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Stabilization of Differential Systems with Hybrid Feedback Control

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Abstract

In this paper two-dimensional systems of differential equations are considered together with their stabilization by a hybrid feedback control. A stabilizing hybrid control for an arbitrary controlled system that belongs to a certain category within two-dimensional systems is constructed as a result of this study and some stabilization proprieties of the system with the obtained hybrid control are presented.

Contents

Chapter 1

Introduction to systems of linear differential equations with control

In this chapter we will introduce some elements of control theory. But first, some notations that are used throughout this paper will be defined.

1. $C(\mathbb{R}^n)$ is the set of all continuous functions $u : [0, \infty) \to \mathbb{R}^n$;
2. $C_s(\mathbb{R}^n)$ is the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^n$;
3. $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of all linear operators from $\mathbb{R}^n$ to $\mathbb{R}^m$;
4. The set of all matrices with real entries of dimension $m \times n$ we denote by $M(m, n, \mathbb{R})$;
5. The euclidean norm $|\cdot|$ in the space $\mathbb{R}^n$ is defined by $|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$;.
6. $\sigma(A)$ is the set of all eigenvalues of a square matrix A, called the spectrum of A.

1.1 Solution of the system $\dot{x} = Ax$ and its exponential estimate

Consider the following linear differential system

$$\dot{x} = Ax, \qquad t \in [0, \infty) \tag{1.1}$$

where $A \in M(n, n, \mathbb{R})$.
Let us present some facts that will be useful for the purpose of this paper. (see [2], [5])

Theorem 1.1.1. *The solution of the linear system* (1.1) *with the initial condition* $x(0) = x_0$ *exists and is uniquely defined by*

$$x(t) = e^{At}x_0.$$

Corollary 1.1.1. *For any solution of the linear system* (1.1)

$$x(t) = e^{A(t-s)}x(s), \qquad t, s \in [0, \infty),$$

where the matrix exponential e^A *is defined by*

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots$$

Theorem 1.1.2. *If the matrix A has pairwise distinct eigenvalues, then any of the solutions of the system* (1.1) *satisfies the exponential estimate*

$$M_- e^{\lambda_- t}|x(0)| \leq |x(t)| \leq M_+ e^{\lambda_+ t}|x(0)|, \qquad t \geq 0, \tag{1.2}$$

where $\lambda_- = \min\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$, $\lambda_+ = \max\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\}$ *and the constants* M_- *and* M_+ *do not depend on* $x(0)$. *Furthermore, the constants* λ_+ *and* λ_- *cannot be improved in the following sense : the inequality on the right is not valid for any constant* $\lambda < \lambda_+$ *and the inequality on the left is not valid for any constant* $\lambda > \lambda_-$.

The constants λ_+ and λ_- are called *upper Lyapunov exponent* and *lower Lyapunov exponent* .

Example 1.1.1. Consider the system (1.1) with $A = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$ where a is a real parameter. This system is called the system of the *generalized harmonic oscillator.* The equation in coordinate form is

$$\begin{cases} \dot{x}_1 = & ax_1 + x_2 \\ \dot{x}_2 = & -x_1 + ax_2 \end{cases}. \tag{1.3}$$

It is more effective to present the system's solution in polar coordinates, (r, φ), giving

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta,$$

because it helps to visualize the system's trajectory.

Converting the variables into polar coordinates we have

$$\begin{cases} \dot{r}\cos\theta - r\dot{\theta}\sin\theta & = & ar\cos\theta + r\sin\theta \\ \dot{r}\sin\theta + r\dot{\theta}\cos\theta & = & -r\cos\theta + ar\sin\theta \end{cases}.$$

Multiplying the first and the second equations of the system by $\cos\theta$ and $\sin\theta$, respectively, and adding we obtain $\dot{r} = ar$.

Now, Multiplying the first and the second equations of the system by $\sin\theta$ and $\cos\theta$, subtracting the second equation from the first, we have $\dot{\theta} = -1$.

So,we get a system, that has the function $r(t)$ only in its' first equation and the function $\theta(t)$ only in the second. Solving this system, we get the solution in polar coordinates:

$$\begin{cases} r(t) = r(0)e^{at} \\ \theta(t) = \theta(0) - t \end{cases} . \tag{1.4}$$

We will draw the trajectories of the system that starts at a position $x(0) \neq 0$ on the phase plane in the figure 1.1. The trajectories will be presented separately for each case for a different sign of the parameter a .

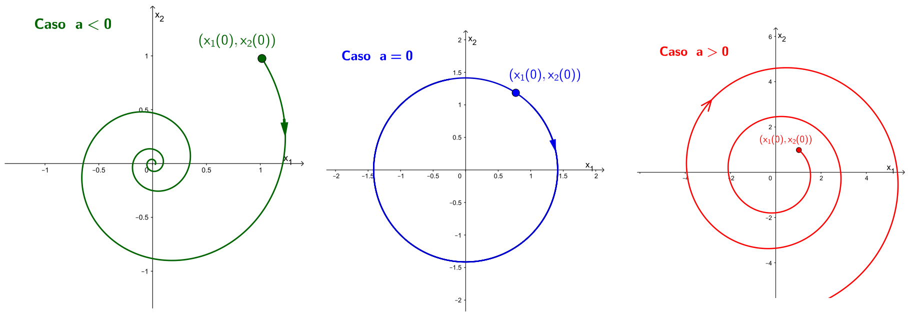

Figure 1.1: The trajectory of the system (1.3) for the cases: (a) $a < 0$, (b) $a = 0$, (c) $a > 0$.

Note that from the solution (1.4) in polar coordinates it is easy to convert to into a solution in cartesian coordinates.

$$\begin{cases} x_1(t) = e^{at}(x_1(0)\cos t + x_2(0)\sin t) \\ x_2(t) = e^{at}(-x_1(0)\sin t + x_2(0)\cos t) \end{cases} . \tag{1.5}$$

According to the theorem 1.1.1 the solution can be presented as $x(t) = e^{At}x(0)$. This means that (1.5) implies, in particular, a form of a matrix A exponential of the system:

$$A = \begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}, \qquad e^{At} = e^{at}\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}.$$

At the end of this example we illustrate Theorem 1.1.2.

Let us calculate the spectrum of the matrix A:

$$\det(A - \lambda I) = \begin{bmatrix} a-\lambda & 1 \\ -1 & a-\lambda \end{bmatrix} = (a-\lambda)^2 + 1 = 0 \quad \Rightarrow \quad \sigma(A) = \{a-i, a+i\}$$

Therefore, the Lyapunov exponents in the estimation of the solution (1.5) are

$$\lambda_- = \lambda_+ = a.$$

This fact agrees with the direct evaluation of the exponents from the first equation of the solution (1.4):

$$r(t) = r(0)e^{at} \qquad \Leftrightarrow \qquad |x(t)| = e^{at}|x(0)|.$$

(in the estimation (1.2) we can take $M_- = M_+ = 1$). From here and from the picture 1.1 it is possible to observe that the system has different asymptotic proprieties for different signs of a:

$$\begin{array}{ll} a < 0, & |x(t)| \to 0 \text{ when } t \to \infty \\ & \qquad \text{(the norm of the solution decreaces exponentially);} \\ a = 0, & |x(t)| \equiv |x(0)| \qquad \text{(the norm of the solution is constant);} \\ a > 0, & |x(t)| \to \infty \text{ when } t \to \infty \\ & \qquad \text{(the norm of the solution increaces exponentially).} \end{array}$$

1.2 Notions of control systems theory

In this section we present some theoretical background of dynamical control systems. For more details consult [17], for example.

Let us suppose that the trivial solution of the system $\dot{x} = Ax$ with $A \in M(n, n, \mathbb{R})$ is not asymptotically stable. Then our goal is, by applying a control to the system, to stabilize it (obtain asymptotic stability of the system).

How can we control the system? The control of the system depends on the nature of the process that the given system describes. One possible choice of control is to sum a vector u to the right part of the matrix equation, obtaining $\dot{x} = Ax + u$. If it were possible to change the vector u in an arbitrary way or in the way of linear dependence from the vector $x \in \mathbb{R}^n$ that characterizes the state of the system, that means in the form Gx, then we would always get the asymptotic stabilization of the system's trivial solution.

However, in practice we usually can only control a part of the state space $\mathbb{R}^n$ or add only a vector from a proper subspace of $\mathbb{R}^n$. Then the equation would take the form $\dot{x} = Ax + Bu$, where B is an $n \times \ell$ matrix, $\ell < n$, also called the control matrix.

Moreover, as a rule, the control u cannot depend on the whole trajectory $x(t)$ but only on a part of it which is the "observable" part and represents the projection of values of $x(t)$ onto a subspace of $\mathbb{R}^n$ with dimension $m < n$ that is also called the observable subspace. The observation $y \in \mathbb{R}^m$ with $m < n$ is linearly dependent on x so that $y = Cx$ where C is a $m \times n$ matrix named the *output matrix*. In general the control u only depends on the observation $y = Cx$.

Now we are going to present some definitions.

Consider the dynamic system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \tag{1.6}$$

where $x \in \mathbb{R}^n$ is the *state vector*, $y \in \mathbb{R}^m$ is the *output vector* that characterizes the observation and $u \in \mathbb{R}^\ell$ is the *input vector* or *control vector*, that characterizes the control of the system.

The trio of matrices (A, B, C) consists of the $n \times n$ *system matrix* A, the $n \times \ell$ *entry matrix* B and of the control $m \times n$ matrix C. This trio of matrices is defined by the nature of the process that the system describes and together with the control u determines completely the controlled system.

Example 1.2.1. A *controlled harmonic oscillator* is a two-dimensional system (1.6) with the trio of matrices

$$(A, B, C) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right) \tag{1.7}$$

that is, the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{cases}. \tag{1.8}$$

In this model we can only control the second equation of the system $\dot{x} = Ax$ (that means, the speed of the second component of the trajectory x_2 affects the angular acceleration $\ddot{\theta}$ of the pendulum) and observe only the first component $x_1 = \theta$

Let us clarify how it is possible to control the system (1.6). From the previous description follows that u is a function defined on the interval $[0, \infty)$ with its values in the *input space* in $\mathbb{R}^\ell$. This function is piecewise continuous (can be discontinuous at the points of an increasing sequence $\{t_i\}_{i=1}^{\infty}$, satisfying $\inf\limits_{t \in \mathbb{N}} (t_{i+1} - t_i) > 0$.) In this case we have the equation

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t \in [0, \infty). \tag{1.9}$$

The *solution of differential equation* (1.6) is a continuous function $x : [0, \infty) \to \mathbb{R}^n$ that is continuously differentiable in the intervals $(0, t_1)$, (t_i, t_{i+1}) $(i \in \mathbb{N})$ and that satisfies the equation in any of these intervals (in the points t_i the function x is continuous but can be not differentiable.)

In the model (1.9) the control u does not depend on the solution x. If $u : [0, \infty) \to \mathbb{R}$ somehow depends on the solution $x : [0, \infty) \to \mathbb{R}$, then that type of control is called *feedback control* and the system (1.6) is called a *feedback system* or a *auto-regulating system.* In the previous description it was defined as a rule that u does not depend on the whole trajectory x, but only on its observable part, that is, on the output y, more precisely, on the output function $y(t) = Cx(t)$. Therefore,in general any feedback control of the system (1.6) is uniquely defined by the operator $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$ so that

$$u(t) = (W_u y)(t), \qquad t \in [0, \infty). \tag{1.10}$$

W_u is called the *control operator*. In this case, the system (1.6) is equivalent to the functional differential equation

$$\dot{x}(t) = Ax(t) + B(W_u Cx)(t), \qquad t \in [0, \infty). \tag{1.11}$$

For any $x \in C(\mathbb{R}^n)$ the right hand side of (1.11) is a piecewise continuous function. The solution of (1.11) can be understood in the same sense as it was defined for the equation (1.9).

Example 1.2.2. For the harmonic oscillator of the example 1.2.1 the equation (1.11) in the coordinate form is

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) + (W_u x_1)(t) \end{cases}.$$

Let us go back to the general model (1.6). In case of the operator W_u that has the form of

$$(W_u y) = g(y(t)), \qquad t \in [0, \infty) \tag{1.12}$$

with some continuous function $g : \mathbb{R}^m \to \mathbb{R}^\ell$ we have a local dependency of the control u on the output y, meaning that in any specific moment of time t^* the value of the function $u : [0, \infty) \to \mathbb{R}^\ell$ at the moment t^* depends only on the value of the function $y : [0, \infty) \to \mathbb{R}^m$ at the same instant and it does not depend on the values of $y(t)$ at $t < t^*$. In this case, the equation (1.11) is an ordinary autonomous differential equation

$$\dot{x}(t) = f(x(t)), \qquad t \in [0, \infty)$$

$f(x) = Ax + Bg(Cx)$.

Particularly, when in (1.12) the function g is linear, that means $g \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$, the control operator has the matrix form

$$(W_u y)(t) = Gy(t), \qquad t \in [0, \infty) \tag{1.13}$$

with some matrix $G \in M(\ell, m, \mathbb{R})$. This type of control is called *standard linear control*. The standard linear control is the most simple of the feedback controls , and in practice it makes sense to test if this control stabilizes the system (1.6). The system (1.6) is equivalent to the differential linear system (particular case of the system (1.12))

$$\dot{x} = (A + BGC)x, \qquad t \in [0, \infty). \tag{1.14}$$

Example 1.2.3. For the controlled harmonic oscillator from the example 1.2.1 we have $\ell = m = 1$, so the matrix $G \in M(1,1)$ is a real number α, so that the equation (1.14) has the following form:

$$\dot{x} = (A+\alpha BC)x \text{ where } A+\alpha BC = \begin{bmatrix} 0 & 1 \\ -1+\alpha & 0 \end{bmatrix}, \text{ or } \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (\alpha - 1)x_1 \end{cases}$$

In general the control operator is not representable in the form of (1.12). In this case the dependency $u = W_u y$ does not have a local character. That means, the value of the function $u : [0, \infty) \to \mathbb{R}^\ell$ at a instance t^* depends not only on the value of the function $y : [0, \infty) \to \mathbb{R}^m$ at the same instance, but also on the values of $y(t)$ at the previous instances $t < t^*$. In this case, the equation (1.11) is not an ordinary differential equation anymore. In general we have a functional differential equation with a delay that depends on the solution. The study of the asymptotic proprieties of the solutions for these systems is impossible only with the methods of the ordinary differential equations theory and also the fact that the delay depends on the solution makes it more difficult to apply the ideas and the methods of the modern theory of functional differential equations ([1],[3]).

1.3 Stabilization of controllable systems. Lyapunov exponents

Let us again consider the controlled system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \tag{1.15}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output vector and $u \in \mathbb{R}^\ell$ is the control vector. According to the expression (1.10), in the system (1.15) we consider the control operator W_u.

Let us denote by $\mathcal{U}^* = \mathcal{U}^*(\ell, m)$ the set of all the possible controls, that means the set of all controls defined by all the operators $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$.

Let us denote the subset of $\mathcal{U}^*$, that consists of all linear controls of the form (1.13) by $\mathcal{LH}_1$.

To find a way to achive the desirable proprieties of the system's (1.15) trajectory with a fixed control $u \in \mathcal{U}^*$ or with one of the controls from a class $\mathcal{U} \subset \mathcal{U}^*$ has been one of the main problems in the control theory. Among the "desirable" proprieties one of the most important are the asymptotic and exponential stabilities of the system with a given upper exponent.

Let us present the definitions for the asymptotic and exponential stabilities.

Definition 1.3.1. The trivial solution of the system (1.15) with the control $u \in \mathcal{U}^*$ *is asymptotically stable* if the following statements hold:

1) $\forall \varepsilon > 0$ exists $\delta > 0$ so that for every solution $x(t)$ of the system (1.15) with the control u, satisfying $|x(0)| < \delta$ it holds that $\sup_{t \geq 0} |x(t)| < \varepsilon$;

2) for any closed and bounded set $K \subset \mathbb{R}^n$ and $\forall \varepsilon > 0$ exists $t_\varepsilon > 0$ so that for any solution $x(t)$ of the system (1.15) with the control u that at $t = 0$ satisfies $x(0) \in K$ it holds that $\sup_{t \geq t_\varepsilon} |x(t)| < \varepsilon$.

Definition 1.3.2. Given $u \in \mathcal{U}^*$, the system (1.15) is called *stabilizable* through the control u, (u*-stabilizable*) if the trivial solution of the system (1.15) with the control $u \in \mathcal{U}$ is asymptotically stable. In that case we also say that the control u *stabilizes* the system (1.15).

Given the set of controls $\mathcal{U} \subset \mathcal{U}^*$, the system (1.15) is called *stabilizable* through the family of controls $\mathcal{U}$ ($\mathcal{U}$*-stabilizable*) if there exists $u \in \mathcal{U}$ so that the system (1.15) is u-stabilizable.

Definition 1.3.3. Let (1.15) be a system with a control $u \in \mathcal{U}$. The infimum of $\lambda \in \mathbb{R}$ with which for every solution of the system it holds:

$$|x(t)| \leq M e^{\lambda t} |x(0)|, \qquad t \in [0, \infty). \tag{1.16}$$

with M positive and independent from the solution constant is called *upper Lyapunov exponent* of the system (1.15) with the control u and is denoted by $\lambda((A, B, C), u)$ (abr. $\lambda(u)$).

If there is no such $\lambda, M \in \mathbb{R}$ so that all the solutions of the system (1.15) with the control u so that (1.16) is valid, then the upper exponent is equal to $+\infty$, which means $\lambda((A, B, C), u) = +\infty$.

Definition 1.3.4. Let $\mathcal{U} \subset \mathcal{U}^*$. *Upper exponent* of the system (1.15) with the family of controls $\mathcal{U}$ is the value $\lambda((A, B, C), \mathcal{U})$ ($\lambda(\mathcal{U})$) defined by

$$\lambda((A, B, C), \mathcal{U}) = \inf_{u \in \mathcal{U}} \lambda((A, B, C), u).$$

Surely, the upper exponent is important because it characterizes the asymptotic behaviour of the solutions. For the family of controls we have the following:

a) if the exponent $\lambda(\mathcal{U})$ is positive, then it shows how it is possible to "limit the speed of growth" of the solution's $|x(t)|$ norm when $t \to \infty$ through the controls of the family $\mathcal{U}$;

b) if the exponent $\lambda(\mathcal{U})$ is negative, it shows how it is possible to achieve a quicker convergence $|x(t)| \to 0$ when $t \to \infty$, by a control from the family $\mathcal{U}$.

In particular, if (1.16) is valid with some negative exponent λ then the system (1.15) is stabilizable by u. To be more specific, the following proposition holds:

Proposition 1.3.1. *1) If $\lambda(u)<0$, then the system* (1.15) *is stabilizable by the control u.*

2) If $\lambda(\mathcal{U}) < 0$, then the system (1.15) *is* stabilizable *by the family of controls $\mathcal{U}$.*

3) The equality $\lambda(\mathcal{U}) = -\infty$ is equivalent to: $\forall R > 0$ (arbitrary big) exist $u \in \mathcal{U}$ and $M > 0$ such that the solution of the system (1.15) *with the control u satisfies the estimate*

$$|x(t)| \leq Me^{-Rt}|x(0)|, \qquad t \in [0, \infty).$$

It is clear, from the point of view of the stabilization of controllable systems, that it is good to find a class of controls $\mathcal{U}$ that is convenient for the application to a practical problem and that also satisfies the condition $\lambda(\mathcal{U}) = -\infty$.

1.4 The insufficiency of a standard linear control for the stabilization of some linear systems

As in the previous section, let us consider:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \tag{1.17}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output vector, $u \in \mathbb{R}^\ell$ is the control vector.

Let us present now the concepts of controllability and observability which are important in the control system theory ([17]).

Definition 1.4.1. The system (1.17) (and also the pair of matrices (A, B)) is called *controllable* if by the means of a sectionally continuous control $u(t)$ it is possible to take the system from any initial state x_0 to a final state x_1 at the end of a finite period of time t_1 .

More precisely, $\forall x_0, x_1 \in \mathbb{R}^n$ and $\forall t_1 > 0$ exists a $u \in C_s(\mathbb{R}^\ell)$ so that the solution $x(t)$ of the equation $\dot{x}(t) = Ax(t) + Bu(t)$ that begins at the point $x(0) = x_0$ satisfies $x(t_1) = x_1$.

Definition 1.4.2. The system (1.17) (and also the pair of matrices (A, C)) is called *observable* if by observing the values of the output $y(t) = Cx(t)$ after a finite period of time t the initial state $x(0)$ of the system can be uniquely determined.

Precisely for any $x_0, \widetilde{x}_0 \in \mathbb{R}^n$ so that $x_0 \neq \widetilde{x}_0$ for the solutions $x(t)$ and $\widetilde{x}(t)$ of the equation $\dot{x} = Ax$ that satisfy the initial conditions $x(0) = x_0$ and $\widetilde{x}(0) = \widetilde{x}_0$, holds: $Cx(t) \not\equiv C\widetilde{x}(t)$ on the interval $[0, \infty)$.

Let us present next some duality proprieties of controllability and observability and the Kalman criterion. The proofs can be found in [17].

Theorem 1.4.1. *The pair* (A, B) *is controllable if and only if the pair* $(A^\top, B^\top)$ *is observable. The pair* (A, C) *is observable if and only if the pair* $(A^\top, C^\top)$ *is controllable.*

Theorem 1.4.2. *The following statements are equivalent:*

1. *The pair* (A, B) *is controllable ;*
2. $\operatorname{rank}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix} = n$*;*
3. *For any set* K *consisting of at most* n *complex numbers satisfying condition* $z \in K \Rightarrow \overline{z} \in K$ *there exists a matrix* $G \in M(\ell, n, \mathbb{R})$ *such that* $\sigma(A + BG) = K$*.*

Theorem 1.4.3. *The following statements are equivalent:*

1. *The pair* (A, C) *is observable;*
2. $\operatorname{rank}\begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix} = n$*;*
3. *For any set* K *consisting of at most* n *complex numbers, satisfying the condition* $z \in K \Rightarrow \overline{z} \in K$ *there exists* $F \in M(n, m, \mathbb{R})$ *such that* $\sigma(A + FC) = K$*.*

The following proposition clarifies the meaning of the concepts of controllability and observability for the stabilization of systems through the standard linear control.

Theorem 1.4.4. *Let one of the following statements be valid:*
1) The pair (A, B) *is controllable and* $\operatorname{rank} C = n$*, or*
2) The pair (A, C) *is observable and* $\operatorname{rank} B = n$*.*
Then, $\lambda((A, B, C), \mathcal{LH}_1) = -\infty$*.*

Remark 1.4.1. The theorem 1.4.4 is basically a generalization of the proposition from [11], p.492 for the planar systems with $n = 2$.

Remark 1.4.2. The theorem 1.4.4 states that in case the system (1.17) is controllable then we can observe all of the trajectory components. For example, when $C = I$, that system can be stabilizable through a standard linear control, so that we can tend the solutions to zero with any negative and arbitrary large exponent by its modulo. The same propriety holds for the observable system and when the input matrix B has o rank n, for example, when $B = I$.

In practice, the situation when both matrices B and C have their ranks inferior to the dimension of the system n, for example $\ell < n$ and $m < n$,is very interesting. In this case the controllability and the observability of the system does not guarantee the stabilization of the system by a standard linear control. Let us present the following example.

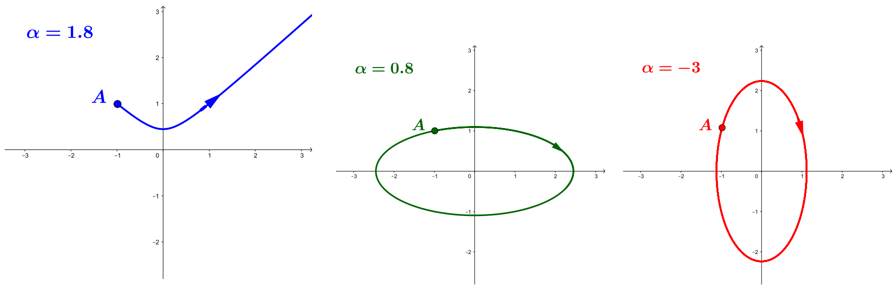

Figure 1.2: Trajectory of the system (1.18) with control $u_\alpha = \alpha y$ starting at point $A = (-1, 1)$, for cases: (a) $\alpha = 1.8$, (b) $\alpha = 0.8$, (c) $\alpha = -3$.

Example 1.4.1. Consider the controlled harmonic oscillator

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{cases} \quad \Leftrightarrow \quad \text{(1.17) with } (A, B, C) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right) \tag{1.18}$$

(see the example 1.2.1). Note that

$$\operatorname{rank}[B \ \ AB] = \operatorname{rank} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2, \qquad \operatorname{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2,$$

so the pair (A, B) is controllable and the pair (A, C) is observable.

As it was analyzed in the example 1.2.3, any $u \in \mathcal{LH}_1$ has the form of $u = \alpha y$, where $\alpha \in \mathbb{R}$, such that the system with control $u_\alpha = \alpha y$ has the form

$$\dot{x} = A_\alpha x \ \text{ where } \ A_\alpha = A + \alpha BC = \begin{bmatrix} 0 & 1 \\ \alpha - 1 & 0 \end{bmatrix}, \ \text{ that is } \ \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (\alpha - 1)x_1 \end{cases}.$$

We have

$$\sigma(A_\alpha) = \begin{cases} \{-\sqrt{\alpha - 1}, \sqrt{\alpha - 1}\} & \text{se } \ \alpha > 1 \\ \{0\} & \text{se } \ \alpha = 1 \\ \{-i\sqrt{1 - \alpha}, i\sqrt{1 - \alpha}\} & \text{se } \ \alpha < 1 \end{cases},$$

therefore $\lambda_+(\alpha) = \max\{\operatorname{Re} \lambda \colon \lambda \in \sigma(A_\alpha)\} \geq 0$. According to theorem 1.1.2 the system (1.18) is not stabilizable by a standard linear control. As an illustration, let us present the trajectories of the system (1.18) on the phase plane with the control $u_\alpha = \alpha y$, for some values of $\alpha \in \mathbb{R}$. See figure 1.2.

Remark 1.4.3. The presented example shows that there exist a two-dimensional controllable and observable systems (1.17) such that cannot be stabilizable by a standard linear control. So, it is important to chose a class of controls that generalizes the standard linear controls and allows the stabilization of the systems of the same type as the harmonic oscillator. In this case, it is pertinent to chose a control that is convenient in practice. This type of control was found in the second half of the 20th century by Z.Artshtein [4] and some other mathematicians of that time and it was given the name of *hybrid feedback control*. Let us proceed with the definition of the hybrid control and the description of some basic results of that control in the next chapter.

Chapter 2

Some elements of the hybrid feedback control theory

2.1 Description of a switching control on the example of the harmonic oscillator

Again, let us consider a system with control called the controllable harmonic oscillator

$$\left\{\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}\right. \quad \text{with} \quad (A, B, C) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right) \tag{2.1}$$

that is, the system

$$\left\{\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + u \\ y = x_1 \end{array}\right.$$

As it was shown in the example 1.4.1, the system (2.1) is not stabilizable by a standard linear control.

The trajectories of the system (2.1) with the controls $u_-, u_d \in \mathcal{LH}_1$ defined by $u_- \equiv 0$ and $u_d = -3y$ are presented in the following way:

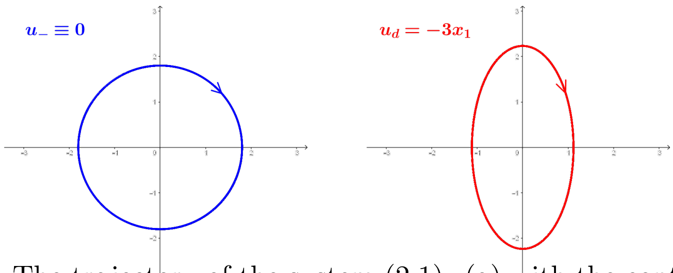

Figure 2.1: The trajectory of the system (2.1): (a) with the control $u_- \equiv 0$; (b) with the control $u_d = -3x_1$.

That way we have a circular and a ellipsoidal trajectories. It is clear that each of the controls u_- and u_d do not stabilize the system.

Let us now suppose that by means of some automaton Δ it would be possible to somehow switch the control u_- to the control u_d and vice versa. That way we obtain two automaton states: q_- and q_d. $Q = \{q_-, q_d\}$, where Q is the set of all the automaton states. When the automaton is at the state q_- we get the system (2.1) with the control u_- and when we have the automaton at the state q_d we get the system (2.1) with the control u_d:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (\alpha - 1)x_1 \end{cases} \Leftrightarrow \ \dot{x} = (A + \alpha BC)x \ \text{ where } \ \alpha = \begin{cases} 0 & \text{when} \quad q = q_- \\ -3 & \text{when} \quad q = q_d \end{cases} . \tag{2.2}$$

The switching system (2.2) is not completely defined because the commutation rule from one state to another of the automaton was not yet defined. The aim is to define it in a way that stabilizes the system. From the figure 2.1 it is evident that for a given system, the stabilization can be achieved if at the instances t that correspond to the point $x(t)$ being at the I and III quadrants of the phase plane the automaton is in the state q_d and when the point $x(t)$ is found at the II and IV quadrants of the phase plane the automaton is at the state q_-. That way we get the following switching system:

$$\dot{x} = (A + \alpha BC)x \qquad \text{where} \qquad \alpha = \begin{cases} 0 & \text{when} \quad x_1(t)x_2(t) \geq 0 \\ -3 & \text{when} \quad x_1(t)x_2(t) < 0 \end{cases} \tag{2.3}$$

Let us examine the trajectory of that system in the figure 2.2.

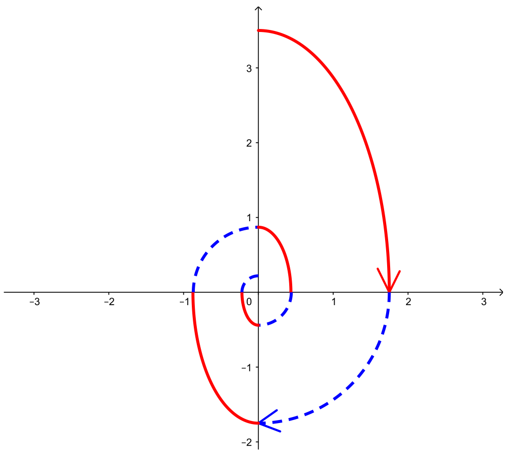

Figure 2.2: Trajectory of the switching system (2.3).

In the figure, when the state q_d is activated, the trajectory $x(t)$ of the system (2.3) is marked by the red solid line and when the q_- state is activated the trajectory is marked by a dashed blue line. The controls related to systems with commutation are usually visualized by diagrams. The diagram of the switching control u that corresponds to the system (2.3) is presented in the following figure:

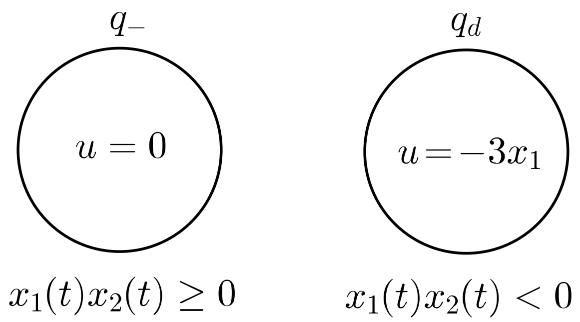

Figure 2.3: Switching control u of the system (2.3).

Clearly, the solution of the system with the control u satisfies $|x(t)| \to 0$ when $t \to +\infty$ so that the convergency is uniform in relation to the initial conditions $|x(0)| \leq R$ for any fixed $R > 0$. That way, the system (2.1) is stabilizable by the control u.

The switching control that was presented is not the only control that stabilizes the system (2.1). For example, it is possible to decrease the number of switching instances at any finite interval of time. That way, we can suggest the switching control $\widetilde{u}$ in which the system is affected by the control at the I, III and IV quadrants and is not affected by control (or affected by a null control) only in the II quadrant.

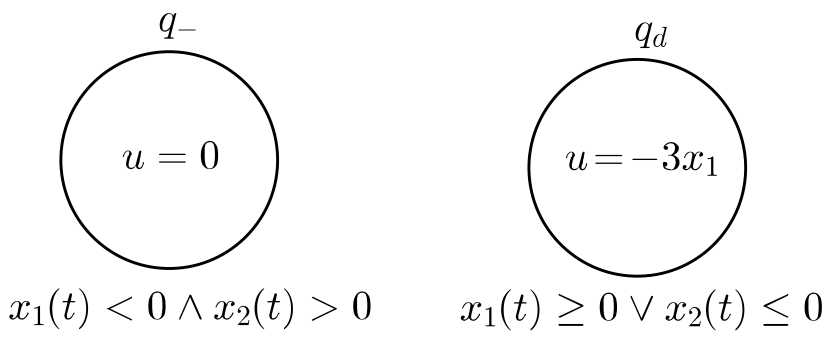

Figure 2.4: Switching control $\widetilde{u}$ of the system (2.1).

It is clear that the switching control $\widetilde{u}$ as the control u stabilizes the harmonic oscillator (2.1), see the figure below.

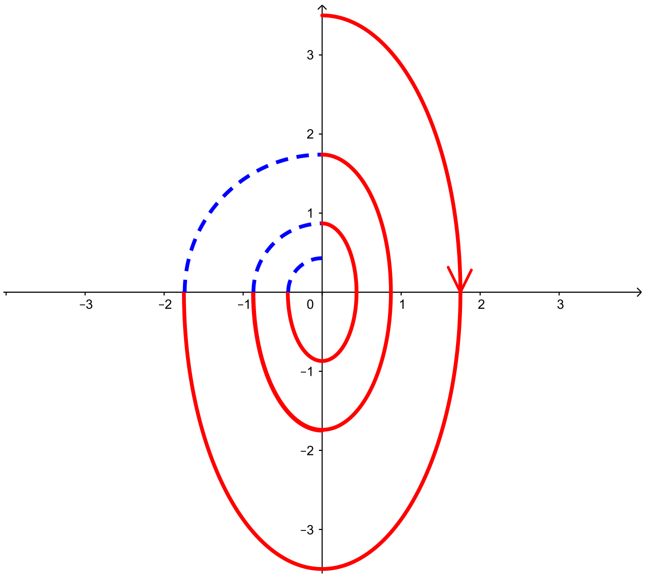

Figure 2.5: Trajectory of the system (2.1) with control $\widetilde{u}$.

2.2 From switching control to hybrid control

In the previous section the switching system and switching control were described and with the example of the harmonic oscillator it was shown that there are linear differential systems that cannot be stabilized by a standard linear control but can be stabilized by a switching control.

Even though it is possible to stabilize the system (2.1) by a control u or $\widetilde{u}$ (see figures 2.3 and 2.4), in practice, the switching control has its disadvantages. Let us describe some of them and also suggest some methods in order to overcome these disadvantages.

a) The continuous observation of the system's state $x(t)$ is not always possible, which makes it impossible to instantly switch the automaton's state from one to another. For example, for the system (2.3), the instant switching of the automaton's states at the moment when the trajectory $x(t)$ intersects the coordinate axes is only possible if the observation of the system's state $x(t)$ is continuous. If we consider the predator-prey models it becomes clear that in practice it is impossible to continuously monitor some animal populations, it is only possible at discrete instances of time.

This way, we can assign to each of the automaton's states $q \in Q$ a fixed positive period $\mathcal{T}_q$. Therefore, a switch of the automaton's states can be only done at the instances t_i that correspond to the end of the respective period of the given state of the automaton, in dependency of the system's state $x(t_i)$ at that moment. That way we have a sequence of switching instances $\{t_i\}$ and that sequence depends on the initial condition $x(0)$.

b) As it was said in the section 1.4, in the applications it is common that the ranks of input matrix B and of control matrix C are less then the systems dimension n, such that the vector Bu and the output vector $y = Cx$ take the values of the proper subspaces of $\mathbb{R}^n$,and the control u only depends on y. This situation can occur, for example, in the case of the controlled harmonic oscillator (2.1) where we can only control the second component of the trajectory $x(t)$ and observe only the first component.In this example we can vary the system by the means of a switching control only by adding a member αx_1 at the right part of the second equation of the system (see (2.2)). However, at the switching controls u and $\widetilde{u}$ in figures 2.3 and 2.4 a dependency on the complete trajectory $x(t)$ was admitted, not only on the first observable component $y(t) = x_1(t)$.

But in practice, it is natural to suppose that at the switching instances and the control u only depend on the observable part of the trajectory. The switching controls u and $\widetilde{u}$ in figures 2.3 and 2.4 do not satisfy this condition, for that they would have to be altered so that the automaton states depend only on the signum of $x_1(t)$. For the predator-prey model the idea of incomplete observation can be justified by the following:it can be easier and cheaper to monitor only the preys or only the predators, but not both.

Paradigm. To modify the switching control in correspondence to a) and b) we need to suppose that each state $q \in Q$ of the automaton has its own fixed period of time $\mathcal{T}_q > 0$. The switch from one state to another can occur exclusively at the moments t_i that correspond to the end of the period of a automaton's state and is dependant only on the output of the trajectory at that instance, that means $y(t_i) = Cx(t_i)$.

The control that satisfies the conditions described above was defined in the article [4] and was given the name of *hybrid feedback control* (HFC). The word *hybrid* means that the system has a continuous-discrete nature (a continuous trajectory $x(t)$ with a discrete sequence $\{t_i\}$ of instances for switching the states of the automaton).

Feedback means that the trajectory $x(t)$, and the state of the automata $q(t)$ at the instant t depend on the trajectory and on the previous automaton's states at the instances $s \leq t$. The exact definition of HFC is given in [8]. In [8]-[15] many results concerning the stabilization of systems by HFC were obtained.

The definition and the presentation of some elements of the theory of systems with HFC can be found in the next sections of this chapter. Let us now show some examples of HFC for the harmonic oscillator.

Consider the system (2.1) with the hybrid control u_h (see [4], [10]), presented in the following diagram:

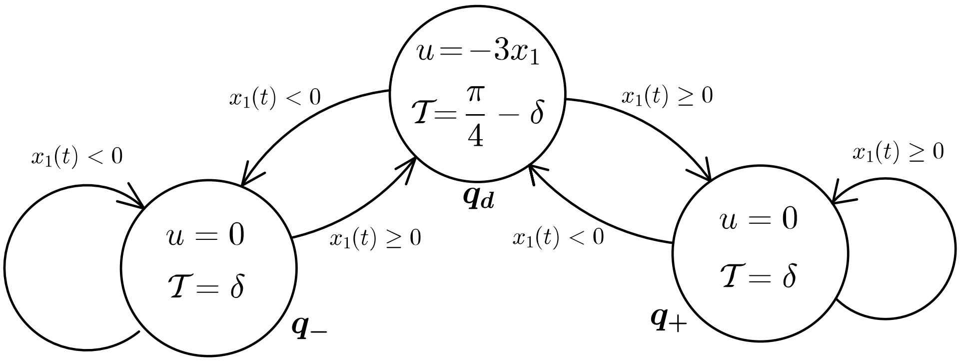

Figure 2.6: Hybrid control u_h of the system (2.3).

The automaton of the control u_h has three states: $Q = \{q_d, q_+, q_-\}$. In the circles it is presented how u is dependent on $y = x_1$ and the period of the correspondent automaton state. The arrows between the circles and the comments represent the way that the switching occurs at the switching moments. For example, if at an instant t_i the automaton switches to the state q_-, then the next switching moment is $t_{i+1} = t_i + \delta$, such that, in dependency of the $\mathrm{sign}(x_1(t))$ in that moment the switch from the state q_- to the state q_d occurs if $x_1(t_{i+1}) \geq 0$ or remains in the state q_- if $x_1(t_{i+1}) < 0$.

Note that regardless of the fact that the formulas for u and $\mathcal{T}$ are the same in the states q_- and q_+, these states are different because the switching conditions at the end of the period $\mathcal{T} = \delta$ are different.

The positive number δ is relatively small, at least $\delta \ll \frac{\pi}{4}$. Let us suppose that $\delta = \pi/20$ and that the initial state q_0 of the automaton is q_d. Let us depict in the figure 2.7 the trajectories of the system (2.1) with the hybrid control u_h that have different starting positions.

As it can be observed, the system's trajectories with the control u_h and the switching instances t_i only depend on the initial condition $x(0)$. However, from a certain instant a certain regularity can be noted: the trajectory u_d prevails at the first and the third quadrants and the solution's norm decreases. But in other periods of time, when the states q_+ and q_- are active, the solution's norm does not alter. Therefore, $|x(t)| \to 0$ when $t \to \infty$. In [13] is proved that for a small $\delta > 0$, $\lambda((A, B, C), u_h) < 0$. That means, the upper Lyapunov exponent of the system (2.1) with the control u_h is negative, in particular, the system (2.1) is u_h-stabilizable (check the definition 1.3.3 and the proposition 1.3.1).

Comparing the switching control u in figure 2.3 and the hybrid control u_h in figure 2.6 we could see that the later can be constructed on base of

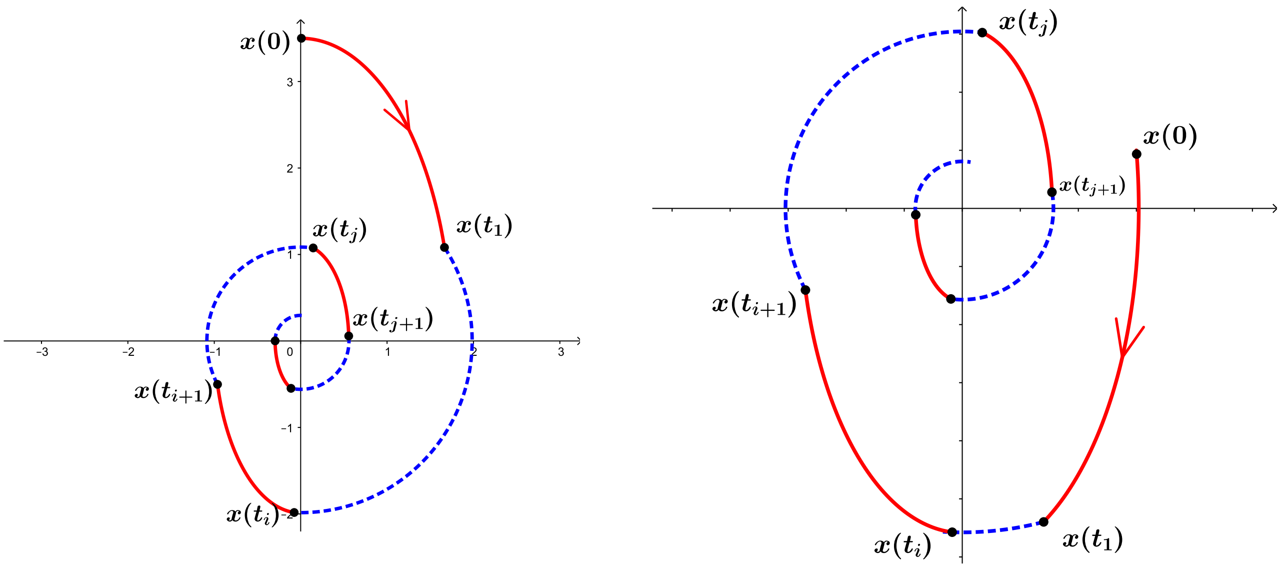

Figure 2.7: Trajectory of the system (2.1) with control u_h and initial condition : (a) $x(0) = (0,\ 3.5)^\top$; (b) $x(0) = (3,\ 0.94)^\top$.

the control u, and the control u can be also intuitively considered as a limit control of hybrid controls u_h when $\delta \to 0$.

In the same way, using the switching control figure 2.4, the hybrid control $\widetilde{u}_h$ can be found, that, in contrast with the control u_h has two automaton states: q_d and q_- (see [6], [12]).

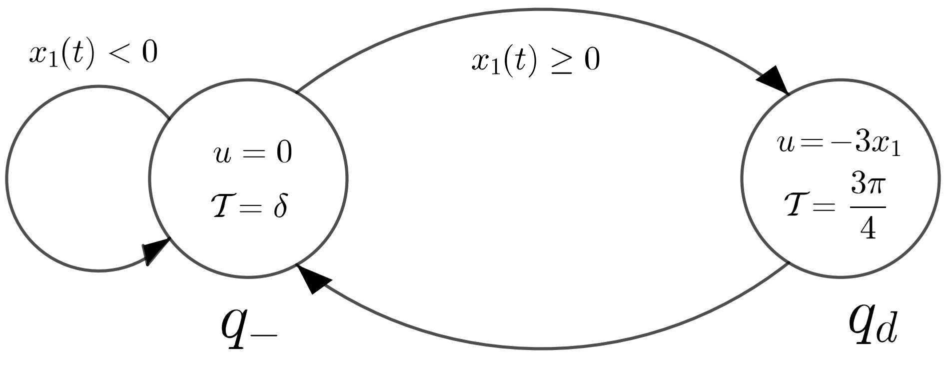

Figure 2.8: Hybrid control $\widetilde{u}_h$ of the system (2.3).

Note that the arrow without any comment and the exit from the state q_d means that at the end of the period of the state q_d the switch to the state q_- is automatic and independent from $y = x_1$.

Supposing that $\delta = \pi/10$ and the initial state q_0 of the automaton is q_d. Let us depict the trajectories of the system (2.1) with the hybrid control $\widetilde{u}_h$

that start at different $x(0) \neq 0$.

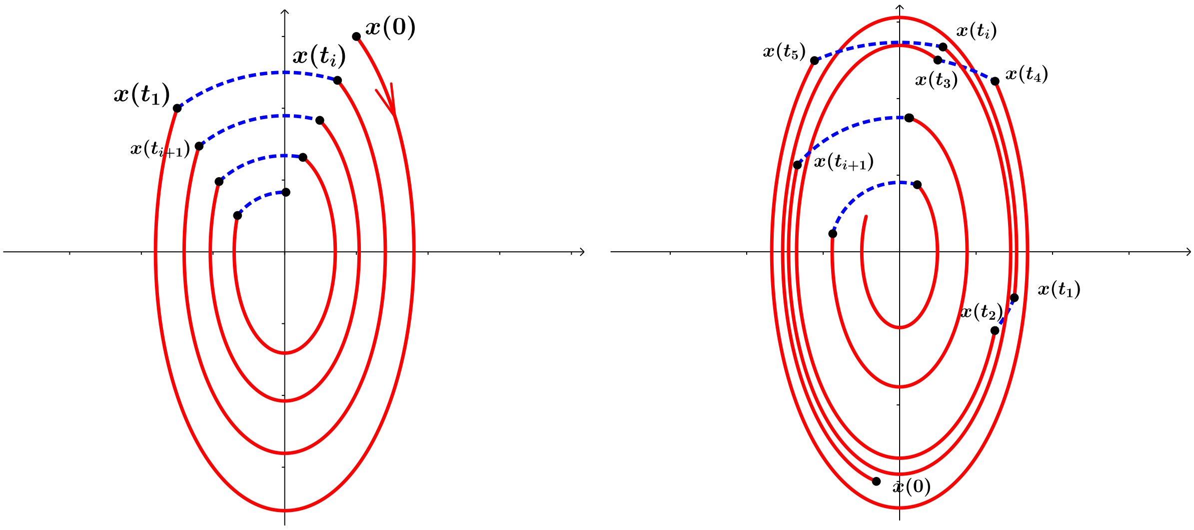

Figure 2.9: Trajectory of the system (2.1) with control $\widetilde{u}_h$ and initial condition: (a) $x(0) = (1,\ 3)^\top$; (b) $x(0) = (-0.3,\ -3)^\top$.

In [12] it was shown that for a sufficiently small $\delta > 0$

$$\lambda((A, B, C), \widetilde{u}_h) < 0,$$

which means that the system (2.1) is $\widetilde{u}_h$-stabilizable.

Furthermore, it follows from the results in [6] and [12], that if the hybrid control $u(R, \delta)$ is considered which is the generalization of the control from the figure 2.8 when the state q_d corresponds to the control $u = -Rx_1$ where $R > 0$ is a positive parameter and the period of q_d is $\mathcal{T} = \dfrac{3\pi}{2\sqrt{1+R}}$, then by changing $R > 0$ and $\delta > 0$ we can achieve the exponential estimate of the solution

$$|x(t)| \leq M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

with the constant $N > 0$ possible to choose arbitrary large. In other words, by considering the class of hybrid controls $\mathcal{A} = \{u(R, \delta)\colon R > 0,\ \delta > 0\}$, for the harmonic oscillator (2.1) we have that $\lambda((A, B, C), \mathcal{A}) = -\infty$. We will consider the HFC class $\mathcal{A}$ with more details in the last section of this chapter.

2.3 Definition of the hybrid feedback control

In this section we will present some definitions from the theory of linear differential systems with hybrid control, including the generalized definition of the HFC which is necessary for the purpose of this paper. The definitions follow from [12], [14]. For more details, consult [10].

Let us consider a controlled system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \tag{2.4}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^m$ is the output vector, $u \in \mathbb{R}^\ell$ is the control vector. The system (2.4) is completely defined by the trio of matrices (A, B, C), where $A \in M(n, n, \mathbb{R})$, $B \in M(n, \ell, \mathbb{R})$ and $C \in M(m, n, \mathbb{R})$.

Definition 2.3.1. A *hybrid automaton* is a set of six objects $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$, where

1) Q is a finite set of all the automaton's states;

2) I is a finite set called the input alphabet;

3) $M : Q \times I \to Q$ is an function that determines a new state of the automaton based on its previous state q and a element from the alphabet $i \in I$ that corresponds to the switching moment of the state;

4) $\mathcal{T} : Q \to (0, \infty)$ is a function that establishes the time period $\mathcal{T}(q)$ between two switching moments, satisfying $\inf\limits_{q \in Q} \mathcal{T}(q) > 0$;

5) $j : \mathbb{R}^m \to I$ is a function that corresponds to the output vector $y \in \mathbb{R}^m$ and the element $j(y)$ of I

6) $q_0 = q(0)$ is the automaton's initial state.

Each hybrid automaton $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ is associated to an operator $F_\Delta : P(\mathbb{R}^m) \to P(Q)$ called the *hybrid operator*. Such that $P(X)$ is a set of functions $v : [0, \infty) \to X$. Let us present the recursive definition of F_Δ.

Definition 2.3.2. For any $y(\cdot) : [0, \infty) \to \mathbb{R}^m$, the function $q(\cdot) = (F_\Delta y)(\cdot) : [0, \infty) \to Q$ is defined by:

a) $q(0) = q_0$, $t_1 = \mathcal{T}(q_0)$, $q(t) = q_0$ $(\forall t \in [0, t_1))$;

b) $q(t_1) = M(q_0, j(y(t_1)))$, $t_2 = t_1 + \mathcal{T}(q(t_1))$, $q(t) = q(t_1)$, $(\forall t \in [t_1, t_2))$;

c) Let $k \in \{2, 3, \ldots\}$. Suppose that $t_0 = 0, t_1, \ldots, t_k$ and that the values of $q(t)$ for $t \in [0, t_k)$ were already defined. Then, t_{k+1} and $q(t)$ for $t \in [t_k, t_{k+1})$ are defined by:

$$q(t_k) = M(q(t_{k-1}),\ j(y(t_k))), \quad t_{k+1} = t_k + \mathcal{T}(q(t_k)), \quad q(t) = q(t_k) \quad (\forall t \in [t_k, t_{k+1})).$$

Note that the sequence $\{t_n\}$ in the definition of F_Δ is a *sequence of switching moments* associated to the function $y(\cdot)$.

Definition 2.3.3. A pair $u = (\Delta, \Phi)$, where $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ is a hybrid automaton and $\Phi : \mathbb{R}^m \times Q \to \mathbb{R}^\ell$ is a function, is called *hybrid feedback control* (HFC).

The *hybrid control operator* $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$ (see (1.10)), associated to the control $u = (\Delta, \Phi)$, is defined by

$$(W_u y)(t) = \Phi(y(t), (F_\Delta y)(t)), \qquad t \in [0, \infty)$$

where F_Δ is the operator that was recursively defined above.

Remark 2.3.1. According to the definition 2.3.3 and to expression (1.11), the linear system (2.4) with the hybrid control $u = (\Delta, \Phi)$ is equivalent to a functional differential equation

$$\dot{x}(t) = Ax(t) + B\Phi(Cx(t), (F_\Delta Cx)(t)), \qquad t \in [0, \infty). \tag{2.5}$$

Example 2.3.1. Let us again consider the controlled harmonic oscillator

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right. \quad \text{with} \quad (A, B, C) = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right) \tag{2.6}$$

with hybrid control $\widetilde{u}_h$ defined by the diagram in figure 2.8. Let us now present the definition of $\widetilde{u}_h$ in correspondence with the definitions 2.3.1, 2.3.3. The hybrid control $\widetilde{u}_h$ is defined by $\widetilde{u}_h = ((Q, I, M, \mathcal{T}, j, q_0), \Phi)$, where

1) $Q = \{q_d, q_-\}$ is a set of two automaton's states ;

2) $I = \{i_+, i_-\}$ is the input alphabet that consist of two elements;

3) the function $M : Q \times I \to Q$ is defined by

$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \qquad M(q_-, i_+) = q_d;$$

4) The function that determines the periods of the automaton $\mathcal{T} : Q \to (0, \infty)$ is defined by $\mathcal{T}(q_d) = \frac{3\pi}{4}$, $\mathcal{T}(q_-) = \delta = \pi/10$;

5) The function $j : \mathbb{R} \to I$ is defined by $j(y) = \left\{ \begin{array}{lll} i_+ & \text{if} & y \geq 0 \\ i_- & \text{if} & y < 0 \end{array} \right.$;

6) $q_0 = q_d$ is the initial state of the automaton;

7) The function $\Phi : \mathbb{R} \times Q \to \mathbb{R}$ is defined by $\Phi(y, q) = \left\{ \begin{array}{lll} -3y & \text{if} & q = q_d \\ 0 & \text{if} & q = q_- \end{array} \right.$.

The system (2.6) with the control $\widetilde{u}_h$, is equivalent to the differential functional equation

$$\dot{x}(t) = (A + \alpha_{(F_\Delta x_1)(t)} BC)x(t), \qquad t \in [0, \infty)$$

where

$$A + \alpha_{(F_\Delta x_1)(t)} BC = \left\{ \begin{array}{lll} A - 3BC & \text{if} & (F_\Delta x_1)(t) = q_d \\ A & \text{if} & (F_\Delta x_1)(t) = q_- \end{array} \right. .$$

2.4 Linear hybrid control. Hybrid trajectory

Definition 2.4.1. Let $u = (\Delta, \Phi)$ be a hybrid control of the system (2.4), where $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$.

The HFC u is called *linear hybrid control* (LHFC) if it satisfies the following conditions:

(a) the function $j : \mathbb{R}^m \to I$, satisfies the condition $j(\lambda y) = j(y)$ for any $y \in \mathbb{R}^m$ and $\lambda > 0$;

(b) the function $\Phi(y, q)$ is linear in relation to y.

We will denote the LHFC class by $\mathcal{LH} = \mathcal{LH}(\ell, m)$.

Of course that LHFC with only one state $Q = \{q\}$ of the automaton represents a linear standard control from the section 1.2. Using the one-to-one correspondence between the set of the linear operators $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^\ell)$ and the set of matrices $M(\ell, m)$ we can reformulate the definition 2.3.3 for LHFC.

Definition 2.4.2. Let $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ a hybrid automaton in which Q and I are finite and $j(\lambda y) = j(y)$ for any $y \in \mathbb{R}^n$ and $\lambda > 0$. A pair $u = (\Delta, \{G_q\}_{q \in Q})$ where $G_q \in M(\ell, m)$ $(q \in Q)$ is called *linear hybrid control* (LHFC).

The *hybrid control operator* $W_u : C(\mathbb{R}^m) \to C_s(\mathbb{R}^\ell)$ associated with $u = (\Delta, \{G_q\}_{q \in Q})$ is defined by

$$(W_u y)(t) = G_{(F_\Delta y)(t)} y(t), t \in [0, \infty).$$

Remark 2.4.1. According to the definition 2.4.2, the linear system (2.4) with LHFC $u = (\Delta, \{G_q\}_{q \in Q})$ is equivalent to the functional differential equation

$$\dot{x}(t) = (A + BG_{(F_\Delta Cx)(t)} C)x(t), \qquad t \in [0, \infty).$$

Definition 2.4.3. In case of the two-dimensional systems (2.4), when $A \in M(2, 2, \mathbb{R})$, $B \in M(2, 1, \mathbb{R})$ and $C \in M(1, 2, \mathbb{R})$ the linear hybrid control is defined as a pair $u = (\Delta, \{\alpha_q\}_{q \in Q})$ where $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ is a hybrid automaton in which the set Q is finite, the set I contains at most three elements, the function $j : \mathbb{R} \to I$ only depends on the sign of y, such that $j(y) = j(\operatorname{sign}(y))$ $(y \in \mathbb{R})$ and $\alpha_q \in \mathbb{R}$ $(q \in Q)$.

The *hybrid control operator* $W_u : C(\mathbb{R}) \to C_s(\mathbb{R})$ associated to the LHFC $u = (\Delta, \{\alpha_q\}_{q \in Q})$ is defined by

$$(W_u y)(t) = \alpha_{(F_\Delta y)(t)} y(t), t \in [0, \infty).$$

The linear system (2.4) with LHFC u is equivalent to the functional differential equation

$$\dot{x}(t) = (A + \alpha_{(F_\Delta Cx)(t)} BC)x(t), \qquad t \in [0, \infty).$$

Remark 2.4.2. $\{t_n\}$ is the sequence of switching moments that is linked to the observation $y(\cdot) = Cx(\cdot)$ of the system (2.4) with LHFC $u = (\Delta, \{G_q\}_{q \in Q})$. Therefore, during each time interval between the switching moments $J_i = (t_i, t_{i+1})$, the automaton does not change its state, so $(F_\Delta Cx)(t)$ is a constant $q[i]$. Thus, during J_i, the dynamics of the hybrid system is simple because the equation (2.4.1) represents a linear differential equation $\dot{x} = A_i x$ with a constant matrix $A_i = A + BG_{q[i]}C$, such that, according to the corollary 1.1.1, the solution of which is defined by

$$x(t) = e^{(A+BG_{q[i]}C)(t-t_i)}x(t_i), \qquad t \in [t_i, t_{i+1}],$$

on this time interval. However, the overall dynamics of the linear system with LHFC on $[0, \infty)$ is complicated, because the function $q(t) = (F_\Delta Cx)(t)$ at the moment t depends on the observable component of the trajectory $y(s) = Cx(s)$ at all moments $s \leq t$. The equation (2.4.1) represents a functional differential equation in which the delay depends on the solution $x(\cdot)$. As it was said in the section 1.2, the study of the asymptotic proprieties of the solutions for these systems is impossible only with the methods of the ordinary differential equations ([2]) or with modern methods of differential equations theory ([1], [3]).

Example 2.4.1. The hybrid control $\widetilde{u}_h$ from the example 2.3.1 (see also the figures 2.8 and 2.9) is a linear hybrid control with two automaton states such that, $\widetilde{u}_h \in \mathcal{LH}(1,1)$. That control is defined by $\widetilde{u}_h = (\Delta, \{\alpha_q\}_{q \in Q})$ where the components of the hybrid automaton $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ were defined in 1)-6) of example 2.3.1 and

$$\alpha_{q_d} = -3, \qquad \alpha_{q_-} = 0.$$

Definition 2.4.4. Given the system (2.4) with hybrid control u. The function $h : [0, \infty) \to \mathbb{R}^n \times Q \times [0, \infty)$ defined by

$$h(t) = (x(t), q(t), \tau(t)), \qquad t \in [0, \infty)$$

is called *hybrid trajectory* of the u-controllable system (2.4) in which the first component x is the system's trajectory, this is, the solution of (2.5), $q(t) = (F_\Delta y)(t)$ the automaton's state at the moment t and the third component $\tau(t)$ is the remaining time until the next state switch.

2.5 Group of transformations GT. Classification of the linear planar systems

Let $\Sigma = M(2, 2, \mathbb{R}) \times (M(2, 1, \mathbb{R}) \setminus \{O\}) \times (M(1, 2, \mathbb{R}) \setminus \{O\})$, this means, Σ is the set of all the trios of matrices (A, B, C) where $A \in M(2, 2, \mathbb{R})$, $B \in M(2, 1, \mathbb{R})$ and and $C \in M(1, 2, \mathbb{R})$, so that B and C are non-zero matrices. Let us denote by $GL(2)$ the multiplicative group of the square non-singular real matrices of order 2.

Definition 2.5.1. We define the applications $T_1(D)$, $T_2(m_1, m_2, m_3)$ and $T_3(\alpha)$ from Σ to Σ by the formulas:

$$\begin{aligned} &T_1(D)(A, B, C) = (DAD^{-1}, DB, CD^{-1}), \quad D \in GL(2); \\ &T_2(m_1, m_2, m_3)(A, B, C) = (m_1 A, m_2 B, m_3 C), \\ &\qquad\qquad\qquad\qquad\qquad\qquad m_1 > 0,\ m_2, m_3 \in \mathbb{R} \setminus \{0\}; \\ &T_3(\alpha)(A, B, C) = (A + \alpha BC, B, C), \quad \alpha \in \mathbb{R}. \end{aligned}$$

Let us consider the set of all the applications defined above:

$$\begin{aligned} GT_0 = \{&T_1(D) \colon D \in GL(2)\} \cup \\ &\{T_2(m_1, m_2, m_3) \colon m_1 > 0;\ m_2, m_3 \in \mathbb{R} \setminus \{0\}\} \cup \{T_3(\alpha) \colon \alpha \in \mathbb{R}\}. \end{aligned}$$

It is clear that any element in $T \in GT_0$ is a bijective function $T : \Sigma \to \Sigma$, this means, is a transformation of the set Σ. Therefore, $GT_0 \subset B(\Sigma)$ where $B(\Sigma)$ is the group of all transformations on Σ with the binary operation that is the composition of transformations. In that way we defined the transformation's group GT, generated by the set GT_0. Consider the following theorem ([9],[14]).

Theorem 2.5.1. *Any transformation $T \in GT$ can be represented as*

$$T = T_1(D) \circ T_2(m_1, m_2, m_3) \circ T_3(\alpha)$$

for some matrix $D \in GL(2)$ and some constants $m_1 > 0$, $m_2, m_3 \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$. The representation $T = T_i(\cdot) \circ T_j(\cdot) \circ T_k(\cdot)$ is also valid for any of the six permutations $\{i, j, k\}$ of the set $\{1, 2, 3\}$.

The meaning of the group GT in the problem of stabilization of planar systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}, \qquad t \in [0, \infty) \tag{2.7}$$

with $\Omega = (A, B, C) \in \Sigma$ is clarified by the theorem below ([11],[14]).

Theorem 2.5.2. *Let $\Omega \in \Sigma$. Each of the predicates $P_i : \Sigma \to \{0, 1\}$ $(i = 1, 2, 3)$ is an invariant of the group GT such that*

$$\begin{aligned} &P_1(\Omega) = \{\lambda(\Omega, \mathcal{LH}_1) < 0\} = \{\Omega \textit{ is stabilizable by a standard linear control}\}, \\ &P_2(\Omega) = \{\lambda(\Omega, \mathcal{LH}) < 0\} = \{\Omega \textit{ is stabilizable by a linear hybrid control}\}, \\ &P_3(\Omega) = \{\lambda(\Omega, \mathcal{LH}) = -\infty\} = \\ &\qquad \{\Omega \textit{ is stabilizable by any LHFC with any upper Lyapunov exponent}\}. \end{aligned}$$

Note that the group GT generates the equivalence relation in Σ, so that $\Omega_1 \sim \Omega_2$ if and only if $\Omega_2 = T(\Omega_1)$ for some $T \in GT$. The theorem 2.5.2 states that in the stabilization problem of the systems through the hybrid control it is of a great importance to find the characteristic propriety of each class in terms of (A, B, C) and to highlight a representant of every class, named the *canonic form*.

Let us present the result of this classification in the form of a table that is analogue to the tables presented in [14] and [15], though by convenience, presenting only the most important proprieties among those found in the cited articles. But first, let us define some functions.

Let $\Sigma_1 = \{\Omega = (A, B, C) \in \Sigma \colon CB \neq 0\}$. The functions $\omega, \eta : \Sigma_1 \to \mathbb{R}$ defined by the formulas:

$$\omega(\Omega) = \operatorname{tr} A - \frac{CAB}{CB}, \qquad \eta(\Omega) = \frac{CAB}{CB} \cdot \omega(\Omega) - \det A.$$

Let $\Sigma_2 = \{\Omega = (A, B, C) \in \Sigma_1 \colon \eta(A, B, C) \neq 0\}$. The function $\psi : \Sigma_2 \to \mathbb{R}$ is defined by

$$\psi(\Omega) = \frac{\omega(\Omega)}{\sqrt{|\eta(\Omega)|}}.$$

Class notation	Characteristic propriety	Canonical trio	Invariants
$S(1,0,\mu)$, $\mu \in \{-1,0,1\}$	$CB{=}CAB{=}0$, $\lambda_1 = \lambda_2$, $\mu = \operatorname{sign} \lambda_1$	$\left(\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,0)$ if $\mu = -1$ $(0,0,0)$ if $\mu \in \{0,1\}$
$S(1,a,-1)$, $a \in \mathbb{R}$	$CB{=}CAB{=}0$, $\lambda_1 < \lambda_2$, $AB = \lambda_1 B$, $a = \frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}$	$\left(\begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, [1 \;\; 1]\right)$	$(1,1,0)$ if $a < -1$ $(0,0,0)$ if $a \geq -1$
$S(1,a,1)$, $a \in \mathbb{R}$	$CB{=}CAB{=}0$, $\lambda_1 < \lambda_2$, $AB = \lambda_2 B$, $a = \frac{\lambda_2+\lambda_1}{\lambda_2-\lambda_1}$	$\left(\begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, [1 \;\; -1]\right)$	$(1,1,0)$ if $a < -1$ $(0,0,0)$ if $a \geq -1$
$S(2,0,\mu)$, $\mu \in \{-1,0,1\}$	$CB = 0$, $CAB \neq 0$, $\mu = \operatorname{sign} \operatorname{tr} A$	$\left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,1)$ if $\mu = -1$ $(0,1,1)$ if $\mu \in \{0,1\}$
$S(3,0,\mu)$, $\mu \in \{-1,0,1\}$	$CB \neq 0$, $\eta(\Omega) = 0$, $\det[B \; AB] = 0$, $\det \begin{bmatrix} C \\ CA \end{bmatrix} = 0$, $\mu = \operatorname{sign} \omega(\Omega)$	$\left(\begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,0)$ if $\mu = -1$ $(0,0,0)$ if $\mu \in \{0,1\}$
$S(3,-1,\mu)$, $\mu \in \{-1,0,1\}$	$CB \neq 0$, $\nu(\Omega) = 0$, $\det[B \; AB] = 0$, $\det \begin{bmatrix} C \\ CA \end{bmatrix} \neq 0$, $\mu = \operatorname{sign} \omega(\Omega)$	$\left(\begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,0)$ if $\mu = -1$ $(0,0,0)$ if $\mu \in \{0,1\}$
$S(3,1,\mu)$, $\mu \in \{-1,0,1\}$	$CB \neq 0$, $\eta(\Omega) = 0$, $\det[B \; AB] \neq 0$, $\det \begin{bmatrix} C \\ CA \end{bmatrix} = 0$, $\mu = \operatorname{sign} \omega(\Omega)$	$\left(\begin{bmatrix} \mu & 0 \\ 1 & \mu \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,0)$ if $\mu = -1$ $(0,0,0)$ if $\mu \in \{0,1\}$
$S(4,-1,a)$, $a \in \mathbb{R}$	$CB \neq 0$, $\eta(\Omega) < 0$ $a = \psi(\Omega)$	$\left(\begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,1)$ if $a < 1$ $(0,1,1)$ if $a \geq 1$
$S(4,1,a)$, $a \in \mathbb{R}$	$CB \neq 0$, $\eta(\Omega) > 0$ $a = \psi(\Omega)$	$\left(\begin{bmatrix} a & 1 \\ 1 & a \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, [1 \;\; 0]\right)$	$(1,1,0)$ if $a < 0$ $(0,0,0)$ if $a \geq 0$

Using the table, let us select all the equivalence classes for which the system (2.7) is $\mathcal{LH}$-stabilizable but not $\mathcal{LH}_1$-stabilizable, in other words, when the system is not stabilizable by any standard linear control but is stabilizable by a hybrid control. These classes are $S(2,0,0)$ (note that the canonic form of this class is the harmonic oscillator), $S(2,0,1)$ and

$S(4,-1,a)$ for $a \geq 0$. A question arises: how, basing ourselves on the results for the canonical trios, find LHFC that would stabilize any system belonging to $S(2,0,a)$ and $S(4,-1,a)$? Currently this is an open problem. In this way the purpose of this paper is presented, which is the described problem for any system of equivalence classes that belong to the category $S(2,0,\mu)$ ($\mu \in \{-1,0,1\}$). This means, it is necessary to find a linear hybrid control for any $(A,B,C) \in \Sigma$ that satisfy the conditions $CB = 0$ and $CAB \neq 0$ for any $N > 0$, so that $\lambda(\Omega(a),u) < -N$. This problem is solved in the next chapter of this paper.

But before proceeding to the presentation of these results, a brief summary of some results for the canonical forms of the classes $S(2,0,\mu)$ published in [6] and [12] is made in the next section.

2.6 Stabilization of the generalized harmonic oscillator through a linear hybrid control

Consider the linear differential system with control:

$$\begin{cases} \dot{x} = A_\mu x + B_0 u \\ y = C_0 x \end{cases} \quad \text{with } \Omega_{[\mu]} = (A_\mu, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right) \tag{2.8}$$

this is, the system

$$\begin{cases} \dot{x}_1 = \mu x_1 + x_2 \\ \dot{x}_2 = -x_1 + \mu x_2 + u \\ y = x_1 \end{cases},$$

called the generalized harmonic oscillator. Again, note that the trio $\Omega_{[\mu]} = (A_\mu, B_0, C_0)$ of the system (2.8) is the canonical trio of the equivalence classes $H_2(2,0,\mu)$ where $\mu \in \{-1,0,1\}$. As in [12] and [6] we will not limit the study of the system to these three values of the parameter μ but will consider the system with an arbitrary parameter $\mu \in \mathbb{R}$. Let us present some basic results on the stabilization of the system (2.8) through a linear hybrid control ([6]).

Let us define LHFC $\mathcal{A}(R,\delta,m) \in \mathcal{LH}$, where $R > 0$, $\delta > 0$ and $m \in \{0,1\}$ by $\mathcal{A}(R,\delta,m) = (\Delta, \{\alpha_q\}_{q\in Q})$ where the components of the hybrid automaton $\Delta = (Q, I, M, \mathcal{T}, j, q_0)$ are given by

$$\begin{aligned}
&Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\}, \\
&M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d, \\
&\mathcal{T}(q_d) = \mathcal{T}_d(R) = \frac{3\pi}{2\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta, \\
&j(y) = \begin{cases} i_+ & \text{if } \; y \geq 0 \\ i_- & \text{if } \; y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if } \; m = 0 \\ q_d & \text{if } \; m = 1 \end{cases}
\end{aligned}$$

and $\{\alpha_q\}_{q\in Q} = \{\alpha_{q_-}, \alpha_{q_d}\}$ where $\alpha_{q_-} = 0$, $\alpha_{q_d} = -R$. The diagram hybrid control $\mathcal{A}(R,\delta,m)$ is presented in the figure 2.10.

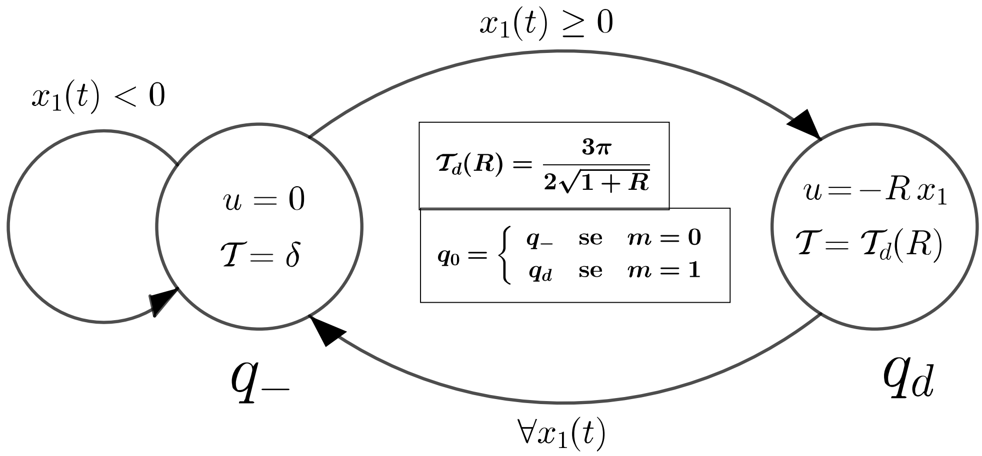

Figure 2.10: Linear hybrid control $\mathcal{A}(R, \delta, m)$ of the system (2.8).

Note that LHFC $\widetilde{u}_h$ defined in the example 2.3.1, corresponding to the figure 2.8 is a special case of the control $\mathcal{A}(R, \delta, m)$ when $R = 3$ and $m = 1$. The system's trajectories (2.8) with $\mu = 0$ and the control $\mathcal{A}(3, \pi/10, 1)$ in the phase plane that have two different initial states $x(0)$ are presented in the figure 2.9.

Let us introduce the hybrid control families

$$\mathcal{A}(R) = \left\{ \mathcal{A}(R, \delta, m) \colon 0 < \delta < \frac{\pi}{4\sqrt{1+R}},\ m \in \{0, 1\} \right\} \ (R > 0), \quad \mathcal{A} = \bigcup_{R>0} \mathcal{A}(R).$$

Of course that $\mathcal{A}(R) \subset \mathcal{A} \subset \mathcal{LH}$.

Let us define the function $\Lambda : (0, \infty) \to (0, \infty)$ by

$$\Lambda(R) = \frac{\sqrt{1+R}\,\ln(1+R)}{\pi\left(3 + \sqrt{1+R}\right)}.$$

Let us now study the assymptotic proprieties of the system's (2.8) trajectories with controls from the class $\mathcal{A}(R, \delta, m)$ ([6], [12]).

Theorem 2.6.1. *For any $R > 0$: $\lambda(\Omega(\mu), \mathcal{A}(R)) = \mu - \Lambda(R)$. If $\mu < \Lambda(R)$, then the system $\Omega(\mu)$ is stabilizable through the family of hybrid controls $\mathcal{A}(R)$, if $\mu > \Lambda(R)$ the system $\Omega(\mu)$ is not stabilizable by the family of hybrid controls $\mathcal{A}(R)$. (see the figure 2.11).*

The theorem 2.6.2 implies the main result of this section.

Theorem 2.6.2. *For any $\mu \in \mathbb{R}$ it is valid: $\lambda(\Omega(\mu), \mathcal{A}) = -\infty$.*

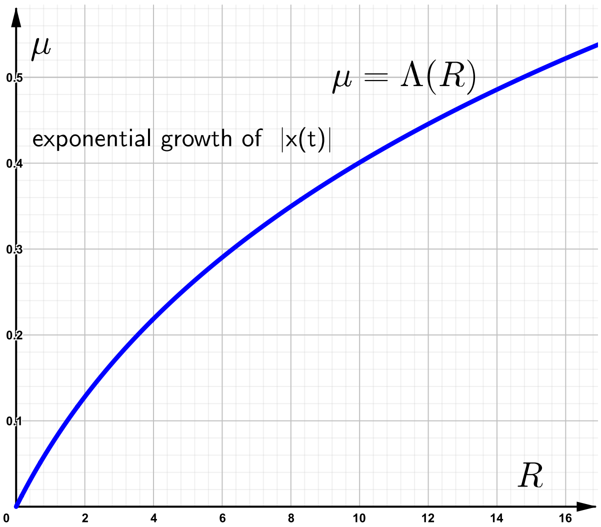

Figure 2.11: Function $\mu = \Lambda(R)$.

Remark 2.6.1. The theorem 2.6.2 states that $\forall \mu \in \mathbb{R}$ the generalized harmonic oscillator (2.8) can be stabilized by a family of controls $\mathcal{A}$, such that a negative upper Lyapunov exponent $-N$ can be chosen as large by the modulo as we define it.

Theorem 2.6.3. *Let $\mu \in \mathbb{R}$ and $N > 0$ be arbitrary constants. Therefore, for any $R > \Lambda^{-1}(\mu + N)$ exists a $\delta_0 = \delta_0(\mu, N, R) > 0$ such that $\forall \delta \in (0, \delta_0)$ and $\forall m \in \{0, 1\}$, any solution $x : [0, \infty) \to \mathbb{R}^2$ of the system $\Omega(\mu)$ governed by LHFC $\mathcal{A}(R, \delta, m)$ satisfies the exponential estimate*

$$|x(t)| \leq M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

where the constant $M = M(\mu, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

Chapter 3

Stabilization of systems for case $CB = 0$, $CAB \neq 0$ with hybrid control

3.1 Formulation of the problem

According to the classification made in the section 2.5, we have categories of systems that can be stabilized by hybrid control and a hybrid control was already constructed for the canonical cases of these categories (sections 2.6 and [12], [6]).

Specifically, the category $S(2, 0, \mu)$, which contains all the trios (A, B, C) that satisfy $BC = 0$, $CAB \neq 0$ will be examined. This category consists of three equivalence classes corresponding to cases when $\mu \in \{-1, 0, 1\}$ and the characteristic propriety of each of these classes is $CB = 0$, $CAB \neq 0$ and $\operatorname{sign} \operatorname{tr} A = \mu$. The canonical form of these classes is

$$\Omega_{[\mu]} = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right).$$

The hybrid controls $\mathcal{A}(R, \delta, m)$ that stabilize the system

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases},$$

with the canonical trio $\Omega_{[\mu]}$ and the results about the estimate of Lyapunov exponents for the system's solutions are presented in the section 2.6. By having an arbitrary trio Ω that satisfies $BC = 0$, $CAB \neq 0$ the goal is to construct a hybrid control with the trio Ω for the corresponding system, using the theorem from the next section. This means, to construct a hybrid control for an arbitrary system that belongs to the category in question. For that it is necessary to determine the parameters of the transformation T

from GT so that $T(\Omega) = \Omega_{[\mu]}$ and with the aid on the inverse transformation T^{-1}, using the results from the sections 2.6 and 3.2–3.4 find the linear hybrid control that stabilizes the system Ω with any upper Lyapunov exponent.

In summary, this chapter contains the solution for the problem described above. This is the main problem of this paper and the results presented are new.

3.2 Relation between hybrid trajectories of equivalent systems

The denotations from the section 2.5, connected to the GT transformation group will the followed.

Proposition 3.2.1. *Let the transformation $T \in GT$ be given and represented in the following form :*

$$T = T_1(D) \circ T_2(m_1, m_2, m_3) \circ T_3(\alpha)$$

for some matrix $D \in GL(2)$ and some constants $m_1 > 0$, $m_2, m_3 \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.Then, the inverse transformation T^{-1} of T is defined by

$$T^{-1} = T_3(-\alpha) \circ T_2\left(m_1^{-1}, m_2^{-1}, m_3^{-1}\right) \circ T_1(D^{-1}).$$

Theorem 3.2.1. *Let the trios $\Omega_i = (A_i, B_i, C_i) \in \Sigma$ $(i = 1, 2)$ be given, such that $\Omega_2 = T(\Omega_1)$, $T \in GT$ can be written as:*

$$T = T_3(\alpha) \circ T_2(m_1, m_2, m_3) \circ T_1(D), \tag{3.1}$$

with some matrix $D \in GL(2)$ and some constants $m_1 > 0$, $m_2, m_3 \in \mathbb{R} \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

Let us consider two controllable systems (S_1) and (S_2):

$$(S_1): \quad \begin{cases} \dot{x} = A_1 x + B_1 u \\ u = C_1 y \end{cases}, \quad \begin{array}{l} \textit{with hybrid control} \\ u_1 = (\Delta_1, \{\alpha_q^{(1)}\}_{q \in Q}) \in \mathcal{LH}(1,1), \\ \textit{where } \Delta_1 = (Q, I, M, \mathcal{T}_1, j_1, q_0), \end{array}$$

$$(S_2): \quad \begin{cases} \dot{x} = A_2 x + B_2 u \\ u = C_2 y \end{cases}, \quad \begin{array}{l} \textit{with hybrid control} \\ u_2 = (\Delta_2, \{\alpha_q^{(2)}\}_{q \in Q})) \in \mathcal{LH}(1,1), \\ \textit{where } \Delta_2 = (Q, I, M, \mathcal{T}_2, j_2, q_0), \end{array}$$

such that the components Q, I, M, q_0 of the hybrid automatons Δ_i are the same and

$$\begin{gathered} \mathcal{T}_2(q) = m_1^{-1} \mathcal{T}_1(q) \quad (\forall q \in Q), \quad j_2(y) = j_1(y \operatorname{sign} m_3) \quad (\forall y \in \mathbb{R}), \\ \alpha_q^{(2)} = \frac{m_1}{m_2 m_3} \alpha_q^{(1)} - \alpha \quad (\forall q \in Q). \end{gathered} \tag{3.2}$$

Consider the hybrid trajectories $h_i(t) = (x^{(i)}(t), q_i(t), \tau_i(t))$, $(t \in [0,\infty))$ of the systems (S_i) $(i = 1,2)$, such that the initial conditions of the components $x^{(i)}$ of these trajectories satisfy the relation $x^{(2)}(0) = Dx^{(1)}(0)$. Then, the following relations take place: $\forall t \in [0,\infty)$

$$x^{(2)}(t) = Dx^{(1)}(m_1 t), \quad q_2(t) = q_1(m_1 t), \quad \tau_2(t) = m_1^{-1}\tau_1(m_1 t).$$

The results of the theorem above follow naturally from the results that are found in [11], however, some changes were necessary because of some inaccuracy found in it.

Corollary 3.2.1. *Let us consider the same systems with hybrid controls (S_1) and (S_2) as in theorem 3.2.1. For any solution $x^{(1)}$ of the system (S_1) the exponential estimate is satisfied:*

$$|x^{(1)}(t)| \leq M_1\, e^{\lambda t}|x^{(1)}(0)|, \qquad t \in [0,\infty) \tag{3.3}$$

such that the constants $\lambda \in \mathbb{R}$ and $M_1 > 0$ that do not depend on the solutions if and only if for any solution $x^{(2)}$ of system (S_2) the exponential estimate is satisfied:

$$|x^{(2)}(t)| \leq M_2\, e^{m_1\lambda t}|x^{(2)}(0)|, \qquad t \in [0,\infty) \tag{3.4}$$

such that $M_2 > 0$ do not depend on the solution and the constant $m_1 > 0$ is the same as in the transformation (3.1)*.*

Proof. By the theorem 3.2.1, a function $x^{(1)} : [0,\infty) \to \mathbb{R}^2$ is a system's solution (S_1) if and only if the function $x^{(2)} : [0,\infty) \to \mathbb{R}^2$ defined by

$$x^{(2)}(t) = Dx^{(1)}(m_1 t), \qquad t \in [0,\infty),$$

which is the solution of the system (S_2). So, from the estimate (3.3) we have:

$$|x^{(2)}(t)| = |Dx^{(1)}(m_1 t)| \leq \|D\|\, |x^{(1)}(m_1 t)| \leq \|D\|\, M_1\, e^{m_1\lambda t}|x^{(1)}(0)| =$$

$$\|D\|\, M_1\, e^{m_1\lambda t}|D^{-1}x^{(2)}(0)| \leq M_2\, e^{m_1\lambda t}|x^{(2)}(0)|, \qquad t \in [0,\infty)$$

where $M_2 = M_1\, \|D\|\, \|D^{-1}\|$. Reciprocally, from the estimate (3.4) we have:

$$|x^{(1)}(t)| = |D^{-1}x^{(2)}(m_1^{-1}t)| \leq \|D^{-1}\|\, |x^{(2)}(m_1^{-1}t)| \leq \|D^{-1}\|\, M_2\, e^{m_1 m_1^{-1}\lambda t}|x^{(2)}(0)|$$

$$= \|D^{-1}\|\, M_2\, e^{\lambda t}|Dx^{(1)}(0)| \leq M_1\, e^{\lambda t}|x^{(1)}(0)|, \qquad t \in [0,\infty)$$

where $M_1 = M_2\, \|D^{-1}\|\, \|D\|$. □

Corollary 3.2.2. *Let us consider the same systems with the hybrid control (S_1) and (S_2) as in the theorem 3.2.1, which means, the systems with the trios $\Omega_i = (A_i, B_i, C_i)$ such that $\Omega_2 = T(\Omega_1)$ where T is defined by* (3.1) *with controls $u_i \in \mathcal{LH}$ connected by* (3.2)*.Then the upper Lyapunov exponents of (S_i) satisfy the relation:*

$$\lambda(\Omega_2, u_2) = m_1\, \lambda(\Omega_1, u_1).$$

The corollary's 3.2.2 proof follows from the corollary 3.2.1 and from the definition 1.3.3.

3.3 Transformation of the trio (A, B, C) in case $BC = 0$, $CAB \neq 0$ into canonical form

In this section the transformation $T \in GT$ will be determined in the form of a composition of the transformations $T_i(\cdot)$ $(i = 1, 2, 3)$ defined in the section 2.5 that transform a trio Ω that satisfies $BC = 0$, $CAB \neq 0$, in the canonical trio

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right), \quad \mu \in \{-1, 0, 1\} \tag{3.5}$$

Let the initial trio Ω be given and defined by

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \;\; c_2] \right)$$

such that $CB = b_1c_1 + b_2c_2 = 0$, $CAB \neq 0$. Let $\mu = \operatorname{sign}(\operatorname{tr} A)$. According to the classification presented in the section 2.5 there exists only one transformation $T \in GT$ such that $T(\Omega) = \Omega_{[\mu]}$. The goal now is to find the representation of this transformation T in terms of elements of matrices A, B and C. The problem is solved in some steps, described bellow.

1) First, the transformation $T_3(\beta)$ is applied, where

$$\beta = \begin{cases} \dfrac{2\det A - \operatorname{tr}^2 A}{2CAB} & \text{if} \quad \operatorname{tr} A \neq 0 \\ \dfrac{\det A - 1}{CAB} & \text{if} \quad \operatorname{tr} A = 0 \end{cases} = \frac{\det A - \frac{1}{2}\operatorname{tr}^2 A + |\mu| - 1}{CAB}. \tag{3.6}$$

We get a new trio

$$T_3(\beta)(\Omega) = T_3(\beta)(A, B, C) = (A + \beta BC, B, C) = (A_1, B_1, C_1) = \Omega_1.$$

As it can be noted, the only matrix that suffers some transformations is the matrix A, such that in the trio Ω_1 the matrices B_1 and C_1 are the same to the matrices B and C, respectively, from the initial trio Ω. Now the form of the matrix A_1 will be determined:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \beta \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} a_{11} + \beta b_1c_1 & a_{12} + \beta b_1c_2 \\ a_{21} + \beta b_2c_1 & a_{22} + \beta b_2c_2 \end{bmatrix}$$

The goal of applying the transformation $T_3(\beta)$ is to obtain the matrix A_1 with two complex eigenvalues which have the same real and imaginary parts by modulo. More precisely, we have

$$\sigma(A_1) = \begin{cases} \left\{ \dfrac{\operatorname{tr} A}{2} - i \cdot \dfrac{\operatorname{tr} A}{2}, \dfrac{\operatorname{tr} A}{2} + i \cdot \dfrac{\operatorname{tr} A}{2} \right\}, & \text{if} \quad \operatorname{tr} A \neq 0 \\ \{-i, i\}, & \text{if} \quad \operatorname{tr} A = 0 \end{cases}$$

Note that the idea of using the transformation $T_3(\beta)$ with the described propriety of the spectrum of A_1 can be found in [15], p.33, however, some changes were necessary due to some inaccuracy in the expressions of β and $\sigma(A_1)$.

2) Next, the transformation $T_2(\nu, 1, 1)$ is applied to the trio Ω_1 with

$$\nu = \begin{cases} \dfrac{2}{|\operatorname{tr} A|}, & \text{if} \quad \mu \in \{-1, 1\} \\ 1, & \text{if} \quad \mu = 0 \end{cases} . \tag{3.7}$$

The trio $\Omega_2 = (A_2.B_2, C_2) = T_2(\nu, 1, 1)(A_1, A_2, A_3)$ is obtained. Being that the two of the last parameters of T_2 are equal to 1, the matrices B and C remain the same. Thus, B_2 and C_2 are the same as B_1 and C_1, that are the matrices B and C from the initial trio Ω. The matrix A_2 has the following form:

$$A_2 = \nu A_1 = \begin{bmatrix} \nu(a_{11} + \beta b_1 c_1) & \nu(a_{12} + \beta b_1 c_2) \\ \nu(a_{21} + \beta b_2 c_1) & \nu(a_{22} + \beta b_2 c_2) \end{bmatrix}$$

The goal of applying the transformation $T_2(\nu, 1, 1)$ is to obtain the spectrum $\sigma(A_2) = \{\mu - i, \mu + i\}$ $(\forall \mu \in \{-1, 0, 1\})$.

3) The goal of this third step is to obtain the canonical matrix $A_{[\mu]}$, defined by (3.5) from the matrix A_2. This transformation was obtained from the theorem 9 in [7], p.299.

Let us determine a eigenvector v of the matrix A_2 associated to the eigenvalue $\lambda = \mu + i$:

$$(A_2 - (\mu + i)I)v = 0 \quad \Rightarrow$$

$$\begin{cases} \big(\nu(a_{11} + \beta b_1 c_1) - (\mu + i)\big)\, v_1 + \nu(a_{12} + \beta b_1 c_2)\, v_2 = 0 \\ \nu(a_{21} + \beta b_2 c_1)\, v_1 + \big(\nu(a_{22} + \beta b_2 c_2) - (\mu + i)\big)\, v_2 = 0 \end{cases} \Rightarrow$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{\nu(a_{12} + \beta b_1 c_2)} \begin{bmatrix} \nu(a_{12} + \beta b_1 c_2) \\ \mu - \nu(a_{11} + \beta b_1 c_1) + i \end{bmatrix},$$

and define a real matrix V by

$$V = [\operatorname{Re} v \;\; \operatorname{Im} v] = \begin{bmatrix} 1 & 0 \\ \dfrac{\mu - \nu(a_{11} + \beta b_1 c_1)}{\nu(a_{12} + \beta b_1 c_2)} & \dfrac{1}{\nu(a_{12} + \beta b_1 c_2)} \end{bmatrix}.$$

Let us now apply the transformation $T_1(D)$ for the trio Ω_2 where

$$D = V^{-1} = \begin{bmatrix} 1 & 0 \\ \nu(a_{11}+\beta b_1 c_1) - \mu & \nu(a_{12}+\beta b_1 c_2) \end{bmatrix}. \tag{3.8}$$

We obtain the trio $\Omega_3 = (A_3, B_3, C_3) = T_1(D)(\Omega_2)$, such that, (see [7],p.299),

$$A_3 = DA_2D^{-1} = V^{-1}A_2V = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}.$$

Note that the matrices B_3 and C_3 are:

$$B_3 = DB = \begin{bmatrix} b_1 \\ \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 \end{bmatrix},$$
$$C_3 = CD^{-1} = CV = \begin{bmatrix} c_1 + \dfrac{c_2(\mu - \nu(a_{11}+\beta b_1 c_1))}{\nu(a_{12}+\beta b_1 c_2)} & \dfrac{c_2}{\nu(a_{12}+\beta b_1 c_2)} \end{bmatrix}.$$

So, by the steps 1), 2) and 3) the matrix $A_3 = A_{[\mu]}$ is obtained from the canonical trio $\Omega_{[\mu]}$. The goal of the next two steps in to find the transformations from the group GT that transform B_3 and C_3, to $B_0 = [0\ \ 1]^\top$ and $C_0 = [1\ \ 0]$,conserving the matrix $A_3 = A_{[\mu]}$.

4) As it was deducted in [15], p.32, the matrix A_3 commutes with any matrix of form

$$L(\varphi, \varepsilon) = \begin{bmatrix} \varphi & \varepsilon \\ -\varepsilon & \varphi \end{bmatrix}$$

such that $L(\varphi,\varepsilon)A_3(L(\varphi,\varepsilon))^{-1} = A_3$. Let us find the values of φ and ε such that $L(\varphi,\varepsilon)B_3 = B_0 = [0\ \ 1]^\top$. Solving the linear system $L(\varphi,\varepsilon)B_3 = B_0$, this means

$$\begin{cases} b_1\,\varphi + \big(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\big)\,\varepsilon = 0 \\ \big(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\big)\,\varphi - b_1\,\varepsilon = 1 \end{cases},$$

in respect of φ and ε, we obtain

$$\varphi = \frac{\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2}, \quad \varepsilon = -\frac{b_1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2} \tag{3.9}$$

Let us now apply the transformation $T_1(L)$, where $L = L(\varphi,\varepsilon)$ with φ and ε defined by (3.9), that means

$$L = \frac{1}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2} \begin{bmatrix} \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 & -b_1 \\ b_1 & \nu(a_{11}b_1 + a_{12}b_2) - \mu b_1 \end{bmatrix}.$$

The trio $\Omega_4 = (A_4, B_4, C_4) = T_1(L)(\Omega_3)$ is obtained, where

$$A_4 = LA_3L^{-1} = A_3 = \begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \quad B_4 = LB_3 = B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_4 = C_3L^{-1} = \begin{bmatrix} \delta & 0 \end{bmatrix},$$

where

$$\delta = \left(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1\right) \cdot \left(c_1 + \frac{c_2(\mu - \nu(a_{11} + \beta b_1 c_1))}{\nu(a_{12} + \beta b_1 c_2)}\right) - \frac{b_1 c_2}{\nu(a_{12} + \beta b_1 c_2)}.$$

Simplifying the expression of δ, according to (3.6), (3.7) and $CB = b_1c_1 + b_2c_2 = 0$, we obtain

$$\delta = \nu \cdot \det[B \;\; AB] \cdot \omega(B, C) \tag{3.10}$$

sendo

$$\omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \quad b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \quad b_1 \neq 0 \end{cases}$$

Note that $-c_1/b_2 = c_2/b_1$ in case of $b_1b_2 \neq 0$, because $CB = 0$. The constant $\omega(B, C)$ has the following geometric interpretation: if consider B and $C^\top$ as vectors in $\mathbb{R}^2$, then we have $\omega(B, C) = |C^\top|/|B|$ if the angle between the vectors B and $C^\top$ are equal to $\pi/2$, and $\omega(B, C) = -|C^\top|/|B|$ if the angle between the vectors B and $C^\top$ is equal to $-\pi/2$.

5) At last, we apply the transformation $T_2(1, 1, \delta^{-1})$, obtaining the canonical trio $\Omega_{[\mu]}$ defined by (3.5).

6) Thus, a resultant transformation is presented:

$$T = T_2(1, 1, \delta^{-1}) \circ T_1(L) \circ T_1(D) \circ T_2(\nu, 1, 1) \circ T_3(\beta)$$

such that $T(\Omega) = \Omega_{[\mu]}$. By applying the propositions of the lema 2.6 from the article [9], the transformation T can be presented in a much compact form:

$$T = T_1(LD) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta).$$

such that the matrices L, D and the real constants ν, δ and β are defined in (3.3), (3.8), (3.7), (3.10) and (3.6), respectively. To conclude the formalization of T, we compute the matrix LD and simplify the expressions of its entries.

Thus, the following theorem has been proved:

Theorem 3.3.1. *Let be given a trio of matrices*

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \;\; c_2] \right)$$

where $CB = 0$ and $CAB \neq 0$ and the trio

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right),$$

where $\mu = \operatorname{sign}(\operatorname{tr} A)$. *Therefore there exists a unique transformation* $T \in GT$ *such that* $T(\Omega) = \Omega_{[\mu]}$ *and that transformation can be represented as following:*

$$T = T_1(P) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta),$$

where

$$\nu = \begin{cases} \dfrac{2}{|\operatorname{tr} A|}, & \text{if} \quad \mu \in \{-1, 1\} \\ 1, & \text{if} \quad \mu = 0 \end{cases}, \qquad \beta = \frac{\det A - \frac{1}{2}\operatorname{tr}^2 A + |\mu| - 1}{CAB},$$

$$\delta = \nu \cdot \det[B \ \ AB] \cdot \omega(B, C) \quad \textit{such that} \quad \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if} \quad b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if} \quad b_1 \neq 0 \end{cases}$$

and the elements of the matrix $P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ *are defined by*

$$p_1 = \frac{\nu(a_{12}b_2 - \beta b_1^2 c_1)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_2 = \frac{-b_1\nu(a_{12} + \beta b_1 c_2)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_3 = \frac{b_1 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)(\nu(a_{11} + \beta b_1 c_1) - \mu)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2},$$

$$p_4 = \frac{\nu(\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)(a_{12} + \beta b_1 c_2)}{b_1^2 + (\nu(a_{11}b_1 + a_{12}b_2) - \mu b_1)^2}.$$

Let us now present three examples of the trios $\Omega = (A, B, C) \in \Sigma$ from the category with the invariant $CB = 0$, $CAB \neq 0$ that belong to the three different equivalence classes $H(2, 0, \mu)$ for $\mu = 1$, $\mu = -1$ and $\mu = 0$, and construct for each of the trios, basing ourselves on the theorem 3.3.1, the transformation T that maps this trio into the canonical trio $\Omega_{[\mu]}$.

Example 3.3.1. Consider the trio of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \ \ 4] \right).$$

Of course that $CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign} 4 = 1$. So, $\Omega \in H(2, 0, 1)$. Also note that $\sigma(A) = \{2 - 3i, 2 + 3i\}$. The transformation T that maps Ω to the canonical form

$$\Omega_{[1]} = \left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right),$$

($T \in GT$, such that $T(\Omega) = (\Omega_{[1]})$) is defined by the formula:

$$T = T_1\left(\begin{bmatrix} \frac{1}{37} & \frac{4}{37} \\ \frac{19}{74} & \frac{1}{37} \end{bmatrix}\right) \circ T_2\left(\frac{1}{2}, 1, \frac{1}{37}\right) \circ T_3\left(\frac{5}{74}\right),$$

Example 3.3.2. Let us consider the trio of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix}\right).$$

$CB = 0$, $CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign}(-1) = -1$. Therefore $\Omega \in H(2, 0, -1)$. Also note that $\sigma(A) = \{-4, 3\}$. The transformation T that maps Ω into a canonical form

$$\Omega_{[-1]} = \left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0]\right),$$

is defined by:

$$T = T_1\left(\begin{bmatrix} 0 & -\frac{1}{10} \\ -1 & \frac{3}{10} \end{bmatrix}\right) \circ T_2\left(2, 1, -\frac{2}{25}\right) \circ T_3(2),$$

Example 3.3.3. Consider the trio

$$\Omega = (A, B, C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix}\right).$$

$CB = 0$,$CAB \neq 0$ and $\mu = \operatorname{sign}(\operatorname{tr} A) = \operatorname{sign} 0 = 0$. So, $\Omega \in H(2, 0, 0)$. Note, $\sigma(A) = \{-5, 5\}$. T that transforms Ω into the canonical form

$$\Omega_{[0]} = \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0]\right),$$

this means, $T \in GT$ tal que $T(\Omega) = (\Omega_{[0]})$,is defined by:

$$T = T_1\left(\begin{bmatrix} -\frac{1}{3+10\sqrt{2}} & \frac{\sqrt{2}}{3(3+10\sqrt{2})} \\ \frac{5}{3+10\sqrt{2}} & \frac{91+15\sqrt{2}}{573} \end{bmatrix}\right) \circ T_2\left(1, 1, \frac{1}{18+60\sqrt{2}}\right) \circ T_3\left(-\frac{13}{9+30\sqrt{2}}\right),$$

3.4 Inverse Transformation

Let $\Omega = (A, B, C)$ be an arbitrary trio, such that $CB = 0$, $CAB = 0$. Having the transformation

$$T_d = T_1(P) \circ T_2(\nu, 1, \delta^{-1}) \circ T_3(\beta),$$

such that $T(\Omega) = \Omega_{[\mu]}$ where $\mu = \operatorname{sign}(\operatorname{tr} A)$ (see the theorem 3.3.1), let us now determine the inverse transformation of T_d, this is, the transformation $T = T_d^{-1}$ such that $T(\Omega_{[\mu]}) = \Omega$.

According to the proposition 3.2.1 the transformation T can be represented in the following form :

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D)$$

where

$$D = P^{-1}, \qquad a = \frac{1}{\nu}, \qquad b = 1, \qquad c = \delta, \qquad \alpha = -\beta.$$

Using the formulas of the theorem 3.3.1, by rewriting the parameters of T in function of the matrices of the trio Ω, we get the following theorem:

Theorem 3.4.1. *Let the trio of matrices*

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \ \ c_2] \right)$$

be given,where $CB = 0$, $CAB \neq 0$ and the trio

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \ \ 0] \right),$$

where $\mu = \operatorname{sign}(\operatorname{tr} A)$. There exists a unique transformation $T \in GT$ such that $T(\Omega_{[\mu]}) = \Omega$ and that transformation can be represented in the following form:

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D),$$

where

$$\alpha = \frac{\frac{1}{2}\operatorname{tr}^2 A - \det A + 1 - |\mu|}{CAB}, \qquad a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu|, \qquad b = 1,$$

$$c = \frac{1}{a} \det[B \ AB]\, \omega(B, C) \ \text{ with } \ \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if } \ b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if } \ b_1 \neq 0 \end{cases}, \tag{3.11}$$

$$D = \begin{bmatrix} \dfrac{(a_{11} - a_{22})b_1 + 2a_{12}b_2}{2a} & b_1 \\ \dfrac{2a_{21}b_1 - (a_{11} - a_{22})b_2}{2a} & b_2 \end{bmatrix}.$$

For each trio from the examples 3.3.1, 3.3.2 and 3.3.3 let us present a transformation T that maps a the canonical trio to these trios. The transformation T can be obtained from the theorem 3.4.1 or by inverting the transformation that was obtained in each of the examples in the section 3.3 with the use of the proposition 3.2.1.

Example 3.4.1. Consider the trio of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \ \ 4] \right).$$

in which $CB = 0$, $CAB \neq 0$ and $\mu = \text{sign}\,(\text{tr}\,A) = 1$. The transformation $T \in GT$ such that $T(\Omega_{[1]}) = \Omega$ is defined by the formula

$$T = T_1 \left(\begin{bmatrix} -1 & 4 \\ \frac{19}{2} & -1 \end{bmatrix} \right) \circ T_2\,(2, 1, 37) \circ T_3 \left(-\frac{5}{74} \right),$$

Example 3.4.2. Consider the trio

$$\Omega = (A, B, C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix} \right).$$

such that $CB = 0$, $CAB \neq 0$ and $\mu = \text{sign}\,(\text{tr}\,A) = -1$. The transformation $T \in GT$ such that $T(\Omega_{[-1]}) = \Omega$ is defined by the formula

$$T = T_1 \left(\begin{bmatrix} -3 & -1 \\ -10 & 0 \end{bmatrix} \right) \circ T_2 \left(\frac{1}{2}, 1, -\frac{25}{2} \right) \circ T_3\,(-2),$$

Example 3.4.3. Consider the trio

$$\Omega = (A, B, C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix} \right).$$

such that $CB = 0$, $CAB \neq 0$ and $\mu = \text{sign}\,(\text{tr}\,A) = 0$. The transformation $T \in GT$ such that $T(\Omega_{[0]}) = \Omega$ is defined by the formula

$$T = T_1 \left(\begin{bmatrix} -3 - 5\sqrt{2} & \sqrt{2} \\ 15 & 3 \end{bmatrix} \right) \circ T_2 \left(1, 1, 18 + 60\sqrt{2} \right) \circ T_3 \left(\frac{13}{9 + 30\sqrt{2}} \right),$$

3.5 Construction of a stabilizing hybrid control for case $CB = 0$, $CAB \neq 0$

Consider the controllable differential linear two-dimensional system:

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1 u \\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2 u \\ y = c_1 x_1 + c_2 x_2 \end{cases} \tag{3.12}$$

where $u(\cdot) : [0, \infty) \to \mathbb{R}$ depends only from the output $u(\cdot) : [0, \infty) \to \mathbb{R}$ by a linear hybrid control. Suppose that the real parameters a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1, c_2 of the system that satisfy the conditions:

$$b_1 c_1 + b_2 c_2 = 0, \qquad a_{11}b_1c_1 + a_{12}b_2c_1 + a_{21}b_1c_2 + a_{22}b_2c_2 \neq 0. \tag{3.13}$$

This section contains the main results of this paper: the control $u \in \mathcal{LH}$ that stabilizes the system (3.12), satisfying (3.13),such that the solution's norm decreases exponentially with any Lyapunov exponent.

Note that the system (3.12) with the conditions (3.13) in its vectorial form is:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{3.14}$$

in which the trio of matrices

$$\Omega = (A, B, C) = \left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, [c_2 \;\; c_2] \right)$$

satisfies $CB = 0$ and $CAB \neq 0$. Thus, we have the trio from the class $H(2, 0, \mu)$ where $\mu = \operatorname{sign}(\operatorname{tr} A) \in \{-1, 0, 1\}$. The canonical form of the class $H(2, 0, \mu)$ is

$$\Omega_{[\mu]} = (A_{[\mu]}, B_0, C_0) = \left(\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [1 \;\; 0] \right),$$

According to the theorem 3.4.1 the transformation $T \in GT$ exists and is unique and $T(\Omega_{[\mu]}) = \Omega$. This transformation can be presented as following:

$$T = T_3(\alpha) \circ T_2(a, b, c) \circ T_1(D) \tag{3.15}$$

such that the constants α, a, b, c and the matrix D are defined by the formulas (3.11).

Let us generalize the results from the section 2.6, concerning the stabilization of the system $\Omega_{[\mu]}$ by a control $\mathcal{A}(R, \delta, m) \in \mathcal{LH}$ defined in (2.6) (see also the figure 2.10), for the system with an arbitrary trio Ω such that $CB = 0$, $CAB \neq 0$. The generalization is based on the theorems 3.2.1 and 3.4.1.

Firstly, let us define the LHFC $\mathcal{H}(\Omega, R, \delta, m) \in \mathcal{LH}$ such that $R > 0$, $\delta > 0$ and $m \in \{0, 1\}$ in the following way. If (S_1) is the system with the trio $\Omega_{[\mu]}$ and control $u_1 = \mathcal{A}(R, \delta, m)$ defined in (2.6) and (S_2) is the system with the trio Ω and control $u_2 = \mathcal{H}(\Omega, R, \delta, m)$, then the parameters of the control u_2 can be expressed by the parameters of the control u_1 using the formulas (3.2) from the theorem 3.2.1 with the use of the expressions (3.11) from the theorem 3.4.1 for the transformation parameters T (T has the form (3.15) such that $T(\Omega_{[\mu]}) = \Omega$).

Definition 3.5.1. Given $\Omega \in \Sigma$ defined by (3.14) where $CB = 0$ and $CAB \neq 0$ and given $R > 0$, $\delta > 0$ and $m \in \{0, 1\}$ the LHFC $\mathcal{H}(\Omega, R, \delta, m) \in \mathcal{LH}$ is defined by $\mathcal{H}(\Omega, R, \delta, m) = (\Delta, \{\alpha_q\}_{q \in Q})$ where the components of the hybrid automaton $\Delta = (Q, I, M, T, j, q_0)$ are given by

$$
\begin{aligned}
& Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\}, \\
& M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d, \\
& \mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{2a\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta, \\
& j(y) = \begin{cases} i_+ & \text{if} \quad \nu y \geq 0 \\ i_- & \text{if} \quad \nu y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \quad m = 0 \\ q_d & \text{if} \quad m = 1 \end{cases}
\end{aligned}
\tag{3.16}
$$

such that

$$
\begin{aligned}
a &= \frac{|\operatorname{tr} A|}{2} + 1 - |\mu|, \quad \text{where} \quad \mu = \operatorname{sign}(\operatorname{tr} A), \\
c &= \frac{1}{a} \det[B\ AB]\, \omega(B, C) \quad \text{where} \quad \omega(B, C) = \begin{cases} -\dfrac{c_1}{b_2}, & \text{if } b_2 \neq 0 \\ \dfrac{c_2}{b_1}, & \text{if } b_1 \neq 0 \end{cases}, \\
\alpha &= \frac{\frac{1}{2}\operatorname{tr}^2 A - \det A + 1 - |\mu|}{CAB}, \quad \nu = \operatorname{sign}(c),
\end{aligned}
\tag{3.17}
$$

and $\{\alpha_q\}_{q \in Q} = \{\alpha_{q_-}, \alpha_{q_d}\}$ where $\alpha_{q_-} = 0$ and $\alpha_{q_d} = -\left(\frac{a}{c} R + \alpha\right)$.

The diagram of the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$ is presented in the figure 3.1.

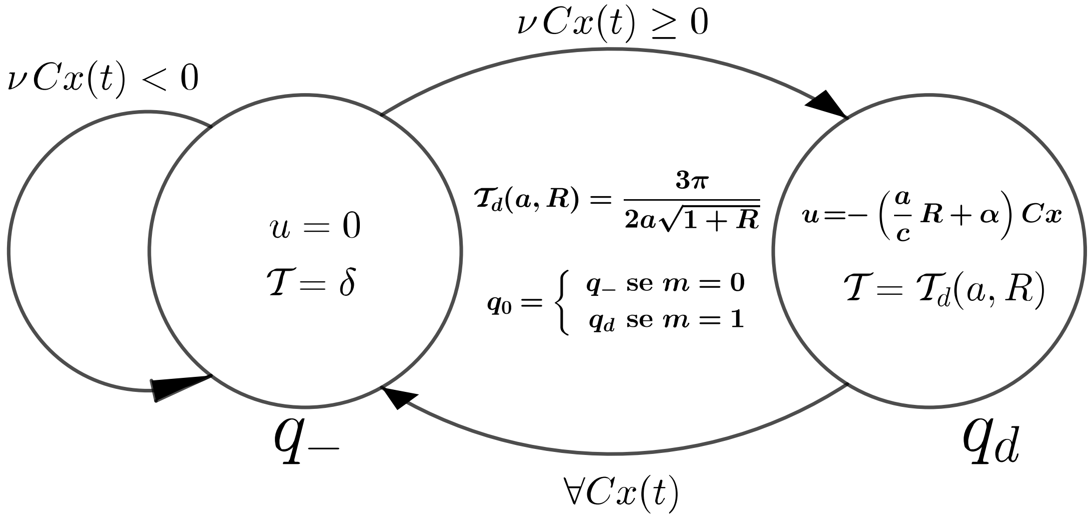

Figure 3.1: Linear hybrid control $\mathcal{H}(\Omega, R, \delta, m)$ where a, c, α, ν are defined in (3.17).

The families of hybrid controls are introduced.

$$\mathcal{H}(\Omega, R) = \left\{ \mathcal{H}(\Omega, R, \delta, m) \colon 0 < \delta < \frac{\pi}{4a\sqrt{1+R}},\ m \in \{0,1\} \right\} \quad (R > 0),$$

$$\mathcal{H}(\Omega) = \underset{R>0}{\cup} \mathcal{H}(\Omega, R).$$

It is clear that $\mathcal{H}(\Omega, R) \subset \mathcal{H}(\Omega) \subset \mathcal{LH}$.

As in the section 2.6 we define the function $\Lambda : (0, \infty) \to (0, \infty)$ by

$$\Lambda(R) = \frac{\sqrt{1+R}\ln(1+R)}{\pi\left(3+\sqrt{1+R}\right)}.$$

We remember that in this section we always consider the system (3.12) satisfying the conditions (3.13), or, indeed, the system (3.14) with trio $\Omega = (A, B, C)$ satisfying the condition $CB = 0$, $CAB \neq 0$. For convenience we designate this system for (S).

The theorems 2.6.1–2.6.3, 3.4.1 and the corollaries 3.2.1–3.2.2 imply the theorems below that consist of the main results of this paper.

Theorem 3.5.1. *For any* $R > 0$, $\lambda(\Omega, \mathcal{H}(\Omega, R)) = a(\mu - \Lambda(R))$*, where* μ *and* a *are defined in* (3.17)*.*

Theorem 3.5.2. *1) If* $\operatorname{tr} A \leq 0$ *(this is, when* $\mu = -1$ *ou* $\mu = 0$*), then* $\forall R > 0$ *the system* (S) *is stabilizable by a family of hybrid controls* $\mathcal{H}(\Omega, R)$*.*

2) If $\operatorname{tr} A > 0$ *(this is, when* $\mu = 1$*), then in case* $R > \Lambda^{-1}(1)$*, the system* (S) *is stabilizable by a family of hybrid controls* $\mathcal{H}(\Omega, R)$ *and in case* $R < \Lambda^{-1}(1)$ *the system* (S) *is not stabilizable by a family of hybrid controls* $\mathcal{A}(\Omega, R)$ *(see the figure 3.2).*

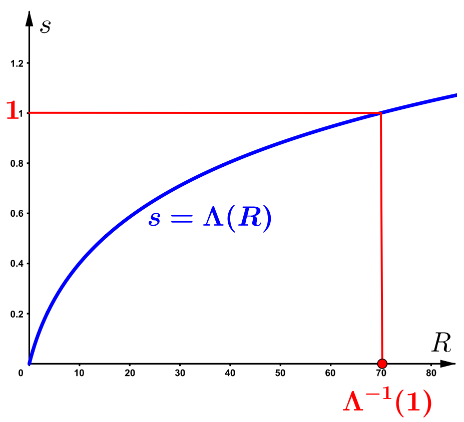

Figure 3.2: Function $s = \Lambda(R)$.

Remark 3.5.1. Note that $\Lambda^{-1}(1) \approx 69.89$.

Theorem 3.5.3. *For any $\Omega \in \Sigma$, such that $CB = 0$, $CAB \neq 0$, $\lambda(\Omega, \mathcal{H}(\Omega)) = -\infty$.*

Remark 3.5.2. According to the theorem 3.5.3, the system (S) is stabilizable by the hybrid controls from the family $\mathcal{H}(\Omega)$, such that the negative upper Lyapunov exponent in the solution estimate can be as large by modulo as we define it.

Let us complement the theorems 3.5.1–3.5.3 with a result which wording is more convenient for the applications. Also note that exists Λ^{-1} : $(0, \infty) \to (0, \infty)$ such that $\lim\limits_{s \to 0+} \Lambda^{-1}(s) = 0$ and $\lim\limits_{s \to +\infty} \Lambda^{-1}(s) = +\infty$. For convenience, let us extend the function Λ^{-1} to any set $\mathbb{R}$ by assigning,by definition $\Lambda^{-1}(s) = 0$ when $s \leq 0$.

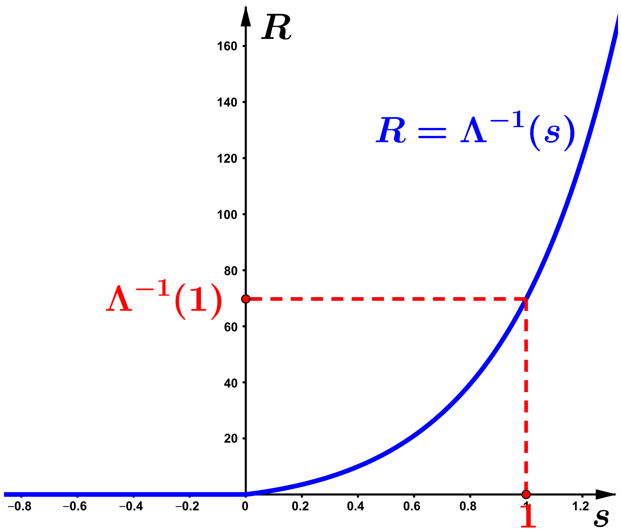

Figure 3.3: Function $R = \Lambda^{-1}(s)$.

Theorem 3.5.4. *Let $N > 0$ be an arbitrary constant.Then, for any positive number R that satisfies*

$$R > \Lambda^{-1}\left(\operatorname{sign}(\operatorname{tr} A) + N\left(\frac{|\operatorname{tr} A|}{2} + 1 - |\mu|\right)^{-1}\right) \tag{3.18}$$

where

$$\Lambda(R) = \frac{\sqrt{1+R}\ln(1+R)}{\pi\left(3+\sqrt{1+R}\right)}$$

exists $\delta_0 = \delta_0(\operatorname{tr} A, N, R) > 0$ such that $\forall \delta \in (0, \delta_0)$ and $\forall m \in \{0, 1\}$ any solution of the equation $x : [0, \infty) \to \mathbb{R}^2$ of the system (S) with LHFC

$\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \leq M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

Remark 3.5.3. The inequality (3.18) is common for the cases $\mu \in \{-1, 0, 1\}$. Of course that in each of these three cases this inequality admits a more simple form:

if $\operatorname{tr} A < 0$, then the condition (3.18) is equivalent to the condition

$$R > \Lambda^{-1}\left(-1 - \frac{2N}{\operatorname{tr} A}\right);$$

if $\operatorname{tr} A = 0$, then the condition (3.18) is equivalent to the condition

$$R > \Lambda^{-1}(N);$$

if $\operatorname{tr} A > 0$, then the condition (3.18) is equivalent to the condition

$$R > \Lambda^{-1}\left(1 + \frac{2N}{\operatorname{tr} A}\right).$$

3.6 Examples of the systems that satisfy $CB = 0$, $CAB \neq 0$ and stabilizing hybrid controls

In this section we will consider three specific systems of type (3.12) that correspond to the trios (A, B, C) considered in the examples from the sections 3.3 and 3.4. For these systems, based on the results of the section 3.5, linear hybrid controls that stabilize it will be presented. Even more, the chosen control parameters are the ones that decrease the solution's norm as in (3.5.4) with a given upper Lyapunov exponent$-N$. Let us first define the function $\Lambda : (0, \infty) \to (0, \infty)$ by

$$\Lambda(R) = \frac{\sqrt{1+R}\,\ln(1+R)}{\pi\left(3 + \sqrt{1+R}\right)} \tag{3.19}$$

(see Figure 3.2). For convenience, consider that the function Λ^{-1} is prolonged to $\mathbb{R}$ where by definition, $\Lambda^{-1}(s) = 0$ when $s \leq 0$ (see Figure 3.3).

Example 3.6.1. Consider the system:

$$\begin{cases} \dot{x}_1 = x_1 - 2x_2 + 4u \\ \dot{x}_2 = 5x_1 + 3x_2 - u \\ y = x_1 + 4x_2 \end{cases} \tag{3.20}$$

or, in the vectorial form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{with} \quad \Omega = (A, B, C) = \left(\begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, [1 \ \ 4] \right). \tag{3.21}$$

We have $CB = 0$ and $CAB \neq 0$. Let us compute the constants μ, a, c, ν, α by the formulas (3.17):

$$\mu = \operatorname{sign}(\operatorname{tr} A) = 1, \quad a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu| = 2, \quad c = \frac{c_2}{ab_1} \det[B \ \ AB] = 37,$$
$$\nu = \operatorname{sign}(c) = 1, \qquad \alpha = \frac{\frac{1}{2} \operatorname{tr}^2 A - \det A + 1 - |\mu|}{CAB} = -\frac{5}{74}. \tag{3.22}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}$ defined in the section 3.5. According to the definition 3.5.1 and the expressions (3.22), the control components are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$
$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$
$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{4\sqrt{1 + R}}, \quad \mathcal{T}(q_-) = \delta,$$
$$j(y) = \begin{cases} i_+ & \text{if} \ \ y \geq 0 \\ i_- & \text{if} \ \ y < 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \ \ m = 0 \\ q_d & \text{if} \ \ m = 1 \end{cases},$$
$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = \frac{1}{74}(5 - 4R),$$

check the diagram in the figure 3.4.

The theorems 3.5.2 and 3.5.4 imply the following conclusions about the system (3.20) with linear hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (3.19) and for all sufficiently small $\delta > 0$ the system (3.20) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(1 + N/2)$ and for all sufficiently small $\delta > 0$, any solution $x : [0, \infty) \to \mathbb{R}^2$ of the system (3.20) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \leq M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

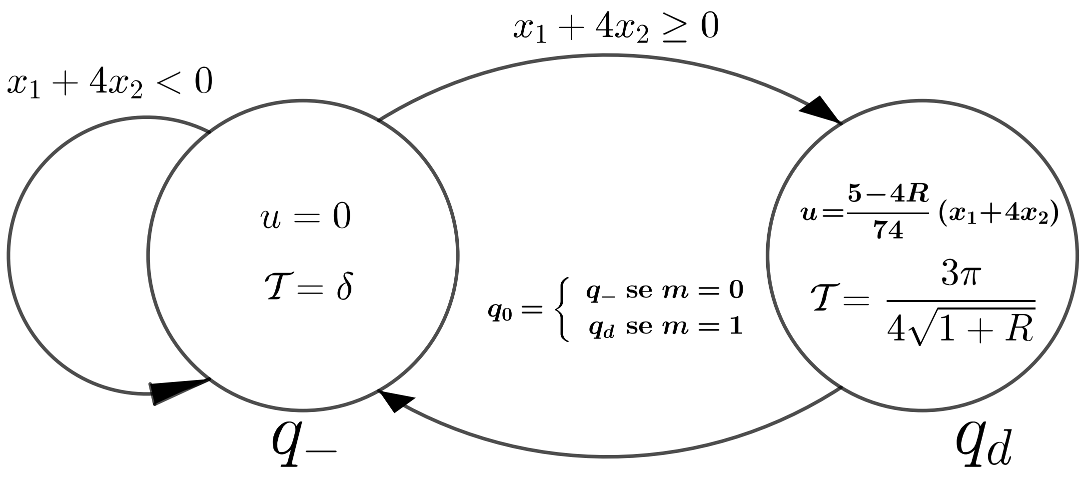

Figure 3.4: Hybrid control $\mathcal{H}(\Omega, R, \delta, m)$ for Ω defined in (3.21).

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(2) \approx 977.35$ and $\delta > 0$ is sufficiently small, then any solution x of the system (3.20) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \leq M\, e^{-2t}|x(0)|, \qquad t \in [0, \infty)$$

where $M > 0$ does not depend on the solution.

Example 3.6.2. Consider the system:

$$\begin{cases} \dot{x}_1 = x_1 + 2x_2 - u \\ \dot{x}_2 = 5x_1 - 2x_2 \\ y = \frac{5}{4}x_2 \end{cases} \tag{3.23}$$

or, in its vectorial form:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \quad \text{with} \quad \Omega = (A, B, C) = \left(\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{5}{4} \end{bmatrix} \right). \tag{3.24}$$

We have $CB = 0$ and $CAB \neq 0$. Let us compute the constants μ, a, c, ν, α by the formulas (3.17):

$$\mu = \operatorname{sign}(\operatorname{tr} A) = -1, \quad a = \frac{|\operatorname{tr} A|}{2} + 1 - |\mu| = \frac{1}{2}, \quad c = \frac{c_2}{ab_1}\det[B \;\; AB] = -\frac{25}{2},$$

$$\nu = \operatorname{sign}(c) = -1, \qquad \alpha = \frac{\frac{1}{2}(\operatorname{tr} A)^2 - \det A + 1 - |\mu|}{CAB} = -2. \tag{3.25}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}_2$ defined in the section 3.5. According to the definition 3.5.1 and the expressions (3.25), the components of this control are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$

$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$

$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{\sqrt{1+R}}, \quad \mathcal{T}(q_-) = \delta,$$

$$j(y) = \begin{cases} i_+ & \text{if} \quad y \leq 0 \\ i_- & \text{if} \quad y > 0 \end{cases}, \qquad q_0 = \begin{cases} q_- & \text{if} \quad m = 0 \\ q_d & \text{if} \quad m = 1 \end{cases},$$

$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = \frac{R}{25} + 2,$$

see the diagram in the figure 3.5.

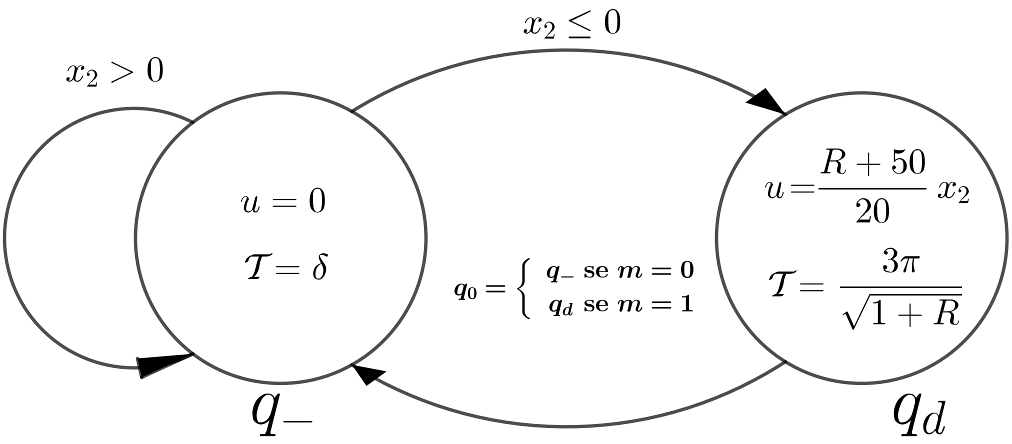

Figure 3.5: O hybrid control $\mathcal{H}(\Omega, R, \delta, m)$ for Ω defined in (3.24).

Theorems 3.5.2 and 3.5.4 imply the following about the system (3.23) with LHFC $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (3.19) and for all sufficiently small $\delta > 0$ the system (3.23) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(2N-1)$ and for all small $\delta > 0$, any solution $x : [0, \infty) \to \mathbb{R}^2$ of the system (3.23) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \leq M\, e^{-Nt} |x(0)|, \qquad t \in [0, \infty)$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(3) \approx 15545$ and $\delta > 0$ is sufficiently small, then any solution x of the system (3.23) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \leq M\, e^{-2t} |x(0)|, \qquad t \in [0, \infty)$$

where $M > 0$ does not depend on the solution.

Example 3.6.3. Consider the system

$$\left\{ \begin{array}{l} \dot{x}_1 = -5x_1 - x_2 + \sqrt{2}\, u \\ \dot{x}_2 = 5x_2 - 2x_2 + 3u \\ y = -6x_1 + 2\sqrt{2}\, x_2 \end{array} \right. \tag{3.26}$$

and in its vectorial form

$$\left\{ \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array} \right. \quad \text{with } \Omega = (A, B, C) = \left(\begin{bmatrix} -5 & -1 \\ 0 & 5 \end{bmatrix}, \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}, \begin{bmatrix} -6 & 2\sqrt{2} \end{bmatrix} \right). \tag{3.27}$$

$CB = 0$ and $CAB \neq 0$. Let us compute the constants μ, a, c, ν, α by the formulas (3.17):

$$\mu! = \text{sign}\,(\text{tr}\, A) = 0,\ a = \frac{|\,\text{tr}\, A|}{2} + 1 - |\mu| = 1,\ c = \frac{c_2}{ab_1} \det[B \;\; AB] = 18 + 60\sqrt{2},$$

$$\nu = \text{sign}\,(c) = 1, \qquad \alpha = \frac{\frac{1}{2}\,\text{tr}^2 A - \det A + 1 - |\mu|}{CAB} = \frac{13}{9 + 30\sqrt{2}}. \tag{3.28}$$

Consider the hybrid control $\mathcal{H}(\Omega, R, \delta, m) = ((Q, I, M, T, j, q_0), \{\alpha_-, \alpha_d\}) \in \mathcal{LH}_2$ defined in the section 3.5. According to the definition 3.5.1 and the expressions (3.28), the components of this control are given by:

$$Q = \{q_d, q_-\}, \qquad I = \{i_+, i_-\},$$

$$M(q_d, i_+) = M(q_d, i_-) = M(q_-, i_-) = q_-, \quad M(q_-, i_+) = q_d,$$

$$\mathcal{T}(q_d) = \mathcal{T}_d(R, a) = \frac{3\pi}{2\sqrt{1 + R}}, \quad \mathcal{T}(q_-) = \delta,$$

$$j(y) = \left\{ \begin{array}{ll} i_+ & \text{if } \; y \geq 0 \\ i_- & \text{if } \; y < 0 \end{array} \right., \qquad q_0 = \left\{ \begin{array}{ll} q_- & \text{if } \; m = 0 \\ q_d & \text{if } \; m = 1 \end{array} \right.,$$

$$\alpha_{q_-} = 0, \qquad \alpha_{q_d} = -\frac{R + 26}{6(3 + 10\sqrt{2})},$$

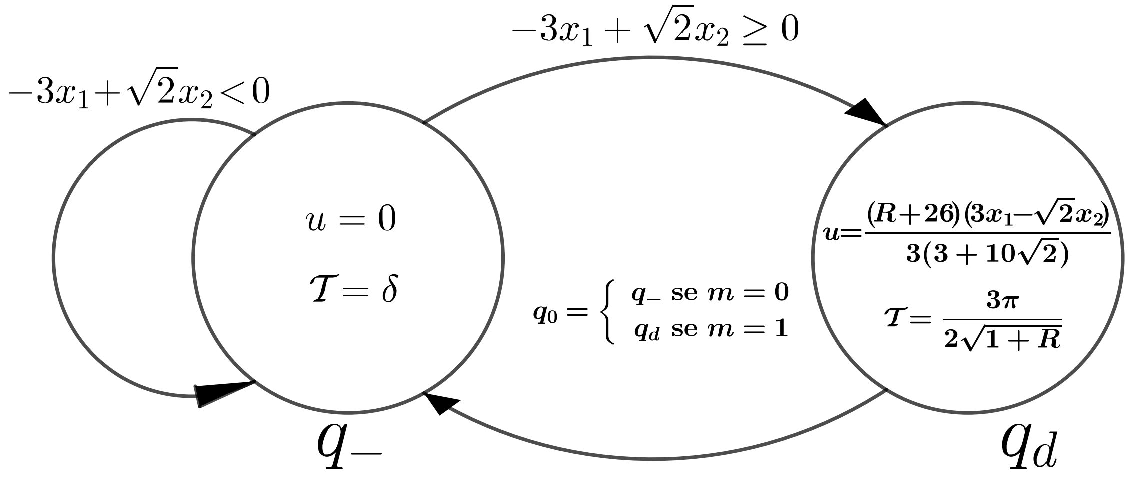

Figure 3.6: O hybrid control $\mathcal{H}(\Omega, R, \delta, m)$ para Ω definido em (3.27).

see the diagram from the figure 3.6.

Theorems 3.5.2 and 3.5.4 imply the following about the system (3.26) with LHFC $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 1. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(1) \approx 69.89$ where Λ is defined in (3.19) and for all sufficiently small $\delta > 0$ the system (3.26) is stabilizable by the hybrid control $\mathcal{H}(\Omega, R, \delta, m)$.

Conclusion 2. Let $N > 0$. For any $m \in \{0, 1\}$, $R > \Lambda^{-1}(N)$ for all small $\delta > 0$ any solution $x : [0, \infty) \to \mathbb{R}^2$ of the system (3.26) with control $\mathcal{H}(\Omega, R, \delta, m)$ satisfies the exponential estimate

$$|x(t)| \leq M\, e^{-Nt}|x(0)|, \qquad t \in [0, \infty)$$

where the constant $M = M(\Omega, R, \delta, m) > 0$ does not depend on the solution $x(\cdot)$.

For example, if a decrease of the solution with $N = 2$ is needed, then we can conclude that if $R > \Lambda^{-1}(2) \approx 977.35$ and $\delta > 0$ is sufficiently small, then any solution x of the system (3.26) with control $\mathcal{H}(\Omega, R, \delta, 0)$ or $\mathcal{H}(\Omega, R, \delta, 1)$ satisfies the condition

$$|x(t)| \leq M\, e^{-2t}|x(0)|, \qquad t \in [0, \infty)$$

where $M > 0$ does not depend on the solution.

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