From Flag Manifolds to Severi-Brauer Varieties: Intersection Theory, Algebraic Cycles and Motives

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Abstract

The study of algebraic varieties originates from the study of smooth manifolds. One of the focal points is the theory of differential forms and de Rham cohomology. It's algebraic counterparts are given by algebraic cycles and Chow groups. Linearizing and taking the pseudo-abelian envelope of the category of smooth projective varieties, one obtains the category of pure motives.

In this thesis, we concentrate on studying the pure Chow motives of Severi-Brauer varieties. This has been a subject of intensive investigation for the past twenty years, with major contributions done by Karpenko, [Kar1], [Kar2], [Kar3], [Kar4]; Panin, [Pan1], [Pan2]; Brosnan, [Bro1], [Bro2]; Chernousov, Merkurjev, [Che1], [Che2]; Petrov, Semenov, Zainoulline, [Pet]; Calmès, [Cal]; Nikolenko, [Nik]; Nenashev, [Nen]; Smirnov, [Smi]; Auel, [Aue]; Krashen, [Kra]; and others. The main theorem of the thesis, presented in sections 4.3 and 4.4, extends the result of Zainoulline et al. in the paper [Cal] by providing new examples of motivic decompositions of generalized Severi-Brauer varieties.

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Contents

Preface

The study of algebraic geometry originates from the study of smooth manifolds. More specifically, in the context of algebraic geometry, one wishes to establish topological invariants of an arbitrary manifold, M, which can be expressed in the language of abstract algebra. One of the important topological invariants of a manifold is the presence of "holes," often higher dimensional and difficult to detect geometrically. One method used to detect the presence of holes requires the use of the theory of differential forms. To classify differential forms on M, one first requires the notions of closed and exact differential forms which form the vector spaces of closed and exact forms, one for every positive integer graded up to the dimension of the manifold. From these, a set of quotient spaces called the de Rham cohomology groups associated to M, and denoted $H^p_{dR}(M)$, can be formed. Although every exact form is closed, the converse, is in general, not true. The de Rham groups measure precisely how exact the closed forms are. Amongst other things, the de Rham groups are invariant (up to isomorphism) under homeomorphisms and homotopy equivalence.

As the first step in the thesis, we generalize the construction of the de Rham cohomology groups to a larger class of objects. That is, we first enlarge the classes of spaces that we work over, generalizing first from manifolds to algebraic varieties, then from algebraic varieties to schemes, giving precise definitions and discussing their properties in section 1.1. In doing so, we also generalize the notion of differential forms. Thus, we introduce the concept of algebraic cycles on an arbitrary scheme, X. These are defined as formal integral sums of classes of subschemes of X, where these cycles play the role on schemes and varieties that differential forms do on manifolds. We define what it means for two algebraic cycles to be rationally equivalent in section 1.3, and form the quotient groups of cycles modulo rational equivalence. In section 1.4, we show that rational equivalence pushes-forward for proper morphisms of schemes. That is, for a proper morphism between two schemes there is an induced homomorphism between their respective groups of cycles modulo rational equivalence. We observe the latter is nothing other than the action of a covariant functor, which we denote by A. Moreover, having already defined the notion of the intersection multiplicity of plane curves in section 1.2, we then generalize the notion of intersection to classes of subschemes of X, which is called an intersection pairing (or product) on the group of cycles. This turns the group into a commutative associative graded ring called the Chow ring of X. In fact, given a class of varieties, $\mathfrak{B}$, an intersection pairing satisfying some axioms, gives rise to an intersection theory on $\mathfrak{B}$, which we present in section 1.6. We conclude chapter 1 by listing some properties of the Chow ring and defining Chern classes.

Having generalized our tools in chapter 1, chapters 2 and 3 serve to describe the objects we are interested in further studying, namely, Grassmannians and Severi-Brauer varieties. Chapter 2 introduces algebraic groups and parabolic subgroups before defining flag manifolds, of which Grassmannians are a special case. We describe some important group actions on flags before identifying flags of the full flag manifold, F_n, with elements of the general linear group, $GL(n, \mathbb{C})$ modulo subgroups which leave the standard flag fixed. In chapter 3, a thorough, early 20th century historical walk-through of the theory of central simple algebras is presented. We begin by defining algebras and some basic concepts such as the nucleus and the commutant of an algebra in section 3.1. From section 3.2 to section 3.4, we gradually add structure to our algebras; from division in 3.2, to simplicity in 3.3, and, among other things, criteria on the centre in 3.4. This culminates in the definition of central simple algebras. We explore relations among the various objects (division algebras, simple rings, etc) that we define along the way, proving many propositions regarding them. In section 3.5, we state and prove the very historically influential Wedderburn's theorem. Sections 3.6 and 3.7 explore the properties of central simple algebras in the context of splitting fields, and in section 3.8 we define the Brauer group. The Brauer group, whose group operation is induced by the tensor product of algebras, serves to classify central simple algebras up to an associative division algebra, which is unique up to isomorphism. We then describe its functorial nature before reaching the main topic of the chapter in section 3.9; generalized Severi-Brauer varieties. Most importantly, we show that Severi-Brauer varieties are just twisted forms of Grassmannians.

As suggested by the work in chapter 1, cohomology theories are not unique. In fact, cohomology theories with coefficients in some ring, R, correspond, roughly speaking, to contravariant functors on subcategories of the category of algebraic varieties over some field k into some R-linear tensor category while satisfying certain properties. The final chapter focuses on a class of cohomology theories called oriented cohomology theories, which are defined and explored in section 4.1. In section 4.2, we linearize the category of smooth projective varieties over some field, k. This involves first enlarging the class of morphisms to include $\sim$-correspondences, where $\sim$ is an equivalence relation on the algebraic cycles. Thus, having chosen an adequate equivalence relation $\sim$, we obtain an oriented cohomology theory A, and we construct the category of A-correspondences. Taking its pseudo-abelian envelope one obtains the category of effective motives over k, denoted $\mathcal{M}ot^{eff}_{\sim}(k)$. Formally inverting the Lefschetz motive, we then obtain the category of pure motives, denoted $\mathcal{M}ot_{\sim}(k)$, a category through which any oriented cohomology theory will factor.

In this thesis, we concentrate on studying the pure Chow motives of Severi-Brauer varieties. This has been a subject of intensive investigation for the past twenty years, with major contributions done by Karpenko, [Kar1], [Kar2], [Kar3], [Kar4]; Panin, [Pan1], [Pan2]; Brosnan, [Bro1], [Bro2]; Chernousov, Merkurjev, [Che1], [Che2]; Petrov, Semenov, Zainoulline, [Pet]; Calmès, [Cal]; Nikolenko, [Nik]; Nenashev, [Nen]; Smirnov, [Smi]; Auel, [Aue]; Krashen, [Kra]; and others. The main theorem of the thesis and its proof, presented in sections 4.3 and 4.4, extends the result of Zainoulline et al. in the paper [Cal] by providing new examples of motivic decompositions of generalized Severi-Brauer varieties. Specifically, we prove the following theorem.

Theorem. *Let $SB_2(A)$ be a generalized Severi-Brauer variety for a central simple k-algebra A of degree 7. Then, there is an isomorphism*

$$\mathcal{M}(SB_2(A)) \simeq (SB_2(A), p) \oplus (SB_2(A), p)^c$$

where p is an idempotent correspondence in $\mathcal{C}or_{CH}(SB_2(A), SB_2(A))$ and the superscript c indicates the complementary object to $(SB_2(A), p)$ in the decomposition of $\mathcal{M}(SB_2(A))$ in the category of pure Chow motives.

The case for $\deg_k(A) = 5$ was considered in [Cal]. Our case for degree 7 is a new result and can be viewed as a major application of our techniques.

Chapter 1

Intersection Theory

Much of the study of the intersection of plane curves begins historically with the theorem of Bézout. Although the notion of algebraically closed fields did not exist in his time, working over the complex numbers, the strength of the theorem was rooted in the fact that the number of points of intersection was ultimately independent of the curves themselves, so long as they intersected transversely. This notion became referred to as the 'preservation of intersection under deformation.' The theorem, originally over affine spaces, was later modified to be considered in the milieu of projective spaces, to tidy up the issue caused by parallel lines, which did not intersect in affine space but did intersect at the point at infinity in projective space. Thus the theorem, originally stated as an upper bound for the number of points of intersection, later became an equality.

Theorem. ***(Bézout's Theorem for Curves)*** *Let $C_1, C_2 \subseteq \mathbb{P}^2_{\mathbb{C}}$, plane curves with deg$(C_1) = d_1$ and deg$(C_2) = d_2$. If $C_1 \pitchfork C_2$, then $|C_1 \cap C_2| = d_1 \cdot d_2$.*

The theorem was later generalized by replacing $\mathbb{P}^2_{\mathbb{C}}$ by any smooth projective variety X of dimension n over an algebraically closed field k and asking whether the cardinality of intersection of any two subvarieties Y, Z of X of complimentary codimension, which intersected transversely, was preserved under continuous deformation in an analogous way to the theorem for curves. The answer is yes, and is the foundation of the development of intersection theory.

In the following sections we give a precise definition to the notion of 'under deformation' in the framework of rational equivalence and we further generalize the theory to remove the requirement of complementary codimension for the intersecting subvarieties. We refer to [Ful] for all the details, arguments and omitted proofs of theorems, specifically chapter 1.

1.1 Schemes and Algebraic Cycles

In this section we begin by introducing algebraic varieties as they were first encountered classically; as the zero sets of some systems of polynomial equations over an algebraically closed field, called affine varieties. We define a suitable topology on the affine algebraic varieties, namely the Zariski topology, which leads us to the notion of the spectrum of a commutative ring. We then enlarge the theory of algebraic varieties by introducing a more general class of objects called schemes. Loosely speaking, schemes allow for the definition of varieties over any commutative ring and therefore play a key role in unifying algebraic geometry with fields such as number theory. Lastly, we formally define the notion of algebraic cycles on schemes as cohomological constructs which are integral linear combinations of classes of subschemes.

We begin by defining the Zariski topology and the spectrum of a commutative ring.

Definition 1.1.1. (The Zariski Topology) Let A be a commutative ring and I an ideal of A. Define

$$V(I) = \{\mathfrak{p} \triangleleft A \mid \mathfrak{p} \text{ is a prime ideal}, \mathfrak{p} \supseteq I\}$$

The sets $V(I)$ satisfy the axioms for closed sets in a topological space. The resulting topology generated by these sets is called the Zariski Topology.

Definition 1.1.2. (The Spectrum of a Commutative Ring) The spectrum of A, denoted $\mathrm{Spec}(A)$, is the set of prime ideals of A, equipped with the Zariski topology, for which the closed sets are the sets given by $V(I) = \{\mathfrak{p} \in \mathrm{Spec}(A) \mid \mathfrak{p} \supseteq I\}$ for any ideal $I \triangleleft A$.

Example 1.1.3. Let k be an algebraically closed field and $A = k[x_1, ..., x_n]$, the polynomial ring in n indeterminates with coefficients in k. Define $\mathbb{A}^n_k$ to be n-tuples of elements of k, i.e. of the form $(a_1, ..., a_n)$, where $a_i \in k$ for all $i = 1, ..., n$. Elements of A are therefore maps from $\mathbb{A}^n_k$ to k. Let $T \subseteq A$ be any subset of A. Define the *zero locus* of T to be $Z(T) = \{p \in \mathbb{A}^n_k \mid f(p) = 0,\ \forall f \in T\}$. $Z(T)$ is a closed subset of $\mathbb{A}^n_k$ and these subsets generate the Zariski topology on $\mathbb{A}^n_k$.

The subsequent definitions in what follows are prerequisite to defining the notion of schemes. We briefly define what is necessary in this regard, observing some important facts along the way. We begin with defining pre-sheafs and then the related concepts of sheafs and covering spaces.

Definition 1.1.4. (Pre-Sheafs on Topological Spaces) Let X be a topological space. A pre-sheaf on X is a contravariant functor $\mathscr{F}$ from the category of open subsets of X and their natural inclusion mappings to a category $\mathcal{C}$, where $\mathcal{C}$ may be the category of sets, groups, rings, modules, etc.

Formally, for each open subset $U \subseteq X$, the functor $\mathscr{F}$ assigns to each object U an object $\mathscr{F}(U)$ in the category $\mathcal{C}$ and to each inclusion map $\iota : U \hookrightarrow V$ of open subsets of X a morphism $\mathscr{F}(\iota) : \mathscr{F}(V) \to \mathscr{F}(U)$ where $\mathscr{F}(\iota) := \mathscr{F}_U^V$. The morphism $\mathscr{F}_U^U$ is the identity morphism in $\mathcal{C}$ and for $U \subseteq V \subseteq W$, we have $\mathscr{F}_U^W = \mathscr{F}_U^V \circ \mathscr{F}_V^W$. The mappings $\mathscr{F}_U^V$ are called restriction homomorphisms.

Definition 1.1.5. (Sheafs on Topological Spaces) A sheaf is a pre-sheaf $\mathscr{F}$ on a topological space X such that for any union of open sets of X, say $\mathcal{U} = \bigcup_{i \in I} U_i$ for some index set I, the following conditions are satisfied:

(i) If on every U_i the restrictions of two elements $s, s' \in \mathscr{F}(\mathcal{U})$ coincide, then $s = s'$.

(ii) If $s_i \in \mathscr{F}(U_i)$ are such that for any pair of indices i and j the restrictions of s_i and s_j to $U_i \cap U_j$ coincide, then there exists an element $s \in \mathscr{F}(\mathcal{U})$ which on each U_i has restriction coinciding with s_i. That is then, there exists $s \in \mathscr{F}(\mathcal{U})$ such that for every $i \in I$, the map $\mathscr{F}_{U_i}^{\mathcal{U}} : \mathscr{F}(\mathcal{U}) \to \mathscr{F}(U_i)$ sends $s \mapsto s|_{U_i} = s_i$.

These conditions are called locality and gluing, respectively.

Definition 1.1.6. (Coverings of Topological Spaces) Let X be a topological space. A covering space of X is a topological space E together with a continuous surjective map $p : E \twoheadrightarrow X$ such that for every point $x \in X$, there exists an open neighbourhood U of x such that the inverse image of U under p, $p^{-1}(U)$, is a union of disjoint open sets in E, each of which is mapped homeomorphically onto U by p.

Here, X is called the base space, E the total space, and p the covering map. Moreover, for any $x \in X$, $p^{-1}(x)$ is a discrete space called the fiber over x. Finally, the neighbourhoods U of x given in the definition are called evenly covered and form an open cover of X.

Remark. Every sheaf on X is isomorphic to the sheaf of continuous sections of a certain covering space $p : E \twoheadrightarrow X$ over X. Therefore, it is common to represent the sheaf with the covering space itself.

We now define ringed spaces and describe the category of ringed spaces.

Definition 1.1.7. (Ringed Spaces) A ringed space is a topological space X with a sheaf of rings $\mathcal{O}(X)$, and so, is denoted by the pair $(X, \mathcal{O}(X))$. The sheaf $\mathcal{O}(X)$ is called the structure sheaf of the ringed space and consists of associative, commutative rings with unity.

Definition 1.1.8. (Morphisms of Ringed Spaces) In the category of ringed spaces, a morphism between objects (ringed spaces) $(X, \mathcal{O}(X))$ and $(Y, \mathcal{O}(Y))$ is a pair $(f, f^\#)$ such that $f : X \to Y$ is a homeomorphism and $f^\# : f^*\mathcal{O}(Y) \to \mathcal{O}(X)$ is a homomorphism of sheaves of rings over X, where $f^*\mathcal{O}(Y)$ is the pull-back sheaf, which transfers unit elements in the stalks to unit elements.

Remark. Defining instead a homomorphism $f_{\#} : \mathcal{O}(Y) \to f_*\mathcal{O}(X)$ of sheaves of rings over Y, where $f_*\mathcal{O}(X)$ is the push-forward sheaf, which transfers unit elements to unit elements in the stalks is equivalent to defining $f^{\#}$.

Next, we introduce the concept of localization of a ring.

Definition 1.1.9. (Localizations of Rings) Let A be a commutative ring with unit, and $S \subset A$ a multiplicative subset. We will assume that $1 \in S$ and $0 \notin S$. If $1 \notin S$, then use $\tilde{S} = \{1\} \cup S$. On $A \times S$ define an equivalence relation $\sim$ by setting $(a, s) \sim (b, t)$ if $\exists\ u \in S$ such that $u(at - bs) = 0$.

The equivalence class of (a, s) is denoted $\frac{a}{s}$ and the set of such classes

$$S^{-1}A = \left\{ \frac{a}{s} \mid a \in A, s \in S \right\}$$

is endowed with the structure of a ring by defining addition and multiplication as

$$\frac{a}{s} + \frac{b}{t} = \frac{at + bs}{st} \qquad \text{and} \qquad \frac{a}{s} \cdot \frac{b}{t} = \frac{ab}{st}$$

respectively.

The ring homomorphism $\sigma : A \to S^{-1}A$ via the identification $a \mapsto (a, 1)$ makes A isomorphic to a subring of $S^{-1}A$. Thus, $S^{-1}A$, also denoted A_S, along with the homomorphism σ is called the localization of A by S.

Example 1.1.10. We have the following examples of localizations.

1. If S is the set of non-zero elements of an integral domain A, then the localization of A by S is the field of fractions of A, denoted $\mathrm{frac}(A)$.

2. If $\mathfrak{p} \triangleleft A$ is a prime ideal of A, we set $S = A \setminus \mathfrak{p}$. The localization of A by S is denoted by $A_{\mathfrak{p}}$. The unique maximal ideal of $A_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$.

3. Suppose $f \in A$ and consider the multiplicative system $\{f^n\}_{n \in \mathbb{N}_0}$. The localization is constructed by inverting powers of f and is denoted A_f. If f is nilpotent, then the localization ring is trivial.

We now have the basic machinery required to define affine schemes and schemes.

Definition 1.1.11. (Affine Schemes) Let A be a commutative ring with a unit element. An affine scheme consists of a topological space $\mathrm{Spec}(A)$ and a sheaf of rings, $\mathcal{O}(\mathrm{Spec}(A))$, on $\mathrm{Spec}(A)$. The topological space $\mathrm{Spec}(A)$ is equipped with the Zariski topology; that is, in terms of a basis of open sets $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(A) | f \notin \mathfrak{p}\}$, where f runs though the elements of the ring A. The sheaf $\mathcal{O}(\mathrm{Spec}(A))$ of local rings is defined by the condition that $\Gamma(D(f), \mathcal{O}(\mathrm{Spec}(A))) = A_f$, where A_f is the localization of the ring A with respect to the multiplicative system $\{f^n\}_{n \in \mathbb{N}_0}$.

Definition 1.1.12. (Morphisms of Affine Schemes) Since an affine scheme is a locally ringed space isomorphic to $\text{Spec}(A)$ for some commutative, unital ring A, then a morphism of affine schemes is just a morphism of locally ringed spaces.

Definition 1.1.13. (Schemes) A scheme is a ringed space that is locally isomorphic to an affine scheme. That is, a scheme consists of a topological space X and a sheaf $\mathcal{O}(X)$ of commutative, unital rings on X with the condition that an open covering $\{X_i\}_{i\in I}$ of X must exist such that $(X_i, \mathcal{O}(X)|_{X_i})$ is isomorphic to the affine scheme $\text{Spec}\,\Gamma(X_i, \mathcal{O}(X))$ of the ring of sections of $\mathcal{O}$ over X_i.

Definition 1.1.14. (Subschemes) An open subscheme of a scheme $(X, \mathcal{O}(X))$ is a scheme $(U, \mathcal{O}(U))$ whose underlying space is the subspace $U \subseteq X$ together with an isomorphism of the structure sheaf $\mathcal{O}(U)$ with the restriction $\mathcal{O}(X)|_U$ of the structure sheaf $\mathcal{O}(X)$ to the subspace U.

Definition 1.1.15. (Algebraic Schemes) An algebraic scheme over a field k is a scheme X, together with a morphism of finite type from X to $\text{Spec}(k)$. In other words, X has a finite covering by affine open sets whose coordinate rings are finitely generated k-algebras.

Finally, to conclude this section, we introduce the concept of algebraic cycles.

Definition 1.1.16. (Algebraic Cycles of Schemes) An algebraic cycle of an arbitrary algebraic variety or scheme X is a finite formal sum $\sum n_V[V]$ of classes of irreducible subvarieties or subschemes V of X, with integer coefficients.

The notion of what classes of irreducible subvarieties or subschemes are and how they are defined will be formalized in the remaining sections of this chapter.

1.2 Intersection Multiplicity of Plane Curves

In this section we expand on the notions introduced in the opening remarks of the chapter. That is, we want to formalize the notion of the number of points of intersection of two transversely intersecting plane curves. We also define a measure of dimension for arbitrary commutative rings in terms of chains of prime ideals. For irreducible affine varieties, which correspond to the zero-locus of some finite family of polynomials which generate a prime ideal in the polynomial ring, this provides a well-defined measure of dimension.

First, we define the criterion for irreducibility.

Definition 1.2.1. (Irreducibility in Topological Spaces) A non-empty subset Y of a topological space X is irreducible if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. Otherwise, Y is said to be reducible.

Next, we define the notions of affine algebraic sets and affine algebraic varieties.

Definition 1.2.2. (Affine Algebraic Sets and Varieties) Let k be an algebraically closed field. A subset $Y \subseteq \mathbb{A}_k^n$ is said to be an affine algebraic set if there exists a subset $T \subseteq k[x_1, ..., x_n]$ such that $Y = Z(T)$. That is, if Y is the zero-locus of T.

An affine algebraic variety over k is the zero-locus in $\mathbb{A}_k^n$ of some finite family of polynomials in $k[x_1, ..., x_n]$ that generate a prime ideal.

Remarks. We make the following remarks.

1. An affine algebraic set results from removing the condition that the family of polynomials generate an ideal that is prime. In other words, affine algebraic varieties are affine algebraic sets that are irreducible. Thus, irreducibility of varieties is equivalent to the prime ideal condition.

2. A Zariski open subvariety of an affine variety is called a quasi-affine variety.

We now define the notions of regular functions and the ring of regular functions on a quasi-affine variety.

Definition 1.2.3. (Regular Functions on Quasi-Affine Varieties) Let k be an algebraically closed field and X a quasi-affine variety in $\mathbb{A}_k^n$. A function $f : X \to k$ is regular at a point $p \in X$ if there exists an open neighbourhood U of p, and polynomials $g, h \in k[x_1, ..., x_n]$ such that both h is nowhere zero on U and $f|_U = (\frac{g}{h})|_U$.

We say f is regular on X if it is regular at every point of X.

Definition 1.2.4. (The Ring of Regular Functions) We denote by $\mathcal{O}(X)$ the ring of all regular functions on X. For $p \in X$, we define the local ring of p on X, denoted $\mathcal{O}_{p,X}$ (or alternatively, just $\mathcal{O}_p$ when X is understood in context), to be the ring of germs of regular functions on X near p. That is, an element of $\mathcal{O}_p$ is a pair (U, f) where U is an open subset of X containing p and f is a regular function on U. We identify two such pairs (U, f) and (V, g) if $f = g$ on $U \cap V$.

Remark. $\mathcal{O}_p$ is a local ring; its maximal ideal is the set of germs of regular functions which vanish at p. For if $f(p) \neq 0$, then $g(p) \neq 0$ and so $\frac{1}{f} = \frac{h}{g}$ is regular in some neighbourhood of p. Thus, f being invertible means it does not belong to the maximal ideal of $\mathcal{O}_p$.

We denote this maximal ideal by $\mathcal{M}_{p,X}$, or simply $\mathcal{M}_p$. Finally, for the residue field $\mathcal{O}_p/\mathcal{M}_p \cong k$.

Definition 1.2.5. (The Coordinate Ring of Affine Varieties) Let X be an affine algebraic variety which is the zero locus of some prime ideal $\mathfrak{p} \subseteq k[x_1, ..., x_n]$. The quotient ring $k[x_1, ..., x_n]/\mathfrak{p}$ is called the coordinate ring of X. This ring is precisely the set of all regular functions on X. In other words, it is the space of global sections of the structure sheaf of X.

Next, we define the Krull dimension of a ring and the height of prime ideals.

Definition 1.2.6. (The Krull Dimension of a Ring) Let A be a commutative ring and $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_n$ be a chain of prime ideals of A. The length of this chain is the number of strict inclusions, in this case n. The Krull dimension of A is the supremum of the lengths of all chains of prime ideals in A.

Definition 1.2.7. (The Height of a Prime Ideal) Let A be a commutative ring and $\mathfrak{p}$ a prime ideal of A. The height of $\mathfrak{p}$, denoted $\text{ht}(\mathfrak{p})$, is the supremum of the lengths of all chains of prime ideals $(\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset ... \subset \mathfrak{p}_n = \mathfrak{p})$ contained in $\mathfrak{p}$. In other words, the height of $\mathfrak{p}$ is the Krull dimension of the localization $A_{\mathfrak{p}}$.

Remark. The Krull dimension of a ring is the supremum of the heights of all the maximal ideals of A.

Finally, we formalize the notion of the intersection multiplicity of two plane curves in affine space over an algebraically closed field.

Definition 1.2.8. (Intersection Multiplicity of Plane Curves) Let k be an algebraically closed field. If $f, g \in k[x, y]$ are two polynomials defining affine plane curves F and G in $\mathbb{A}^2_k$, then the intersection scheme Z is a subscheme of $\mathbb{A}^2_k$ defined by the ideal $\langle f, g\rangle \lhd k[x, y]$ generated by f and g. If $p = (a, b) \in \mathbb{A}^2_k$ is a point in affine 2-space over k, then the intersection multiplicity of F and G at p is defined to be

$$i(p, F \cdot G) = \dim_k(\mathcal{O}_{p,Z}) = \dim_k(\mathcal{O}_{p,\mathbb{A}^2_k}/\langle f, g\rangle)$$

The intersection number satisfies the following properties:

(1) $i(p, F \cdot G) = i(p, G \cdot F)$.

(2) $i(p, (F_1 + F_2) \cdot G) = i(p, F_1 \cdot G) + i(p, F_2 \cdot G)$, where $F_1 + F_2$ is the curve defined by $f_1 f_2$ with f_i defining F_i.

(3) If F' is defined by $f + gh$ for some $h \in k[x, y]$, then $i(p, F' \cdot G) = i(p, F \cdot G)$.

(4) $i(p, F \cdot G) = \infty$ if F and G have a common component through p. Otherwise, $i(p, F \cdot G)$ is finite and positive.

(5) $i(p, F \cdot G) = 0$ if $p \notin F \cap G$.

(6) $i(p, F \cdot G) = 1$ if $f = x - a$ and $g = y - b$ or more generally, if the Jacobian

$$\left.\left(\frac{\partial(f, g)}{\partial(x, y)}\right)\right|_p \neq 0$$

at p.

(7) If p is a simple point on F, and F has no common component with G or H through p, then

$$i(p, G \cdot H) \geq \min\{i(p, F \cdot G), i(p, F \cdot H)\}$$

1.3 Cycles and Rational Equivalence

In this section we establish the fact that cycles on an arbitrary scheme (or algebraic variety) X are finite formal integral linear combinations of classes of subschemes (or subvarieties) of X. Moreover, any rational function r on any subscheme (or subvariety) of X determines a cycle $[\text{div}(r)]$. Cycles differing by a sum of such cycles are defined to be rationally equivalent. We conclude the section by defining the group of k-cycles modulo rational equivalence, the algebraic counterpart to de Rham cohomology.

We first define the order of vanishing of an invertible rational function on an algebraic variety along a subvariety of codimension one.

Definition 1.3.1. (Order of Vanishing of Invertible Rational Functions) Let X be an algebraic variety and V a subvariety of codimension one. The local ring $A = \mathcal{O}_{V,X}$ is a one-dimensional local domain. Let $r \in R(X)^*$, where $R(X)$ is the field of rational functions on X and the set of non-zero elements of this field forms the multiplicative group $R(X)^*$. We will define the order of vanishing of r along V, denoted $\text{ord}_V(r)$, by setting

$$\text{ord}_V(r) = l_A(A/(r))$$

where l_A denotes the length of the A-module in parentheses. This map is a group homomorphism, i.e. it satisfies

$$\text{ord}_V(rs) = \text{ord}_V(r) + \text{ord}_V(s)$$

for $r, s \in R(X)^*$.

Remark. As any $r \in R(X)^*$ may be written as a ratio $r = a/b$, for some $a, b \in A$, then we must also have

$$\text{ord}_V(r) = \text{ord}_V(a/b) = \text{ord}_V(a) - \text{ord}_V(b)$$

The fact that $\text{ord}_V : R(X)^* \to \mathbb{Z}$ is a well-defined group homomorphism is found in Appendix A.3 in [Ful].

We now define k-cycles and the group of k-cycles.

Definition 1.3.2. (The Group of k-Cycles) Let X be an algebraic scheme and $k \in \mathbb{N}$. A k-cycle on X is a finite formal sum $\sum n_i[V_i]$ where the V_i are k-dimensional subvarieties of X and $n_i \in \mathbb{Z}$ for all i. The coefficient n_i is often referred to as the multiplicity of V_i in V.

The set of k-cycles, denoted Z_kX forms a free abelian group on the k-dimensional subvarieties of X where to each k-dimensional subvariety V of X, we have a corresponding cycle $[V]$ of Z_kX.

Definition 1.3.3. (Normal Rings) A normal ring is an integral domain that is integrally closed in its field of fractions. Thus, a variety X is normal if $\mathcal{O}_x$ is a normal ring for all $x \in X$.

Definition 1.3.4. (Prime Divisors on Schemes) Let X be an algebraic scheme. A prime divisor on X is an irreducible subvariety of X of codimension 1.

We now define an important class of algebraic cycles called principal Weil divisors that will allow us to establish the notion of rational equivalence in the group of k-cycles.

Definition 1.3.5. (Principal Weil Divisors) Let W be a $(k+1)$-dimensional subvariety of X and $r \in R(W)^*$. Define a k-cycle, $[\mathrm{div}(r)]$ on X called a principal Weil divisor by setting

$$[\mathrm{div}(r)] = \sum \mathrm{ord}_V(r)[V]$$

where the sum runs over all the codimension-1 subvarieties V of W and ord_V is the order function on $R(W)^*$ defined by the local ring $\mathcal{O}_{V,W}$.

Definition 1.3.6. (Cycles Rationally Equivalent to Zero) Let α be a k-cycle. We say that α is rationally equivalent to zero, written as $\alpha \sim 0$, if there exists a finite number of $(k+1)$-dimensional subvarieties W_i of X and invertible rational functions $r_i \in R(W_i)^*$ such that

$$\alpha = \sum [\mathrm{div}(r_i)]$$

Thus, if for two k-cycles α and β we have

$$\alpha - \beta = \sum [\mathrm{div}(r_i)]$$

for some $r_i \in R(W_i)^*$, then we say α and β are rationally equivalent.

Since $[\mathrm{div}(r_i)^{-1}] = -[\mathrm{div}(r_i)]$, the cycles rationally equivalent to zero form a subgroup of Z_kX denoted Rat_kX.

We are now ready to define the key concept of this section; the group of k-cycles modulo rational equivalence.

Definition 1.3.7. (The Group of k-Cycles Modulo Rational Equivalence) The factor group

$$A_kX = Z_kX/\mathrm{Rat}_kX$$

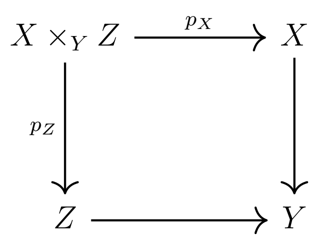
is called the group of k-cycles modulo rational equivalence.

1.4 Push-forwards of Rational Cycles

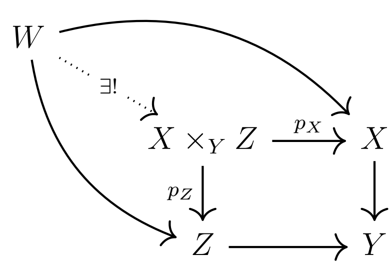
One of the most fundamental notions of intersection theory is that rational equivalence pushes forward. That is, for a proper morphism of schemes there is an induced homomorphism between the groups of cycles modulo rational equivalence. In this section we define this push-forward map, describe its functorial nature and define the degree map on the group of rational cycles modulo rational equivalence. We also give an alternative definition to the notion of rational equivalence in terms of dominant morphisms over $\mathbb{P}^1$.

We begin with the definition of a fiber product, also referred to as the categorical pull-back or Cartesian square.

Definition 1.4.1. (Fiber Products of Schemes) For any morphisms of schemes $X \to Y$ and $Z \to Y$, there exists a scheme $X \times_Y Z$ with morphisms to X and Z making the diagram



commute, and which is universal with that property. That is, for any scheme W with morphisms to X and Z whose compositions to Y are equal, there is a unique morphism from W to $X \times_Y Z$ that makes the diagram commute.



As always with universal properties, if it exists, then this condition determines the scheme $X \times_Y Z$ uniquely, up to isomorphism.

The fiber product of the morphism $X \to Y$, for any other given morphism $Z \to Y$, is the object $X \times_Y Z$ and the maps $X \times_Y Z \to X$ and $X \times_Y Z \to Z$.

We next want to define proper morphisms of schemes as this will be critical in discussing push-forwards of rational cycles. We require however, some preliminary definitions.

Definition 1.4.2. (Separated Morphisms and Schemes) Let $f : X \to Y$ be a morphism of schemes. Write $\Delta : X \to X \times_Y X$ for the diagonal morphism. The morphism f is called separated if $\Delta(X)$ is a closed subscheme of $X \times_Y X$; in other words, if the diagonal map is a closed immersion.

A scheme X is called separated if the terminal morphism $X \to \text{Spec}(\mathbb{Z})$ is separated.

Definition 1.4.3. (Morphisms of Rings of Finite Type) A ring homomorphism $R \to S$ is said to be of finite type if S is isomorphic to a quotient of $R[x_1, ..., x_n]$ as an R-algebra for some $n \in \mathbb{N}$.

Definition 1.4.4. (Morphisms of Schemes of Finite Type) Let $f : X \to Y$ be a morphism of schemes. We say that f is:

(i) of finite type at a point $x \in X$ if there exists an affine open neighbourhood $\text{Spec}(S) = U \subseteq X$ of x and an affine open set $\text{Spec}(R) = V \subseteq Y$ with $f(U) \subseteq V$ such that the induced ring map $R \to S$ is of finite type.

(ii) locally of finite type if it is of finite type at every point of X.

(iii) of finite type if it is locally of finite type and quasi-compact.

Definition 1.4.5. (Universally Closed Morphisms of Schemes) A morphism $f : X \to Y$ of schemes is called universally closed if for every scheme Z with a morphism $Z \to Y$, the projection from the fiber product $X \times_Y Z \to Z$ is a closed map of the underlying topological spaces.

We are now ready to define proper morphisms of schemes.

Definition 1.4.6. (Proper Morphisms of Schemes) A morphism of schemes is called proper if it is separated, of finite type, and universally closed.

Definition 1.4.7. (Schemes Defined over Schemes) Let S be a scheme. We say X is a scheme over S if X comes equipped with a morphism of schemes $X \to S$. The morphism $X \to S$ is sometimes called the structure morphism.

Definition 1.4.8. (The Degree of Subvarieties for Proper Morphisms) Let $f : X \to Y$ be a proper morphism. For any irreducible subvariety V of X, the image $f(V) = W$ is a closed subvariety of Y. If W has the same dimension as V, then the induced embedding $R(W)$ into $R(V)$ is a finite field extension. We define

$$\deg(V/W) = \begin{cases} [R(V) : R(W)] & \text{if} \quad \dim(W) = \dim(V) \\ 0 & \text{if} \quad \dim(W) < \dim(V) \end{cases}$$

where $[R(V) : R(W)]$ denotes the degree of the field extension.

We are now ready to define the push-forward map of the group of k-cycles on an arbitrary scheme X. We then define the push-forward for principal Weil divisors which will induce a push-forward map at the level of k-cycles modulo rational equivalence.

Definition 1.4.9. (The Push-Forward of the Group $Z_k(X)$) The push-forward of the group of k-cycles on X by the proper morphism f is defined by setting

$$f_*[V] = \deg(V/W)[W]$$

and which extends to a homomorphism $f_* : Z_kX \to Z_kY$ by linearity.

Remark. If $g : Y \to Z$ is also a proper morphism, then $(g \circ f)_* = g_* \circ f_*$, which follows from the multiplicity of degrees of field extensions. Thus, these induced homomorphisms exhibit the behaviour of being under the action of a covariant functor.

Proposition 1.4.10. ***(Push-Forwards of Principal Weil Divisors)***
Let $f : X \to Y$ be a proper, surjective morphism of varieties and let $r \in R(X)^*$. Then,

$$\begin{aligned} &(a)\ f_*[div(r)] = 0 && if && dim(Y) < dim(X) \\ &(b)\ f_*[div(r)] = [div(N(r))] && if && dim(Y) = dim(X) \end{aligned}$$

where $N(r)$ is the norm of r, i.e. the determinant of the $R(Y)$-linear endomorphism of $R(X)$ given by multiplication of r.

Theorem 1.4.11. ***(The Push-Forward of the Group $Rat_k(X)$)*** *If $f : X \to Y$ is a proper morphism, and α is a k-cycle on X which is rationally equivalent to zero, then $f_*\alpha$ is rationally equivalent to zero on Y.*

Definition 1.4.12. (The Push-Forward of the Group $A_k(X)$) By Proposition 1.4.10, there is therefore an induced homomorphism,

$$f_* : A_kX \to A_kY$$

implying the action of a covariant functor for proper morphisms, which will be denoted by A_*. This homomorphism is called the push-forward of the group of k-cycles modulo rational equivalence.

Definition 1.4.13. (The Fundamental Cycle of a Scheme) Let X be any scheme, and let $X_1, ..., X_t$ be the irreducible components of X. The local rings $\mathcal{O}_{X_i,X}$ are all zero-dimensional (Artinian, see Chapter 4). The geometric multiplicity m_i of X_i in X is defined to be the length of $\mathcal{O}_{X_i,X}$, that is;

$$m_i = l_{\mathcal{O}_{X_i,X}}(\mathcal{O}_{X_i,X})$$

The fundamental cycle $[X]$ of X is the cycle

$$[X] = \sum_{i=1}^{t} m_i [X_i]$$

This is regarded as an element of $Z_* X$. However, we also write $[X]$ for its image in $A_* X$. If moreover, $\dim(X_i) = k$ for all $i = 1, ..., t$, then $[X] \in Z_k X$. In this case $Z_k X = A_k X$ is the free abelian group on $[X_1], ..., [X_t]$.

We now define the notion of the degree of a cycle.

Definition 1.4.14. (The Degree of an Algebraic Cycle) Let X be a complete scheme over a field K. In other words, X is proper over $S = \mathrm{Spec}(K)$. Furthermore, let $\alpha = \sum_P n_P [P]$ be a zero-cycle on X. The degree of α, denoted $\deg(\alpha)$ or $\int_X \alpha$ is defined by setting

$$\deg(\alpha) = \int_X \alpha = \sum_P n_P [R(P) : K]$$

Equivalently, let p denote the structure morphism from X to S. We could have also defined $\deg(\alpha) = p_*(\alpha)$, where $A_0(S) = \mathbb{Z} \cdot [S]$ is identified with $\mathbb{Z}$.

By Theorem 1.4.11, rationally equivalent cycles have the same degree so we can extend the degree homomorphism to all of $A_* X$ as

$$\int_X : A_* X \to \mathbb{Z}$$

by defining $\int_X \alpha = 0$ if $\alpha \in A_k X$ for $k > 0$.

Thus, for any morphism of complete schemes $f : X \to Y$, and any $\alpha \in A_* X$, we have:

$$\int_X \alpha = \int_Y f_*(\alpha)$$

a special case of functoriality.

Definition 1.4.15. (Dominant Morphisms of Schemes) A morphism $f : X \to S$ of schemes is called dominant if the image of f is a dense subset of S.

Finally, we conclude this section by giving an alternative definition for the notion of rational equivalence.

Definition 1.4.16. (Alternative Definition of Rational Equivalence)
Let X be a scheme, and consider the Cartesian product $X \times \mathbb{P}^1$. Let $\pi_1 : X \times \mathbb{P}^1 \to X$ be the projection onto the first factor and $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection onto the second. Let V be a $(k+1)$-dimensional subvariety of $X \times \mathbb{P}^1$ such that π_2 induces a dominant morphism $f : V \to \mathbb{P}^1$.

For any point $p \in \mathbb{P}^1$ that is rational over the ground field k, the scheme-theoretic fiber $f^{-1}(p) \subseteq V$ is a subscheme of $X \times \{p\}$, which π_1 maps isomorphically to a subscheme of X, which we denote by $V(p)$.

Note, the push-forward of the map π_1 is $\pi_{1*} : Z_k(X \times \mathbb{P}^1) \to Z_k X$ and

$$\pi_{1*}[f^{-1}(p)] = [V(p)]$$

in $Z_k X$. Moreover, $f : V \to \mathbb{P}^1$ determines a rational function $f \in R(V)^*$ from which we deduce

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [\mathrm{div}(f)]$$

where $0 = (1 : 0)$ and $\infty = (0 : 1)$ are the usual zero and infinity points of $\mathbb{P}^1$.

Therefore;

$$[V(0)] - [V(\infty)] = \pi_{1*}[\mathrm{div}(f)]$$

which is rationally equivalent to zero on X.

1.5 Pull-Backs of Rational Cycles

In this section we briefly define the flat pull-back homomorphisms on the groups of k-cycles modulo rational equivalence which are induced by flat morphisms of schemes.

First, we introduce some preliminary definitions on the concepts of exactness of sequences of modules, exact functors, and the flatness criterion for both modules and ring homomorphisms.

Definition 1.5.1. (Exact Sequences and Functors) Let R be a commutative ring.

(i) A sequence of R-modules and R-homomorphisms

$$\ldots \to M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \to \ldots$$

is said to be exact at M_i if $\mathrm{Im}(f_i) = \ker(f_{i+1})$. The sequence is exact if it is exact at each M_i.

(ii) An exact functor is a functor that preserves exact sequences.

Definition 1.5.2. (Flat Modules and Ring Homomorphisms) Let R and A be rings and M an R-module.

(i) M is flat if the functor $- \otimes_R M : \text{Mod}_R \to \text{Mod}_R$ is exact.

(ii) The homomorphism $R \to A$ of rings is flat, if A is flat as an R-module.

We can now define the flatness condition and the relative dimension of a morphism of schemes.

Definition 1.5.3. (Flat Morphisms of Schemes) Let $f : X \to S$ be a morphism of schemes. Let $\mathcal{F}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-modules.

(i) We say f is flat at a point $x \in X$ if the local ring $\mathcal{O}_{x,X}$ is flat over the local ring $\mathcal{O}_{f(x),S}$. Moreover, f is flat if it is flat at every point of X.

(ii) We say that $\mathcal{F}$ is flat over S at a point $x \in X$ if the stalk $\mathcal{F}_x$ is flat as an $\mathcal{O}_{f(x),S}$-module. Moreover, $\mathcal{F}$ is flat over S if $\mathcal{F}$ is flat over S at every point of X.

Thus, we see that f is flat if and only if the structure sheaf $\mathcal{O}_X$ is flat over S.

Definition 1.5.4. (The Relative Dimension of a Morphism of Schemes) Let $f : X \to S$ be a morphism of schemes and assume f is locally of finite type. Then, we say that f is of relative dimension d if all non-empty fibres X_S are equidimensional of dimension d.

The relative dimension of a morphism f is also sometimes referred to as the codimension of f.

We can now define the pull-back homomorphisms on the groups of k-cycles induced by flat morphisms of schemes (or varieties).

Definition 1.5.5. (The Pull-Back of the Group $\boldsymbol{Z_k(X)}$) Let $f : X \to Y$ be a flat morphism of relative dimension n. For any subvariety V of Y set

$$f^*[V] = [f^{-1}(V)]$$

where $f^{-1}(V)$ is the inverse image scheme, a subscheme of X of pure dimension equal to $\dim(V) + n$, and $[f^{-1}(V)]$ is its cycle. This extends by linearity to pull-back homomorphisms

$$f^* : Z_k Y \to Z_{k+n} X$$

Lemma 1.5.6. ***(Pull-Backs of Subschemes)*** *If $f : X \to Y$ is flat, then for any subscheme Z of Y*

$$f^*[Z] = [f^{-1}(Z)]$$

Theorem 1.5.7. ***(The Pull-Back of the Group $Rat_k(X)$)*** *Let $f : X \to Y$ be a flat morphism of relative dimension n, and α a k-cycle on Y which is rationally equivalent to zero. Then, $f^*\alpha$ is rationally equivalent to zero in $Z_{k+n}X$.*

Definition 1.5.8. (The Pull-Back of the Group $A_k(X)$) The induced homomorphisms, the flat pull-backs of the group of k-cycles modulo rational equivalence,

$$f^* : A_kY \to A_{k+n}X$$

imply the action of a contravariant functor for flat morphisms, denoted by A^*.

1.6 Intersection Theories and the Chow Ring

We conclude the chapter by defining what it means to be an intersection theory. We define the criteria for a suitable intersection pairing which turns the group $A(X)$ into a commutative associative graded ring with identity called the Chow ring of X. We then look at the case when the given class of varieties that the theory is devised over is the class of nonsingular quasi-projective varieties and we list some of its properties. Finally, we define Chern classes and also list some of their properties. For definitions of Cartier divisors, Weil divisors, and the Picard group, see Chapter 2 in [Ful]. For a more detailed presentation of the material covered in this section see Appendix A.1, A.2, and A.3 in [Har].

Definition 1.6.1. (Intersection Theories)

Let $\mathfrak{B}$ be a given class of varieties. An intersection theory on $\mathfrak{B}$ consists of giving a pairing

$$\cup : A^r(X) \times A^s(X) \to A^{r+s}(X)$$

for each r, s and for all $X \in \mathfrak{B}$, subject to the following axioms. Note, if $V \in A^r(X)$ and $W \in A^s(X)$, we denote the intersection cycle class by $V \cup W$.

(**A1**) The intersection pairing makes $A(X)$ into a commutative associative graded ring with identity, for every $X \in \mathfrak{B}$. It is called the Chow ring of X.

(**A2**) For any morphism $f : X \to Y$ of varieties in $\mathfrak{B}$, $f^* : A(Y) \to A(X)$ is a ring homomorphism. If $g : Y \to Z$ is another morphism, then $(g \circ f)^* = f^* \circ g^*$.

(**A3**) For any proper morphism of varieties $f : X \to Y$ in $\mathfrak{B}$, $f_* : A(X) \to A(Y)$ is a homomorphism of graded groups (which shifts degrees). If $g : Y \to Z$ is another proper morphism, then $(g \circ f)_* = g_* \circ f_*$.

(**A4**) Projection Formula: Let $f : X \to Y$ be a proper morphism. If $\alpha \in A(X)$ and $\beta \in A(Y)$, then

$$f_*(\alpha \cup f^*\beta) = f_*(\alpha) \cup \beta$$

(**A5**) Reduction to the Diagonal: If V_1 and V_2 are cycles on X, and $\Delta : X \to X \times X$ is the diagonal morphism, then

$$V_1 \cup V_2 = \Delta^*(V_1 \times V_2)$$

where $\Delta^* : A(X \times X) \to A(X)$.

(**A6**) Local Nature: If V_1 and V_2 are subvarieties of X which intersect properly (meaning that every irreducible component of $V_1 \cap V_2$ has codimension equal to $\operatorname{codim}(V_1) + \operatorname{codim}(V_2)$), then we can write

$$V_1 \cup V_2 = \sum i(V_1, V_2; W_j) W_j$$

where the sum runs over the irreducible components W_j of $V_1 \cap V_2$ and where the integer $i(V_1, V_2; W_j)$ depends only on a neighbourhood of the generic point of W_j on X. We call $i(V_1, V_2 : W_j)$ the local intersection multiplicity of V_1 and V_2 along W_j.

(**A7**) Normalization: If V is a subvariety of X and W is an effective Cartier divisor meeting V properly, then $V \cup W$ is just the cycle associated to the Cartier divisor $V \cap W$ on V, which is defined by restricting the local equation of W to V. (This implies in particular that transversal intersections of non-singular subvarieties have multiplicity 1).

Definition 1.6.2. (Properties of the Chow Ring)

For any nonsingular quasi-projective variety we now consider the Chow ring $A(X)$, and list some of its properties.

(**A8**) Since the cycles in codimension 1 are just Weil divisors, and rational equivalence is the same as linear equivalence for them, and X is nonsingular, we have $A^1(X) \cong \operatorname{Pic}(X)$.

(**A9**) For any affine space $\mathbb{A}^m$, the projection $\pi_1 : X \times \mathbb{A}^m \to X$ induces an isomorphism

$$\pi_1^* : A(X) \to A(X \times \mathbb{A}^m)$$

(**A10**) Exactness: If V is a nonsingular closed subvariety of X, and $W = X \setminus V$, then there is an exact sequence

$$A(V) \xrightarrow{i_*} A(X) \xrightarrow{j^*} A(W) \to 0$$

where $i : V \to X$ and $j : W \to X$ are the inclusion maps of V and W into X.

(**A11**) Let $\mathscr{E}$ be a locally free sheaf of rank r on X, let $\mathbb{P}(\mathscr{E})$ be the associated projective space bundle, and let $\xi \in A^1(\mathbb{P}(\mathscr{E}))$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the projection. Then, π^* makes $A(\mathbb{P}(\mathscr{E}))$ into a free $A(X)$ module generated by $1, \xi, \xi^2, ..., \xi^{r-1}$.

Definition 1.6.3. (Chern Classes) Let $\mathscr{E}$ be a locally free sheaf of rank r on a nonsingular quasi-projective variety X. For each $i = 0, 1, ..., r$, we define the i^{th} Chern class $c_i(\mathscr{E}) \in A^i(X)$ by the requirement that $c_0(\mathscr{E}) = 1$ and

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathscr{E}) \cup \xi^{r-i} = 0$$

in $A^r(\mathbb{P}(\mathscr{E}))$, using the notation of **(A11)**.

This makes sense, since by **(A11)**, we can express ξ^r as a unique linear combination of $1, \xi, ..., \xi^{r-1}$, with coefficients in $A(X)$, via π^*.

Definition 1.6.4. (Properties of the Chern Classes) The following are some properties of the Chern classes. For convenience, we define the total Chern class as

$$c(\mathscr{E}) = c_0(\mathscr{E}) + c_1(\mathscr{E}) + ... + c_r(\mathscr{E})$$

and the Chern polynomial

$$c_t(\mathscr{E}) = c_0(\mathscr{E}) + c_1(\mathscr{E})t + ... + c_r(\mathscr{E})t^r$$

(C1) If $\mathscr{E} = \mathscr{L}(D)$ for a divisor D, then $c_t(\mathscr{E}) = 1 + Dt$. Indeed, in this case $\mathbb{P}(\mathscr{E}) = X$, $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) = \mathscr{L}(D)$, so $\xi = D$. Thus, by definition, we have

$$1 \cup \xi - c_1(\mathscr{E}) \cup 1 = 0$$

so $c_1(\mathscr{E}) = D$.

(C2) If $f : X' \to X$ is a morphism and $\mathscr{E}$ is a locally free sheaf on X, then for each i,

$$c_i(f^*\mathscr{E}) = f^* c_i(\mathscr{E})$$

This follows immediately from the functoriality properties of the $\mathbb{P}(\mathscr{E})$ construction and f^*.

(C3) If $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is an exact sequence of locally free sheaves on X, then

$$c_t(\mathscr{E}) = c_t(\mathscr{E}') \cdot c_t(\mathscr{E}'')$$

(C4) We say that $\mathscr{E}$ splits if it has a filtration $\mathscr{E} = \mathscr{E}_0 \supseteq \mathscr{E}_1 \supseteq ... \supseteq \mathscr{E}_r = 0$ whose successive quotients are all invertible sheaves.

If $\mathscr{E}$ splits, and the filtration has the invertible sheaves $\mathscr{L}_1, ..., \mathscr{L}_r$ as quotients, then

$$c_t(\mathscr{E}) = \prod_{i=1}^{r} c_t(\mathscr{L}_i)$$

(C5) Let $\mathscr{E}$ have rank r, and let $\mathscr{F}$ have rank s. Write

$$c_t(\mathscr{E}) = \prod_{i=1}^{r}(1 + a_i t)$$

and

$$c_t(\mathscr{F}) = \prod_{j=1}^{s}(1 + b_j t)$$

where $a_1, ..., a_r, b_1, ..., b_s$ are just formal symbols. Then, we have

$$\begin{aligned} c_t(\mathscr{E} \otimes \mathscr{F}) &= \prod_{i,j}(1 + (a_i + b_j)t) \\ c_t(\Lambda^p \mathscr{E}) &= \prod_{1 \leq i_1 \leq ... \leq i_p \leq r} (1 + (a_{i_1} + ... + a_{i_p})t) \\ c_t(\mathscr{E}^\vee) &= c_{-t}(\mathscr{E}) \end{aligned}$$

(C6) Let $s \in \Gamma(X, \mathscr{E})$ be a global section of a locally free sheaf $\mathscr{E}$ of rank r on X. Then, s defines a homomorphism $\mathcal{O}_X \to \mathscr{E}$ by sending 1 to s. We define the scheme of zeros of s to be the closed subscheme Y of X defined by the exact sequence

$$\mathscr{E}^\vee \xrightarrow{s^\vee} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

where $s^\vee$ is the dual of the map s. Let Y also denote the associated cycle of Y. Then, if Y has codimension r, we have $c_r(\mathscr{E}) = Y$ in $A^r(X)$. This generalizes the fact that a section of an invertible sheaf gives the corresponding divisor.

(C7) Self-Intersection Formula: Let Y be a nonsingular subvariety of X of codimension r, and let $\mathcal{N}$ be the normal sheaf. Let $\iota : Y \hookrightarrow X$ be the inclusion map. Then,

$$\iota^* \iota_*(1_Y) = c_r(\mathcal{N})$$

Therefore, applying the projection formula **(A4)**, we have

$$\iota_*(c_r(\mathcal{N})) = Y \cup Y$$

on X.

Chapter 2

Algebraic Groups and Flag Manifolds

In the present chapter, we briefly recall several basic facts regarding algebraic groups, group actions, and homogeneous spaces. In particular, we are interested in transitive group actions on flag manifolds. We also briefly define adjoint representations of Lie and algebraic groups. A more detailed exposition of these concepts can be found in [Knu], [Bor], [Hum], and [Mil].

2.1 Algebraic Groups and Parabolic Subgroups

In this section, we define algebraic groups and study various closely related objects; namely affine algebraic groups, linear algebraic groups and simple algebraic groups. We define what it means for a group to be solvable, and then define Borel subgroups of an algebraic group. Finally, we define parabolic subgroups and homogeneous varieties.

We first define algebraic groups and group varieties.

Definition 2.1.1. (Algebraic Groups) Let k be a field. An algebraic group over k, or algebraic k-group, is a group object in the category of algebraic schemes over k.

More precisely, let G be an algebraic scheme over k and let $m : G \times G \to G$ be a regular map. The pair (G, m) is an algebraic group over k if there exist regular maps

$$e : \mathrm{Spec}(k) \to G, \qquad \mathrm{inv} : G \to G$$

such that the following diagrams commute:

(i) associativity

$$\begin{CD} G \times G \times G @>{\mathrm{id}\times m}>> G \times G \\ @V{m\times \mathrm{id}}VV @VV{m}V \\ G \times G @>>{m}> G \end{CD}$$

(ii) existence of an identity element

$$\begin{array}{ccccc} \mathrm{Spec}(k) \times G & \xrightarrow{e\times \mathrm{id}} & G \times G & \xleftarrow{\mathrm{id}\times e} & G \times \mathrm{Spec}(k) \\ & \underset{\simeq}{\searrow} & \Big\downarrow{\scriptstyle m} & \underset{\simeq}{\swarrow} & \\ & & G & & \end{array}$$

making (G, m) a monoid in the category of algebraic schemes over k, and

(iii) existence of inverses

$$\begin{CD} G @>{(\mathrm{inv},\mathrm{id})}>> G \times G @<{(\mathrm{id},\mathrm{inv})}<< G \\ @VVV @VV{m}V @VVV \\ \mathrm{Spec}(k) @>>{e}> G @<<{e}< \mathrm{Spec}(k) \end{CD}$$

Remark. When G is an algebraic variety, we call the pair (G, m) a group variety.

Next, we define homomorphisms of algebraic groups and algebraic subgroups.

Definition 2.1.2. (Morphisms of Algebraic Groups) Let (G, m) and (G', m') be algebraic k-groups. A homomorphism of algebraic groups, $\phi : (G, m) \to (G', m')$, is a regular map $\phi : G \to G'$ such that $\phi \circ m = m' \circ (\phi \times \phi)$. In other words, that is that the following diagram commutes:

$$\begin{CD} G \times G @>{m}>> G \\ @V{\phi\times\phi}VV @VV{\phi}V \\ G' \times G' @>>{m'}> G' \end{CD}$$

Definition 2.1.3. (Algebraic Subgroups) Let (G, m_G) be an algebraic k-group. An algebraic subgroup of (G, m_G) is an algebraic group (H, m_H) over k such that H is a k-subscheme of G and the inclusion map is a homomorphism of algebraic groups.

We are now ready to define affine algebraic groups, linear algebraic groups, and simple algebraic groups, providing also a basic, yet important, example along the way.

Definition 2.1.4. (Affine Algebraic Groups) An affine k-group is a representable functor on the category of commutative associative k-algebras, $G : \mathrm{Alg}_k \to \mathrm{Set}$ together with a natural transformation $m : G \times G \to G$ such that for any k-algebra A, the map

$$m(A) : G(A) \times G(A) \to G(A)$$

is a group structure on $G(A)$. Moreover, if G is represented by a finitely presented k-algebra, then it is called an affine algebraic group. A homomorphism $G \to H$ of affine k-groups is a natural transformation preserving group structures.

Definition 2.1.5. (Linear Algebraic Groups) An algebraic group, G is linear if it admits a faithful finite-dimensional representation.

Since such a representation of G is an isomorphism of G onto a closed algebraic subgroup of $GL(V)$, then an algebraic group is linear if and only if it can be realized as an algebraic subgroup of $GL(V)$ for some finite-dimensional vector space V.

Definition 2.1.6. (Simple Algebraic Groups) An algebraic k-group is simple if it is non-commutative and has no non-trivial closed connected normal subgroups.

Furthermore, for an algebraic group satisfying the above conditions, the terminology *almost simple* will be used to emphasize that the group need not be simple as an abstract group.

Example 2.1.7. The special linear group, $SL_n(k)$ is a classic example of both a simple algebraic group and a linear algebraic group.

The next result shows that there is a one-to-one correspondence between linear algebraic groups and affine algebraic groups.

Theorem 2.1.8. *Every linear algebraic group is affine, and since the regular representation has a faithful finite-dimensional sub-representation, the converse is also true. Therefore, the linear algebraic groups over k are exactly the affine algebraic groups over k.*

For a proof of this theorem see Chapter 4, section d., 'Affine algebraic groups are linear', Theorem 4.9 in [Mil] and Chapter II, section 8.6, 'Linearization of Affine Groups' in [Hum].

We now move to describing parabolic subgroups and homogeneous spaces. However, we need some preliminary definitions and results on solvable groups first.

Definition 2.1.9. (The Derived Series of an Abstract Group) Let G be an abstract group. The derived series of G is defined inductively by setting $\mathscr{D}^0 G = G$ and for all $n \in \mathbb{N}_0$, $\mathscr{D}^{n+1}G = (\mathscr{D}^n G, \mathscr{D}^n G)$, the commutator group of $\mathscr{D}^n G$ with itself.

Definition 2.1.10. (Solvable Groups) An abstract group G is solvable if it is a finite iterated extension of an abelian group by abelian groups. In other words, if there exists a finite sequence

$$\{1\} = G_0 \subset G_1 \subset G_2 \subset ... \subset G_k = G$$

in which G_{j-1} is normal in G_j and G_j/G_{j-1} is abelian for all $j = 1, ..., k$.

Equivalently, one says that G is solvable if for some $n \in \mathbb{N}_0$, we have $\mathscr{D}^n G = \{e\}$.

Proposition 2.1.11. ***(Solvable Algebraic Groups)*** *Let G be an algebraic k-group. Then, G is solvable if and only if there is a chain $G = G_0 \supset G_1 \supset ... \supset G_n = \{e\}$ of closed subgroups defined over k such that $(G_i, G_i) \subset G_{i+1}$ for all $i = 0, 1, ..., n-1$.*

For a proof, see [Bor], Chapter I, § 2.4, 'Solvable and nilpotent groups', Corollary 3.

Definition 2.1.12. (Complete Algebraic Varieties) Let X be an algebraic variety. Then, X is a complete algebraic variety if for any variety Y, the projection morphism $X \times Y \to Y$ is a closed map.

Remark. In particular, all projective varieties are complete.

We are now ready to define Borel subgroups, parabolic subgroups and homogeneous varieties.

Definition 2.1.13. (Borel Subgroups of Connected Algebraic Groups) Let G be an arbitrary connected algebraic k-group. A subgroup $B \subseteq G$ is said to be a Borel subgroup if it is maximal, with respect to inclusion, among all the Zariski closed connected solvable subgroups.

Definition 2.1.14. (Parabolic Subgroups of Connected Algebraic Groups) Let G be a connected algebraic k-group. A parabolic subgroup of G is a Zariski closed subgroup $P \subset G$, for which the quotient space G/P is a complete algebraic variety.

Proposition 2.1.15. *If P is a closed subgroup of G, then G/P is a projective variety if and only if P contains a Borel subgroup.*

For a proof, see [Bor], Chapter IV, § 11, 'Borel Subgroups', the first Corollary on page 148 or [Hum], Chapter VIII, section 21.3, 'Conjugacy of Borel Subgroups and Maximal Tori', Corollary B.

Definition 2.1.16. (Homogeneous Varieties) Let G be a linear algebraic k-group and P a parabolic subgroup of G. A homogeneous variety is an algebraic variety of the form G/P, which is a smooth projective variety.

2.2 Adjoint Representations of Lie (or Algebraic) Groups

In this short section, we define adjoint representations of Lie and algebraic groups. That is, given any Lie or algebraic group, G, we want a construction which assigns to every element $g \in G$, an element of the automorphism group of the Lie algebra, $\mathfrak{g}$, associated to G.

Definition 2.2.1. (Adjoint Representations of Lie or Algebraic Groups) Let G be a Lie (or algebraic) group and $\mathrm{Aut}(G)$ the automorphism group of G. Let

$$\Psi : G \to \mathrm{Aut}(G)$$

be the mapping defined by $g \mapsto \Psi_g$, where for each $g \in G$, $\Psi_g \in \mathrm{Aut}(G)$ is an automorphism of Lie (or algebraic) groups

$$\Psi_g : G \to G$$

given by the inner automorphism $\Psi_g(h) = ghg^{-1}$ for all $h \in G$.

Let X be a vector field on G, or in other words, a section of the map $\pi : TG \to G$, where TG is the tangent bundle of G. That is, the map $X : G \to TG$ is defined by setting $h \mapsto X_h := X(h) \in T_hG \subseteq TG$. We have the following commutative diagram:

$$\begin{array}{ccc} TG & \xrightarrow{d\Psi_g} & TG \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle X} \\ G & \xrightarrow[\Psi_g]{} & G \end{array}$$

That is, $d\Psi_g(X_h) = X(\Psi_g(h))$ for all $h \in G$. Now, let e be the identity element of the group G and consider T_eG, the tangent space of G at e, where we denote $\mathfrak{g} := T_eG$ and call it the Lie algebra $\mathfrak{g}$ of G. If we consider the pair (G, e), then for each $g \in G$, we obtain a commutative diagram:

$$\begin{array}{ccc} T_eG & \xrightarrow{(d\Psi_g)_e} & T_eG \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle X} \\ (G,e) & \xrightarrow[\Psi_g]{} & (G,e) \end{array}$$

fixing base points since $\Psi_g(e) = geg^{-1} = e$ and $X_e \in T_eG$, and depending only on the vector field X.

Furthermore, for each $g \in G$, define Ad_g to be the derivative of Ψ_g at the identity, i.e.

$$\mathrm{Ad}_g := (d\Psi_g)_e : T_eG \to T_eG$$

Since Ψ_g is an automorphism of Lie Groups then Ad_g is a Lie algebra automorphism. That is, Ad_g is an invertible linear transformation of $\mathfrak{g}$ to itself that preserves the Lie bracket. The map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$$

defined by sending $g \mapsto \mathrm{Ad}_g$ is a group representation called the adjoint representation of G. Finally, since $\mathrm{Aut}(\mathfrak{g})$ is a closed Lie subgroup of $GL(\mathfrak{g})$, then the map Ad is a Lie Group homomorphism.

2.3 Transitive Actions on Flag Manifolds

In the present section we first introduce the notion of group actions on sets. Then, we discuss how this group action, called a G-action on the set X induces an equivalence relation. The classes formed by this equivalence relation are called orbits and we specify what it means for these actions to be transitive. Finally, we formalize group actions on manifolds, and in particular, on flag manifolds.

We first define group actions on sets and the natural equivalence relation induced by the action.

Definition 2.3.1. (Group Actions on Sets) Let X be a non-empty set and G a group. A left-action of G on X is a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

such that $e \cdot x = x$, where e is the identity element of G, and $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ for all $x \in X$ and for all $g_1, g_2 \in G$.

We call the left group action on X, a left G-action on X, for short.

Definition 2.3.2. (Orbits Induced by Group Actions) A G-action on X induces an equivalence relation on X given by

$$x \sim y \iff \exists\ g \in G \text{ such that } g \cdot x = y$$

These equivalence classes are called orbits of X and are denoted

$$O_x = \{y \in X | x \sim y\} = \{g \cdot x | g \in G\}$$

We now introduce transitive group actions and what it means for a set X to be a homogeneous G-space.

Definition 2.3.3. (Transitive Group Actions) Let G be a left group action on X. Then, G is said to be a transitive group action on X if for all $x, y \in X$, there exists $g \in G$ such that $y = g \cdot x$.

Definition 2.3.4. (Homogeneous Spaces) Let X be a non-empty set and G a group. Then, equipped with the action of G on X, we call X a G-space. A homogeneous space is a G-space, X, on which G acts transitively.

Remark. Note that G acts bijectively on the set X. If X belongs to some category, then G acts on X by automorphisms in the same category.

Example 2.3.5. We have the following examples regarding the remark.

(i) If X is a topological space, then the elements of the group G act as homeomorphisms on X. The structure of a G-space is a group homomorphism $\varphi : G \to \mathrm{Homeo}(X)$ into the homeomorphism group of X.

(ii) If X is a vector space over a field, then G acts by linear automorphisms and we obtain a representation of G (or a G-representation).

For an example of a transitive group action, consider the following.

Example 2.3.6. The general linear group $GL(2, \mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$. To see why this is the case, take any nonzero $v = (a, b) \in \mathbb{R}^2$. We show that we can find a matrix $A \in GL(2, \mathbb{R})$ such that $Ae_1 = v$, where $e_1 = (1, 0)$ is the standard basis vector in $\mathbb{R}^2$. That is, if $a \neq 0$, then set

$$A = \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix}$$

If, on the other hand, $a = 0$, then $b \neq 0$, so set

$$A = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$$

It is easy to verify that both of these matrices are invertible and each map $e_1 \mapsto v$.

As the next step we recall the definition, followed by some basic examples of flag manifolds.

Definition 2.3.7. (Complete and Partial Flags) Let V be a finite dimensional vector space over a field k. A flag is a proper increasing sequence of non-zero subspaces of V. Thus, we have:

$$0 \neq V_1 \subset V_2 \subset ... \subset V_n = V$$

Denoting $\dim(V_i) = d_i$, naturally we obtain:

$$0 < d_1 < d_2 < ... < d_n = \dim(V)$$

where because each subspace is proper, we have $n \leq \dim(V)$.

A flag is called a complete flag or full flag if $d_i = i$ for all $1 \leq i \leq n$, otherwise it is called a partial flag. The signature of a flag is the sequence $(d_1, d_2, ..., d_n)$.

Definition 2.3.8. (Flag Varieties) A flag variety is a homogeneous space whose points are flags of a finite-dimensional vector space V over a field k.

Remark. When $k = \mathbb{R}$ (or $\mathbb{C}$), a flag variety is a smooth real (or complex) manifold, and is called a real (or complex) flag manifold.

Example 2.3.9. We have the following examples.

(i) The complex full flag manifold, F_n, is the space consisting of all complete flags

$$V_1 \subset V_2 \subset ... \subset V_n = \mathbb{C}^n,$$

where V_j is a complex linear subspace of $\mathbb{C}^n$ and $\dim(V_j) = j$ for all $j = 1, ..., n$.

(ii) Fix $1 \leq d \leq n$. The Grassmannian, $Gr(d, n)$, is the manifold consisting of all complex linear subspaces $V \subset \mathbb{C}^n$ with $\dim(V) = d$.

(iii) Fix $1 \leq k_1 < k_2 < ... < k_p \leq n$. The space $F_{k_1,k_2,...,k_p}$ consisting of all sequences $V_1 \subset V_2 \subset ... \subset V_p \subseteq \mathbb{C}^n$, where V_j is a complex linear subspace of $\mathbb{C}^n$ and $\dim(V_j) = k_j$ for all $j = 1, ..., p$ is called a complex flag manifold of which examples (i) and (ii) are special cases.

We are now ready to discuss transitive group actions on flag manifolds.

Definition 2.3.10. (Standard Full Flags) The standard full flag of $\mathbb{C}^n$, which we will denote by $E_\bullet$, is the sequence

$$\text{span}\{e_1\} \subset \text{span}\{e_1, e_2\} \subset ... \subset \text{span}\{e_1, e_2, ..., e_n\} = \mathbb{C}^n$$

where $\{e_1, e_2, ..., e_n\}$ is the standard orthonormal unit vector basis of $\mathbb{C}^n$. Notice, $E_\bullet \in F_n$.

Remark. The general linear group, $GL(n, \mathbb{C})$ acts transitively on F_n. That is, for any sequence

$$V_\bullet := V_1 \subset V_2 \subset ... \subset V_n = \mathbb{C}^n$$

there exists an element $g \in GL(n, \mathbb{C})$ such that $gE_\bullet = V_\bullet$. Formally, we can associate to g its matrix representation, and $g(\text{span}\{e_1, e_2, ..., e_j\}) = V_j$ for all $j = 1, ..., n$.

If we further restrict the choice of the matrix g so that $\{ge_1, ge_2, ..., ge_n\}$ forms an orthonormal basis of $\mathbb{C}^n$ with respect to the standard Hermitian product $\langle\ ,\ \rangle$ on $\mathbb{C}^n$, then $g \in U(n)$ since $\langle g\zeta, g\xi\rangle = \langle \zeta, \xi\rangle$ for all $\zeta, \xi \in \mathbb{C}^n$.

Example 2.3.11. Flags of F_n can be identified with elements of the group $GL(n, \mathbb{C})$ (respectively, $U(n)$), modulo their subgroups which leave the standard flag fixed. Indeed,

(i) for $GL(n, \mathbb{C})$, the subgroup B_n, which consists of all invertible upper triangular matrices, leaves $E_\bullet$ fixed.

(ii) for $U(n)$, the subgroup T^n, which consists of all diagonal matrices in $U(n)$, leaves $E_\bullet$ fixed.

Elements of T^n are of the form $\text{Diag}(z_1, z_2, ..., z_n)$, where $z_j \in \mathbb{C}$ and $|z_j| = 1$ for all $j = 1, ..., n$. That is, $T^n \simeq \prod_{i=1}^{n} S^1$ as groups, and so, T^n is appropriately called the n-torus. As such, we obtain:

$$F_n \simeq GL_n(\mathbb{C})/B_n \simeq U(n)/T^n.$$

Chapter 3

Severi-Brauer Varieties

In 1907, J. H. Maclagan Wedderburn would submit his doctoral thesis, titled "On Hypercomplex Numbers," in which he classified all finite-dimensional simple and semisimple algebras. His thesis, [Wed], appearing in the Proceedings of the London Mathematical Society the following year, elegantly characterized every finite-dimensional simple algebra as a matrix algebra over some division ring. Richard Brauer, one of the key mathematicians picking up on this work, showed in 1932, [Bra1], that the isomorphism classes of these algebras can be used to form an abelian group, which was later termed the Brauer group. The properties of the Brauer group, in turn, gave back tremendous insight into the structure of simple algebras and was used, in particular, to prove the longstanding conjecture that every rational division algebra is cyclic over its centre. The Wedderburn theorem was later generalized to classify all semisimple Artinian rings in the Artin-Wedderburn theorem.

In the following sections we classify central simple algebras and define the Brauer group, including its modern mathematical formalism in terms of category theory. Along the way, we introduce an assortment of concepts, keeping in mind the goal of this chapter, which is to define Severi-Brauer varieties and their close relationship to Grassmannians, as these varieties are the objects of focus in Chapter 4.

3.1 Preliminaries on Algebras

In this section we introduce some basic preliminary notions for the study of algebras. We begin by providing two different definitions for what an algebra is, along with examples according to each definition. We then show that these definitions coincide under certain criteria. Finally, we introduce the notion of an opposite algebra, a natural object that will play a key role later in this chapter.

We begin with the notion of an algebra as a module with bilinear product, followed by some examples.

Definition 3.1.1. (Algebras) Let R be a commutative ring and A an R-module. Define an additional binary operation $* : A \times A \to A$ by $(x, y) \mapsto x * y$, often referred to as *multiplication* in A. Then, A is an algebra over R (or simply an R-algebra) if the multiplication operation is an R-bilinear product on A. In other words, that is if $\forall x, y, z \in A$ and $\forall r \in R$:

$$(x + y) * z = (x * z) + (y * z)$$
$$x * (y + z) = (x * y) + (x * z)$$
$$(r \cdot x) * y = r \cdot (x * y) = x * (r \cdot y)$$

To be unambiguous, we often denote an R-algebra by the pair $(A, *)$.

Remark. Although it is not required, when the operation $*$ is also associative, A is endowed with the structure of a ring.

Remark. When the commutative ring R is replaced with a field k, one obtains the familiar notion of a k-algebra as a vector space with an additional bilinear product.

Example 3.1.2. $\mathbb{R}^3$ with the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is an $\mathbb{R}$-algebra which is neither associative, nor commutative. To see why it is not associative, take any non-zero $v_1, v_2 \in \mathbb{R}^3$ such that $v_1 \perp v_2$. Then, $(v_1 \times v_1) \times v_2 = 0 \times v_2 = 0$ but $v_1 \times v_2$ is non-zero and perpendicular to v_1, so $v_1 \times (v_1 \times v_2) \neq 0$.

Example 3.1.3. Let V be a non-trivial vector space over a field k and denote by $\text{End}(V)$, the set of all linear transformations of V to itself. Then, $\text{End}(V)$ with multiplication defined by $\circ : \text{End}(V) \times \text{End}(V) \to \text{End}(V)$ sending $(T_1, T_2) \mapsto T_1 \circ T_2$ (the usual composition of functions) is an associative, unital k-algebra.

The following examples are of particular interest since they allow us to construct new algebras from existing ones.

Example 3.1.4. Let $(A, *_A)$ and $(B, *_B)$ be R-algebras. Then, $A \oplus B$ and $A \otimes_R B$ with respective multiplicative products defined by $(a + b) *_{A \oplus B} (a' + b') = a *_A a' + b *_B b'$ and $(a \otimes b) *_{A \otimes_R B} (a' \otimes b') = a *_A a' \otimes b *_B b'$ (extended to non-simple tensors by distributivity of $*_{A \otimes_R B}$ over addition), are also R-algebras.

We now define the nucleus, the commutant, and the center of an algebra.

Definition 3.1.5. (The Nucleus of an Algebra) Let A be an R-algebra. Define the associator on A to be the R-trilinear map $[\cdot, \cdot, \cdot] : A \times A \times A \to A$ sending $(x, y, z) \mapsto [x, y, z] = (x * y) * z - x * (y * z)$. The set of all associative elements of A

$$N(A) = \{x \in A \mid [x, y, z] = [y, x, z] = [y, z, x] = 0 \text{ for all } y, z \in A\}$$

is called the associative center or nucleus of A. Note that A is an associative algebra whenever $N(A) = A$.

Definition 3.1.6. (The Commutant of an Algebra) Let A be an R-algebra. Define the commutator on A to be the R-bilinear map $[\cdot,\cdot] : A \times A \to A$ sending $(x, y) \mapsto [x, y] = x * y - y * x$. The set of all commutative elements of A denoted

$$C(A) = \{x \in A \mid x * y = y * x \text{ for all } y \in A\}$$

is called the commutant or centralizer of A and A is commutative whenever $C(A) = A$.

Definition 3.1.7. (The Center of an Algebra) Let A be an R-algebra. Define the centre of A as the intersection

$$Z(A) = N(A) \cap C(A) = \{x \in N(A) \mid x * y = y * x \text{ for all } y \in A\}$$

The centre is precisely the subset of all the associative, commutative elements of A, and is an associative, commutative sub-algebra of A.

Next, we proceed by providing an alternate definition to the notion of an algebra in the sense of a homomorphism of rings. We then give some examples according to this definition.

Definition 3.1.8. (Alternative Definition of an Algebra) Let R be a commutative ring with unity. An R-algebra is a ring A (with unity), together with a ring homomorphism $f : R \to A$ such that:

$$(1)\ f(1_R) = 1_A$$
$$(2)\ f(R) \subseteq Z(A)$$

The pair (A, f) will also be called an R-algebra.

Remark. All algebras satisfying this definition are associative unital algebras.

Example 3.1.9. Let A be any ring and $R \subseteq Z(A)$ a subring of the centre of A. Then, (A, ι) is an R-algebra, where $\iota : R \hookrightarrow A$ is the inclusion of R into A. In particular, this says that every ring is an associative algebra over its centre.

Example 3.1.10. Any ring A with unity is a $\mathbb{Z}$-algebra by the following construction. Define $f : \mathbb{Z} \to A$ by setting $f(n) = n \cdot 1_A = \underbrace{1_A + ... + 1_A}_{\text{n times}}$ for all $n \in \mathbb{Z}$.

Proposition 3.1.11. *The definitions of an algebra in 3.1.1 and 3.1.8 are equivalent in the case of associative unital algebras, that is, when the binary operation in 3.1.1 is also associative and unital.*

Proof: $(\implies)$: Suppose that $(A, *)$ is an R-algebra as in the first definition and that the binary operation is associative and unital. For any $a \in A$ and $r \in R$, denote the R-action on A by $(r, a) \mapsto r \cdot a$. Now define $f : R \to A$ by $f(r) = r \cdot 1_A$.

(i) f is a ring homomorphism as both:

$$\begin{aligned} f(rs) &= 1_A * f(rs) \\ &= 1_A * ((rs) \cdot 1_A) \\ &= 1_A * (r \cdot (s \cdot 1_A)) \\ &= r \cdot 1_A * s \cdot 1_A \\ &= f(r) * f(s) \end{aligned}$$

and

$$\begin{aligned} f(r+s) &= (r+s) \cdot 1_A \\ &= (r \cdot 1_A) + (s \cdot 1_A) \\ &= f(r) + f(s) \end{aligned}$$

(ii) $f(1_R) = 1_R \cdot 1_A = 1_A$ by the properties of the R-action on A
(iii) finally, $f(R) \subseteq Z(A)$ since for any $a \in A$ and any $r \in R$ we have:

$$a * f(r) = a * (r \cdot 1_A) = r \cdot (a * 1_A) = r \cdot (1_A * a) = (r \cdot 1_A) * a = f(r) * a$$

where (i) and (iii) use the fact that $* : A \times A \to A$ is a bilinear mapping to move the scalars around as needed.

$(\impliedby)$: Conversely, let $f : R \to A$ be a ring homomorphism such that $f(1_R) = 1_A$ and $f(R) \subseteq Z(A)$ as in the second definition. Define the R-action on A by sending $(r, a) \mapsto r \cdot a = f(r) * a$, where $*$ is the multiplicative operation in the ring A. We verify that this operation makes A an R-module. That is, for all $a, b \in A$, for all $r, s \in R$, we have:

(i)

$$\begin{aligned} r \cdot (a+b) &= f(r) * (a+b) \\ &= (f(r) * a) + (f(r) * b) \\ &= (r \cdot a) + (r \cdot b) \end{aligned}$$

(ii)

$$\begin{aligned} (r+s) \cdot a &= f(r+s) * a \\ &= (f(r) + f(s)) * a \\ &= (f(r) * a) + (f(s) * a) \\ &= (r \cdot a) + (s \cdot a) \end{aligned}$$

(iii)

$$\begin{aligned}(rs)\cdot a &= f(rs)*a\\ &= (f(r)*f(s))*a\\ &= f(r)*(f(s)*a)\\ &= f(r)*(s\cdot a)\\ &= r\cdot(s\cdot a)\end{aligned}$$

(iv)

$$\begin{aligned}1_R\cdot a &= f(1_R)*a\\ &= 1_A*a\\ &= a\end{aligned}$$

which verifies $(A,\cdot)$ is a left R-module. However, since $f(r)\in Z(A)$, then for all $r\in R$ and all $a\in A$, we have $r\cdot a = f(r)*a = a*f(r) = a\cdot r$, so $(A,\cdot)$ is a two-sided R-module.

Finally, multiplication in A is R-bilinear since $\forall a,b,c\in A$ and $\forall r\in R$, we have both

(i)
$$(a+b)*c = a*c+b*c$$

(ii)
$$a*(b+c) = a*b+a*c$$

by distributivity of multiplication over addition in the ring structure of A, and

(iii)

$$\begin{aligned}(r\cdot a)*b &= (f(r)*a)*b\\ &= f(r)*(a*b)\\ &= r\cdot(a*b)\\ &= f(r)*a*b\\ &= a*f(r)*b\\ &= a*(f(r)*b)\\ &= a*(r\cdot b)\end{aligned}$$

using the fact that A is associative in its ring structure and $f(R)\subseteq Z(A)$. ∎

We conclude this section by defining opposite algebras, showing the opposite of an algebra A is isomorphic to the A-linear homomorphisms of A as k-algebras and then proving a key proposition on opposite matrix algebras that will be used later in the chapter in the proofs of some deeper results.

Definition 3.1.12. (Opposite Algebras) Let k be a field and A an algebra over k. Define A^{op} to have the same vector space structure as A but with multiplication $*^{\text{op}} : A \times A \to A$ defined by $a *^{\text{op}} b = b * a$, where $*$ is the multiplication in A.

Proposition 3.1.13. *If A is associative, unital or commutative, then so is A^{op}.*

Proof: For all $a, b, c \in A$:

(i) if A is associative, then A^{op} is also associative since

$$a *^{\text{op}} (b *^{\text{op}} c) = a *^{\text{op}} (c * b) = (c * b) * a = c * (b * a) = (b * a) *^{\text{op}} c = (a *^{\text{op}} b) *^{\text{op}} c$$

(ii) if A is unital with unit element 1_A, then A^{op} is also unital with unit element 1_A since

$$a *^{\text{op}} 1_A = 1_A * a = a = a * 1_A = 1_A *^{\text{op}} a$$

(iii) if A is commutative, then A^{op} is also commutative since

$$a *^{\text{op}} b = b * a = a * b = b *^{\text{op}} a$$

∎

Proposition 3.1.14. *Let A be a unital algebra over k. Then, $A^{op} \simeq Hom_A(A, A)$ as k-algebras, where $Hom_A(A, A)$ is the set of A-linear k-algebra homomorphisms from A to itself.*

Proof: Let $\phi \in \text{Hom}_A(A, A)$. Notice that ϕ is completely determined by $\phi(1_A)$ since for any $a \in A$, we have $\phi(a) = \phi(a * 1_A) = a * \phi(1_A)$. Now define a map $\tau : A^{\text{op}} \to \text{Hom}_A(A, A)$ by sending $a \mapsto R_a$, where $R_a(1_A) = a$ and in general, R_a is right multiplication by a.

We first show that the map τ is a homomorphism of vector spaces. That is, for all $a, b \in A^{\text{op}}$ and all $\lambda \in k$, we have:

$$(i)\ \tau(a + b) = R_{a+b} = R_a + R_b = \tau(a) + \tau(b)$$

since $R_{a+b}(1_A) = a + b = R_a(1_A) + R_b(1_A) = (R_a + R_b)(1_A)$, and

$$(ii)\ \tau(\lambda \cdot a) = R_{\lambda \cdot a} = \lambda \cdot R_a = \lambda \cdot \tau(a)$$

since $R_{\lambda \cdot a}(1_A) = \lambda \cdot a = \lambda \cdot R_a(1_A) = (\lambda \cdot R_a)(1_A)$. Thus, τ is a homomorphism of vector spaces.

We now show τ is an isomorphism. Suppose that for some $a, b \in A^{\mathrm{op}}$ we have $\tau(a) = \tau(b)$. That is, $R_a = R_b$. But, from rearranging, we obtain

$$0 = \tau(a) - \tau(b) = \tau(a - b) = R_{a-b}$$

where 0 is the homomorphism R_0, since R_0 is the map sending $1_A \mapsto 0$, i.e. for any $a \in A$, $R_0(a) = a * R_0(1_A) = a * 0 = 0$. However, $R_{a-b} = R_0$ as A-linear homomorphisms of A if and only if

$$R_{a-b}(1_A) = a - b = 0 = R_0(1_A)$$

That is, if and only if $a = b$. So τ is one to one.

Now suppose $\phi \in \mathrm{Hom}_A(A, A)$, and say, $\phi(1_A) = a$, for some $a \in A$, which completely determines ϕ. Notice that $\phi(b) = \phi(b * 1_A) = b * \phi(1_A) = b * a$ for any $b \in A$, and therefore, $\phi = R_a = \tau(a)$. Hence, τ is also onto which makes τ is an isomorphism of vector spaces over k.

Finally, for all $a, b \in A^{\mathrm{op}}$, we have

$$\tau(a *^{\mathrm{op}} b) = \tau(b * a) = R_{b*a} = R_a \circ R_b = \tau(a) \circ \tau(b)$$

since $R_a \circ R_b(1_A) = R_a(b) = b * a = R_{b*a}(1_A)$. ∎

Corollary 3.1.15. *In particular, if A is also associative, then A is an A-module and $A^{op} \simeq End_A(A)$ as k-algebras.*

Proof: Using the premises and notation from the previous proposition, we need only show that composition of k-algebra homomorphisms in $\mathrm{Hom}_A(A, A)$ is associative. That is, for any $a, b, c \in A$, we have

$$\begin{aligned}
R_a \circ (R_b \circ R_c) &= \tau(a) \circ (\tau(b) \circ \tau(c)) \\
&= \tau(a) \circ \tau(b *^{\mathrm{op}} c) \\
&= \tau(a *^{\mathrm{op}} (b *^{\mathrm{op}} c)) \\
&= \tau((a *^{\mathrm{op}} b) *^{\mathrm{op}} c) \\
&= \tau(a *^{\mathrm{op}} b) \circ \tau(c) \\
&= (\tau(a) \circ \tau(b)) \circ \tau(c) \\
&= (R_a \circ R_b) \circ R_c
\end{aligned}$$

∎

Proposition 3.1.16. *For any ring R, $M_n(R^{op}) = M_n(R)^{op}$.*

Proof: Let $*^{\mathrm{op}}$ be the multiplication in R^{op} and $*_M^{\mathrm{op}}$ the multiplication in $M_n(R)^{\mathrm{op}}$. Consider the transpose map $t : M_n(R)^{\mathrm{op}} \to M_n(R^{\mathrm{op}})$ sending $A \mapsto A^t$. Clearly, $(\alpha A)^t = \alpha A^t$ and $(A+B)^t = A^t + B^t$. Finally, when we consider the matrices A^t and B^t as matrices over R^{op}, then we also have the following. Since

$$AB = \left(\sum_{l=1}^{n} a_{il} * b_{lj} \right)_{1 \leq i,j \leq n}$$

then $\sum_{l=1}^{n} a_{il} * b_{lj}$ being the $(i,j)^{th}$ entry in AB is the $(j,i)^{th}$ entry in $(AB)^t$. Moreover, the $(j,i)^{th}$ entry of $B^t A^t$ is $\sum_{l=1}^{n} b_{lj} *^{\mathrm{op}} a_{il} = \sum_{l=1}^{n} a_{il} * b_{lj}$. Thus, $(B *_M^{\mathrm{op}} A)^t = (AB)^t = B^t A^t$. Hence, t is an isomorphism of R-algebras. ∎

3.2 Associative Division Algebras

In this section we introduce the concept of an associative division algebra. As the name suggests, we consider algebras as those from the previous section which are associative and are also endowed with a division-like structure. We follow this by a series of propositions which establish the proximity of division algebras to familiar concepts from ring theory, i.e. we ask the question "how close are division algebras to such things like domains and division rings," establishing precisely what is required for these notions to coincide. Finally, we characterize associative division algebras over the real numbers and over fields which are algebraically closed.

We begin by defining the notion of an associative division algebra.

Definition 3.2.1. (Associative Division Algebras) Let k be a field and D a non-zero k-algebra. Then, we say D is an associative division algebra if D is associative and for any $a, b \in D$ with $b \neq 0$ we have that there exists unique elements $x, y \in D$ such that $a = b * x$ and $a = y * b$.

We proceed by relating associative division algebras to ring-based structures with several propositions in this regard.

Proposition 3.2.2. *Let k be a field and A an algebra over k. Then, A is a unital associative division algebra if and only if it is a division ring, that is, if for all non-zero $x \in A$, there exists $y \in A$ such that $xy = 1 = yx$.*

Proof: $(\Longleftarrow)$: Suppose A is a division ring. That is, A is unital, associative, and for all non-zero $x \in A$, there exists a unique $y \in A$ such that $xy = 1 = yx$. We want to show that A is an associative division algebra. So take any $a, b \in A$ with $b \neq 0$. Since $b \neq 0$, there exists a unique $c \in A$ such that $bc = 1$ and $cb = 1$. Thus, by multiplying the first equation by a from the right and the second equation by a from the left we obtain $bca = a$ and $acb = a$, respectively. Since A is associative, then $b(ca) = a$ and $(ac)b = a$ where the elements $x = ca$ and $y = ac$ are unique.

$(\Longrightarrow)$: Suppose A is a unital associative division algebra. That is, for any $a, b \in A$ with $b \neq 0$ there exist unique $x, y \in A$ such that $a = bx$ and $a = yb$. In particular, $1 \in A$ (as A is unital) and so, for all non-zero $b \in A$ there exist unique $x, y \in A$ such that $1 = bx$ and $1 = yb$. We show that $x = y$. That is,

$$x = 1x = (yb)x = y(bx) = y1 = y$$

using the fact that A is associative. Hence, $bx = 1 = xb$ and so A is a division ring. ∎

Corollary 3.2.3. *Wedderburn's little theorem states that every finite division ring is commutative, therefore is a finite field. Thus, any finite unital associative division algebra over k is a finite field.*

Proposition 3.2.4. *An associative division algebra A over k is a domain, that is, A has no zero divisors.*

Proof: Assume the contrary. That is, assume that for some non-zero $a, b \in A$ we have $ab = 0$. But if this is the case, then for $0, b \in A$ with $b \neq 0$, the element a satisfying $0 = ab$ is not unique. That is, we also have that $0 = 0b$. Since $a \neq 0$, this is a contradiction to the fact that A is an associative division algebra. ∎

Proposition 3.2.5. *Let A be a finite-dimensional unital k-algebra. Then, A is an associative division algebra if and only if it is a domain.*

Proof: $(\Longrightarrow)$: By Proposition 3.2.4, if A is an associative division algebra, then A is a domain. Moreover, since A is unital, then by Proposition 3.2.2, it is in fact more than a domain, it is a division ring.

$(\Longleftarrow)$: Now suppose A is a domain. Take any non-zero $x \in A$ and suppose x is not invertible. Since A is a finite-dimensional vector space over k, then for some $N \in \mathbb{N}$, the elements $1, x, x^2, ..., x^N$ are linearly dependent. In fact, there can be at most $\dim_k(A)$ linearly independent powers of x. Let n be the minimal integer for which the relation $x^n + a_1 x^{n-1} + ... + a_n = 0$ holds for some $a_i \in k$ not all zero. Notice that if $a_n = 0$, then $(x^{n-1} + a_1x^{n-2} + ... + a_{n-1})x = 0$. Since A is a domain, then either $x^{n-1} + a_1x^{n-2} + ... + a_{n-1} = 0$ or $x = 0$. But $x^{n-1} + a_1x^{n-2} + ... + a_{n-1} \neq 0$ by the minimality of n and $x \neq 0$ by assumption, so this would imply a contradiction to the fact that A is a domain. Hence, $a_n \neq 0$. As such, we can rewrite the equation as $-a_n = x^n + a_1x^{n-1} + ... + a_{n-1}x = (x^{n-1} + a_1x^{n-2} + ... + a_{n-1})x$. Since k is a field, $-a_n$ is invertible and we have $1 = -a_n^{-1}(x^{n-1} + a_1x^{n-2} + ... + a_{n-1})x$. Therefore, $x^{-1} = -a_n^{-1}(x^{n-1} + a_1x^{n-2} + ... + a_{n-1})$, a contradiction.

Thus, A is a division ring and by Proposition 3.2.2, it is a division algebra. ∎

Corollary 3.2.6. *Let A be a finite-dimensional unital k-algebra. Then, A is a domain if and only if it is a division ring.*

Example 3.2.7. The Frobenius Theorem characterizes all finite-dimensional associative division algebras over the real numbers up to isomorphism. They are:

- $\mathbb{R}$ - the real vector space as a dimension 1 space over itself
- $\mathbb{C}$ - the complex numbers as a dimension 2 real vector space
- $\mathbb{H}$ - the quaternions as a dimension 4 real vector space

Finally, we consider finite-dimensional associative division algebras over algebraically closed fields.

Proposition 3.2.8. *The only finite-dimensional associative division algebra over an algebraically closed field k is k itself.*

Proof: Let D be a finite-dimensional associative division algebra over k. As in the proof of Proposition 3.2.5 let n be the minimal integer for which the relation $\alpha^n + a_1\alpha^{n-1} + ... + a_n = 0$ holds for some $a_i \in k$ not all zero. Notice, since $a_i \in k$ for all $i = 1, ..., n$, then $f(x) = x^n + a_1x^{n-1} + ... + a_n \in k[x]$ is a polynomial of minimal degree for which α is a root. Moreover, as k is an algebraically closed field, then there exists $\lambda \in k$ such that $f(\lambda) = 0$ and thus, f factors uniquely as $f(x) = (x - \lambda)g(x)$ where the degree of g is $n-1$. Notice that since the degree of f is minimal, $g(\alpha) \neq 0$. But then, $0 = f(\alpha) = (\alpha - \lambda)g(\alpha)$, where because D is a division algebra, $(\alpha - \lambda)$ is the unique element of D such that $(\alpha - \lambda)g(\alpha) = 0$. Since it is also the case that $0g(\alpha) = 0$, then we must have $\alpha - \lambda = 0$. Thus, $\alpha = \lambda \in k$ for all $\alpha \in D$, so $D = k$. ∎

3.3 Simple Structures and Schur's Lemma

This section is devoted to the study of simple structures such as simple rings, simple modules and simple algebras. We define all three and prove some important facts related to these concepts. The major result presented in this section is Schur's Lemma which is a powerful result that characterizes the set of homomorphisms between any two simple modules over a ring.

We begin with the definitions of simple rings and simple algebras.

Definition 3.3.1. (Simple Rings) A simple ring is a non-zero ring that has no two-sided ideals besides the zero ideal and itself.

Definition 3.3.2. (Simple Algebras) An algebra A is simple if it contains no two-sided ideals besides the zero ideal and itself, and the multiplication operation is not zero, that is, $A^2 = \{a * b \mid a, b \in A\} \neq \{0\}$.

Next, we show some of the characteristics of simple rings through the following series of propositions.

Proposition 3.3.3. *Let R be a unital commutative ring. Then, R is simple if and only if R is a field.*

Proof: $(\Longleftarrow)$: Since R is commutative, all ideals are two-sided. Moreover, since R is a field, the only ideals of R are (0) and R itself. Thus, R is trivially simple.

$(\Longrightarrow)$: Conversely, let R be a simple unital commutative ring. By commutativity, every ideal of R is a two-sided ideal. Moreover, since R is simple, it has no non-trivial two-sided proper ideals. That is, its only two-sided ideals are (0) and R. Since R is unital, then for all non-zero $a \in R$, for the ideal generated by a, we have $(a) = (1)$. Therefore, there exists $b \in R$ such that $ab = 1 = ba$. Thus, R is a field. ∎

Proposition 3.3.4. *Any quotient of a ring by a maximal ideal is a simple ring.*

Proof: Suppose $\mathfrak{m}$ is a maximal ideal of a ring R and let $\pi : R \twoheadrightarrow R/\mathfrak{m}$ be the projection map. Let J be a non-zero ideal in $R/\mathfrak{m}$. Then, $\pi^{-1}(J)$ is an ideal of R strictly containing $\mathfrak{m}$. Since $\mathfrak{m}$ is maximal in R, then $\pi^{-1}(J) = R$. Hence, $J = R/\mathfrak{m}$. Thus, $R/\mathfrak{m}$ is simple. ∎

Proposition 3.3.5. *If R is a simple ring, then so is $M_n(R)$ for any $n \in \mathbb{N}$.*

Proof: We first show that every ideal of $M_n(R)$ is of the form $M_n(J)$ for some unique ideal $J \lhd R$. (Note that, here, ideal is used to mean two-sided ideal).

Uniqueness. Suppose $\mathcal{J}$ is an ideal of $M_n(R)$ and that J_1 and J_2 are two ideals of R for which both $\mathcal{J} = M_n(J_1)$ and $\mathcal{J} = M_n(J_2)$. By definition, $A \in M_n(J)$ if and only if $A_{i,j} \in J$ for all $1 \leq i, j \leq n$. Now, if $J_1 \neq J_2$, then there exists an $x \in R$ for which $x \in J_1 \setminus J_2$ or $x \in J_2 \setminus J_1$. But then, there exists a matrix A containing x as its $(i,j)^{th}$ entry (that is, $x = A_{i,j}$ and zeros elsewhere, for example) for which $A \in M_n(J_1)\backslash M_n(J_2)$ or $A \in M_n(J_2)\backslash M_n(J_1)$, a contradiction, since $M_n(J_1) = M_n(J_2)$ implies both of these sets are empty.

Existence. Suppose $\mathcal{J}$ is an ideal of $M_n(R)$. Let

$$J(i,j) = \{x \in R \mid x = A_{i,j} \text{ for some } A \in \mathcal{J}\}$$

where $1 \leq i, j \leq n$. We first show that $J(i,j)$ is an ideal of R. That is, for all $r \in R$ and $j_1, j_2 \in J(i,j)$, where $j_1 = A_{i,j}$ for some $A \in \mathcal{J}$ and $j_2 = B_{i,j}$ for some $B \in \mathcal{J}$, we have

$$\begin{aligned} (i)\ & j_1 + j_2 \in J(i,j) \text{ since } j_1 + j_2 = A_{i,j} + B_{i,j} \text{ for } A + B \in \mathcal{J} \\ (ii)\ & rj_1 \in J(i,j) \text{ since } rj_1 = rA_{i,j} \text{ for } (rI_n)A \in \mathcal{J} \\ (iii)\ & j_1 r \in J(i,j) \text{ since } j_1 r = A_{i,j} r \text{ for } A(rI_n) \in \mathcal{J} \end{aligned}$$

Thus, $J(i,j)$ is an ideal of R. Moreover, since for any matrix $A \in \mathcal{J}$ we can apply permutation matrices on both the left and right and still get an element of $\mathcal{J}$, then it follows that $J(i,j)$ is independent of the choices of i and j. Denote

$$J = J(1,1) = J(1,2) = ... = J(n,n)$$

We claim that $\mathcal{J} = M_n(J)$.

Denote by $E_{i,j}$ the matrix with 1 in entry (i,j) and zeros elsewhere. For all matrices $A \in M_n(R)$ and for all $1 \leq i, j, k, l \leq n$, we have $E_{i,j} A E_{k,l} = A_{j,k} E_{i,l}$. If moreover $A \in \mathcal{J}$, then $A_{j,k} E_{i,l} \in \mathcal{J}$ and so $A_{j,k} \in J(i,l) = J$. But this is for all $1 \leq j, k \leq n$, so $A \in M_n(J)$. Thus, $\mathcal{J} \subseteq M_n(J)$.

Conversely, let $A \in M_n(J)$. By definition of $M_n(J)$, for any (i,l), $A_{i,l} \in J$. Moreover, by definition of J, there exists a matrix $M \in \mathcal{J}$ such that $M_{1,1} = A_{i,l} \in J = J(1,1)$. Now, $A_{i,l} E_{i,l} = M_{1,1} E_{i,l} = E_{i,1} M E_{1,l} \in \mathcal{J}$. Thus, $A = \sum_{i=1}^{n} \sum_{l=1}^{n} A_{i,l} E_{i,l} \in \mathcal{J}$, and so $M_n(J) \subseteq \mathcal{J}$.

Now if R is a simple ring, then the only two-sided ideals of R are (0) and itself. Thus, the only two-sided ideals of $M_n(R)$ are $M_n((0)) = (0)$ and $M_n(R)$ itself. Hence, $M_n(R)$ is simple. ∎

Proposition 3.3.6. *Any division ring is a simple ring, but not vice versa.*

Proof: Let R be a division ring. However, assume on the contrary that R is not simple. That is, there exists a non-trivial two-sided proper ideal $I \triangleleft R$. Since I is non-trivial, then $\exists a \in I$ such that $a \neq 0$. Moreover, since R is a division ring, then $\exists b \in R$ such that $ab = 1 = ba$. But $a \in I$ implies both $ab \in I$ and $ba \in I$, and thus that $1 \in I$, a contradiction to the fact that $I \neq R$.

For an example of a simple ring which is not a division ring take $M_n(R)$ where R is simple and $n \geq 2$. ∎

We now examine some of the characteristics of simple algebras.

Theorem 3.3.7. *Every unital associative k-algebra can be embedded into a simple algebra with the same unit element.*

For a proof of this theorem see [Mik], Chapter 4, in the section titled 'Embeddings into simple associative algebras' on page 108 and [Bok], Chapter 1, pages 13 and 14 leading up to Theorem 1.3.9.

Proposition 3.3.8. *Every unital associative division algebra A over k is simple.*

Proof: We know that every unital associative division algebra A over k is a division ring. Now apply Proposition 3.3.6. ∎

We are now ready to define simple modules. We show some equivalences to the property of being simple before stating and proving Schur's Lemma. We then conclude the section with an important proposition regarding left $M_n(D)$-modules, where D is a division ring, which will be used in the proofs of theorems presented later in the chapter.

Definition 3.3.9. (Simple Modules) Let R be a ring with unity. A left R-module M is simple if it is non-zero and does not admit a proper non-zero R-submodule.

Theorem 3.3.10. *Let R be a ring with unity and M a left R-module. Then, the following are equivalent:*

(i) M *is simple*

(ii) $Rm = M$ *for every non-zero* $m \in M$

(iii) $M \simeq R/\mathfrak{m}$ *for some maximal left ideal* $\mathfrak{m} \triangleleft R$.

Proof: $(i) \implies (ii)$: By contrapositive. If for some non-zero $m \in M$ we have $Rm \neq M$, (i.e. $Rm \subset M$), then Rm is a proper non-zero R-submodule of M since for all $r \in R$ and all $m_1, m_2 \in Rm$ where $m_1 = r_1 m$ and $m_2 = r_2 m$ for some $r_1, r_2 \in R$ we have

$$(i)\ m_1 + m_2 = r_1 m + r_2 m = (r_1 + r_2)m \in Rm$$
$$(ii)\ rm_1 = r(r_1 m) = (rr_1)m \in Rm$$

However, this contradicts the fact that M is simple.

$(ii) \implies (iii)$: Consider R as a left module over itself and define the left R-module homomorphism $\phi_m : R \twoheadrightarrow M$ by setting $r \mapsto rm$ for some fixed non-zero $m \in M$. By the first isomorphism theorem for modules, we have $R/\ker(\phi_m) \simeq M$. Now, if $\ker(\phi_m)$ is not a maximal left ideal, then $\ker(\phi_m) \subset \mathfrak{m}$ for some maximal left ideal $\mathfrak{m} \triangleleft R$ and in such a case, the quotient ring $\mathfrak{m}/\ker(\phi_m)$ is a left ideal of $R/\ker(\phi_m)$. That is, $\mathfrak{m}/\ker(\phi_m)$ is a left R-submodule of $R/\ker(\phi_m)$. Thus, $\mathfrak{m}/\ker(\phi_m)$ is isomorphic to its image, a left R-submodule of M which we denote by N. But then, for any non-zero $n \in N$, $Rn = M$, so either $N = M$ or $N = 0$. That is, either $\mathfrak{m} = R$ or $\mathfrak{m} = \ker(\phi_m)$, in either case a contradiction. Thus, $\ker(\phi_m)$ must be maximal, and we obtain the result.

$(iii) \implies (i)$: We know from Proposition 3.3.4 that $R/\mathfrak{m}$ is a simple ring because in particular, there are no non-zero proper left ideals in the quotient ring $R/\mathfrak{m}$. Moreover, left R-submodules of R coincide with left ideals of R since in both instances we have, by definition, $RI \subseteq I$ for a subgroup I of R. Thus, if we again consider R as a left module over itself, then $\mathfrak{m}$ is an R-submodule of R and therefore, $R/\mathfrak{m}$ is the quotient of the left R-module R by the left R-submodule $\mathfrak{m}$. This results in a quotient left R-module $R/\mathfrak{m}$ that contains no non-zero proper left R-submodules. That is, $M \simeq R/\mathfrak{m}$ is a simple module. ∎

Lemma 3.3.11. ***(Schur's Lemma)*** *Let M and N be simple R-modules. Then:*

(i) if $M \not\simeq N$, then $Hom_R(M, N) = 0$.

(ii) $End_R(M)$ is a division ring.

Proof: (i) By the contrapositive statement suppose that $\mathrm{Hom}_R(M, N) \neq 0$. That is, there exists a non-zero R-module homomorphism $f : M \to N$. Since $\ker(f)$ is an R-submodule of M, and the only R-submodules of M are 0 and itself (since M is simple), then $\ker(f) = 0$ as $\ker(f) = M$ would contradict the fact that f is a non-zero R-module homomorphism. Similarly, since $\mathrm{im}(f)$ is an R-submodule of N, and the only R-submodules of N are 0 and itself (since N is also simple), then

$\text{im}(f) = N$, as $\text{im}(f) = 0$ would also contradict the fact that f is a non-zero R-module homomorphism. But then, $\ker(f) = 0$ and $\text{im}(f) = N$ implies that f is an isomorphism between M and N.

(ii) Let $f \in \text{End}_R(M)$ such that $f \neq 0$. By (i) f is an isomorphism and thus, is invertible in $\text{End}_R(M)$. ∎

Proposition 3.3.12. *Let D be a division ring. Then, any simple left $M_n(D)$-module is isomorphic to D^n.*

Proof: First, pick $j \in \{1, ..., n\}$ and fix the column C_j in the ring of matrices $M_n(D)$. Identify each element $v \in D^n$ with the matrix having $C_j = v$ and zeros elsewhere. It is important to note that the reasoning in the proof is independent of the choice of j as all of the resulting identifications of D^n are isomorphic via a permutation of the columns. Identified in this way, D^n is a minimal left ideal of $M_n(D)$. Therefore, D^n is a simple left $M_n(D)$-module. Now let M be any simple left $M_n(D)$-module. By Theorem 3.3.10 part (ii), M is generated by every non-zero $m \in M$. That is, say, $M = M_n(D) \cdot m$ for some non-zero $m \in M$. Moreover, $D^n \simeq M_n(D) \cdot e_{1,1}$ where $e_{1,1}$ is the first standard basis element of $M_n(D)$. The $M_n(D)$-module homomorphism $\phi : M \to D^n$ sending $m \mapsto e_{1,1}$ is non-zero, so by Schur's Lemma, is an isomorphism. ∎

Remark. Just as all the simple left $M_n(D)$-modules are isomorphic, so too, are all the simple right $M_n(D)$-modules.

3.4 Central Simple Algebras

In this section, we introduce the notion of central simple algebras. We then state and prove a series of propositions which serve to further characterizes their properties and introduce a very important class of examples of such objects: the generalized quaternions. Central simple algebras and their classification will then be the focus of most of the remainder of the chapter.

We begin by defining central simple algebras and exploring some of their characteristics.

Definition 3.4.1. (Central Simple Algebras) A central simple algebra over a field k is a finite-dimensional simple unital associative k-algebra which has center k.

Remark. We often use the abbreviations C.S.A or CSA when referring to central simple algebras.

Proposition 3.4.2. *Any field is a central simple algebra over itself.*

Proof: Let k be any field. As a vector space over itself, the dimension of k is 1, thus finite-dimensional. Moreover, since k is a field, its multiplication is associative and unital, it is simple, and $Z(k) = k$. ∎

Proposition 3.4.3. *Any k-central finite-dimensional unital associative division algebra A over k is a CSA over k.*

Proof: Clearly, A is finite-dimensional and has centre k. Since A is also a unital associative division algebra over k, then A is a division ring and thus, simple. ∎

Remark. The requirement of k-central is not only sufficient, but necessary as $\mathbb{C}$ is a dimension 2 associative unital division algebra over $\mathbb{R}$, is simple, but has $Z(\mathbb{C}) = \mathbb{C}$.

Proposition 3.4.4. *The algebra of $n \times n$ matrices with entries in k, namely $M_n(k)$ is a CSA over k.*

Proof: Since k is simple, then so is $M_n(k)$. Moreover, $\dim_k(M_n(k)) = n^2$, thus finite-dimensional. The multiplication in $M_n(k)$ is associative and finally, the centre of $M_n(k)$ is $k * I_n$ which is isomorphic to k, thus $Z(M_n(k)) = k$. ∎

Proposition 3.4.5. *More generally, if A is a CSA over k, then so is $M_n(A)$.*

Proof: From Proposition 3.3.5, $M_n(A)$ is a simple ring. Moreover, it is finite-dimensional as $\dim_k(M_n(A)) = n^2 \cdot \dim_k(A)$. The multiplication in $M_n(A)$ is associative as the entries of a matrix which is the product of matrices in $M_n(A)$ are just sums of products of elements of A, which is associative. Thus, by regrouping the multiplication in each entry, we can regroup the multiplication of the matrices (relatively easy, but tedious exercise). Finally, $Z(M_n(A)) = \{a \cdot I_n \in M_n(A) \mid a \in Z(A)\}$ since if $a \notin Z(A)$, then there exists $b \notin Z(A)$ for which $ab \neq ba$. Thus, we would have

$$(a \cdot I_n) * (b \cdot I_n) = (ab) \cdot I_n \neq (ba) \cdot I_n = (b \cdot I_n) * (a \cdot I_n)$$

Therefore, $Z(M_n(A)) \simeq Z(A)$. Since A is k-central, then so is $M_n(A)$. Thus, $M_n(A)$ is a CSA over k. ∎

Proposition 3.4.6. *If A is a CSA over k, then so is A^{op}.*

Proof: By Proposition 3.1.13, A^{op} is associative and unital. Moreover, A^{op} is finite-dimensional over k since $\dim_k(A^{\text{op}}) = \dim_k(A)$. As any left (respectively, right) ideal of A is a right (respectively, left) ideal of A^{op}, then any two-sided ideal of A is a two-sided ideal of A^{op}. Therefore, as A is simple, then so, too, is A^{op}. Finally, we show that $Z(A) = Z(A^{\text{op}})$. That is, if $a \in Z(A)$, then for any $b \in A^{\text{op}}$, we have $a *^{\text{op}} b = b * a = a * b = b *^{\text{op}} a$. Hence, $a \in Z(A^{\text{op}})$ and $Z(A) \subseteq Z(A^{\text{op}})$. Conversely, if $a \in Z(A^{\text{op}})$, then for any $b \in A$, we have $a * b = b *^{\text{op}} a = a *^{\text{op}} b = b * a$. Hence, $a \in Z(A)$ and $Z(A^{\text{op}}) \subseteq Z(A)$. Therefore, $Z(A^{\text{op}}) = Z(A) = k$, so A^{op} is k-central. Thus, A^{op} is a CSA over k. ∎

We now establish some key facts regarding algebras over a field in order to show that the tensor product of CSA's is also a CSA.

Proposition 3.4.7. *Let A and B be two finite-dimensional unital k-algebras. Then:*

(i) $Z(A \otimes_k B) = Z(A) \otimes_k Z(B)$

(ii) If A and B are simple and $Z(A) = k$, then $A \otimes_k B$ is also simple with center $Z(B)$.

Proof: (i) Let $\alpha \in Z(A)$, $\beta \in Z(B)$ so that $\alpha \otimes \beta \in Z(A) \otimes_k Z(B)$. For any $a \otimes b \in A \otimes_k B$, we have

$$(\alpha \otimes \beta) *_\otimes (a \otimes b) = (\alpha *_A a) \otimes (\beta *_B b) = (a *_A \alpha) \otimes (b *_B \beta) = (a \otimes b) *_\otimes (\alpha \otimes \beta)$$

where $*_\otimes$ is the multiplication in $A \otimes_k B$ and where, by linearity of $*_\otimes$ over k, we can extend to non-simple tensors in $A \otimes_k B$. Hence, we have that $\alpha \otimes \beta \in Z(A \otimes_k B)$ and thus, $Z(A) \otimes_k Z(B) \subseteq Z(A \otimes_k B)$.

Conversely, as B is finite-dimensional over k, say of dimension n, let $\beta_1, ..., \beta_n$ be a k-basis for B. Then,

$$A \otimes_k B = A \otimes_k \left(\bigoplus_{i=1}^{n} k \cdot \beta_i \right) \simeq \bigoplus_{i=1}^{n} (A \otimes_k \beta_i)$$

as vector spaces over k. Thus, any element $\xi \in A \otimes_k B$ can be uniquely written as $\xi = a_1 \otimes \beta_1 + ... + a_n \otimes \beta_n$ where $a_i \in A$ for all $i = 1, ..., n$. In particular, if $\xi \in Z(A \otimes_k B)$, then for any $a \in A$, we have

$$(a \otimes 1_B) *_\otimes \xi = \xi *_\otimes (a \otimes 1_B)$$

That is;

$$\begin{aligned}0 &= (a \otimes 1_B) *_\otimes \xi - \xi *_\otimes (a \otimes 1_B)\\ &= (a \otimes 1_B) *_\otimes (a_1 \otimes \beta_1 + ... + a_n \otimes \beta_n) - (a_1 \otimes \beta_1 + ... + a_n \otimes \beta_n) *_\otimes (a \otimes 1_B)\\ &= \sum_{i=1}^{n} (a *_A a_i) \otimes (1_B *_B \beta_i) - \sum_{i=1}^{n} (a_i *_A a) \otimes (\beta_i *_B 1_B)\\ &= \sum_{i=1}^{n} (a *_A a_i) \otimes \beta_i - \sum_{i=1}^{n} (a_i *_A a) \otimes \beta_i\\ &= \sum_{i=1}^{n} (a *_A a_i - a_i *_A a) \otimes \beta_i\end{aligned}$$

and so, $a *_A a_i - a_i *_A a = 0$ for all $i = 1, ..., n$ by the uniqueness of the representation of the element 0. Thus, $a *_A a_i = a_i *_A a$ which implies that $a_i \in Z(A)$, for all $i = 1, ..., n$. Thus, $\xi \in Z(A) \otimes_k B$. Reproducing the proof, letting $\alpha_1, ..., \alpha_r$ be a k-basis of A, we obtain $\xi \in Z(A \otimes_k B) \implies \xi \in A \otimes_k Z(B)$. Combining these results, one obtains

$$\xi \in Z(A \otimes_k B) \implies \xi \in (Z(A) \otimes_k B) \cap (A \otimes_k Z(B)) = Z(A) \otimes_k Z(B)$$

Thus, $Z(A \otimes_k B) \subseteq Z(A) \otimes_k Z(B)$.

(ii) Let I be a non-zero two sided ideal of $A \otimes_k B$. Suppose first that there exists a non-zero simple tensor $a \otimes b \in I$. Since A is simple, then the ideal generated by a is equal to A, that is $(a) = (1_A)$. Thus, there exists $a_i, a_i' \in A$ such that $\sum_{i=1}^{n} a_i a a_i' = 1_A$. Thus,

$$1_A \otimes b = \sum_{i=1}^{n} (a_i \otimes 1_B) *_\otimes (a \otimes b) *_\otimes (a_i' \otimes 1_B) \in I$$

Reversing the roles of A and B, we conclude that $1_A \otimes 1_B \in I$. Thus, $I = A \otimes_k B$.

Now suppose there exists a non-zero, non-simple tensor $\xi = a_1 \otimes b_1 + ... + a_n \otimes b_n \in I$ for which $n \in \mathbb{N}$ is minimal. We may assume that the b_i are linearly independent over k. Otherwise, if say, $b_n = \lambda_1 b_1 + ... + \lambda_{n-1} b_{n-1}$ where $\lambda_i \in k$ for all $i = 1, ..., n-1$, then we could write $\xi = (a_1 + \lambda_1 a_n) \otimes b_1 + ... + (a_{n-1} + \lambda_{n-1} a_n) \otimes b_{n-1}$, contradicting the minimality of n. By the same reasoning, we may also assume that the a_i are also linearly independent over k. Moreover, applying the reasoning from the special case above, we can also assume that $a_1 = 1_A$.

Now suppose $n > 1$. We have $a_2 \notin k$, since otherwise a_1 and a_2 would be linearly dependent over k. Since $Z(A) = k$, there exists an $a \in A$ such that $a *_A a_2 \neq a_2 *_A a$. Consider

$$(a \otimes 1_B) *_\otimes \xi - \xi *_\otimes (a \otimes 1_B) = \sum_{i=2}^{n} (a *_A a_i - a_i *_A a) \otimes (1_B *_B b_i) \in I$$

Since the b_i are linearly independent over k and $a *_A a_2 - a_2 *_A a \neq 0$, then this element we've produced is a non-zero element of I, which contradicts the minimality of n. Therefore, $n = 1$ and by the special case, we are done.

Finally, to see why $A \otimes_k B$ has centre equal to $Z(B)$, just apply part (i). ∎

Corollary 3.4.8. *Let A and B be CSA's over k. Then, $A \otimes_k B$ is a CSA over k.*

Proposition 3.4.9. *Let A be a CSA over k. Then, $A \otimes_k A^{op} \simeq End_k(A)$.*

Proof: For each $a \in A$ define $L_a : A \to A$ as the map given by left multiplication by a, that is, for all $b \in A$, $L_a(b) = a*b$. Similarly, define $R_a : A \to A$ by $R_a(b) = b*a$ for all $b \in A$, right multiplication by a.

By this construction we have the injective algebra homomorphisms $\sigma : A \to \mathrm{End}_k(A)$ sending $a \mapsto L_a$ and $\tau : A^{\mathrm{op}} \to \mathrm{End}_k(A)$ sending $b \mapsto R_b$. Notice that for any $a, b \in A$, L_a and R_b commute, as for all $c \in A$ we have,

$$L_a \circ R_b(c) = L_a(c * b) = a * (c * b) = (a * c) * b = R_b(a * c) = R_b \circ L_a(c)$$

Thus, we get a k-algebra homomorphism

$$\phi : A \otimes_k A^{\mathrm{op}} \to \mathrm{End}_k(A)$$

where $a \otimes b \mapsto L_a \circ R_b = R_b \circ L_a$ and extending to elements of the form $\sum \lambda_i(a_i \otimes b_i)$, where $\lambda_i \in k$ for all i, by linearity. That is, $\sum \lambda_i(a_i \otimes b_i) \mapsto \sum \lambda_i(L_{a_i} \circ R_{b_i})$. This map is well-defined since $\phi(\lambda \cdot a \otimes b) = L_{\lambda \cdot a} \circ R_b = L_a \circ R_{\lambda \cdot b} = \phi(a \otimes \lambda \cdot b)$. That is, for all $c \in A$, we have

$$\begin{aligned} L_{\lambda\cdot a} \circ R_b(c) &= L_{\lambda\cdot a}(c * b) \\ &= (\lambda \cdot a) * (c * b) \\ &= a * (c * (\lambda \cdot b)) \\ &= L_a(c * (\lambda \cdot b)) \\ &= L_a \circ R_{\lambda\cdot b}(c) \end{aligned}$$

In fact, $\phi(\lambda \cdot a \otimes b) = \phi(\lambda \cdot (a \otimes b)) = \phi(a \otimes \lambda \cdot b)$ as

$$\begin{aligned} (\lambda \cdot (L_a \circ R_b))(c) &= \lambda \cdot (L_a \circ R_b)(c) \\ &= \lambda \cdot L_a(c * b) \\ &= \lambda \cdot (a * (c * b)) \\ &= (\lambda \cdot a) * (c * b) \\ &= a * (c * (\lambda \cdot b)) \end{aligned}$$

It preserves multiplication since

$$\begin{aligned}\phi((a\otimes b)*_{A\otimes_k A^{\mathrm{op}}}(a'\otimes b')) &= \phi(a*a'\otimes b*^{\mathrm{op}} b')\\ &= L_{a*a'}\circ R_{b*^{\mathrm{op}}b'}\\ &= L_{a*a'}\circ R_{b'*b}\\ &= (L_a\circ R_b)\circ(L_{a'}\circ R_{b'})\\ &= \phi(a\otimes b)\circ\phi(a'\otimes b')\end{aligned}$$

since for all $c\in A$, we have

$$\begin{aligned}L_{a*a'}\circ R_{b'*b}(c) &= L_{a*a'}(c*b'*b)\\ &= a*a'*c*b'*b\\ &= (L_a\circ R_b)(a'*c*b')\\ &= (L_a\circ R_b)\circ(L_{a'}\circ R_{b'})(c)\end{aligned}$$

Moreover, since $1_A\otimes 1_A\mapsto L_{1_A}\circ R_{1_A}=R_{1_A}\circ L_{1_A}=\mathrm{id}_A$, then $\ker(\phi)\neq A\otimes_k A^{\mathrm{op}}$. As $A\otimes A^{\mathrm{op}}$ is simple, and $\ker(\phi)$ is an ideal of $A\otimes_k A^{\mathrm{op}}$, then $\ker(\phi)=(0)$. So ϕ is one to one.

Finally, $\dim_k(A\otimes_k A^{\mathrm{op}})=(\dim_k(A))^2=\dim_k(\mathrm{End}_k(A))$, so ϕ is an isomorphism. ∎

Proposition 3.4.10. *Let A be a CSA over k. Then, for every field extension $k'\supseteq k$, $A\otimes_k k'$ is a CSA over k'.*

Proof: Let A be a CSA over k and $k'\supseteq k$ a field extension. We want to show that $A\otimes_k k'$ is a CSA over k'.

First, notice that $A\otimes_k k'$ is finite-dimensional over k' as

$$\dim_{k'}(A\otimes_k k')=\dim_k(A)$$

Moreover, $A\otimes_k k'$ is simple by Proposition 3.4.7 part (ii), noting that we can consider k' as a k-algebra and that the proof does not require B (k' in our case) to be finite-dimensional over k.

Now clearly,

$$Z(A)\otimes_k Z(k')=k\otimes_k k'=k'\subseteq Z(A\otimes_k k')$$

For the other inequality, suppose that

$$\xi=\sum_{i=1}^{n}a_i\otimes b_i\in Z(A\otimes_k k')$$

where we can always arrange the sum so that the b_i are part of a k-basis for k' and thus, that the elements $1_A \otimes b_i$ are part of a basis for $A \otimes_k k'$ viewed as a free A-module. For any $a \in A$, we have $(a \otimes 1_{k'}) *_\otimes \xi = \xi *_\otimes (a \otimes 1_{k'})$, and so

$$\begin{aligned}
0 &= (a \otimes 1_{k'}) *_\otimes \xi - \xi *_\otimes (a \otimes 1_{k'}) \\
&= (a \otimes 1_{k'}) *_\otimes \left(\sum_{i=1}^{n} a_i \otimes b_i \right) - \left(\sum_{i=1}^{n} a_i \otimes b_i \right) *_\otimes (a \otimes 1_{k'}) \\
&= \sum_{i=1}^{n} (a *_A a_i) \otimes (1_{k'} *_{k'} b_i) - \sum_{i=1}^{n} (a_i *_A a) \otimes (b_i *_{k'} 1_{k'}) \\
&= \sum_{i=1}^{n} (a *_A a_i) \otimes b_i - \sum_{i=1}^{n} (a_i *_A a) \otimes b_i \\
&= \sum_{i=1}^{n} (a *_A a_i - a_i *_A a) \otimes b_i
\end{aligned}$$

Hence, we have $a *_A a_i - a_i *_A a = 0$ for all $i = 1, ..., n$. Since this is for all $a \in A$, then $a_i \in Z(A) = k$ for all $i = 1, ..., n$. Moreover, as the tensor product is k-linear, then we can simplify ξ and write $\xi = 1_A \otimes b$ for some $b \in k'$. So $\xi \in 1_A \otimes_k k' \subseteq Z(A) \otimes_k Z(k')$. Thus, $Z(A \otimes_k k') = Z(A) \otimes_k Z(k') = k'$.

Therefore, $A \otimes_k k'$ is a CSA over k'. ▮

We conclude the section by introducing arguably the most famous example of a central simple algebra, namely Hamilton's quaternions, and generalizing this construct.

Proposition 3.4.11. *The Quaternions* ($\mathbb{H}$), *an* $\mathbb{R}$*-algebra of dimension 4, with basis* $\{1, i, j, ij\}$ *which satisfy* $i^2 = -1$, $j^2 = -1$ *and* $ij = -ji$, *is a CSA over* $\mathbb{R}$.

Proof: $\mathbb{H}$ is finite-dimensional as a vector space over $\mathbb{R}$ and its multiplication is associative. Moreover, $\mathbb{H}$ is a division ring, thus it is simple. Finally, $Z(\mathbb{H}) = \mathbb{R}$. ▮

Theorem 3.4.12. ***(Generalized Quaternions)*** *Let k be any field of characteristic not equal to 2. Take any non-zero $a, b \in k$ and let $(a, b)_k$ be the k-algebra with basis $\{1, i, j, ij\}$ such that $i^2 = a$, $j^2 = b$, and $ij = -ji$. Then, $(a, b)_k$ is a CSA over k.*

Notice, for the special case of quaternions, $\mathbb{H} = (-1, -1)_\mathbb{R}$.

For a proof of this theorem, see [Lam], section 3, Proposition 1.1.

3.5 Wedderburn's (Original) Theorem

In this section we state and prove the famous result from Wedderburn's 1907 thesis, [Wed], now known as Wedderburn's theorem.

We first require some preliminaries which includes defining Artinian rings and showing that every CSA is Artinian.

Definition 3.5.1. (The Descending Chain Condition for Posets) A partially ordered set P, is said to satisfy the descending chain condition if every strictly descending sequence of elements eventually terminates, i.e., there is no infinite descending chain. Equivalently, one can say every descending sequence, $a_1 \geq a_2 \geq a_3 \geq \ldots$ of elements of P, eventually stabilizes.

Definition 3.5.2. (Artinian Rings) An Artinian ring is a ring that satisfies the descending chain condition on ideals (ordered by inclusion).

Proposition 3.5.3. *Let A be an associative algebra over a field k. If $dim_k(A)$ is finite, then A is Artinian.*

Proof: First notice that because A is associative, it is a ring. Moreover, every left (and equivalently, right) ideal of A, by definition, is a k-subalgebra of A. Thus, any strictly descending chain of left (or right) ideals (ordered by inclusion) corresponds to a strictly descending chain of subalgebras. As A is finite-dimensional over k, then all such sequences must terminate. Thus, A is Artinian. ∎

Corollary 3.5.4. *Any CSA over k is Artinian.*

Proposition 3.5.5. *Let D be a division ring. Then, $M_n(D)$ is left-Artinian.*

Proof: Let L_i be the set of all $n \times n$ matrices in $M_n(D)$ whose entries are all 0 outside of the i^{th} column. Then, L_i is a minimal left ideal for each $i = 1, ..., n$. As $M_n(D) = \bigoplus_{i=1}^{n} L_i$, then $M_n(D)$ (as a left module over itself) has composition series

$$0 \subsetneq L_1 \subsetneq L_1 \oplus L_2 \subsetneq ... \subsetneq L_1 \oplus ... \oplus L_n = M_n(D)$$

unique up to permutation of indices. ∎

Remark. By Proposition 3.1.16, $M_n(D)$ is also right-Artinian.

We are now ready to state and prove Wedderburn's theorem.

Theorem 3.5.6. ***(Wedderburn's Theorem)*** *Let A be a CSA over a field k. Then, there exists a unique $n \in \mathbb{N}$ and a unital associative division algebra D over k, unique up to isomorphism, such that*

$$A \simeq M_n(D)$$

as k-algebras.

Note that D must be finite-dimensional over k and is k-central since

$$Z(D) \simeq Z(M_n(D)) \simeq Z(A) = k$$

In fact, by Proposition 3.2.2, D is a division ring. Moreover, D is uniquely determined, up to isomorphism, by setting $D = \text{End}_A(I)$ for any non-zero minimal left ideal I of A (which are all isomorphic to D^n by Proposition 3.3.12).

Proof: By Corollary 3.5.4, we know that A is Artinian and therefore, every strictly descending chain of ideals has a minimal element. Let I be any minimal non-zero left ideal of A and let $D = \text{End}_A(I)$. Since I is a minimal non-zero ideal of A, then I is simple as a left A-module. Thus, by applying Schur's Lemma, we have that $D = \text{End}_A(I)$ is a division ring.

We can also consider I as a vector space over D by defining the scalar multiplication to be $\phi \cdot x = \phi(x)$ for all $\phi \in D$ and $x \in I$. Since there is a natural embedding $D = \text{End}_A(I) \hookrightarrow \text{End}_k(I) \simeq M_{\dim_k(I)}(k)$, then

$$\dim_k(D) = \dim_k(\text{End}_A(I)) \leq \dim_k(\text{End}_k(I)) = \dim_k(M_{\dim_k(I)}(k)) = (\dim_k(I))^2 < \infty$$

As both $\dim_k(I)$ and $\dim_k(D)$ are finite, then $\dim_D(I)$ is also finite, say $\dim_D(I) = n$. Then, $I \simeq D^n$ (non-canonically since it depends on the choice of a basis) and so, $\text{End}_D(I) \simeq M_n(D)$.

Consider the map $\rho : A \to \text{End}_D(I)$ given by $a \mapsto L_a$ where $L_a : I \to I$ denotes left multiplication by a. The map L_a is indeed D-linear as

$$L_a(\phi \cdot x) = L_a(\phi(x)) = a * \phi(x) = \phi(a * x) = \phi \cdot (L_a(x))$$

for all $\phi \in D$ and for all $x \in I$.

We now show that ρ is a ring homomorphism. That is, for all $a, b \in A$ and all $\lambda \in k$:

$$\rho(a + b) = L_{a+b} = L_a + L_b = \rho(a) + \rho(b)$$

since for all $x \in I$, we have

$$\begin{aligned} L_{a+b}(x) &= (a + b) * x \\ &= a * x + b * x \\ &= L_a(x) + L_b(x) \\ &= (L_a + L_b)(x) \end{aligned}$$

Moreover,

$$\rho(a * b) = L_{a*b} = L_a \circ L_b = \rho(a) \circ \rho(b)$$

since for all $x \in I$, we have

$$\begin{aligned} L_{a*b}(x) &= (a * b) * x \\ &= a * (b * x) \\ &= a * L_b(x) \\ &= L_a(L_b(x)) \\ &= (L_a \circ L_b)(x) \end{aligned}$$

Finally,

$$\rho(\lambda \cdot a) = L_{\lambda \cdot a} = \lambda \cdot L_a = \lambda \cdot \rho(a)$$

since for all $x \in I$, we have

$$\begin{aligned} L_{\lambda \cdot a}(x) &= (\lambda \cdot a) * x \\ &= \lambda \cdot (a * x) \\ &= \lambda \cdot L_a(x) \\ &= (\lambda \cdot L_a)(x) \end{aligned}$$

Now, since $\ker(\rho)$ is an ideal of A, and A is simple, then $\ker(\rho) = \{0\}$. Otherwise, $\ker(\rho) = A$ would imply that the operation of left-multiplication in A is zero. That is, $L_a = 0$ as maps, for all $a \in A$, or in other words, $L_a(b) = a * b = 0$ for all $a, b \in A$, which would contradict the fact that A is a simple algebra. Hence, ρ is injective.

We are left to show that ρ is surjective. Since the unit element L_{1_A} of $\mathrm{End}_D(I)$ is in $\rho(A)$, it is enough to show that $\rho(A)$ is a left ideal of $\mathrm{End}_D(I)$.

We first show that $\rho(I)$ is a left ideal of $\mathrm{End}_D(I)$. Before showing this, notice that for any fixed $y \in I$, the map $R_y : I \to I$ sending $x \mapsto x * y$ is an A-linear endomorphism of I as for any $a \in A$, we have

$$R_y(a * x) = (a * x) * y = a * (x * y) = a * R_y(x)$$

Thus, $R_y \in \mathrm{End}_A(I)$. Moreover, since every element in $\rho(I)$ is of the form L_x, for some $x \in I$, then for any $L_x \in \rho(I)$, any $\phi \in \mathrm{End}_D(I)$, and any $y \in I$, we have

$$\phi \circ L_x(y) = \phi(x * y) = \phi(R_y(x)) = R_y(\phi(x)) = \phi(x) * y = L_{\phi(x)}(y)$$

since $R_y \in D$ and ϕ is D-linear. Thus, $\phi \circ L_x = L_{\phi(x)}$ as elements of $\mathrm{End}_D(I)$. Since $\phi(x) \in I$ for all $x \in I$, then $L_{\phi(x)} \in \rho(I)$. Hence, $\phi \circ L_x \in \rho(I)$ for all $\phi \in \mathrm{End}_D(I)$ and $L_x \in \rho(I)$. Therefore, $\rho(I)$ is a left ideal of $\mathrm{End}_D(I)$.

Now consider the two-sided ideal IA of A. Since A is simple, then $IA = A$. Thus, we have that $\rho(A) = \rho(IA) = \rho(I)\rho(A)$ is also a left ideal of $\mathrm{End}_D(I)$. Thus, ρ is surjective and therefore, $A \simeq \mathrm{End}_D(I) \simeq M_n(D)$.

Finally, by Proposition 3.3.12, all minimal left ideals I of A are isomorphic, and therefore, D is uniquely determined, up to isomorphism, by setting $D = \mathrm{End}_A(I)$. ∎

As an important consequence, we obtain the following.

Corollary 3.5.7. *If k is an algebraically closed field, then every central simple algebra A over k is isomorphic to $M_n(k)$.*

Proof: By Proposition 3.2.8, the only finite-dimensional associative division algebra over an algebraically closed field k is k itself. Thus, applying Wedderburn's theorem, we have that $D = k$ and so, $A \simeq M_n(k)$. ∎

Finally, we provide a result characterizing all CSA's over the real numbers.

Proposition 3.5.8. *The only CSA's over $\mathbb{R}$ are matrix rings over $\mathbb{R}$ or $\mathbb{H}$.*

Proof: Frobenius' theorem tells us that the only finite-dimensional associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. However, $\mathbb{C}$ is not $\mathbb{R}$-central as $Z(\mathbb{C}) = \mathbb{C}$. Combining this with Wedderburn's Theorem, we obtain the result. ∎

3.6 Splitting Fields for CSA's

In this section we define the notion of a splitting field for a central simple algebra. We follow this by some important propositions regarding splitting fields, in particular, showing that for any CSA a splitting field always exists. This then allows us to characterize the dimension of any CSA.

We begin by defining splitting fields for a CSA and then showing that for every CSA one always exists.

Definition 3.6.1. (Splitting fields for CSA's) Let A be a CSA over k. A splitting field for A is a field extension $k' \supseteq k$ such that $A \otimes_k k' \simeq M_n(k')$.

Proposition 3.6.2. *Splitting fields always exist. In particular, the algebraic closure $\overline{k}$ is always one.*

Proof: Let k be a field and $\overline{k}$ its algebraic closure. Since A is a CSA over k, then by Proposition 3.4.10, $A \otimes_k \overline{k}$ is a CSA over $\overline{k}$. By the Corollary to Wedderburn's theorem (Corollary 3.5.7), $A \otimes_k \overline{k} \simeq M_n(\overline{k})$. ∎

We now show that every field extension of a splitting field is also a splitting field.

Proposition 3.6.3. *Let A be a CSA over k. If an extension $k' \supseteq k$ splits A, then so does every field extension of k'.*

Proof: Suppose $k' \supseteq k$ splits A and let $l \supseteq k'$ be a field extension of k'. We have

$$A \otimes_k l \simeq (A \otimes_k k') \otimes_{k'} l \simeq M_n(k') \otimes_{k'} l \simeq M_n(k' \otimes_{k'} l) \simeq M_n(l)$$

∎

The next proposition shows that we can determine if a unital, associative k-algebra is a CSA over k, simply by knowing whether it splits over some field extension of k.

Proposition 3.6.4. *Let A be a unital associative k-algebra. If there exists an extension $k' \supseteq k$ splitting A, then A is a CSA over k.*

Proof: Suppose that $k' \supseteq k$ splits A. That is, $A \otimes_k k' \simeq M_n(k')$ for some $n \in \mathbb{N}$. If $I \triangleleft A$ is a two sided ideal of A, then $I \otimes_k k'$ is a two-sided ideal of $A \otimes_k k' \simeq M_n(k')$ of dimension equal to $\dim_k(I) \cdot \dim_k(k')$. In particular, I is proper if and only if $I \otimes_k k'$ is proper. Since $A \otimes_k k'$ is simple (as it is isomorphic to $M_n(k')$), then it follows that A has no proper ideals and is therefore also simple.

By Proposition 3.4.7 part (i), we know that $Z(A) \otimes_k Z(k') \subseteq Z(A \otimes_k k')$, (which applies here as this inequality does not require that the algebras be finite-dimensional, see its proof) where $Z(k') = k'$ and

$$Z(A \otimes_k k') \simeq Z(M_n(k')) = \{\alpha \cdot I_n \mid \alpha \in k'\} \simeq k' \simeq 1_A \otimes_k k'$$

Now clearly, $k \subseteq Z(A)$. If, however, there exists $a \in Z(A)$ such that $a \notin k$, then for any $\lambda \in k'$,

$$a \otimes \lambda \notin 1_A \otimes_k k' \simeq Z(A \otimes_k k'),$$

a contradiction. So $Z(A) = k$.

Finally, as $A \otimes_k k' \simeq M_n(k')$, then,

$$\dim_k(A) \cdot \dim_k(k') = \dim_k(A \otimes_k k') = \dim_k(M_n(k')) = n^2 \cdot \dim_k(k')$$

That is, $\dim_k(A) = n^2$. Thus, A is a CSA over k. ∎

We can now characterize the dimension of CSA's.

Proposition 3.6.5. *The dimension of a CSA over a field is always a square.*

Proof: Let k be a field, $\overline{k}$ its algebraic closure, and A a CSA over k. Then

$$\dim_k(A) = \dim_{\overline{k}}(A \otimes_k \overline{k})$$

By Wedderburn's theorem, the latter is isomorphic to $M_n(\overline{k})$, for some n, which has dimension n^2 over $\overline{k}$. ∎

Remark. If A is a CSA over k of dimension n^2, then we say the degree of A over k is equal to n. Moreover, we say that the Schur index of a CSA is the degree of the unique (up to isomorphism) associative division algebra D over k attributed to A by Wedderburn's theorem. Notationally, we write $\deg_k(A) = \sqrt{\dim_k(A)}$ and $\operatorname{ind}_k(A) = \deg_k(D) = \sqrt{\dim_k(D)}$.

We now relate the splitting field for a central simple algebra to the associative division algebra attributed to it by Wedderburn's theorem.

Proposition 3.6.6. *Let A be a CSA over k. Then, l is a splitting field for A if and only if l is a splitting field for D, where D is, up to isomorphism, the unique unital associative division algebra over k for which $A \simeq M_n(D)$.*

Proof: $(\Longleftarrow)$: Suppose l splits D. That is, $D \otimes_k l = M_m(l)$. Then,

$$A \otimes_k l \simeq M_n(D) \otimes_k l \simeq M_n(D \otimes_k l) \simeq M_n(M_m(l)) \simeq M_{mn}(l)$$

Therefore, l splits A.

$(\Longrightarrow)$: Suppose l splits A. That is, $A \otimes_k l = M_m(l)$. Then,

$$M_m(l) \simeq A \otimes_k l \simeq M_n(D) \otimes_k l \simeq M_n(D \otimes_k l)$$

By Proposition 3.4.10, $D \otimes_k l$ is a CSA over l. Applying Wedderburn's theorem, we have that $D \otimes_k l \simeq M_p(D')$ for some unique $p \in \mathbb{N}$ and unique, up to isomorphism, division algebra D'. But then,

$$M_n(D \otimes_k l) \simeq M_n(M_p(D')) \simeq M_{np}(D')$$

Now, by Proposition 3.4.5, $M_n(D \otimes_k l)$ is a CSA over l. Since Wedderburn's theorem says that the given natural number is unique and the division algebra is unique up to isomorphism, then $m = np$ and $D' \simeq l$. Therefore, $D \otimes_k l \simeq M_p(l)$, so l splits D. ∎

The next goal of this section is to present a theorem relating splitting fields of central simple algebras to separable field extensions. We first define separable polynomials and separable field extensions.

Definition 3.6.7. (Separable Polynomials) Let k be an arbitrary field. An irreducible polynomial $f(x) \in k[x]$ is separable over k if and only if it has distinct roots in any field extension of k, that is, if and only if it can be factored into a product of distinct linear factors over an algebraic closure of k.

Definition 3.6.8. (Separable Field Extensions) Let $l \supseteq k$ be an algebraic field extension. We say that such an extension is separable if for every $\alpha \in l$, the minimal polynomial of α over k, denoted $\text{minpoly}_k(\alpha)(x) \in k[x]$ is a separable polynomial.

We are now ready to present the theorem.

Theorem 3.6.9. *Let A be a CSA over k. Then, there exists a splitting field $k' \supseteq k$ for A which is separable over k.*

For a proof of this theorem, see [Gil], Proposition 2.2.5.

We conclude this section by defining subfields of algebras, including maximal subfields, and presenting a theorem which characterizes the degree of maximal subfields for unital associative division algebras.

Definition 3.6.10. (Subfields of an Algebra) Let A be a k-algebra. A subfield of A is a k-subalgebra which is also a field. A maximal subfield is a subfield F of A such that there does not exist any subfield L of A with $F \subsetneq L \subsetneq A$.

Theorem 3.6.11. *Let D be a unital associative division algebra over a field, k. Then, D is a division ring and thus, the centre of D is a field. In fact, $k \to Z(D)$, by the second definition of an algebra, or where for any $a \in k$ we have $a * 1_A \in Z(D)$, in the first algebra definition, so $Z(D)$ is a field extension of k.*

Now, denote the dimension of a maximal sub-field L of D over its centre by $[L : Z(D)]$. If the dimension of the algebra over its centre, $[D : Z(D)]$ is finite, then

$$[D : Z(D)] = [L : Z(D)]^2$$

Note: If $Z(D) = k$, then we say D is a k-central associative division algebra.

For a proof of this theorem, see [Pie], section 31.1, 'Maximal Subfields', Corollary b.

3.7 Semisimple Structures and the Wedderburn-Artin Theorem

The main purpose of this section is to generalize Wedderburn's theorem to semisimple rings and semisimple algebras. This is the so-called Wedderburn-Artin theorem. We define the notions of semisimple rings and algebras, along with semisimple modules and semiprime rings. We also state and prove Brauer's Lemma.

We begin by defining the Jacobson radical of a ring and semisimple algebras. We then give the definition of a semisimple module in the form of a series of equivalent characterizations.

Definition 3.7.1. (The Jacobson Radical of a Ring) Let R be a ring. The Jacobson Radical of R, denoted $\mathfrak{J}_l(R)$ (or $\mathfrak{J}_r(R)$ for right, as the notion is left-right symmetric) is the intersection of all the maximal left ideals of R. That is,

$$\mathfrak{J}_l(R) = \bigcap_{i \in \Omega_l} \mathfrak{m}_i$$

where $\mathfrak{m}_i$ is a maximal left ideal of R and the intersection is taken over the set of all maximal left ideals, Ω_l. We define $\mathfrak{J}_r(R)$ similarly, over the set of all maximal right ideals, Ω_r.

Remark. The Jacobson Radical is itself an ideal, and it consists precisely of those elements which annihilate any simple left R-module (or any simple right R-module).

Definition 3.7.2. (Semisimple Algebras) A semisimple algebra is an artinian associative algebra over a field k which has trivial Jacobson radical (only the zero element of the algebra is in the Jacobson radical).

Theorem 3.7.3. *A module is semisimple if it satisfies any of the following equivalent conditions:*

(i) it is a sum of simple submodules

(ii) it is a direct sum of simple submodules

(iii) every submodule has a complement

Proof: (ii) $\Longrightarrow$ (i): This is obvious.

(i) $\Longrightarrow$ (iii): Let M be a sum of simple submodules and let N be a submodule of M. By Zorn's lemma, we can choose a submodule P of M which is maximal with respect to the following two properties: it is the sum of simple modules and it intersects N trivially. If $N \oplus P \subsetneq M$, then there is a simple submodule S of M not entirely contained in $N+P$. In fact, by the simplicity of S, we have $S \cap (N+P) = 0$. As $N \subset N+P$, then $S \cap N = 0$. Hence, $N \cap (P+S) = 0$. Since $P \subsetneq P+S$ and $P+S$ is the sum of simple modules, then this contradicts the maximality of P. Thus, $N \oplus P = M$ and since $N \cap P = 0$, then $P = N^{\perp}$.

(iii) $\implies$ (ii): Suppose every submodule of M has a complement. Choose, by Zorn, a maximal collection $\mathcal{C}$ of simple submodules of M whose sum is their direct sum, and denote this sum by N. Suppose that $N \subsetneq M$. Take any non-zero $x \in N$ and consider the submodule generated by it, $\langle x \rangle$. Choose, by Zorn, a maximal submodule P of $\langle x \rangle$ and let S be the complement to P in $\langle x \rangle$. The existence of S follows from the fact that P is also a submodule of M. That is, since by hypothesis every submodule of M has a complement, then there exists a submodule $P^{\perp} \subseteq M$ such that $P \oplus P^{\perp} = M$. Therefore,

$$\langle x \rangle = M \cap \langle x \rangle = (P \oplus P^{\perp}) \cap \langle x \rangle = (P \cap \langle x \rangle) \oplus (P^{\perp} \cap \langle x \rangle) = P \oplus (P^{\perp} \cap \langle x \rangle)$$

and so, $S = P^{\perp} \cap \langle x \rangle$ is a submodule complement to P in $\langle x \rangle$. Since $S \simeq \langle x \rangle / P$, then it is simple. However, as S is simple, its existence is a contradiction to the maximality of the collection $\mathcal{C}$. ∎

Having defined semisimple modules, we can now define semisimple rings. We briefly characterize them through a few theorems and propositions.

Definition 3.7.4. (Semisimple Rings) A ring R is left semisimple if it is semisimple as a left module over itself, or equivalently, if every left R-module is semisimple.

Remark. The distinction between left and right in the previous definition is unnecessary as a left semisimple ring is also right semisimple.

Theorem 3.7.5. *Let R be a ring. Then, R is semisimple if and only if it is simple and left-Artinian.*

For a proof of this theorem, see [Gri], Chapter 9, section 3, 'The Artin-Wedderburn Theorem', Theorem 3.8.

Proposition 3.7.6. *Let R be a ring. Then, R is semisimple if and only if every left R-module is semisimple.*

Proof: $(\Longleftarrow)$: By the definition of a semisimple ring.

$(\Longrightarrow)$: If R is semisimple, then in particular, R is semisimple as a left module over itself. Therefore, every free left R-module,

$$M \simeq \bigoplus_{\omega \in \Omega} R_{\omega}$$

where Ω is some index set, is also semisimple. Finally, every left R-module is semisimple since it is isomorphic to a quotient module of a free module. ∎

Proposition 3.7.7. *Let R be a semisimple ring. Then, every finitely generated R-module is semisimple.*

Proof: We first show that every cyclic R-module is semisimple. So suppose $M = Rm$ for some non-zero $m \in M$. Then, the homomorphism of left R-modules given by $R_m : R \to M$ sending $r \mapsto rm$ is surjective. Since R is semisimple (as a left module over itself), then $R = \sum_{i \in I} S_i$ where I is an index set and each S_i is a simple R-submodule. Thus, we can write:

$$M = R_m\left(\sum_{i \in I} S_i\right) = \sum_{i \in I} R_m(S_i)$$

Now, consider the map R_m restricted to S_i. That is, $(R_m)_{|S_i} : S_i \twoheadrightarrow R_m(S_i)$. Since the kernel of $(R_m)_{|S_i}$ is an R-submodule of S_i, and S_i is simple, then we have that either $\ker((R_m)_{|S_i}) = \{0\}$ or $\ker((R_m)_{|S_i}) = S_i$. That is, either $R_m(S_i) \simeq S_i$ or $R_m(S_i) = 0$, so in either case $R_m(S_i)$ is simple. Since this is true for all $i \in I$, we have that $M = \sum_{i \in I} R_m(S_i)$ is a sum of simple R-modules, thus semisimple.

Now suppose M is finitely generated. That is, $M = Rm_1 + ... + Rm_n$ for some $m_1, ..., m_n \in M$. Since each Rm_i is semisimple, then, so too is M. ∎

We can now state and prove the following useful result.

Proposition 3.7.8. *Let D be a division ring. Then, every finitely generated left $M_n(D)$-module, M is isomorphic to a direct sum of copies of D^n. That is, $M \simeq (D^n)^s$ for some $s \in \mathbb{N}$.*

Proof: By Proposition 3.3.5, $M_n(D)$ is a simple ring. Since by Proposition 3.5.5, $M_n(D)$ is also left-Artinian, then it is a semisimple ring by Theorem 3.7.5. By Proposition 3.7.7, every finitely generated left $M_n(D)$-module is semisimple, that is, it is a sum of simple left $M_n(D)$-submodules. As any simple left $M_n(D)$-module is isomorphic to D^n, Proposition 3.3.12, then every finitely generated $M_n(D)$-module is isomorphic to $(D^n)^s$ for some $s \in \mathbb{N}$. ∎

Next, we define semiprime rings and then state and prove Brauer's Lemma.

Definition 3.7.9. (Semiprime Rings) Let R be a ring. Then, R is semiprime if $I^2 \neq 0$ for every non-zero ideal $I \triangleleft R$.

Lemma 3.7.10. ***(Brauer's Lemma)*** *Let R be a ring and $I \triangleleft R$ a minimal left ideal such that $I^2 \neq 0$. Then, $I = Re = \langle e \rangle$ where $e^2 = e \in R$ and eRe is a division ring.*

Proof: Since $I^2 \neq 0$, then for some non-zero $x \in I$, $Ix \neq 0$. Now $0 \subseteq Ix \subseteq I$, and as I is minimal and $Ix \neq 0$, then $Ix = I$. Thus, for some $e \in I$, we have $ex = x$. If $y \in I$, then $ye - y \in \text{ann}_I(x) = \{a \in I \mid ax = 0\}$. Now $\text{ann}_I(x)$ is a left ideal and $0 \subseteq \text{ann}_I(x) \subseteq I$. Since $e \notin \text{ann}_I(x)$ as $ex = x \neq 0$, then $\text{ann}_I(x) \neq I$. As I is minimal, this implies $\text{ann}_I(x) = 0$. Thus, $ye = y$ for all $y \in I$, and in particular, $e^2 = e$. Clearly, $I = \langle e \rangle = Re$.

Now take any non-zero $z \in eRe$. Then, $0 \neq Rz \subseteq ReRe = Re$. Since Re is minimal, then $Rz = Re = \langle e \rangle$. Then, $e = rz$ for some $r \in R$. Noting that e is the identity element of eRe, we have,

$$(ere)z = er(ez) = er(z) = e(rz) = e^2 = e$$

so z has a left inverse in eRe. As ere is a non-zero element of eRe, it, too, has a left inverse in eRe, say $er'e$. That is, $(er'e)(ere) = e$. However,

$$z = ez = ((er'e)(ere))z = (er'e)((ere)z) = (er'e)e = er'e$$

so the element ere is both a left and a right inverse to z in eRe. As this is for all non-zero $z \in eRe$, it follows that eRe is a division ring. ∎

Corollary 3.7.11. *Every non-zero left ideal in a semiprime, left-Artinian ring contains a non-zero idempotent.*

Proof: Let R be a semiprime, left-Artinian ring. If J is a non-zero left ideal of R, then there exists a minimal left ideal $I \subseteq J$. Now, multiplying I by R on the right to obtain a two sided ideal IR, and using the fact that R is semiprime, then $(IR)^2 \neq 0$. Thus, $0 \neq (IR)^2 = IRIR \subseteq I^2R$ implies that $I^2 \neq 0$. Now apply Brauer's Lemma. ∎

Theorem 3.7.12. ***(Wedderburn-Artin Theorem)***

(i) If R is a semiprime left-Artinian ring, then

$$R \simeq Mat_{n_1}(D_1) \times \cdots \times Mat_{n_m}(D_m)$$

where each D_i is a division ring, unique up to isomorphism, $n_i \in \mathbb{N}$ and the pairs (D_i, n_i) are unique up to permutation.

(ii) If A is a semisimple k-algebra, then

$$A \simeq Mat_{n_1}(D_1) \times \cdots \times Mat_{n_m}(D_m)$$

where each D_i is an associative division algebra, unique up to isomorphism, $n_i \in \mathbb{N}$ and the pairs (D_i, n_i) are unique up to permutation.

For a proof of this theorem, see [Nic].

3.8 The Brauer Group

In this section we define an equivalence relation for central simple algebras which we call similarity. Taking the group of central simple algebras modulo similarity, one obtains a group of equivalence classes of central simple algebras called the Brauer group. We define the Brauer group and show that the multiplication in the group is induced by the tensor product of central simple algebras, which also makes the group abelian. Furthermore, we prove some interesting results regarding central simple algebras from the perspective of the Brauer group.

We begin by defining what it means for two central simple algebras to be similar. Then, for any field, we show that the field is similar to all matrix algebras over it.

Definition 3.8.1. (Similarity of CSA's) Let A and B be CSA's over a field k. Then, A and B are called similar, denoted $A \sim B$, if

$$A \otimes_k M_m(k) \simeq B \otimes_k M_n(k)$$

for some $m, n \in \mathbb{N}$.

An equivalent characterization is given by the following. We call the associative division algebra D, attributed to the CSA by Wedderburn's theorem, the division ring component of A. Thus, the CSA's A and B are similar if their respective attributed division rings are k-isomorphic (a k-isomorphism is a ring isomorphism that fixes k).

We denote the equivalence class of A under the relation $\sim$ by $[A]$. Later, we will refer to two central simple algebras over k that are similar as being Brauer-equivalent.

Proposition 3.8.2. *For any field k, $k \sim M_n(k)$ for all $n \in \mathbb{N}$.*

Proof: Just choose $M_{ns}(k)$ and $M_s(k)$. Thus, $k \otimes_k M_{ns}(k) \simeq M_n(k) \otimes_k M_s(k)$ which are both isomorphic to $M_{ns}(k)$. ∎

We are now ready to formally define the Brauer group. After doing so, we then prove that the construction of Brauer groups is functorial.

Proposition 3.8.3. *The set of equivalence classes of CSA's over k forms an abelian group $Br(k)$, called the Brauer group of k with multiplication induced by the tensor product of algebras with identity $[k]$ and inverses given by $[A]^{-1} = [A^{op}]$.*

Proof: We know that if A and B are CSA's over k, then $A \otimes_k B$ is also a CSA over k. Thus, we have a well-defined multiplication of classes given by $[A][B] = [A \otimes_k B]$. This operation is associative since

$$\begin{aligned}([A][B])[C] &= [A \otimes_k B][C] \\ &= [(A \otimes_k B) \otimes_k C] \\ &= [A \otimes_k (B \otimes_k C)] \\ &= [A][B \otimes_k C] \\ &= [A]([B][C])\end{aligned}$$

and commutativity follows from the fact that $A \otimes_k B \simeq B \otimes_k A$, which implies $[A \otimes_k B] = [B \otimes_k A]$. Moreover, since $[A][k] = [A \otimes_k k] = [A] = [k \otimes_k A] = [k][A]$, the identity element in $Br(k)$ is $[k]$. Finally, we show that $[A]^{-1} = [A^{\mathrm{op}}]$. Since by Proposition 3.4.9, we have that $A \otimes_k A^{\mathrm{op}} \simeq \mathrm{End}_k(A) \simeq M_{\dim_k(A)}(k)$, then

$$[A][A^{\mathrm{op}}] = [A \otimes_k A^{\mathrm{op}}] = [\mathrm{End}_k(A)] = [M_{\dim_k(A)}(k)] = [k]$$

as required. ∎

Theorem 3.8.4. *$Br()$: Fields $\to \mathcal{A}b$ is a covariant functor from the category of fields to the category of abelian groups by assigning to every field k, the abelian group $Br(k)$ and to every morphism of fields (i.e. field extension) $\iota : k \hookrightarrow l$ the homomorphism of abelian groups $Br(\iota) : Br(k) \to Br(l)$ which sends $[A] \mapsto [A \otimes_k l]$.*

Proof: We first show that $Br(\iota)$ is indeed an abelian group homomorphism. That is:

$$\begin{aligned}Br(\iota)([A][B]) &= Br(\iota)([A \otimes_k B]) \\ &= [(A \otimes_k B) \otimes_k l] \\ &= [(A \otimes_k l) \otimes_l (B \otimes_k l)] \\ &= [A \otimes_k l][B \otimes_k l] \\ &= Br(\iota)([A])Br(\iota)([B])\end{aligned}$$

Moreover,

$$\begin{aligned} Br(\iota)([A]^{-1}) &= Br(\iota)([A^{\mathrm{op}}]) \\ &= [A^{\mathrm{op}} \otimes_k l] \\ &= [(A \otimes_k l)^{\mathrm{op}}] \\ &= [A \otimes_k l]^{-1} \\ &= (Br(\iota)([A]))^{-1} \end{aligned}$$

Now, for every field k, $Br(\mathrm{id}_k) = \mathrm{id}_{Br(k)}$ since for every $[A] \in Br(k)$ we have:

$$Br(\mathrm{id}_k)([A]) = [A \otimes_k k] = [A] = \mathrm{id}_{Br(k)}([A])$$

Finally, for all $\iota_1 : k \hookrightarrow k'$, $\iota_2 : k' \hookrightarrow l$ (i.e. for well-defined compositions $\iota_2 \circ \iota_1 : k \hookrightarrow l$), we have for all $[A] \in Br(k)$:

$$\begin{aligned} Br(\iota_2 \circ \iota_1)([A]) &= [A \otimes_k l] \\ &= [(A \otimes_k k') \otimes_{k'} l] \\ &= Br(\iota_2)([A \otimes_k k']) \\ &= Br(\iota_2)(Br(\iota_1)([A])) \\ &= (Br(\iota_2) \circ Br(\iota_1))([A]) \end{aligned}$$

Thus, $Br(\iota_2 \circ \iota_1) = Br(\iota_2) \circ Br(\iota_1)$. ∎

The rest of this section is devoted to studying some of the properties of Brauer groups. We do this by stating and proving the following propositions.

Proposition 3.8.5. *If k is algebraically closed, then $Br(k)$ is trivial.*

Proof: Let A be any CSA over an algebraically closed field k. By Corollary 3.5.7, $A \simeq M_n(k)$, where n is the degree of A over k. Since $[M_n(k)] = [k]$ for any $n \in \mathbb{N}$, then $Br(k) = \{[k]\}$. ∎

Proposition 3.8.6. $\mathbb{H} \simeq \mathbb{H}^{op}$.

Proof: Since $Br(\mathbb{R}) \simeq \mathbb{Z}_2$, then $[\mathbb{H}]$ has order 2 in $Br(\mathbb{R})$. Thus, $[\mathbb{H}][\mathbb{H}] = [\mathbb{R}]$ and so $[\mathbb{H}] = [\mathbb{H}^{\mathrm{op}}]$ implying $\mathbb{H} \simeq \mathbb{H}^{\mathrm{op}}$. ∎

Proposition 3.8.7. *If $[A] = [B]$ in $Br(k)$ and $dim_k(A) = dim_k(B)$, then $A \simeq B$.*

Proof: Because $[A] = [B]$ in $Br(k)$, then the respective division rings attributed to each by Wedderburn's theorem are isomorphic. Let $A \simeq M_n(D)$ and $B \simeq M_r(D')$, where $D \simeq D'$. We have,

$$\dim_k(A) = \dim_k(M_n(D)) = n^2 \dim_k(D) = r^2 \dim_k(D') = \dim_k(M_r(D')) = \dim_k(B)$$

which implies $n = r$. But then, $M_n(D) \simeq M_n(D')$, so $A \simeq B$. ∎

Proposition 3.8.8. *Let $k' \supseteq k$ be an extension of fields. Denote the kernel of the map $Br(k) \to Br(k')$ by $Br(k'/k)$. Then, $Br(\overline{k}/k) = Br(k)$, where $\overline{k}$ is the algebraic closure of k.*

Proof: If A is a CSA over k, then by Proposition 3.4.10, $A \otimes_k \overline{k}$ is a CSA over $\overline{k}$. By Corollary 3.5.7, $A \otimes_k \overline{k} \simeq M_n(\overline{k})$, where n is the degree of A over k. As every $[A] \in Br(k)$ is mapped to $[A \otimes_k \overline{k}] = [M_n(\overline{k})] = [\overline{k}] \in Br(\overline{k})$, then $Br(\overline{k}/k) = Br(k)$. ∎

Proposition 3.8.9. *$[A] \in Br(k'/k)$ if and only if k' splits A.*

Proof: $(\implies)$: Suppose $[A] \in Br(k'/k)$. Thus, $[A] \mapsto [k']$ under the map $Br(k) \to Br(k')$. That is, $[A \otimes_k k'] = [k'] = [M_n(k')]$, where $n^2 = \dim_k(A)$. Hence, $A \otimes_k k' \simeq M_n(k')$, so k' splits A.

$(\impliedby)$: Suppose $A \otimes_k k' \simeq M_n(k')$. Then, $[A] \mapsto [A \otimes_k k'] = [M_n(k')] = [k']$. Hence, $[A] \in Br(k'/k)$. ∎

Discussion 3.8.10. Let A be a CSA over k. By Wedderburn's Theorem, we know that $A \simeq M_n(D)$ for some unique $n \in \mathbb{N}$ and associative division algebra D, unique up to isomorphism. Let M be a finitely generated left A-module. By Proposition 3.7.8, $M \simeq (D^n)^s$ for some $s \in \mathbb{N}$ and we have,

$$\begin{aligned}
\dim_k(M) &= \dim_k((D^n)^s) \\
&= ns \cdot \dim_k(D) \\
&= ns \cdot \deg_k(D)^2 \\
&= ns \cdot \deg_k(D) \cdot \text{ind}_k(A) \\
&= s \cdot \deg_k(A) \cdot \text{ind}_k(A)
\end{aligned}$$

Moreover, we can consider M by the left $M_n(D)$-module isomorphism $M \simeq M_{n,s}(D)$ so that the $M_n(D)$-action on M is realized as left matrix multiplication.

Definition 3.8.11. (Reduced Dimension of Modules over CSA's) The reduced dimension of M, the left A-module from the previous discussion is defined as:

$$\mathrm{rdim}_A(M) = \frac{\dim_k(M)}{\deg_k(A)} = s \cdot \mathrm{ind}_k(A)$$

Proposition 3.8.12. *Let A be a CSA over k. Every left A-module, M of finite type has a natural structure of right module over $E = End_A(M)$, so that M is therefore an A-E-bimodule. If $M \neq \{0\}$, then the algebra E is also a CSA over k Brauer-equivalent to A. Moreover;*

$$\begin{aligned} (i)&\ \ deg_k(E) = rdim_A(M) \\ (ii)&\ \ rdim_E(M) = deg_k(A),\ \ and \\ (iii)&\ \ A = End_E(M) \end{aligned}$$

Conversely, if A and E are Brauer-equivalent CSA's over k, then there exists an A-E-bimodule $M \neq \{0\}$ such that (i), (ii), and (iii) hold and $E = End_A(M)$.

Proof: As the endomorphisms of left modules are written on the right of the arguments, the first statement is clear. Now, by Wedderburn, $A \simeq M_n(D)$ for some unique $n \in \mathbb{N}$ and some associative division algebra D, unique up to isomorphism. As a result of Proposition 3.3.12, every minimal left ideal of $A \simeq M_n(D)$, i.e. every simple left $M_n(D)$-module is isomorphic to D^n and thus, $D \simeq \mathrm{End}_A(D^n)$. Moreover, since M is a left $M_n(D)$-module of finite type, then by Proposition 3.7.8, $M \simeq (D^n)^s$ for some $s \in \mathbb{N}$. Therefore,

$$E = \mathrm{End}_A(M) \simeq \mathrm{End}_A((D^n)^s) \simeq M_s(\mathrm{End}_A(D^n)) \simeq M_s(D)$$

Thus, E is a CSA over k, Brauer-equivalent to A. Moreover,

$$\deg_k(E) = \deg_k(M_s(D)) = s \cdot \deg_k(D) = s \cdot \mathrm{ind}_k(A) = \mathrm{rdim}_A(M)$$

Hence,

$$\mathrm{rdim}_E(M) = \frac{\dim_k(M)}{\deg_k(E)} = \frac{ns \cdot \dim_k(D)}{s \cdot \deg_k(D)} = n \cdot \deg_k(D) = \deg_k(A)$$

Since M is an A-E-bimodule, then there is a natural embedding $A \hookrightarrow \mathrm{End}_E(M)$ sending $a \mapsto L_a$, left multiplication by a. We show that L_a is indeed an element of $\mathrm{End}_E(M)$, i.e. that it is indeed E-linear. Recalling that endomorphisms of the right module M_E can now go to the left of the argument, then for any endomorphism $\phi \in E = \mathrm{End}_A(M)$, for any $m \in M$, we have:

$$\begin{aligned} L_a((m)\phi) &= a \cdot ((m)\phi) \\ &= (a \cdot m)\phi \qquad \text{since } \phi \text{ is } A\text{-linear} \\ &= (L_a(m))\phi \end{aligned}$$

Finding the degree of $\mathrm{End}_E(M)$ as we did for $\deg_A(M)$, we obtain

$$\deg_k(\mathrm{End}_E(M)) = \deg_k(A)$$

Hence, this natural embedding is surjective and so $A \simeq \mathrm{End}_E(M)$.

To help visualize all of what's going on, we first make the following identifications: $M \simeq M_{n,s}(D)$, $A \simeq M_n(D)$, and as was proven above, $E = \mathrm{End}_A(M) \simeq M_s(D)$. Identify the action of A on M by matrix multiplication from the left by $M_n(D)$ and identify the action of E on M by matrix multiplication from the right by $M_s(D)$. A rough illustration which abuses notation by representing the generality of the entries as the division ring D itself, could be viewed as:

$$\underset{n \times n}{\overset{A}{\begin{bmatrix} D & \dots & D \\ \vdots & \ddots & \vdots \\ D & \dots & D \end{bmatrix}}} \quad \underset{n \times s}{\overset{M}{\begin{bmatrix} D & \dots & \dots & D \\ \vdots & \ddots & \vdots & \vdots \\ D & \dots & \dots & D \end{bmatrix}}} \quad \underset{s \times s}{\overset{E}{\begin{bmatrix} D & \dots & \dots & D \\ \vdots & \ddots & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ D & \dots & \dots & D \end{bmatrix}}}$$

To describe $\mathrm{End}_A(M)$, one seeks to identify the ring of all endomorphisms of M which are A-linear. Clearly, any matrix which multiplies M from the right is indeed trivially A-linear since matrix multiplication is associative. That is, for all $\alpha \in A$, $\mu \in M$, and $\epsilon \in E$, we have $(\alpha\mu)\epsilon = \alpha(\mu\epsilon)$, which when considering ϵ as an endomorphism of M is exactly the condition for A-linearity. Similarly, to describe $\mathrm{End}_E(M)$, one seeks to identify the ring of all endomorphisms of M which are E-linear. Clearly, any matrix which multiplies M from the left is trivially E-linear, so it is no surprise that $\mathrm{End}_E(M) \simeq A$. We now return to the proof.

For the converse, suppose that A and E are Brauer-equivalent CSA's over k. Thus, for some division algebra D, we have $A \simeq M_n(D)$ and $E \simeq M_s(D)$ for some unique $n, s \in \mathbb{N}$, respectively. Let $M = M_{n,s}(D)$, the group of $n \times s$ matrices over D. Matrix multiplication by A from the left and E from the right, respectively, endows M with an A-E-bimodule structure. Thus, we have natural embeddings:

$$(i)\ A \hookrightarrow \mathrm{End}_E(M)$$
$$(ii)\ E \hookrightarrow \mathrm{End}_A(M)$$

Since $\dim_k(M) = ns \cdot \dim_k(D)$, then

$$\mathrm{rdim}_E(M) = \frac{\dim_k(M)}{\deg_k(E)} = \frac{ns \cdot \dim_k(D)}{s \cdot \deg_k(D)} = n \cdot \deg_k(D) = \deg_k(A)$$
$$\mathrm{rdim}_A(M) = \frac{\dim_k(M)}{\deg_k(A)} = \frac{ns \cdot \dim_k(D)}{n \cdot \deg_k(D)} = s \cdot \deg_k(D) = \deg_k(E)$$

The first part of the proposition then yields

$$\deg_k(\mathrm{End}_E(M)) = \mathrm{rdim}_E(M) = \deg_k(A)$$
$$\deg_k(\mathrm{End}_A(M)) = \mathrm{rdim}_A(M) = \deg_k(E)$$

Hence, $A \simeq \mathrm{End}_E(M)$ and $E \simeq \mathrm{End}_A(M)$. ∎

3.9 Generalized Severi-Brauer Varieties

In the final section of this chapter, we arrive at the objects we are most interested in studying; generalized Severi-Brauer varieties of central simple algebras. We define these objects and provide some criteria characterizing when these objects have a rational point. Finally, we show that these generalized Severi-Brauer varieties are indeed, just twisted forms of Grassmannians.

We begin the section by defining Severi-Brauer varieties and generalized Severi-Brauer varieties for central simple algebras. For a more detailed exposition of these objects see [Châ], [Gil], [Kar1], and [Knu].

Definition 3.9.1. (Severi-Brauer Varieties) A Severi-Brauer variety over a field k is a projective algebraic variety X over k such that the base extension $X_l := X \times_k l$ becomes isomorphic to $\mathbb{P}_l^{n-1}$ for some finite field extension $l \supseteq k$. The field l is called a splitting field for X.

Definition 3.9.2. (The Severi-Brauer Variety of a CSA) Let A be a CSA over k of dimension n^2. Among the n-dimensional subspaces of A are the right ideals $I \triangleleft A$, subspaces of A which are invariant under right multiplication by A.

Denote the collection of right ideals of A which are n-dimensional over k by $SB_1(A)$, called the Severi-Brauer variety of A.

Definition 3.9.3. (The Generalized Severi-Brauer Variety of a CSA) Let A be a CSA over k of degree n. Then, for any $1 \leq d \leq n$, the d^{th} generalized Severi-Brauer variety of A, denoted $SB_d(A)$, is the variety of right ideals of dimension nd over k in A. For an ideal $I \in SB_d(A)$, the integer d is called the reduced dimension of I. Thus, $SB_d(A)$ is the variety of right ideals of A of reduced dimension d.

Example 3.9.4. In reduced dimension 1, the Severi-Brauer varieties are conics. In particular, let $A = (a, b)_k$, a generalized quaternion algebra. Then,

$$SB_1(A) \simeq \{ax^2 + by^2 - z^2 = 0\} \subseteq \mathbb{P}_k^2$$

For a more detailed exposition of this fact, see [Gil], Example 5.2.4.

We now characterize precisely when a generalized Severi-Brauer variety over some field has a rational point (over some field extension of that field).

Proposition 3.9.5. *Let A be a CSA over k of degree n and let $k' \supseteq k$ be an extension of fields. Then, $SB_d(A)$ has a rational point over k' if and only if $ind(A \otimes_k k') \mid d$. In particular, $SB_1(A)$ has a rational point over k' if and only if k' splits A.*

Proof: $(\implies)$: By the definition of $SB_d(A)$, it follows that $SB_d(A)$ has a k'-rational point if and only if $A \otimes_k k'$ contains a right ideal of reduced dimension d. Since the reduced dimension of any finitely generated right $A \otimes_k k'$-module is a multiple of $\text{ind}_k(A \otimes_k k')$, then it follows that $\text{ind}_k(A \otimes_k k')$ divides d if $SB_d(A)$ has a rational point over k'.

$(\impliedby)$: Suppose that $d = m \cdot \text{ind}(A \otimes_k k')$ for some integer m and let $A \otimes_k k' \simeq M_r(D)$ for some unique $r \in \mathbb{N}$ and some division algebra D, unique up to isomorphism. The set of matrices in $M_r(D)$ whose bottom $r - m$ rows are all zero is a right ideal of reduced dimension d, hence $SB_d(A)$ has a rational point over k'. ∎

Before heading to the main result of the chapter, we first formally define what it means for two quasi-projective varieties to be twisted forms of one another.

Definition 3.9.6. (Twisted Quasi-Projective Varieties)

Let k be a field and let X and Y be quasi-projective varieties over k. Let $l \supseteq k$ be any field extension. Then, X and Y are twisted forms of each other with respect to the field extension $l \supseteq k$ if, as quasi-projective varieties,

$$X \times_k l \simeq Y \times_k l$$

Finally, we arrive at the main result of the chapter. That is, we state and prove the following theorem which shows that generalized Severi-Brauer varieties are simply twisted forms of Grassmannians.

Theorem 3.9.7. *Let V be a finite-dimensional vector space over a field k. For $A = End_k(V)$, there is a natural isomorphism $SB_d(A) \simeq Gr(d, V)$. In particular, for $d = 1$, we have $SB_1(A) \simeq \mathbb{P}(V)$.*

Proof: Let V be a vector space of dimension n over k and let $V^* = \text{Hom}_k(V, k)$ be the dual space of V. Under the natural isomorphism, $A := \text{End}_k(V) \simeq V \otimes_k V^*$, the multiplication in A is defined by setting $(v \otimes \phi) * (w \otimes \psi) = (v \otimes \psi)\phi(w)$, and extending to arbitrary tensors by linearity.

The right ideals of reduced dimension d in A are of the form $\mathrm{Hom}_k(V, U) \simeq U \otimes_k V^*$ where U is a d-dimensional subspace of V.

We now show that the correspondence $U \leftrightarrow \mathrm{Hom}_k(V, U)$ between d-dimensional subspaces of V and right ideals of reduced dimension d in A induces an isomorphism of varieties $Gr(d, V) \simeq SB_d(A)$.

For any vector space W of dimension p over k, there is a morphism

$$Gr(d, V) \to Gr(dp, V \otimes_k W)$$

sending $U \mapsto U \otimes_k W$ where $U \subseteq V$ is a vector subspace of dimension d, and so $U \otimes_k W \subseteq V \otimes_k W$ is a vector subspace of dimension dp. In particular, when $W = V^*$, we get a morphism

$$\Phi : Gr(d, V) \to SB_d(A)$$

which maps $U \mapsto \mathrm{Hom}_k(V, U) \simeq U \otimes_k V^*$.

Now, consider the affine covering of $Gr(d, V)$ where for each subspace $S \subseteq V$ of codimension d, we assign a set

$$\mathcal{U}_S = \{U \subseteq V \mid U \oplus S = V\}$$

of subspaces complementary to S. The set $\mathcal{U}_S$ is an affine open subset of $Gr(d, V)$ as each element in $\mathcal{U}_S$ is a d-dimensional subset of V. In other words, if U_0 is a fixed complimentary subspace of S, there is an isomorphism:

$$\Psi : \mathrm{Hom}_k(U_0, S) \to \mathcal{U}_S$$

which sends $f \in \mathrm{Hom}_k(U_0, S)$ to $U_f = \{x + f(x) \mid x \in U_0\}$.

Analogously, we may also consider $\mathcal{U}_{S \otimes_k V^*} \subseteq Gr(dn, A)$. The image of Φ restricted to $\mathcal{U}_S$ is

$$\Phi(\mathcal{U}_S) = \{U \otimes_k V^* \subseteq V \otimes_k V^* \mid (U \otimes_k V^*) \oplus (S \otimes_k V^*) = V \otimes_k V^*\} = \mathcal{U}_{S \otimes_k V^*} \cap SB_d(A)$$

Moreover, there is a commutative diagram:

$$\begin{array}{ccc}
\mathcal{U}_S & \xrightarrow{\Phi|_{\mathcal{U}_S}} & \mathcal{U}_{S \otimes_k V^*} \\
\simeq \downarrow & & \downarrow \simeq \\
\mathrm{Hom}_k(U_0, S) & \xrightarrow[\phi]{} & \mathrm{Hom}_k(U_0 \otimes_k V^*, S \otimes_k V^*)
\end{array}$$

where $\phi(f) = f \otimes \mathrm{Id}_{V^*}$. Since ϕ is linear and injective, it is an isomorphism of varieties between $\mathrm{Hom}_k(U_0, S)$ and its image. Therefore, the restriction of Φ to $\mathcal{U}_S$ is an isomorphism $\mathcal{U}_S \simeq \mathcal{U}_{S \otimes_k V^*} \cap SB_d(A)$.

Since the open sets $\mathcal{U}_S$ form a covering of $Gr(d, V)$, it follows that Φ is an isomorphism. ∎

Chapter 4

Motives

The study of algebraic geometry involves many different cohomology theories. Often, it is the case that these theories share common properties. A cohomology theory with coefficients in a ring R is given by a contravariant functor from the category of algebraic varieties over an arbitrary field k (or subcategories such as smooth, projective, or quasi-projective varieties, etc.) to a R-linear tensor category. The functor, which we will denote by A, should satisfy certain properties, but most importantly, algebraic cycles on the variety X should map to elements in $A(X)$, and the intersection product of cycles on X should be reflected in the structure of $A(X)$. Examples of such cohomology theories includes De Rham cohomology, Étale cohomology and Betti cohomology, among others.

The commonality of these cohomology theories led Grothendieck to the idea of a universal cohomology theory, and as such, to the formulation of the theory of motives. This theory involves constructing a contravariant functor H from the category of algebraic varieties over k to a category $\mathcal{M}(k)$ through which any cohomology theory will factor. Thus, for any oriented cohomology theory A, there should be a realization functor Υ_A defined on $\mathcal{M}(k)$ such that for any algebraic variety X, the following equality $A(X) = \Upsilon_A(H(X))$ holds.

The following will focus on motives of smooth projective varieties over an arbitrary base field k, called pure motives. The construction of the category of pure motives depends however, on the choice of an equivalence relation on the algebraic cycles on varieties over k. Thus, beginning with such an equivalence relation, denoted by $\sim$, satisfying certain properties, one first enlarges the class of morphisms in the category of smooth projective varieties over k to include $\sim$-correspondences. This linearizes the category of smooth projective varieties over k to an additive category $\mathcal{C}or_\sim$. Taking the pseudo-abelian envelope of $\mathcal{C}or_\sim$, one obtains the category of effective motives over k, denoted $\mathcal{M}ot^{eff}_\sim(k)$. The tensor product structure in $\mathcal{M}ot^{eff}_\sim(k)$ is induced by the product of cycles in the category of smooth projective varieties over k, where the identity $\mathbb{1}_k$ corresponds to Spec(k). Finally, the projective line $\mathbb{P}^1_k$ decomposes in $\mathcal{M}ot^{eff}_{\sim}(k)$ as $\mathbb{1}_k \oplus \mathbb{L}_k$, where $\mathbb{L}_k$ is the Lefschetz motive. The category of pure motives $\mathcal{M}ot_{\sim}(k)$ is obtained from the category $\mathcal{M}ot^{eff}_{\sim}(k)$ by formally inverting $\mathbb{L}_k$.

4.1 Oriented Cohomology Theories

In this section we introduce a class of cohomology theories called oriented cohomology theories. We mainly follow the constructions presented in [Nen], [Pan1], and [Pan2].

We first introduce the category of smooth pairs before defining a contravariant functor into the category of $\mathbb{Z}$-graded abelian groups as in [Nen], section 2.1.

Definition 4.1.1. (The Category of Smooth Pairs)

Let k be any field. Let $\boldsymbol{\mathcal{S}m\mathcal{P}}$ denote the category of smooth pairs. Objects of $\boldsymbol{\mathcal{S}m\mathcal{P}}$ are pairs (X, U) consisting of a smooth quasi-projective variety X over k and an open subset $U \subseteq X$. A morphism from an object (X, U) to another object (X', U') in $\boldsymbol{\mathcal{S}m\mathcal{P}}$ is a morphism of pairs, i.e. a morphism of smooth quasi-projective varieties $f : X \to X'$ such that $f(U) \subseteq U'$.

Remark. We make the following remarks.

1. The category of smooth quasi-projective varieties over k embeds into the category of smooth pairs, $\boldsymbol{\mathcal{S}m\mathcal{P}}$ by sending $X \mapsto (X, \emptyset)$.

2. The terminal object in $\boldsymbol{\mathcal{S}m\mathcal{P}}$ is $(\mathrm{Spec}(k), \mathrm{Spec}(k))$, which will be denoted simply by $pt = \mathrm{Spec}(k)$.

Definition 4.1.2. (Pull-Back Morphisms of the Graded Components of A^*)

Let $A^* : \boldsymbol{\mathcal{S}m\mathcal{P}}^{op} \to \boldsymbol{\mathcal{A}b}^{\mathbb{Z}}$ be a contravariant functor from the category of smooth pairs over a field k to the category of $\mathbb{Z}$-graded abelian groups. Let A^i denote the i^{th} graded component of A^*. Note that for a given morphism in the category of smooth pairs, $f : (X, X \setminus Z) \to (X', X' \setminus Z')$ the induced map $f^* : A^i(X', X' \setminus Z') \to A^i(X, X \setminus Z)$ is called a pull-back morphism and that pull-backs preserve grading.

Next, we define the notion of a ring cohomology theory as in [Nen], section 2.2.

Definition 4.1.3. (Ring Cohomology Theories)

A ring cohomology theory is a contravariant functor $A^* : \boldsymbol{\mathcal{S}m\mathcal{P}}^{op} \to \boldsymbol{\mathcal{A}b}^{\mathbb{Z}}$ together with a functorial morphism of degree +1

$$\partial : A^*(U, \emptyset) \to A^{*+1}(X, U)$$

and a cup-product

$$\cup : A^i(X, X \setminus Z) \times A^j(X, X \setminus Z') \to A^{i+j}(X, X \setminus (Z \cap Z'))$$

with usual properties (see [Pan1], Definition 2.13) which turns $A^*(X)$ into a $\mathbb{Z}$-graded commutative ring. That is, given homogeneous elements $\alpha, \beta \in A^*(X)$, we have

$$\alpha \cup \beta = (-1)^{\deg(\alpha)\deg(\beta)} \beta \cup \alpha$$

We now define transversal squares as in [Pan2], Definition 2.1, and thereafter, the notion of integration on a ring cohomology theory, which Panin refers to as a trace structure, [Pan2], section 2.

Definition 4.1.4. (Transversality of Squares in the Category of Schemes)
Let $i : X \hookrightarrow Y$ be a closed embedding of smooth varieties. Consider the following Cartesian square consisting of smooth varieties (in the category of schemes):

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{i}} & \tilde{Y} \\ {\scriptstyle \tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[i]{} & Y \end{array}$$

That is, $\tilde{X} \simeq \tilde{Y} \times_Y X$. Let N and $\tilde{N}$ be the normal bundles to X in Y and $\tilde{X}$ in $\tilde{Y}$ respectively. Then, this square is called transversal if the canonical morphism of normal bundles $\tilde{\varphi}^*(N) \to \tilde{N}$ is an isomorphism.

Definition 4.1.5. (Integrations on a Ring Cohomology Theory)
Let A^* be a ring cohomology theory. An integration on A^* is a rule assigning to every projective morphism of smooth varieties $f : X \to Y$ of codimension c a two-sided $A^*(X)$-module operator:

$$f_* : A^*(X) \to A^{*+c}(Y)$$

satisfying the following properties:

1. For any projective morphisms $f : X \to Y$ and $g : Y \to Z$ of smooth varieties $(g \circ f)_* = g_* \circ f_*$
2. Given the transversal square from the previous section, the following diagram

commutes:

$$\begin{array}{ccc} A^*(\tilde{X}) & \xrightarrow{\tilde{i}_*} & A^{*+(\tilde{n}-\tilde{m})}(\tilde{Y}) \\ \uparrow{\scriptstyle \tilde{\varphi}^*} & & \uparrow{\scriptstyle \varphi^*} \\ A^*(X) & \xrightarrow[i_*]{} & A^{*+(n-m)}(Y) \end{array}$$

where $m = \dim(X)$, $n = \dim(Y)$, $\tilde{m} = \dim(\tilde{X})$ and $\tilde{n} = \dim(\tilde{Y})$.

3. For any morphism of smooth varieties $f : X \to Y$, the following diagram commutes:

$$\begin{array}{ccc} A^*(\mathbb{P}^n \times Y) & \xrightarrow{(\mathrm{id} \times f)^*} & A^*(\mathbb{P}^n \times X) \\ \downarrow{\scriptstyle (p_Y)_*} & & \downarrow{\scriptstyle (p_X)_*} \\ A^{*-n}(Y) & \xrightarrow[f^*]{} & A^{*-n}(X) \end{array}$$

4. Normalization: For any smooth variety X, $(\mathrm{id}_X)_* = \mathrm{id}_{A(X)}$

5. Localization: For any closed embedding of smooth varieties $i : Y \hookrightarrow X$ where $m = \dim(X)$ and $n = \dim(Y)$ and the inclusion $j : X \setminus Y \hookrightarrow X$, the following sequence

$$A^*(Y) \xrightarrow{i_*} A^{*+(m-n)}(X) \xrightarrow{j^*} A^{*+(m-n)}(X \setminus Y)$$

is exact. Note that this sequence is often referred to as a Gysin sequence.

Remark. We make the following remarks.

1. Every morphism between smooth projective varieties is projective.

2. The notation $A^i_Z(X)$ will be used in place of $A^i(X, X \setminus Z)$ henceforth.

We are now ready to define what it means to be an orientation of a ring cohomology theory as in [Pan1], section 3, and [Pan2], Definition 1.9.

Definition 4.1.6. (Orientations of a Ring Cohomology Theory)

An orientation of a ring cohomology theory A^* is a rule assigning to each smooth quasi-projective variety X and closed subset $Z \hookrightarrow X$ and each vector bundle E/X of rank r an operator $th^E_Z : A^*_Z(X) \to A^{*+r}_Z(E)$ which is a two-sided $A^*(X)$-module isomorphism and satisfies the following properties:

1. Invariance: For every isomorphism of rank r vector bundles $\varphi : E \to F$, the following diagram commutes:

$$\begin{CD} A^*_Z(X) @>{th^F_Z}>> A^{*+r}_Z(F) \\ @V{\mathrm{id}^*_X}VV @VV{\varphi^*}V \\ A^*_Z(X) @>>{th^E_Z}> A^{*+r}_Z(E) \end{CD}$$

2. Base change: For every morphism of pairs, $f : (X', X' \setminus Z') \to (X, X \setminus Z)$ in $\boldsymbol{\mathcal{S}m\mathcal{P}}$, with $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ closed subsets and every vector bundle E/X (with say, rank r), for its pull-back E' over X' and the projection morphism $g : E' = E \times_X X' \to E$ the following diagram commutes:

$$\begin{CD} A^*_Z(X) @>{th^E_Z}>> A^{*+r}_Z(E) \\ @V{f^*}VV @VV{g^*}V \\ A^*_{Z'}(X') @>>{th^{E'}_{Z'}}> A^{*+r}_{Z'}(E') \end{CD}$$

3. Additive property: For all vector bundles $p : E \to X$ and $q : F \to X$ of rank r and s respectively, the following diagram commutes:

$$\begin{CD} A^*_Z(X) @>{th^E_Z}>> A^{*+r}_Z(E) \\ @V{th^F_Z}VV @VV{th^{p^*(F)}_Z}V \\ A^{*+s}_Z(F) @>>{th^{q^*(E)}_Z}> A^{*+(r+s)}_Z(E \oplus F) \end{CD}$$

   and both compositions coincide with the operator $th^{E \oplus F}_Z$.

The operators th^E_Z will be referred to as *Thom isomorphisms*. The theory A^* is called orientable if there exists an orientation of A^*. If an orientation is chosen and fixed, then A^* is called an oriented cohomology theory.

Remark. There are several examples of oriented cohomology theories, but we are particularly interested in Chow groups, K-theory, and motivic cohomology.

The following is a list of some, but not all, of the important properties of an oriented cohomology theory. These properties are further described in [Nen], sections 2.3-2.9, and [Pan1], sections 2 and 3.

Definition 4.1.7. (Properties of Oriented Cohomology Theories)

(i) Integration Structure: For every oriented cohomology theory A^*, there exists a unique integration structure. That is, for any projective morphism of smooth varieties $f : Y \to X$ of codimension c there is a given two-sided $A*(X)$-module operator

$$f_* : A^*(Y) \to A^{*+c}(X)$$

called a push-forward of f satisfying properties listed in Definition 4.1.5.

Conversely, an integration on a ring cohomology theory gives rise to an orientation and these two constructions are inverse to each other.

Note that this fact is stated as Theorem 2.5 in section 2.1 in [Pan2] and its proof constitutes all of section 2.

(ii) Projective Bundle Theorem: Let X be a smooth variety and $Z \hookrightarrow X$ a closed subset. Let E/X be a vector bundle of rank $n+1$ over X. The map

$$(1, \xi, ..., \xi^n) \cup - : \bigoplus_{i=0}^{n} A_Z^{*-i}(X) \to A^*_{\mathbb{P}(E_Z)}\mathbb{P}(E)$$

is an isomorphism, where $\xi = c(\mathcal{O}_E(-1))$ and $E_Z = E\mid_Z$ is the restriction of E to Z.

(iii) Projection Formula: Let $f : Y \to X$ be a projective morphism of smooth varieties. Then, for any $\alpha \in A(Y)$ and any $\beta \in A(X)$, we have:

$$f_*(\alpha \cup f^*(\beta)) = f_*(\alpha) \cup \beta$$

(iv) Localization Sequence: Let X be a smooth variety and let $Z \hookrightarrow Y \hookrightarrow X$ be closed subsets, respectively. Then, there is an exact sequence of $A^*(\mathrm{pt})$-modules

$$... \to A^*_Z(X) \to A^*_Y(X) \to A^*_{Y\setminus Z}(X \setminus Z) \xrightarrow{\partial} A^{*+1}_Z(X)$$

(v) Strong Homotopy Invariance: Let X be a smooth variety and $Z \hookrightarrow X$ a closed subset. Let $p : E \to X$ be an affine bundle over X. Then, the pull-back $p^* : A^*_Z(X) \to A^*_{p^{-1}(Z)}(E)$ is an isomorphism.

Remark. Properties (iv) and (v) still hold if A^* is simply a ring cohomology theory.

4.2 Constructing the Category of A-Motives

By fixing an adequate equivalence relation $\sim$ for the algebraic cycles on a variety, we can linearize the category of smooth projective varieties over a field k. Since cycles modulo this chosen equivalence relation form an abelian group, and correspondences between varieties possess all the formal properties of morphisms, constructing a category whose objects are smooth projective varieties over k, but whose morphisms are correspondences will make it such that it is an additive category. In this section, we construct the category of A-motives following such a recipe.

We begin by defining the category of A-correspondences, as in [Nen], section 5.1, where A is an oriented cohomology theory.

Definition 4.2.1. (The Category of A - Correspondences)

Let k be a field, A an oriented cohomology theory, and let $\mathcal{C}or_A$ denote the category of A-correspondences defined as follows. The objects of $\mathcal{C}or_A$ are just smooth projective varieties over k and the set of morphisms from X to Y is defined by setting $\mathcal{C}or_A(X,Y) = A(X \times Y)$. Thus, an element from the ring $A(X \times Y)$ is called a correspondence between X and Y. Consider first the following maps:

$$
\begin{aligned}
p_{1,2} &: X \times Y \times Z \to X \times Y \\
p_{1,3} &: X \times Y \times Z \to X \times Z \\
p_{2,3} &: X \times Y \times Z \to Y \times Z
\end{aligned}
$$

which induce the following pull-backs and push-forwards:

$$
\begin{aligned}
p_{1,2}^* &: A(X \times Y) \to A(X \times Y \times Z) \\
p_{2,3}^* &: A(Y \times Z) \to A(X \times Y \times Z) \\
(p_{1,3})_* &: A(X \times Y \times Z) \to A(X \times Z)
\end{aligned}
$$

Then, for any correspondences $\alpha \in A(X \times Y)$ and $\beta \in A(Y \times Z)$, the correspondence given by the composition of the morphisms α and β

$$\beta \circ \alpha = (p_{1,3})_*(p_{1,2}^*(\alpha) \cup p_{2,3}^*(\beta)) \in A(X \times Z) = \mathcal{C}or_A(X,Z)$$

is called the correspondence product of the cycles α and β.

Remark. We make the following remarks about A-correspondences.

1. The push-forward $(p_{1,3})_*$ exists, since A is oriented.

2. Since the theory A is also $\mathbb{Z}$-graded, we can decompose morphisms in $\mathcal{C}or_A$ into graded components by setting $\mathcal{C}or_A^d(X,Y) = A^{\dim(X)+d}(X \times Y)$. Thus,

a correspondence α is said to have degree d if it is an element of the group $\mathcal{C}or_A^d(X,Y)$.

In particular, if $\alpha \in \mathcal{C}or_A^r(X \times Y)$ and $\beta \in \mathcal{C}or_A^s(Y \times Z)$ are homogeneous correspondences of degree r and s respectively, then $\beta \circ \alpha \in \mathcal{C}or_A^{r+s}(X \times Z)$.

3. Let X and Y be smooth projective varieties over k. Consider the decomposition of X into irreducible components:

$$X = \coprod_{i \in I} X_i$$

Then,

$$\mathcal{C}or_A^r(X,Y) = \bigoplus_{i \in I} A^{r+d_i}(X_i \times Y)$$

where $d_i = \dim(X_i)$ for each $i \in I$.

4. The identity morphism in $\mathcal{C}or_A(X,X)$ is the class of the image of the diagonal morphism $[\Delta_X] \subseteq A(X \times X)$. That is, for any other morphism $\alpha \in \mathcal{C}or_A(X,Y)$, we have:

$$\begin{aligned}\alpha \circ [\Delta_X] &= (p_{1,3})_*(p_{1,2}^*([\Delta_X]) \cup p_{2,3}^*(\alpha)) \\ &= (p_{1,3})_*([\Delta_X] \times 1_Y \cup 1_X \times \alpha) \\ &= \alpha \in A(X \times Y) = \mathcal{C}or_A(X,Y)\end{aligned}$$

and any other morphism $\beta \in \mathcal{C}or_A(W,X)$, we have:

$$\begin{aligned}[\Delta_X] \circ \beta &= (p_{1,3})_*(p_{1,2}^*(\beta) \cup p_{2,3}^*([\Delta_X])) \\ &= (p_{1,3})_*(\beta \times 1_X \cup 1_W \times [\Delta_X]) \\ &= \beta \in A(W \times X) = \mathcal{C}or_A(W,X)\end{aligned}$$

Note that when the context is understood, $[\Delta_X]$ may be written as just Δ_X.

We now describe some of the properties of $\mathcal{C}or_A$, following [Nen], sections 5.2-5.5. In particular, we show that $\mathcal{C}or_A$ is self-dual and we describe the induced tensor product structure on $\mathcal{C}or_A$.

Definition 4.2.2. (Self-Duality of $\mathcal{C}or_A$)

The category $\mathcal{C}or_A$ is a self-dual category, by acknowledging the following construction. Let $\alpha \in \mathcal{C}or_A(X,Y)$ and consider the twisting map $A(X \times Y) \to A(Y \times X)$. The image of α under this map is called the transpose of α and is denoted by α^t. Clearly, $\alpha^t \in \mathcal{C}or_A(Y,X)$ and $(\alpha^t)^t = \alpha$. Moreover, if α has degree d, then α^t has degree $\dim(X) + d - \dim(Y)$.

Definition 4.2.3. (Tensor Product Structure on $\mathcal{C}or_A$)

For any $\alpha \in \mathcal{C}or_A(X_1, X_3)$ and any $\beta \in \mathcal{C}or_A(X_2, X_4)$, the correspondence $\alpha \times \beta$, called the product of α and β, is given by

$$\alpha \times \beta = p_{1,3}^*(\alpha) \cup p_{2,4}^*(\beta) \in \mathcal{C}or_A((X_1 \times X_2), (X_3 \times X_4))$$

where $p_{1,3}^*$ and $p_{2,4}^*$ are respectively induced by

$$p_{1,3} : X_1 \times X_2 \times X_3 \times X_4 \to X_1 \times X_3$$
$$p_{2,4} : X_1 \times X_2 \times X_3 \times X_4 \to X_2 \times X_4$$

Observe that the product of varieties and correspondences induces a tensor product structure on $\mathcal{C}or_A$ where $X \otimes Y := X \times_k Y$ and $\alpha \otimes \beta := \alpha \times \beta$.

Definition 4.2.4. (The Functor $c : \mathcal{S}m\mathcal{P}roj^{op} \to \mathcal{C}or_A$)

Let $f : X \to Y$ be a morphism of smooth projective varieties. The graph of f is the image of the morphism

$$\Gamma_f : X \xrightarrow{(f,\mathrm{id}_X)} Y \times X$$

which is a closed embedding. Notice here that the notation Γ_f is used to represent two notions. Firstly, $\Gamma_f = (f, \mathrm{id}_X)$ as morphisms and $\Gamma_f = \mathrm{im}(f, \mathrm{id}_X) = \Gamma_f(X) \subseteq Y \times X$ as varieties, and both of these will be referred to as the graph of f in various contexts.

We define the contravariant functor $c : \mathcal{S}m\mathcal{P}roj^{op} \to \mathcal{C}or_A$ from the category of smooth projective varieties to the category of A-correspondences by assigning $X \mapsto X$ and $f \mapsto c(f) = (\Gamma_f)_*(1_{A(X)}) \in A(Y \times X)$. Notice $(\Gamma_f)_* : A^*(X) \to A^{*+n}(Y \times X)$ is well-defined as Γ_f is a projective morphism, where $n = \dim(Y)$.

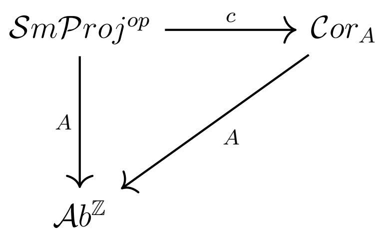
We now show that oriented cohomology theories factor through $\mathcal{C}or_A$.

Proposition 4.2.5. ***(Oriented Cohomology Theories Factor Through $\mathcal{C}or_A$)***

Let $\alpha \in \mathcal{C}or_A(Y, X)$. Define the realization of α as the map $A(\alpha) : A(Y) \to A(X)$ with the following construction. Identify $A(Y)$ with $A(pt \times Y) = \mathcal{C}or_A(pt, Y)$, $A(X)$ with $A(pt \times X) = \mathcal{C}or_A(pt, X)$ and $A(Y \times X)$ with $A(pt \times Y \times X) = \mathcal{C}or_A(pt, Y \times X)$. Setting $A(\alpha)(\beta) = \alpha \circ \beta$ for all $\beta \in A(pt \times Y)$, where the correspondence product of α and β is given by $\alpha \circ \beta = (p_X)_*(p_Y^*(\beta) \cup \alpha) \in A(pt \times X) = A(X)$, where

$$p_X : pt \times Y \times X \to pt \times X$$
$$p_Y : pt \times Y \times X \to pt \times Y$$

This yields a covariant functor $\mathcal{C}or_A \to \mathcal{A}b^{\mathbb{Z}}$ which is also denoted by A. Thus, the functor A restricted to projective varieties factors through $\mathcal{C}or_A$ and we obtain the *following diagram of functorial maps:*

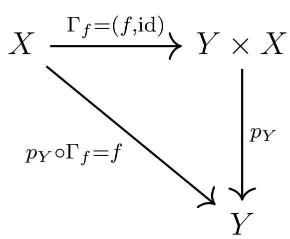


which commutes.

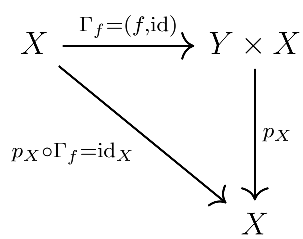
Proof: For every morphism of smooth projective varieties $f : X \to Y$ and correspondence $\beta \in A(\mathrm{pt} \times Y)$, we have:

$$\begin{aligned} A(c(f))(\beta) &= A((\Gamma_f)_*(1_{A(X)}))(\beta) \\ &= (\Gamma_f)_*(1_{A(X)}) \circ \beta \\ &= (p_X)_*(p_Y^*(\beta) \cup (\Gamma_f)_*(1_{A(X)})) \\ &= (p_X)_*((\Gamma_f)_*(1_{A(X)} \cup \Gamma_f^*(p_Y^*(\beta)))) \\ &= (p_X)_*((\Gamma_f)_*(\Gamma_f^*(p_Y^*(\beta)))) \end{aligned}$$

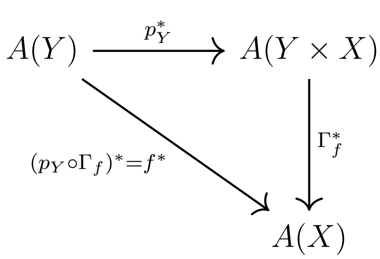
Now notice the commutative diagrams,



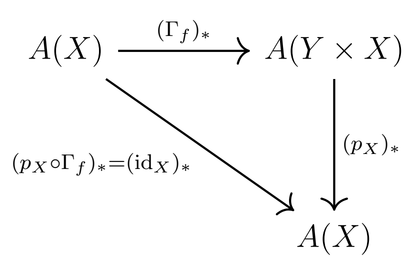
and



respectively induce the following diagrams, which are also commutative,

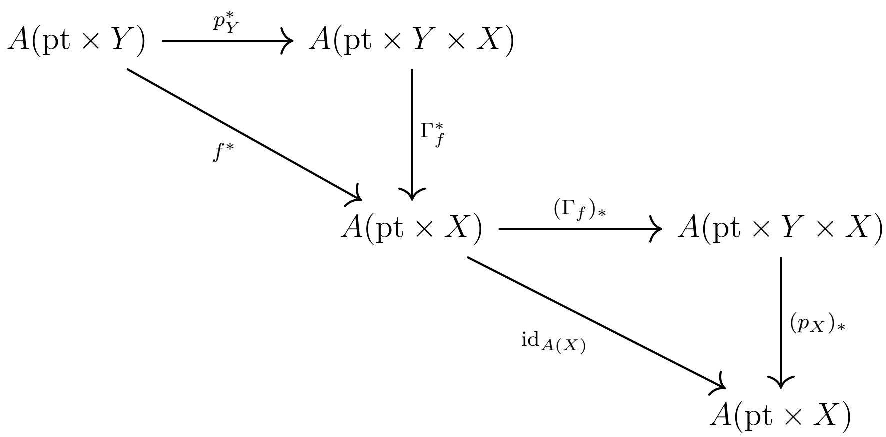


and



Thus, by placing the latter set of diagrams together, and associating $A(Y)$ with

$A(\mathrm{pt} \times Y)$, and so on as above:



we see that for any $\beta \in A(\mathrm{pt} \times Y)$, we have:

$$(p_X)_*((\Gamma_f)_*(\Gamma_f^*(p_Y^*(\beta)))) = \mathrm{id}_{A(X)}(f^*(\beta)) = f^*(\beta) \in A(\mathrm{pt} \times X)$$

Hence, $A(c(f)) = f^*$. ∎

We now define what it means for a category to be pseudo-abelian. Moreover, we show that additive categories may be given a pseudo-abelian completion.

Definition 4.2.6. (Pseudo-Abelian Categories)
Let $\mathcal{C}$ be an additive category. Then, $\mathcal{C}$ is called pseudo-abelian if for any projector $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, there exists a kernel, $\ker(p)$, and the canonical homomorphism

$$\ker(p) \oplus \ker(\mathrm{id}_X - p) \to X$$

is an isomorphism. Notice that $\mathrm{id}_X - p$ is also a projector element in $\mathrm{Hom}_{\mathcal{C}}(X, X)$.

Definition 4.2.7. (The Pseudo-Abelian Completion of Additive Categories)
Let $\mathcal{C}$ be an additive category. The pseudo-abelian completion of $\mathcal{C}$ is the category $\tilde{\mathcal{C}}$ defined as follows. The objects in $\tilde{\mathcal{C}}$ are pairs (X, p) where $X \in \mathcal{O}b(\mathcal{C})$ and the morphism $p \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ is a projector. The morphisms in $\tilde{\mathcal{C}}$ are given by

$$\mathrm{Hom}_{\tilde{\mathcal{C}}}((X, p), (Y, q)) = \frac{\{\alpha \in \mathrm{Hom}(X, Y) \mid \alpha \circ p = q \circ \alpha\}}{\{\alpha \in \mathrm{Hom}(X, Y) \mid \alpha \circ p = q \circ \alpha = 0\}}$$

That is, the category $\tilde{\mathcal{C}}$ is obtained by formally adding kernels of idempotent endomorphisms in $\mathcal{C}$. The composition of morphisms in $\tilde{\mathcal{C}}$ is induced from the composition of morphisms in $\mathcal{C}$. The functor $\Psi_{\mathcal{C}} : \mathcal{C} \to \tilde{\mathcal{C}}$ sending an object $X \mapsto (X, \mathrm{id}_X)$ and a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ to its class in $\mathrm{Hom}_{\mathcal{C}}(X, Y)/(0)$ is fully faithful.

Moreover, given any category $\mathcal{C}$, one can always construct a pseudo-abelian category $\tilde{\mathcal{C}}$ into which $\mathcal{C}$ embeds fully faithfully via a functor $\Psi_{\mathcal{C}} : \mathcal{C} \to \tilde{\mathcal{C}}$ which is universal with that property. That is, given any additive functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, where $\mathcal{D}$ is a pseudo-abelian category, there exists an additive functor $\tilde{\mathcal{F}} : \tilde{\mathcal{C}} \to \mathcal{D}$ for which $\tilde{\mathcal{F}} \circ \Psi_{\mathcal{C}} = \mathcal{F}$. In other words, $\mathcal{F}$ factors through $\tilde{\mathcal{C}}$.

Remark. If $p_0, p_1, ..., p_n \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ are projectors such that $p_i \circ p_j = 0$ for all $i \neq j$ and $\sum_{i=0}^{n} p_i = \mathrm{Id}_X$, then in the category $\tilde{\mathcal{C}}$, we have:

$$(X, \mathrm{Id}_X) \simeq \bigoplus_{i=0}^{n} (X, p_i)$$

We are now ready to define the category of degree 0 correspondences.

Definition 4.2.8. (The Category of Degree 0 Correspondences, $\mathcal{C}or_A^0$)

As seen from the remarks following Definition 4.2.1, the composition of degree 0 correspondences is degree 0. $\mathcal{C}or_A^0$ is therefore the category whose objects are smooth projective varieties over k and whose morphisms are degree 0 correspondences.

The category $\mathcal{C}or_A^0$ has direct sums defined by $X \oplus Y = X \coprod Y$ and tensor products defined by $X \otimes Y = X \times_k Y$. Most importantly, it is an additive category.

The rest of the section is devoted to constructing the category of pure motives. We begin by defining the category of effective motives.

Definition 4.2.9. (The Category of Effective Motives, $\mathcal{M}ot_{\sim}^{eff}(k)$)

The category of effective motives, denoted $\mathcal{M}ot_{\sim}^{eff}(k)$ is the pseudo-abelian completion of the category $\mathcal{C}or_A$. By construction, $\mathcal{M}ot_{\sim}^{eff}(k)$ is also a k-linear tensor category. Notice that its objects are pairs, (X, p), where X is a smooth projective variety over k and $p \in \mathcal{C}or_A(X, X) = A(X \times X)$ is an idempotent correspondence. Since for any such idempotent, we have:

$$X = \ker(p) \oplus \ker(\Delta_X - p)$$

in $\mathcal{M}ot_{\sim}^{eff}(k)$, then we see that effective motives over k are just direct sums of smooth projective varieties over k. Finally, for any two objects (X, p) and (Y, q), the morphisms in $\mathcal{M}ot_{\sim}^{eff}(k)$ are given by

$$\mathrm{Hom}_{\mathcal{M}ot_{\sim}^{eff}(k)}((X, p), (Y, q)) = \{\alpha \in \mathcal{C}or_A(X, Y) \mid \alpha \circ p = \alpha = q \circ \alpha\}$$

We now formally introduce the Lefschetz motive and decompose the projective line, $\mathbb{P}^1_k$ in the category of effective motives.

Definition 4.2.10. Fix an equivalence relation, $\sim$, on the algebraic cycles on smooth projective varieties over a field k. Choose a rational point in $\mathbb{P}^1_k$ and denote by e its class modulo $\sim$. Then,

$$A(\mathbb{P}^1_k) = \mathbb{Z} \oplus \mathbb{Z}e$$

Remark. Later on, we will denote the class of a rational point simply by [pt] or even pt when the context is understood.

Proposition 4.2.11. *Consider $\mathbb{P}^1_k$. Then, the correspondences $1 \times e$ and $e \times 1$ in $A(\mathbb{P}^1_k \times \mathbb{P}^1_k)$ are idempotent.*

Proof: By a straight-forward calculation,

$$\begin{aligned}(1 \times e) \circ (1 \times e) &= (p_{1,3})_*(p_{1,2}^*(1 \times e) \cup p_{2,3}^*(1 \times e)) \\ &= (p_{1,3})_*((1 \times e \times 1) \cup (1 \times 1 \times e)) \\ &= (p_{1,3})_*(1 \times e \times e) \\ &= 1 \times e\end{aligned}$$

and

$$\begin{aligned}(e \times 1) \circ (e \times 1) &= (p_{1,3})_*(p_{1,2}^*(e \times 1) \cup p_{2,3}^*(e \times 1)) \\ &= (p_{1,3})_*((e \times 1 \times 1) \cup (1 \times e \times 1)) \\ &= (p_{1,3})_*(e \times e \times 1) \\ &= e \times 1\end{aligned}$$

where the pull-back and the push-forward morphisms are induced by the projection maps $p_{i,j} : \mathbb{P}^1_k \times \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^1_k$ onto the i^{th} and j^{th} copies of $\mathbb{P}^1_k$. ∎

Proposition 4.2.12. *The effective motives $(\mathbb{P}^1_k, e \times 1)$ and $\mathbb{1}_k = (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)})$ are isomorphic.*

Proof: By the construction outlined in Definition 4.2.4, the structure morphism $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ gives rise to a correspondence $\alpha \in \mathcal{C}or_A(\mathrm{Spec}(k), \mathbb{P}^1_k)$. On the other hand, the rational point e is a morphism $\mathrm{Spec}(k) \to \mathbb{P}^1_k$ which gives rise to a correspondence $\beta \in \mathcal{C}or_A(\mathbb{P}^1_k, \mathrm{Spec}(k))$.

Now, $\beta \circ \alpha = \Delta_{\mathrm{Spec}(k)} \in \mathcal{C}or_A(\mathrm{Spec}(k), \mathrm{Spec}(k))$ and the composition $\alpha \circ \beta$ corresponds to the graph of the composition $\mathbb{P}^1_k \to \mathrm{Spec}(k) \to \mathbb{P}^1_k$, which is the element $e \times 1$ in $\mathcal{C}or_A(\mathbb{P}^1_k, \mathbb{P}^1_k)$. As

$$\alpha = \alpha \circ \Delta_{\mathrm{Spec}(k)} = \alpha \circ (\beta \circ \alpha) = (\alpha \circ \beta) \circ \alpha = (e \times 1) \circ \alpha$$

and

$$\beta = \Delta_{\mathrm{Spec}(k)} \circ \beta = (\beta \circ \alpha) \circ \beta = \beta \circ (\alpha \circ \beta) = \beta \circ (e \times 1)$$

then, we have that

$$\alpha \in \mathrm{Hom}_{\mathcal{M}ot^{eff}_{\sim}(k)}((\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)}), (\mathbb{P}^1_k, e \times 1))$$

and

$$\beta \in \mathrm{Hom}_{\mathcal{M}ot^{eff}_{\sim}(k)}((\mathbb{P}^1_k, e \times 1), (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)}))$$

are mutually inverse morphisms. ∎

Definition 4.2.13. (The Lefschetz Motive)

We define the Lefschetz motive over k to be the effective motive given by

$$\mathbb{L}_k := (\mathbb{P}^1_k, 1 \times e)$$

Proposition 4.2.14. $\mathbb{P}^1_k = \mathbb{1}_k \oplus \mathbb{L}_k$ *in* $\mathcal{M}ot^{eff}_{\sim}(k)$, *where* $\mathbb{1}_k = (\mathrm{Spec}(k), \Delta_{\mathrm{Spec}(k)})$.

Proof: As $e \times 1 \in A(\mathbb{P}^1_k \times \mathbb{P}^1_k)$ is an idempotent correspondence, then in $\mathcal{M}ot^{eff}_{\sim}(k)$, $(\mathbb{P}^1_k, \Delta_{\mathbb{P}^1_k}) = (\mathbb{P}^1_k, e \times 1) \oplus (\mathbb{P}^1_k, \Delta_{\mathbb{P}^1_k} - e \times 1) = \mathbb{1}_k \oplus \mathbb{L}_k$ as $\Delta_{\mathbb{P}^1_k} = 1 \times e + e \times 1$. ∎

Remark. More generally,

$$\mathbb{P}^n_k = \mathbb{1}_k \oplus \mathbb{L}^1_k \oplus ... \oplus \mathbb{L}^n_k$$

where $\mathbb{L}^i_k = \underbrace{\mathbb{L}_k \otimes ... \otimes \mathbb{L}_k}_{i \text{ times}}$.

We now formally invert the Lefschetz motive and subsequently define the category of pure motives.

Definition 4.2.15. (Inversion of the Lefschetz Motive)

Taking the tensor product with regards to the Lefschetz motive induces a functor, which is fully faithful, from the category of effective motives to itself, sending objects $(X,p) \mapsto (X,p) \otimes \mathbb{L}_k$ and morphisms $\alpha \mapsto \alpha \otimes \mathrm{id}_{\mathbb{L}_k}$, where $\mathrm{id}_{\mathbb{L}_k} = 1 \times e$. The tensor product of the objects is given by

$$(X,p) \otimes \mathbb{L}_k = (X \otimes \mathbb{P}^1_k, p \otimes (1 \times e)) = (X \times \mathbb{P}^1_k, p \times (1 \times e))$$

where $p \times (1 \times e) = p^*_{1,3}(p) \cup p^*_{2,4}(1 \times e) \in \mathcal{C}or_A((X \times \mathbb{P}^1_k), (X \times \mathbb{P}^1_k))$ and the pull-back morphisms $p^*_{1,3}$ and $p^*_{2,4}$ are induced by the corresponding projections

$$p_{1,3} : X \times \mathbb{P}^1_k \times X \times \mathbb{P}^1_k \to X \times X$$
$$p_{2,4} : X \times \mathbb{P}^1_k \times X \times \mathbb{P}^1_k \to \mathbb{P}^1_k \times \mathbb{P}^1_k$$

respectively. Moreover, for any $\alpha \in \mathrm{Hom}_{\mathcal{M}ot^{eff}_{\sim}(k)}((X,p),(Y,q))$ the tensor product of the morphisms is given by

$$\alpha \otimes (1 \times e) = \alpha \times (1 \times e) = p_{1,3}^{*}(\alpha) \cup p_{2,4}^{*}(1 \times e) \in \mathcal{C}or_A((X \times \mathbb{P}^1_k),(Y \times \mathbb{P}^1_k))$$

such that $(\alpha \times (1 \times e)) \circ (p \times (1 \times e)) = \alpha \times (1 \times e) = (q \times (1 \times e)) \circ (\alpha \times (1 \times e))$.

In particular, given two effective motives (X,p) and (Y,q), and integers m,n,N such that $m,n \leq N$, the k-space

$$\mathrm{Hom}_{\mathcal{M}ot^{eff}_{\sim}(k)}((X,p) \otimes \mathbb{L}_k^{N-m}, (Y,q) \otimes \mathbb{L}_k^{N-n})$$

is independent of the choice of N. Therefore, in this way, we can formally invert $\mathbb{L}_k$.

Definition 4.2.16. (The Category of Pure Motives, $\mathcal{M}ot_{\sim}(k)$)

Define the category of pure motives, denoted $\mathcal{M}ot_{\sim}(k)$, as the category whose objects are triples (X,p,n), where the pair (X,p) is an effective motive, where $n \in \mathbb{Z}$, and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{M}ot_{\sim}(k)}((X,p,m),(Y,q,n)) = \mathrm{Hom}_{\mathcal{M}ot^{eff}_{\sim}(k)}((X,p) \otimes \mathbb{L}_k^{N-m}, (Y,q) \otimes \mathbb{L}_k^{N-n})$$

where $m,n \leq N$.

Remark. We make the following remarks.

(i) The category $\mathcal{M}ot_{\sim}(k)$ has a tensor product structure given by

$$(X,p,m) \otimes (Y,q,n) = ((X,p) \otimes (Y,q), m+n) = (X \times Y, p \times q, m+n)$$

(ii) We can embed the category of effective motives into the category of pure motives by sending $(X,p) \mapsto (X,p,0)$.

(iii) For the pure motive $(X,\Delta_X,0)$, we will use the shorthand notation $\mathcal{M}(X)$.

We conclude the section by defining the Tate motive and Tate twisting.

Definition 4.2.17. (The Tate Motive)

Let $\mathbb{T}_k^1 := (\mathbb{1}_k, -1) \in \mathcal{M}ot_{\sim}(k)$. More generally, for any $n \in \mathbb{Z}$, write $\mathbb{T}_k^n = (\mathbb{1}_k, -n)$. Then, $\mathbb{T}_k^0 = \mathbb{1}_k$, and there is a canonical isomorphism $\mathbb{T}_k^{-1} \simeq \mathbb{L}_k$. The object $\mathbb{T}_k^1$ is called the Tate motive.

Definition 4.2.18. (Tate Twisting)

Let $(X,p) \in \mathcal{M}ot^{eff}_{\sim}(k)$ and $n \in \mathbb{Z}$. By the embedding $\mathcal{M}ot^{eff}_{\sim}(k) \hookrightarrow \mathcal{M}ot_{\sim}(k)$, we can send $(X,p) \mapsto (X,p,0)$. Define the operation of Tate twisting to be the tensor product $(X,p)(n) := (X,p,0) \otimes \mathbb{T}_k^n \in \mathcal{M}ot_{\sim}(k)$.

Remark. By construction, any pure motive can be written as the direct sum of Tate twists of some effective motives with corresponding integers.

4.3 Motivic Decomposition of $SB_2(A)$

For the remainder of this chapter, we fix the equivalence relation $\sim$ to be rational equivalence as described in section 1.3. In this case, the oriented cohomology theory A corresponds to Chow cohomology and is denoted by CH. By definition, for any two smooth projective varieties X and Y over k, we have $\mathcal{C}or_{CH}(X,Y) := CH(X \times Y)$, and after taking the pseudo-abelian completion and formally inverting the Lefschetz motive, the resulting category $\mathcal{M}ot_{\mathrm{rat}}(k)$ is the category of pure Chow motives.

We are now ready to state our main result.

Theorem 4.3.1. *Let $SB_2(A)$ be a generalized Severi-Brauer variety for a central simple k-algebra A of degree 7. Then, there is an isomorphism*

$$\mathcal{M}(SB_2(A)) \simeq (SB_2(A), p) \oplus (SB_2(A), p)^c$$

where p is an idempotent correspondence in $\mathcal{C}or_{CH}(SB_2(A), SB_2(A))$ and the superscript c indicates the complementary object to $(SB_2(A), p)$ in the decomposition of $\mathcal{M}(SB_2(A))$ in the category of pure Chow motives.

The case for a central simple k-algebra A of degree 5 was considered in [Cal]. Our case for degree 7 is a new result. The rest of this section is devoted to building up the tools necessary for proving this theorem, and its proof will be presented in the next section.

We begin by defining relative cellular spaces as in [Kar3] and describing some properties of their Chow rings.

Definition 4.3.2. (Relative Cellular Spaces)

Let X be a smooth projective variety over a field k. Then, we say that X is a relative cellular space if there exists a finite increasing filtration by closed (not necessarily smooth) subvarieties

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_{n-1} \subset X_n = X$$

such that for each complement $X_{i\setminus i-1} := X_i \setminus X_{i-1}$ there is a map $p_i : X_{i\setminus i-1} \to Y_i$ for some smooth projective variety Y_i over k over which $X_{i\setminus i-1}$ is an affine bundle.

The varieties Y_i are called the bases of the cells of X and the union

$$Y = \coprod_{i=0}^{n} Y_i$$

is called the total base of X.

Theorem 4.3.3. ***(Chow Decomposition of G-Homogeneous Varieties)***

Let G be a split linear algebraic group over a field k and X a projective G-homogeneous variety. That is, $X = G/P$, where P is a parabolic subgroup of G. Then, X has a cellular filtration and the generators of the Chow groups of the bases of this filtration correspond to the free additive generators of $CH(X)$.

For a proof of this theorem, we refer the reader to [Köc].

Definition 4.3.4. (Correspondence Products for Homogeneous Varieties)

Let X and Y be two projective homogeneous varieties over a field k. Then, the variety $X \times Y$ also has a cellular filtration and moreover, $\mathrm{CH}^*(X \times Y) \simeq \mathrm{CH}^*(X) \otimes \mathrm{CH}^*(Y)$ as graded rings. In such a case, the formula for the correspondence product of two cycles $\alpha = \alpha_X \times \alpha_Y \in \mathrm{CH}(X \times Y)$ and $\beta = \beta_Y \times \beta_X \in \mathrm{CH}(Y \times X)$ is given by

$$\beta \circ \alpha = (\beta_Y \times \beta_X) \circ (\alpha_X \times \alpha_Y) = \deg(\alpha_Y \cdot \beta_Y)(\alpha_X \times \beta_X) \in \mathrm{CH}(X \times X)$$

where $\deg : \mathrm{CH}(Y) \to \mathrm{CH}(\{\mathrm{pt}\}) = \mathbb{Z}$ is the degree map.

For further details, see [Bon], specifically Lemma 5.

We now define as in [Cal] rational cycles in the Chow ring of the variety $X_S := X \times_k k_S$. Then, we state a few versions of the Rost Nilpotence Theorem in varying contexts emphasizing some of its corollaries in the context of projective homogeneous varieties.

Definition 4.3.5. (Rational Cycles on $\boldsymbol{CH(X \times_k k^{\mathrm{sep}})}$)

Let X be a projective variety over a field k with separable closure k^{sep}. Consider the variety $X_S := X \times_k k^{\mathrm{sep}}$ constructed by extending scalars, where the morphism between projective varieties $p_X : X_S \to X$ induces the pull-back homomorphism between graded rings $p_X^* : \mathrm{CH}(X) \to \mathrm{CH}(X_S)$. Then, we say that a cycle class $\alpha \in \mathrm{CH}(X_S)$ is rational if $\alpha \in \mathrm{im}(p_X^*)$.

Remark. We make the following remarks.

(i) Linear combinations, intersections, and correspondence products of rational cycles are rational.

(ii) The class of the diagonal morphism Δ_{X_S} is a rational cycle on $\mathrm{CH}(X_S \times X_S)$.

Theorem 4.3.6. ***(Rost Nilpotence Theorem)***

Let k be a field with algebraic closure $\bar{k}$. Suppose Q is a smooth quadric over k and let $f \in End(\mathcal{M}(Q))$ be an endomorphism of its integral Chow motive. Then, if $f \otimes \bar{k} = 0$ in $End(\mathcal{M}(Q \otimes \bar{k}))$, f is nilpotent.

For a proof of this theorem, we refer the reader to [Bro1].

Theorem 4.3.7. ***(Rost Nilpotence for Projective Homogeneous Varieties)***

Let X be a projective homogeneous variety over a field k. Then, for every field extension l/k, the kernel of the natural ring homomorphism $End(\mathcal{M}(X)) \to End(\mathcal{M}(X_l))$ consists of nilpotent correspondences.

For a proof of this theorem, we refer the reader to [Che1], Theorem 8.2.

Corollary 4.3.8. ***(Existence of Projectors in End($\mathcal{M}(X)$))***

Let X be a projective homogeneous variety over a field k. Suppose that for a field extension l/k, the image of the ring homomorphism $End(\mathcal{M}(X)) \to End(\mathcal{M}(X_l))$ sending $f \to f_l = f \otimes_k l$, contains a projector (idempotent) q. Then, there exists a projector $p \in End(\mathcal{M}(X))$ such that $p \otimes_k l = q$.

For a proof of this corollary, we refer the reader to [Che1], Corollary 8.3.

Corollary 4.3.9. ***(Existence of Rational Projectors in $CH^n(X \times X)$)***

In particular, let p_S be a non-trivial rational projector (idempotent correspondence) in $CH^n(X_S \times X_S)$. That is, $p_S \circ p_S = p_S$. Then, there exists a non-trivial projector (idempotent correspondence) $p \in CH^n(X \times X)$ such that $p \times_k k^{sep} = p_S$.

Remark. The existence of a non-trivial rational projector p_S in $\mathrm{CH}(X_S \times X_S)$ induces the decomposition of the pure Chow motive of X, where

$$\mathcal{M}(X) \simeq (X, p) \oplus (X, \Delta_X - p)$$

Next, we define isomorphisms between twisted motives.

Definition 4.3.10. (Isomorphisms Between Twisted Motives)

An isomorphism between the twisted motives (X, p, m) and (Y, q, l) is given by correspondences $j_1 \in CH^{\dim(X)-l+m}(X \times Y)$ and $j_2 \in CH^{\dim(Y)-m+l}(Y \times X)$ such that

$$j_1 \circ p = q \circ j_1$$
$$j_2 \circ q = p \circ j_2$$
$$p = j_2 \circ j_1 \text{ and } q = j_1 \circ j_2$$

Proposition 4.3.11. *If X and Y lie in the category $\mathcal{M}(G, \mathbb{Z})$, then by the Rost nilpotence theorem, it suffices to give rational j_1 and some j_2 satisfying these conditions over separable closure. (Note that j_2 will automatically be rational).*

For a proof of this Proposition, we refer the reader to [Pet], section 2, specifically Proposition 2.6 and Lemma 2.10.

We now shift our attention to describing Grassmann varieties. We begin by defining the Segre map and Segre varieties, followed by Segre embeddings. This will allow us to emded the product of Grassmannians into another Grassmannian.

Definition 4.3.12. (The Segre Map and Segre Varieties)

The Segre map $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^{(m+1)(n+1)-1}$ is defined as the map which sends

$$([x_0 : ... : x_m], [y_0 : ... : y_n]) \mapsto [x_0y_0 : x_0y_1 : ... : x_0y_n : x_1y_0 : ... : x_my_n]$$

The image of σ is called a Segre variety and is denoted $\Sigma_{m,n}$. Denoting the homogeneous coordinates of $\mathbb{P}^{(m+1)(n+1)-1}$ as $z_{i,j}$, the variety $\Sigma_{m,n}$ is given as the zero set of polynomials of the form $z_{a,b}z_{c,d} - z_{c,b}z_{a,d} = 0$ as a, b, c, d vary. Thus, the Segre variety is an example of a determinantal variety, i.e. it is the zero-locus of the set of 2×2 minors of the matrix $(z_{i,j})$.

Definition 4.3.13. (Segre Embeddings) Let k be a field and U, V vector spaces over k. There is a natural way to map the Cartesian product $U \times V$ to the tensor product $U \otimes V$ given by $\phi : U \times V \to U \otimes V$ where $\phi(u, v) = u \otimes v$.

In general, this map is not injective as for any $u \in U$, $v \in V$, and $c \in k^\times$ we have:

$$\phi(u, v) = u \otimes v = cu \otimes c^{-1}v = \phi(cu, c^{-1}v)$$

However, for the underlying projective spaces $\mathbb{P}(U)$ and $\mathbb{P}(V)$, the map

$$\sigma : \mathbb{P}(U) \times \mathbb{P}(V) \to \mathbb{P}(U \otimes V)$$

is a morphism of varieties. That is, it is a closed immersion since one can give a set of equations for its image. The morphism σ is called the Segre embedding.

Remark. The Segre embedding allows us to construct products of varieties explicitly. That is, if V_1 and V_2 are quasi-projective varieties (they can be affine, projective, open subsets of either, etc.), then $V_1 \subseteq \mathbb{P}^m$ and $V_2 \subseteq \mathbb{P}^n$ for some $m, n \in \mathbb{N}$. Hence, the image of the Segre embedding restricted to $V_1 \times V_2$ is a quasi-projective variety in $\mathbb{P}^{(m+1)(n+1)-1}$ which can be described explicitly by a set of equations and is isomorphic to $V_1 \times V_2$. Thus, in particular, products of quasi-projective varieties exist and are also quasi-projective. In light of this fact, $\sigma : \mathbb{P}^m \times \mathbb{P}^n \to \Sigma_{m,n}$ is an isomorphism of projective varieties.

We now formalize the notion of universal quotient bundles on Grassmannians.

Definition 4.3.14. (Universal Quotient Bundles on Grassmannians)

Let k be a field and $Gr(d, n)$ be the Grassmannian of d-dimensional subspaces of k^n. There is an obvious vector bundle of rank n on $Gr(d, n)$, namely $\mathcal{O}^n := Gr(d, n) \times k^n$ whose fiber at every point is the vector space k^n.

Define the vector bundle τ_d on $Gr(d, n)$ as the rank d subbundle of $\mathcal{O}^n$ whose fiber at a point $V \in Gr(d, n)$ is the subspace V itself. τ_d is called the universal subbundle on $Gr(d, n)$, but it is also often referred to as the tautological bundle.

The rank $n-d$ quotient bundle $Q_{n-d} = \mathcal{O}^n/\tau_d$ is called the universal quotient bundle on $Gr(d,n)$ and there is an exact sequence

$$0 \to \tau_d \to \mathcal{O}^n \to Q_{n-d} \to 0$$

Proposition 4.3.15. *Let k be a field and consider the Grassmann variety $Gr(d,n+1)$, where $1 \leq d \leq n$. That is, consider the space of all d-planes contained in the affine space $\mathbb{A}_k^{n+1}$. A twisted form of $Gr(d,n+1)$ is a generalized Severi-Brauer variety $SB_d(A)$ for a central simple k-algebra A of degree $n+1$, as shown in Theorem 3.9.7.*

Then, for any $1 \leq d, d' \leq n$, there is a fiber product diagram (as in Definition 1.4.1)

$$\begin{array}{ccc} Gr(d,n+1) \times Gr(d',n+1) & \xrightarrow{Seg} & Gr(dd',(n+1)^2) \\ \downarrow & & \downarrow \\ SB_d(A) \times SB_{d'}(A^{op}) & \xrightarrow[Seg]{} & SB_{dd'}(A \otimes A^{op}) \end{array}$$

where the horizontal arrows are the Segre embeddings given by the tensor product of ideals (respectively, linear subspaces) and the vertical arrows are canonical maps induced by the scalar extension k^{sep}/k.

Moreover, the fiber product diagram induces via the pull-back homomorphisms, Definition 1.5.8, the following commutative diagram of rings

$$\begin{array}{ccc} CH(Gr(d,n+1) \times Gr(d',n+1)) & \xleftarrow{Seg^*} & CH(Gr(dd',(n+1)^2)) \\ \uparrow{\scriptstyle res} & & \simeq \uparrow{\scriptstyle res} \\ CH(SB_d(A) \times SB_d(A^{op})) & \xleftarrow[Seg^*]{} & CH(SB_{dd'}(A \otimes A^{op})) \end{array}$$

Note that there is an isomorphism on the right as $A \otimes A^{op}$ splits, Proposition 3.4.9.

Now consider a vector bundle $\mathscr{E}$ over $Gr(dd',(n+1)^2)$. The total Chern class of $\mathscr{E}$ is $c(\mathscr{E}) \in CH(Gr(dd',(n+1)^2))$. Notice, that $Seg^*(c(\mathscr{E}))$ is a rational cycle on $CH(Gr(d,n+1) \times Gr(d',n+1)) \simeq CH(Gr(d,n+1)) \otimes CH(Gr(d',n+1))$, that is, $Seg^*(c(\mathscr{E}))$ is an element of $im(res) = res(CH(SB_d(A)) \times SB_{d'}(A^{op}))$. This is clear as $Seg^*(c(\mathscr{E})) = res(Seg^*(res^{-1}(c(\mathscr{E}))))$.

In particular, if $\mathscr{E} = \tau_{dd'}$ is the tautological bundle of $Gr(dd',(n+1)^2)$, then the total Chern class $c(pr_1^*(\tau_d) \otimes pr_2^*(\tau_{d'}))$ of the tensor product of the pull-back bundles over $Gr(d,n+1) \times Gr(d',n+1)$ of the tautological bundles τ_d and $\tau_{d'}$ over $Gr(d,n+1)$ and $Gr(d',n+1)$ respectively, is rational.

Proof: The pull-back bundles $\mathrm{pr}_1^*(\tau_d)$ and $\mathrm{pr}_2^*(\tau_{d'})$ are given by the commutative diagrams

$$\begin{array}{ccc} \mathrm{pr}_1^*(\tau_d) & \xrightarrow{\pi_2} & \tau_d \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \rho} \\ Gr(d, n+1) \times Gr(d', n+1) & \xrightarrow[\mathrm{pr}_1]{} & Gr(d, n+1) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{pr}_2^*(\tau_{d'}) & \xrightarrow{\pi_2'} & \tau_{d'} \\ \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle \rho'} \\ Gr(d, n+1) \times Gr(d', n+1) & \xrightarrow[\mathrm{pr}_2]{} & Gr(d', n+1) \end{array}$$

where all the arrows in both diagrams are projection maps and where

$$\mathrm{pr}_1^*(\tau_d) = \{((V, W), V) \in (Gr(d, n+1) \times Gr(d', n+1)) \times \tau_d \mid \mathrm{pr}_1(V, W) = \rho(V)\}$$

and

$$\mathrm{pr}_2^*(\tau_{d'}) = \{((V, W), W) \in (Gr(d, n+1) \times Gr(d', n+1)) \times \tau_{d'} \mid \mathrm{pr}_2(V, W) = \rho'(W)\}$$

Now, the tensor product of the vector bundles $\mathrm{pr}_1^*(\tau_d)$ and $\mathrm{pr}_2^*(\tau_{d'})$ is the bundle over $Gr(d, n+1) \times Gr(d', n+1)$ whose fiber over any point is the tensor product of the vector spaces of its respective fibers in $\mathrm{pr}_1^*(\tau_d)$ and $\mathrm{pr}_2^*(\tau_{d'})$. That is,

$$\mathrm{pr}_1^*(\tau_d) \otimes \mathrm{pr}_2^*(\tau_{d'}) = \{((V, W), V \otimes W) \in (Gr(d, n+1) \times Gr(d', n+1)) \times \tau_{dd'}\}$$

which is nothing other than the pullback bundle of $\tau_{dd'}$ to $Gr(d, n+1) \times Gr(d', n+1)$ by the map Seg given by the commutative diagram

$$\begin{array}{ccc} \mathrm{Seg}^*(\tau_{dd'}) = \mathrm{pr}_1^*(\tau_d) \otimes \mathrm{pr}_2^*(\tau_{d'}) & \xrightarrow{\pi_2''} & \tau_{dd'} \\ \downarrow{\scriptstyle \pi_1''} & & \downarrow{\scriptstyle \rho''} \\ Gr(d, n+1) \times Gr(d', n+1) & \xrightarrow[\mathrm{Seg}]{} & Gr(dd', (n+1)^2) \end{array}$$

From this, we obtain that $c(\mathrm{pr}_1^*(\tau_d) \otimes \mathrm{pr}_2^*(\tau_{d'})) = c(\mathrm{Seg}^*(\tau_{dd'})) = \mathrm{Seg}^*(c(\tau_{dd'}))$ and thus, $c(\mathrm{pr}_1^*(\tau_d) \otimes \mathrm{pr}_2^*(\tau_{d'})) = \mathrm{res}(\mathrm{Seg}^*(\mathrm{res}^{-1}(c(\tau_{dd'}))))$ by commutativity of the second diagram in the proposition. Therefore, $c(\mathrm{pr}_1^*(\tau_d) \otimes \mathrm{pr}_2^*(\tau_{d'}))$ is rational. ∎

4.4 Decomposition of $\mathcal{M}(SB_2(A))$ for $\deg(A) = 7$

We are now ready to prove Theorem 4.3.1. That is, we provide an example of the motivic decomposition of $SB_2(A)$ where A is a CSA over k with $\deg_k(A) = 7$. Thus, we restrict our focus to the case where $n = 6$, $d = 2$, and $d' = 1$ from Proposition 4.3.15. This means we are considering the Grassmannian $Gr(2,7)$ and the projective space $\mathbb{P}^6 = Gr(1,7)$.

First, we need to describe the Chow ring $CH(Gr(2,7))$. We follow the methods in [Ful] in the section on Schubert Calculus, section 14.7. The reader should take note of the fact that $Gr(d+1,n+1) = \mathbb{G}r(d,n)$ for Grassmannians of projective spaces as per the notation in Fulton's book. The Chow ring $CH(Gr(2,7))$ is generated (as a ring with some relations) by the special Schubert Cycles corresponding to the Chern classes $\sigma_m = c_m(Q)$ for $m = 1,2,...,5$ where $Q = \mathcal{O}^7/\tau_2$ is the universal quotient bundle of rank 5 over $Gr(2,7)$.

Moreover, the Schubert cycles $\Delta_\lambda(\sigma)$ that are parameterized by the set of partitions $\lambda = (\lambda_0, \lambda_1)$ where $5 \geq \lambda_0 \geq \lambda_1 \geq 0$ form a $\mathbb{Z}$-module basis for $CH(Gr(2,7))$. Using the formula

$$\{\lambda_0, \lambda_1\} = \Delta_\lambda(\sigma) = \det(\sigma_{\lambda_i+j-i})_{0\leq i,j\leq 1} = \begin{vmatrix} \sigma_{\lambda_0} & \sigma_{\lambda_0+1} \\ \sigma_{\lambda_1-1} & \sigma_{\lambda_1} \end{vmatrix}$$

and recalling that $\sigma_0 = 1$ and $\sigma_r = 0$ for $r \notin \{0,1,...,5\}$, we obtain the following Schubert classes for $CH(Gr(2,7))$:

$$\begin{array}{ll}
\{0,0\} = \Delta_{(0,0)} = Gr(2,7) & \{1,1\} = \Delta_{(1,1)} = \sigma_1^2 - \sigma_2 \\
\{1,0\} = \Delta_{(1,0)} = \sigma_1 & \{2,1\} = \Delta_{(2,1)} = \sigma_2\sigma_1 - \sigma_3 \\
\{2,0\} = \Delta_{(2,0)} = \sigma_2 & \{3,1\} = \Delta_{(3,1)} = \sigma_3\sigma_1 - \sigma_4 \\
\{3,0\} = \Delta_{(3,0)} = \sigma_3 & \{4,1\} = \Delta_{(4,1)} = \sigma_4\sigma_1 - \sigma_5 \\
\{4,0\} = \Delta_{(4,0)} = \sigma_4 & \{5,1\} = \Delta_{(5,1)} = \sigma_5\sigma_1 \\
\{5,0\} = \Delta_{(5,0)} = \sigma_5 &
\end{array}$$

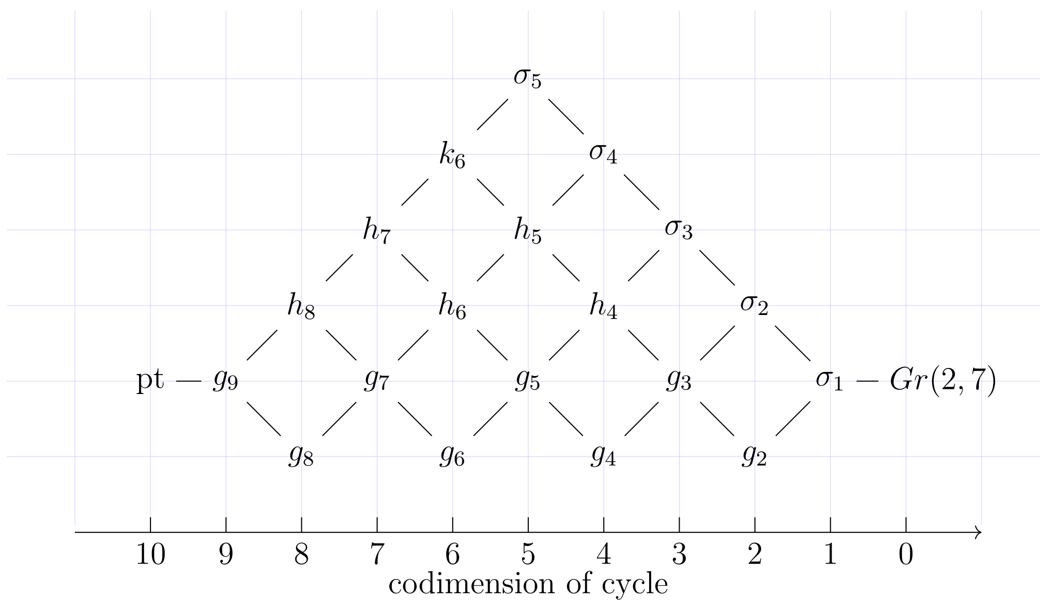
$$\begin{array}{ll}
\{2,2\} = \Delta_{(2,2)} = \sigma_2^2 - \sigma_1\sigma_3 & \{3,3\} = \Delta_{(3,3)} = \sigma_3^2 - \sigma_2\sigma_4 \\
\{3,2\} = \Delta_{(3,2)} = \sigma_3\sigma_2 - \sigma_1\sigma_4 & \{4,3\} = \Delta_{(4,3)} = \sigma_4\sigma_3 - \sigma_2\sigma_5 \\
\{4,2\} = \Delta_{(4,2)} = \sigma_4\sigma_2 - \sigma_1\sigma_5 & \{5,3\} = \Delta_{(5,3)} = \sigma_5\sigma_3 \\
\{5,2\} = \Delta_{(5,2)} = \sigma_5\sigma_2 &
\end{array}$$

$$\begin{array}{ll}
\{4,4\} = \Delta_{(4,4)} = \sigma_4^2 - \sigma_3\sigma_5 & \{5,5\} = \Delta_{5,5} = [\mathrm{pt}] \\
\{5,4\} = \Delta_{(5,4)} = \sigma_5\sigma_4 &
\end{array}$$

We use the following shorthand notation:

$$\begin{array}{rllll} 1 = Gr(2,7) = \{0,0\} & \sigma_1 = \{1,0\} & g_2 = \{1,1\} & h_4 = \{3,1\} & k_6 = \{5,1\} \\ \text{pt} = \{5,5\} & \sigma_2 = \{2,0\} & g_3 = \{2,1\} & h_5 = \{4,1\} & \\ & \sigma_3 = \{3,0\} & g_4 = \{2,2\} & h_6 = \{4,2\} & \\ & \sigma_4 = \{4,0\} & g_5 = \{3,2\} & h_7 = \{5,2\} & \\ & \sigma_5 = \{5,0\} & g_6 = \{3,3\} & h_8 = \{5,3\} & \\ & & g_7 = \{4,3\} & & \\ & & g_8 = \{4,4\} & & \\ & & g_9 = \{5,4\} & & \end{array}$$

where the lower index is indicative of the codimension of the Schubert class. In fact, these generating cycles correspond to the vertices in the Hasse diagram of $Gr(2,7)$:



The multiplication of classes in $CH(Gr(2,7))$ is determined by Pieri's formula:

$$\Delta_{(\lambda_0,\lambda_1)} \cdot \sigma_m = \sum_{\mu} \Delta_\mu$$

where the sum is taken over all partitions $\mu = (\mu_0, \mu_1)$ such that

$$5 \geq \mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq 0 \qquad \text{and} \qquad |\mu| = |\lambda| + m$$

where $|\mu| = \mu_0 + \mu_1$ and $|\lambda| = \lambda_0 + \lambda_1$.

Example 4.4.1. For example, we show the calculation of $g_3 \cdot \sigma_2$. We have:

$$\begin{aligned} g_3 \cdot \sigma_2 &= \Delta_{(2,1)} \cdot \sigma_2 \\ &= \sum_{\mu} \Delta_\mu \qquad \text{where } 5 \geq \mu_0 \geq 2 \geq \mu_1 \geq 1 \text{ and } |\mu| = |\lambda| + m = 5 \\ &= \Delta_{(4,1)} + \Delta_{(3,2)} \\ &= h_5 + g_5 \end{aligned}$$

For a complete set of multiplicative products of the Schubert classes of $Gr(2,7)$, see Appendix A.

We now determine the Chern classes of the tautological bundle of $Gr(2,7)$ and find some relations among the special Schubert classes. Notice that $CH(Gr(2,7))$ is not a domain, i.e. it contains zero divisors. However, whenever two classes have non-empty intersection, we have that a codimension l_1 class and a codimension l_2 class intersect to form a class of codimension $l_1 + l_2$. In particular, pairs of special Schubert classes always intersect non-trivially. We can determine the total Chern class of τ_2 as follows:

$$\begin{aligned} c(\tau_2) = c(Q)^{-1} &= \frac{1}{1 + c_1(Q) + ... + c_5(Q)} \\ &= \frac{1}{1 - (-\sigma_1 - ... - \sigma_5)} \\ &= \sum_{n=0}^{\infty} (-1)^n (\sigma_1 + ... + \sigma_5)^n \\ &= 1 \underbrace{- \sigma_1}_{c_1(\tau_2)} + \underbrace{\sigma_1^2 - \sigma_2}_{c_2(\tau_2)} \underbrace{- \sigma_1^3 + 2\sigma_1\sigma_2 - \sigma_3}_{c_3(\tau_2)} \\ &\quad + \underbrace{\sigma_1^4 - 3\sigma_1^2\sigma_2 + 2\sigma_1\sigma_3 + \sigma_2^2 - \sigma_4}_{c_4(\tau_2)} \\ &\quad \underbrace{- \sigma_1^5 + 4\sigma_1^3\sigma_2 - 3\sigma_1^2\sigma_3 + 2\sigma_1\sigma_4 - 3\sigma_1\sigma_2^2 + 2\sigma_2\sigma_3 - \sigma_5}_{c_5(\tau_2)} \\ &\quad + \; ... \end{aligned}$$

Since $c_m(\tau_2) = 0$ for $m > 2$, we have the following relations:

$$\begin{aligned} \sigma_3 &= 2\sigma_1\sigma_2 - \sigma_1^3 \\ \sigma_4 &= 2\sigma_1\sigma_3 - 3\sigma_1^2\sigma_2 + \sigma_2^2 + \sigma_1^4 \\ \sigma_5 &= 2\sigma_1\sigma_4 + 2\sigma_2\sigma_3 - 3\sigma_1^2\sigma_3 - 3\sigma_1\sigma_2^2 + 4\sigma_1^3\sigma_2 - \sigma_1^5 \end{aligned}$$

and most importantly, the Chern classes:

$$\begin{aligned} c_1(\tau_2) &= -\sigma_1 \\ c_2(\tau_2) &= \sigma_1^2 - \sigma_2 = g_2 \end{aligned}$$

To describe the Chow ring $CH(\mathbb{P}^6)$, we can identify $\mathbb{P}^6$ with the factor ring $\mathbb{Z}[H]/(H^7)$ where $H = c_1(\mathcal{O}(1))$ is the class of a hyperplane section. Thus, the first Chern class of the tautological bundle of $\mathbb{P}^6$ is equal to $c_1(\tau_1) = c_1(\mathcal{O}(-1)) = -H$.

Using the formula, (Example 3.2.2 following Remark 3.2.3 (b) in [Ful])

$$c_p(\mathscr{E} \otimes \mathscr{L}) = \sum_{i=0}^{p} \binom{q-i}{p-i} c_i(\mathscr{E}) c_1(\mathscr{L})^{p-i}$$

for the p^{th} Chern class of the tensor product of a vector bundle $\mathscr{E}$ of rank q and line bundle $\mathscr{L}$, we have the following rational cycles in the Chow ring $CH^*(Gr(2,7) \times \mathbb{P}^6)$:

$$\begin{aligned}
r = c_1(\mathrm{pr}_1^*(\tau_2) \otimes \mathrm{pr}_2^*(\tau_1)) &= \sum_{i=0}^{1} \binom{2-i}{1-i} c_i(\mathrm{pr}_1^*(\tau_2)) c_1(\mathrm{pr}_2^*(\tau_1))^{1-i} \\
&= 2c_1(\mathrm{pr}_2^*(\tau_1)) + c_1(\mathrm{pr}_1^*(\tau_2)) \\
&= 2\mathrm{pr}_2^*(c_1(\tau_1)) + \mathrm{pr}_1^*(c_1(\tau_2)) \\
&= 2\mathrm{pr}_2^*(-H) + \mathrm{pr}_1^*(-\sigma_1) \\
&= 2(-1 \times H) + (-\sigma_1 \times 1) \\
&= -2 \times H + (-\sigma_1 \times 1) \\
\rho = c_2(\mathrm{pr}_1^*(\tau_2) \otimes \mathrm{pr}_2^*(\tau_1)) &= \sum_{i=0}^{2} \binom{2-i}{2-i} c_i(\mathrm{pr}_1^*(\tau_2)) c_1(\mathrm{pr}_2^*(\tau_1))^{2-i} \\
&= c_1(\mathrm{pr}_2^*(\tau_1))^2 + c_1(\mathrm{pr}_1^*(\tau_2)) c_1(\mathrm{pr}_2^*(\tau_1)) + c_2(\mathrm{pr}_1^*(\tau_2)) \\
&= \mathrm{pr}_2^*(c_1(\tau_1))^2 + \mathrm{pr}_1^*(c_1(\tau_2)) \mathrm{pr}_2^*(c_1(\tau_1)) + \mathrm{pr}_1^*(c_2(\tau_2)) \\
&= \mathrm{pr}_2^*(-H)^2 + \mathrm{pr}_1^*(-\sigma_1)\mathrm{pr}_2^*(-H) + \mathrm{pr}_1^*(g_2) \\
&= (-1 \times H)^2 + (-\sigma_1 \times 1)(-1 \times H) + (g_2 \times 1) \\
&= 1 \times H^2 + \sigma_1 \times H + g_2 \times 1
\end{aligned}$$

Notice that there is also an equivalence relation on cycles in $CH(Gr(2,7) \times \mathbb{P}^6)$. For two cycles α and β we shall write $\alpha \equiv \beta$ if there exists a cycle γ such that $\alpha - \beta = 7\gamma$. The equivalence relation $\equiv$ preserves rationality.

We have the following rational cycles in $CH(Gr(2,7)\times\mathbb{P}^6)$:

$$\begin{aligned}\rho^2 &= (1\times H^2+\sigma_1\times H+g_2\times 1)^2\\ &= (1\times H^2)^2+(\sigma_1\times H)^2+(g_2\times 1)^2+2(1\times H^2)(\sigma_1\times H)\\ &\qquad +2(1\times H^2)(g_2\times 1)+2(\sigma_1\times H)(g_2\times 1)\\ &= (1\times H^4)+(\sigma_1^2\times H^2)+(g_2^2\times 1)+(2\sigma_1\times H^3)+(2g_2\times H^2)+(2g_2\sigma_1\times H)\\ &= (1\times H^4)+(2\sigma_1\times H^3)+(\sigma_2+3g_2\times H^2)+(2g_3\times H)+(g_4\times 1)\end{aligned}$$

$$\begin{aligned}\rho^3 &= [(1\times H^4)+(2\sigma_1\times H^3)+(\sigma_2+3g_2\times H^2)+(2g_3\times H)+(g_4\times 1)]\\ &\qquad \cdot[(1\times H^2)+(\sigma_1\times H)+(g_2\times 1)]\\ &= (1\times H^6)+(\sigma_1\times H^5)+(g_2\times H^4)+(2\sigma_1\times H^5)+(2\sigma_1^2\times H^4)+(2g_2\sigma_1\times H^3)\\ &\quad +(\sigma_2+3g_2\times H^4)+(\sigma_2\sigma_1+3g_2\sigma_1\times H^3)+(g_2\sigma_2+3g_2^2\times H^2)+(2g_3\times H^3)\\ &\qquad +(2g_3\sigma_1\times H^2)+(2g_3g_2\times H)+(g_4\times H^2)+(g_4\sigma_1\times H)+(g_4g_2\times 1)\\ &\equiv (1\times H^6)+(3\sigma_1\times H^5)+(3\sigma_2+6g_2\times H^4)\\ &\quad +(\sigma_3+g_3\times H^3)+(6g_4+3h_4\times H^2)+(3g_5\times H)+(g_6\times 1)\end{aligned}$$

$$\begin{aligned}\rho^5 &\equiv \rho^3\cdot\rho^2\\ &\equiv (5\sigma_4+g_4+3h_4\times H^6)+(\sigma_5+5g_5+3h_5\times H^5)+(g_6+3h_6+5k_6\times H^4)\\ &\qquad +(5g_7+3h_7\times H^3)+(g_8+3h_8\times H^2)+(5g_9\times H)+(\mathrm{pt}\times 1)\end{aligned}$$

Now consider the correspondence product $(\rho^3)^t\circ\rho^5\in CH(Gr(2,7)\times Gr(2,7))$ where $(\rho^3)^t$ denotes the rational correspondence obtained from the transpose map. We calculate, recalling that $\deg(\alpha)=0$ for $\alpha\in CH^j(\mathbb{P}^6)$ such that $j\neq\dim(\mathbb{P}^6)$. That is, we only write terms in the expansion with factors of $\deg(\eta H^6)$ where $\eta\in\mathbb{Z}$.

$$\begin{aligned}&(\rho^3)^t\circ\rho^5\\ &\quad\equiv [(H^6\times 1)+(H^5\times 3\sigma_1)+(H^4\times 3\sigma_2+6g_2)+(H^3\times\sigma_3+g_3)\\ &\qquad +(H^2\times 6g_4+3h_4)+(H\times 3g_5)+(1\times g_6)]\circ[(5\sigma_4+g_4+3h_4\times H^6)\\ &\qquad +(\sigma_5+5g_5+3h_5\times H^5)+(g_6+3h_6+5k_6\times H^4)+(5g_7+3h_7\times H^3)\\ &\qquad +(g_8+3h_8\times H^2)+(5g_9\times H)+(\mathrm{pt}\times 1)]\\ &\quad\equiv \deg(H^6\cdot 1)(\mathrm{pt}\times 1)+\deg(H^5\cdot H)(5g_9\times 3\sigma_1)\\ &\qquad +\deg(H^4\cdot H^2)(g_8+3h_8\times 3\sigma_2+6g_2)+\deg(H^3\cdot H^3)(5g_7+3h_7\times\sigma_3+g_3)\\ &\qquad +\deg(H^2\cdot H^4)(g_6+3h_6+5k_6\times 6g_4+3h_4)\\ &\qquad +\deg(H\cdot H^5)(\sigma_5+5g_5+3h_5\times 3g_5)+\deg(1\cdot H^6)(5\sigma_4+g_4+3h_4\times g_6)\\ &\quad\equiv (\mathrm{pt}\times 1)+(g_9\times\sigma_1)+(3g_8-5h_8\times\sigma_2+2g_2)+(5g_7-4h_7\times\sigma_3+g_3)\\ &\qquad +(3g_6-5h_6+k_6\times 2g_4+h_4)+(3\sigma_5+g_5+2h_5\times g_5)+(5\sigma_4+g_4+3h_4\times g_6)\end{aligned}$$

We show that the rational cycle $(\rho^3)^t \circ \rho^5 \in CH^{10}(Gr(2,7) \times Gr(2,7))$ is indeed a projector (idempotent correspondence). For this case, we omit terms in the expansion containing $\deg(\alpha)$ for which $\alpha \in CH^j(Gr(2,7))$, $j \neq \dim(Gr(2,7)) = 10$. That is;

$$
\begin{aligned}
&((\rho^3)^t \circ \rho^5) \circ ((\rho^3)^t \circ \rho^5) \\
&\quad \equiv [(\mathrm{pt} \times 1) + (g_9 \times \sigma_1) + (3g_8 - 5h_8 \times \sigma_2 + 2g_2) + (5g_7 - 4h_7 \times \sigma_3 + g_3) \\
&\qquad + (3g_6 - 5h_6 + k_6 \times 2g_4 + h_4) + (3\sigma_5 + g_5 + 2h_5 \times g_5) + (5\sigma_4 + g_4 + 3h_4 \times g_6)] \\
&\qquad \circ [(\mathrm{pt} \times 1) + (g_9 \times \sigma_1) + (3g_8 - 5h_8 \times \sigma_2 + 2g_2) + (5g_7 - 4h_7 \times \sigma_3 + g_3) \\
&\qquad + (3g_6 - 5h_6 + k_6 \times 2g_4 + h_4) + (3\sigma_5 + g_5 + 2h_5 \times g_5) + (5\sigma_4 + g_4 + 3h_4 \times g_6)] \\
&\quad \equiv \deg(\mathrm{pt} \cdot 1)(\mathrm{pt} \times 1) + \deg(g_9 \cdot \sigma_1)(g_9 \times \sigma_1) \\
&\qquad + \deg((3g_8 - 5h_8) \cdot (\sigma_2 + 2g_2))(3g_8 - 5h_8 \times \sigma_2 + 2g_2) \\
&\qquad + \deg((5g_7 - 4h_7) \cdot (\sigma_3 + g_3))(5g_7 - 4h_7 \times \sigma_3 + g_3) \\
&\qquad + \deg((3g_6 - 5h_6 + k_6) \cdot (2g_4 + h_4))(3g_6 - 5h_6 + k_6 \times 2g_4 + h_4) \\
&\qquad + \deg((3\sigma_5 + g_5 + 2h_5) \cdot g_5)(3\sigma_5 + g_5 + 2h_5 \times g_5) \\
&\qquad + \deg((5\sigma_4 + g_4 + 3h_4) \cdot g_6)(5\sigma_4 + g_4 + 3h_4 \times g_6) \\
&\quad \equiv \deg(\mathrm{pt})(\mathrm{pt} \times 1) + \deg(\mathrm{pt})(g_9 \times \sigma_1) + \deg(\mathrm{pt})(3g_8 - 5h_8 \times \sigma_2 + 2g_2) \\
&\qquad + \deg(\mathrm{pt})(5g_7 - 4h_7 \times \sigma_3 + g_3) + \deg(\mathrm{pt})(3g_6 - 5h_6 + k_6 \times 2g_4 + h_4) \\
&\qquad + \deg(\mathrm{pt})(3\sigma_5 + g_5 + 2h_5 \times g_5) + \deg(\mathrm{pt})(5\sigma_4 + g_4 + 3h_4 \times g_6) \\
&\quad \equiv (\mathrm{pt} \times 1) + (g_9 \times \sigma_1) + (3g_8 - 5h_8 \times \sigma_2 + 2g_2) + (5g_7 - 4h_7 \times \sigma_3 + g_3) \\
&\qquad + (3g_6 - 5h_6 + k_6 \times 2g_4 + h_4) + (3\sigma_5 + g_5 + 2h_5 \times g_5) + (5\sigma_4 + g_4 + 3h_4 \times g_6) \\
&\quad \equiv ((\rho^3)^t \circ \rho^5)
\end{aligned}
$$

Since $((\rho^3)^t \circ \rho^5)$ is an integral rational projector (over $\mathbb{Z}$) in $CH^{10}(Gr(2,7) \times Gr(2,7))$, then by Corollary 4.3.9, it has the form $p_S := p \times_k k^{\mathrm{sep}}$, where p is a projector in $CH^{10}(SB_2(A) \times SB_2(A))$ and therefore, also in $\mathrm{End}(\mathcal{M}(SB_2(A)))$. Hence, we have an object $(SB_2(A), p)$ in the category $\mathcal{M}(G, \mathbb{Z})$, and thus,

$$\mathcal{M}(SB_2(A)) \simeq (SB_2(A), p) \oplus (SB_2(A), p)^c$$

Appendix A

Schubert Multiplication Table

The following tables give the complete set of multiplicative products of the Schubert classes of $Gr(2,7)$.

$\cdot$	1	σ_1	σ_2	σ_3	σ_4	σ_5	g_2	g_3	g_4
1	1	σ_1	σ_2	σ_3	σ_4	σ_5	g_2	g_3	g_4
σ_1	σ_1	$\sigma_2 + g_2$	$\sigma_3 + g_3$	$\sigma_4 + h_4$	$\sigma_5 + h_5$	k_6	g_3	$g_4 + h_4$	g_5
σ_2	σ_2	$\sigma_3 + g_3$	$\sigma_4 + g_4 + h_4$	$\sigma_5 + g_5 + h_5$	$h_6 + k_6$	h_7	h_4	$g_5 + h_5$	h_6
σ_3	σ_3	$\sigma_4 + h_4$	$\sigma_5 + g_5 + h_5$	$g_6 + h_6 + k_6$	$g_7 + h_7$	h_8	h_5	$h_6 + k_6$	h_7
σ_4	σ_4	$\sigma_5 + h_5$	$h_6 + k_6$	$g_7 + h_7$	$g_8 + h_8$	g_9	k_6	h_7	0
σ_5	σ_5	k_6	h_7	h_8	g_9	pt	0	0	0
g_2	g_2	g_3	h_4	h_5	k_6	0	g_4	g_5	g_6
g_3	g_3	$g_4 + h_4$	$g_5 + h_5$	$h_6 + k_6$	h_7	0	g_5	$g_6 + h_6$	g_7
g_4	g_4	g_5	h_6	h_7	0	0	g_6	g_7	g_8
g_5	g_5	$g_6 + h_6$	$g_7 + h_7$	h_8	0	0	g_7	$g_8 + h_8$	g_9
g_6	g_6	g_7	h_8	0	0	0	g_8	g_9	pt
g_7	g_7	$g_8 + h_8$	g_9	0	0	0	g_9	pt	0
g_8	g_8	g_9	0	0	0	0	pt	0	0
g_9	g_9	pt	0	0	0	0	0	0	0
h_4	h_4	$g_5 + h_5$	$g_6 + h_6 + k_6$	$g_7 + h_7$	h_8	0	h_6	$g_7 + h_7$	h_8
h_5	h_5	$h_6 + k_6$	$g_7 + h_7$	$g_8 + h_8$	g_9	0	h_7	h_8	0
h_6	h_6	$g_7 + h_7$	$g_8 + h_8$	g_9	0	0	h_8	g_9	0
h_7	h_7	h_8	g_9	pt	0	0	0	0	0
h_8	h_8	g_9	pt	0	0	0	0	0	0
k_6	k_6	h_7	h_8	g_9	pt	0	0	0	0
pt	pt	0	0	0	0	0	0	0	0

$\cdot$	g_5	g_6	g_7	g_8	g_9	h_4	h_5	h_6	h_7	h_8	k_6	pt
1	g_5	g_6	g_7	g_8	g_9	h_4	h_5	h_6	h_7	h_8	k_6	pt
σ_1	$g_6 + h_6$	g_7	$g_8 + h_8$	g_9	pt	$g_5 + h_5$	$h_6 + k_6$	$g_7 + h_7$	h_8	g_9	h_7	0
σ_2	$g_7 + h_7$	h_8	g_9	0	0	$g_6+h_6+k_6$	$g_7 + h_7$	$g_8 + h_8$	g_9	pt	h_8	0
σ_3	h_8	0	0	0	0	$g_7 + h_7$	$g_8 + h_8$	g_9	pt	0	g_9	0
σ_4	0	0	0	0	0	h_8	g_9	0	0	0	pt	0
σ_5	0	0	0	0	0	0	0	0	0	0	0	0
g_2	g_7	g_8	g_9	pt	0	h_6	h_7	h_8	0	0	0	0
g_3	$g_8 + h_8$	g_9	pt	0	0	$g_7 + h_7$	h_8	g_9	0	0	0	0
g_4	g_9	pt	0	0	0	h_8	0	0	0	0	0	0
g_5	pt	0	0	0	0	g_9	0	0	0	0	0	0
g_6	0	0	0	0	0	0	0	0	0	0	0	0
g_7	0	0	0	0	0	0	0	0	0	0	0	0
g_8	0	0	0	0	0	0	0	0	0	0	0	0
g_9	0	0	0	0	0	0	0	0	0	0	0	0
h_4	g_9	0	0	0	0	$g_8 + h_8$	g_9	pt	0	0	0	0
h_5	0	0	0	0	0	g_9	pt	0	0	0	0	0
h_6	0	0	0	0	0	pt	0	0	0	0	0	0
h_7	0	0	0	0	0	0	0	0	0	0	0	0
h_8	0	0	0	0	0	0	0	0	0	0	0	0
k_6	0	0	0	0	0	0	0	0	0	0	0	0
pt	0	0	0	0	0	0	0	0	0	0	0	0

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