22. Using either Algorithm 4 or the brute force approach given in Appendix A for generating the digitally convex sets of a particular graph, we find that $n_{\mathscr{D}}(S_{3,9}) = 14 = (3+1+1) \cdot 2^2 - 6$. Note that setting $k = 1$ gives the spiderstars, S_n, and setting $k = 2$ gives the 2-spiderstars constructed in Section 3.1.

We now conjecture that the proposed generalization of the lower bounds on the number of digitally convex sets of trees and 2-trees gives a lower bound on the number of digitally convex sets for a k-tree with $k \geq 1$. We also conjecture that the k-spiderstars attain this lower bound for $k \geq 1$.

Conjecture 3.23. *Let G be a k-tree of order $n = (k+1)\ell + p$, where $n \equiv p \pmod{k+1}$. Then $n_{\mathscr{D}}(G) \geq (k+1+p) \cdot 2^{\ell} - 2k$, with this lower bound attained by the k-*

spiderstars, $S_{k,n}$.

Unlike with the upper bound on the number of digitally convex sets of trees and 2-trees, there appears to be no simple extension of the lower bound for trees to 2-trees. While the proposed outline for 2-trees is similar to that for trees and uses induction on n, there are more cases to consider for 2-trees. Several additional lemmas are required to apply the induction hypothesis in the case of 2-trees than are required in the case of trees. There is not an obvious generalization to k-trees, so we leave this lower bound as a conjecture.

3.3 Simple clique 2-trees

A subclass of k-trees for which the enumeration of digitally convex sets is not obvious is the class of simple clique k-trees, defined by Markenzon, Justel and Paciornik [17]. In this section, we focus only on the case when $k = 2$. We give an upper bound on the number of digitally convex sets in a simple clique 2-tree, which differs from the upper bound for 2-trees when $n \geq 5$, and note that a sharp lower bound likely also differs from the conjectured sharp lower bound on the number of digitally convex sets of 2-trees for larger values of n, though we do not give an explicit formula for such a lower bound. We begin with the definition of a simple clique k-tree, from which it is obvious that this graph class is a subclass of the k-trees.

Definition 3.24 (Markenzon, Justel and Paciornik [17]). *For $k \geq 1$, a* simple clique k-tree (*or* SC k-tree) *is a graph defined as follows: a $k+1$ clique, K_{k+1}, is an SC k-tree, and an SC k-tree of order $n > k+1$ is constructed by adding to an SC k-tree of order $n-1$ a vertex v adjacent to a k-clique that belongs to exactly one $k+1$-clique in the existing SC k-tree, and only to these vertices.*

Note that, when $k = 1$, the simple clique 1-trees are equivalent to the paths. Figure 3.23 shows a simple clique 2-tree of order eight.

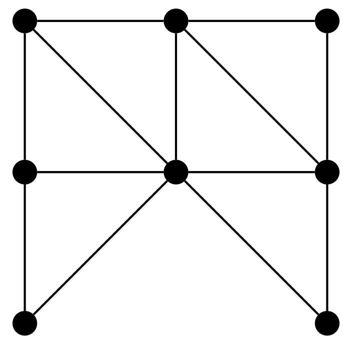


Figure 3.23: A simple clique 2-tree of order eight

We also include the following definition and result, as they will be useful in the proofs in this section.

Definition 3.25 (Markenzon, Justel and Paciornik [17])**.** *The* k-line graph *of a graph* G *is a graph whose vertices are the cliques of size* k *in* G*. Two vertices in the* k*-line graph are adjacent if and only if their corresponding cliques in* G *have* $k - 1$ *vertices in common.*

Theorem 3.26 (Markenzon, Justel and Paciornik [17])**.** *A* k*-tree* G *is a simple clique* k*-tree if and only if its* $(k + 1)$*-line graph is a tree.*

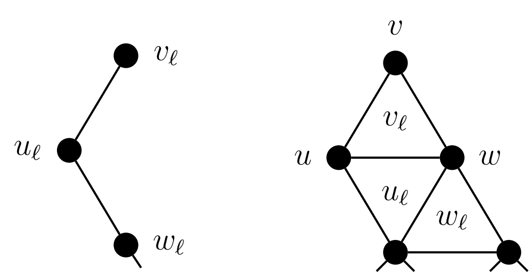
We note that for SC 2-trees in particular, the vertices of the 3-line graph can have degree at most 3, as a given 3-clique can share each of its edges with at most one other 3-clique in an SC 2-tree.

In Section 3.1 we saw that, as with 2-trees, not all SC 2-trees of the same order have the same number of digitally convex sets. This fact can be seen using the same graphs used to show this fact for 2-trees, P_6^2 and $P_5 + K_1$ which have six and eight digitally convex sets, respectively. In Theorem 3.4 and Proposition 3.5, we describe the subclass of 2-trees, $K_2 + \overline{K}_{n-2}$, that attain the upper bound on the number of digitally convex sets in a 2-tree. However, these graphs are not SC 2-trees for $n \geq 5$, as the 2-clique, xy, belongs to every 3-clique in the graph. These graphs indicate that the upper bound given in Theorem 3.4 may not be sharp for SC 2-trees. In this section, we show that the upper bound can be improved when restricted to SC 2-trees. Before we do this, we require the following lemma.

Lemma 3.27. *Let G be an SC 2-tree of order $n > 4$. Then G has a vertex of degree 2 adjacent to a vertex of degree 3, or G has two vertices of degree 2 with a common neighbour of degree 4.*

Proof. Let G_ℓ be the 3-line graph of G. By Theorem 3.26, we know that G_ℓ is a tree. The leaves of G_ℓ correspond to the 3-cliques in G that contain a vertex of degree 2, as only two vertices in the 3-clique are contained in a second 3-clique in G.

Suppose there is a leaf v_ℓ of G_ℓ adjacent to a vertex u_ℓ of degree 2 in G_ℓ. Then, in G, v_ℓ corresponds to a 3-clique containing a vertex v of degree 2 and whose other two vertices, say u and w, are also contained in the 3-clique corresponding to u_ℓ. The clique corresponding to u_ℓ shares an edge with the clique corresponding to v_ℓ and with one other clique corresponding to a vertex w_ℓ of G_ℓ (see Figure 3.24). So one of u or w, say u, is not in the clique corresponding to w_ℓ. Then u has degree 3 in G.



(a) G_ℓ (b) G

Figure 3.24: The 3-line graph G_ℓ corresponding to the SC 2-tree G

Suppose instead that there is no leaf of G_ℓ with a neighbour of degree 2. Then, there must be two leaves, x_ℓ and y_ℓ, both adjacent to a vertex z_ℓ of degree 3 in G_ℓ.

Since x_ℓ and y_ℓ are leaves in G_ℓ, their corresponding 3-cliques in G each contain a vertex, x and y, respectively, of degree 2. Then, each of the 3-cliques corresponding to x_ℓ and y_ℓ share an edge with the 3-clique corresponding to z_ℓ (see Figure 3.25). There is therefore a vertex z that is contained in each of the 3-cliques corresponding to x_ℓ, y_ℓ and z_ℓ. The vertex z is a neighbour of both x and y and has two other neighbours in the clique corresponding to z_ℓ. Thus, z is a common neighbour of x and y with degree 4. □

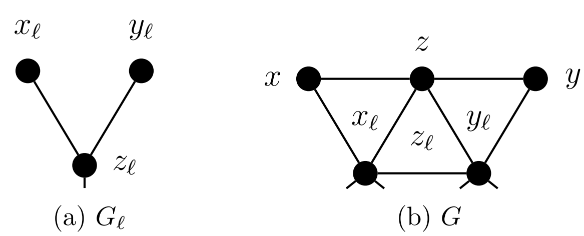


Figure 3.25: The 3-line graph G_ℓ corresponding to the SC 2-tree G

Theorem 3.28. *Let G be a simple clique 2-tree of order $n \geq 3$. Then $n_{\mathscr{D}}(G) \leq 2f_{n-1}$, where f_{n-1} is the $(n-1)^{th}$ Fibonacci number, as defined in Chapter 2.*

Proof. We prove this by induction on n. If $n = 3$, then $G \cong K_3$. So $n_{\mathscr{D}}(G) = 2 = 2(1) = 2f_2$. If $n = 4$, then G is the graph in Figure 3.26. So $n_{\mathscr{D}}(G) = 4 = 2(2) = 2f_3$.

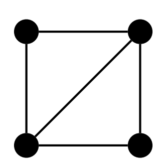


Figure 3.26: Base case for Theorem 3.28

Now, suppose that there exists a $k \geq 5$ such that $n_{\mathscr{D}}(G_n) \leq 2f_{n-1}$ for all SC 2-trees G_n of order $3 \leq n < k$. Let G be a graph of order k. By Lemma 3.27, the graph G has a vertex of degree 2 adjacent to a vertex of degree 3, or has two vertices of degree 2 with a common neighbour of degree 4. We consider each of these cases separately.

Suppose G has a vertex v of degree 2 with a neighbour u of degree 3. Let w be the other neighbour of v in G. We now show $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G-\{v\})+n_{\mathscr{D}}(G-\{u,v\})$. To do this, we partition the digitally convex sets of G into two types. The sets S' of type one are those that contain none of v, u, w or have $v \notin S'$ and $u \notin N_G[S']$. The sets of type two are all of the remaining digitally convex sets of G. We now show an injection from the digitally convex sets of G of type one to $\mathscr{D}(G-\{v\})$ and an injection from those of type two to $\mathscr{D}(G-\{u,v\})$.

Let $S \in \mathscr{D}(G)$. If $v, u, w \in S$, then $S-\{v\}$ is digitally convex in $G-\{v\}$, as $N_G[v]$ is dominated by both u and w. Thus, every vertex in $(G-\{v\})-S$ has the same private neighbour with respect to $S-\{v\}$ in $G-\{v\}$ as with respect to S in G.

If $v \notin S$ and $u \notin N_G[S]$, then S is digitally convex in $G-\{v\}$, as u is a private neighbour for itself and for w with respect to S. Every other vertex not in S has the same private neighbour with respect to S in $G-\{v\}$ as in G.

If $v \in S$ and $u, w \notin S$, then $S-\{v\}$ is digitally convex in $G-\{u,v\}$, as w has a private neighbour with respect to S in G that is in $V(G-\{u,v\})$. Every other vertex has the same private neighbour with respect to $S-\{v\}$ in $G-\{u,v\}$ as in G.

If $v \notin S$, $u \in N_G[S]$ and $N_G[w] \subseteq N_G[S \cup \{v\}]$, then $S \cup \{w\}$ is digitally convex in $G-\{u,v\}$, as every vertex $a \in N_{G-\{u,v\}}[w]$ is also in $N_{G-\{u,v\}}[S]$. So every vertex not in $S \cup \{w\}$ has the same private neighbour with respect to $S \cup \{w\}$ in $G-\{u,v\}$ as with respect to S in G.

Finally, if $v \notin S$, $u \in N_G[S]$ and $N_G[w] \nsubseteq N_G[S \cup \{v\}]$, then S is digitally convex in $G-\{u,v\}$, as w has a private neighbour with respect to S in G that is in $V(G-\{u,v\})$. Since $u \in N_G[S]$, u is not a private neighbour with respect to S for any vertex in G. Every other vertex has the same private neighbour with respect to

S in $G - \{u, v\}$ as in G.

We have shown an injective mapping from the digitally convex sets of G of type one to the digitally convex sets of $G - \{v\}$ and an injective mapping from the digitally convex sets of G of type two to the digitally convex sets of $G - \{u, v\}$. Thus, $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{v\}) + n_{\mathscr{D}}(G - \{u, v\})$. By the induction hypothesis, we have

$$n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{v\}) + n_{\mathscr{D}}(G - \{u, v\})$$

$$\leq 2f_{n-2} + 2f_{n-3} = 2f_{n-1}.$$

Now suppose G has two vertices x and y, both of degree 2, with a common neighbour z of degree 4. Let x' be the other neighbour of x and y' the other neighbour of y. Similar to above, we show that $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{x\}) + n_{\mathscr{D}}(G - \{x, y\})$. To do this, we partition the digitally convex sets of G into two types. The sets S' of type one are those that contain x and at least one of x' or z, or do not contain x and satisfy one of $x' \notin N_G[S']$ or $z \notin N_G[S']$. The sets of type two are all of the remaining digitally convex sets of G. We now show an injection from the digitally convex sets of G of type one to $\mathscr{D}(G - \{x\})$ and an injection from those of type two to $\mathscr{D}(G - \{x, y\})$.

If $x \in S$ and $x' \in S$ or $z \in S$, then $S - \{x\}$ is digitally convex in $G - \{x\}$, as $N_G[x]$ is dominated by both z and x'. So every vertex has the same private neighbour with respect to $S - \{x\}$ in $G - \{x\}$ as with respect to S in G.

If $x \notin S$ and $x' \notin N_G[S]$ or $z \notin N_G[S]$, then S is digitally convex in $G - \{x\}$, as every vertex has the same private neighbour with respect to S in $G - \{x\}$ as in G.

If $x \in S$ and $x', z \notin S$, then $S - \{x\}$ is digitally convex in $G - \{x, y\}$, as x' must have a private neighbour with respect to S, and hence $S - \{x\}$, that is in $V(G - \{x, y\})$. It must be the case that $z \notin N_G[S - \{x\}]$, as $y' \in S$ means $y \in S$ since y' dominates

y, but x and y together dominate z. So $y \in S$ contradicts $z \notin S$. Thus, z is a private neighbour for itself and for y' in $G - \{x, y\}$ with respect to $S - \{x\}$.

If $x \notin S$ and $x', z \in N_G[S]$, then it must be the case that $y \in S$, as $x' \in S$ or $z \in S$ implies $x \in S$, which is a contradiction, and $y' \in S$ implies $y \in S$. So x is the only private neighbour of z, i.e. $N_G[z] \subseteq N_G[S \cup \{x\}]$. If, in addition, $N_G[x'] \subseteq N_G[S \cup \{x\}]$, then $(S - \{y\}) \cup \{x', z\}$ is digitally convex in $G - \{x, y\}$. Since $y \in S$, it is not a private neighbour for any vertex in G with respect to S. Every other vertex has the same private neighbour with respect to $(S - \{y\}) \cup \{x', z\}$ in $G - \{x, y\}$ as in G.

Similarly, if $x \notin S$, $x', z \in N_G[S]$ and $N_G[z] \subseteq N_G[S \cup \{x\}]$ but $N_G[x'] \nsubseteq N_G[S \cup \{x\}]$, then $(S - \{y\}) \cup \{z\}$ is digitally convex in $G - \{x, y\}$. The vertex x' has a private neighbour with respect to S in G that is not x or z, so it has this same private neighbour with respect to $(S - \{y\}) \cup \{z\}$ in $G - \{x, y\}$. Every other vertex has the same private neighbour with respect to $(S - \{y\}) \cup \{z\}$ in $G - \{x, y\}$ as in G.

As above, we have shown an injective mapping from the digitally convex sets of G of type one to the digitally convex sets of $G - \{x\}$ and an injective mapping from the digitally convex sets of G of type two to the digitally convex sets of $G - \{x, y\}$. Thus, $n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{x\}) + n_{\mathscr{D}}(G - \{x, y\})$. By the induction hypothesis, we have

$$n_{\mathscr{D}}(G) \leq n_{\mathscr{D}}(G - \{x\}) + n_{\mathscr{D}}(G - \{x, y\})$$

$$\leq 2f_{n-2} + 2f_{n-3} = 2f_{n-1}.$$

□

Note that this upper bound matches the number of digitally convex sets of a path, P_{n-1}, of order $n - 1$, given in Proposition 2.4. The following results show the connection between the digitally convex sets of SC 2-trees and those of paths.

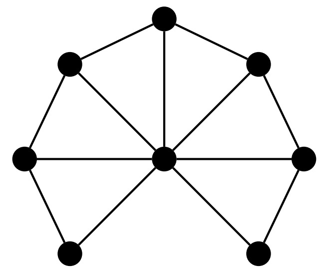


Figure 3.27: $P_7 + K_1$

Proposition 3.29. *Let G be a graph with v a universal vertex. Then $n_{\mathscr{D}}(G) = n_{\mathscr{D}}(G-v)$.*

Proof. Let $S \in \mathscr{D}(G-v)$. If $S \neq V(G-v)$, then there is some $x \in V(G-v)-N[S]$ that is a private neighbour of v with respect to S in G. Every other vertex $y \in V(G-v)-S$ has the same private neighbour with respect to S in G as in $G-v$. Thus, S is digitally convex in G. If $S = V(G-v)$, then S is a dominating set in G, and is therefore not digitally convex. However, $S \cup \{v\} = V(G)$ is digitally convex in G. Thus, $n_{\mathscr{D}}(G) \geq n_{\mathscr{D}}(G-v)$.

Now, let $S \in \mathscr{D}(G)$. Since v is a universal vertex, any set containing v is a dominating set in G, so the only digitally convex set containing v is $V(G)$. So if $S \neq V(G)$, then $v \notin S$. Since $v \in N[S]$ for any $S \neq \emptyset$, it is not a private neighbour for any vertex $x \in V(G)-S$. Thus, every vertex $x \in V(G)-S$ has a private neighbour in $V(G-v)$, and so S is digitally convex in $G-v$. Therefore, $n_{\mathscr{D}}(G) = n_{\mathscr{D}}(G-v)$. □

Using this result, we can easily construct an SC 2-tree that attains the bound given in Theorem 3.28, using the fact that $n_{\mathscr{D}}(P_{n-1}) = 2f_{n-1}$.

Proposition 3.30. *The upper bound on the number of digitally convex sets in a simple clique 2-tree, given in Theorem 3.28, is attained by the graph $P_{n-1} + K_1$.*

This is easily shown by applying Proposition 3.29 to the graph $P_{n-1} + K_1$ to show that it has the same number of digitally convex sets as P_{n-1}. By Proposition 2.4,

$n_{\mathscr{D}}(P_{n-1}) = 2f_{n-1}$, as desired. Figure 3.27 shows $P_7 + K_1$, which has $2f_7 = 26$ digitally convex sets.

Since the SC 2-trees form a subclass of the 2-trees, the lower bound on the number of digitally convex sets conjectured in Conjecture 3.6 also holds for SC 2-trees if it holds for 2-trees. However, 2-spiderstars are not SC 2-trees for larger values of n, as their 3-line graphs are not trees when $n \geq 9$. Therefore, as with the upper bound, the lower bound in Conjecture 3.6 may not be sharp for SC 2-trees.

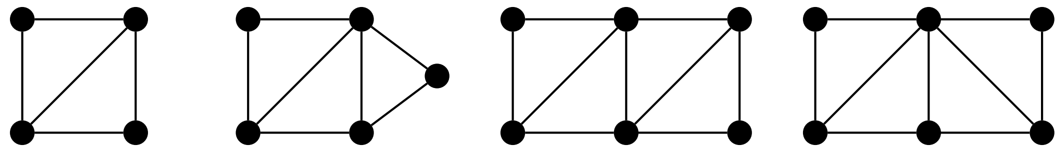
2-Path graphs

Of particular interest are the 2-trees that contain exactly two vertices of degree 2. These are called 2-*path graphs*, denoted $2P_n$ and form a subclass of the simple clique 2-trees [17]. Note that the 3-line graph of a 2-path graph must be a path, as it must be a tree with two leaves, one for each of the 3-cliques containing a vertex of degree 2. An example of a 2-path graph of order n is the square of the path of order n, P_n^2. We now show that the lower bound on the number of digitally convex sets in a 2-tree, conjectured in Conjecture 3.6, holds for 2-path graphs, but is not sharp for 2-path graphs of order $n = 9$.

Theorem 3.31. *Let G be a 2-path graph of order $n \geq 4$. Then,*

$$n_{\mathscr{D}}(G) \geq \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof. We prove this result using induction on n. Figure 3.28 shows all 2-path graphs of orders 4, 5 and 6. If $n = 4$, then $n_{\mathscr{D}}(G) = 4 = 4 \cdot 2^{\frac{4-1}{3}} - 4$. If $n = 5$, then $n_{\mathscr{D}}(G) = 6 = 5 \cdot 2^{\frac{5-2}{3}} - 4$. If $n = 6$, then $n_{\mathscr{D}}(G) \geq 8 = 3 \cdot 2^{\frac{6}{3}} - 4$. Therefore, the result holds for $4 \leq n \leq 6$.



(a) $n_{\mathscr{D}}(G) = 4$ (b) $n_{\mathscr{D}}(G) = 6$ (c) $n_{\mathscr{D}}(G) = 8$ (d) $n_{\mathscr{D}}(G) = 10$

Figure 3.28: All 2-path graphs of orders 4, 5 and 6

Now, suppose there exists a $k \geq 7$ such that the result holds for 2-path graphs G_n of order n, with $4 \leq n < k$. Now, suppose G is a 2-path graph of order k.

Claim: The graph G has a vertex v of degree 2, adjacent to two vertices u and w of degree 3 and 4, respectively, or G has a vertex b of degree 3 with neighbours a and c of degree 2 and 3, respectively.

Let G_ℓ be the 3-line graph of G. Then G_ℓ must be a path of order $n-2$ [17]. So G_ℓ has vertices x_ℓ, y_ℓ and z_ℓ such that x_ℓ is a leaf of G_ℓ, $d(y_\ell) = d(z_\ell) = 2$ and $x_\ell y_\ell, y_\ell z_\ell \in E(G_\ell)$. The vertex x_ℓ in G_ℓ corresponds to a 3-clique in G with a vertex x_1 of degree 2. The other two vertices, x_2 and x_3, in the 3-clique corresponding to x_ℓ are both also contained in the 3-clique corresponding to y_ℓ. Let y be the other vertex in the 3-clique corresponding to y_ℓ. Now, y and one of x_2 or x_3, say x_2, are also in the 3-clique corresponding to z_ℓ, since y_ℓ has degree 2 in G_ℓ. Thus, x_3 has degree 3 in G. Moreover, z_ℓ has degree 2 in G_ℓ so the 3-clique corresponding to z_ℓ in G shares an edge with a second 3-clique, w_ℓ. Exactly one of x_2 or y is in w_ℓ. If x_2 is in w_ℓ, then y is only in the 3-cliques y_ℓ and z_ℓ and thus has degree 3 in G. If y is in w_ℓ, then x_2 is only in the 3-cliques x_ℓ, y_ℓ and z_ℓ and this has degree 4 in G. This completes the proof of the claim.

Now, if G has a vertex v of degree 2, adjacent to two vertices u and w of degree 3 and 4, respectively, then we can apply the inequality in Lemma 3.7. So $n_{\mathscr{D}}(G) \geq$

$2n_{\mathscr{D}}(G - \{u, v, w\}) + 4$. By the induction hypothesis,

$$n_{\mathscr{D}}(G) \geq 2n_{\mathscr{D}}(G - \{u, v, w\}) + 4 \geq \begin{cases} 2(3 \cdot 2^{\frac{n}{3}-1} - 4) + 4, & \text{if } n \equiv 0 \pmod 3 \\ 2(4 \cdot 2^{\frac{n-1}{3}-1} - 4) + 4, & \text{if } n \equiv 1 \pmod 3 \\ 2(5 \cdot 2^{\frac{n-2}{3}-1} - 4) + 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$$= \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

as desired.

Similarly, if G has a vertex b of degree 3 with neighbours a and c of degree 2 and 3, respectively, then we can apply the inequality in Lemma 3.8. So $n_{\mathscr{D}}(G) \geq 2n_{\mathscr{D}}(G - \{a, b, c\}) + 4$. By the induction hypothesis,

$$n_{\mathscr{D}}(G) \geq 2n_{\mathscr{D}}(G - \{a, b, c\}) + 4 \geq \begin{cases} 2(3 \cdot 2^{\frac{n}{3}-1} - 4) + 4, & \text{if } n \equiv 0 \pmod 3 \\ 2(4 \cdot 2^{\frac{n-1}{3}-1} - 4) + 4, & \text{if } n \equiv 1 \pmod 3 \\ 2(5 \cdot 2^{\frac{n-2}{3}-1} - 4) + 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

$$= \begin{cases} 3 \cdot 2^{\frac{n}{3}} - 4, & \text{if } n \equiv 0 \pmod 3 \\ 4 \cdot 2^{\frac{n-1}{3}} - 4, & \text{if } n \equiv 1 \pmod 3 \\ 5 \cdot 2^{\frac{n-2}{3}} - 4, & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

as desired. □

As shown in the proof above, this lower bound is sharp for 2-path graphs of order $n \leq 6$. The square of the path of order 7, P_7^2, also attains this bound. From Theorem 3.3, we have $n_{\mathscr{D}}(P_7^2) = 12 = 4 \cdot 2^{\frac{7-1}{3}} - 4$. The 2-path G of order 8 shown in Figure 3.29 satisfies $n_{\mathscr{D}}(G) = 16 = 5 \cdot 2^{\frac{8-2}{3}} - 4$.

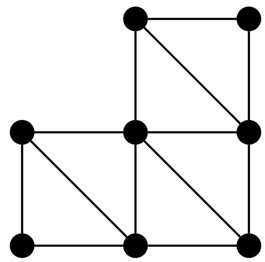


Figure 3.29: A 2-path of order 8 that attains the lower bound in Theorem 3.31

When $n = 9$, however, the lower bound in Theorem 3.31 is no longer sharp. Every 2-path of order 9 has at least $22 > 20 = 3 \cdot 2^{\frac{9}{3}} - 4$ digitally convex sets. This indicates that the lower bound on the number of digitally convex sets of a 2-path can likely be improved.

Chapter 4

Cycles and Cartesian Products

In Chapters 2 and 3, we saw that the digitally convex sets of paths, as well as of powers of paths, satisfy recurrence relations that make it possible to enumerate the digitally convex sets using only the number of digitally convex sets in those graphs of smaller orders. In this chapter, we examine other classes of graphs whose digitally convex sets can be counted using recurrence relations, including cycles and Cartesian products of paths. We show that, for some of these graph classes, their digitally convex sets can be counted directly or using the digitally convex sets of graphs of smaller orders while, for others, we use a bijection between the digitally convex sets and other mathematical objects whose enumeration satisfies the same recurrence.

4.1 Digitally convex sets of cycles

Throughout this section, for a cycle C_n, we denote the vertices by $v_1, v_2, \ldots, v_n$ with $v_i v_{i+1} \in E(C_n)$ for $i = 1, 2, \ldots, n-1$ and $v_1 v_n \in E(C_n)$.

A cycle of order n, unlike a path of order n, does not contain a cycle of a smaller order as an induced subgraph. Despite this, the number of digitally convex sets of the cycle C_n can be computed using the number of digitally convex sets in the cycles C_{n-1}, C_{n-2} and C_{n-4}, as we show in the following result.

Theorem 4.1. *Let C_n be the cycle of order n. Then $n_{\mathscr{D}}(C_3) = 2$, $n_{\mathscr{D}}(C_4) = 6$, $n_{\mathscr{D}}(C_5) = 12$, $n_{\mathscr{D}}(C_6) = 20$ and, for $n \geq 7$,*

$$n_{\mathscr{D}}(C_n) = 2n_{\mathscr{D}}(C_{n-1}) - n_{\mathscr{D}}(C_{n-2}) + n_{\mathscr{D}}(C_{n-4}).$$

Proof. We first prove the initial conditions, shown in Figure 4.1. If $n = 3$, then the only digitally convex sets are $\emptyset$ and $V(C_3)$, so $n_{\mathscr{D}}(C_3) = 2$. If $n = 4$, then the digitally convex sets are $\emptyset$, $V(C_4)$ and $\{v_i\}$ for $1 \leq i \leq 4$, so $n_{\mathscr{D}}(C_4) = 6$. If $n = 5$, then the digitally convex sets are $\emptyset$, $V(C_5)$, $\{v_i\}$ for $1 \leq i \leq 5$ and $\{v_j, v_k\}$ for $v_jv_k \in E(C_5)$. So $n_{\mathscr{D}}(C_5) = 12$. Finally, if $n = 6$, then the digitally convex sets are $\emptyset$, $V(C_6)$, $\{v_i\}$ for $1 \leq i \leq 6$, $\{v_j, v_k\}$ for $v_jv_k \in E(C_6)$ and $\{v_i, v_j, v_k\}$ for $v_iv_j, v_jv_k \in E(C_6)$. So $n_{\mathscr{D}}(C_6) = 20$.

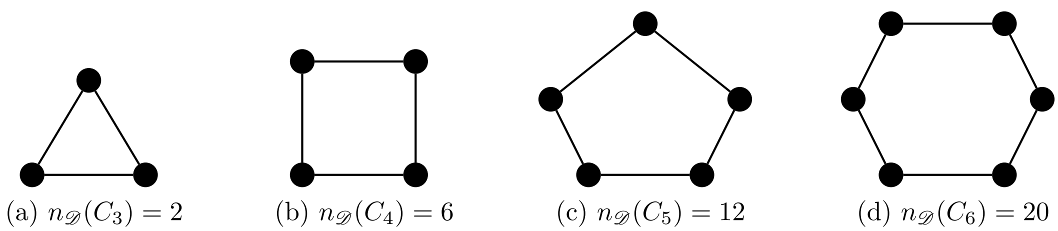


Figure 4.1: Base cases for Theorem 4.1

Now, suppose $n \geq 7$ and consider C_n. To show the recurrence, we show a bijection between the digitally convex sets of the cycle C_n and the number of cyclic binary n-bit strings with no alternating substring of length greater than 2 (i.e. 101 and 010 are not substrings). These cyclic binary n-bit strings satisfy the same recurrence, which is the On-Line Encyclopedia of Integer Sequences (OEIS) sequence A007039 [23]. To start, we label the edges of the cycle by assigning label i to the edge v_iv_{i+1}, for $1 \leq i < n$, and label n to the edge v_1v_n.

Given a digitally convex set $S \in \mathscr{D}(C_n)$, we get a corresponding cyclic binary n-bit string S^* by setting the i^{th} bit to be 1 if edge i is incident with a vertex in the set S and 0 otherwise. As an example, the digitally convex set $S = \{v_4, v_5\}$, shown in Figure 4.2, corresponds to the cyclic binary string (0011100), as the edges 3, 4 and 5 are incident with vertices in S.

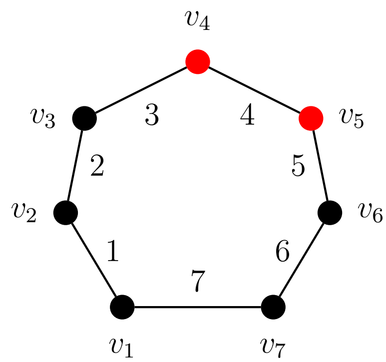
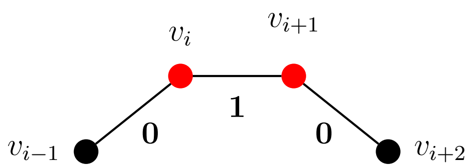


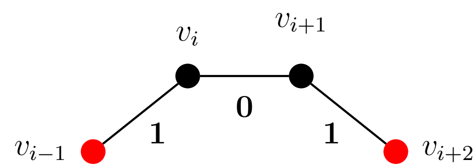
Figure 4.2: The digitally convex set $S = \{v_4, v_5\}$ is indicated in red

Now, we show that the cyclic binary n-bit string S^* has no alternating substring of length greater than 2. Note that if any edge label or vertex index (e.g. $i+1$ or $i-1$) is not in $\{1, 2, \ldots, n\}$, then we consider that label or index mod n. Suppose the binary substring of S^* corresponding to positions $i-1, i, i+1$ is 010. Then edge i is incident with a vertex in the set S, but neither edges $i-1$ or $i+1$ are. The vertices incident with edge i are v_i and v_{i+1}, indicated in red in Figure 4.3(a). So one of these must be in S. However, each of these is incident with one of the edges $i-1$ or $i+1$. So neither vertex can be in S, which is a contradiction. Suppose the binary substring of S^* corresponding to positions $i-1, i, i+1$ is 101. Then edge i is not incident with a vertex in the set S, so $v_i, v_{i+1} \notin S$. The edges $i-1$ and $i+1$ are each incident with a vertex in S, so $v_{i-1}, v_{i+2} \in S$, indicated in red in Figure 4.3(b). However, then the vertices v_i and v_{i+1} have no private neighbour with respect to S, contradicting the fact that S is digitally convex. Thus, neither 010 nor 101 can appear as a substring of S^*.

We now reverse this process to show a bijection. Given a cyclic binary n-bit string S^* without an alternating substring of length greater than 2, we construct the



(a) 010



(b) 101

Figure 4.3: Neither 010 nor 101 can appear as a substring

corresponding subset S of vertices in the cycle in the following way. If bits i and $i+1$ are both 1, then add vertex v_{i+1} to the set S, for $1 \leq i < n$. If bit 1 and bit n are both 1, then add the vertex v_1 to S. Since S^* does not contain the substring 010, each 1 in S^* is either preceded by a 1 or followed by a 1. So each substring of k consecutive 1's in S^* corresponds to $k-1$ consecutive vertices being added to S, for $k < n$, or all n vertices being added to S in the case that $k = n$. This means that there is a one-to-one map from the cyclic n-bit binary strings without alternating substrings of length greater than 2 to the family of sets of vertices S on the n-cycle. It remains to be shown that S is a digitally convex set in C_n.

Suppose S is not digitally convex. Then there is some $v_i \notin S$ such that $N[v_i] \subseteq N[S]$. So at least one of v_{i-1} and v_{i+1} is in S. Also, if $v_{i-1} \notin S$, then $v_{i-2} \in S$ and if $v_{i+1} \notin S$, then $v_{i+2} \in S$.

If $v_{i-1}, v_{i+1} \in S$ then, in S^*, bits $i-2,\ i-1,\ i$ and $i+1$ must all be 1. Then, in the construction of S, since both bits $i-1$ and i are 1, the vertex v_i is added to S, which is a contradiction. If $v_{i+1} \notin S$ and $v_{i-1}, v_{i+2} \in S$, then, in S^*, bits $i-2,\ i-1,\ i+1$ and $i+2$ must all be 1, and bit i must be 0. However, the bits $i-1,\ i,\ i+1$ then form a substring 101, which is forbidden in S^*, so we have a contradiction. Similarly, if $v_{i-1} \notin S$ and v_{i-2}, v_{i+1}, then, in S^*, the bits $i-1,\ i,\ i+1$ form a substring 101, which is a contradiction.

Thus, the set S is digitally convex in C_n, and we have a bijection between the digitally convex sets of C_n and the cyclic binary n-bit strings with no alternating substring of length greater than 2, which satisfy the desired recurrence [23]. □

The number of digitally convex sets in a cycle satisfies the same recurrence as the number of cyclic binary n-bit strings with no alternating substring of length greater than 2, so the two sequences have the same generating function [23]

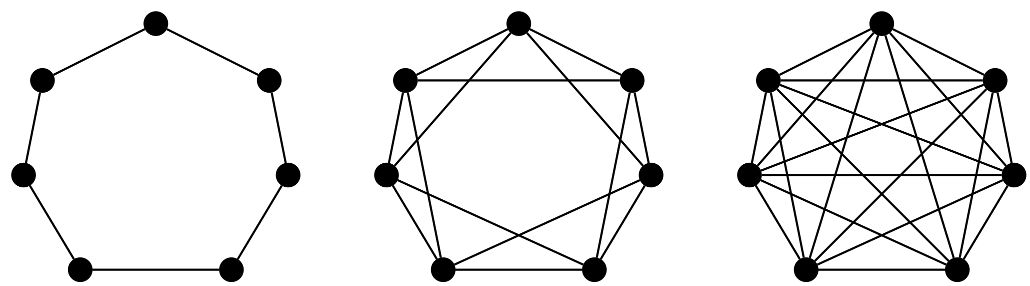
$$\frac{2x(1+x)(1-2x+2x^2)}{(1-x+x^2)(1-x-x^2)}.$$

Notice that this expands to

$$\begin{gathered} 2x + 2x^2 + 2x^3 + 6x^4 + 12x^5 + 20x^6 + 30x^7 + 46x^8 + 74x^9 \\ +122x^{10} + 200x^{11} + \cdots + 1362x^{15} + \cdots + 15126x^{20} + \cdots + 103684x^{24} \\ +167762x^{25} + \cdots + 1149852x^{29} + 1860500x^{30} + \ldots \end{gathered}$$

indicating that C_5 is the smallest cycle with more than ten digitally convex sets, C_{10} is the smallest cycle with more than 100, C_{15} is the smallest cycle with more than 1000, and C_{20} is the smallest cycle with more than 10 000. However, C_{24}, not C_{25}, is the smallest cycle with more than 100 000 digitally convex sets and, similarly, C_{29} is the smallest cycle with more than 1 000 000. This pattern suggests that $n_{\mathscr{D}}(C_{5k}) \geq 10^k$, but not that C_{5k} is the smallest such cycle satisfying this inequality.

Just as with paths, we can generalize the recurrence for the number of digitally convex sets of a cycle to the number of digitally convex sets of the k^{th} power of a cycle. Figure 4.4 shows the first three powers of the cycle C_7. Note that for $3 \leq n \leq 2k+1$, C_n^k is a complete graph. A vertex v_i in C_{2k+1}^k is adjacent to the vertices $v_{i-k}, v_{i-k+1}, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_{i+k}$, taking each subscript mod n if necessary so that it falls in the set $\{1, 2, \ldots, n\}$. Thus, each vertex has $2k$ distinct neighbours and $C_{2k+1}^k \cong K_{2k+1}$.



(a) $k = 1$ (b) $k = 2$ (c) $k = 3$

Figure 4.4: The graph C_7^k for $k = 1, 2, 3$

To enumerate the digitally convex sets of a power of a cycle, we identify a bijection between the digitally convex sets and cyclic binary strings avoiding blocks (maximal runs of 0's or 1's) of certain lengths, as in the proof of Theorem 4.1.

For $k \geq 2$, we let $\mathscr{B}_{k,n}$ be the set of cyclic binary strings of length n in which each block (maximal run of 0's or 1's) has length at least k, if $n \geq k$, or length exactly n, if $n < k$. Let $a_k(n) = |\mathscr{B}_{k,n}|$. Note that $\mathscr{B}_{2,n}$ is the set of cyclic binary strings with no alternating substring of length greater than 2, used in the proof of Theorem 4.1. Thus, $a_2(n)$ is the number of digitally convex sets in C_n. Now, before we show a bijection between the digitally convex sets of the k^{th} power of a cycle and the cyclic binary strings in $\mathscr{B}_{k,n}$, we first establish a recurrence relation satisfied by $a_k(n)$.

Munarini and Salvi [18] use the Schützenberger symbolic method to do this for $k = 2$. Symbolic methods used to enumerate combinatorial objects are described in detail in [13]. In particular, symbolic methods provide a way of breaking down a cyclic binary string into substring components to facilitate enumeration. Munarini and Salvi break the cyclic binary strings into the substrings 01, 10 and the remaining substrings of consecutive 0's and 1's. For example, the cyclic binary string 0011000011111 can be represented as (0)(01)(10)(00)(01)(1111). Then, from [13], the generating function for cyclic binary strings is the product of the generating functions for each of the substring components. We now use this method to generalize the result of Munarini and Salvi to any $k \geq 2$.

Lemma 4.2. *For some* $k \geq 2$, $a_k(i) = 2$ *for* $3 \leq i \leq 2k-1$, $a_k(j) = 2 + j(j-2k+1)$ *for* $2k \leq j \leq 2k+2$ *and, for* $n \geq 2k+3$,

$$a_k(n) = 2a_k(n-1) - a_k(n-2) + a_k(n-2k).$$

Proof. First, we show the initial conditions. If $3 \leq n < k$, then the only cyclic binary strings of length n with each block of length exactly n are clearly $(00\ldots0)$ and $(11\ldots1)$. Thus, $a_k(n) = 2$ for $3 \leq n < k$. Now suppose $k \leq n \leq 2k-1$. Any cyclic binary string with at least two blocks must have one block of length $\ell \leq \frac{n}{2} \leq \frac{2k-1}{2} < k$. So the only strings in $\mathscr{B}_{k,n}$ are $(00\ldots0)$ and $(11\ldots1)$ when $k \leq n \leq 2k-1$.

If $2k \leq n \leq 2k+2$, then clearly both $(00\ldots0)$ and $(11\ldots1)$ are cyclic binary strings in $\mathscr{B}_{k,n}$. The remaining strings in $\mathscr{B}_{k,n}$ are those with two blocks, one of length ℓ, with $k \leq \ell \leq n-k$, and the other of length $n-\ell$. Without loss of generality, let the block of length ℓ be ℓ consecutive 1's. There are n distinct cyclic shifts of these two blocks, giving n distinct cyclic binary strings with ℓ consecutive 1's. There are $n-2k+1$ possible values of ℓ, so there are $n(n-2k+1)$ cyclic binary strings in $\mathscr{B}_{k,n}$ with exactly two blocks. Overall, we have $a_k(n) = 2 + n(n-2k+1)$ for $2k \leq n \leq 2k+2$. Therefore, the initial conditions hold.

Now, we find the generating function to show the desired recurrence. To do this, we uniquely decompose the strings in $\mathscr{B}_{k,n}$ into the smaller strings $(0\ldots0)$, $(1\ldots1)$, (01), and (10). The latter two types of strings will be called *principal blocks*. Then, the strings in $\mathscr{B}_{k,n}$ containing a principal block can be decomposed in one of the following ways:

$$(0\ldots0)(01)(1^{k_1*})(10)(0^{k_2*})\ldots$$

$$(1\ldots1)(10)(0^{k_1*})(01)(1^{k_2*})\ldots$$

where $k_i{*} \geq k-2$ and an exponent of $k_i{*}$ means that the indicated bit is repeated a total of $k_i{*}$ times. So each principal block (01) that does not appear at the beginning of the string must be preceded by a string of type (0^{k_i*}) and followed by a string of type $(1^{k_{i+1}*})$, and, similarly, each principal block (10) must be preceded by a string of type (1^{k_j*}) and followed by a string of type $(0^{k_{j+1}*})$. Now, we can break these two cases down further into those containing an even number of principal blocks and those containing an odd number. The strings in $\mathscr{B}_{k,n}$ containing $2\ell > 0$ principal blocks have one of the forms

$$(0^{p_1})(01)(1^{k_1*})(10)(0^{k_2*})\ldots(01)(1^{k_{2\ell-1}*})(10)(0^{p_2})$$

$$(1^{q_1})(10)(0^{k_1*})(01)(1^{k_2*})\ldots(10)(0^{k_{2\ell-1}*})(01)(1^{q_2})$$

where each $k_i{*} \geq k-2$, $p_1+p_2 \geq k-2$ and $q_1+q_2 \geq k-2$.

Similarly, the strings in $\mathscr{B}_{k,n}$ containing $2\ell+1$ principal blocks have one of the forms

$$(0^{r_1})(01)(1^{k_1*})(10)(0^{k_2*})\ldots(01)(1^{r_2})$$

$$(1^{s_1})(10)(0^{k_1*})(01)(1^{k_2*})\ldots(10)(0^{s_2})$$

where, as above, each $k_i{*} \geq k-2$, and $r_1, r_2, s_1, s_2 \geq k-1$.

Then, we can express $\mathscr{B}_k$, the set of all cyclic binary strings with all blocks of length at least k, using the symbolic method described above. First, we let $0^* = \{\varepsilon, 0, 00, \ldots\}$ (ε denotes the empty string), $1^* = \{\varepsilon, 1, 11, \ldots\}$, $0^+ = 0^* - \{\varepsilon\}$ and $1^+ = 1^* - \{\varepsilon\}$. Then,

$$\mathscr{B}_k = 0^+ \cup 1^+ \bigcup_{p=0}^{k-2} \Big(\bigcup_{\ell=1}^{\infty} \underbrace{0^p(01)1^{k-2}1^*(10)0^{k-2}0^* \ldots (01)1^{k-2}1^*(10)0^{k-2-p}0^*}_{2\ell \textit{ principal blocks}} \Big)$$

$$\bigcup_{\ell=1}^{\infty} \underbrace{0^*0^{k-2}(01)1^{k-2}1^*(10)0^{k-2}0^* \ldots (01)1^{k-2}1^*(10)0^*}_{2\ell \textit{ principal blocks}}$$

$$\bigcup_{q=0}^{k-2} \Big(\bigcup_{\ell=1}^{\infty} \underbrace{1^q(10)0^{k-2}0^*(01)1^{k-2}1^* \ldots (10)0^{k-2}0^*(01)1^{k-2-q}1^*}_{2\ell \textit{ principal blocks}} \Big)$$

$$\bigcup_{\ell=1}^{\infty} \underbrace{1^*1^{k-2}(10)0^{k-2}0^*(01)1^{k-2}1^* \ldots (10)0^{k-2}0^*(01)1^*}_{2\ell \textit{ principal blocks}}$$

$$\bigcup_{\ell=0}^{\infty} \underbrace{0^*0^{k-1}(01)1^{k-2}1^*(10)0^{k-2}0^* \ldots (01)1^{k-1}1^*}_{2\ell+1 \textit{ principal blocks}}$$

$$\bigcup_{\ell=0}^{\infty} \underbrace{1^*1^{k-1}(10)0^{k-2}0^*(01)1^{k-2}1^* \ldots (10)0^{k-1}0^*}_{2\ell+1 \textit{ principal blocks}}$$

Note that we divide the strings with 2ℓ principal blocks into two cases: those beginning with fewer than $k-1$ 0's (or $k-1$ 1's) and those beginning with at least $k-1$ 0's (resp. 1's). We divide in this way because, in the first case, there is a minimum number of 0's (resp. 1's) that must be at the end of the string so that it is contained in $\mathscr{B}_{k,n}$. There is no such minimum in the second case.

Now, we can use the symbolic method to find the generating function for $a_k(n)$. The generating function for a string 0^* or 1^* is $\frac{1}{1-x}$, the generating function for a string 0^i or 1^i is x^i, and the generating function for a principal block (01) or (10) is x^2. From [13], we multiply the generating functions for the substrings to get a generating function for $a_k(n)$, using the same deconstruction as above.

$$B(x) = \sum_{n=0}^{\infty} a_k(n)x^n = 2\frac{x}{1-x}$$
$$+\sum_{p=0}^{k-2}\sum_{\ell=1}^{\infty}\left(x^p\frac{x^{k-2-p}}{1-x}(x^2)^{2\ell}\frac{(x^{k-2})^{2\ell-1}}{(1-x)^{2\ell-1}}\right)+\sum_{\ell=1}^{\infty}\left(\frac{x^{k-1}}{1-x}\frac{1}{1-x}(x^2)^{2\ell}\frac{(x^{k-2})^{2\ell-1}}{(1-x)^{2\ell-1}}\right)$$
$$+\sum_{q=0}^{k-2}\sum_{\ell=1}^{\infty}\left(x^q\frac{x^{k-2-q}}{1-x}(x^2)^{2\ell}\frac{(x^{k-2})^{2\ell-1}}{(1-x)^{2\ell-1}}\right)+\sum_{\ell=1}^{\infty}\left(\frac{x^{k-1}}{1-x}\frac{1}{1-x}(x^2)^{2\ell}\frac{(x^{k-2})^{2\ell-1}}{(1-x)^{2\ell-1}}\right)$$
$$+\sum_{\ell=0}^{\infty}\left((x^2)^{2\ell+1}\frac{(x^{k-2})^{2\ell}}{(1-x)^{2\ell}}\frac{(x^{k-1})^2}{(1-x)^2}\right)+\sum_{\ell=0}^{\infty}\left((x^2)^{2\ell+1}\frac{(x^{k-2})^{2\ell}}{(1-x)^{2\ell}}\frac{(x^{k-1})^2}{(1-x)^2}\right)$$

$$= \frac{2x}{1-x} + 2\sum_{p=0}^{k-2}\sum_{\ell=1}^{\infty}\left(\frac{x^{2k}}{(1-x)^2}\right)^{\ell} + \frac{2x}{1-x}\sum_{\ell=1}^{\infty}\left(\frac{x^{2k}}{(1-x)^2}\right)^{\ell} + \frac{2x^{2k}}{(1-x)^2}\sum_{\ell=0}^{\infty}\left(\frac{x^{2k}}{(1-x)^2}\right)^{\ell}$$

Each series in terms of ℓ in $B(x)$ is a geometric series. So we can write the function in closed form and simplify.

$$B(x) = \frac{2x}{1-x} + 2\sum_{p=0}^{k-2}\left(\frac{\frac{x^{2k}}{(1-x)^2}}{1-\frac{x^{2k}}{(1-x)^2}}\right) + \frac{2x}{1-x}\left(\frac{\frac{x^{2k}}{(1-x)^2}}{1-\frac{x^{2k}}{(1-x)^2}}\right) + \frac{2x^{2k}}{(1-x)^2}\left(\frac{1}{1-\frac{x^{2k}}{(1-x)^2}}\right)$$

$$= \frac{2x}{1-x} + \sum_{p=0}^{k-2}\left(\frac{2x^{2k}}{1-2x+x^2-x^{2k}}\right) + \frac{2x^{2k+1}}{(1-x)(1-2x+x^2-x^{2k})} + \frac{2x^{2k}}{1-2x+x^2-x^{2k}}$$

$$= \frac{2x}{1-x} + \frac{2(k-1)x^{2k}}{1-2x+x^2-x^{2k}} + \frac{2x^{2k+1}}{(1-x)(1-2x+x^2-x^{2k})} + \frac{2x^{2k}}{1-2x+x^2-x^{2k}}$$

$$= \frac{2x-4x^2+2x^3+2kx^{2k}-2kx^{2k+1}}{(1-x)(1-2x+x^2-x^{2k})}$$

$$= \frac{2x-2x^2+2kx^{2k}}{1-2x+x^2-x^{2k}}.$$

From the form of the generating function, we know that

$$a_k(n) - 2a_k(n-1) + a_k(n-2) - a_k(n-2k) = 0.$$

Rearranging this, we get the desired recurrence. □

Theorem 4.3. *Let C_n^k be the k^{th} power of the cycle C_n, $k \geq 1$. Then $n_{\mathscr{D}}(C_i^k) = 2$ for $3 \leq i \leq 2k+1$, $n_{\mathscr{D}}(C_j^k) = 2 + j(j-2k-1)$ for $2k+2 \leq j \leq 2k+4$ and, for $n \geq 2k+5$,*

$$n_{\mathscr{D}}(C_n^k) = 2n_{\mathscr{D}}(C_{n-1}^k) - n_{\mathscr{D}}(C_{n-2}^k) + n_{\mathscr{D}}(C_{n-2k-2}^k).$$

Proof. To prove the recurrence, we show a bijection between the digitally convex sets in $\mathscr{D}(C_n^k)$ and the cyclic binary n-bit strings in $\mathscr{B}_{k+1,n}$. If $n < k+1$, then these are the cyclic binary strings with blocks of length exactly n, i.e. $(0\dots0)$ and $(1\dots1)$. Clearly, since $k+1 \leq 2k+1$, there are exactly two digitally convex sets in C_n^k when $n < k+1$, the sets $\emptyset$ and $V(C_n^k)$. These sets get mapped to $(0\dots0)$ and $(1\dots1)$, respectively.

Now, suppose $n \geq k+1$. Then $\mathscr{B}_{k+1,n}$ is the set of cyclic binary n-bit strings whose maximal blocks each have length at least $k+1$. Given a digitally convex set $S \in \mathscr{D}(C_n^k)$, we get a corresponding cyclic binary n-bit string S^* in the following way. For each vertex $v_i \in S$, set bits $i, i+1, \dots, i+k$ in S^* to be 1, taking the index mod n if $i+j > n$. After repeating this for each vertex in S, set the remaining bits in S^* to 0. As an example, the digitally convex set $S = \{v_1, v_7\}$ in C_7^2, shown in Figure 4.5, corresponds to the cyclic binary string $S^* = (1110001)$.

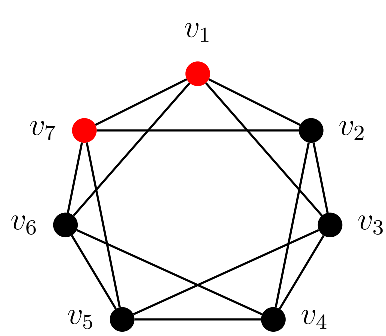


Figure 4.5: The digitally convex set $S = \{v_1, v_7\}$ of C_7^2 is indicated in red

It is clear from the construction of S^* that each block of 1's must have length at least $k+1$. We show now that each block of 0's in S^* must also have length at least $k+1$. Suppose that there is a block of 0's with length $\ell \leq k$, say bits $i, i+1, \ldots, i+\ell-1$. Then, the vertex $v_i \notin S$, since bit i is 0 in S^* and $v_{i+\ell} \in S$, since bit $i+\ell$ is 1 in S^*. In C_n^k, we must have $v_i v_{i+\ell} \in E(C_n^k)$, as $\ell \leq k$, as well as $v_{i+j} v_{i+\ell} \in E(C_n^k)$ for $j = 1, 2, \ldots, \ell - 1$. In S^*, bit $i-1$ is also 1. So, by the construction of S^*, the vertex $v_{i-k-1} \in S$. This vertex is adjacent to the vertices $v_{i-k}, v_{i-k+1}, \ldots, v_{i-1}$ in C_n^k. Thus, $N[v_i] \subseteq N[\{v_{i+\ell}, v_{i-k-1}\}] \subseteq N[S]$, contradicting the fact that S is digitally convex in C_n^k.

We now show that there is an injective map from $\mathscr{B}_{k+1,n}$ to the set of digitally convex sets of the k^{th} power of C_n. Let $S^* \in \mathscr{B}_{k+1,n}$. If $S^* = (00\ldots0)$, then let $S = \emptyset$. If $S^* = (11\ldots1)$, then let $S = V(C_n^k)$. Both of these are clearly digitally convex. Otherwise, let $B_1, B_2, \ldots, B_r$ be the distinct blocks of at least $k+1$ 1's in S^*. Say bits $i, i+1, \ldots, i+k+\ell-1$ are the bits of B_1. Then, let $S_1 = \{v_i, v_{i+1}, \ldots, v_{i+\ell-1}\}$. Define $S_2, S_3, \ldots, S_r$ similarly. Finally, let $S = S_1 \cup S_2 \cup \cdots \cup S_r$. It is clear that S would be mapped to S^* using the above mapping. For example, if $S^* = (1110001)$ and $k = 2$, then bits $7, 1, 2, 3$ are the bits of the only block of 1's. This string would be mapped to the set of vertices $\{v_7, v_1\}$, reversing the example of the mapping shown earlier in the proof. We show now that each such set S is a digitally convex set in $\mathscr{D}(C_n^k)$. Suppose otherwise, i.e. that S is not digitally convex. Then, there must be some $v_j \notin S$ such that $N[v_j] = \{v_{j-k}, v_{j-k+1}, \ldots, v_j, v_{j+1}, \ldots, v_{j+k}\} \subseteq N[S]$.

Since $v_j \notin S$ and $v_{j-k} \in N[S]$, we must have one of the vertices in $N[v_{j-k}] - \{v_j\}$ in S, say v_{j-k+m} for some $m \in \{-k, -k+1, \ldots, k-1\}$. Then, by definition of S, the bits $j-k+m$, $j-k+m+1, \ldots, j+m$ are all 1 in S^*. None of these vertices is adjacent to v_{j+k}, so one of the vertices in $N[v_{j+k}] - \{v_j\}$ is in S, say v_{j+k+p}, for some $p \in \{-k+1, -k+2, \ldots, k\}$. So, again by definition, the bits $j+k+p$, $j+k+p+1, \ldots, j+2k+p$ are each 1 in S^*. In addition, these two vertices can be chosen so that each of the vertices $v_{j-k}, v_{j-k+1}, \ldots, v_j, v_{j+1}, \ldots, v_{j+k}$ appears in the closed neighbourhood of v_{j-k+m} or v_{j+k+p}. Thus, the maximum possible difference between $j-k+m$ and $j+k+p$ is $2k+1$. Then, the longest block of 0's in S^* between bits $j+m$ and $j+k+p$ has length at most k, contradicting the fact that $S^* \in \mathscr{B}_{k+1,n}$. Therefore, S is digitally convex in C_n^k.

We now have a bijection between the digitally convex sets in $\mathscr{D}(C_n^k)$ and the cyclic binary strings in $\mathscr{B}_{k+1,n}$. So they satisfy the same recurrence. Therefore, $n_{\mathscr{D}}(C_n^k) = 2n_{\mathscr{D}}(C_{n-1}^k) - n_{\mathscr{D}}(C_{n-2}^k) + n_{\mathscr{D}}(C_{n-2(k+1)}^k)$, with $n_{\mathscr{D}}(C_i^k) = 2$, for $3 \leq i \leq 2(k+1)-1$, and $n_{\mathscr{D}}(C_j^k) = 2 + j(j - 2(k+1) + 1)$, for $2(k+1) \leq j \leq 2(k+2)$. Simplifying, we get the desired recurrence. □

4.2 Cartesian Products

For some graph parameters, knowing their value for graphs G and H gives their value for a product of G and H. For other parameters, such a relationship is not known. In this section, we examine the digitally convex sets of the Cartesian products of complete graphs and of paths. In both of these cases, we see that there is not an obvious relationship between the digitally convex sets in the product and in the constituent graphs. We begin by giving the definition of the Cartesian product.

Definition 4.4. *Let G and H be graphs. Then, the* Cartesian product*, $G \Box H$, is the graph with vertex set $V(G\Box H) = V(G) \times V(H)$ and an edge $(u,v)(x,y) \in E(G \Box H)$ if and only if $u = x$ in G and $vy \in E(H)$, or $ux \in E(G)$ and $v = y$ in H.*

A digitally convex set in the Cartesian product $G\Box H$ is not necessarily digitally convex when restricted to G or to H. In other words, if $S \in \mathscr{D}(G\Box H)$ then the set $S_G = \{x \in V(G) \mid (x,y) \in S\}$ is not necessarily digitally convex in G. As an

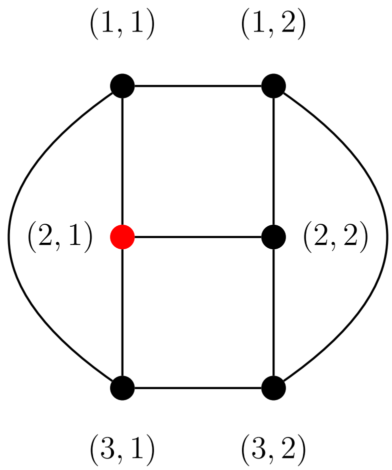


Figure 4.6: The set $\{(2,1)\} \in \mathscr{D}(K_3 \square K_2)$ is indicated in red

example, the set $\{(2,1)\}$, shown in Figure 4.6, is digitally convex in $K_3 \square K_2$ but $\{2\} \notin \mathscr{D}(K_3)$ and $\{1\} \notin \mathscr{D}(K_2)$, as the only digitally convex sets in a complete graph are the empty set and the entire vertex set. This example shows that, even in small graphs, the digitally convex sets of a Cartesian product of graphs G and H cannot be obviously computed from those of G and H. We begin by examining the number of digitally convex sets in the Cartesian product of complete graphs, $K_n \square K_2$, to show how different this number is from the number of digitally convex sets in either of the constituent graphs of the product.

Proposition 4.5. *For any $n \geq 1$, $n_{\mathscr{D}}(K_n \square K_2) = 2^{n+1} - 2$.*

Proof. First, we denote the vertices of $K_n \square K_2$ by $v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$, with $v_i v_j \in E(K_n \square K_2)$ and $u_i u_j \in E(K_n \square K_2)$ for all $i \neq j$ and $v_i u_i \in E(K_n \square K_2)$ for each $i = 1, 2 \ldots, n$.

For any i, j, the set $\{v_i, u_j\}$ is a dominating set for $K_n \square K_2$, so the only digitally convex set containing both of these vertices is $V(K_n \square K_2)$. The rest of the digitally convex sets must be subsets of $\{v_1, v_2, \ldots, v_n\}$ or of $\{u_1, u_2, \ldots, u_n\}$. Since $\{v_1, v_2, \ldots, v_n\}$, and similarly $\{u_1, u_2, \ldots, u_n\}$, is a dominating set for $K_n \square K_2$, it is not digitally convex. Consider $S \subsetneq \{v_1, v_2, \ldots, v_n\}$. Then, for any $v_j \notin S$, we have $u_j \notin N[S]$, so u_j is a private neighbour for v_j and for every u_i. Therefore, S is a digitally convex set. There are $2^n - 1$ such digitally convex sets.

Similarly, any set $S \subsetneq \{u_1, u_2, \ldots, u_n\}$ is a digitally convex set. However, the empty set was counted in the previous case, so here we get only $2^n - 2$ additional digitally convex sets.

Overall, this gives $n_{\mathscr{D}}(K_n \square K_2) = 1 + 2^n - 1 + 2^n - 2 = 2^{n+1} - 2$. $\square$

By taking the Cartesian product of two larger complete graphs, the number of digitally convex sets quickly increases, while the number of digitally convex sets of a single complete graph is independent of the order of the graph. As an example, the product $K_3 \square K_2$, in Figure 4.6, has $2^4 - 2 = 14$ digitally convex sets. The graph $K_3 \square K_3$, however, has 38 digitally convex sets, as we show below, which is more than twice the number in $K_3 \square K_2$.

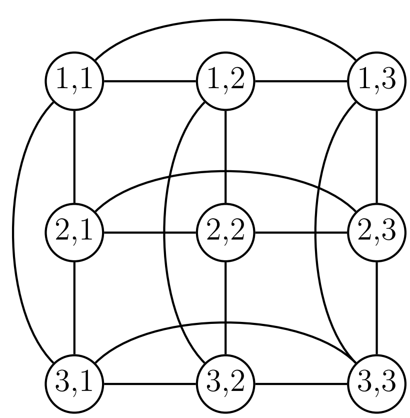


Figure 4.7: $K_3 \square K_3$

To find the digitally convex sets of $K_3 \square K_3$, let $\emptyset \neq S_1 \subsetneq \{1, 2, 3\}$ and $\emptyset \neq S_2 \subsetneq \{1, 2, 3\}$. Then, the set $S = \{(x, y) \mid x \in S_1, y \in S_2\}$ is digitally convex. There are $(2^3 - 2)(2^3 - 2) = 36$ such sets. These sets, along with $\emptyset$ and $V(K_3 \square K_3)$, form the 38 digitally convex sets in $K_3 \square K_3$. This method of generating digitally convex sets extends to the Cartesian product of two complete graphs of any order, giving the following result.

Theorem 4.6. *For any* $m, n \geq 1$, $n_{\mathscr{D}}(K_n \Box K_m) = 2 + (2^n - 2)(2^m - 2)$.

Proof. We begin by denoting the vertices of $K_n \Box K_m$ by (v_i, u_j) for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Now, let $\emptyset \neq S_1 \subsetneqq \{v_1, v_2, \ldots, v_n\}$ and $\emptyset \neq S_2 \subsetneqq \{u_1, u_2, \ldots, u_m\}$. Then, $S = S_1 \times S_2 = \{(v_i, u_j) \mid v_i \in S_1, u_j \in S_2\}$ is digitally convex in $K_n \Box K_m$. Consider $(v_x, u_y) \notin S$. If $v_x \notin S_1$ and $u_y \notin S_2$, then $(v_x, u_y) \notin N[S]$. If $v_x \in S_1$ and $u_y \notin S_2$, then there is some $v_z \notin S_1$ such that $(v_x, u_y)(v_z, u_y) \in E(K_n \Box K_m)$ and $(v_z, u_y) \notin N[S]$. If $v_x \notin S_1$ and $u_y \in S_2$, then there is some $u_w \notin S_2$ such that $(v_x, u_y)(v_x, u_w) \in E(K_n \Box K_m)$ and $(v_x, u_w) \notin N[S]$. Thus, (v_x, u_y) has a private neighbour with respect to S, and so S is digitally convex. There are $(2^n - 2)(2^m - 2)$ such sets S.

Any set of vertices containing a set of type $\{(v_1, u_{i_1}), (v_2, u_{i_2}), \ldots, (v_n, u_{i_n})\}$, where each $i_j \in \{1, 2, \ldots, m\}$, is a dominating set in $K_n \Box K_m$. Similarly, any set of vertices containing a set of type $\{(v_{j_1}, u_1), (v_{j_2}, u_2), \ldots, (v_{j_m}, u_m)\}$, where each $j_k \in \{1, 2, \ldots, n\}$, is a dominating set. Therefore, the only digitally convex set containing any of these sets of vertices is $V(K_n \Box K_m)$.

Two vertices (v_x, u_y) and (v_w, u_z) dominate the neighbourhoods of both of the vertices (v_x, u_z) and (v_w, u_y). So any digitally convex set containing the former pair of vertices must also contain the latter pair. Therefore, every nonempty digitally convex set in $K_n \Box K_m$ must be $V(K_n \Box K_m)$ or must take on the form $S_1 \times S_2$, where S_1 and S_2 are defined as above.

Therefore, along with the empty set, the graph $K_n \Box K_m$ has a total of $2 + (2^n - 2)(2^m - 2)$ digitally convex sets. □

We turn now to the Cartesian product of paths, beginning with $P_n \Box P_2$. As an example, consider $P_3 \square P_2$, shown in Figure 4.8, which has the following collection of 16 digitally convex sets:

$$\mathscr{D}(P_3 \square P_2) = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{u_1\}, \{u_2\}, \{u_3\}, \{v_1, v_3\}, \{v_1, u_1\}, \{v_3, u_3\},$$
$$\{u_1, u_3\}, \{v_1, v_2, u_1\}, \{v_1, u_1, u_2\}, \{v_2, v_3, u_3\}, \{v_3, u_2, u_3\}, V(P_3 \square P_2)\}$$

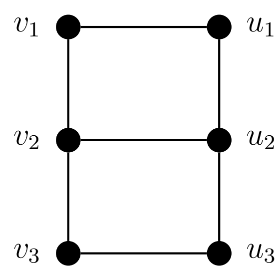


Figure 4.8: $P_3 \square P_2$

As with the Cartesian product of complete graphs, many of the digitally convex sets in the product of paths are no longer digitally convex when restricted to one of the constituent graphs. Thus, there is no obvious method of using the digitally convex sets of the constituent graphs to generate those of the product. We can, however, use the digitally convex sets of the graphs $P_{n-1} \square P_2$, $P_{n-2} \square P_2$ and $P_{n-3} \square P_2$ to determine those of $P_n \square P_2$.

Theorem 4.7. *Let P_n be the path of order n. Then $n_{\mathscr{D}}(P_1 \square P_2) = 2$, $n_{\mathscr{D}}(P_2 \square P_2) = 6$ and $n_{\mathscr{D}}(P_3 \square P_2) = 16$ and, for $n \geq 4$,*

$$n_{\mathscr{D}}(P_n \square P_2) = n_{\mathscr{D}}(P_{n-1} \square P_2) + 3n_{\mathscr{D}}(P_{n-2} \square P_2) + 2n_{\mathscr{D}}(P_{n-3} \square P_2).$$

Proof. First, we denote the vertices of $P_n \square P_2$ by $v_1, v_2 \ldots, v_n, u_1, u_2, \ldots, u_n$, with $v_i v_{i+1} \in E(P_n \square P_2)$ and $u_i u_{i+1} \in E(P_n \square P_2)$ for $i = 1, 2, \ldots, n-1$ and $v_j u_j \in E(P_n \square P_2)$ for $j = 1, 2, \ldots, n$.

Now, we prove the initial conditions. Since $P_1 \Box P_2 \cong K_2$, we have $n_{\mathscr{D}}(P_1 \Box P_2) = n_{\mathscr{D}}(K_2) = 2$. Similarly, $P_2 \Box P_2 \cong C_4$, so $n_{\mathscr{D}}(P_2 \Box P_2) = n_{\mathscr{D}}(C_4) = 6$. Finally, the 16 digitally convex sets of $P_3 \Box P_2$ are listed above. So $n_{\mathscr{D}}(P_3 \Box P_2) = 16$.

Suppose $n \geq 4$. We begin by showing $n_{\mathscr{D}}(P_n \Box P_2) \geq n_{\mathscr{D}}(P_{n-1} \Box P_2) + 3n_{\mathscr{D}}(P_{n-2} \Box P_2) + 2n_{\mathscr{D}}(P_{n-3} \Box P_2)$. We now construct three pairwise disjoint families $\mathscr{D}_i$, $i = 1, 2, 3$, of digitally convex sets in $\mathscr{D}(P_n \Box P_2)$ such that $|\mathscr{D}_i| = c_i n_{\mathscr{D}}(P_{n-i} \Box P_2)$, where $c_1 = 1, c_2 = 3, c_3 = 2$.

To construct $\mathscr{D}_1$, let $S \in \mathscr{D}(P_{n-1} \Box P_2)$. If $v_{n-1}, u_{n-1} \notin S$, then S is digitally convex in $P_n \Box P_2$, because the vertices v_n and u_n are each a private neighbour for themselves with respect to S. Then, we add S to $\mathscr{D}_1$. If $v_{n-1} \in S$ or $u_{n-1} \in S$, then $S \cup \{v_n, u_n\}$ is digitally convex in $P_n \Box P_2$, because each vertex in $V(P_{n-1} \Box P_2) - S$ must have a private neighbour with respect to S in $V(P_{n-1} \Box P_2) - \{v_{n-1}, u_{n-1}\}$, which is also a private neighbour with respect to $S \cup \{v_n, u_n\}$ in $P_n \Box P_2$. Then, we add $S \cup \{v_n, u_n\}$ to $\mathscr{D}_1$. Note that $|\mathscr{D}_1| = n_{\mathscr{D}}(P_{n-1} \Box P_2)$, as desired.

To construct $\mathscr{D}_2$, let $S \in \mathscr{D}(P_{n-2} \Box P_2)$. If $v_{n-2}, u_{n-2} \in S$, then S is digitally convex in $P_n \Box P_2$, because the vertices u_n and v_n are private neighbours for themselves, as well as for v_{n-1} and u_{n-1}, with respect to S in $P_n \Box P_2$. Then, we add S to $\mathscr{D}_2$. This set is not digitally convex in $P_{n-1} \Box P_2$, as the vertices v_{n-1} and u_{n-1} have no private neighbours with respect to S. The set $S \cup \{v_{n-1}\}$ is also digitally convex in $P_n \Box P_2$, because the vertex u_n is a private neighbour for itself, v_n and u_{n-1} with respect to S in $P_n \Box P_2$. Then, we add $S \cup \{v_1\}$ to $\mathscr{D}_2$. Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \Box P_2$, so we add it to $\mathscr{D}_2$.

If $v_{n-2} \in S$ and $u_{n-2} \notin S$, then $S \cup \{v_n\}$ is digitally convex in $P_n \Box P_2$, because the vertex u_{n-1} is a private neighbour for itself, v_{n-1} and u_n with respect to $S \cup \{v_n\}$ in $P_n \Box P_2$. Then, we add $S \cup \{v_n\}$ to $\mathscr{D}_2$. The set $S \cup \{v_{n-1}\}$ is also digitally convex in $P_n \Box P_2$, because the vertex u_n is a private neighbour for itself, v_n and u_{n-1} with respect to $S \cup \{v_{n-1}\}$. Then, we add $S \cup \{v_{n-1}\}$ to $\mathscr{D}_2$. Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \Box P_2$, so we add it to $\mathscr{D}_2$.

If $v_{n-2} \notin S$ and $u_{n-2} \in S$ then, by the same argument as above, $S \cup \{v_{n-1}\}$, $S \cup \{u_{n-1}\}$ and $S \cup \{u_n\}$ are all digitally convex in $P_n \Box P_2$. We add them all to $\mathscr{D}_2$.

If $v_{n-2}, u_{n-2} \notin S$, then $S \cup \{v_n\}$ is digitally convex in $P_n \Box P_2$ because the vertex u_{n-1} is a private neighbour for itself, u_n and v_{n-1} with respect to $S \cup \{v_n\}$ in $P_n \Box P_2$. Then, we add $S \cup \{v_n\}$ to $\mathscr{D}_2$. Similarly, $S \cup \{u_n\}$ is digitally convex in $P_n \Box P_2$, so we add it to $\mathscr{D}_2$. If, in addition, $v_{n-3}, u_{n-3} \notin S$, then $S \cup \{v_n, u_n\}$ is digitally convex in $P_n \Box P_2$, because both $u_{n-2}, v_{n-2} \notin N[S \cup \{v_n, u_n\}]$ in $P_n \Box P_2$. So u_{n-2} and v_{n-2} are private neighbours for u_{n-1} and v_{n-1} with respect to $S \cup \{v_n, u_n\}$. Then, we add $S \cup \{v_n, u_n\}$ to $\mathscr{D}_2$. If $v_{n-3} \in S$ and $u_{n-3} \notin S$, then $S \cup \{v_{n-1}\}$ is digitally convex in $P_n \Box P_2$, because the vertex u_{n-2} is a private neighbour for itself and for v_{n-2}, and the vertex u_n is a private neighbour for itself, v_n and u_{n-1} with respect to $S \cup \{v_{n-1}\}$ in $P_n \Box P_2$. Then, we add $S \cup \{v_{n-1}\}$ to $\mathscr{D}_2$. Similarly, if $v_{n-3} \notin S$ and $u_{n-3} \in S$, then $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \Box P_2$. So we add it to $\mathscr{D}_2$. Now, we have $|\mathscr{D}_2| = 3n_{\mathscr{D}}(P_{n-2} \Box P_2)$, as desired.

Finally, to construct $\mathscr{D}_3$, let $S \in \mathscr{D}(P_{n-3} \Box P_2)$. If $v_{n-3}, u_{n-3} \notin S$, then $S \cup \{v_{n-1}\}$ is digitally convex in $P_n \Box P_2$, because the vertex u_{n-2} is a private neighbour for itself, v_{n-2} and u_{n-1}, and the vertex u_n is a private neighbour for itself and v_n with respect to $S \cup \{v_{n-1}\}$. Then, we add $S \cup \{v_{n-1}\}$ to $\mathscr{D}_3$. Similarly, $S \cup \{u_{n-1}\}$ is digitally convex in $P_n \Box P_2$, so we add it to $\mathscr{D}_3$.

If $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S \cup \{v_{n-2}, v_n\}$ is digitally convex in $P_n \Box P_2$, because the vertex u_{n-1} is a private neighbour for itself, u_{n-2}, v_{n-1} and u_n with respect to $S \cup \{v_{n-2}, v_n\}$ in $P_n \Box P_2$. Then, we add $S \cup \{v_{n-2}, v_n\}$ to $\mathscr{D}_3$. Similarly, $S \cup \{u_{n-2}, u_n\}$ is digitally convex in $P_n \Box P_2$, so we add it to $\mathscr{D}_3$. Now, we have $|\mathscr{D}_3| = 2n_{\mathscr{D}}(P_{n-3} \Box P_2)$, as desired.

Now, we have $\mathscr{D}_i \cap \mathscr{D}_j = \emptyset$ for $i \neq j$, and each $\mathscr{D}_i$, $i = 1, 2, 3$, is a subset of $\mathscr{D}(P_n \square P_2)$. Thus $n_{\mathscr{D}}(P_n \square P_2) \geq |\mathscr{D}_1| + |\mathscr{D}_2| + |\mathscr{D}_3| = n_{\mathscr{D}}(P_{n-1} \square P_2) + 3n_{\mathscr{D}}(P_{n-2} \square P_2) + 2n_{\mathscr{D}}(P_{n-3} \square P_2)$.

Now, to show the reverse inequality, let $S \in \mathscr{D}(P_n \square P_2)$.

(a) Suppose $v_n, u_n \in S$. If $v_{n-1} \in S$ or $u_{n-1} \in S$, then each $x \notin S$ has a private neighbour with respect to S in $V(P_{n-1} \square P_2)$. Thus, $S - \{v_n, u_n\}$ is digitally convex in $P_{n-1} \square P_2$. If $v_{n-1}, u_{n-1} \notin S$, then $v_{n-2}, u_{n-2} \notin N[S]$. Thus, $v_{n-3}, u_{n-3} \notin S$ and $S - \{v_n, u_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(b) Suppose $v_n \in S$ and $u_n \notin S$. Then, $u_{n-1} \notin N[S]$, so $v_{n-1}, u_{n-1}, u_{n-2} \notin S$. If $v_{n-2} \in S$ and $v_{n-3}, u_{n-3} \notin S$, then it must be the case that $u_{n-3} \notin N[S]$. So $S - \{v_n\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-2} \in S$ and $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S - \{v_n, v_{n-2}\}$ is digitally convex in $P_{n-3} \square P_2$. If $v_{n-2} \notin S$, then at most one of v_{n-3} and u_{n-3} can be in S. So either $v_{n-2} \notin N[S]$ or $u_{n-2} \notin N[S]$. Then, $S - \{v_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(c) Suppose $v_n \notin S$ and $u_n \in S$, then $v_{n-1} \notin N[S]$, so $v_{n-1}, u_{n-1}, v_{n-2} \notin S$. If $u_{n-2} \in S$ and $v_{n-3}, u_{n-3} \notin S$, then $S - \{u_n\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-2} \in S$ and $v_{n-3} \in S$ or $u_{n-3} \in S$, then $S - \{u_n, u_{n-2}\}$ is digitally convex in $P_{n-3} \square P_2$. If $u_{n-2} \notin S$, then $S - \{u_n\}$ is digitally convex in $P_{n-2} \square P_2$.

(d) Suppose $v_n, u_n \notin S$. Then, at most one of v_{n-1} and u_{n-1} can be in S. If $v_{n-1} \in S$ and at least one of v_{n-2} and u_{n-2} is in S, then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-1} \in S$ and $v_{n-2}, u_{n-2} \notin S$, then it must be the case that $u_{n-2} \notin N[S]$. So $u_{n-3} \notin S$. If $v_{n-3} \in S$, then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $v_{n-3} \notin S$, then $S - \{v_{n-1}\}$ is digitally convex in $P_{n-3} \square P_2$. Similarly, if $u_{n-1} \in S$ and at least one of v_{n-2} and u_{n-2} is in S, then $S - \{u_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-1} \in S$, $v_{n-2}, u_{n-2} \notin S$ and $u_{n-3} \in S$, then

$S - \{u_{n-1}\}$ is digitally convex in $P_{n-2} \square P_2$. If $u_{n-1} \in S$, $v_{n-2}, u_{n-2} \notin S$ and $u_{n-3} \notin S$, then $S - \{u_{n-1}\}$ is digitally convex in $P_{n-3} \square P_2$.

Each digitally convex set in $P_{n-1} \square P_2$ has been counted here at most once, each digitally convex set in $P_{n-2} \square P_2$ at most three times, and each digitally convex set in $P_{n-3} \square P_2$ at most twice. Refer to Table 4.1 for a summary of which digitally convex sets in $P_{n-2} \square P_2$ and $P_{n-3} \square P_2$ are counted in each part of the above argument. Therefore, $n_{\mathscr{D}}(P_n \square P_2) \leq n_{\mathscr{D}}(P_{n-1} \square P_2) + 3n_{\mathscr{D}}(P_{n-2} \square P_2) + 2n_{\mathscr{D}}(P_{n-3} \square P_2)$. $\square$

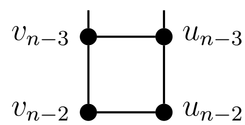
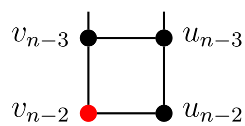
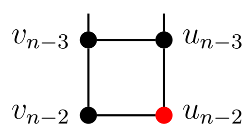
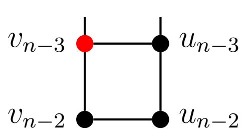
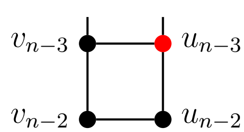
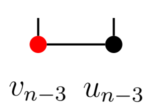
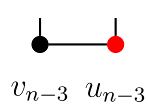
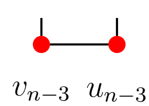
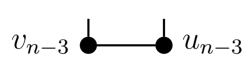
$P_{n-2} \square P_2$	
	(a), (b), and (c)
	(b), and twice in (d)
	(c), and twice in (d)
 and 	(b), (c), and (d)
$P_{n-3} \square P_2$	
 and  and 	(b) and (c)
	twice in (d)

Table 4.1: A summary of the counting argument in Theorem 4.7

From the proof of Theorem 4.7, we get an algorithm for generating the digitally convex sets of $P_n \Box P_2$.

Algorithm 5. *Generating the collection $\mathscr{D}(P_n \Box P_2)$ of all digitally convex sets of $P_n \Box P_2$ for $n \geq 1$.*

1. *If $n = 1$, then $\mathscr{D}(P_1 \Box P_2) = \{\emptyset, V(P_1 \Box P_2)\}$.*
2. *If $n = 2$, then $\mathscr{D}(P_2 \Box P_2) = \{\emptyset, \{v_1\}, \{v_2\}, \{u_1\}, \{u_2\}, V(P_2 \Box P_2)\}$*
3. *If $n = 3$, then $\mathscr{D}(P_3 \Box P_2) = \{\emptyset, \{v_1\}, \{v_2\}, \{v_3\}, \{u_1\}, \{u_2\}, \{u_3\}, \{v_1, v_3\}, \{v_1, u_1\}, \{v_3, u_3\}, \{u_1, u_3\}, \{v_1, v_2, u_1\}, \{v_1, u_1, u_2\}, \{v_2, v_3, u_3\}, \{v_3, u_2, u_3\}, V(P_3 \Box P_2)\}$*
4. *Suppose $n \geq 4$. Use this algorithm to generate the collections $\mathscr{D}(P_{n-1} \Box P_2)$, $\mathscr{D}(P_{n-2} \Box P_2)$ and $\mathscr{D}(P_{n-3} \Box P_2)$. Obtain $\mathscr{D}(P_n \Box P_2)$ as follows: Set $\mathscr{S}_n = \emptyset$.*
   (a) *For each $S \in \mathscr{D}(P_{n-1} \Box P_2)$*
      i. *if $v_{n-1} \in S$ or $u_{n-1} \in S$, then add $S \cup \{u_n, v_n\}$ to $\mathscr{S}_n$.*
      ii. *if $v_{n-1}, u_{n-1} \notin S$, then add S to $\mathscr{S}_n$.*

   (b) *For each $S \in \mathscr{D}(P_{n-2} \Box P_2)$*
      i. *if $v_{n-2}, u_{n-2} \in S$, then add $S \cup \{v_{n-1}\}$, $S \cup \{u_{n-1}\}$ and S to $\mathscr{S}_n$.*
      ii. *if $v_{n-2} \in S$ and $u_{n-2} \notin S$, then add $S \cup \{v_{n-1}\}$, $S \cup \{u_{n-1}\}$ and $S \cup \{v_n\}$ to $\mathscr{S}_n$.*
      iii. *if $v_{n-2} \notin S$ and $u_{n-2} \in S$, then add $S \cup \{v_{n-1}\}$, $S \cup \{u_{n-1}\}$ and $S \cup \{u_n\}$ to $\mathscr{S}_n$.*
      iv. *if $v_{n-2}, u_{n-2}, v_{n-3}, u_{n-3} \notin S$, then add $S \cup \{v_n\}$, $S \cup \{u_n\}$ and $S \cup \{v_n, u_n\}$ to $\mathscr{S}_n$.*
      v. *if $v_{n-2}, u_{n-2}, u_{n-3} \notin S$ and $v_{n-3} \in S$, then add $S \cup \{v_n\}$, $S \cup \{u_n\}$ and $S \cup \{v_{n-1}\}$ to $\mathscr{S}_n$.*

vi. if $v_{n-2}, u_{n-2}, v_{n-3} \notin S$ *and* $u_{n-3} \in S$*, then add* $S \cup \{v_n\}$*,* $S \cup \{u_n\}$ *and* $S \cup \{u_{n-1}\}$ *to* $\mathscr{S}_n$*.*

(c) For each $S \in \mathscr{D}(P_{n-3} \square P_2)$

i. if $v_{n-3} \in S$ *or* $u_{n-3} \in S$*, then add* $S \cup \{v_{n-2}, v_n\}$ *and* $S \cup \{u_{n-2}, u_n\}$ *to* $\mathscr{S}_n$*.*

ii. if $v_{n-3}, u_{n-3} \notin S$*, then add* $S \cup \{v_{n-1}\}$ *and* $S \cup \{u_{n-1}\}$ *to* $\mathscr{S}_n$*.*

(d) Then, $\mathscr{D}(P_n \square P_2) = \mathscr{S}_n$*.*

The number of digitally convex sets of $P_n \square P_2$ follows the OEIS sequence A217631, which is also the number of $n \times 2$ arrays of the minimum value of corresponding elements and their horizontal and vertical neighbours in a random $n \times 2$ binary array [23]. Examples of these are given below. This sequence has the generating function

$$\frac{2x + 4x^2 + 4x^3}{1 - x - 3x^2 - 2x^3}.$$

More generally, given a set $\mathscr{A}$ of $n \times m$ binary arrays, we let $\mathscr{A}^*$ be the set of arrays obtained as follows. For each array A in $\mathscr{A}$, construct a new array A^* by taking the minimum value of corresponding elements of A and their horizontal and vertical neighbours. In other words, each element of A^* is the minimum value over the closed neighbourhood of the corresponding element of A.

As an example, consider the following array A

$$\begin{matrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1. \end{matrix}$$

Each of the entries on the diagonal have value 1, as does each of their horizontal and vertical neighbours. The entries off the diagonal each either have value 0 or have a neighbour of value 0. Taking the minimum over the closed neighbourhood of each element in the array produces the array A^*:

$$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1. \end{array}$$

Note that in this process, two distinct arrays A_1 and A_2 can produce the same array A^*. Consider the following two 3×2 arrays.

$$\begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{array} \qquad \begin{array}{cc} 1 & 1 \\ 1 & 0 \\ 0 & 1. \end{array}$$

In both of these arrays, only the element in the first row and column has a closed neighbourhood with minimum value 1. Each other entry either has value 0 or has a neighbour with value 0. Thus, for both of the above arrays, taking the minimum value over the closed neighbourhood of each element will produce the array:

$$\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 0. \end{array}$$

By letting $\mathscr{A}_{3,2}$ be all 3×2 binary arrays, we see that $\mathscr{A}^*_{3,2}$ contains 16 distinct arrays. Each of these arrays corresponds to a digitally convex set in $P_3 \Box P_2$, with the 1's in the array indicating the vertices contained in the digitally convex set. For example, the array

$$\begin{matrix} 1 & 1 \\ 1 & 0 \\ 0 & 0 \end{matrix}$$

corresponds to the digitally convex set $\{v_1, u_1, v_2\}$ in $P_3 \square P_2$. The same correspondence exists between the digitally convex sets of $P_n \square P_m$ and distinct arrays in $\mathscr{A}^*_{n,m}$, where $\mathscr{A}_{n,m}$ is the set of all $n \times m$ binary arrays.

Theorem 4.8. *Let $\mathscr{A}_{n,m}$ be the set of all $n \times m$ binary arrays. Then, $n_{\mathscr{D}}(P_n \square P_m) = |\mathscr{A}^*_{n,m}|$.*

Proof. First, we label the vertices of the product $P_n \square P_m$. Let the vertices of P_n be $u_1, u_2, \ldots, u_n$, with $u_i u_{i+1} \in E(P_n)$ for $i = 1, 2, \ldots, n-1$, and let the vertices of P_m be $v_1, v_2, \ldots, v_m$, with $v_j v_{j+1} \in E(P_m)$ for $j = 1, 2, \ldots, m-1$. Then, the vertices of $P_n \square P_m$ have the form (u_i, v_j).

Now we show a bijection between the digitally convex sets in $\mathscr{D}(P_n \square P_m)$ and the arrays in $\mathscr{A}^*_{n,m}$. Let $A^* \in \mathscr{A}^*_{n,m}$ and consider the set $S = \{(u_i, v_j) \mid a^*_{i,j} = 1\}$. Each vertex $(u_x, v_y) \notin S$ corresponds to an entry $a^*_{x,y}$ that has value 0 in A^*. Then, either the corresponding entry in A also has value 0, or it has value 1 and has a horizontal or vertical neighbour with value 0. In the first case, every entry in the closed neighbourhood of $a^*_{x,y}$ also has value 0 in A^*. In $P_n \square P_m$, this means that none of the vertices in $N[(u_x, v_y)]$ is in S, so (u_x, v_y) is its own private neighbour. In the second case, there is an entry $a_{w,z}$ in the closed neighbourhood of $a_{x,y}$ which has value 0 in A. Then, in A^*, every entry in the closed neighbourhood of $a^*_{w,z}$ has value 0, including $a^*_{x,y}$. In $P_n \square P_m$, this means that none of the vertices in $N[(u_w, v_z)]$ is in S and $(u_w, v_z)(u_x, v_y) \in E(P_n \square P_m)$, so the vertex (u_w, v_z) is a private neighbour for (u_x, v_y) with respect to S. Therefore, S is digitally convex in $P_n \square P_m$.

It is clear from the construction of S that this mapping from $\mathscr{A}^*_{n,m}$ to $\mathscr{D}(P_n \square P_m)$

is injective. It remains to be shown that the mapping is surjective. Consider $S \in \mathscr{D}(P_n \square P_m)$ and let B be the $n \times m$ array with $b_{i,j} = 1$ if $(u_i, v_j) \in S$ and $b_{i,j} = 0$ otherwise. Then, let C be the $n \times m$ array whose entries are the maximum over the closed neighbourhood of the corresponding entry in B. In other words, $c_{i,j} = 1$ if any of the entries in the closed neighbourhood of $b_{i,j}$ has value 1, and $c_{i,j} = 0$ otherwise. Clearly, $C \in \mathscr{A}$ and now we show that $C^* = B$. By construction of C, each entry of C^* whose corresponding entry in B has value 1 also has value 1 in C^*. So if $C^* \neq B$, then there is some i, j with $c^*_{i,j} = 1$ and $b_{i,j} = 0$. This means that, in C, each entry in the closed neighbourhood of $c_{i,j}$ has value 1. However, the entries in C are defined to be 1 because their corresponding entry in B has a 1 in its closed neighbourhood. In other words, the entries in the closed neighbourhood of $b_{i,j}$ each either have value 1 or have a horizontal or vertical neighbour with value 1. In terms of the set S, this corresponds to a vertex (u_i, v_j) with every vertex in $N[(u_i, v_j)]$ in $N[S]$, i.e. (u_i, v_j) has no private neighbour with respect to S in $P_n \square P_m$. This contradicts S being digitally convex and thus $C^* = B$.

It is clear that B gets mapped to the digitally convex set S, using the mapping described above. Therefore, $n_{\mathscr{D}}(P_n \square P_m) = |\mathscr{A}^*_{n,m}|$. □

The number of digitally convex sets of $P_n \square P_m$ follows the OEIS sequence A217637 [23]. The OEIS notes an observation from Andrew Howroyd that this sequence also enumerates the maximal independent sets in the graph $P_n \square P_m \square P_2$. Euler, Oleksik and Skupień [11] prove this equivalence for $m = 2$ and for $m = 3$. However, the correspondence between the maximal independent sets in $P_n \square P_m \square P_2$ and the digitally convex sets in $P_n \square P_m$ is not clear, even for very small values of n and m.

Chapter 5

Block graphs

In this chapter, we return to a class of graphs that is closely related to trees. The class of block graphs, graphs in which every biconnected component is a clique, contains the trees as a subclass. Every edge in a tree, along with its endpoints, is a block and, in fact, a 2-clique, and these are the only blocks in trees. In this chapter, we give an upper bound on the number of digitally convex sets in a block graph and enumerate the digitally convex sets of some subclasses of block graphs. We then conjecture a lower bound on the number of digitally convex sets in a block graph and a subclass of block graphs that attains it.

Throughout this chapter, we enumerate the digitally convex sets of a block graph in terms of the number of blocks it contains, instead of in terms of the order of the graph. The reason for this difference is that the upper and lower bounds on the number of digitally convex sets of a block graph of order n are trivial if they are given in terms of n. Both the complete graph K_n and its complement $\overline{K}_n$ are block graphs. The former gives a lower bound on the number of digitally convex sets and the latter gives an upper bound. So for a block graph G of order n, we obtain the bounds $2 \leq n_{\mathscr{D}}(G) \leq 2^n$. However, these bounds give very little information on the structure of the digitally convex sets of a block graph if its number of digitally convex sets falls between the upper and lower bounds. The following upper bound, stated in terms of the number of blocks, gives an indication of how to generate digitally convex sets of a block graph.

Theorem 5.1. *Let G be a block graph with blocks $B_1, B_2, \ldots, B_k$. Then $n_{\mathscr{D}}(G) \leq 2^k$.*

Proof. We show that any digitally convex set in G has the form $V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$ for some $\mathscr{B} \subseteq \{B_1, B_2, \ldots, B_k\}$. Let $S \in \mathscr{D}(G)$ and let $\{u_1, u_2, \ldots, u_r\} = V(G) - S$. We now describe, for each u_j, $1 \leq j \leq r$, a (possibly empty) family $\mathscr{B}_j$ of blocks such that no vertex in any of the blocks in $\mathscr{B}_j$ belongs to S and such that $S = V(G) - \bigcup\{V(B_j) \mid B_j \in \bigcup_{j=1}^{r} \mathscr{B}_j\}$.

If u_j is not a cut vertex in G, then $N[u_j]$ is a clique, i.e. $N[u_j] = V(B_j)$ for some $j \in \{1, 2, \ldots, k\}$. Since each neighbour of u_j dominates $N[u_j]$, it must be the case that $N[u_j] \cap S = \emptyset$ or, equivalently, $V(B_j) \cap S = \emptyset$. In this case, let $\mathscr{B}_j = \{B_j\}$. As an example, in Figure 5.1, if $u_1 \notin S$, then $\mathscr{B}_1 = \{B\}$, as $N[u_1] = V(B)$.

If u_j is a cut vertex in G, then let $B_{j_1}, B_{j_2}, \ldots, B_{j_\ell}$ be the blocks containing u_j, so that $N[u_j] = \bigcup_{i=1}^{\ell} V(B_{j_i})$. If $u_j \notin N[S]$, then it must be the case that $V(B_{j_i}) \cap S = \emptyset$ for each $i = 1, 2, \ldots, \ell$. In this case, let $\mathscr{B}_j = \{B_{j_i} \mid 1 \leq j \leq \ell\}$. If $u_j \in N[S]$, then there is at least one block B_{j_i} containing a vertex $u_{j_i} \notin N[S]$. In this case, $u_j \in V(B_{j_i})$ and $B_{j_i} \in \mathscr{B}_{j_i}$. In this case, let $\mathscr{B}_j = \emptyset$. As an example, in Figure 5.1, if $u_2 \notin S$, then $\mathscr{B}_2 = \{A, B\}$ if $u_2 \notin N[S]$, and $\mathscr{B}_2 = \emptyset$, otherwise.

Now, let $\mathscr{B} = \bigcup_{j=1}^{r} \mathscr{B}_j$ and $V_{\mathscr{B}} = \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$. It is clear from construction that $u_j \in V_{\mathscr{B}}$ for each $j = 1, 2, \ldots, r$ and that $S \cap V_{\mathscr{B}} = \emptyset$. Therefore, $S = V(G) - V_{\mathscr{B}} = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$, where $\mathscr{B} = \bigcup_{j=1}^{r} \mathscr{B}_j \subseteq \{B_1, B_2, \ldots, B_k\}$.

There are 2^k possible subsets of $\{B_1, B_2, \ldots, B_k\}$, so there can be at most 2^k distinct digitally convex sets of G. □

Note that this upper bound matches those for the number of digitally convex sets

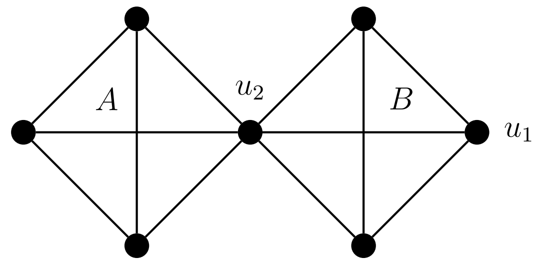


Figure 5.1: Any digitally convex set containing u_2 must also contain u_1

of subclasses of block graphs that we examined in previous chapters. A complete graph has one block and satisfies $n_{\mathscr{D}}(K_n) \leq 2^1 = 2$. Similarly, the complement of a complete graph has n blocks and satisfies $n_{\mathscr{D}}(\overline{K}_n) \leq 2^n$. A tree T has $n-1$ blocks and satisfies $n_{\mathscr{D}}(T) \leq 2^{n-1}$, as stated in Theorem 2.5. For these subclasses, this upper bound is sharp. The following result gives a more general subclass of block graphs for which this upper bound on the number of digitally convex sets is sharp.

Theorem 5.2. *Let G be a block graph with blocks $B_1, B_2 \ldots, B_k$. For $1 \leq i \leq k$, let $V^*(B_i)$ be the set of vertices in the block B_i that are not cut vertices of G. If $V^*(B_i) \neq \emptyset$ for each $i = 1, 2, \ldots, k$, then $n_{\mathscr{D}}(G) = 2^k$.*

Proof. As shown in the proof of Theorem 5.1, each digitally convex set $S_{\mathscr{B}}$ of G has the form $S_{\mathscr{B}} = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$ for some $\mathscr{B} \subseteq \{B_1, B_2, \ldots, B_k\}$. We show first that, for every subset $\mathscr{B}$ of $\{B_1, B_2, \ldots, B_k\}$, the set $V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$ is digitally convex. We then show that for two different subsets, $\mathscr{B}_1$ and $\mathscr{B}_2$, of $\{B_1, B_2, \ldots, B_k\}$, we have $S_{\mathscr{B}_1} \neq S_{\mathscr{B}_2}$.

Let $\mathscr{B} \subseteq \{B_1, B_2, \ldots, B_k\}$ and consider $S = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$. Let $v \notin S$. Then $v \in V(B_i)$ for some block $B_i \in \mathscr{B}$. If $v \in V^*(B_i)$, then $N[v] = V(B_i)$ and $V(B_i) \cap S = \emptyset$. So $v \notin N[S]$. If $v \notin V^*(B_i)$, then there exists some $u \in N[v]$ with $u \in V^*(B_i)$, since $V^*(B_i) \neq \emptyset$. As shown above, $u \notin N[S]$ so u is a private neighbour for v with respect to S. Therefore, S is digitally convex in G.

Now, let $\mathscr{B}_1, \mathscr{B}_2 \subseteq \{B_1, B_2, \ldots, B_k\}$ with $\mathscr{B}_1 \neq \mathscr{B}_2$, and consider $S_{\mathscr{B}_\ell} = V(G) -$

$\bigcup\{V(B_i) \mid B_i \in \mathscr{B}_\ell\}$ for $\ell = 1, 2$. So either $\mathscr{B}_1 \backslash \mathscr{B}_2$ or $\mathscr{B}_2 \backslash \mathscr{B}_1$ is nonempty, say the former. Let $B_j \in \mathscr{B}_1 \backslash \mathscr{B}_2$ and let $z \in V^*(B_j)$. Then $z \notin B_p$ for any $p \neq j$ because blocks intersect only at cut vertices. So $z \in (\bigcup\{V(B_i) \mid B_i \in \mathscr{B}_1\}) \setminus (\bigcup\{V(B_i) \mid B_i \in \mathscr{B}_2\})$. Equivalently, $z \in S_{\mathscr{B}_2} \backslash S_{\mathscr{B}_1}$, so $S_{\mathscr{B}_1} \neq S_{\mathscr{B}_2}$.

Therefore, the collection $\mathscr{D}(G)$ of all digitally convex sets $S_{\mathscr{B}}$ (as defined above) contains 2^k distinct digitally convex sets. □

As an example, consider the block graph G shown in Figure 5.2. Each of the five blocks, labelled with the letters A through E, contains at least one vertex that is not a cut vertex of the graph. Thus, by Theorem 5.2, $n_{\mathscr{D}}(G) = 2^5 = 32$. As shown in the proof of Theorem 5.2, a digitally convex set S can be generated by taking a subset $\mathscr{B}$ of the set of blocks, $\{A, B, C, D, E\}$, and removing from $V(G)$ the vertices contained in the blocks of $\mathscr{B}$. For example, taking $\mathscr{B} = \{B, E\}$ results in the digitally convex set indicated in red in Figure 5.2. The vertex in B that is not a cut vertex is a private neighbour for all of the vertices in this block, and similarly for E.

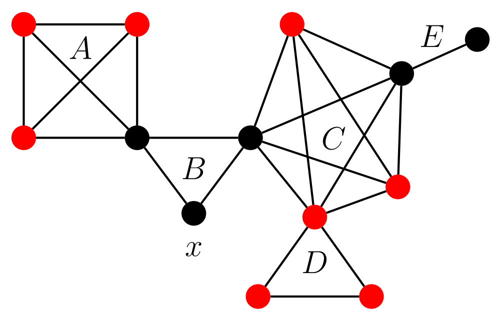


Figure 5.2: A digitally convex set is highlighted in red

However, removing only the vertex x from G reduces the number of digitally convex sets without reducing the number of blocks. Then, the remaining vertices in the block B no longer have a private neighbour with respect to the set S, highlighted in red in Figure 5.2. All vertices in B are cut vertices so, using the notation of Theorem 5.2, $V^*(B) = \emptyset$. Thus, the set $\mathscr{B} = \{B, E\}$ does not correspond to a digitally convex set in G.

Furthermore, the sets $\mathscr{B}_1 = \{A, C, D, E\}$ and $\mathscr{B}_2 = \{A, B, C, D, E\}$ both give the same digitally convex set, $S_{\mathscr{B}_1} = \emptyset = S_{\mathscr{B}_2}$. The following result gives a formula for the number of digitally convex sets in a graph with one block of order 2 whose vertices are both cut vertices of the graph.

Theorem 5.3. *Let G be a block graph with blocks $B_1, B_2, \ldots, B_k$. For $1 \leq i \leq k$, let $V^*(B_i)$ be the set of vertices in the block B_i that are not cut vertices of G. Suppose for $i = 1, 2, \ldots, k-1$, we have $V^*(B_i) \neq \emptyset$, and suppose that $B_k \cong K_2$ with vertices x and y contained in $k_1 > 0$ and $k_2 > 0$ other blocks, respectively (i.e. $V^*(B_k) = \emptyset$). Then, $n_{\mathscr{D}}(G) = 2^k - 2^{k-(k_1+k_2+1)}[(2^{k_1} - 1)(2^{k_2} - 1) + ((2^{k_1} - 1) + (2^{k_2} - 1) - 1)]$.*

Proof. Let $B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}$ be the blocks, along with B_k, that contain x and $B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}$ be the blocks, along with B_k, that contain y. From the proofs of Theorem 5.1 and Theorem 5.2, we know that all digitally convex sets in G are of the form $S = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$, for some $\mathscr{B} \subseteq \{B_1, B_2, \ldots, B_k\}$. We count the number of these sets $\mathscr{B}$ that give digitally convex sets in G.

First, suppose that $B_k \in \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} = \emptyset$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \not\subseteq \mathscr{B}$. Consider $S_1 = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$. Since $B_k \in \mathscr{B}$, we have $x, y \notin S_1$. We show that $N[x] \subseteq N[S_1]$, i.e. x has no private neighbour with respect to S_1. There exists a block $B_{y_i} \notin \mathscr{B}$ with $V^*(B_{y_i}) \neq \emptyset$, so there is a vertex $y_i \in V^*(B_{y_i}) \subseteq S_1$ with y_i adjacent to y in G. Thus, $y \in N[S_1]$. For $1 \leq j \leq k$, we have $B_{x_j} \notin \mathscr{B}$ and $V^*(B_{x_j}) \neq \emptyset$. So there exists a vertex $x_j \in V^*(B_{x_j}) \subseteq S_1$ and $N[x_j] = V(B_{x_j})$. Since $N[x] = \left(\bigcup_{j=1}^{k_1} V(B_{x_j})\right) \bigcup\{y\}$, we have $N[x] \subseteq N[S_1]$, and thus S_1 is not digitally convex in G. There are $2^{k-(k_1+k_2+1)}(2^{k_2} - 1)$ such sets $\mathscr{B}$, since there are $2^{k_2} - 1$ ways to choose at least one set B_{y_i} to omit from $\mathscr{B}$ and $2^{k-(k_1+k_2+1)}$ ways to choose any number of blocks not containing x or y to omit from $\mathscr{B}$. Similarly, there are $2^{k-(k_1+k_2+1)}(2^{k_1} - 1)$ sets $\mathscr{B}$ with $B_k \in \mathscr{B}$, $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} = \emptyset$

and $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \not\subseteq \mathscr{B}$ such that $S_2 = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$ is not digitally convex in G. However, the $2^{k-(k_1+k_2+1)}$ sets $\mathscr{B}$ containing B_k and none of $B_{x_1}, B_{x_2}, \ldots, , B_{x_{k_1}}, B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}$ are counted in both cases. So there is a total of $2^{k-(k_1+k_2+1)}[(2^{k_1} - 1) + (2^{k_2} - 1) - 1]$ sets $\mathscr{B}$ satisfying one of these two cases.

Suppose now that $B_k \notin \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} \neq \emptyset$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} \neq \emptyset$. Consider $S_3 = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$. There exist blocks B_{x_i} and B_{y_j}, with $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, with $B_{x_i}, B_{y_j} \in \mathscr{B}$. Since $x \in B_{x_i}$ and $y \in B_{y_j}$, we have $x, y \in \bigcup\{V(B_\ell) \mid B_\ell \in \mathscr{B}\}$. Thus, $\bigcup\{V(B_\ell) \mid B_\ell \in \mathscr{B}\} = \bigcup\{V(B_{\ell^*}) \mid B_{\ell^*} \in \mathscr{B} \cup \{B_k\}\}$. So $S_3 = V(G) - \bigcup\{V(B_{\ell^*}) \mid B_{\ell^*} \in \mathscr{B} \cup \{B_k\}\}$. There are $2^{k-(k_1+k_2+1)}(2^{k_1} - 1)(2^{k_2} - 1)$ such sets $\mathscr{B}$, as there are $2^{k_1} - 1$ and $2^{k_2} - 1$ ways to choose nonempty subsets of $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\}$ and of $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\}$, respectively. As before, there are $2^{k-(k_1+k_2+1)}$ ways to choose any number of the blocks not containing x or y to omit from $\mathscr{B}$. Thus, to avoid double counting, we subtract these $2^{k-(k_1+k_2+1)}(2^{k_1} - 1)(2^{k_2} - 1)$ sets from the 2^k possible digitally convex sets of G.

Now we show that each of the other possibilities for $\mathscr{B} \subseteq \{B_1, B_2, \ldots, B_k\}$ corresponds to a digitally convex set $S_4 = V(G) - \bigcup\{V(B_i) \mid B_i \in \mathscr{B}\}$ in G.

If $B_k \in \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} \neq \emptyset$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} \neq \emptyset$, then there exist blocks B_{x_i} and B_{y_j}, with $1 \leq i \leq k_1$ and $1 \leq j \leq k_2$, such that $B_{x_i}, B_{y_j} \in \mathscr{B}$. Since $V^*(B_{x_i}) \neq \emptyset$ and $V^*(B_{y_j}) \neq \emptyset$, there exist vertices $v_i \in V^*(B_{x_i})$ and $v_j \in V^*(B_{y_j})$, so $v_i, v_j \notin N[S_4]$. Thus, v_i is a private neighbour for every vertex in B_{x_i}, including x. Similarly, v_j is a private neighbour for every vertex in B_{y_j}, including y. In every other block $B_\ell \in \mathscr{B}$, the vertices in $V^*(B_\ell) \neq \emptyset$ are private neighbours for all vertices in $V(B_\ell)$ with respect to S_4. Thus, for the sets $\mathscr{B}$ described above, S_4 is digitally convex in G.

If $B_k \in \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \subseteq \mathscr{B}$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} = \emptyset$, then $x \notin N[S_4]$, since all of the blocks containing x are in $\mathscr{B}$. So x is a private neighbour for itself, for y and for all vertices in $V(B_{x_i})$, with $i = 1, 2, \ldots, k_1$. For every other $B_\ell \in \mathscr{B}$, the vertices in $V^*(B_\ell) \neq \emptyset$ are private neighbours for the vertices in $V(B_\ell)$. Thus, S_4 is digitally convex in G. Similarly, if $B_k \in \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} = \emptyset$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \subseteq \mathscr{B}$, then $y \notin N[S_4]$ and S_4 is digitally convex in G.

Finally, if $B_k \notin \mathscr{B}$ and $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} = \emptyset$, then we consider two cases. In the first case, if $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} = \emptyset$, then $x, y \in S_4$. In the second case, if $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} \neq \emptyset$, then there is some $v_i \in V^*(B_{y_i})$ with $B_{y_i} \in \mathscr{B}$. So $v_i \notin N[S_4]$. In the latter case, $y \notin S_4$ and v_i is a private neighbour for y with respect to S_4. As above, for every other $B_\ell \in \mathscr{B}$, the vertices in $V^*(B_\ell) \neq \emptyset$ are private neighbours for the vertices in $V(B_\ell)$. Thus, S_4 is digitally convex in G. Similarly, if $B_k \notin \mathscr{B}$, $\{B_{x_1}, B_{x_2}, \ldots, B_{x_{k_1}}\} \cap \mathscr{B} \neq \emptyset$ and $\{B_{y_1}, B_{y_2}, \ldots, B_{y_{k_2}}\} \cap \mathscr{B} = \emptyset$, then S_4 is digitally convex in G.

This gives a total of $2^k - 2^{k-(k_1+k_2+1)}[(2^{k_1}-1)(2^{k_2}-1) + ((2^{k_1}-1) + (2^{k_2}-1) - 1)]$ digitally convex sets in G. □

The cases in this proof depend only on the blocks in G that contain the vertices x and y. If there are multiple blocks B_i containing no cut vertices and satisfying the condition $B_i \cong K_2$, then the principle of inclusion-exclusion can be used to count the digitally convex sets, provided that there is no block containing vertices from at least two of these blocks B_i. We state this result without proof, due to its similarity to that of Theorem 5.3.

Theorem 5.4. *Let G be a block graph with blocks $B_1, B_2, \ldots, B_k$, and let $m \geq 0$. For $1 \leq i \leq k$, let $V^*(B_i)$ be the set of vertices in the block B_i that are not cut vertices of G. Suppose that for $i = 1, 2, \ldots, m$, $B_i \cong K_2$ with vertices x_i and y_i, and for $j = m+1, m+2, \ldots, k$, we have $V^*(B_j) \neq \emptyset$. Suppose, for each $1 \leq i \leq m$, that x_i and y_i are contained in $k_{i_1} > 0$ and $k_{i_2} > 0$ other blocks, respectively (i.e. $V^*(B_i) = \emptyset$),* *and that for every* $p = 1, 2, \ldots, k$, *we have* $V(B_p) \cap V(B_q) \neq \emptyset$ *for at most one* $q \in \{1, 2, \ldots, m\}$. *Let* $M_i = (2^{k_{i_1}} - 1)(2^{k_{i_2}} - 1)$, $A_j = (2^{k_{i_1}} - 1) + (2^{k_{i_2}} - 1) - 1$, *and* $k_i = k_{i_1} + k_{i_2}$. *Then*

$$n_{\mathscr{D}}(G) = \sum_{S \subseteq [m]} (-1)^{|S|} 2^{k - |S| - (\sum_{i \in S} k_i)} \left[\sum_{T \subseteq S} \left(\prod_{i \in T} M_i \prod_{j \in S - T} A_j \right) \right].$$

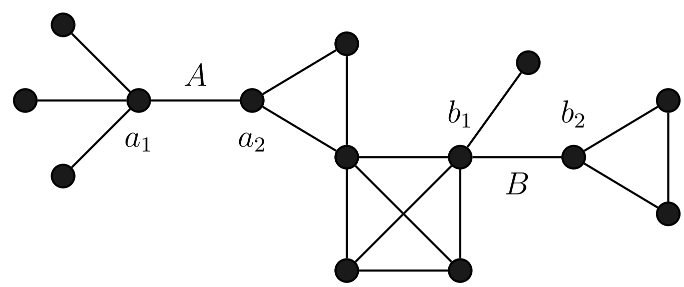


Figure 5.3: Both of the vertices in blocks A and B are contained in other blocks

As an example, consider the block graph G in Figure 5.3. This graph has a total of nine blocks, with every block except for A and B containing a vertex that is not a cut vertex of G. The vertices a_2 and b_2 are each contained in one other block, the vertex b_1 is contained in two other blocks, and the vertex a_1 is contained in three others. Thus, by Theorem 5.4,

$$n_{\mathscr{D}}(G) = 2^9 - 2^4(14) - 2^5(6) + 2^0(84) = 180.$$

However, when there are larger blocks whose vertices are all contained in other blocks, or when these blocks share vertices, the calculations quickly get very complicated. For example, in the graph in Figure 5.4, each of the vertices in the blocks A and B is a cut vertex. If we use the arguments above to determine which subsets $\mathscr{B}$ of the set of blocks, $\{A, B, C, D, E\}$, correspond to a digitally convex set, we see that the restrictions can become complicated and hard to generalize. In this case,

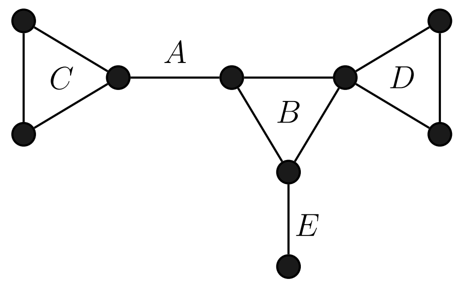


Figure 5.4: Every vertex in blocks A and B is contained in another block

we cannot have $A \in \mathscr{B}$ without also having one of B or C in $\mathscr{B}$. However, to have $B \in \mathscr{B}$, one of A, D or E must be in $\mathscr{B}$. By increasing the size or number of these blocks that need special consideration, the number of cases that need to be considered would make any formula for the number of digitally convex sets very complicated.

Similar to the upper bound on the number of digitally convex sets in a block graph, we can use the lower bound on the number of digitally convex sets in a tree in Theorem 2.5, as well as the fact that a tree has $n-1$ blocks, to identify a potential lower bound on the number of digitally convex sets in a block graph. Note that, as in Theorem 2.5, we consider two cases: when the number of blocks k is even and when it is odd.

Conjecture 5.5. *Let G be a block graph with k blocks. Then*

$$n_{\mathscr{D}}(G) \geq \begin{cases} 2 \cdot 2^{\frac{k+1}{2}} - 2, & \text{if } k \text{ is odd} \\ 3 \cdot 2^{\frac{k}{2}} - 2, & \text{if } k \text{ is even} \end{cases}$$

We now construct a subclass of block graphs G_k that contains the spiderstars and attains the conjectured lower bound. We begin with the spiderstar S_{k+1}, which has k blocks. Let u be a leaf in S_{k+1} and let u' be its neighbour in S_{k+1}. Then, we add the remaining $n-k-1$ vertices $v_1, v_2, \ldots, v_{n-k-1}$ and edges v_iv_j for every $i \neq j$, uv_i and $u'v_i$ for every $i = 1, 2, \ldots, n-k-1$ to form G_k. Figure 5.5 shows the graph G_5

of order nine.

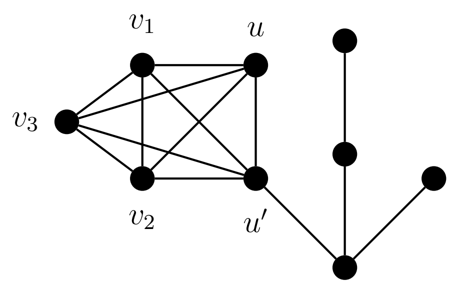


Figure 5.5: The block graph G_5 of order nine

To show that the graphs G_k attain the lower bound, we show that G_k of order $n \geq k+1$ has the same number of digitally convex sets as the spiderstar S_{k+1}. Let $S \in \mathscr{D}(S_{k+1})$. If $u \in S$, then $S \cup \{v_1, v_2, \ldots, v_{n-k-1}\}$ is digitally convex in G_k, since $N_{G_k}[\{v_1, v_2, \ldots, v_{n-k-1}\}] = N_{G_k}[u]$. If $u \notin S$, then S is also digitally convex in G_k, since it must be the case that $u \notin N_G[S]$. So $v_1, v_2, \ldots, v_{n-k-1} \notin N_{G_k}[S]$. Note that any digitally convex set containing any of $v_1, v_2, \ldots, v_{n-k-1}$ must also contain u, as each vertex v_i dominates $N_{G_k}[u]$. Moreover, since u' dominates $N_{G_k}[v_i]$ for each $i = 1, 2, \ldots, n-k-1$, any digitally convex set not containing $v_1, v_2, \ldots, v_{n-k-1}$ cannot contain u' and therefore must also be digitally convex in S_{k+1}. Thus, $n_{\mathscr{D}}(G_k) = n_{\mathscr{D}}(S_{k+1})$. By Theorem 2.5, the spiderstars attain the bound in Conjecture 5.5. So the graphs G_k attain this bound as well.

Chapter 6

Conclusion and Future Work

In this thesis, we have explored the generation and enumeration of digitally convex sets in various classes of graphs. In Chapter 3, we extended several of the results of Lafrance, Oellermann and Pressey [15] from trees to k-trees. We gave an algorithm for generating the digitally convex sets of a k-tree and provided an upper bound on the number of digitally convex sets in a 2-tree and then generalized the upper bound to k-trees. Furthermore, we conjectured a lower bound on the number of digitally convex sets in a 2-tree and then conjectured a generalization to k-trees. In Chapter 4, we used the enumeration of other mathematical objects, such as cyclic binary strings and binary arrays, to enumerate the digitally convex sets of cycles, powers of cycles, and Cartesian products of complete graphs and of paths. Finally, in Chapter 5, we enumerated the digitally convex sets of block graphs, in terms of the number of blocks instead of the number of vertices. Several of the proofs given in this thesis also provide algorithms for generating the collection of digitally convex sets for the corresponding class of graphs.

We conclude with a discussion of directions for future research in this area. In Chapter 3, there are several bounds on the number of digitally convex sets in k-trees and simple clique 2-trees that we conjecture or are unknown. We conjecture a generalization of the lower bound on the number of digitally convex sets of trees [15] to 2-trees and then to k-trees. In the case of 2-trees, we provided an outline of a possible proof of the lower bound. Completion of the proof of this lower bound requires a proof of Conjectures 3.12, 3.13 and 3.14. In the case of simple clique 2-trees, considered in Section 3.3, the lower bound on the number of digitally convex sets of an SC 2-tree of order n only matches that of a 2-tree of order n for $n < 10$. For $n \geq 10$, that of an SC 2-tree appears to be larger. This was calculated using a brute force approach. The code for this brute force approach is given in Appendix A. The problem of finding a sharp lower bound on the number of digitally convex sets of SC 2-trees and, specifically, for the subclass of 2-path graphs remains an open problem, as does the problem of finding both upper and lower bounds on the number of digitally convex sets of SC k-trees.

In Chapter 4, we enumerate the digitally convex sets of Cartesian products of complete graphs and of paths. Although there does not appear to be an obvious connection between the digitally convex sets of a graph product $G\Box H$ and those of its constituent graphs, a possible direction for future research is to find an exact formula, or upper/lower bounds on $n_{\mathscr{D}}(G\Box H)$ in terms of the number of digitally convex sets in the constituent graphs, i.e. $n_{\mathscr{D}}(G)$ and $n_{\mathscr{D}}(H)$. Another possible direction is to do the same for other graph products, such as the strong product or categorical product.

In Chapter 5, we showed an upper bound on the number of digitally convex sets in a block graph with k blocks. This upper bound is both sharp and matches the upper bound on the number of digitally convex sets in a tree, which has $n-1$ blocks. We then conjectured a lower bound on the number of digitally convex sets in a block graph in terms of the number of blocks. As with the upper bound, the conjectured lower bound for block graph matches the lower bound on the number of digitally convex sets in a tree, given in Theorem 2.5. The proof of Conjecture 5.5 remains open.

There are several other problems related to the digital convexity that remain unexplored. It is known that a graph G of order n has at least 2 and at most 2^n digitally convex sets, and that $n_{\mathscr{D}}(G)$ must be even. However, it is not obvious for a given value of n whether, for each even integer $2k$ between 2 and 2^n, there exists a graph G such that $n_{\mathscr{D}}(G) = 2k$. The same question can be posed for particular classes of graphs. For example, for each even integer $2k$ between 2 and 2^n, does there exist a cograph G of order n such that $n_{\mathscr{D}}(G) = 2k$? If this is not the case, then for which even integers $2k$ does there exist a cograph G such that $n_{\mathscr{D}}(G) = 2k$? The same question can be asked for trees, with a restriction to even integers between the lower and upper bounds given in Theorem 2.5.

A digitally convex set is defined in terms of the neighbourhoods of the vertices in the graph, so there is the question of whether the number of digitally convex sets in a graph is changed when the edge set of the graph is altered. In the case of a complete graph K_n, the removal of any single edge $e = uv$ results in a graph that has $n_{\mathscr{D}}(K_n - e) = 4$ digitally convex sets: $\emptyset$, $V(K_n - e)$, $\{u\}$ and $\{v\}$. In other words, removing any single edge from a complete graph increases the number of digitally convex sets. A possible direction for future research would be to identify other graphs for which the removal of any edge increases the number of digitally convex sets.

However, in the graph P_4, the removal of the edge that is incident with the two vertices of degree 2 results in the disjoint union of two K_2's, a graph that has four digitally convex sets. It is known that $n_{\mathscr{D}}(P_4) = 6$ so, in this case, the removal of an edge from the graph decreases the number of digitally convex sets. If an edge incident with a leaf in P_4 is removed instead, the resulting graph is the disjoint union of P_3 and K_1, which has eight digitally convex sets. Thus, the removal of a single edge from P_4 may increase or decrease the number of digitally convex sets, depending on which edge is removed. Whether there exists a graph for which the removal of any single edge decreases the number of digitally convex sets remains an open problem, as does a method of identifying whether the removal of a given edge will increase or decrease the number of digitally convex sets in the graph.

In [16], Lafrance, Oellermann and Pressey explore the problem of reconstructing a tree from its collection of digitally convex sets. They show that any tree T can be uniquely reconstructed from the sets in $\mathscr{D}(T)$. However, it is unknown whether this is possible for any general graph G, or for any class of graphs other than trees, or whether there are any two non-isomorphic labelled graphs G_1 and G_2 with $\mathscr{D}(G_1) = \mathscr{D}(G_2)$. This area would be useful to research, as the ability to reconstruct a graph or class of graphs from its digitally convex sets may give valuable information about the structure of the graph or class of graphs.

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Appendix A

Code: Generating Digitally Convex Sets of a Graph

```
// ************************************************************
// Generates the number of digitally convex sets in a given graph,
// with the option to print the digitally convex sets.
//
// Input:  - int n indicating the number of vertices in the graph
//     - n x n x k array of integers with the first two dimensions
// containing the adjacency matrix for the desired graph.  The 3rd
// dimension allows the function to be easily implemented in a
// loop to generate the number of digitally convex sets in a
// collection of graphs.
//     - int tree indicating which adjacency matrix in adj should
// be used.  Set tree to be 0 if only one graph is being used.
//     - boolean print indicates whether the digitally convex sets
// should be printed.  If true, the digitally convex sets will be
// printed with one set per line and a space between each element.
// Vertices are labeled from 1 to n.
//
// Output:  int indicating the number of digitally convex sets in
// adj[][][tree]
// ************************************************************

public class twotrees {
  public static int numDigConv(int n, int[][][] adj, int tree,
          boolean print)
```

```
{
  if (n < 1)
  {
    return 1;
  }
  int num = 0;

  // Stores the current potential convex set
  int[] current = new int[n];

  // Generate each subset of [n]
  for (int i = 0; i < (1 << n); i++)
  {
    int k = 0;
    int m = 1;
    for (int j = 0; j < n; j++)
    {
      if ((i & m) > 0)
      {
        current[k] = j;
        k++;
      }
      m = m << 1;
    }
    boolean convex = true;

    // Stores the neighbours of vertices in current[]
    // Some vertices may be added multiple times
    int[] nbr_current = new int[n*n];
    int nbr_k = 0;
    // Find neighbourhood of vertices in current[]
    for (int p = 0; p < k; p++)
    {
      nbr_current[nbr_k] = current[p];
      nbr_k++;
      for (int q = 0; q < current[p]; q++)
      {
        if (adj[q][current[p]][tree] == 1)
        {
          nbr_current[nbr_k] = q;
          nbr_k++;
        }
      }
```

```
    for (int q = current[p]+1; q < n; q++)
    {
      if (adj[current[p]][q][tree] == 1)
      {
        nbr_current[nbr_k] = q;
        nbr_k++;
      }
    }
  }
  for (int v = 0; v < n; v++)
  {
    boolean priv_nbr = false;

    // Check if each vertex has a private neighbour
    for (int x = 0; x < k; x++)
    {

      // If vertex is in the set, doesn't need a private nbr
      if (v == current[x])
      {
        priv_nbr = true;
      }
    }

    // Find closed neighbourhood of a vertex
    if (!priv_nbr)
    {
      int[] nbr_v = new int[n];
      nbr_v[0] = v;
      int v_k = 1;
      for (int y = 0; y < v; y++)
      {
        if (adj[y][v][tree] == 1)
        {
          nbr_v[v_k] = y;
          v_k++;
        }
      }
      for (int y = v+1; y < n; y++)
      {
        if (adj[v][y][tree] == 1)
        {
          nbr_v[v_k] = y;
```

```
            v_k++;
          }
        }

        // Check neighbours of vertex to see if they're private
        for (int a = 0; a < v_k; a++)
        {
          boolean nbr = false;
          for (int b = 0; b < nbr_k; b++)
          {
            if (nbr_v[a] == nbr_current[b])
            {
              nbr = true;
            }
          }
          if (!nbr)
          {
            priv_nbr = true;
          }
        }
      }

      // Sets convex to be false if vertex doesn't have private nbr
      if (!priv_nbr)
      {
        convex = false;
      }
    }
    if (convex)
    {

      // Prints current convex set
      if (print)
      {
        for (int b = 0; b < k; b++)
        {
          System.out.print(current[b] + 1 + " ");
        }
        System.out.println();
      }
      // Increase count if set is convex
      num++;
    }
```

```
    }
    return num;
  }
}
```