On some periodic solutions of discrete vibro-impact systems with a unilateral contact condition

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Sur des solutions périodiques d'oscillateurs discrets à vibro-impact avec une unique condition de contact unilatéral

Thèse dirigée par Stéphane JUNCA

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ON SOME PERIODIC SOLUTIONS OF DISCRETE VIBRO-IMPACT OSCILLATORS WITH A UNIQUE UNILATERAL CONTACT CONDITION

Abstract

The purpose of this thesis is to investigate N degree-of-freedom vibro-impact oscillators with an unilateral contact. The dynamics is linear in the absence of contact; it is governed by an impact law otherwise. Trajectories that display a sticking phase are identified. The First Return Map is a fundamental tool to explore such periodic solutions. The Poincaré section is tangent to grazing orbits and thus yields the well-known square-root singularity, as already reported in Mechanics, which is here revisited in a rigorous mathematical framework. Another important singularity is exhibited: the discontinuity of the First Return Time. Finally, the square-root dynamics near the linear grazing orbits which may lead to the instability of these linear grazing orbits is studied. It is found that the square-root dynamics emerges from the square-root singularity of the First Return Time if one of the coefficients related to a linear map does not vanish. Under a generic condition of the matrix of eigenvectors, the square-root dynamics near a linear grazing orbit is proven to exist.

Keywords: nonsmooth analysis, vibro-impact systems, unilateral contact, periodic solutions, sticking phase, linear grazing orbit, poincaré map

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SUR DES SOLUTIONS PÉRIODIQUES D'OSCILLATEURS DISCRETS À VIBRO-IMPACT AVEC UNE UNIQUE CONDITION DE CONTACT UNILATÉRAL

Résumé

Le but de cette thèse est d'étudier des oscillateurs à vibro-impact à N degrés de liberté avec une condition de contact unilatéral. La dynamique est linéaire en l'absence de contact ; elle est régie par une loi d'impact autrement. Des trajectoires présentant une phase de contact collant sont identifiées. L'application de premier retour de Poincaré est un outil fondamental pour étudier la dynamique près de solutions périodiques. La section de Poincaré est tangente aux orbites rasantes et conduit à une singularité en « racine carrée », déjà connue en Mécanique, singularité revisitée dans un cadre mathématique rigoureux. Elle implique la discontinuité du temps de premier retour. Enfin, la dynamique en racine carrée près des modes linéaires rasants, qui peut conduire à l'instabilité de ces modes, est abordée. On constate que la dynamique en racine carrée émerge de la singularité en racine carrée du temps de premier retour si l'un des coefficients liés à une carte linéaire ne s'annule pas. Pour une condition générique de la matrice des vecteurs propres, la dynamique en racine carrée près d'une mode linéaire rasants est prouvée.

Mots clés : analyse non lisse, systèmes discrets à vibro-impact, contact unilatéral, solutions périodiques, mode linéaire rasant, l'application de poincaré

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LE THI Huong

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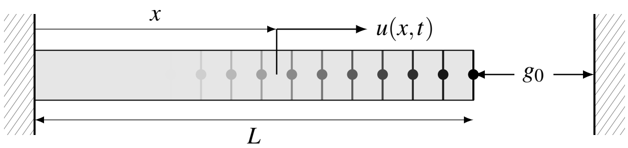
Introduction

Undergraduate and graduate students who learn physics and differential equations learn how to solve analytically the dynamics of a single mass attached to a wall by a spring. Such oscillating systems form the foundation for more complex behaviours as found in real applications. For instance, various models have been proposed to model the motion of blades in turbomachinery applications, see Figure 1(a). A simplified continuous model for the blade could be a thin longitudinal elastic rod attached to a wall and which stretches until it interacts with a constraint, which plays the role of the casing, see Figure 1(b). In the framework of the continuous model, there exists a methodology to perform modal analysis of elastic structures [24]. Moreover, numerical procedures capable of performing nonsmooth modal analysis of an elastic bar subject to a unilateral contact constraint are proposed in the literature [52, 51]. Another approach is to use discrete models since they can exhibit a variety of behaviors that bring important insight on nonsmoothness [6, 12, 20]. Hence, the present work focuses on a discrete model where the rod is replaced by a vibrating chain of masses connected by springs, one of which being attached to a wall. Depending on the initial conditions, this chain may hit the constraint. This model is commonly referred to as a vibro-impact system with a unilateral constraint; the unilateral constraint comes from the presence of a rigid foundation.

It is known from linear theory that the modes of vibration of a discrete or continuous system can be used to decouple the equations of motion, and that any arbitrary oscillation can be expressed as superposition of the modal responses. Any forced resonances of the system under external harmonic forces always occur in neighborhoods of normal modes. Moreover, linear normal modes are neutrally stable. A mode is unstable if a small perturbation of the initial conditions that are required for the realization of the modal motion leads to the elimination of the mode oscillation [48]. These properties are however not necessarily enjoyed by the so-called nonlinear normal modes arising from the study of nonlinear systems [43, 44]. Nonlinear modes are defined as continuous one-parameter families of periodic orbits [22, 49, 39]. For the discrete N-dof system undergoing an energy-preserving impact law on one of its masses, nonlinear modes of vibration with non-grazing impact are explored in [38, 35, 19, 23, 25, 32, 46, 45]. It was shown in [25] that the families of periodic orbits featuring one impact per period exist and lie on two-dimensional invariant manifolds in the state-space. The stability of the such periodic orbits

(a)



(b)

FIGURE 1 – (a) Turbomachinery blade and casing. (b) Thin rod with a unilateral constraint

lying on the invariant manifolds is also established. Further research on stability properties of non-smooth dynamical systems is found in [27].

In the same aspiration of seeking for nonlinear modes with non-grazing impacts, the author identifies the nonlinear modes that display long “sticking” contact phases. The term “sticking” here means that the impacting mass will rest on the rigid foundation for a finite or infinite amount of time during its motion even though there is no “sticking force” arising from the rigid foundation and acting on the impacting mass. Results indicate that the periodic solutions involving sticking phases are isolated, which is different from the nonlinear modes involving non-zero velocity impacts.

Another main task of the thesis is to investigate and understand the dynamics via the First Return Map (FRM) [18]. In smooth analysis, this map is defined in a so-called Poincaré section which is naturally chosen to be transverse to the flow of the system. The first return map then associates the points in Poincaré section with the first return of the flow to this Poincaré section. The impact hyperplane is the natural choice for the Poincaré section in the study of impacts with non-zero pre-impact velocity [25, 21]. While treating the grazing trajectory, the transversality condition is lost. Nordmark et al [7, 17, 33] introduced a discontinuity mapping on a suitable Poincaré section lying outside the contact interface to keep the transversality condition. For a one-dof forced oscillator that impacts a rigid obstacle, Chillingworth [9, 10] used an alternative geometric approach which replaces the picture of curved orbits intersecting a flat obstacle with and equivalent picture of straightened out orbits intersecting a curved obstacle: the impact surface. Particular interest in bifurcations is investigated near the grazing trajectory [3, 7, 17, 29, 30, 26, 50, 11, 15]. In the present work, the FRM, which is the fundamental tool to explore periodic solutions, is extended to the investigation of other potential solutions in order to improve the understanding of the global dynamics. Since the Poincaré section is a subset of the contact interface in the phase-space, it can be tangent to orbits which yields the well-known square-root singularity [7, 33]. This singularity is here revisited in a rigorous mathematical framework. Moreover, the study of this singularity implies a more important singularity: the discontinuity of the First Return Time (FRT).

The thesis is organized as follows:

- Chapter 1 is devoted to proving the existence of the solution to the N-dof vibro-impact system that is based on the fundamental existence result given by Schatzman [40]. The well-posedness for a general problem in $\mathbb{R}^n$ without friction and under strong restrictions on the regularity of the data was shown by Ballard [5]. In this chapter, the difficulty in giving a simpler proof on the uniqueness of the solution is also proposed.
- Chapter 2 concerns a class of periodic solutions which involve sticking phases. As a minimal model, periodic solutions with one sticking phase-per-period (1-SPP) of a two-dof vibro impact system are considered. A method to obtain such periodic solutions is proposed: it provides conditions on the existence of 1-SPP as well as closed-form expressions of these periodic solutions. In particular, the set of 1-SPP is shown to be at most a countable set of isolated periodic orbits. These solutions are then compared to the nonlinear modes of vibration involving one impact per period (1-IPP), where it is shown that there is an equivalence between 1-SPP and a special set of isolated 1-IPP. It is also demonstrated that the pre-stressed system features sticking phases of infinite duration.
- The first return map is explored in Chapter 3 in a rigorous mathematical framework to describe the square-root singularity for a linear vibro-impact system with a unilateral contact. The Poincaré section is chosen at the contact interface at the cost of losing of the transversality condition. New singular behaviors are observed. In particular, the discontinuity of the First Return Time is displayed along grazing orbits. This discontinuity is largely undesirable because it may cause the instability of the periodic solutions with grazing contacts. Finally, a general condition is given to verify the square-root dynamics in the vicinity of the linear grazing modes (LGM), the periodic solutions with one grazing contact per period, which are known to be the source of many branches of nonlinear modes (e.g. k-impact-per-period [25]). From this square-root dynamics might emanate the instability of these linear grazing modes.

Conclusions and remaining open questions are listed in the Conclusion. Appendices, which contain alternative proofs or detailed calculations, follow.

Chapter 1

On solutions to vibro-impact oscillators

Summary The aim of this chapter is to give a simple proof for the existence and discuss around the uniqueness of the solution to a vibro-impact system with a unilateral constraint and without source terms. It is also confirmed that the impacts are isolated, which is known from [5].

1.1 Introduction

In this chapter, the existence of the solutions of the N-degree-of-freedom (dof) vibro-impact system is given. There is no such theorem like the Cauchy-Lipschitz theorem in the case of a unilateral constraint [5]. The first investigation on the existence and uniqueness of solutions for such systems is that of Schatzman [40] and the Italian school. The study was limited on the configuration space $\mathbb{R}^d$ and to the case of an elastic impact. The proof on the existence of solution is based on the Yosida regularization and compactness arguments. The uniqueness is however proved only for a very specific case. She also gave examples where the loss of regularity can affect the uniqueness. For further investigation on the uniqueness of a solution, Percivale is the first to introduce the analyticity hypothesis [36]. However, the results only apply to a very specific case. The problem with completely inelastic impacts has been studied by Moreau. The one-degree-of-freedom problem with arbitrary impact constitutive law is studied by Schatzman. The uniqueness is proved under the analyticity of the data. Ballard [5] gave a complete proof on the existence and uniqueness of solution in a general setting for N-dof system where it is required that all the data are analytic to ensure the existence and uniqueness.

By Proposition 19 in [5, pp. 248-249], it is proven that infinitely many impacts can accumulate at the left of a given instant but *not at the right*. This is a specific feature of the analytical setting that is lost in the C^∞-setting. Moreover, in the case of purely elastic impacts, the instants of impact are *isolated*.

The system of interest is a second-order system

$$\begin{cases} \mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{r} & (1.1a) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & (1.1b) \\ u_N(t) \leq d, \quad R(t) \leq 0, \quad (u_N(t) - d)\, R(t) = 0 & (1.1c) \\ \dot{\mathbf{u}}^+(t)^\top \mathbf{M}\dot{\mathbf{u}}^+(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t)) = E(\mathbf{u}(0), \dot{\mathbf{u}}(0)), & (1.1d) \end{cases}$$

with

$$\mathbf{M} = \mathbf{diag}(m_j)_{j=1}^N; \quad \mathbf{K} = (k_{ij})_{i,j=1}^N; \quad \mathbf{u}(t) = (u_j)_{j=1}^N; \quad \mathbf{r}(t) = (0, \dots, 0, R(t)) \tag{1.2}$$

where u_j, $\dot{u}_j$ and $\ddot{u}_j$ represent the displacement from its equilibrium, the velocity and acceleration of the mass j, $j = 1, \dots, N$, respectively. Condition (1.1c) says that mass N is constrained on the right side by a rigid obstacle at a distance $d > 0$ from its equilibrium. The N^{th} mass is assumed to be unilaterally constrained by the presence of a rigid foundation. The other masses are not constrained in any way. The quantity $R(t)$ is the reaction force induced by the obstacle on mass N at the time of gap closure. In general $R(t)$ is a measure, but for solutions with sticking phases, it is a Lipschitz function that is as regular as $\ddot{u}_N$ [28].

Matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite. There is then a matrix $\mathbf{P}$ of eigenmodes that simultaneously diagonalizes both of them, that is $\mathbf{P}^\top\mathbf{MP} = \mathbf{I}$ and $\mathbf{P}^\top\mathbf{KP} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2)|_{i=1}^N$ where $\mathbf{I}$ is the $N \times N$ identity matrix and ω_i^2, $i = 1, \dots, N$ are the eigenfrequencies of the linear system without unilateral contact condition.

Equation (1.1d) reflects the conservative nature of the system. The energy is preserved during a motion. Equation (1.1d) commonly implies the existence of a perfectly elastic impact law of the form $\dot{u}_N^+ = -e\dot{u}_N^-$ with $e = 1$ where $\dot{u}_N^-$ and $\dot{u}_N^+$ respectively stand for the pre- and post-impact velocities of the N^{th} mass.

It is known [5] and will be proven in the next section that there exists a solution $\mathbf{u}$ to System (1.1) which belongs to the space of motions with measure acceleration (MMA). For our system in which there is only one potential contact point between the N^{th} mass and the obstacle, there is more regularity for $u_1, \dots, u_{N-1}$ than u_N. In the two-dof vibro-impact system modelling a chain of two masses, investigated in Chapter 2, it is shown that u_1 belongs to BV^4 while $u_2 \in$ MMA (or BV^2) where the space BV^n is defined as follows: $\mathbf{u} \in \mathrm{BV}^n$ means $\frac{\mathrm{d}^n u}{\mathrm{d}t^n}$ is a measure (or $\frac{\mathrm{d}^{n-1} u}{\mathrm{d}t^{n-1}} \in \mathrm{BV}$), i.e it is a function of bounded variation for every integer $n \geq 1$.

This chapter is organized as follows. The proof of the solution existence to the solutions of the N-dof vibro-impact system is stated in Section 1.2. In Section 1.3, a discussion on the uniqueness is given.

1.2 On the existence

We now give the theorem on the existence of solutions to System (1.1). At the time t when $u_N(t) = d$, the total energy $E(\mathbf{u}(t), \dot{\mathbf{u}}(t))$ is conserved at $t+$ and $t-$. Hence, we can assume $E(\mathbf{u}(t), \dot{\mathbf{u}}^-(t)) = E(\mathbf{u}(0), \dot{\mathbf{u}}(0))$, $\forall t$.

Theorem 1.1. Consider System (1.1). There exists a solution $\mathbf{u} \in \mathrm{MMA}([0\,;\infty[, \mathbb{R}^N)$ if $u_N(0) \leq d$. Moreover, $\dot{u}_N \in \mathrm{BV}_{\mathrm{loc}}([0\,;+\infty[, \mathbb{R}^N)$.

Besides, the other components u_k, $k \neq N$, are more regular than u_N.

Proof. To show existence, Yosida regularization is used. For $\epsilon > 0$, let us consider the system with the initial data satisfying the constraint condition

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r}_\epsilon \tag{1.3}$$

$$(\mathbf{u}(0), \dot{\mathbf{u}}(0)) = (\mathbf{u}_0, \dot{\mathbf{u}}_0) \in \mathbb{R}^N \times \mathbb{R}^N \tag{1.4}$$

$$u_N(0) \leq d \tag{1.5}$$

where $\mathbf{r}_\epsilon = [0, \ldots, 0, -(u_N(t) - d)_+/\epsilon]^\top$ is a Lipschitz function, with $x_+ = \max(0, x)$. Via the Cauchy-Lipschitz theorem, there exists a unique solution $\mathbf{u}_\epsilon \in C^2([0\,;\infty[)$. Since the initial data satisfies (1.5), the total energy of the system reads

$$E_\epsilon(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) = E(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) + \frac{1}{\epsilon}\left((\mathbf{u}_{\epsilon N}(t) - d)_+\right)^2 \tag{1.6}$$

$$= E_\epsilon(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) = E(\mathbf{u}_0, \dot{\mathbf{u}}_0) \tag{1.7}$$

where $\mathbf{u}_{\epsilon N}$ is the N^{th} coordinate of $\mathbf{u}_\epsilon$. Then the total energy is constant and independent of ϵ. In particular, $E(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) \leq E(\mathbf{u}_0, \dot{\mathbf{u}}_0)$ and $((\mathbf{u}_{\epsilon N}(t) - d)_+)^2 \leq \epsilon E(\mathbf{u}_0, \dot{\mathbf{u}}_0)$ and thus

$$\mathbf{u}_{\epsilon N}(t) \leq d + \sqrt{\epsilon E(\mathbf{u}_0, \dot{\mathbf{u}}_0)}. \tag{1.8}$$

Since the energy $E_\epsilon(\mathbf{u}(t), \dot{\mathbf{u}}_\epsilon(t))$ is conserved, for $\mu > 0$ and $\kappa > 0$, we get

$$E(\mathbf{u}_0, \dot{\mathbf{u}}_0) \geq E_\epsilon(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) \geq \mathbf{u}_\epsilon^\top(t)\mathbf{K}\mathbf{u}_\epsilon(t) \geq \kappa\|\mathbf{u}_\epsilon(t)\|, \tag{1.9}$$

$$E(\mathbf{u}_0, \dot{\mathbf{u}}_0) \geq E_\epsilon(\mathbf{u}_\epsilon(t), \dot{\mathbf{u}}_\epsilon(t)) \geq \dot{\mathbf{u}}_\epsilon^\top(t)\mathbf{M}\dot{\mathbf{u}}_\epsilon(t) \geq \mu\|\dot{\mathbf{u}}_\epsilon(t)\|. \tag{1.10}$$

Accordingly, $\mathbf{u}_\epsilon(t)$ is bounded in $C^1([0\,;+\infty[, \mathbb{R}^N)$. Using Ascoli's theorem to the family of $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ defined on $[0\,;T[$ with $T > 0$, we obtain that $\{\mathbf{u}_\epsilon\}_{\epsilon>0}$ is relatively compact in $C^0\left([0\,;T[, \mathbb{R}^N\right)$. As a consequence, there exists a subsequence $\{\mathbf{u}_n\} := \{\mathbf{u}_{\epsilon_n}\}_n$ that uniformly converges to $\mathbf{u} \in \mathrm{Lip}([0\,;T])$.

We now verify that $\mathbf{u}(t)$ satisfies the conditions of the original system (1.1).

- $u_N(t) \leq d, \forall t$ — From Inequality (1.8), by taking the limit as $\epsilon_n \to 0$, we have $u_N(t) \leq d$.
- $R(t) \leq 0$ — Let $R_\epsilon(t) = -(u_{\epsilon N}(t) - d)_+/\epsilon$, it follows that $R_\epsilon(t) \leq 0$. Integrating the system (1.1a) from 0 to t with $0 < t < T$ we have

$$\int_0^t (\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t))\, dt = \int_0^t \mathbf{r}_\epsilon(t)dt. \tag{1.11}$$

Consider the last equation

$$m_N\dot{u}_N\Big|_0^t + \sum_{i=1}^{N} k_{Ni} \int_0^t u_i(s)ds = \int_0^t R_\epsilon(s)ds, \tag{1.12}$$

that is

$$O(1) + O(T) = \int_0^t R_\epsilon(s)ds. \tag{1.13}$$

It follows that $\int_0^t |R_\epsilon(s)|ds \leq O(1+T)$. Therefore, R_ϵ is uniformly bounded in $L^1([0\,;T[)$ and consequently in $\mathcal{M}^1([0\,;T[)$. Hence R_ϵ weakly converges to $R \in \mathcal{M}^1([0\,;T[)$. Thus $R(t) \leq 0$.
- $(u_N(t) - d)\,R(t) = 0$, i.e. $\operatorname{supp} R(t) \subset \{t : u_N(t) = d\}$ — If $u_N(t) = d$ then $(u_N(t) - d)\,R(t) = 0$. Otherwise, if $u_N(t_0) < d$, for $t_0 \in I = [t_{-1}\,; t_{+1}]$ (an arbitrary closed interval). Since u_N is continuous, there exists $\delta > 0$ such that $u_N(t) < d - \delta$, $\forall t \in I$. For $t \in I$, $u_{\epsilon_n N} \to u_N$ uniformly in L^∞. Therefore, $\exists n_0$ such that $\forall n > n_0$, $u_{\epsilon_n N}(t) < d - \delta/2$ on I. It follows that $-(u_{\epsilon_n N}(t) - d)_+/\epsilon_n = 0$, for $n \geq n_0$ and $t \in I$. In other words, $R_{\epsilon_n}(t) = 0$, for all $n \geq n_0$, and $t \in I$. Hence, $R(t) = \lim_{\epsilon_n \downarrow 0} R_{\epsilon_n}(t) = 0$, $\forall t \in I$. This shows that $(u_N(t) - d)R(t) = 0$.
- The energy is conserved — Set $\alpha_\epsilon(t) = ((u_{\epsilon N}(t) - d)_+)^2/\epsilon \leq E(\mathbf{u}_0, \dot{\mathbf{u}}_0)$. We will show that $\alpha_n := \alpha_{\epsilon_n}$ converges to 0 as n tends to infinity. The definition of R_ϵ and α_ϵ yields

$$|R_\epsilon(t)| = \frac{(u_{\epsilon N}(t) - d)_+}{\epsilon}, \tag{1.14}$$

$$0 \leq \alpha_\epsilon(t) = |R_\epsilon(t)|(u_{\epsilon N}(t) - d)_+, \tag{1.15}$$

$$0 \leq \int_0^T \alpha_\epsilon(t)dt \leq \left(\int_0^T |R_\epsilon(t)|dt\right) \sup_{0 \leq t \leq T} (u_{\epsilon N}(t) - d)_+ \to 0, \tag{1.16}$$

since $\int_0^T |R_\epsilon(t)|dt$ is uniformly bounded, and $u_{\epsilon N}$ uniformly converges to d. Thus $\alpha_{\epsilon_n} \to 0$ in L^1_{loc}. It is up to a subsequence $\alpha_{\epsilon_{n_{n'}}} = \alpha_{n'}(t) \to 0$ for almost all $t \in [0\,;T[$. From (1.6), it follows that, when ϵ tends to 0, $E_\epsilon(\mathbf{u}(t), \dot{\mathbf{u}}(t))$ converges to a constant, which is exactly the energy $E(\mathbf{u}(t), \dot{\mathbf{u}}(t))$ of (1.1). We will now show that E is conserved. For all time t outside the impact time, i.e. for all t such that $u_N(t) < d$, total energy is conserved: $E(\mathbf{u}(t), \dot{\mathbf{u}}(t)) = E(\mathbf{u}_0, \dot{\mathbf{u}}_0)$. We will now prove that for any impact time t,

$E(\mathbf{u}(t), \dot{\mathbf{u}}^-(t)) = E(\mathbf{u}_0, \dot{\mathbf{u}}_0) = E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t))$, where $E(\mathbf{u}(t), \dot{\mathbf{u}}^-(t))$ and $E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t))$ are the pre- and post-impact energies of the system. From the last equation

$$m_N \ddot{u}_N + \sum_{i=1}^{N} k_{Ni} u_i = R(t), \tag{1.17}$$

and knowing that $\mathbf{u} \in C^0([0\,;T[) \subset L^1([0\,;T[) \subset \mathcal{M}^1([0\,;T[)$, and $R(t) \in \mathcal{M}^1([0\,;T[)$, it follows that $\ddot{u}_N \in \mathcal{M}^1([0\,;T[)$, i.e. $\dot{u}_N \in \mathrm{BV}([0\,;T[)$ where BV is the space of functions with bounded variations. Therefore, $\dot{u}_N^-(t)$ and $\dot{u}_N^+(t)$ exist for all time t. These are the only factors that change and affect the energy of the system. Recall that the set of impacts, i.e. $\{t : \dot{u}_N^-(t) \neq \dot{u}_N^+(t)\}$ is countable since $\dot{u}_N$ is a BV function. Hence, $E(\mathbf{u}(t), \dot{\mathbf{u}}^-(t)) = E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t))$, for all t. □

1.3 On the uniqueness

This section is devoted to a brief discussion on the uniqueness of the solution of (1.1) . A long proof is provided in [5] for a general framework and under the assumption of analyticity of all data. In our specific framework with linear stiffness, a simpler proof is expected. Nevertheless, the uniqueness of the solution in the presence of sticking contact is not easy to prove. Let us explain further this difficulty. Given an initial data (1.1b), the solution of (1.1) is unique as long as there is no contact. The question on the uniqueness arises when contacts are taken into account.

The local uniqueness of the solutions is addressed in the right neighborhood of an impact which and without loss of generality, is assumed to be at $t = 0$. It is proven that there exists a solution $\mathbf{U}_a = [\mathbf{u}_a, \dot{\mathbf{u}}_a]$ in $\mathrm{MMA}([0\,;t_0[)$ of (1.1), where $t_0 > 0$.

If the impact is with non-zero velocity, i.e. $\dot{u}_{aN}^-(0) > 0$, then by (1.1d), the energy of the system is conserved right before and right after the impact time $t = 0$:

$$E(\mathbf{u}(0), \dot{\mathbf{u}}^+(0)) = E(\mathbf{u}(0), \dot{\mathbf{u}}^-(0)), \tag{1.18}$$

$$\dot{\mathbf{u}}_a^+(0)^\top \mathbf{M} \dot{\mathbf{u}}_a^+(0) + \mathbf{u}_a(0)^\top \mathbf{K} \mathbf{u}_a(0) = \dot{\mathbf{u}}_a^-(0)^\top \mathbf{M} \dot{\mathbf{u}}_a^-(0) + \mathbf{u}_a(0)^\top \mathbf{K} \mathbf{u}_a(0). \tag{1.19}$$

This gives $(\dot{u}_{aN}^+(0))^2 = (\dot{u}_{aN}^-(0))^2$. It follows that the velocity of mass N after the impact is $\dot{u}_{aN}^+(0) = -\dot{u}_{aN}^-(0) < 0$. Otherwise, the mass penetrates the obstacle. Indeed, mass N bounces back immediately since

$$u_{aN}(t) = u_{aN}(0) + \int_0^t \dot{u}_{aN}(s)ds = u_{aN}(0) + t\,\dot{u}_{aN}^+(0) + o(t) \tag{1.20}$$

$$= d + t(\dot{u}_{aN}^+(0) + o(1)) < d \text{ for } t \gtrsim 0 \tag{1.21}$$

where $t > 0$, $\dot{u}^+_{aN}(0) < 0$. After the impact, the N masses are driven by linear dynamics similar to (1.1a) where $\mathbf{r} = 0$ with the new initial data at 0: $[\mathbf{u}_a(0), \dot{\mathbf{u}}^+_a(0)]$. Via the Cauchy theorem, uniqueness is guaranteed for $t \in [0, \epsilon]$ where $\epsilon > 0$ is small enough so that there is no contact between the N^{th} mass and the obstacle.

If the impact is with zero velocity, then $\dot{u}^+_{aN}(0) = -\dot{u}^-_{aN}(0) = 0$. Let $\mathbf{z}$ be the solution of the linear system with the initial data $[\mathbf{z}(0), \dot{\mathbf{z}}(0)] = [\mathbf{u}_a(0), \dot{\mathbf{u}}^+_a(0)]$. Let us divide this into two cases:

Case 1: Suppose that there is no sticking phase at $t = 0$, i.e. $u_{aN}(t) < d$ for all $t \in]0\,;\delta[$, where $\delta > 0$ is small enough so that there is no other contact between $t = 0$ and $t = \delta$. Hence, for all $t \in]0\,;\delta[$, $\mathbf{u}_a$ is analytic and satisfies

$$\mathbf{M}\ddot{\mathbf{u}}_a + \mathbf{K}\mathbf{u}_a = \mathbf{0}. \tag{1.22}$$

Assume that there is another solution $\mathbf{U}(t) = [\mathbf{u}(t), \dot{\mathbf{u}}(t)]$ of (1.1) for $t \in [0\,;\delta]$. In Section 1.2, it is proven that $\dot{\mathbf{u}} \in \mathrm{BV}(]0\,;\delta[)$. Let $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}_a(t)$, $t \in [0\,;\delta]$, it follows that $\dot{\mathbf{v}} \in \mathrm{BV}(]0\,;\delta[)$ and in the interval $[0\,;\delta]$, $\mathbf{v}(t)$ satisfies

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{r} \tag{1.23}$$

together with $\mathbf{v}(0) = \mathbf{0}$ and $\dot{\mathbf{v}}(0) = \mathbf{0}$. Let us define the quantity $\bar{\dot{u}}_N = (\dot{u}^+_N + \dot{u}^-_N)/2$. Take note that the notation $\bar{\dot{u}}_N$ coincides with the classical notation of derivative of u_N at any time away from impacts. Since the energy of the system is conserved, it follows that $\dot{u}^+_N(t) = -\dot{u}^-_N(t)$ where t is the impact time and

$$\bar{\dot{u}}_N(t) = \begin{cases} 0 & \text{if } u_N(t) = d, \\ \dot{u}_N(t) & \text{if } u_N(t) < d. \end{cases} \tag{1.24}$$

Accordingly, $\bar{\dot{v}}_N(t) = \bar{\dot{u}}_N - \dot{u}_{aN}$ satisfies

$$\bar{\dot{v}}_N(t) = \begin{cases} 0 & \text{if } u_N(t) = d \\ \dot{v}_N(t) & \text{if } u_N(t) < d. \end{cases} \tag{1.25}$$

Pre-multiplying both sides of (2.1a) by $\bar{\dot{\mathbf{v}}} = [\dot{v}_1, \ldots, \dot{v}_{N-1}, \bar{\dot{v}}_N]^\top$ yields

$$\bar{\dot{\mathbf{v}}}^\top \mathbf{M}\ddot{\mathbf{v}} + \bar{\dot{\mathbf{v}}}^\top \mathbf{K}\mathbf{v} = \bar{\dot{v}}_N(t)R(t), \tag{1.26}$$

$$\dot{\mathbf{v}}^\top \mathbf{M}\dot{\mathbf{v}} + \mathbf{v}^\top \mathbf{K}\mathbf{v} = \int_0^t \bar{\dot{v}}_N(s)R(s)ds. \tag{1.27}$$

By denoting $E(\mathbf{v}(t), \dot{\mathbf{v}}(t)) = \dot{\mathbf{v}}^\top(t)\mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{v}^\top(t)\mathbf{K}\mathbf{v}(t)$, it is seen that $E(\mathbf{v}(0), \dot{\mathbf{v}}(0)) = 0$.

Hence,

$$E(\mathbf{v}(t), \dot{\mathbf{v}}(t)) = E(\mathbf{v}(t), \dot{\mathbf{v}}(t)) - E(\mathbf{v}(0), \dot{\mathbf{v}}(0)) = \int_0^t \bar{v}_N(s)R(s)ds. \tag{1.28}$$

From (1.1c) and (1.25), it follows that $\bar{v}_N(s)R(s) = 0$, $\forall s \in [0\,;\delta]$. Therefore, $E(\mathbf{v}(t)) = 0$, $\forall t \in]0\,;\delta[$. It follows that $\mathbf{v}(t) = 0$, $\forall t \in]0\,;\delta[$, or $\mathbf{v} \equiv \mathbf{0}$. In other words, $\mathbf{u} \equiv \mathbf{u}_a$, which shows the uniqueness of the solution of (1.1) on the right neighborhood of the impact when there is no sticking phase.

Case 2: Suppose that there is a sticking phase starting at $t = 0$, i.e. $u_{aN}(t) = d$, $\forall t \in [0\,;\tau]$, where τ is the end of the sticking phase. For all $t \in]0\,;\delta[$, $0 < \delta < \tau$, let $\mathbf{U}(t)$ be another solution of (1.1) on $[0\,;\delta]$. Then $\mathbf{u}(t)$ and $\mathbf{u}_a(t)$ satisfy (1.1a), (1.1b). The difference $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}_a(t)$ then satisfies

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{r} - \mathbf{r}_a, \tag{1.29}$$

with $\mathbf{v}(0) = \mathbf{0}$ and $\dot{\mathbf{v}}(0) = \mathbf{0}$ where $\mathbf{r}_a(t) = [0, \ldots, 0, R_a(t)]^\top$, with $R_a(t) < 0$ as the reaction of the wall during the sticking phase. Premultiplying both sides of (1.1a) by $\bar{\dot{\mathbf{v}}}$ leads to

$$\bar{\dot{\mathbf{v}}}^\top(t)\mathbf{M}\ddot{\mathbf{v}}(t) + \bar{\dot{\mathbf{v}}}^\top(t)\mathbf{K}\mathbf{v}(t) = \bar{\dot{v}}_N(t)(R(t) - R_a(t)), \tag{1.30}$$

$$E(\mathbf{v}(t), \dot{\mathbf{v}}(t)) = \dot{\mathbf{v}}^\top(t)\mathbf{M}\dot{\mathbf{v}}(t) + \mathbf{v}^\top(t)\mathbf{K}\mathbf{v}(t) = \int_0^t \bar{\dot{v}}_N(s)\,(R(s) - R_a(s))\,ds. \tag{1.31}$$

Notice that $\bar{\dot{v}}_N(s) = \bar{\dot{u}}_N(s) - \dot{u}_{aN}(s) = \bar{\dot{u}}_N(s)$ for all $s \in]0\,;\delta[$ since $\dot{u}_{aN}(s) = 0$. Hence, $\bar{\dot{v}}_N(s) = 0$ if there is an impact at the time s. As before, $\bar{\dot{v}}_N(s)R(s) = 0$ since the mean velocity vanishes when $u_N = d$ and the measure R is also concentrated when $u_N = d$. Thus,

$$E(\mathbf{v}(t), \dot{\mathbf{v}}(t)) = -\int_0^t \bar{\dot{u}}_N(s)R_a(s)ds. \tag{1.32}$$

Notice that $R_a \leq 0$. To show the local uniqueness in this case it suffices to prove that $\bar{\dot{u}}_N(s) \geq 0$. This, however, is not trivial and remains yet to be proven.

Chapter 2

Periodic solutions with sticking phases of a two-degree-of-freedom oscillator

Summary This chapter explores the free dynamics of a simple two-degree-of-freedom vibro-impact oscillator. One degree-of-freedom is limited by the presence of a rigid obstacle and periodic solutions involving one sticking phase per period (1-SPP) are targeted. A method to obtain such orbits is proposed: it provides conditions on the existence of 1-SPP as well as closed-form solutions. It is shown that 1-SPP might not exist for a given combination of masses and stiffnesses. The set of 1-SPP is at most a countable set of isolated periodic orbits. The construction of 1-SPP requires numerical developments that are illustrated on a few relevant examples. Comparison with nonlinear modes of vibration involving one impact per period (1-IPP) is also considered. Interestingly, an equivalence between 1-SPP and a special set of isolated 1-IPP is established. It is also demonstrated that the prestressed system features sticking phases of infinite duration.

2.1 Introduction

In the context of vibration and modal analysis of vibro-impact oscillators, nonlinear modes of vibration with non-grazing impact are explored in [38, 35, 19, 25, 46, 45]. In these works, only one degree-of-freedom (dof) is unilaterally constrained by the presence of a rigid foundation: the dynamics is purely linear when the contact constraint is not active, and governed by an impact law otherwise. Accordingly, the contact force arising when the system interacts with the foundation is a periodic distribution of Dirac deltas.

Instead, the present work pays attention to periodic solutions involving long "sticking"contact phases, thus discarding impulse-driven dynamics reported in the previous works, during which the contacting mass rests against the obstacle for a finite amount of time. Let us clarify the terminology now: as explained later, "sticking" here means that the impacting mass will rest on the rigid foundation for a finite or infinite amount of time during its motion even though there is no "sticking force" arising from the rigid foundation and acting on the impacting mass. An equivalent terminology would be "lasting non-impulsive" contact phases. This investigation is motivated by the fact that in a continuous setting in space and time, unilateral contact forces are known to be discontinuous functions at most [16] while they become impulsive after a semi-discretization in space via the Finite Element technique. The question is then: are there non-impulsive solutions in the Finite Element framework and alike?

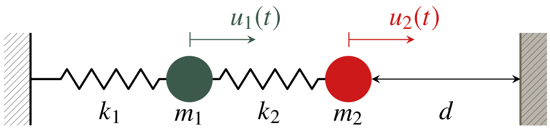


FIGURE 2.1 – Two degree-of-freedom vibro-impact system, $d > 0$

The system of interest is an oscillator with two masses m_1 and m_2 linearly connected through two springs of stiffness k_1 and k_2 respectively, as depicted in Figure 2.1. The dynamics of interest reads

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} \tag{2.1a}$$
$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 \tag{2.1b}$$
$$u_2(t) \leq d, \quad R(t) \leq 0, \quad (u_2(t) - d)R(t) = 0, \quad \forall t \tag{2.1c}$$
$$\dot{\mathbf{u}}^+(t)^\top \mathbf{M}\dot{\mathbf{u}}^+(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t)) = E(\mathbf{u}(0), \dot{\mathbf{u}}(0)), \tag{2.1d}$$

where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad ; \quad \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \quad ; \quad \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \quad ; \quad \mathbf{r}(t) = \begin{pmatrix} 0 \\ R(t) \end{pmatrix}. \tag{2.2}$$

Above, u_j, $\dot{u}_j$, and $\ddot{u}_j$ represent the displacement, velocity, and acceleration of mass j, $j = 1, 2$, respectively. The gap d is defined between the obstacle and the equilibrium position of the second mass. It is the algebraic *distance* between mass 2 at rest without any external force and the rigid wall and might thus be negative in the prestressed case. The quantity $R(t)$ is the reaction force of the wall on mass 2.

Matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite so that there is a matrix $\mathbf{P}$ of eigenmodes which simultaneously diagonalizes both of them, that is $\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}$ and $\mathbf{P}^\top\mathbf{K}\mathbf{P} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_i^2)|_{i=1,2}$ where $\mathbf{I}$ is the 2×2 identity matrix and ω_i^2, $i = 1, 2$ are the eigenfrequencies of the linear system without unilateral contact.

Equation (2.1d) reflects the conservative nature of the system. The energy is preserved during a motion. Equation (2.1d) implies the existence of a perfectly elastic impact law of the form $\dot{u}_2^+ = -e\dot{u}_2^-$ with $e = 1$ where $\dot{u}_2^-$ and $\dot{u}_2^+$ respectively stand for the pre- and post-impact velocities of mass 2. As detailed later, 1-SPP orbits are defined to be independent of the restitution coefficient e. Thus 1-SPP still exist for $e \in [0, 1]$. However, the framework $e = 1$ is chosen for comparison with another class of periodic solutions with nonzero velocity at the impact. Moreover, the structure of a general solution is simpler when $e = 1$. For the well-posedness of the initial-value problem with conserved energy, the reader is referred to [36, 5].

The sticking phase is known to appear as a limit of a chattering sequence [4, 7, 8, 34]. Here, there is no source term but sticking periodic solutions can still occur.

The chapter is organized as follows. In Section 2.2, the "sticking phase" definition is provided and the conditions on its occurrence are stated. Then, necessary conditions satisfied by periodic solutions with one sticking phase per period (1-SPP) are given on the period through the free flight duration s which appears to be a key parameter as well as a root of an explicit nonlinear function. Furthermore, when s is known, the corresponding 1-SPP is expressed in closed form. The method and numerical examples are described in Section 2.3 in order to find all 1-SPP.

Mathematical proofs and comments are detailed in Section 2.4, 2.5 and 2.6. More precisely, Section 2.4 deals with the structure of the solution space with a sticking phase. Section 2.5 is devoted to prove Theorem 2.17 on 1-SPP. The existence result of an infinite set of admissible initial data satisfying the constraint $u_2 \leq d$ near the sticking phase is proven in Section 2.6. The existence of 1-SPP satisfying the constraint $u_2 \leq d$ during the whole period remains an open problem. The prestressed case with $d \leq 0$ is discussed in Section 2.7. Section 2.10 concludes the chapter. In this work, all numerical simulations are performed using the parameters $m_1 = m_2 = 1$ and $k_1 = k_2 = 1$, unless indicated otherwise.

2.2 Main results

The occurrence of a sticking phase is first defined with necessary and sufficient conditions in Section 2.2.1. Then, in Section 2.2.2, periodic solutions with one sticking phase per period, the so-called 1-SPP, are characterized through necessary conditions to exhibit them all. Throughout the current work, internal resonances are discarded, i.e. the linear periods satisfy $\cap_{j=1}^{2} T_j \mathbb{N} = \emptyset$, unless stated otherwise.

2.2.1 Sticking phase

A sticking phase occurs when the mass number 2 stays at $u_2 = d$ on a proper time interval.

Definition 2.1 — Sticking phase and its duration. Let $\mathbf{u}$ be a solution to System (2.1). A *sticking phase* arises if there exists $t_0 \in \mathbb{R}$ and $\tau > 0$ such that

$$u_2(t) = d, \quad \forall t \in [t_0\,; t_0 + \tau]. \tag{2.3}$$

Moreover, when there exists $0 < \delta \ll 1$ such that $\forall t \in]0\,;\delta[$

$$u_2(t_0 - t) < d \quad \text{and} \quad u_2(t_0 + \tau + t) < d, \tag{2.4}$$

then t_0 is the starting time and τ is the duration of the sticking phase.

The central reference [5] is used throughout the chapter: existence is recalled, uniqueness and continuous dependence to the initial data is proved. Moreover, in the conservative case, it is shown that there is no impact accumulation. Accordingly, before and after the sticking phase, condition (2.97) above is sufficient to properly define the beginning and the end of a sticking phase of finite duration. More precisely, it is proven in Proposition 19 of [5] that the impact times are isolated for the perfectly elastic impact law. It is not the same when $0 < e < 1$ since chattering might occur [7].

In the present work, the finite duration of the sticking phase for $d > 0$ is a consequence of Theorem 2.17. This is not always true, as for example with the prestressed case $d \le 0$ detailed in Section 2.7. Such conditions are well known to be related to the sign of the acceleration [7] just before impact and are precisely established for our two-dof system as in [19].

Theorem 2.2 — Sticking contact. Assume $d > 0$. There exists a sticking phase exactly starting at time t_0 and persisting on its right neighbourhood if and only if:

1. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $u_1(t_0) > d$, or
2. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $u_1(t_0) = d$, $\dot{u}_1(t_0) > 0$.

The second case where $u_1(t_0) = d$ and $\dot{u}_1(t_0) > 0$ corresponds to the beginning of the sticking phase. The duration of the sticking phase τ then only depends on

$$v = \dot{u}_1(t_0), \quad \text{and}$$

$$\tau(v) = \frac{2}{\omega}\arctan(\xi v) \quad \text{where} \quad \omega = \sqrt{\frac{k_1 + k_2}{m_1}} \quad \text{and} \quad \xi = \frac{\sqrt{(k_1 + k_2)m_1}}{k_1\, d}. \tag{2.5}$$

The state of system at the end of the sticking phase is:

$$u_2(t_0 + \tau) = d, \quad \dot{u}_2^-(t_0 + \tau) = 0, \quad u_1(t_0 + \tau) = d, \quad \dot{u}_1(t_0 + \tau) = -v. \tag{2.6}$$

Moreover, the regularity of the curve $\{(u_2(t), \dot{u}_2(t)), t \in \mathbb{R}\}$ is $C^{1.5}$ at the point $(u_2, \dot{u}_2) = (d, 0)$ which corresponds to the time interval $[t_0, t_0 + \tau]$.

The starting of the sticking phase corresponding to the second case stated in Theorem 2.17 is only true for $e = 1$ since there is the uniqueness of solution in the past for $e = 1$ [5]. For $e < 1$, the beginning of the sticking phase can be for $u_1(t_0) > d$. This is easy to see for $e = 0$. The proof of this theorem is written in Section 2.4. The space C^s is the Sobolev space $W^{s,\infty}$ when s is a fractional number [2] which is a generalization of Hölder spaces for $s > 1$. It should also be highlighted that there is no unilateral constraint on mass 1. The expression $u_1 > d$ sometimes used below does not mean that the mass 1 hits wall since u_1 is the displacement and not the position of mass 1. In the sequel, we do not restrain u_1.

The loss of regularity at the sticking point is quite clear in Figure 2.2.

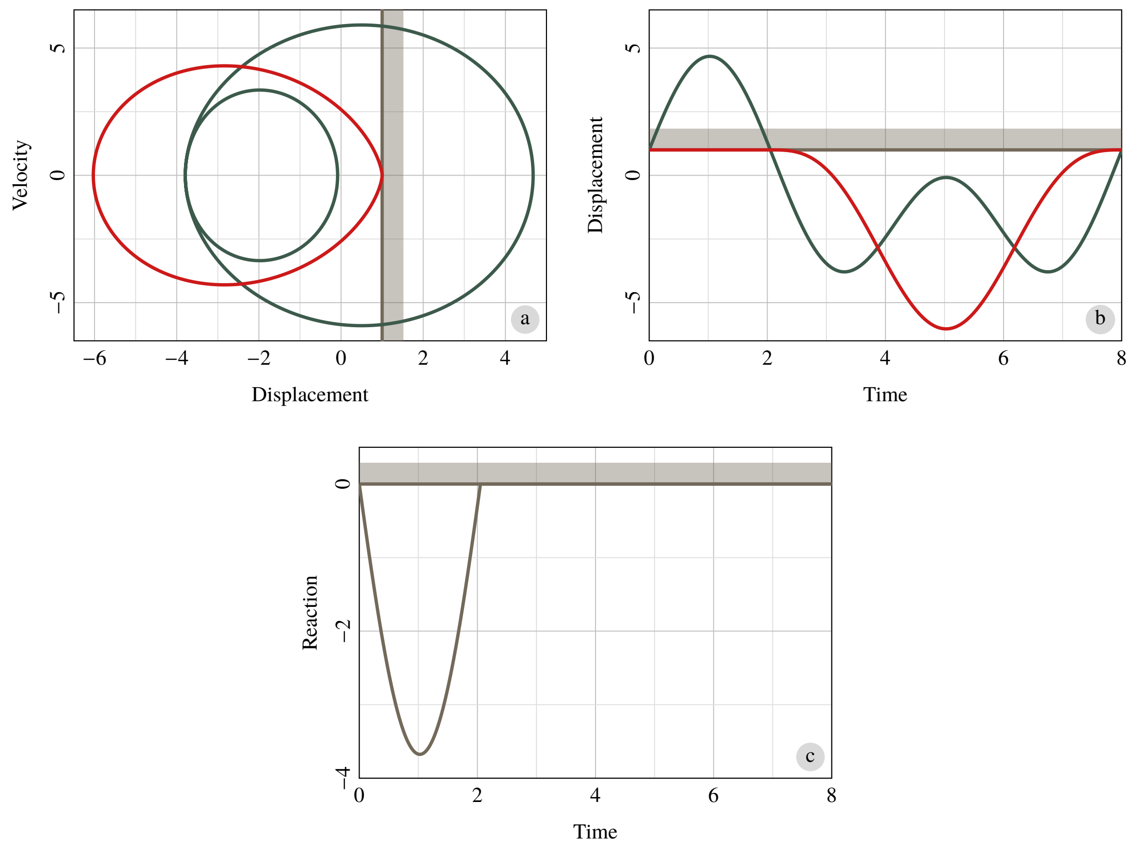


FIGURE 2.2 – Admissible 1-SPP with a singularity 1.5 at the intersection between red line and wall, $d = 1$ and initial data: $U(0) = [1, 1, 5.8624394, 0]$. (a) Orbits $\{(u_i(t), \dot{u}_i(t)),\ t \in \mathbb{R}\}$, $i = 1, 2$. (b) Displacements $t \mapsto u_i(t)$, $i = 1, 2$. (c) Reaction of the wall $t \mapsto R(t)$

This is the least smooth point on the curve $\{(u_2(t), \dot{u}_2(t)),\ t \in \mathbb{R}\}$. Locally, the orbit is very similar to the graph of $t \mapsto (d - |t|^{1.5}, t)$. Elsewhere, the curve is analytic.

Incidentally, the proof of Theorem 2.17 leads to an explicit classification of all possible contact patterns. There is no accumulation of impacts and only three distinct contact configurations arise. At $t = t_0$, let us assume $u_2(t_0) = d$, then contact is

1. an impact if $\dot{u}_2^-(t_0) > 0$;
2. a grazing contact if $\dot{u}_2(t_0) = 0$ and $R(t) = 0$ for all $t \approx t_0$;
3. a sticking contact if $\dot{u}_2(t_0) = 0$ and $R(t) < 0$ for some $t \approx t_0$.

Since the velocity of mass 2 shall be discontinuous, $\dot{u}_2^\pm(t_0)$ denotes its left/right limit. Due to energy conservation, $\dot{u}_2^+(t_0) = -\dot{u}_2^-(t_0)$ in our model; in particular, when the incoming velocity vanishes, that is $\dot{u}_2^-(t_0) = \dot{u}_2^+(t_0) = \dot{u}_2(t_0) = 0$, the velocity is continuous.

Sticking contact and grazing contact are the two possible contact occurrences [7] for an impact with zero velocity. In general, it is challenging to know whether a zero pre-velocity impact generates a reaction of the wall without further information. For the considered two-dof system, simple criteria are given on the position and velocity of mass 1 to distinguish a grazing contact from a sticking contact:

1. An impact yields an instantaneous rebound with $\dot{u}_2^+(t_0) = -\dot{u}_2^-(t_0) < 0$,
2. A grazing contact means that the trajectory would not change irrespective of the presence of the wall. As a corollary of Theorem 2.17, the data at time t_0 is either $u_1(t_0) < d$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$ or $u_1(t_0) = d$, $\dot{u}_1(t_0) \leq 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. Moreover, there exists $\varepsilon > 0$ such that $R(t) = 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0 + \varepsilon[$;
3. A sticking contact can be divided into three sequential steps:
   (a) the beginning of a sticking contact: $u_1(t_0) = d$, $\dot{u}_1(t_0) > 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) = 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0[$ and $R(t) < 0$ for all $t \in \,]t_0\,; t_0 + \varepsilon[$;
   (b) the resting phase of a sticking contact: $u_1(t_0) > d$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) < 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0 + \varepsilon[$;
   (c) the end of a sticking contact: $u_1(t_0) = d$, $\dot{u}_1(t_0) < 0$, $u_2(t_0) = d$, and $\dot{u}_2(t_0) = 0$. There exists $\varepsilon > 0$ such that $R(t) < 0$ for all $t \in \,]t_0 - \varepsilon\,; t_0[$ and $R(t) = 0$ for all $t \in \,]t_0\,; t_0 + \varepsilon[$.

2.2.2 Periodic solutions with one sticking phase per period (1-SPP)

The main results of this chapter is concerned with the possible existence and computation of 1-SPP.

Definition 2.3 — One sticking phase per period solution. A function $\mathbf{u}$ is called a 1-SPP, if there exists $0 < \tau < T$ such that $\mathbf{u}$ is a T-periodic solution to (2.1) with one sticking phase per period, that is to say up to a time translation that

1. $u_2 = d$ on $[0\,; \tau]$,
2. $u_2 < d$ on $]\tau\,; T[$, and
3. $\mathbf{u}(T) = \mathbf{u}(0)$ and $\dot{\mathbf{u}}^-(T) = \dot{\mathbf{u}}^-(0)$.

Condition (2) above can be relaxed to $u_2(t) \leq d$ on $]\tau\,; T[$ only. This yields admissible periodic solutions with potentially many grazing contacts and sticking phases.

In order to find and characterize all 1-SPP, the following notations are needed:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad \mathbf{B} = \mathbf{P}^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{2.7}$$

$$\Phi_j(s) = \frac{\sin(\omega_j s)}{\omega_j(1-\cos(\omega_j s))} = \frac{1}{\omega_j}\cot\Big(\frac{\omega_j s}{2}\Big), \tag{2.8}$$

$$a_{kj} = -P_{kj}B_{j1}, \quad b_{kj} = \frac{a_{kj}}{\omega_j}, \quad \alpha_j = b_{1j} - b_{2j}, \quad \beta_j = b_{1j}, \tag{2.9}$$

$$w_k(s) = \sum_{j=1}^{2} a_{kj}\Phi_j(s) = \sum_{j=1}^{2} b_{kj}\cot\Big(\frac{\omega_j s}{2}\Big), \tag{2.10}$$

with $j = 1, 2$ and $k = 1, 2$. Note that the interaction coefficients a_{kj} in [25] and in this work have opposite sign. If a 1-SPP exists, then there is only one control parameter, the duration of the free flight $s = T - \tau$, which uniquely determines the 1-SPP through Theorem 2.4. The initial data and the period T are functions of s. Conversely, such initial data may generate a ghost solution [38] if u_2 becomes greater than d during the free flight.

The natural way to obtain periodic solutions for System (2.1) is to look for the fixed points of the associated first return map (FRM). From a quasi-explicit expression of the FRM, it appears that closed forms can be obtained except for an unknown parameter: the free flight time which is a root of an explicit function h. The roots of h are carefully explored to obtain a countably infinite set of initial data which yields 1-SPP if and only if the constraint $u_2 < d$ is satisfied during the free flight. This approach generalizes to the prestressed case $d \leq 0$.

Theorem 2.4 — 1-SPP characterization. Assume $\mathbf{u}$ is a 1-SPP of System (2.1), then:

1. The duration of the free flight $s > 0$ is necessarily a root of:

$$h(s) = w_1(s) - w_2(s) = \sum_{j=1}^{2} \alpha_j \cot\Big(\frac{\omega_j s}{2}\Big) = 0, \tag{2.11}$$

2. the solution $\mathbf{u}$ corresponds to the initial data

$$[u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)] = [d, d, v, 0], \quad \text{where} \quad v = v(s) = d/w_1(s). \tag{2.12}$$

3. The period T of $\mathbf{u}$ is a function of s: $T(s) = s + \tau(v(s))$, where τ is defined in (2.5).
4. The orbit is symmetric: $\mathbf{u}(\theta + t) = \mathbf{u}(\theta - t)$, $\forall t$, where $\theta := \tau + s/2$.

The proof of this theorem is given in Sections 2.5.1 and 2.5.2.

Remark 2.5. Instead of s, the parameter characterizing a 1-SPP could be the velocity v of the first mass at the beginning of the sticking phase. Choosing v specifies all the initial data (2.12) at the beginning of the sticking phase for a 1-SPP and the 1-SPP for all time. Accordingly, there is a one-to-one correspondence between the set of 1-SPP and the set of initial velocity $v = \dot{u}_1(t_0)$ which yields a 1-SPP. However, the 1-SPP closed-form expressions are simpler with s which is kept in the remainder.

Remark 2.6. The set of 1-SPP is at most countable and corresponds to a subset of roots of the analytic function $h(\cdot)$ defined in Equation (2.11).

The roots of the quasi-periodic function $h(\cdot)$ are the first quantities to be carefully investigated to find 1-SPP. In addition, the velocity of the first mass at the beginning of the sticking phase has to be positive, see Theorem 2.17. The sign of this velocity is governed by the sign of $w_1(s)$.

The sticking phase can now be exactly computed. Without loss of generality, assume $t_0 = 0$. The end of the sticking phase is the beginning of the free flight. By denoted $\underline{\mathbf{u}}$ the free flight portion of the solution with $\tau = \tau(s)$ leads to

$$(\underline{\mathbf{u}}, \dot{\underline{\mathbf{u}}})(\tau) := (\mathbf{u}, \dot{\mathbf{u}})(\tau) \tag{2.13}$$

$$\mathbf{M}\ddot{\underline{\mathbf{u}}}(t) + \mathbf{K}\underline{\mathbf{u}}(t) = 0, \quad \forall t \in]\tau\,; T[. \tag{2.14}$$

A solution to Equations (2.13)-(2.14) is a physically admissible solution to System (2.1) if it satisfies the constraint :

$$\underline{u}_2(t) < d, \quad \tau < t < T. \tag{2.15}$$

If condition (2.15) is violated, then the 1-SPP is not admissible: this is a “ghost” solution [38]. Hence, introducing the following sets:

$$Z = \{s > 0,\ h(s) = 0\}, \tag{2.16}$$

$$Z^- = \{s \in Z \text{ and } w_1(s) < 0\}, \tag{2.17}$$

$$Z^0 = \{s \in Z \text{ and } w_1(s) = 0\}, \tag{2.18}$$

$$Z^+ = \{s \in Z \text{ and } w_1(s) > 0\}, \tag{2.19}$$

$$Z^{\text{ad}} = \{s \in Z^+ \text{ and (2.15) is satisfied}\} \subset Z^+ \tag{2.20}$$

and $Z = Z^+ \cup Z^0 \cup Z^-$, the admissible free flight times s belongs to Z^+ which also corresponds to the “admissible” initial data. Furthermore, from the admissible initial data, the set of admissible 1-SPP has a one-to-one correspondence with Z^{ad}. Is Z^{ad} empty or not? Answering is not straightforward due to the global constraint (2.15) during the full free flight. However, we can quantify the size of Z^+ which leads to solutions satisfying (2.15) at least near the sticking phase. The following assumption 2.7 is needed to avoid that $Z^+ = \emptyset$, see Section 2.6 below.

Assumption 2.7. $\det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} \neq 0.$

Notice that if Assumption 2.7 is violated and $(\alpha_1, \alpha_2) \neq (0, 0)$, $(\beta_1, \beta_2) \neq (0, 0)$ then $h(\cdot)$ and $w_1(\cdot)$ have the same roots, i.e. $Z = Z^0$, and $Z^+ = \emptyset$ and there is no 1-SPP.

Theorem 2.8 — Countable infinity of Z^+. If $\omega_1/\omega_2 \notin \mathbb{Q}$ then Z is countably infinite. Moreover, if Assumption 2.7 holds, Z^+ is also countably infinite.

The proof of this theorem is exposed in Section 2.6. It is straightforward to show that Z is countably infinite when $\omega_1/\omega_2 \notin \mathbb{Q}$ since $h(\cdot)$ is quasi-periodic with many vertical asymptotes. The challenging part in Theorem 2.8 is to prove that Z^+ is also infinite. Incidentally, it turns out that Z^- is also infinite and more precisely that $\mathrm{card}(Z^+ \cap [0\,;A]) \sim \mathrm{card}(Z^- \cap [0\,;A])$ for large A. It means that many roots of h do not correspond to 1-SPP. Not only a 1-SPP is a rare object but among the roots of the function h, only a few correspond to an admissible 1-SPP.

In the next Section, the procedure to find 1-SPP is detailed.

2.3 Examples

To construct 1-SPP, Theorems 2.17 and 2.4 are interpreted as follows: let $s > 0$ satisfy $h(s) = 0$ and $w_1(s) > 0$. Such s is a candidate to construct a 1-SPP $\mathbf{u}$ to System (2.1) corresponding to the initial data

$$[u_1(0), u_2(0), \dot{u}_1(0), \dot{u}_2(0)]^\top = [d, d, +v, 0]^\top \text{ where } v = d/w_1(s), \tag{2.21}$$

with a sticking phase on $[0\,;\tau]$ and then a free-flight on $[\tau\,;\tau + s]$ with $\tau = \tau(s)$; more precisely:

- sticking phase for $t \in [0\,;\tau]$: mass 2 sticks to the wall and mass 1 behaves as a 1-dof linear oscillator.
- free flight for $t \in\,]\tau\,;\tau + s[$: System (2.1a) is solved with "initial" data at time $\tau = \tau(s)$:

$$[u_1(\tau), u_2(\tau), \dot{u}_1(\tau), \dot{u}_2(\tau)]^\top = [d, d, -v, 0]^\top. \tag{2.22}$$

The condition $u_2(t) < d$ is to be checked on the interval $]\tau\,;\tau + s[$ to obtain a real solution to System (2.1). Otherwise, an impact emerges before $\tau + s$ and the assumption of a free flight is violated on $]\tau\,;\tau + s[$ so that the corresponding $\mathbf{u}(t)$ is not a 1-SPP.

Accordingly, building a 1-SPP requires two numerical steps:

1. Compute the roots of $h(\cdot)$: Figure 2.3 depicts the set of roots as the intersections of $h(\cdot)$ and the x-axis.
2. Check the admissibility of the associated solution: check if $v > 0$ and if $u_2(t) < d$ for all $t \in\,]\tau(s)\,;\tau(s) + s[$. From the symmetry of the solution during the free flight, it is sufficient to check $u_2(t) < d$ for all $t \in\,]\tau\,;\tau + s/2[$.

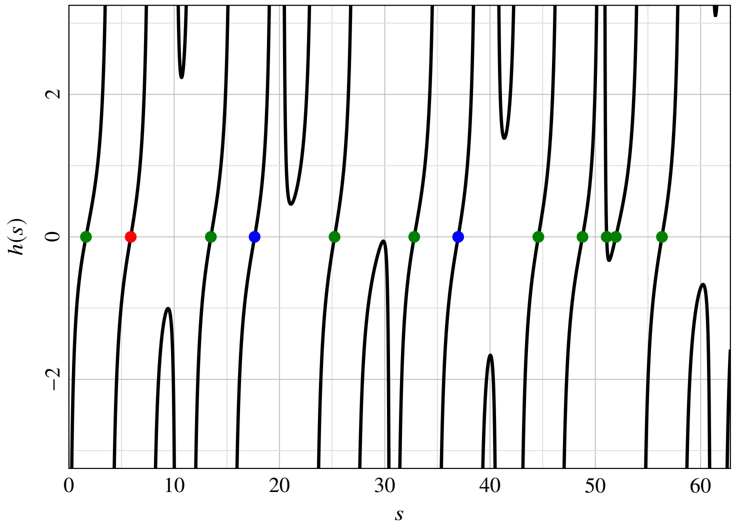


FIGURE 2.3 – $s \in Z = \{h(s) = 0\}$: red point: $s \in Z^{\mathrm{ad}}$; blue points: $s \in Z^+$ but $s \notin Z^{\mathrm{ad}}$; green points: $s \in Z^-$. The set of s corresponding to the admissible initial data are points in blue or red but only one point correspond to a 1-SPP: the red point.

First numerical examples are provided with $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 1$ N/m. Hence, the two natural periods of the unconstrained linear System (2.1a) are $T_1 \approx 10.17$ s and $T_2 \approx 3.88$ s.

Figure 2.2 shows the simplest 1-SPP one can find: only one loop for the orbit of the second mass. This orbit is very smooth except at one point corresponding to the whole sticking phase. At this *sticking point*, only a $C^{1.5}$-regularity is achieved as discussed in Section 2.4.3. Various examples featuring other responses are introduced in Figures 2.4 and 2.5. Many roots of h

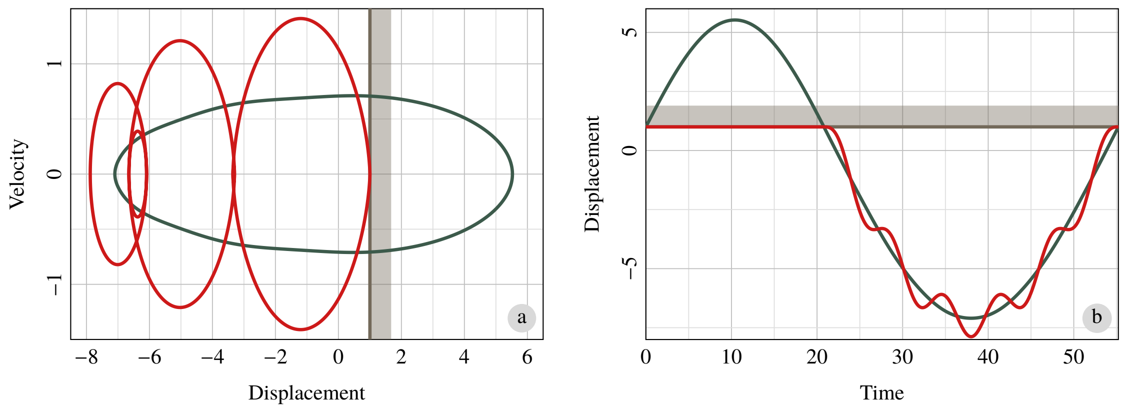


FIGURE 2.4 – Admissible 1-SPP with $k_1 = k_2 = 1; m_1 = 100; m_2 = 1$, initial data $U(0) = [1, 1, 0.7070682, 0]$: $s \approx 34.412$ s and $\tau \approx 20.804$ s. (a) Orbits. (b) Displacements

belonging to Z^+ do not correspond to 1-SPP. For instance, for $s \approx 17.97 \in Z^+$, the free-flight is not acceptable since the second mass penetrates the rigid obstacle, as pictured in Figure 2.6. The condition $s \in Z^+$ only stipulates that the non-penetration constraint (2.15) is satisfied near the sticking phase. Although Z^+ is countably infinite, it is challenging to find the set $Z^{\mathrm{ad}} \subset Z^+$

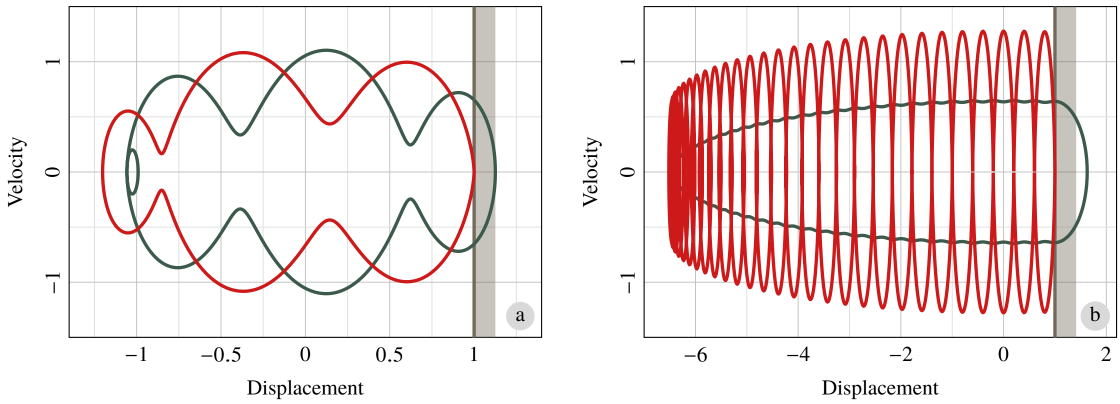


FIGURE 2.5 – 1-SPP Orbits. (a) $k_1 = 1$, $k_2 = 10k_1$ and $m_1 = m_2 = 1$, initial data $U(0) = [1, 1, 0.6525913, 0]$. (b) $k_1 = 1, k_2 = 100k_1$ and $m_2 = 1, m_1 = 100m_2$, initial data $U(0) = [1, 1, 0.6409175, 0]$.

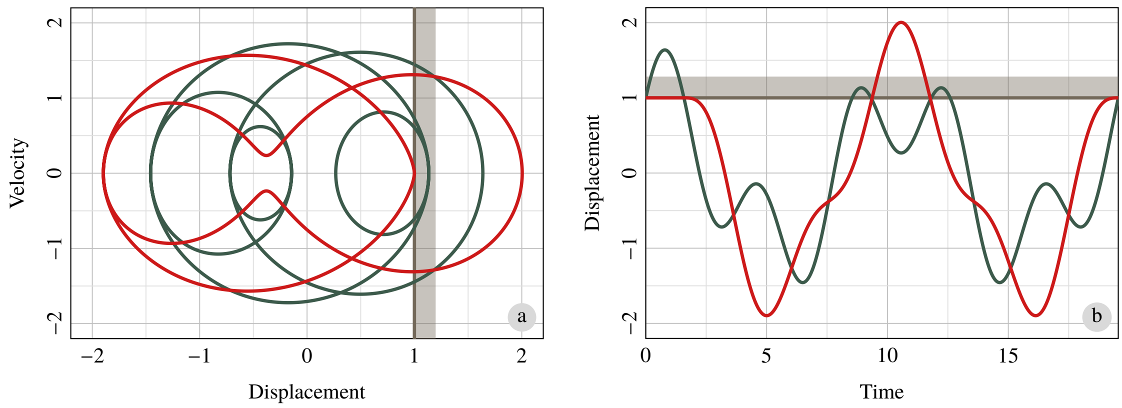


FIGURE 2.6 – Non-admissible 1-SPP with initial data $U(0) = [1, 1, 1.4447006, 0]$: mass 2 penetrates the wall during the free flight. (a) Orbits. (b) Displacements.

yielding 1-SPP. At first, for large s, the free-flight lasts a long period of time and the possibility that u_2 exceeds d seems to increase. Nevertheless, 1-SPP with large s are found in Figures 2.4-2.5.

2.4 Sticking contact

This section is devoted to the mathematical proof of Theorem 2.17 concerned with the necessary and sufficient conditions on the occurrence of a sticking phase[1]. The theory for such systems with impacts can be found in [1, 5, 7, 6, 40].

In order to experience a sticking phase, the first necessary condition is a zero-velocity of the contacting mass when the gap is being closed, see below or [5, 7]. Then, the sticking phase holds whenever there is a positive force generated from mass 1 on mass 2. This force is explicit through

[1]This theorem was also stated in [19].

the last equation of (2.1a). Next, the sticking system is presented. Via Lemma 3.16, it is clear that the energy of the unconstrained linear free flight system is conserved by the sticking system which becomes simply a 1-dof problem: the closed-form solution as well as the explicit duration of sticking phase are obtained.

The solution is analytic away from the beginning and the end of the sticking phase [5]. It can be seen from Figure 2.2(c) that the contact force is only Lipschitz at the beginning and at the end of the sticking phase. Thus, $\ddot{u}_2$ is a Lipschitz function, so the function u_2 belongs to the Sobolev space $W^{3,\infty}$. The smoother function u_1 belongs to $C^4 \cap W^{5,\infty}$: for both functions u_1 and u_2, the singularity is located on the boundary of the sticking phase. The orbit $\{(u_2(t), \dot{u}_2(t)),\ t \in \mathbb{R}\}$ has only a $C^{1.5}$-regularity at the sticking point. The singularity $C^{1.5}$ is visible in Figure 2.2 and is caused by the zero velocity and zero acceleration of mass 2 exactly when the sticking phase starts and ends. This loss of regularity is explored at the end of this Section. The prestressed case $d \le 0$ is covered in Section 2.7 with the occurrence of sticking phases of infinite duration.

2.4.1 Occurrence of a sticking phase

The proof for the necessary and sufficient conditions for a sticking phase to occur as stated in Theorem 2.17 is now given for $d > 0$. The case $d < 0$ is explained just after this proof.

Proof. The right and left analyticity of the solution for the perfect elastic rebound is used (Proposition 19 in [5]). Notice that the condition of a closed contact, *i.e.* $u_2(0) = d$, with zero velocity $\dot{u}_2^-(0) = 0$ is mandatory. Otherwise $\dot{u}_2^-(0) > 0$, $\dot{u}_2^+(0) = -\dot{u}_2^-(0) < 0$ and the mass immediately leaves the wall, that is $u_2(t) < d$ for $t > 0$ and $t \approx 0$ such that there is no sticking phase. The second equation of System (2.1a) is rewritten with the aforementioned initial data for mass 2 only:

$$\begin{aligned} &m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t)) + R(t), \\ &u_2(0) = d, \quad \dot{u}_2^-(0) = 0, \quad R(t) \le 0. \end{aligned} \tag{2.23}$$

During a sticking phase $u_2(t) \equiv d$ so $\ddot{u}_2(t) \equiv 0$ and Equation (2.23) yields the relation between the reaction $R(t)$ and the displacement $u_1(t)$:

$$R(t) = k_2(d - u_1(t)), \tag{2.24}$$

which is non-positive if and only if

$$u_1(t) \ge d. \tag{2.25}$$

Let us emphasize that Condition (2.25) is important in this work even though there is no unilateral constraint on u_1 in the formulation of System (2.1). During the sticking phase, Inequality (2.25) is satisfied. As a consequence, $\dot{u}_2(0) = 0$ is not a sufficient condition to ensure the existence of a sticking phase starting at time $t = 0$. Various situations depending on the state $(u_1(0), \dot{u}_1(0))$ should be scrutinized:

$u_1(0) < d$ The left-hand side of (2.23) is strictly negative and so is $\ddot{u}_2^+(0)$. There is no sticking phase. More precisely, u_2 is a piecewise analytic function [5] and its Taylor series in the right neighbourhood of 0 is:

$$u_2(t) = d + t^2 \frac{\ddot{u}_2^+(0)}{2} + O(t^3) < d. \tag{2.26}$$

$u_1(0) > d$ Since u_1 is continuous, it remains larger than d in a right neighbourhood $[0\,;\varepsilon_2[$ of $t = 0$. Thus, there is a positive force $F(t) = k_2(u_1(t) - u_2(t))$ acting on mass 2, and by Newton's third law, there exists a reaction $R(t)$ such that $R(t) = -F(t)$. Substitution into (2.23) yields $\ddot{u}_2(t) = 0$, $\forall t \in [0\,;\varepsilon_2[$. Hence $u_2(t) = d$, $\forall t \in [0\,;\varepsilon_2[$, *i.e.* a sticking phase emerges.

$u_1(0) = d$ In this case one has $\ddot{u}_2^+(0) = 0$ and there are three possibilities for the velocity of mass 1:

1. If $\dot{u}_1(0) > 0$, $u_1(t)$ becomes immediately larger than d for $t > 0$ small enough. This is similar to the previous case where a sticking phase occurs.
2. If $\dot{u}_1(0) < 0$ then $u_1(t)$ becomes immediately smaller than d and no sticking phase occurs. More precisely from Equation (2.23), $m_2\dddot{u}_2^+(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) < 0$ and the Taylor series of u_2 in the right neighbourhood of 0 is

$$u_2(t) = u_2(0) + t\dot{u}_2(0) + \frac{t^2}{2}\ddot{u}_2^+(0) + \frac{t^3}{6}\dddot{u}_2^+(0) + O(t^4) \tag{2.27}$$

$$= d + 0 + 0 + t^3 \frac{\dddot{u}_2^+(0)}{6} + O(t^4) < d. \tag{2.28}$$

3. If $\dot{u}_1(0) = 0$, then $\ddot{u}_1^+(0) = -k_1 d/m_1 < 0$. Thus $m_2\dddot{u}_2^+(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) = 0$ and $m_2 u_2^{(4)+}(0) = k_2(\ddot{u}_1^+(0) - \ddot{u}_2^+(0)) < 0$. Similarly, a Taylor series of u_2 in the right neighbourhood of 0 shows that $u_2(t) < d$ for $t > 0$ and $t \approx 0$. Thus, there is no sticking phase.

Note that only the last case $u_1(0) = d$ and $\dot{u}_1(0) = 0$ crucially depends on the sign of d. It is further discussed in Section 2.7 when $d \leq 0$. Moreover, all piecewise analytic solutions presented above preserve energy. It is clear for the grazing case since $R \equiv 0$. When sticking occurs, energy conservation is a consequence of Lemma 3.16. In conclusion, every introduced case corresponds to the unique solution preserving energy [5]. □

Remark 2.9. The duration of the sticking phase directly relates to the reaction $R(t)$. The sign of $R(t)$ is given by the sign of $d - u_1(t)$ via (2.24). Thus, for $d < 0$, an infinite sticking phase appears when the solution u_1 of the sticking equation (2.31) satisfies $u_1(t) \geq d = -|d|$ for all time, which is possible as detailed in Section 2.7.

2.4.2 Sticking system

It is now shown that the solution with a sticking phase expounded in Section 2.4.1 is the unique solution which preserves the total energy, see [5].

The notion of sticking system is explained in [7]. It is the reduced system during the sticking phase. Since the last mass is at rest, the system loses one degree of freedom. From the previous developments, the sticking system complemented by the initial data at the beginning of a sticking phase is explicitly derived as

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_2 = 0,\ u_1(0) = d,\ \dot{u}_1(0) = v > 0, \tag{2.29}$$

$$m_2\ddot{u}_2 = 0,\ u_2(0) = d,\ \dot{u}_2(0) = 0. \tag{2.30}$$

The sticking system (3.23), (3.24) becomes simply a sticking equation (2.31). The initial data for mass 1 has to be clarified. If $u_1(0) > d$ then this inequality is also valid locally in the past, and the sticking phase exists before $t = 0$. If $u_1(0) = d$ and $\dot{u}_1(0) > 0$, then there exists $\eta > 0$ such that $u_1(t) < d$ for $-\eta < t < 0$ so there is no sticking phase just before $t = 0$, in other words, $t = 0$ is the beginning of the sticking phase. The grazing contact case $\dot{u}_1(0) = 0$ and the case where constraint (2.15) is violated, $\dot{u}_1(0) < 0$, do not have to be considered.

During free-flight, there is a total energy (2.1d) for the symmetric system (2.14). However, during the sticking phase, system (3.23-3.24) is not symmetric. Accordingly, the question of conservation of energy during the sticking phase is not obvious. It is proven in the next lemma.

Lemma 2.10. The solution to Equations (3.23-3.24) conserves the energy E (2.1d).

Proof. Assume that $t = 0$ is the beginning of a sticking phase and $t = \tau$, the end. During this sticking phase on the interval $[0\,;\tau]$, the governing equations are

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 = k_2d, \tag{2.31}$$

$$u_2 = d. \tag{2.32}$$

The first equation conserves the energy around the new equilibrium $\overline{u}_1 = k_2d/(k_1 + k_2)$:

$$\mathbf{E}_1(u_1(t), \dot{u}_1(t)) = m_1\dot{u}_1^2(t) + (k_1 + k_2)(u_1(t) - \overline{u}_1)^2 = \mathbf{E}_1(u_1(0), \dot{u}_1(0)). \tag{2.33}$$

Moreover, since $u_1(t) = (u_1(t) - \overline{u}_1) + \overline{u}_1$ and $u_2(t) = d$, an easy computation yields:

$$(k_1 + k_2)u_1^2(t) = (k_1 + k_2)(u_1(t) - \overline{u}_1)^2 + 2k_2u_1(t)u_2(t) + C, \tag{2.34}$$

and $C = -(k_1 + k_2)\overline{u}_1^2$. The energy of System (2.1) can be calculated. Since $\dot{\mathbf{u}}$ is continuous along a sticking phase, the exponents ± are dropped:

$$E(\mathbf{u}(t),\dot{\mathbf{u}}(t)) = \dot{\mathbf{u}}^\top(t)\mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) \tag{2.35}$$

$$= m_1\dot{u}_1^2(t) + (k_1 + k_2)u_1^2(t) + m_2\dot{u}_2^2(t) + k_2u_2^2(t) - 2k_2u_1(t)u_2(t) \tag{2.36}$$

$$= m_1\dot{u}_1^2(t) + (k_1 + k_2)(u_1(t) - \overline{u}_1)^2 + 2k_2u_1(t)u_2(t) + C + k_2d^2 - 2k_2u_1(t)u_2(t) \tag{2.37}$$

$$= \mathbf{E}_1(u_1(t), \dot{u}_1(t)) + C + k_2d^2 = \mathbf{E}_1(u_1(0), \dot{u}_1(0)) + C + k_2d^2 = E(\mathbf{u}(0), \dot{\mathbf{u}}(0)). \tag{2.38}$$

This ends the proof: the total energy of the system is constant for all 1-SPP. □

The sticking system is now solved and the sticking time is explicitly exhibited: this is an interesting feature of our 2-dof mechanical system. The 1-dof linear oscillator problem with a constant force (2.31) has the explicit solution

$$u_1(t) = A\cos(\omega t + \phi) + \frac{k_2}{k_1 + k_2}d \quad \text{where} \quad \omega = \sqrt{\frac{k_1 + k_2}{m_1}}. \tag{2.39}$$

The expression of the constants A and ϕ stems from the initial condition $[u_1(0), \dot{u}_1(0)]^\top = [d, v]^\top$ as follows

$$A = \frac{k_1 d}{(k_1 + k_2)\cos(\phi)} \quad \text{and} \quad \phi = -\arctan(\xi v) \quad \text{with} \quad \xi = \frac{\sqrt{(k_1 + k_2)m_1}}{k_1 d} \tag{2.40}$$

and τ is the first positive time satisfying $u_1(\tau) = d$, that is $\tau = 2\arctan(\xi v)/\omega$. This is due to the symmetry of the solution to the 1-dof Problem (2.31) with respect to the u_1 axis in the plane $(u_1, \dot{u}_1)$, $u_1(\tau) = d$ and $\dot{u}_1(\tau) = -v$ which also means that τ is the end of the sticking phase via Theorem 2.17.

2.4.3 1.5-singularity at the sticking point

The following Proposition states precisely the regularity near a sticking phase, essentially C^2 and almost C^3. The lower $C^{1.5}$-regularity of the orbit is obtained at the end of the Section.

Proposition 2.11 — Regularity of solutions. Assume $\mathbf{u}(\cdot)$ is a solution of System (2.1) on $[T_0\,;T_1]$ with only a sticking phase on $[0\,;\tau]$ and a free flight elsewhere with $T_0 < 0 < \tau < T_1$. Then $u_1 \in C^4([T_0\,;T_1]) \cap W^{5,\infty}([T_0\,;T_1])$ and $u_2 \in C^2([T_0\,;T_1]) \cap W^{3,\infty}([T_0\,;T_1])$.

Proof. Away from the strict beginning and end of the sticking phase, the solution is regular: analytic outside $[0\,;\tau]$, u_1 is analytic and u_2 is constant inside $]0\,;\tau[$. The solution regularity at

$t = 0$ and $t = \tau$ is of higher interest. Only the case $t = 0$ is considered since the other case $t = \tau$ is quite similar. The initial data at $t = 0$ is $[\mathbf{u}(0)^\top, \dot{\mathbf{u}}(0)^\top]^\top = [d, d, v, 0]^\top$. The second Equation within (2.1a) is

$$m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t)) + R(t). \tag{2.41}$$

During the sticking phase, $0 < t < \tau$, $u_2(t) = d$ so $\ddot{u}_2(t) = 0$ and $\ddot{u}_2^+(0) = 0$ and before the sticking phase, $t < 0$, since $u_2(t) < d$, $R(t) = 0$ and $\lim_{0>t\to 0} u_2(t) = d = \lim_{0>t\to 0} u_1(t)$ so from Equation (2.41) $\ddot{u}_2^-(0) = 0$, thus $\ddot{u}_2$ is continuous at time $t = 0$ and $\ddot{u}_2(0) = 0$. However, the third derivative of u_2 on the left of $t = 0$ does not vanish since $m_2\dddot{u}_2^-(0) = k_2(\dot{u}_1(0) - \dot{u}_2(0)) = k_2 v > 0$ and $\dddot{u}_2$ is then bounded. Hence, $u_2 \in C^2([0\,;T]) \cap W^{3,\infty}([0\,;T])$.

The regularity of u_1 is investigated from the first Equation of (2.1a) which reads

$$m_1\ddot{u}_1 + (k_1 + k_2)u_1 = k_2 u_2, \tag{2.42}$$

and shows that $\ddot{u}_1$ and u_2 have the same regularity. Accordingly, $\ddot{u}_1 \in C^2([0\,;T]) \cap W^{3,\infty}([0\,;T])$ that is $u_1 \in C^4([0\,;T]) \cap W^{5,\infty}([0\,;T])$. □

We now prove the $C^{1.5}$-regularity of the orbit without using explicit formula.

Proof. The $C^{1.5}$-smoothness of the projection of the orbit on the last component, more precisely the regularity of the set

$$\Gamma_2 = \{\gamma(t) = (u_2(t), \dot{u}_2(t)),\ 0 \le t \le T\} \subset \mathbb{R}^2 \tag{2.43}$$

is explored. By T-periodicity, this parametrization is defined for all time. During the sticking phase, the last mass rests against the foundation, $\gamma(t) = \gamma(0) = (d, 0)$, $\dot{\gamma}(t) = (0, 0)$ for $0 \le t \le \tau$ and the parametrization is then singular. Instead, a regular parametrization of Γ_2 is thus proposed in the form $\tilde{\gamma}(t) = \gamma(t - \tau)$, $\tau \le t \le T$. In other words, $\tilde{\gamma}$ is γ where the sticking phase has been removed. Also, $\tilde{\gamma}$ is defined for all time through s-periodicity with $s = T - \tau$. The set $\tilde{\Gamma}_2 = \tilde{\gamma}([0\,;s])$ is exactly Γ_2. The curve is analytic away from the sticking point $(d, 0)$. A precise study of $\tilde{\gamma}(t)$, $|t| < \varepsilon$ should now be undertaken for $\varepsilon > 0$ sufficiently small. To this end, the left and right derivatives are computed since the solution is left and right analytic at the sticking point [5]:

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\tilde{\gamma}^-(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^-(0), \quad \frac{\mathrm{d}^k}{\mathrm{d}t^k}\tilde{\gamma}^+(0) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\gamma^+(\tau). \tag{2.44}$$

To compute the successive left and right derivatives, the ODE

$$m_2\ddot{u}_2(t) = k_2(u_1(t) - u_2(t)) \tag{2.45}$$

is used just before the sticking phase and just after the sticking phase. Recall that $u_1(0) = d$ and $\dot{u}_1(0) = v > 0$, $u_2(0) = d$ and $\dot{u}_2(0) = 0$, $u_1(\tau) = d$ and $\dot{u}_1(\tau) = -v < 0$, $u_2(\tau) = d$ and $\dot{u}_2(\tau) = 0$. The ODE gives $m_2\ddot{u}_2^-(0) = k_2(d-d) = 0$, $\ddot{u}_2^+(\tau) = 0$, so $\dot{\tilde{\gamma}}^{\pm}(0) = (0,0)$. The parametrization is still singular and higher derivatives of u_2 are computed by differentiating the ODE, that is

$$m_2\dddot{u}_2(t) = k_2(\dot{u}_1(t) - \dot{u}_2(t)) \tag{2.46}$$

$$\ddot{\tilde{\gamma}}^-(0) = (0,\beta), \quad \ddot{\tilde{\gamma}}^+(0) = (0,-\beta), \quad \beta = k_2 v/m_2 > 0 \tag{2.47}$$

$$m_2\ddddot{u}_2(t) = k_2(\ddot{u}_1(t) - \ddot{u}_2(t)) \tag{2.48}$$

$$\dddot{\tilde{\gamma}}^-(0) = (\beta,\delta), \quad \dddot{\tilde{\gamma}}^+(0) = (-\beta,\delta), \quad \delta = -k_2k_1d/(m_2m_1) < 0 \tag{2.49}$$

where the second derivative of u_1 comes from the equation $m_1\ddot{u}_1(t) = -k_1u_1(t) - k_2(u_1(t) - u_2(t))$: $\ddot{u}_1(0) = -k_1d/m_1 < 0$. The local behaviour at $t = 0$ is then for $\pm t > 0$:

$$\tilde{\gamma}(t) = (d,0) + \frac{1}{2}\operatorname{sign}(t)(0,\beta)t^2 + \frac{1}{6}(\operatorname{sign}(t)\beta,\delta)t^3 + O(t^4). \tag{2.50}$$

There are two singularities for this parametrization: the left and right expansions for $\pm t > 0$, and the more important $\dot{\tilde{\gamma}}(0) = (0,0)$. To clearly identify the regularity of the curve at $t = 0$, a last change of variable is performed [38]: $\tau = \operatorname{sign}(t)t^2$ and $\hat{\gamma}(\tau) = \tilde{\gamma}(t)$ such that:

$$\hat{\gamma}(\tau) = (d,0) + \frac{1}{2}(0,\beta)\tau + \frac{1}{6}(\beta,\operatorname{sign}(\tau)\delta)|\tau|^{1.5} + O(\tau^2). \tag{2.51}$$

The $C^{1.5}$-regularity is then identified since $\dot{\hat{\gamma}}(0) \neq (0,0)$ and this is optimal. □

2.5 Building 1-SPP

This Section addresses the construction of the 1-SPP developed in Section 2.2.2. An explicit formula for τ is obtained and the set of admissible initial data is derived. The initial velocity of the first mass depends on the free flight time s and it is proven that s can be found in the infinite set of roots of $h(\cdot)$. The symmetry of the solutions is also discussed.

2.5.1 Initial data

Without loss of generality, the initial data is defined in the Poincaré section $u_2 = d$. The problem is to find a periodic function $\mathbf{u}$ generated by the initial data $[d,d,v,0]^\top$ such that there is one sticking phase per period. As explained previously, T and τ are parameterized by s. The sticking solution and the sticking time $\tau > 0$ are calculated explicitly in Section 2.4.2.

By denoting $\mathbf{U} = [\mathbf{u}^\top, \dot{\mathbf{u}}^\top]^\top$, a free flight starts with the following initial data at time τ:

$$\mathbf{U}(\tau) = [d, d, -v, 0]^\top. \tag{2.52}$$

It can be written as

$$\mathbf{U}(\tau) = \mathbf{S}\mathbf{U}(0) \quad \text{where} \quad \mathbf{S} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{bmatrix} \quad \text{and} \quad \mathbf{L} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.53}$$

Away from the sticking phase, system (2.1a) simplifies to

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{2.54}$$

Through the change of variable $\mathbf{u} = \mathbf{P}\mathbf{q}$ where $\mathbf{P}$ is defined in (2.7), Equation (2.54) becomes

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{\Omega}^2\mathbf{q} = \mathbf{0} \tag{2.55}$$

which features the following block matrix solution

$$\mathbf{Q}(t) = \begin{bmatrix} \mathbf{q}(t) \\ \dot{\mathbf{q}}(t) \end{bmatrix} = \mathbf{R}(t-\tau) \begin{bmatrix} \mathbf{q}(\tau) \\ \mathbf{q}(\tau) \end{bmatrix}, \quad \forall t \in \,]\tau\,; T[\tag{2.56}$$

where

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{cos}(t\,\mathbf{\Omega}) & \mathbf{\Omega}^{-1}\,\mathbf{sin}(t\,\mathbf{\Omega}) \\ -\mathbf{\Omega}\,\mathbf{sin}(t\,\mathbf{\Omega}) & \mathbf{cos}(t\,\mathbf{\Omega}) \end{bmatrix}. \tag{2.57}$$

We shall find the solution $\mathbf{u}$ and the period T such that

$$\mathbf{Q}(T) = \mathbf{Q}(0). \tag{2.58}$$

By denoting $s = T - \tau$, Equation (2.58) projected onto modal coordinates reads

$$\mathbf{R}(s)\tilde{\mathbf{S}}\mathbf{Q}(0) = \mathbf{Q}(0) \quad \text{where} \quad \tilde{\mathbf{S}} = [\mathbf{B}]\,\mathbf{S}\,[\mathbf{P}] \quad \text{and} \quad [\mathbf{P}] = \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{P} \end{bmatrix} \tag{2.59}$$

which can be expressed as $(\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I})\mathbf{Q}(0) = \mathbf{0}$ with

$$\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I} = \begin{bmatrix} \mathbf{cos}(\mathbf{\Omega}s) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(\mathbf{\Omega}s)\mathbf{BLP} \\ -\mathbf{\Omega}\,\mathbf{sin}(\mathbf{\Omega}s) & \mathbf{cos}(\mathbf{\Omega}s)\mathbf{BLP} - \mathbf{I} \end{bmatrix} \tag{2.60}$$

where $\mathbf{B}$ is defined in (2.7). The computations are similar to those introduced in [25]. This similarity will be explained later through the relationship between 1-SPP and the one-Impact-Per-Period solutions (1-IPP) detailed in [25][2].

The duration s is assumed not to be a period of the linear differential system, $s \notin \cup_{j=1}^{2} T_j\mathbb{Z}$ where $T_j = 2\pi/\omega_j$, $j = 1, 2$ are frequencies of the linear system. Then, the following quantities are well defined:

$$\mathbf{\Phi}(s) = (\mathbf{I} - \mathbf{cos}(\mathbf{\Omega}s))^{-1}\mathbf{\Omega}^{-1}\,\mathbf{sin}(\mathbf{\Omega}s), \tag{2.61}$$

$$\mathbf{w}(s) = \mathbf{P\Phi}(s)\mathbf{BLe}_1, \quad \mathbf{e}_1 = (1, 0)^\top, \tag{2.62}$$

$$w_1(s) = \mathbf{e}_1^\top \mathbf{w}(s). \tag{2.63}$$

The solution set of initial data yielding 1-SPP, possibly "ghost" solutions if constraint (2.15) is violated, is described explicitly via the following lemma:

Lemma 2.12. If $s \notin \cup_{j=1}^{2} T_j\mathbb{Z}$ then the system

$$\mathbf{R}(s)\tilde{\mathbf{S}}\mathbf{Q}(0) = \mathbf{Q}(0) \tag{2.64}$$

defines a one dimensional vector space parametrized by $c \in \mathbb{R}$ given in variables

$$\begin{bmatrix} \mathbf{u}(0) \\ \dot{\mathbf{u}}(0) \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{q}(0) \\ \dot{\mathbf{q}}(0) \end{bmatrix} = c \begin{bmatrix} \mathbf{w}(s) \\ \mathbf{e}_1 \end{bmatrix} \tag{2.65}$$

Proof. Compute $\ker(\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I})$ by blocks (see [25]):

$$\begin{bmatrix} \mathbf{cos}(\mathbf{\Omega}s) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(\mathbf{\Omega}s)\mathbf{BLP} \\ -\mathbf{\Omega}\,\mathbf{sin}(\mathbf{\Omega}s) & \mathbf{cos}(\mathbf{\Omega}s)\mathbf{BLP} - \mathbf{I} \end{bmatrix} \sim \begin{bmatrix} \mathbf{cos}(\mathbf{\Omega}s) - \mathbf{I} & \mathbf{\Omega}^{-1}\,\mathbf{sin}(\mathbf{\Omega}s)\mathbf{BLP} \\ \mathbf{0} & (\mathbf{L} + \mathbf{I})\mathbf{P} \end{bmatrix} \tag{2.66}$$

because the matrix $(\mathbf{I} - \mathbf{cos}(\mathbf{\Omega}s))^{-1}\mathbf{B}$ is invertible. Since $\dot{\mathbf{u}} = \mathbf{P}\dot{\mathbf{q}}$, the right lower block in (2.66) simplifies to $(\mathbf{L} + \mathbf{I})\dot{\mathbf{u}} = \mathbf{0}$, that is $\dot{\mathbf{u}} = c\mathbf{e}_1$ with $c \in \mathbb{R}$. Similarly, the upper block provides the expression $\mathbf{q} = c(\mathbf{I} - \mathbf{cos}(\mathbf{\Omega}s))^{-1}\mathbf{\Omega}^{-1}\,\mathbf{sin}(\mathbf{\Omega}s)\mathbf{BLe}_1$. □

The assumption of lemma 2.12 is always valid and proven in the next lemma: the free flight duration of any 1-SPP is never a (multiple of a) period of the linear system.

Lemma 2.13. For a 1-SPP, the duration s of the free-flight is not a linear period: $s \notin \cup_{j=1}^{2} T_j\mathbb{Z}$.

Proof. It is proven that there is no 1-SPP involving a free-flight duration as a linear period. In other words, if $s \in \cup_{j=1}^{2} T_j\mathbb{Z}$, i.e. there exists $k, \ell \in \mathbb{Z}$ such that $s = kT_1$ or $s = \ell T_2$, then the corresponding solutions must be linear grazing orbits.

[2]Note that there is a change of sign in $\mathbf{w}(s)$ due to the coefficient $a_{kj} = -P_{kj}B_{j1}$ instead of $P_{kj}B_{jN}$ in [25].

Firstly, let us emphasize that all the components of the matrix $\mathbf{P} = (P_{ij})_{i,j=1}^2$ are nonzero in our case of study, a chain of two masses. The matrix $\mathbf{P}$ of eigenvectors can be computed explicitly and has the following formula

$$\mathbf{P} = \begin{bmatrix} \dfrac{a+\sqrt{a^2+b^2}}{\sqrt{m_1((a+\sqrt{a^2+b^2})^2+b^2)}} & \dfrac{a-\sqrt{a^2+b^2}}{\sqrt{m_1((a-\sqrt{a^2+b^2})^2+b^2)}} \\ \dfrac{b}{\sqrt{m_2((a+\sqrt{a^2+b^2})^2+b^2)}} & \dfrac{b}{\sqrt{m_2((a-\sqrt{a^2+b^2})^2+b^2)}} \end{bmatrix} \tag{2.67}$$

where

$$a = \frac{1}{2}\Big(\frac{k_2}{m_2} - \frac{k_1+k_2}{m_1}\Big), \quad b = \frac{k_2}{\sqrt{m_1 m_2}}. \tag{2.68}$$

Hence, $P_{ij} \neq 0$ for all $i, j = 1, 2$.

Secondly, one shows that if $s = kT_1$ then the periodic solutions with the free-flight duration s must be the first linear grazing orbit. Consider $s = kT_1$, then the matrix $\mathbf{R}(s)\tilde{\mathbf{S}} - \mathbf{I}$ becomes

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C_2-1 & S_2/\omega_2 A_{21} & S_2/\omega_2 A_{22} \\ 0 & 0 & A_{11}-1 & A_{12} \\ 0 & -\omega_2 S_2 & C_2 A_{21} & C_2 A_{22}-1 \end{bmatrix} \tag{2.69}$$

where $C_2 = \cos(2\pi\omega_2/\omega_1)$, $S_2 = \sin(2\pi\omega_2/\omega_1)$, and $\mathbf{A} = \mathbf{BLP}$. With the assumption of no internal resonance, $C_2 \neq 0$, $C_2 - 1 \neq 0$ and $S_2 \neq 0$. Hence, after simple calculations, the last three equations become

$$\begin{bmatrix} C_2-1 & S_2/\omega_2 A_{21} & S_2/\omega_2 A_{22} \\ 0 & A_{11}-1 & A_{12} \\ -\omega_2 S_2 & C_2 A_{21} & C_2 A_{22}-1 \end{bmatrix} \sim \begin{bmatrix} C_2-1 & S_2/\omega_2 A_{21} & S_2/\omega_2 A_{22} \\ 0 & A_{11}-1 & A_{12} \\ 0 & A_{21} & A_{22}+1 \end{bmatrix} \tag{2.70}$$

The last two equations can be rewritten as $(\mathbf{A}+\mathbf{L})\,\dot{\mathbf{q}}(0) = \mathbf{0}$ or $(\mathbf{LP}+\mathbf{PL})\dot{\mathbf{q}}(0) = \mathbf{0}$. Since $\mathbf{LP}+\mathbf{PL} = \mathbf{diag}(-2P_{11}, 2P_{22})$, where P_{11} and P_{22} are nonzero, this yields $\dot{\mathbf{q}}(0) = \mathbf{0}$. By substituting into the second equation, it follows that $q_2(0) = 0$ or $u_1(0) = -B_{22}/B_{21}d$. Therefore, the initial data is $[u_1(0), d, 0, 0]$ which corresponds to the first linear grazing orbit.

Similarly, if $s = \ell T_2$ then the periodic solutions with the free-flight duration s must be the second linear grazing orbit. One obtains a similar expression $(\mathbf{A}-\mathbf{L})\dot{\mathbf{q}}(0) = \mathbf{0}$ or $(\mathbf{LP}-\mathbf{PL})\dot{\mathbf{q}}(0) = \mathbf{0}$.

Since

$$\mathbf{LP} - \mathbf{PL} = \begin{bmatrix} 0 & -2P_{12} \\ -2P_{21} & 0 \end{bmatrix} \tag{2.71}$$

where both P_{12} and P_{21} are both non-vanishing terms, then $\dot{\mathbf{q}}(0) = \mathbf{0}$. Consequently, the initial data obtained correspond to the second linear grazing orbit. □

The parameter c is identified from the third row of (2.65), that is $c = \dot{u}_1^+(0) = v$. By expressing the initial condition, it follows that

$$\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix} = \begin{bmatrix} d \\ d \end{bmatrix} = v \begin{bmatrix} w_1(s) \\ w_2(s) \end{bmatrix} = c\,\mathbf{P\Phi}(s)\mathbf{BLe}_1. \tag{2.72}$$

System (2.72) simplifies to $w_1(s) = w_2(s)$ or

$$h(s) = w_1(s) - w_2(s) = \sum_{j=1}^{2} \alpha_j \cot\left(\frac{\omega_j s}{2}\right) = 0, \tag{2.73}$$

and the initial velocity of the first mass is found from

$$v = \frac{d}{w_1(s)}, \quad w_1(s) > 0. \tag{2.74}$$

If $\omega_1/\omega_2 \notin \mathbb{Q}$, the function $h(\cdot)$ exhibits a countably infinite set of roots. Moreover the set of s such that $h(s) = 0$ and $v(s) > 0$ is also countably infinite by Theorem 2.8. The particular case when $\omega_1/\omega_2 \in \mathbb{Q}$ is discussed in Section 2.6.

2.5.2 Symmetry

To conclude the validation of Theorem 2.4, the symmetry of 1-SPP is proven.

Proof. Through periodicity, it is sufficient to check the symmetry of the solutions on one period only. The symmetry is satisfied during the sticking phase and the free flight. Since only the first mass oscillates during the sticking phase, the solution is symmetric. Let us check the symmetry of solutions during the free flight time $t \in [\tau\,; T]$ where $\mathbf{u}(\tau) = \mathbf{u}(T)$ and $\dot{\mathbf{u}}^+(\tau) = -\dot{\mathbf{u}}^-(T)$. Denoting $\theta = (T + \tau)/2$, it is sufficient to show that $\mathbf{u}(\theta + t) = \mathbf{u}(\theta - t)$, $\forall t \in I = [-s/2\,; s/2]$. Let $\mathbf{z}_+$ be the function defined on I such that $\mathbf{z}_+(t) = \mathbf{u}(\theta + t)$. Then $\mathbf{z}_+$ is a well defined smooth function on I with $\mathbf{z}_+(s/2) = \mathbf{u}(T)$ and $\dot{\mathbf{z}}_+(s/2) = \dot{\mathbf{u}}^-(T)$. Similarly, by defining the function $\mathbf{z}_-(t) := \mathbf{u}(\theta - t)$, for $t \in I$, it can be checked that $\mathbf{z}_-(s/2) = \mathbf{u}(\tau)$ and $\dot{\mathbf{z}}_-(s/2) = -\dot{\mathbf{u}}^+(\tau)$. Furthermore, both $\mathbf{z}_+$ and $\mathbf{z}_-$ are solutions to the linear differential system $\mathbf{M}\ddot{\mathbf{z}} + \mathbf{Kz} = \mathbf{0}$ on I. Notice that $\mathbf{z}_+$ and $\mathbf{z}_-$ have the same initial data $\mathbf{z}_+(s/2) = \mathbf{z}_-(s/2)$ and $\dot{\mathbf{z}}_+(s/2) = \dot{\mathbf{z}}_-(s/2)$.

Hence, by the uniqueness of the initial value problem, it is deduced that $\mathbf{z}_+(t) \equiv \mathbf{z}_-(t)$ on I. This completes the proof on the symmetry of solutions. □

2.5.3 Relationship between 1-SPP and 1-IPP

For this two-degree-of-freedom vibro-impact system, a relationship between one-sticking-phase-per-period and one-impact-per-period solutions [25] is exhibited. It clarifies the similarities and differences of such periodic solutions.

Consider the two Figures 2.2(a) and 2.2(b) showing a single loop in Γ_2. Figures 2.7(a) and 2.7(b) are then obtained by "deleting" the sticking phase on the whole interval $]0\,;\tau[$ such that a 1-IPP solution [19, 25] is identified, where the jump occurs on mass 1 (instead of mass 2) when $u_1(0) = d$ as well as $u_2(0) = d$. Take note that $u_1(t) \leq d$ for all time. It is important to note that this 1-IPP is "unique" is the sense that it satisfies $u_2(0) = d$; it is denoted 1-IPP$_\text{p}$ in the remainder. The correspondence between 1-SPP and this particular 1-IPP$_\text{p}$ is now detailed. To

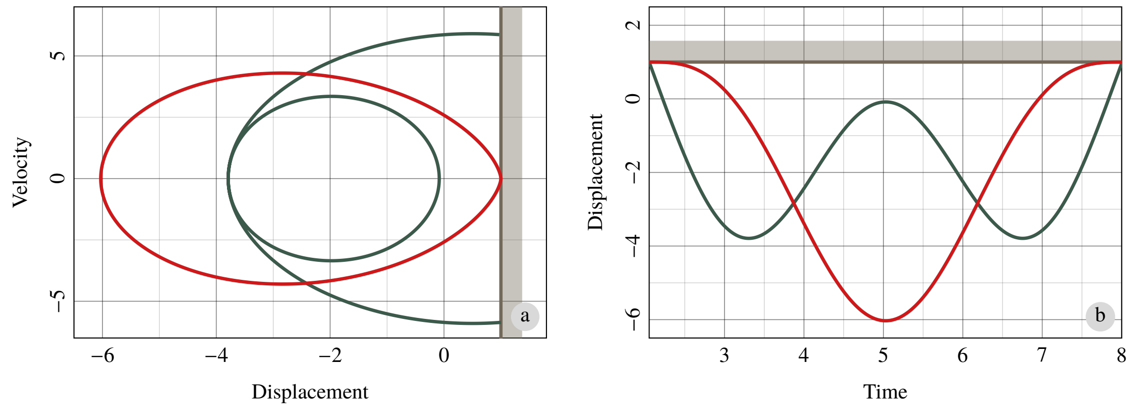


FIGURE 2.7 – (a) Is a 1-IPP or a 1-SPP without sticking phase drawn? (b) Displacements

this end, generalized 1-SPP and 1-IPP$_\text{p}$, *ie* G1-SPP and G1-IPP$_\text{p}$ respectively, are first defined: they are unconstrained 1-SPP and 1-IPP$_\text{p}$ during the free flight and u_1 as well as u_2 might exceed d[3]. By definition, a G1-SPP $\mathbf{u}$ satisfies the following requirements:

1. $s \in Z$,
2. $T = s + \tau$ is the fundamental period with $\tau = \tau(s)$,
3. a sticking phase on $]0\,;\tau(s)[$ with $u_1(0) = d$, $\dot{u}_1(0) = v = v(s)$, $u_2(0) = d$, $\dot{u}_2(0) = 0$,
4. a free flight on $]\tau\,;T[$ with $u_1(\tau) = d$, $\dot{u}_1(\tau) = -v$, $u_2(\tau) = d$, $\dot{u}_2(\tau) = 0$, $\mathbf{Mu} + \mathbf{Ku} = \mathbf{0}$.

[3]The G1-IPP in this chapter has a counterpart in [25]: it is a G1-IPP where the jump in velocity affects the second mass instead of the first mass here. As such, we know that there is a unique G1-IPP for all positive periods. This is the reason why the formulas in Section 2.5 are slightly different from [25]. The condition on mass 1 in Theorem 2.17 corresponds to an elastic impact for mass 1.

The one-to-one correspondence from Figure 2.2 to Figure 2.7 is formalized as

$$\widetilde{\mathbf{u}}(t) = \mathbf{u}(\tau + t), \quad 0 < t < s, \tag{2.75}$$

where $\widetilde{\mathbf{u}}$ is taken s-periodic so that $\widetilde{\mathbf{u}}^-(0) = \mathbf{u}(0)$, $\widetilde{\mathbf{u}}^+(0) = \mathbf{u}(\tau)$. As a consequence, $\widetilde{\mathbf{u}}$ satisfies

1. s is the fundamental period,
2. $\widetilde{u}_1^{\pm}(0) = d$, $\dot{\widetilde{u}}_1^-(0) = \dot{u}_1(0) = v$, $\dot{\widetilde{u}}_1^+(0) = \dot{u}_1(\tau) = -v$,
3. $\widetilde{u}_2^{\pm}(0) = d$, $\dot{\widetilde{u}}_2^{\pm}(0) = 0$,
4. a free flight on $]0\,;s[$: $\mathbf{M}\ddot{\widetilde{\mathbf{u}}} + \mathbf{K}\widetilde{\mathbf{u}} = \mathbf{0}$.

We can check that $\tilde{\mathbf{u}}$ is a G1-IPP$_\text{p}$. The only surprising condition is $\dot{\widetilde{u}}_2(0) = 0$ but zero velocity is automatically achieved by a G1-IPP [5, 25].

Proposition 2.14 — G1-SPP ⇔ G1-IPP$_\text{p}$. There is a one-to-one correspondence between G1-SPP with a sticking phase for the mass 2 and G1-IPP$_\text{p}$.

Proof. This is a brief sketch. G1-SPP ⇒ G1-IPP$_\text{p}$ was explained previously. Conversely, from a given G1-IPP$_\text{p}$, it is possible to build a sticking phase as in the proof of Theorem 2.17 in Section 2.4 with a free flight duration s to then define a unique G1-SPP. □

The key parameter s appears to be simply the period of the associated G1-IPP$_\text{p}$. This proposition shows that the set Z corresponds exactly to the set of all G1-IPP$_\text{p}$. Let us state briefly the correspondence between Z^+ and Z^{ad} and the corresponding subset of all G1-IPP$_\text{p}$.

Concerning generalized solutions with a positive velocity at the impact ($v > 0$), it can be said that for all $s \in Z^+$ there exists a unique G1-SPP and a corresponding unique G1-IPP$_\text{p}$ which has a physical initial data at the impact time (no violation of the constraint near the impact time). Conversely, if a G1-IPP$_\text{p}$ is such that, at the impact time, the incoming velocity of mass 1 is positive then the period belongs to Z^+ which corresponds to a unique G1-SPP.

Finally, a 1-SPP, *i.e.* a G1-SPP satisfying the constraint $u_2(t) \leq d$ for all time, is in a unequivocal correspondence with a G1-IPP$_\text{p}$ satisfying the same constraint. This condition is not the constraint to be a 1-IPP since the constraint for 1-IPP is on the mass 1. Figures 2.2 and 2.7 show a perfect and rare correspondence between a 1-SPP and a 1-IPP since the associated G1-IPP$_\text{p}$ satisfies the two constraints $u_k(t) \leq d$ for all time and $k = 1, 2$. As a consequence, 1-SPP are isolated solutions. The reason lies in the fact that the space of G1-IPP is a one-dimensional manifold which intersects $\widetilde{u}_2 = d$ on a discrete set such that the G1-IPP become isolated. Another possible consequence of Proposition 2.14, which is not further discussed here, is to prove the existence of 1-SPP through the proof of the existence of such particular G1-IPP.

2.6 The countable set $Z^{\pm}$

In order to find admissible solutions, the set $Z^{\pm}$ are defined to encompass the corresponding set of initial data of admissible solution: $\mathcal{V}_0^{ad}$. Admissible initial data is defined by a constraint on the associated solution: $u_2(t) \leq d$ at least locally near the sticking phase. The problem to satisfy this constraint is only at the end and the beginning of the sticking phase but not for all time. Thus, a question of interest emerges: how large is this set "admissible initial data"? The set of admissible initial data $\mathcal{V}_0^+$ contains$\mathcal{V}_0^{\mathrm{ad}}$. In this Section, the sets Z^+ and Z^- are proven to be countably infinite if some generic assumptions are fulfilled. The proof of Theorem 2.8 is similar for the two sets and only the proof for Z^+ is provided.

2.6.1 $Z^{\pm}$ is infinite with no resonance

Before stating the main proof with $\omega_1/\omega_2 \notin \mathbb{Q}$, we start with Lemma 2.15 below. First, the orbit O is defined in the torus $\Pi = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$:

$$O = \{(x, y) = (\bar{t}, \overline{\rho t}) | t > 0\} \tag{2.76}$$

where $\bar{t} = t + 2\pi\mathbb{Z}$ and ρ, a constant.

Lemma 2.15 — Transversality and density. Let f be a $2\pi-$periodic continuously differentiable function from $[0\,;2\pi[$ to $[0\,;2\pi[$. For any irrational number ρ, if $(x_0, y_0 = f(x_0))$ located on the curve C defined by the graph of f satisfies the transversal condition between C and O, that is

$$\dot{f}(x_0) \neq \rho \tag{2.77}$$

then $\forall \varepsilon > 0$, $\exists t > 0$ such that $\overline{\rho t} = f(\bar{t})$ and $|\bar{t} - x_0| < \varepsilon$.

In other words, every point on the curve C at which the tangent is transverse to the orbit O is an accumulation point of $O \cap C$, see Figure 2.8. Precisely, the set $O \cap C$ is dense in $\{(x, f(x)) | \dot{f}(x) \neq \rho\}$. Moreover, for all $A > 0$, the set $O_A = \{(x, y) = (\bar{t}, \overline{\rho t}) | t > A\}$ shares the same property.

Proof. Assume $\rho > 0$, the cases $\rho = 0$ and $\rho < 0$ follow immediately.

Since $\dot{f}(x_0) \neq \rho$ and $\dot{f}$ is continuous, there exists $\varepsilon_0 > 0$ small enough such that $\dot{f}(x) \neq \rho$for all $x \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$. Without loss of generality, assume that $\dot{f}(x) > \rho$, for all $x \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$. Since O is dense in Π, $\forall \varepsilon > 0$, there exists $t_0 > 0$ such that $z = (\overline{t_0}, \overline{\rho t_0})$ belongs to O close enough to (x_0, y_0), *i.e.* $|\overline{t_0} - x_0| < \varepsilon$ and $|\overline{\rho t_0} - y_0| < \varepsilon$: if z is on the curve C then t is chosen to be t_0, else z is above the curve C, *i.e.* $\overline{\rho t_0} > f(\overline{t_0})$.

We will show that the orbit O intersects the curve C inside the box $]x_0 - \varepsilon_0\,; x_0 + \varepsilon_0[\times]y_0 - k\varepsilon_0\,; y_0 + k\varepsilon_0[$ where k is the maximum of $|\dot{f}|$ on $[x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$ as shown in Figure 2.9. For

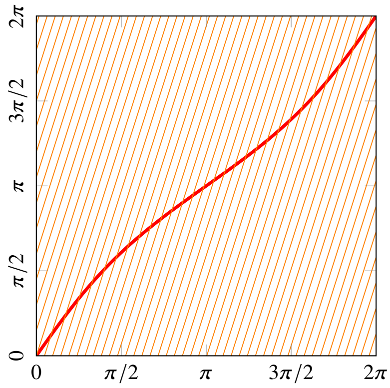


FIGURE 2.8 – Density of $\mathbf{O} \cap \mathbf{H}$ in $\mathbf{H}$

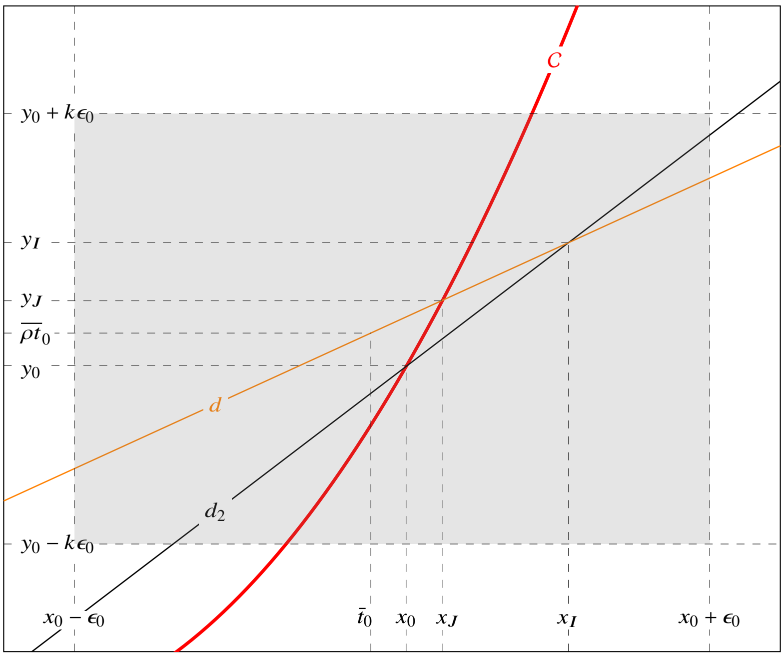


FIGURE 2.9 – Zoom in the box $]x_0 - \varepsilon_0\,; x_0 + \varepsilon_0[\times]y_0 - k\varepsilon_0\,; y_0 + k\varepsilon_0[$ when $\dot{\psi}(x_0) > \rho$

this purpose, we use a line d_2 under the curve on the right of (x_0, y_0). From $p = \min_{[x_0-\varepsilon_0\,;x_0+\varepsilon_0]} \dot{f}$, the equation of the line d_2 with slope p passing through (x_0, y_0) is $y = p(x - x_0) + y_0$. The line d with slope ρ passing through $(\overline{t_0}, \overline{\rho t_0})$ and associated to the orbit O is defined by $y = \rho(x - \overline{t_0}) + \overline{\rho t_0}$. Let $I(x_I, y_I)$ be the intersection of those two lines. Since $p > \rho$, we have

$$x_I = \frac{px_0 - y_0 - \rho\overline{t_0} + \overline{\rho t_0}}{p - \rho}. \tag{2.78}$$

Choosing ε small enough such that $\varepsilon < \varepsilon_0(|p - \rho|)/(\rho + k)$ implies $x_I \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$.

Consider the two curves d_2 and C intersecting at (x_0, y_0) and satisfying $\dot{f}(x) > p$ for all $x \in [x_0 - \varepsilon_0\,; x_0 + \varepsilon_0]$. Since $p > \rho$, d intersects d_2 at I. Hence, there exists an intersection of C and d in the interval $]x_0\,; x_I[$. In other words, there exists $t > 0$ such that $\overline{\rho t} = f(\bar{t})$ and

$|\bar{t} - x_0| < \varepsilon_0$. The proof for the case z is under the curve C is similar. □

The proof of Theorem 2.8 starts by showing that the set $Z = \{s > 0,\ h(s) = 0\}$ is countably infinite. It is true for the set $\{(\omega_1 s, \omega_2 s),\ h(s) = 0\}$ and will be useful to prove that the set Z^+ of free flight times s with admissible initial velocity $v(s) > 0$ is also countably infinite.

Proof. Set $\varphi(t) = \cot(t/2)$, then $h(s) = \alpha_1\varphi(\omega_1 s) + \alpha_2\varphi(\omega_2 s)$ and $w_1(s) = \beta_1\varphi(\omega_1 s) + \beta_2\varphi(\omega_2 s)$ where $\beta_j = b_{1j}$, and α_j, b_{kj} are defined in Equation (2.9). For every $(x, y) \in \Pi = \mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$, the two functions $H(x, y) = \alpha_1\varphi(x)+\alpha_2\varphi(y)$ and $W(x, y) = \beta_1\varphi(x)+\beta_2\varphi(y)$ correspond to $h(s) = H(\omega_1 s, \omega_2 s)$, $w_1(s) = W(\omega_1 s, \omega_2 s)$. In order to simplify, the sets $\mathbf{O} = \{(\overline{\omega_1 s}, \overline{\omega_2 s}) | s > 0\}$, $\mathbf{H} = \{(x, y) \in \Pi | H(x, y) = 0\}$, and $\mathbf{W} = \{(x, y) \in \Pi | W(x, y) = 0\}$ are defined on the torus Π; $\mathbf{W}^+$, $\mathbf{W}^-$ are denoted as the domains of Π where $W(x, y) > 0$ and < 0, respectively.

The set $\mathbf{O}$ is equal to $\mathcal{O}$ with $\rho = \omega_2/\omega_1$. Consider the map $\gamma : \mathbb{R}^+ \to \mathbf{O}$, $s \mapsto (\overline{\omega_1 s}, \overline{\omega_2 s})$, then γ is bijective for $\omega_2/\omega_1 \notin \mathbb{Q}$ and

$$\gamma(Z) = \mathbf{O} \cap \mathbf{H} \tag{2.79}$$

$$\gamma(Z^+) = \mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+ \tag{2.80}$$

Hence, instead of proving the set Z is countably infinite, the stronger result $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$ is proven.

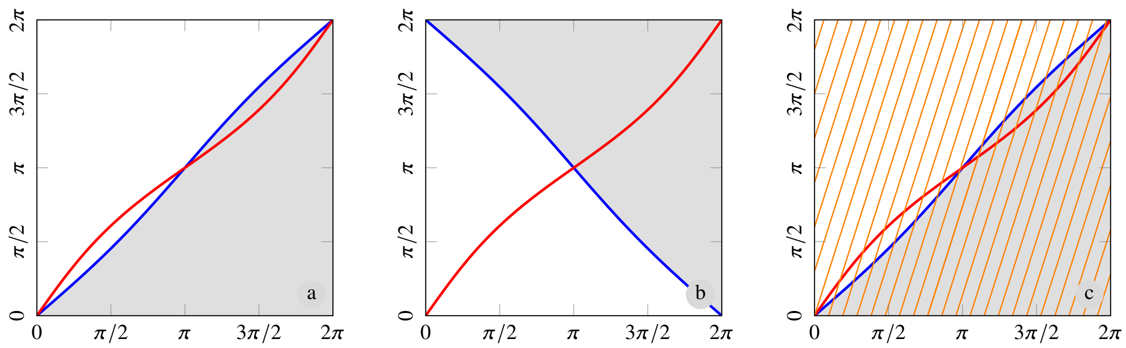


Figure 2.10 – (a) or (b) $\mathbf{H} \cap \mathbf{W}^+$ is the half of the red curve which lies in the grey domain; (c) The set $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is the set of all intersections between the red curve and the orange lines within the grey domain

This implies $\mathbf{O} \cap \mathbf{H}$ is countably infinite. This stronger result shows that Z^+ is countably infinite by pointing out the density of $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ in $\mathbf{H} \cap \mathbf{W}^+$ and the countable infinity of $\mathbf{H} \cap \mathbf{W}^+$.

To show that $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, assume $\alpha_2 \neq 0$, rewrite $H(x, y) = 0$ to have $y = \psi(x)$ where $\psi = \varphi^{-1}(r\varphi)$ and $r = -\alpha_1/\alpha_2$.

1. We show that $\dot{\psi} \neq \rho$ almost everywhere. Since ψ is an analytic function on $I =]0\,;2\pi[$, so

is $\dot{\psi}$. After simplification, the derivative of ψ becomes

$$\dot{\psi} = \frac{r(1+\varphi^2)}{1+r^2\varphi^2} = \frac{1}{r}\left(1 + \frac{r^2-1}{1+r^2\varphi^2}\right) \tag{2.81}$$

which degenerates to a constant function for $r = \pm 1$. Otherwise, $\dot{\psi}$ is not a constant function and the set $\{x \in I | \dot{\psi}(x) = \rho\}$ is empty or countable. Hence, $\dot{\psi} \neq \rho$ holds almost everywhere. It is still true if $\alpha_2 = 0$ since $H(x, y)$ becomes a periodic function of x, and $\mathbf{H}$ then degenerates to a vertical line in the torus Π.

2. Through Lemma 2.15 where $f = \psi$ is periodic of period 2π, the set O is $\mathbf{O}$ where ρ is the ratio ω_2/ω_1 and $\mathbf{O} \cap \mathbf{H}$ is dense in $\{(x, y) \in \mathbf{H} \,|\, \dot{\psi}(x) \neq \rho\}$ follows. In addition, it is proven above that $\dot{\psi} \neq \rho$ almost everywhere, thus $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, since $\mathbf{H}$ is infinite, thus $\mathbf{O} \cap \mathbf{H}$ is countably infinite. In particular, there is a countably infinite set of $s > 0$ such that $h(s) = 0$.

To complete the proof, we show that Z^+ is countably infinite by proving that $\gamma(Z^+) = \mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is countably infinite. In a similar manner, it is sufficient to show that $\overline{\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+} = \overline{\mathbf{H} \cap \mathbf{W}^+}$ and $\mathbf{H} \cap \mathbf{W}^+$ is an infinite set. If $\beta_2 \neq 0$, denote $\kappa = -\beta_1/\beta_2$, the curve $\mathbf{W}$ then corresponds to the function $y = \tilde{\psi}(x)$ where $\tilde{\psi} = \varphi^{-1}(\kappa\varphi)$ which has the same properties with ψ. The result still holds if $\beta_2 = 0$ since $\mathbf{W}$ degenerates to the vertical line in Π. By Assumption 2.7, $r \neq \kappa$ and it follows that H and W cannot coincide and the determinant of the coefficient matrix of the homogeneous system

$$H(x, y) = 0 \tag{2.82}$$

$$W(x, y) = 0 \tag{2.83}$$

is nonzero. Therefore, it has a trivial solution $\varphi(x) = \varphi(y) = 0$, *i.e.* $(\bar{\pi}, \bar{\pi})$ is one intersection between $\mathbf{H}$ and $\mathbf{W}$. Assumption 2.7 is optimal to have Z^+ is infinite. Otherwise, if Assumption 2.7 does not hold, $\mathbf{H} = \mathbf{W}$, thus $\mathbf{H} \cap \mathbf{W}^+ = \varnothing$ and Z^+ is empty.

Assumption 2.7 implies that $\dot{\psi}(\bar{\pi}) \neq \dot{\tilde{\psi}}(\bar{\pi})$ since $\dot{\psi}(\bar{\pi}) = r$ and $\dot{\tilde{\psi}}(\bar{\pi}) = \kappa$, thus the curves are transverse. Moreover, since the signs of the derivatives of ψ and $\tilde{\psi}$ depend on the signs of r and κ, respectively, ψ and $\tilde{\psi}$ are monotonic functions, in which case, $\mathbf{H} \cap \mathbf{W}^+$ is a half of the curve $\mathbf{H}$ which lies in the region $\mathbf{W}^+$ (see Figures 2.10(a) and 2.10(b)).

As $\mathbf{W}^+$ is an open set and $\overline{\mathbf{O} \cap \mathbf{H}} = \mathbf{H}$, then $\overline{\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+} = \overline{\mathbf{H} \cap \mathbf{W}^+}$. It follows that $\mathbf{O} \cap \mathbf{H} \cap \mathbf{W}^+$ is infinite (Figure 2.10c.). Hence, Z^+ is infinite which concludes the proof. □

2.6.2 Internal resonances

The situation is much simpler when the ratio ω_1/ω_2 is rational. All the functions involved are periodic with the same period and the set $\mathbf{O} \cap \mathbf{H}$ is finite or empty. Thus, the set of initial velocity $\{v(s),\ s \in Z\}$ is finite which also means that the set of generalized 1-SPP is finite. Z^+ can be an empty set for instance if $Z = \emptyset$: with the parameters $\alpha_1 = 1$, $\alpha_2 = -1$, and $\omega_1/\omega_2 = 2$, the graph of function $h(s)$ is depicted in Figure 2.11. As a consequence, 1-SPP do not exist in this case.

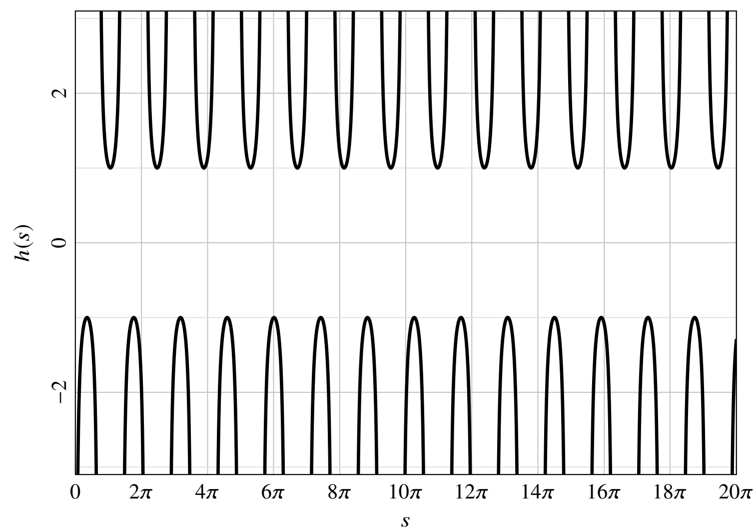


FIGURE 2.11 – Function $h(s)$ when $\omega_1/\omega_2 = 2 \in \mathbb{Q}$. To be compared to Figure 2.3.

2.7 Prestressed structure

In this Section, the structure of the 1-SPP when $d \leq 0$ is discussed. An argument on the occurrence of the sticking phase is stated in Proposition 2.16. Precisely, sticking phases of unbounded duration can arise besides the solutions with finite sticking phases, when the initial velocity of m_1 is zero. 1-SPP for $d < 0$ and $d = 0$ are also explored and illustrated through appropriate numerical examples.

Proposition 2.16. Assume $d \leq 0$. Up to a time translation, periodic solutions with a permanent sticking phase have a one-to-one correspondence to the solutions with initial data:

$$u_2(0) = d, \quad \dot{u}_2(0) = 0, \quad d \leq u_1(0) \leq \overline{u}_1 := \frac{k_2}{k_1 + k_2} d, \quad \text{and} \quad \dot{u}_1(0) = 0. \tag{2.84}$$

Otherwise, if

$$u_2(0) = d, \quad \dot{u}_2(0) = 0, \quad u_1(0) = d, \quad \text{and} \quad \dot{u}_1(0) > 0, \tag{2.85}$$

then a sticking phase with finite duration occurs.

The above proposition calls for a few comments:

- for $d = 0$, there is only one periodic solution with infinite duration displayed below in Figure 2.14.

- for $d < 0$, the set of periodic solutions with infinite duration is infinite (continuous set).
- It suffices to start at time $t = 0$ but not necessarily when u_1 reaches its minimum. More precisely, a solution has a sticking phase for all $t > 0$ if and only if the minimum of u_1 during the sticking phase is greater or equal to d. An easy computation of the 1-dof dynamics of u_1 during the sticking phase can be written with the engergy E_1 (see in the proof of Lemma 3.16) as follows.

$$u_2(0) = d,\ \dot{u}_2(0) = 0, \text{ and } m_1(\dot{u}_1(0))^2 + (k_1 + k_2)(u_1(0) - \overline{u}_1)^2 \leq \frac{(k_1 d)^2}{k_1 + k_2}. \quad (2.86)$$

Proof. The proof is a consequence of the proof of Theorem 2.17 together with Remark 2.9 and the 1-dof periodic dynamics of u_1 around $\overline{u}_1$ during the sticking phase. The conditions given in (2.84) (or (2.86)) for u_1 are just the conditions such that $u_1(t) \geq d$ during the period of the "sticking equation" (2.31) then $u_1(t) \geq d$ for all $t \geq 0$ and mass 2 remains sticked for all time in the future. For the periodic solution, up to a time translation, it is only assumed that u_1 reaches its minimum at $t = 0$.

In general, the conditions $u_1(0) > d$ or $\{u_1(0) = d$ and $\dot{u}_1(0) > 0\}$ are necessary to enjoy the existence of a sticking phase. However, the latter implies that $u_1(t)$ gets strictly smaller than d in the past and by the periodicity in the future, the sticking phase stops. □

2.7.1 Strictly prestressed structure

The dynamics is explored with $d < 0$.

Sticking phase of finite duration

From Proposition 2.16, at the beginning of the sticking phase, the initial data is $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [d, d, v, 0]^\top$ where $v > 0$. It directly follows, from Equation (2.74) when $d < 0$, that the admissible initial data is found in the set Z^- instead of Z^+. Z^- is also countably infinite as stated in Theorem 2.8. In a manner similar to the case $d > 0$, an infinite set of admissible initial data is expected when $d < 0$.

A 1-SPP is depicted in Figure 2.12 where $d = -1$; the positive initial velocity is $v \approx 2.26$. With a period $T \approx 5.42$, the sticking phase occurs until $\tau \approx 1.58$ and is then followed by a free flight of duration $s \approx 3.84$.

Sticking phase of infinite duration

The initial data corresponding to the periodic solution with the largest u_1 in magnitude is $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [d, d, 0, 0]^\top$. The first mass then follows the oscillation around its new equilibrium

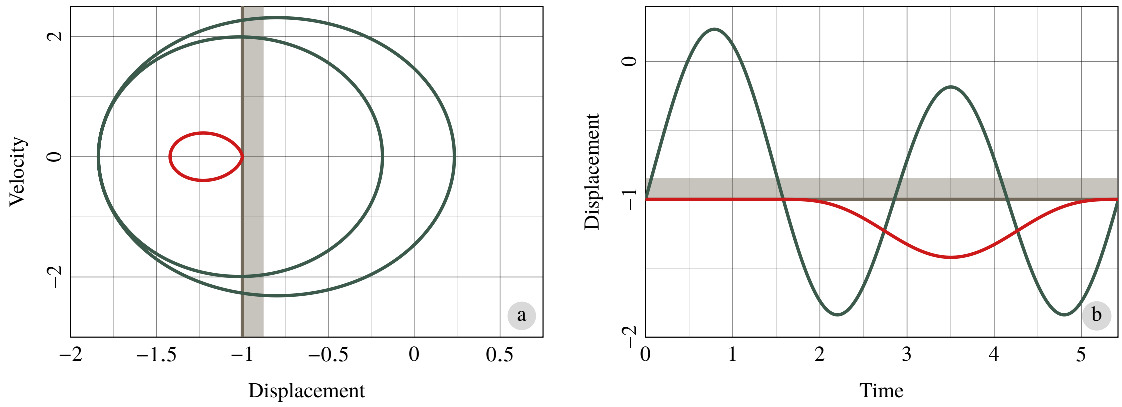


FIGURE 2.12 – 1-SPP with finite sticking phase for $d = -1$ and $v > 0$, $m_1 = 1.0; m_2 = 6.0; k_1 = 1.0; k_2 = 4.0$ and initial data $U(0) = [-1, -1, 2.2686626, 0]$: (a) Orbits. (b) Displacements

$\overline{u}_1$. Moreover, 0 is the minimum point of u_1, thus $u_1(t) \geq d$ for all t. By Theorem 2.17, it follows that the sticking phase never ends. This argument is illustrated in Figure 2.13.

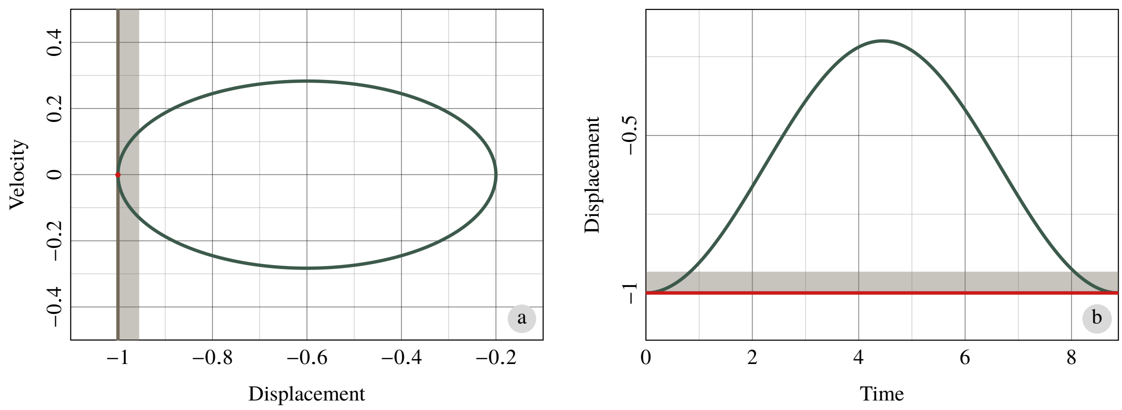


FIGURE 2.13 – Sticking phase of infinite duration for $d = -1 < 0$ and $v = 0$, $m_1 = 10; m_2 = 6; k_1 = 2; k_2 = 3$, initial data $U(0) = [-1, -1, 0, 0]$. (a) Orbits. (b) Displacements

2.7.2 Statically grazing system

The dynamics is explored with $d = 0$.

Finite duration sticking phase

From Proposition 2.16, the sticking phase of finite duration exists if the initial data satisfies $u_2(0) = 0$, $\dot{u}_2(0) = 0$, $u_1(0) = 0$, and $\dot{u}_1(0) = v > 0$. The set of free flight time s is found from Equation (2.72) where $d = 0$, *i.e.* $vw_1(s) = 0$ and $vw_2(s) = 0$. Hence, v is arbitrarily positive and s satisfies $h(s) = w_1(s) - w_2(s) = 0$ and $w_2(s) = 0$ or

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \begin{bmatrix} \varphi(\omega_1 s/2) \\ \varphi(\omega_2 s/2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.87}$$

Through Assumption 2.7, this linear system has the unique solution

$$\varphi(\omega_1 s/2) = 0, \tag{2.88}$$

$$\varphi(\omega_2 s/2) = 0, \tag{2.89}$$

where $\varphi(t) = \cot(t/2)$. It follows that

$$\frac{\omega_1}{\omega_2} = \frac{2k+1}{2l+1} \quad \text{with} \quad k, l \in \mathbb{Z}, \tag{2.90}$$

condition which loosely speaking represents half of the rationals. It should be satisfied to observe a sticking phase of finite duration when $d = 0$ while the initial velocity of mass m_1 can be chosen arbitrarily positive. Such a 1-SPP when $\omega_1/\omega_2 = 1/5$ is shown in Figure 2.14.

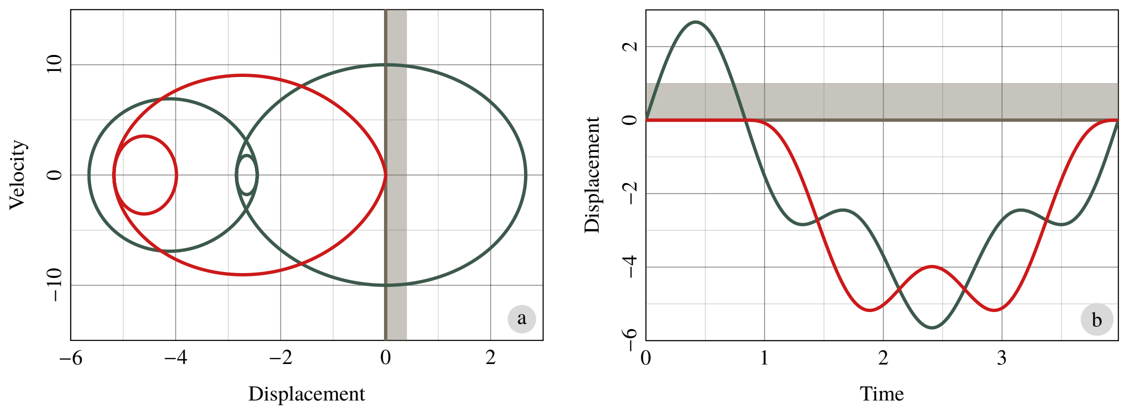


Figure 2.14 – 1-SPP with finite sticking phase for $d = 0$ and $v = 10$ (any arbitrary positive number is acceptable). (a) Orbits. (b) Displacements.

Infinite duration sticking phase

The unique corresponding initial data in this case is $[\mathbf{u}(0), \dot{\mathbf{u}}(0)]^\top = [0, 0, 0, 0]^\top$. The equilibrium $\mathbf{u} \equiv \mathbf{0}$ is a solution in which mass m_2 always grazes with the wall since the two masses stay at their equilibrium position when $d = 0$ and $\dot{u}_1(0) = 0$.

It should thus be noted that generically, there is no 1-SPP except the equilibrium when $d = 0$.

2.8 A general two-degree-of-freedom vibro-impact system

Consider a general two-dof vibro-impact system with a diagonal mass matrix $\mathbf{M}$ and a stiffness matrix

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix}$$

with $k_1, k_3 > 0$ and $0 < |k_2| \leq \sqrt{k_1 k_3}$. The 1-SPP is found in a similar way, except the condition is changed due to the change of $\mathbf{K}$. The theorem 2.17 stating the occurrence of the sticking phase becomes:

Theorem 2.17 — Sticking contact. There exists a sticking phase exactly starting at time t_0 and persisting on its right neighbourhood if and only if:

1. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $-k_2 u_1(t_0) > k_3 d$, or
2. $u_2(t_0) = d$, $\dot{u}_2^-(t_0) = 0$, $-k_2 u_1(t_0) = k_3 d$, $\dot{u}_1(t_0) > 0$.

The second case where $-k_2 u_1(t_0) = k_3 d$ and $\dot{u}_1(t_0) > 0$ corresponds to the beginning of the sticking phase.

All the conditions are kept except the one on mass 1, this is due to the second equation

$$m_2 \ddot{u}_2 = -(k_2 u_1 + k_3 u_2) + R(t). \tag{2.91}$$

The sticking phase only occurs when $u_2 = d$, $\dot{u}_2 = 0$, and $-(k_2 u_1 + k_3 u_2) \geq 0$ or $-k_2 u_1 \geq k_3 u_2$. Hence, the occurrence of sticking phase happens if $u_2 = d$, $\dot{u}_2 = 0$ and

$$\{-k_2 u_1 > k_3 u_2\} \text{ or } \{-k_2 u_1 = k_3 u_2 \text{ and } \dot{u}_1 > 0\}. \tag{2.92}$$

Based on the sticking equation

$$m_1 \ddot{u}_1 + k_1 u_1 = -k_2 d, \tag{2.93}$$

it follows that the new equilibrium of the first mass is $\bar{u}_1 = -\frac{k_2}{k_1} d$. Hence, the solution with infinite duration of sticking phase may occur only if $\bar{u}_1$ satisfies the condition in Theorem 2.17, i.e.

$$-k_2 \bar{u}_1 > k_3 d, \tag{2.94}$$

when $d > 0$, this is equivalent to $k_2^2 > k_1 k_3$, which contradicts the positive definiteness of $\mathbf{K}$. Hence, the 1-SPP with a sticking phase of an infinite duration is expected to exist only in the prestressed case, i.e. $d < 0$.

Remark 2.18. Function $h(s)$ in theorem 2.4 becomes $h(s) = w_1(s) + \frac{k_3}{k_2} w_2(s)$ and the initial velocity of the first mass is $v = \frac{d}{w_2(s)}$.

2.9 On 1-SPP of N-degree-of-freedom vibro-impact systems with $N > 2$

A general N-degree-of-freedom vibro-impact system is considered,

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} \tag{2.95a}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 \tag{2.95b}$$

$$u_N(t) \leq d, \quad R(t) \leq 0, \quad (u_N(t) - d)R(t) = 0, \quad \forall t \tag{2.95c}$$

$$\dot{\mathbf{u}}(t)^\top \mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = E(\mathbf{u}(t), \dot{\mathbf{u}}(t)) = E(\mathbf{u}(0), \dot{\mathbf{u}}(0)), \tag{2.95d}$$

where $\mathbf{M} = \text{diag}(m_j)_{j=1}^N$ and $\mathbf{K} = (k_{ij})_{i,j=1}^N$ are the mass and stiffness matrices of the system, $\mathbf{u}(t) = (u_j(t))_{j=1}^N$ is the displacement of the masses, $\mathbf{r}(t) = (0, \ldots, 0, R(t))^\top$, with $R(t)$ is the reaction of the obstacle. It is always assumed that there is one degree-of-freedom in contact, which is the N^{th} mass. Similarly, the condition to have a sticking phase is then studied.

2.9.1 The occurrence of the sticking phase

Assume that $\mathbf{u}(t)$ is a solution of (2.95). A general definition of a sticking phase is given.

Definition 2.19 — Sticking phase and its duration. Let $\mathbf{u}$ be the solution to System (2.95). A *sticking phase* arises if there exist $t_0 \in \mathbb{R}$ and $\tau > 0$ such that

$$u_N(t) = d, \quad \forall t \in [t_0\,; t_0 + \tau]. \tag{2.96}$$

Moreover, when there exists $0 < \delta < 1$ such that $\forall t \in \,]0\,; \delta[$

$$u_N(t_0 - t) < d \quad \text{and} \quad u_N(t_0 + \tau + t) < d, \tag{2.97}$$

then t_0 is the starting time and τ is the duration of the sticking phase.

The necessary and sufficient conditions to have sticking phase is then studied similarly as in the 2-dof system.

Remark 2.20 — N-dof sticking contact. There exists a sticking phase starting exactly at time $t_0 = 0$ and persisting on its right neighbourhood if and only if:

1. $u_N(t_0) = d$, $\dot{u}_N^-(t_0) = 0$, $F(t_0) < 0$ or
2. $u_N(t_0) = d$, $\dot{u}_N^-(t_0) = 0$, $F(t_0) = 0$, $F(t_0 + \delta) < 0$ for some $\delta > 0$.

where $F(t) = \mathbf{e}_N^\top \mathbf{K}\,\mathbf{u}(t)$.

The is remark can be proven in a similar way as the proof of Theorem 2.17. The sticking system is a non-homogeneous $(N-1)$-dof system

$$\underline{\mathbf{M}}\ddot{\underline{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = \underline{\mathbf{C}}, \tag{2.98}$$

where $\underline{\mathbf{M}}$ and $\underline{\mathbf{K}}$ are $(N-1)\times(N-1)$ matrices obtained by neglecting the last row and last column of $\mathbf{M}$ and $\mathbf{K}$, respectively; $\underline{\mathbf{C}} = d\,\underline{\mathbf{l}}_N^\top$ with $\underline{\mathbf{l}}_N = [k_{Nj}]_{j=1}^{N-1}$. It will be proven in Section 3.3 of Chapter 3, that when d is positive, the sticking phase ends after a finite time. Base on the occurrence of the sticking phase in Proposition 2.20, the sticking duration τ is then implicitly found from:

$$F(\tau) = 0,\; F(\tau + \delta) > 0,\; \text{for some } \delta \in\,]0\,;\delta_0[,\, \delta_0 > 0. \tag{2.99}$$

The feature of the 2-dof system is the 1-dof sticking system which exhibits the symmetry of the solution during the sticking phase. This is the main reason why the author have not succeeded to obtain the 1-SPP of the N-dof system.

2.9.2 A 3-dof chain with a non prestressed gap $d > 0$

Consider a chain of three masses in which the third mass is constrained by a wall at a distance d from its equilibrium. The system of interest is (2.1) where

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix};\; \mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix};\; \mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix};\; \mathbf{r}(t) = \begin{pmatrix} 0 \\ 0 \\ R(t) \end{pmatrix}. \tag{2.100}$$

Let us explain in details the difficulty when looking for 1-SPP.

The occurrence of the sticking phase

There are following possibilities for a sticking phase to occurs:

- $u_3(0) = d$, $\dot{u}_3(0) = 0$, $u_2(0) > d$, or
- $u_3(0) = d$, $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) > 0$, or
- $u_3(0) = d$, $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) = 0$, $u_1(0) > d$, or
- $u_3(0) = d$, $\dot{u}_3(0) = 0$, $u_2(0) = d$, $\dot{u}_2(0) = 0$, $u_1(0) = d$, $\dot{u}_1(0) > 0$.

The end of sticking phase at the time τ is replaced by one of the following conditions

- $u_3(\tau) = d$, $\dot{u}_3(0) = 0$, $u_2(\tau) = d$, $\dot{u}_2(\tau) < 0$, or
- $u_3(\tau) = d$, $\dot{u}_3(\tau) = 0$, $u_2(\tau) = d$, $\dot{u}_2(\tau) = 0$, $u_1(\tau) < d$, or
- $u_3(\tau) = d$, $\dot{u}_3(\tau) = 0$, $u_2(\tau) = d$, $\dot{u}_2(\tau) = 0$, $u_1(\tau) = d$, $\dot{u}_1(\tau) < 0$.

The existence of solution with sticking phase

Sticking phase in general Assume that there exists a sticking phase at t_0. The sticking system is then a non-homogeneous 2-dof system with the new equilibrium

$$[u_{1e}, u_{2e}] = \left[\frac{k_2\, k_3}{k_1\, k_2 + k_2\, k_3 + k_3\, k_1} d, \frac{k_1\, k_3 + k_2\, k_3}{k_1\, k_2 + k_2\, k_3 + k_3\, k_1} d\right]. \tag{2.101}$$

Hence, for $d > 0$, the duration of sticking phase is always finite since $u_{2e} < d$.

For $d < 0$, it follows that $u_{2e} > d$, hence with an energy small enough, $u_2(t)$ can be always greater than d, which follows the existence of the sticking phase with infinite duration.

Periodic solution with 1 sticking phase per period Assume that the initial state of the system is $\mathbf{U}(0) = [\mathbf{u}_0, \dot{\mathbf{u}}_0]^\top = [u_{10}, d, d, \dot{u}_{10}, v_{20}, 0]^\top$ with 3 parameters $u_{10}, \dot{u}_{10}, v_{20}$.

Assume that the sticking phase ends when $u_2(\tau) = d$ where $u_2(t)$ is the solution of the 2-dof sticking system. When τ is found, we have the data for the system at τ which is the initial data for the free flight phase after the sticking.

Then solving the system during free flight from τ to T with the condition to have a periodic solution $\mathbf{U}(T) = \mathbf{U}(0)$. It follows 5 equations of 3 parameters to solve. It seems to be over-determined.

2.10 Conclusion

The free dynamics of a two-degree-of-freedom linear oscillator subject to a unilateral constraint on one mass is investigated. Generically, a Newton-like impact law has to be incorporated in this type of formulation. In this work, periodic orbits with one sticking phase per period (1-SPP) are considered: it is shown that they are independent of the impact law. They might not always exist and whenever they exist, they are isolated as opposed to one-impact-per-period solutions (1-IPP) known to be organized on manifolds [25]. Also, they cannot be obtained through usual perturbation methods.

The whole set of 1-SPP is characterized by only one parameter belonging to a discrete set: the free flight duration. This parameter belongs to a countable set which can be empty, or even infinite in some circumstances.

A systematic numerical procedure designed to find all possible 1-SPP is expounded. It involves two numerical steps:

1. finding the roots of an explicit quasi-periodic function, and
2. checking that the corresponding closed-form trajectory satisfies the unilateral condition on the whole period of motion.

Many examples are presented but the existence proof of 1-SPP remains an open problem. In addition, parameter values leading to the non-existence of 1-SPP are provided. However, under generic assumptions on the mass and stiffness matrices, a countable infinite set of initial data including all the initial data of 1-SPP can be exhibited. The closed-forms emanating from this set (of initial data) satisfy the unilateral constraint at least near the sticking phase. The prestressed structure is also explored. The picture is similar except that 1-SPP with infinite sticking time are also found.

Extension to N degrees-of-freedom with $N > 2$ is not straightforward: the symmetry $\mathbf{u}(t) = \mathbf{u}(-t)$, property heavily used in this work, is unknown and the sticking dynamics is more complicated, the sticking system has dimension $N - 1 > 1$.

Chapter 3

On the first return time of N-degree-of-freedom vibro-impact oscillators

Summary For an N-degree-of-freedom linear vibro-impact system with a unilateral contact it is known in structural mechanics that orbits near grazing contacts lead to square-root instability [7, 33]. In this chapter, the first return map is revisited in a rigorous mathematical framework to properly describe this singularity. The Poincaré section is chosen at the contact to understand the dynamics but with a loss of the usual transversality condition. The domain of the first return map is precisely explored. The square-root singularity and new singular behaviors are highlighted. In particular, the discontinuity of the first return time is the worst singular behavior of some grazing contacts. Finally, a general condition is stated to check the square-root dynamics near the linear grazing orbits which may lead to their instability.

3.1 Introduction

Recent results on nonlinear modes for vibro-impact discrete structural systems suggest that the first return map (FRM) should be defined on the hyperplane of the unilateral constraint in the phase-space [7, 33]. This approach distinguishes itself from the “discontinuity mapping" [7, 33] where the Poincaré section lies outside the contact interface. In the context of serial mass-spring systems, simplified versions of the FRM already exist; they give access to special periodic solutions with closed-form expression [25] and to reduced-order systems of nonlinear equations [45, 28] together with the companion stability analysis. In this work the FRM, which is the fundamental tool to explore periodic solutions, is extended to the investigation of other potential solutions in order to improve the understanding of the global dynamics. Since the Poincaré section is a subset of the contact interface in the phase-space, the natural transversality condition is lost. This means that the Poincaré section can be tangent to orbits which yields the well-known square-root singularity [7, 33]. This singularity is here revisited in a rigorous mathematical framework.

The system of interest is a second-order system

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{r} & (3.1a) \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0 & (3.1b) \\ u_N(t) \leq d, \quad R(t) \leq 0, \quad (u_N(t) - d)\, R(t) = 0 & (3.1c) \\ \dot{\mathbf{u}}^+(t)^\top \mathbf{M}\dot{\mathbf{u}}^+(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) = E(\mathbf{u}(t), \dot{\mathbf{u}}^+(t)) = E(\mathbf{u}(0), \dot{\mathbf{u}}(0)), & (3.1d) \end{cases}$$

with

$$\mathbf{M} = \mathbf{diag}(m_j)_{j=1}^N; \qquad \mathbf{K} = (k_{ij})_{i,j=1}^N; \qquad \mathbf{u}(t) = (u_j)_{j=1}^N; \quad \mathbf{r}(t) = (0, \ldots, 0, R(t)) \quad (3.2)$$

where $\dot{u}_j$ and $\ddot{u}_j$ represent the velocity and acceleration of the mass j, $j = 1, \ldots, N$, respectively. Condition (3.1c) says that mass N is constrained on the right side by a rigid obstacle at a distance $d > 0$ from its equilibrium. There is only one constraint on the mass N. The other masses are not constrained. The quantity $R(t)$ is the reaction force induced by the obstacle on mass N at the time of gap closure. In general $R(t)$ is a measure, but for solution with sticking phases, it is a Lipschitz function which is as regular as $\ddot{u}_N$ [28].

System (3.1) without (3.1d) is not well posed: it is known that uniqueness might not be ensured for the initial value problem. An impact law is usually incorporated into the formulation. Here, since we are interested in the non-dissipative dynamics, condition (3.1d) is enforced: the total energy is preserved during the motion. This implies the existence of a perfectly elastic impact law of the form $\dot{u}_N^+ = -e\dot{u}_N^-$ with $e = 1$ where $\dot{u}_N^-$ and $\dot{u}_N^+$ respectively stand for the pre- and post-impact velocities of mass N. For the well-posedness of the initial-value problem with conserved energy, see [5, 40].

Matrices $\mathbf{M}$ and $\mathbf{K}$ are symmetric positive definite. Hence, there exists a matrix $\mathbf{P}$ of $\mathbf{M}$-orthogonal eigenmodes which diagonalizes both $\mathbf{M}$ and $\mathbf{K}$, that is $\mathbf{P}^\top\mathbf{M}\mathbf{P} = \mathbf{I}$ and $\mathbf{P}^\top\mathbf{K}\mathbf{P} = \mathbf{\Omega}^2 = \mathbf{diag}(\omega_j^2)|_{j=1,\ldots,N}$ where $\mathbf{I}$ is the $N \times N$ identity matrix; ω_j^2 (resp., T_j) are the eigenfrequencies (resp., the associated linear periods) where $\omega_j T_j = 2\pi$, $j = 1, \ldots, N$.

The chapter is organized as follows: in Section 3.2, the main results are stated including the choice of a suitable Poincaré section. Are shown also the square-root term in the first return time near the first return time of a grazing orbit and the square-root dynamics near a linear grazing orbit. The following sections detail the above results. Section 3.3 deals with the exact subset of initial data at the contact constraint $u_N = d$ where the trajectory will come back to the constraint, that is the domain of the FRM. The square-root singularity is focused on in Section 3.4 in a mathematical framework only where the implicit function theorem is used in a degenerate case.

In particular, it implies that the first return time is discontinuous. The dynamics induced by the square-root singularity is studied in Section 3.5. Some coefficients of the asymptotic expansion of the FRM activating the square-root instability are identified. The stability of grazing periodic solutions is discussed in Section 3.6.

3.2 Main results

The Poincaré section is first defined in Section 3.2.1. The theorems on the square-root singularity that comes from the grazing contact are in Section 3.2.2. The last section provides a result on the dynamics near a periodic solution with one grazing contact occurrence.

3.2.1 Poincaré section

In smooth analysis, a Poincaré section can always be locally defined transversally to the flow away from fixed-points. This section corresponds to the domain of definition of the corresponding First Return Map. In nonsmooth analysis where the vector field governing the dynamics is piecewise smooth, the hyperplane $\mathcal{H} = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathbb{R}^{2N},\ u_N = d\}$ of the phase-space defined by $u_N = d$ is a natural choice for the Poincaré section when targeting trajectories with non-vanishing pre-impact velocities [25] since they hit the section transversally. However, for grazing trajectories, the transversality condition is lost. A discontinuity mapping on another suitable section for which the transversality condition is recovered shall be used instead [7, 17, 33]. In this work, the Poincaré section is still a subset of the hyperplane $\mathcal{H}$. This is the simplest cross section to describe the dynamics with only two phases: contact dynamics and free-flight dynamics. This does not have adverse effect of introducing a second free-flight dynamics as the discontinuity mapping does. Nevertheless, our critical choice necessitates a very careful delimitation of the domain of definition of the FRM. This is stated in the following theorem that categorizes the initial data generating orbits which will always come back to the section.

With the above Newton's impact law, two types of closing contacts can be reported: contacts with non-zero pre-impact velocity (or simply impacts) and contacts with zero pre-impact velocity. The second type can be divided as follows: a grazing contact if the mass leaves the obstacle right after the contact time, or sticking if the mass stays in contact with the obstacle for a finite time interval.

Proposition 3.1 — Finite sticking duration. When d is positive, the sticking phase of a solution to (3.1) is of finite duration.

This is proven in [28] for a 2-dof vibro-impact system. A general proof for the N-dof system is given in Section 3.3.1. This property is used to show that a solution to (3.1) has zero, one or an infinite number of closing contacts with the hyperplane $\mathcal{H}$. Before stating this result, the following assumption is needed.

Assumption 3.2 — No internal resonances. The linear periods of system (3.1) satisfy $\cap_{j=1}^{N} T_j\mathbb{N} = \emptyset$.

By this assumption, the internal resonances are discarded in the current work.

Theorem 3.3 — Zero, one or infinite number of closing contacts on $\mathcal{H}$. Let $\mathbf{u}(t)$ be a solution to (3.1). Under Assumption 3.2 of no internal resonances, the solution is such that

Case 1: linear solution the N^{th} mass never hits $\mathcal{H}$, i.e. $u_N(t) < d$ for all t.

Case 2: linear solution the N^{th} mass experiences only one closing contact, i.e. there exists t_0 such that $u_N(t_0) = d$ and $u_N(t) < d$ for all $t \neq t_0$.

Case 3: nonlinear solution the N^{th} mass experiences a countably infinite number of isolated closing contacts on $\mathcal{H}$.

This theorem is proven in Section 3.3.2. The affine space $\mathcal{H}$ is of dimension $2N - 1$. It is divided into three disjoint subsets:

$$\mathcal{H}^- = \{[\mathbf{u}, \dot{\mathbf{u}}^-]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N^- > 0\} \tag{3.3}$$

$$\mathcal{H}^+ = \{[\mathbf{u}, \dot{\mathbf{u}}^+]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N^+ < 0\} \tag{3.4}$$

$$\mathcal{H}^0 = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathbb{R}^{2N},\ u_N = d \text{ and } \dot{u}_N = 0\} \tag{3.5}$$

By conservation of energy, it can be seen that the solution which has a contact with non-zero velocity will always experience a later closing contact. The problem of whether there will be a subsequent closing contact or not emerges only on $\mathcal{H}^0$. Theorem 3.3 implies that $\mathcal{H}^0$ is the union of the two subsets

$$\mathcal{H}^0_\infty = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathcal{H}^0,\ \text{the associated orbit has an infinite number of contacts}\} \tag{3.6}$$

$$\mathcal{H}^0_1 = \{[\mathbf{u}, \dot{\mathbf{u}}]^\top \in \mathcal{H}^0,\ \text{the associated orbit has only one grazing contact}\} \tag{3.7}$$

Since $\mathcal{H}^0_1$ includes solutions with only one grazing contact, it is sufficient to discard this set from the definition of the Poincaré section.

Definition 3.4 — Poincaré section. The Poincaré section $\mathcal{H}_\mathcal{P} \subset \mathcal{H}$ is the union of the set of initial data with non-zero velocity contacts and the set of initial data with zero velocity contact that gives rise to an infinite number of closing contacts:

$$\mathcal{H}_\mathcal{P} = \mathcal{H}^- \cup \mathcal{H}^0_\infty. \tag{3.8}$$

Remark 3.5. There are two options for the choice of the Poincaré section in $\mathcal{H}$ depending on whether one wants to start right before or right after the contact occurrence. The former gives the Poincaré section defined as above, and the latter gives $\mathcal{H}_{\mathcal{P}}^{+} = \mathcal{H}^{+} \cup \mathcal{H}_{\infty}^{0}$ as the Poincaré section.

The set $\mathcal{H}_{\infty}^{0}$ can also be split into two subsets: $\mathcal{H}_{\mathcal{S}}^{0}$ including all the initial data belonging to $\mathcal{H}_{\infty}^{0}$ such that the solution starts by a sticking contact and $\mathcal{H}_{\mathcal{G}}^{0}$ of initial data such that the solution starts by a grazing contact:

$$\mathcal{H}_{\infty}^{0} = \mathcal{H}_{\mathcal{S}}^{0} \cup \mathcal{H}_{\mathcal{G}}^{0}. \tag{3.9}$$

3.2.2 The square-root singularity

The first return time is known to be generically analytic: let $\mathsf{W}_0 \in \mathcal{H}^{-}$, if the first return to $\mathcal{H}$, named W_1, belongs to $\mathcal{H}^{-}$ then the FRM is analytic near W_0 [7, 25]. Nonetheless, if $\mathsf{W}_1 \in \mathcal{H}^{0}$, then there is a grazing contact and it is known that a square-root singularity appears [17, 33].

By definition of $\mathcal{H}_{\mathcal{P}}$, there exists a time such that the orbit emanating from $\mathsf{W} \in \mathcal{H}_{\mathcal{P}}$ comes back to $\mathcal{H}_{\mathcal{P}}$. The first return time is then well defined as follows.

Definition 3.6 — First return time. Let $\mathbf{u}(t)$ be a solution to (3.1) with the initial data $\mathsf{W} = (W_i)_{i=1}^{2N} \in \mathcal{H}_{\mathcal{P}}$ at the initial time $t = 0$, i.e. $u_N(0) = \mathbf{e}_N^{\top}\mathbf{u}(0) = d$. The first return time $T = T(\mathsf{W}) > 0$ is defined by

$$T(\mathsf{W}) = \begin{cases} \min\{t > 0 : u_N(t) = d\} \text{ if there is no sticking phase at } t = 0 & (3.10a) \\ \min\{t > \tau(\mathsf{W}) : u_N(t) = d\} \text{ if there is sticking phase at } t = 0 & (3.10b) \end{cases}$$

where $\tau(\mathsf{W})$ is the sticking duration.

This definition can be shorten by saying that $T(\mathsf{W}) = \min\{t > \tau(\mathsf{W}) : u_N(t) = d\}$, with the convention that $\tau(\mathsf{W}) = 0$ if there is no sticking at $t = 0$. For the case when a sticking phase occurs at $t = 0$, Proposition 3.1 ensures that the first return time is well defined since the duration of the sticking phase is finite.

Let $\mathbf{u}(t, \mathsf{W})$ be the solution associated with the initial data $\mathsf{W} \in \mathcal{H}_{\mathcal{P}}$. By the definition of $\mathcal{H}_{\mathcal{P}}$, $W_N = d$, hence W is viewed as a vector of $2N - 1$ variables. The displacement of the N^{th} mass is $u_N(t, \mathsf{W}) = \mathbf{e}_N^{\top}\mathbf{u}(t, \mathsf{W})$ which is a function of time variable t and the initial data W. Consider the smooth function Φ defined for all $t \in \mathbb{R}$ and for all $\mathsf{W} \in \mathbb{R}^{2N}$:

$$\Phi(t, \mathsf{W}) = \mathbf{e}_N^{\top}\mathbf{R}(t)\mathbf{S}\mathsf{W}, \tag{3.11}$$

where $\mathbf{S} = \mathbf{diag}(1, \ldots, 1, -1)$ is a $2N \times 2N$ diagonal matrix with last entry -1 to reflect the impact law. The operator $\mathbf{R}(t)$ describes the free-flight dynamics:

$$\mathbf{R}(t) = \begin{bmatrix} \mathbf{P}\cos(t\boldsymbol{\Omega})\mathbf{P}^{-1} & \mathbf{P}\boldsymbol{\Omega}^{-1}\sin(t\boldsymbol{\Omega})\mathbf{P}^{-1} \\ -\mathbf{P}\boldsymbol{\Omega}\sin(t\boldsymbol{\Omega})\mathbf{P}^{-1} & \mathbf{P}\cos(t\boldsymbol{\Omega})\mathbf{P}^{-1} \end{bmatrix}. \tag{3.12}$$

The dynamics between two successive closing contacts is smooth, except at the beginning and the end of the contact occurrences. The function $\Phi(t, \mathsf{W})$ coincides with $u_N(t, \mathsf{W})$ as long as $u_N(t, \mathsf{W}) < d$.

Assumption 3.7 — Non-zero acceleration. Let W_0 be the initial data leading to an orbit which has a grazing contact at the first time T_0. Assume that $\partial_{tt}\Phi(T_0, \mathsf{W}_0) \neq 0$.

This is an important assumption to activate the square-root singularity near a grazing contact. With this assumption and the assumption that for some $1 \leq k \leq 2N$, $\partial_{W_k}\Phi(T_0, \mathsf{W}_0) \neq 0$, the following scalar is well defined and non-zero:

$$\gamma_k = -\frac{\partial_{tt}\Phi(T_0, \mathsf{W}_0)}{2\,\partial_{W_k}\Phi(T_0, \mathsf{W}_0)}. \tag{3.13}$$

The expression of the first return time near a grazing contact is then given in the following theorem. The implicit function theorem is applied on the smooth function Φ in place of the nonsmooth function u_N. It is a crucial ingredient of the proof.

Theorem 3.8 — Square-root singularity near a grazing contact. Let $\mathsf{W}_0 = (W_{0i})_{i=1}^{2N} \in \mathcal{H}^-$ be the initial data generating an orbit which has a grazing contact at the first return time $T_0 = T(\mathsf{W}_0)$. Let W_k be a component of W such that $\partial_{W_k}\Phi(T_0, \mathsf{W}_0) \neq 0$. Such a component always exists. Let $\underline{\mathsf{W}} \in \mathbb{R}^{2N-1}$ be the reduced vector obtained from W by removing W_k. Moreover, if Assumptions 3.2 and 3.7 hold then:

1. There exist two neighborhoods V_{T_0} and $V_{\mathsf{W}_0} \subset \mathcal{H}_{\mathcal{P}}$ of T_0 and W_0, respectively as well as two smooth scalar functions η and α defined on $V_{\underline{\mathsf{W}}_0} = \{\underline{\mathsf{W}}, \mathsf{W} \in V_{\mathsf{W}_0}\}$ such that the set

$$S_c = \{(t, \mathsf{W}) \in \mathbb{R} \times \mathbb{R}^{2N},\ \Phi(t, \mathsf{W}) = d \text{ and } \partial_t\Phi(t, \mathsf{W}) = 0\}, \tag{3.14}$$

   where the square-root singularity will emerge, is locally parameterized as follows:

$$S_c \cap \{V_{T_0} \times V_{\mathsf{W}_0}\} = \{(\eta(\underline{\mathsf{W}}), \alpha(\underline{\mathsf{W}}), \underline{\mathsf{W}}),\ \underline{\mathsf{W}} \in V_{\underline{\mathsf{W}}_0}\}, \tag{3.15}$$

   where $\eta(\underline{\mathsf{W}}_0) = T_0$ and $\alpha(\underline{\mathsf{W}}_0) = W_{0k}$.
2. Let $s_k = \text{sign}(\gamma_k)$ and the set

$$\mathcal{B}_k = \{\mathsf{W} \in V_{\mathsf{W}_0},\ s_k(W_k - \alpha(\underline{\mathsf{W}})) \geq 0\}. \tag{3.16}$$

Let $\sigma = \text{sign}(\ddot{u}_N^-(T_0, \mathsf{W}_0))$. There exists a smooth function ψ such that $\psi(0, \underline{\mathsf{W}}_0) = 0$, $\partial_1\psi(0, \underline{\mathsf{W}}_0) = 1/\sqrt{|\gamma_k|}$ and for all $\mathsf{W} \in \mathcal{B}_k$, the first return time T is given by

$$T(\mathsf{W}) = \eta(\underline{\mathsf{W}}) + \psi(\sigma\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}))}, \underline{\mathsf{W}}). \tag{3.17}$$

Moreover, the set $\{\mathsf{W} \in \mathcal{H}_{\mathcal{P}}, (T(\mathsf{W}), \mathsf{W}) \in V_{T_0} \times V_{\mathsf{W}_0}\}$ is exactly $\mathcal{B}_k$.

A mathematical proof of this theorem will be given in Section 3.6.1. Take note that the square-root dependence on the initial data only appears on the set $\mathcal{B}_k$ which is simply the region above or below the hypersurface $W_k = \alpha(\underline{\mathsf{W}})$. The square-root term acts in one direction. This is reported by Nordmark and Fredriksson [17] and also in references [7, 33, 47].

The presence of the square-root singularity is the consequence of the non-vanishing acceleration $\ddot{u}_N^-$ at T_0. While studying the other contacts with zero-velocity, if, instead, $\dot{u}_N^-(T_0) = \ddot{u}_N^-(T_0) = 0$ and $\dddot{u}_N^-(T_0) \neq 0$ then a cube-root singularity is expected. More generally, if $u_N^{(l)-}(T_0) = 0$ for all $0 < l < n$ and $u_N^{(n)-}(T_0) \neq 0$, then a n^{th}-root singularity is expected, where $u_N^{(l)-}$ refers to the left time derivative of order l^{th} of u_N.

Orbits involving grazing contacts are expected to be less frequent than the ones with non-zero velocity contacts. Studying the grazing orbits is however important since the dynamics near such a grazing contact is diversified and many criteria can be exhibited as explained further in the next section.

Remark 3.9 — Discontinuous first return time (FRT). Since the set $\mathcal{B}_k$ is a one-sided set with respect to an hypersurface, the set of initial data with their first return time near T_0 is not a neighborhood of W_0. This means that, in the vicinity of W_0, there exist a lot of initial data such that the first return time is far from T_0. In other words, the first return time $T(\mathsf{W})$ is discontinuous at W_0, see Section 3.4.3.

Theorem 3.8 also has to be generalized for an initial data $\mathsf{W}_0 \in \mathcal{H}_{\mathcal{G}}^0$ for instance the initial data of a linear grazing orbit (LGO). The neighborhood V_{W_0} has to be replaced by the one half neighborhood $V_{\mathsf{W}_0}^+ = \{\mathsf{W} \in V_{\mathsf{W}_0}, \mathbf{e}_{2N}^\top \mathsf{W} \geq 0\}$ where the initial velocity is non-positive and the set $\mathcal{B}_k$ is replaced by the smaller set

$$\mathcal{B}_k^+ = \mathcal{B}_k \cap V_{\mathsf{W}_0}^+ = \{\mathsf{W} \in V_{\mathsf{W}_0}, \gamma_k(W_k - \alpha(\underline{\mathsf{W}})) \geq 0,\ W_{2N} \geq 0\}. \tag{3.18}$$

The following theorem is stated with the notations of Theorem 3.8.

Theorem 3.10 — Square-root singularity for two successive grazing contacts. Assume that $\mathsf{W}_0 \in \mathcal{H}_{\mathcal{G}}^0$, $\ddot{u}_N^+(0, \mathsf{W}_0) < 0$ and $\mathsf{W}(T_0) \in \mathcal{H}^0$ where $T_0 = T(\mathsf{W}_0)$. Let k belong to $\{1, \ldots, 2N\}$ such that $\partial_{W_k}\Phi(T_0, \mathsf{W}_0) \neq 0$ and $s_k = \text{sign}(\gamma_k)$ together with $\sigma = \text{sign}(\ddot{u}_N^-(T_0, \mathsf{W}_0))$.

If Assumptions 3.2 and 3.7 hold, then there exist two neighborhoods V_{T_0} and $V_{\mathsf{W}_0} \subset \mathcal{H}_{\mathcal{P}}$ of T_0 and W_0, respectively as well as two smooth scalar functions η and α defined on $V_{\underline{\mathsf{W}}_0}$

containing $\underline{\mathsf{W}}_0$ where $\eta(\underline{\mathsf{W}}_0) = T_0$ and $\alpha(\underline{\mathsf{W}}_0) = W_{0k}$. There also exists a smooth function ψ such that $\psi(0, \underline{\mathsf{W}}_0) = 0$, $\partial_1\psi(0, \underline{\mathsf{W}}_0) = |\gamma_k|^{-1/2}$ and for all $\mathsf{W} \in \mathcal{B}_k^+$, the first return time T is given by

$$T(\mathsf{W}) = \eta(\underline{\mathsf{W}}) + \psi(\sigma\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}))}, \underline{\mathsf{W}}). \tag{3.19}$$

Moreover, the set $\{\mathsf{W} \in \mathcal{H}_{\mathcal{P}}, (T(\mathsf{W}), \mathsf{W}) \in V_{T_0} \times V^+_{\mathsf{W}_0}\}$ is exactly $\mathcal{B}_k^+$.

The difference with Theorem 3.8 is the smaller validity of the square-root fomula for the first return time. This is due to the fact that $\mathcal{H}^0_{\mathcal{G}}$ is at the boundary of the cross section $\mathcal{H}_{\mathcal{P}}$ so not all perturbations of W_0 are admissible. The condition $\ddot{u}^+_N(0, \mathsf{W}_0) < 0$ insures that solutions with initial data near W_0 and with a non-negative velocity for the last mass have a first return time near T_0. The case $\mathsf{W}_0 \in \mathcal{H}^0_{\mathcal{S}}$ can also be considered and can add another singularity due to the sticking phase.

3.2.3 Dynamics near a linear grazing orbit

The square-root dynamics near the periodic solutions with one grazing contact per period is addressed. For an N-dof system without internal resonances, there are N such periodic solutions which are called linear grazing orbits. It is known [25, 46] that linear grazing orbits are the source of many branches of periodic solutions with k Impact-Per-Period (k-IPP).

Recall that we define the j^{th} linear grazing orbit as a periodic trajectory $\mathbf{u}$ associated to the j^{th} linear mode which satisfies $\max_{t\in\mathbb{R}} u_N(t) = d$, i.e. the contacts are at most of grazing type. A essential tool to study the dynamics near a linear grazing orbit is the first return map [7]. This map is well defined on $\mathcal{H}_{\mathcal{P}}$.

Definition 3.11 — First return map. Suppose $\mathsf{W} \in \mathcal{H}_{\mathcal{P}}$ and $T = T(\mathsf{W}) > 0$ is the first return time to $\mathcal{H}_{\mathcal{P}}$ of the orbit emanating from W. The map which associates points in $\mathcal{H}_{\mathcal{P}}$ to their first return images to $\mathcal{H}_{\mathcal{P}}$ is called the first return map $\mathcal{F}$. To be more precise, $\mathcal{F} : \mathcal{H}_{\mathcal{P}} \to \mathcal{H}_{\mathcal{P}}$,

$$\mathcal{F}(\mathsf{W}) = \begin{cases} \mathbf{R}(T(\mathsf{W}))\,\mathbf{S}\,\mathsf{W} & \text{if } \mathsf{W} \in \mathcal{H}^- \cup \mathcal{H}^0_{\mathcal{G}}, \quad (3.20\text{a}) \\ \mathbf{R}(s(\mathbf{U}(\tau(\mathsf{W})))\,\mathbf{U}(\tau(\mathsf{W})) & \text{if } \mathsf{W} \in \mathcal{H}^0_{\mathcal{S}}, \quad (3.20\text{b}) \end{cases}$$

where the matrix $\mathbf{S}$ describes the impact law; $\mathbf{U}(t) = [\mathbf{u}(t), \dot{\mathbf{u}}(t)]$ is the state of the system at the time t; τ and s are the duration of the sticking phase and of the free-flight phase, respectively.

Let us investigate formula (3.20b). If there is a sticking phase, i.e. $\mathsf{W} \in \mathcal{H}^0_{\mathcal{S}}$, then $\tau = \tau(\mathsf{W})$ is the sticking duration and $s = T(\mathsf{W}) - \tau(\mathsf{W})$ denotes the duration of the free-flight after the sticking phase until the next contact. The state of the system at the end of the sticking phase is called $\mathbf{U}(\tau(\mathsf{W}))$. If $\mathsf{W} \in \mathcal{H}^- \cup \mathcal{H}^0_{\mathcal{G}}$, i.e. there is no sticking at $t = 0$, then τ is assumed to be 0, hence, $s \equiv T$, and $\mathbf{U}(\tau(\mathsf{W})) \equiv \mathsf{W}$. In other words, formula (3.20b) is valid for all the cases. An explicit formula of τ for a two-degree-of-freedom system is exposed in [28].

The square-root singularity in formula (3.17) of the first return time in the vicinity of a grazing contact is an element showing that the periodic orbits including grazing contacts yield complex dynamics [17, 33]. This square-root term may produce the so-called square-root dynamics via the coefficients C_k as defined below. In the particular framework of Section 3.5, the instability of the linear grazing orbit is expected.

Definition 3.12 — Square-root dynamics coefficients. Suppose that $\mathsf{W}_0 \in \mathcal{H}_{\mathcal{P}}$ generates an orbit with the first contact at T_0 being a grazing contact , i.e. $u_N(T_0) = d$ and $\dot{u}_N^-(T_0) = 0$. Under Assumption 3.7, for each $k \in \{1, \ldots, 2N\}$ satisfying

$$\partial_{W_k}\Phi(T_0, \mathsf{W}_0) \neq 0, \tag{3.21}$$

the square-root dynamics coefficient C_k is defined by

$$C_k = \begin{cases} \sqrt{|\gamma_k|}\, \mathbf{e}_k^\top\, \mathbf{P}\, \dot{\mathbf{q}}(T_0) & \text{if } 1 \leq k < N, \quad (3.22a) \\ -\sqrt{|\gamma_k|}\, \mathbf{e}_{k-N}^\top\, \mathbf{M}\, \mathbf{P}\, \mathbf{\Omega}^2\, \mathbf{q}(T_0) & \text{if } N < k \leq 2N, \quad (3.22b) \end{cases}$$

where $[\mathbf{q}, \dot{\mathbf{q}}]$ are modal coordinates following from $\mathbf{u} = \mathbf{P}\mathbf{q}$.

The square-root dynamics is then defined as follows.

Definition 3.13 — Square-root dynamics. System (3.1) is said to feature square-root dynamics near the grazing periodic solution associated to the initial condition W_0 if there exists at least a coefficient C_k, $1 \leq k \leq 2N$, $k \neq N$ which does not vanish.

Near a linear grazing orbit, the square-root dynamics is shown to exist under a generic condition as stated in the following theorem. This theorem is proven in Section 3.6.2.

Theorem 3.14 — Square-root dynamics near linear grazing orbits. Consider the j^{th} linear grazing orbit of (3.1) associated to the initial state $\mathsf{W}_0 \in \mathcal{H}_{\mathcal{P}}$ and T_j, its period. Assumption 3.2 and 3.7 hold. If there exists an index $N < i \leq 2N$ such that $\partial_{W_i} u_N(T_j, \mathsf{W}_0) \neq 0$ and $P_{ij} \neq 0$, then there exists a square-root dynamics near the j^{th} linear grazing orbit.

If, instead, $C_k = 0$ for all k then the square-root dynamics is not activated and the dynamics near the periodic orbit is similar to that of smooth dynamics.

This theorem gives a hint on how to study the instability of the linear grazing orbits. For instance, if after every closing contact, the orbit stays in the same regime where the square-root singularity is preserved, then the dynamics follows the framework stated in Section 3.5 where the instability of the associated fixed-point of the first return map is elaborated.

The condition $P_{ij} \neq 0$ comes from the formula of C_i, where $N < i \leq 2N$. Despite the fact that $P_{ij} \neq 0$ is a generic property, it may be violated for a chain of masses. Hence, in order to have such a $C_i \neq 0$, we need to give a condition on P_{ij}. When $N = 2$ (see [28]), it will be shown that $P_{ij} \neq 0$ for all $i, j = 1, 2$, therefore, there is always a square-root dynamics near the linear grazing orbits.

3.3 Domain of definition of the first return map

This section details a comprehensive investigation of the Poincaré section.

3.3.1 Contact times

The contact times are defined and categorized. In particular, if a sticking phase starts, then it is of finite duration. Moreover, the total energy of the system is conserved during the sticking phase. These features will be used to show that there is a countably infinite number of closing contacts after a sticking phase, further details are found in Section 3.3.2.

Definition 3.15 — Contact time. Assume that $\mathbf{u}(t)$ is the solution to system (3.1); T is a contact time if $u_N(T) = d$ and there exists $\delta > 0$ such that $u_N(T - t) < d$ for all $0 < t < \delta$.

In other words, a contact time T is the time when the mass N touches the rigid obstacle after a free-flight phase. Invoking [5], the contact time is well defined in the conservative case which is without chattering.

Contact times are classified into three categories:

1. It is a contact with non-zero pre-velocity if $u_N(T) = d$ and $\dot{u}_N^-(T) > 0$.
2. It is a contact with zero pre-velocity if $u_N(T) = d$ and $\dot{u}_N^-(T) = 0$, with two new possibilities:
   (a) a grazing contact if the mass leaves the obstacle right after the contact time; or
   (b) a sticking contact if the mass stays in contact with the obstacle.

In this chapter, the term "closing contact" indifferently refers to either an impulsive impact or a grazing contact or the beginning of a sticking contact phase. The sticking system [7] dictates the dynamics during the sticking phase. Since the last mass is at rest, the system "loses" one degree-of-freedom. The sticking system complemented by the initial data at the beginning of a sticking phase is explicitly derived as

$$\underline{\mathbf{M}}\ddot{\underline{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = -d\,\underline{\mathbf{l}}_N^\top \tag{3.23}$$

$$m_N \ddot{u}_N = 0, \quad u_N(0) = d, \quad \dot{u}_N(0) = 0. \tag{3.24}$$

where $\underline{\mathbf{M}}$, $\underline{\mathbf{K}}$ are the mass and stiffness matrix after removing the last row and last column, $\underline{\mathbf{u}}$ is a $N - 1$ vector solution to the sticking dynamics, $\underline{\mathbf{l}}_N^\top$ is the last column of $\mathbf{K}$ where the last entry k_{NN} has been removed.

Proposition 3.1 can now be proven.

Proof. Assume that the sticking phase never ends. The new equilibrium $\underline{\mathbf{u}}_e$ of the sticking system satisfies

$$\underline{\mathbf{K}}\underline{\mathbf{u}}_e = -d\,\underline{\mathbf{l}}_N^\top \text{ or } \underline{\mathbf{u}}_e = -d\,\underline{\mathbf{K}}^{-1}\,\underline{\mathbf{l}}_N^\top. \tag{3.25}$$

During the sticking phase, the last equation reduces to $m_N\ddot{u}_N + \underline{\mathbf{l}}_N\underline{\mathbf{u}} + k_{NN}d = R(t)$, or $m_N\ddot{u}_N = -F(t) + R(t)$, with

$$F(t) = \underline{\mathbf{l}}_N\underline{\mathbf{u}}(t) + k_{NN}d \leq 0. \tag{3.26}$$

It should be understood that $F(t)$ cannot be positive during the sticking phase. Otherwise, there exists t_0 during the sticking phase such that $F(t_0) > 0$. By continuity of the function F, it is strictly positive on an open interval including t_0. By integrating the acceleration, $u_N(t_0) < d$ which contradicts that $u_N(t) = d$ during the sticking phase.

The solution $\underline{\mathbf{u}}$ of the sticking system is quasi-periodic and continuous and its mean value $\langle\underline{\mathbf{u}}\rangle$ is the equilibrium $\underline{\mathbf{u}}_e$. Therefore, the mean value of the scalar quasi-periodic function F is $\langle F\rangle = \underline{\mathbf{l}}_N\langle\underline{\mathbf{u}}\rangle + k_{NN}d = \underline{\mathbf{l}}_N\underline{\mathbf{u}}_e + k_{NN}d$. Since F is continuous, there exits $t_0 > 0$ such that $F(t_0) = \langle F\rangle = -d\,\underline{\mathbf{l}}_N\underline{\mathbf{K}}^{-1}\,\underline{\mathbf{l}}_N^\top + k_{NN}d = d\mathbf{X}^\top\,\mathbf{K}\,\mathbf{X}$ with $\mathbf{X} = [\underline{\mathbf{K}}^{-1}\underline{\mathbf{l}}_N^\top, -1]^\top$. $F(t_0)$ is positive because of $d > 0$ and the positive definiteness of the matrix $\mathbf{K}$, which contradicts (3.26). Therefore, the sticking duration is finite. □

A priori, the conservation of energy during the sticking phase is not straightforward since the sticking system is a different system.

Lemma 3.16 — Energy during sticking phase. The solution to System (3.23)-(3.24) preserves the total energy E (3.1d).

Proof. Assume that $t = 0$ is the beginning of a sticking phase and $t = \tau$, the end. During this sticking phase on the interval $[0\,;\tau]$, the governing equations become $\underline{\mathbf{M}}\ddot{\overline{\mathbf{u}}} + \underline{\mathbf{K}}\overline{\mathbf{u}} = \mathbf{0}$, with $\overline{\mathbf{u}} = \underline{\mathbf{u}} - \underline{\mathbf{u}}_e$ where $\underline{\mathbf{u}}_e = -d\,\underline{\mathbf{K}}^{-1}\,\underline{\mathbf{l}}_N^\top$ is the new equilibrium of the sticking system. This sticking system conserves the energy around the new equilibrium $\underline{\mathbf{u}}_e$:

$$\overline{E}(t) = \dot{\overline{\mathbf{u}}}^\top(t)\underline{\mathbf{M}}\dot{\overline{\mathbf{u}}}(t) + \overline{\mathbf{u}}^\top(t)\underline{\mathbf{K}}\overline{\mathbf{u}}(t) = \overline{E}(0). \tag{3.27}$$

Moreover, since $\underline{\mathbf{u}} = \overline{\mathbf{u}} + \underline{\mathbf{u}}_e$, an easy computation yields:

$$\underline{E}(t) = \dot{\underline{\mathbf{u}}}^\top(t)\underline{\mathbf{M}}\dot{\underline{\mathbf{u}}}(t) + \underline{\mathbf{u}}^\top(t)\underline{\mathbf{K}}\underline{\mathbf{u}}(t) = \overline{E}(t) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}}\underline{\mathbf{u}}_e + 2\overline{\mathbf{u}}^\top(t)\underline{\mathbf{K}}\underline{\mathbf{u}}_e. \tag{3.28}$$

In particular,

$$\underline{E}(0) = \overline{E}(0) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}}\underline{\mathbf{u}}_e + 2\overline{\mathbf{u}}^\top(0)\underline{\mathbf{K}}\underline{\mathbf{u}}_e. \tag{3.29}$$

The total energy of System (3.1) can be now calculated. Since $\dot{\mathbf{u}}$ is continuous along a sticking phase, the exponents ± are dropped.

$$
\begin{aligned}
E(t) &= \dot{\mathbf{u}}^\top(t)\mathbf{M}\dot{\mathbf{u}}(t) + \mathbf{u}^\top(t)\mathbf{K}\mathbf{u}(t) && (3.30)\\
&= \underline{\dot{\mathbf{u}}}^\top(t)\underline{\mathbf{M}\dot{\mathbf{u}}}(t) + \underline{\mathbf{u}}^\top(t)\underline{\mathbf{K}\mathbf{u}}(t) + k_{NN}u_N^2(t) + 2u_N(t)\underline{\mathbf{l}}_N\underline{\mathbf{u}}(t) && (3.31)\\
&= \underline{E}(t) + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N\underline{\mathbf{u}}(t) && (3.32)\\
&= \overline{E}(t) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}\mathbf{u}}_e + 2\overline{\mathbf{u}}^\top(t)\underline{\mathbf{K}\mathbf{u}}_e + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N(\overline{\mathbf{u}}(t) + \underline{\mathbf{u}}_e) && (3.33)\\
&= \overline{E}(t) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}\mathbf{u}}_e - 2d\,\overline{\mathbf{u}}^\top(t)\underline{\mathbf{l}}_N^\top + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N(\overline{\mathbf{u}}(t) + \underline{\mathbf{u}}_e) && (3.34)\\
&= \overline{E}(t) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}\mathbf{u}}_e + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N\underline{\mathbf{u}}_e. && (3.35)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E(0) &= \underline{E}(0) + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N\underline{\mathbf{u}}(0) && (3.36)\\
&= \overline{E}(0) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}\mathbf{u}}_e + 2\overline{\mathbf{u}}^\top(0)\underline{\mathbf{K}\mathbf{u}}_e + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N\underline{\mathbf{u}}(0) && (3.37)\\
&= \overline{E}(0) + \underline{\mathbf{u}}_e^\top\underline{\mathbf{K}\mathbf{u}}_e + k_{NN}d^2 + 2d\,\underline{\mathbf{l}}_N\underline{\mathbf{u}}_e. && (3.38)
\end{aligned}
$$

This follows that $E(t) = E(0)$, i.e. the total energy of the system is a constant during the sticking phase. □

3.3.2 Zero, one or infinity?

This section is devoted to the study of the number of closing contacts. The main results are stated in Theorem 3.3. The number of closing contacts can only be either 0, 1 or countably infinite. Consequently, if there are at least two closing contacts, then mass N will come in contact with the obstacle an infinite number of times. As a consequence, it is impossible to have a solution with $2, 3, \ldots, m$ closing contacts.

The proof of Theorem 3.3 is mainly grounded on the relative position of mass N with the wall. In case 1 of the theorem, system (3.1) is linear since the last mass moves within a free-flight and there is a unique expression for u_N. Similarly, in case 2 with only one grazing contact, the nonlinear term from the contact with zero velocity does not affect the solution of the linear system.

Between every two consecutive closing contacts, say $]t_j\,;t[_{j+1}$, system (3.1a) with $r(t) = 0$ is linear. There is a unique solution associated with the initial data at the exit time t_j. Hence, u_N has the unique form of a quasi-periodic function with distinct coefficients on each interval $]t_j\,;t_{j+1}[$.

The proof is divided into two steps. We first consider the quasi-periodic function φ defined on $\mathbb{R}$ which coincides with u_N on a free-flight interval. The properties of almost periodic functions is called to study the behavior of φ. Then, the results obtained for φ are applied to obtain the behavior of the solution $\mathbf{u}(t)$. The first step is contained in the following lemma.

Lemma 3.17 — Maximum of a quasi-periodic function. Let $\varphi(t)$ be a quasi-periodic function defined on $\mathbb{R}$ such that

$$\varphi(t) = \sum_{j=1}^{N} \big(c_j \cos(\omega_j t) + s_j \sin(\omega_j t)\big) \quad \text{and} \quad \varphi(0) = \sum_{j=1}^{N} c_j = d > 0. \tag{3.39}$$

If $(\omega_1, \dots, \omega_N)$ are $\mathbb{Z}$-independent, then there are two possibilities:

1. If $s_j = 0$ and $c_j \geq 0$ for all $j = 1, \dots, N$ then $\sup_{\mathbb{R}} \varphi = d$. Moreover, if φ is not periodic, i.e there exist at least two coefficients c_j and c_k with $j \neq k$ such that $c_j > 0$ and $c_k > 0$, then $\varphi(t) < d$ for all $t \neq 0$ and $\varphi(t) = d$ for only $t = 0$.
2. Otherwise, if there exists at least one $l \in \{1, \dots, N\}$ such that $s_l \neq 0$ or $c_l < 0$ then $\sup_{\mathbb{R}} \varphi > d$.

The latter case is equivalent to saying that the converse of the first case is true. In the first case, $\varphi(t) = \sum_{j=1}^{N} c_j \cos(\omega_j t)$ and because of the $\mathbb{Z}$-independence of $\{\omega_j\}$, φ is periodic if and only if there exists a unique $c_j > 0$ with $c_k = 0$ for all $k \neq j$. Then, the set $\{t : \varphi(t) = d\}$ is the infinite set $\{k\,T_j,\ k \in \mathbb{Z}\}$ where $T_j = 2\pi/\omega_j$. This case corresponds to the solution to system (3.1) with many grazing contacts. Discarding the periodic case, the value of φ can be very close to d but will never equal it again. This argument helps prove the case when the response of system (3.1) has only one grazing contact and never reaches the obstacle again.

Later, it is proven that the function φ with a supremum strictly greater than d corresponds to u_N of a solution with many closing contacts. Now, Lemma 3.17 is proven.

Proof.

1. Consider $s_j = 0$ and $c_j \geq 0$ for all $j = 1, \dots, N$, then $\varphi(t) = \sum_{j=1}^{N} c_j \cos(\omega_j t)$ and $\sup_{\mathbb{R}} \varphi = \sum_{j=1}^{N} |c_j| = \sum_{j=1}^{N} c_j = d$. If $\exists t > 0$ such that $\varphi(t) = d$, i.e. $\sum_{j=1}^{N} c_j \cos(\omega_j t) = d$, then $\sum_{j=1}^{N} c_j(1 - \cos(\omega_j t)) = 0$ where $c_j(1 - \cos(\omega_j t)) \geq 0$ for all $j = 1, \dots, N$. Hence, $\cos(\omega_j t) = 1$ for all $j = 1, \dots, N$. Thus, $\omega_j t = k_j\, 2\pi, k \in \mathbb{Z}$, which contradicts the $\mathbb{Z}$-independence assumption. Hence, φ is always smaller than d for $t > 0$.
2. Otherwise, if there exists $l \in \{1, \dots, N\}$ such that $s_l \neq 0$, or $c_l < 0$, then $\sup_{\mathbb{R}} \varphi = \sum_{j=1}^{N} \sqrt{c_j^2 + s_j^2} \geq \max(\sum_{j=1}^{N} \sqrt{c_j^2 + s_j^2}, \sum_{j=1}^{N} |c_j|) > \sum_{j=1}^{N} c_j = d$. Hence, there exists $t > 0$ such that $\varphi(t) = d$. □

Now, using Lemma 3.17, Theorem 3.3 is proven.

Proof. The solution to (3.1a) when $R(t) = 0$ is for $t \in \mathbb{R}$

$$\begin{aligned}\mathbf{\Phi}(t) &= \mathbf{P}\cos(t\mathbf{\Omega})\mathbf{P}^{-1}\mathbf{u}(0) + \mathbf{P}\mathbf{\Omega}^{-1}\sin(t\mathbf{\Omega})\mathbf{P}^{-1}\dot{\mathbf{u}}(0),\\ &= \sum_{j=1}^{N}(c_j\cos(\omega_j t) + s_j\sin(\omega_j t))\mathbf{P}\,\mathbf{e}_j,\end{aligned} \tag{3.40}$$

where $\mathbf{P}\,\mathbf{e}_j$, $j = 1,\dots,N$ are eigenvectors corresponding to eigenvalues ω_j^2 of $\mathbf{M}^{-1}\mathbf{K}$. We can choose $\mathbf{P}\,\mathbf{e}_j$ such that $P_{Nj} = 1$, $j = 1,\dots,N$. It is clear that $\mathbf{u}(t) = \mathbf{\Phi}(t)$ as long as $u_N(t) < d$.

Henceforth, consider the solution $\mathbf{u}(t)$ to the vibro-impact system (3.1). If $u_N(t) < d$, for all $t \geq 0$, then the solution of the linear problem does not impact the wall.

Since the system is autonomous, the impacting time is chosen to be $t = 0$, i.e. $u_N(0) = d$. Two possibilities based on the velocity of the N^{th} mass at the contact time $t = 0$ are considered:

If $\dot{u}_N^-(0) > 0$, then $\dot{u}_N^+(0) < 0$. Let φ be a function defined on $\mathbb{R}$ such that $\varphi(t) = u_N(t)$ on $\{t,\ u_N(t) < d\}$. It follows that $\varphi(0) = d$ and $\dot{\varphi}(0) < 0$, thus there exists $\tau > 0$ such that $\varphi(t) > d$ for $t \in\,]-\tau\,;0[$. Therefore, $\sup_{\mathbb{R}^-}\varphi \geq \sup_{]-\tau\,;0[}\varphi > d$. For an almost periodic function φ, the supremum taken on $\mathbb{R}^-$ is also the one taken on $\mathbb{R}^+$ [13], this yields $\sup_{\mathbb{R}^+}\varphi > d$. By Lemma 3.17, the first instant $t_1 > 0$ such that $\varphi(t_1) = d$ exists, i.e. mass N will come back to the obstacle at time t_1.

If $\dot{u}_N^-(0) = 0$ then $\dot{u}_N^+(0) = 0$ and $\sum_{j=1}^N c_j = d$, $\sum_{j=1}^N s_j\omega_j = 0$. Let φ be the function defined on $\mathbb{R}$ so that it coincides with u_N when $u_N(t) < d$ before the contact. If $s_j = 0$ and $c_j \geq 0$ for all $j = 1,\dots,N$, by Lemma 3.17, it follows that $\sup_{\mathbb{R}}\varphi = d$ and $\varphi(t) < d$ for all $t > 0$. Thus, the solution has only one grazing contact and it can be seen as the counterpart of the linear system without contact.

Otherwise, again, Lemma 3.17 shows that $\sup_{\mathbb{R}}\varphi > d$. It follows that there is a sticking phase from 0 to τ, i.e. $u_N(t) = d$ for all $t \in [0,\tau]$ and $u_N(t) < d$ for $\tau < t < \tau + \delta$, where $\delta > 0$. Assume that $\mathbf{w}$ is the solution to (3.1a) after the sticking time, associated to the new initial data at $t = \tau$. Thus, w_N has the expression

$$w_N(t) = \sum_{j=1}^{N}\left(\underline{c}_j\cos(\omega_j(t-\tau)) + \underline{s}_j\sin(\omega_j(t-\tau))\right), \quad t \geq \tau. \tag{3.41}$$

Let $\underline{\varphi}$ be the function defined on $\mathbb{R}$ satisfying $\underline{\varphi}(t) = w_N(t)$ on $\{t : w_N(t) < d\}$. Consider the solution $\mathbf{u}$ to the vibro-impact system (3.1) just before and just after the sticking phase, $u_N(t)$ can be written as

$$u_N(t) = \begin{cases}\varphi(t) & \text{for } t \lesssim 0,\\ d & \text{for } t \in [0\,;\tau],\\ \underline{\varphi}(t) & \text{for } t \gtrsim \tau.\end{cases} \tag{3.42}$$

We then show that $\sup_{[\tau;\infty[} \underline{\varphi} > d$. If not, $\sup_{\mathbb{R}} \underline{\varphi} = d$, then $\underline{\varphi}(t) < d$, for all $t \in \mathbb{R}$ and $\underline{\varphi}(t) = d$ for $t = \tau$ only. By reversibility and uniqueness of the solution, the solution to (3.1) is always without contact and there is only one grazing contact at τ. It is in contradiction with the fact that the solution involved the sticking phase. As a consequence, $\sup_{\mathbb{R}} \underline{\varphi} > d$. Hence, there exists $t > \tau$ such that $\underline{\varphi}(t) = d$.

The process can be continued in both cases, thus the set of closing contacts is countably infinite. □

3.3.3 Poincaré section

This part deals with the construction of the Poincaré section on which the first return map is well-defined. Consider an orbit $[\mathbf{u}, \dot{\mathbf{u}}] \subset \mathbb{R}^{2N}$ of (3.1). To study the dynamics near such an orbit, we use the first return map (or Poincaré map). This map is defined on a Poincaré section which is classically a $(2N-1)$-dimensional manifold in $\mathbb{R}^{2N}$ that contains a point $\mathbf{U}(t) = [\mathbf{u}(t), \dot{\mathbf{u}}(t)]$ of the orbit and is transverse to the orbit at $\mathbf{U}(t)$. In the current work, the transversality is lost on $\mathcal{H}^0$. An orbit starting on $\mathcal{H}^0$ may not intersect $\mathcal{H}$ again. Hence, an important task is to eliminate the set of data such that the associated orbit does not intersect $\mathcal{H}_\varphi$ again. This is achieved by investigating the structure of $\mathcal{H}^0$. By Theorem 3.3, there are two possibilities after the grazing contact: the orbit never comes in contact again ($\mathcal{H}^0_1$) or the orbit comes in contact an infinite number of time ($\mathcal{H}^0_\infty$). The explicit description of $\mathcal{H}^0_1$ is studied in the proposition below. Recall that $\mathbf{B} = (B_{ij})_{i,j=1}^N$ denotes $\mathbf{P}^{-1}$.

Proposition 3.18 — $\mathcal{H}^0_1$ and solutions with only one contact. The set $\mathcal{H}^0_1$ of initial data on $\mathcal{H}$ such that the associated orbits have only one contact is the subset

$$\{[\mathbf{u}^\top, \dot{\mathbf{u}}^\top]^\top \in \mathbb{R}^{2N},\ u_N = d,\ \dot{u}_k = 0,\ k = 1, \ldots, N\} \tag{3.43}$$

of a $(N-1)$-dimensional affine subspace, such that the $N-1$ components $u_1, u_2, \ldots, u_{N-1}$ satisfy N inequalities:

$$P_{Nk} \sum_{j=1}^{N-1} B_{kj} u_j \geq -P_{Nk} B_{kN}\, d, \qquad \forall k = 1, \ldots, N \tag{3.44}$$

of which at least two are strict inequalities.

Note that $\mathcal{H}^0_1$ can be empty.

Proof. The solution to (3.1) with a unique grazing contact is analytic and $u_N(t)$ has a closed form-expression:

$$u_N(t) = \mathbf{e}_N^\top \left(\mathbf{P}\cos(t\boldsymbol{\Omega})\mathbf{P}^{-1}\mathbf{u}(0) + \mathbf{P}\boldsymbol{\Omega}^{-1}\sin(t\boldsymbol{\Omega})\mathbf{P}^{-1}\dot{\mathbf{u}}(0)\right), \tag{3.45}$$

$$= \sum_{k=1}^{N}(c_k\cos(\omega_k t) + s_k\sin(\omega_k t)), \tag{3.46}$$

where $c_k = P_{Nk}\,\mathbf{e}_k^\top\,\mathbf{P}^{-1}\,\mathbf{u}(0)$, $s_k = P_{Nk}\,\omega_k^{-1}\,\mathbf{e}_k^\top\,\mathbf{P}^{-1}\,\dot{\mathbf{u}}(0)$, $k = 1,\dots,N$. Since $[\mathbf{u}^\top, \dot{\mathbf{u}}^\top]^\top$ belongs to $\mathcal{H}_1^0$, this corresponds to case 1 of Lemma 3.17, thus, $u_N(0) = d$ and $\dot{u}_N(0) = 0$, and the coefficients c_k, s_k satisfy

$$c_k \geq 0, \quad \forall k = 1,\dots,N, \quad \exists l \neq m : c_l > 0, c_m > 0, \tag{3.47}$$

$$s_k = 0, \quad \forall k = 1,\dots,N. \tag{3.48}$$

This gives

$$P_{Nk}\,\mathbf{e}_k^\top\,\mathbf{P}^{-1}\,\mathbf{u}(0) \geq 0, \quad \forall k = 1,\dots,N, \quad \exists l \neq m :\; c_l > 0, c_m > 0, \tag{3.49}$$

$$P_{Nk}\omega_k^{-1}\,\mathbf{e}_k^\top\,\mathbf{P}^{-1}\,\dot{\mathbf{u}}(0) = 0, \quad \forall k = 1,\dots,N. \tag{3.50}$$

The second condition yields a linear system $\mathbf{P}^{-1}\,\dot{\mathbf{u}}(0) = \mathbf{0}$, since $\det(\mathbf{P}^{-1}) \neq 0$, it follows that $\dot{\mathbf{u}}(0) = \mathbf{0}$ and $u_1,\dots,u_{N-1}$ satisfy the inequalities (3.49). This gives the explicit formula of $\mathcal{H}_1^0$ stated in Proposition 3.18. □

An immediate consequence of Proposition 3.18 is that most orbits which belong to $\mathcal{H}^0$ are in $\mathcal{H}_\infty^0$. To be precise, the set of initial data such that the associated orbits belong to $\mathcal{H}_1^0$ has a $(2N-2)$-dimensional zero measure in $\mathcal{H}^0$. As a consequence, the Poincaré section chosen in Definition 3.4 is reasonable.

Corollary 3.19 — Domain of definition of the first return map. The maximal subset of $\mathcal{H}$ where the first return map $\mathcal{F}$ is well-defined is $\mathcal{H}_{\mathcal{P}} = \mathcal{H}^- \cup \mathcal{H}_\infty^0$.

Some consequences of Proposition 3.18 are now stated.

Corollary 3.20. The orbits including a sticking phase have an infinite number of closing contacts.

Proof. An orbit including a sticking phase intersects $\mathcal{H}^0$. As proven in Proposition 3.18, the initial data must be in $\mathcal{H}_\infty^0$ since $\mathcal{H}_1^0$ only involves the data on $\mathcal{H}$ such that the orbits have a unique grazing contact but no sticking phase. Hence, it belongs to the set with an infinite number of closing contacts. □

Corollary 3.21 — Infinite number of closing contacts. If the orbit intersects $\mathcal{H}^-$ then it intersects $\mathcal{H}^-$ infinitely many times in the future and in the past.

Proof. By Theorem 3.3, the orbit corresponding to the initial data in $\mathcal{H}^-$ experiences at least one impact at $t = 0$. This eliminates the possibilities of case 1 and case 2 in the theorem. Hence, the orbit belongs to the third category, which means that there will be an infinite number of closing contacts. However, if the set of closing contacts involves only grazing contacts in the future, the system becomes linear with $u_N(t) \leq d$ for all $t > 0$. Let φ be a function defined on $\mathbb{R}$ with $\varphi(t) = u_N(t)$, it follows that $\sup_{\mathbb{R}} \varphi = \sup_{t>0} \varphi = d$. This contradicts the fact that the supremum of φ must be greater than d since there is at least one impact at $t = 0$. The process can be repeated to get another closing contact and so on. □

Denote by $\mathcal{H}^{--}$ a subset of $\mathcal{H}^-$, containing all the data such that the corresponding orbits have impacts only. It seems to be dense in $\mathcal{H}^-$.

About the set $\mathcal{H}_1^0$ for 2-dof systems

Corollary 3.22 — $\mathcal{H}_1^0$ for a 2-dof system. Consider a 2-dof system (3.1), then from Proposition 3.18, $\mathcal{H}_1^0$ is defined as

$$\mathcal{H}_1^0 = \{[\mathbf{u}^\top, \dot{\mathbf{u}}^\top]^\top \in \mathbb{R}^4 : P_{2k}B_{k1}u_1 > -P_{2k}B_{k2}\, d,\ k = 1, 2,\ u_2 = d, \dot{u}_1 = \dot{u}_2 = 0\} \tag{3.51}$$

$$= D_{u_1} \times \{d\} \times \{0\} \times \{0\} \tag{3.52}$$

where D_{u_1} is the subset of $\mathbb{R}$ including all the values u_1 satisfying the two strict inequalities:

$$P_{21}B_{11}u_1 > -P_{21}B_{12}d \text{ and } P_{22}B_{21}u_1 > -P_{22}B_{22}d. \tag{3.53}$$

Therefore, the set D_{u_1} can be either void or an open interval of the form (b, ∞), $(-\infty, a)$, or (a, b), with $a = \min\{\alpha_1, \alpha_2\}$, $b = \max\{\alpha_1, \alpha_2\}$, where $\alpha_1 = -dB_{22}/B_{21}$ and $\alpha_2 = -dB_{12}/B_{11}$.

Remark 3.23 — Linear gazing modes of a 2-dof system and the boundary of $\mathcal{H}_1^0$. The scalars α_1 and α_2 are the distinct initial values of u_1 corresponding to the first and the second linear grazing orbit.

The set $\mathcal{H}_1^0$ is an interval in the two-dimensional space $\mathcal{H}^0$. This means that the set of initial data such that the associated orbits have only one grazing contact is a very small subset of all the initial data such that orbits contain zero velocity contacts.

3.4 Implicit function theorem and power-root singularity

This section is divided into two parts. The square-root singularity is studied in the first part. This singularity implies the square-root dynamics near the linear grazing orbits as stated in Section 3.6. A general power-root singularity is investigated in the second part.

3.4.1 The square-root singularity

The first return time T is implicitly defined from the equation $f(T, \mathsf{W}) = 0$ meaning $u_N(T) = d$ in Section 3.6. Unfortunately, if the contact at T_0 is grazing, $\partial_t f(T_0, \mathsf{W}_0)$ vanishes. The square-root singularity is expected along the intersection of the hypersurfaces defined by $f = 0$ and $\partial_t f = 0$. This section gives a general approach for a general function f to show this emergence of the square-root singularity. The main tool is to use implicit function theorem on a variable other than t and then use Taylor expansion of the function obtained from the implicit function theorem to get back to the t variable as a function of W. We state the results in two cases corresponding to the function f defined in a two-dimensional and a $m + 2$-dimensional spaces with $m \geq 1$.

In two dimensions

Let f be a function of two variables x and y, where the relation of x with respect to y is implicitly given by $f(x, y) = 0$. Let us write x locally as a function of y when f satisfies some unusual conditions as follows.

Theorem 3.24 — Square-root singularity in two dimensions. Suppose that $f(x, y) \in C^3(\mathbb{R}^2, \mathbb{R})$ satisfies the following conditions at (x_0, y_0):

1. $f(x_0, y_0) = 0$,
2. $\partial_x f(x_0, y_0) = 0$,
3. $\partial_{xx} f(x_0, y_0) \neq 0$,
4. $\partial_y f(x_0, y_0) \neq 0$.

Denote the ratio $2\gamma = -\partial_{xx} f(x_0, y_0)/\partial_y f(x_0, y_0) \neq 0$ and $s_\gamma = \operatorname{sign} \gamma$. There exist two intervals I and J containing y_0 and x_0, respectively, as well as a function $\psi \in C^3(I_\gamma, J)$ defined on the subinterval

$$I_\gamma = \{y \in I,\ s_\gamma(y - y_0) \geq 0\} \tag{3.54}$$

such that the set $\{(x, y) \in I \times J,\ f(x, y) = 0\}$ is form by the following two branches near (x_0, y_0):

$$x = x_0 + \psi(\sqrt{s_\gamma(y - y_0)}) \text{ and } x = x_0 + \psi(-\sqrt{s_\gamma(y - y_0)}). \tag{3.55}$$

Moreover, ψ satisfies $\psi(0) = 0$ and $\dot{\psi}(0) = 1/\sqrt{|\gamma|} \neq 0$.

An illustration of this theorem is given in Figure 3.1. Note that ψ is not defined on the whole open interval I but only on half of it which is determined by the sign of γ.

Proof. Without loss of generality, assume that (x_0, y_0) is at the origin. If not, consider the new variables $x^* = x - x_0$ and $y^* = y - y_0$, and note that at the point $(0, 0)$, the function

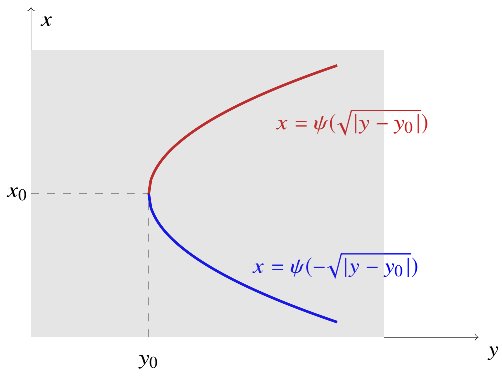


FIGURE 3.1 – Two branches on the right of the line $y = y_0$ when $\gamma > 0$.

$f^*(x^*, y^*) = f(x_0 + x^*, y_0 + y^*)$ satisfies the conditions of the theorem. The conclusions are then modified accordingly.

The implicit function theorem ensures the existence of the interval V_x of $x_0 = 0$ and V_y of $y_0 = 0$, and a unique function $y = \varphi(x)$ for $x \in V_x$ and $y \in V_y$, such that $f(x, \varphi(x)) = 0$ for $x \in V_x$, $\varphi(0) = 0$, $\dot{\varphi}(0) = -\partial_x f(0, 0)/\partial_y f(0, 0) = 0$ and $\ddot{\varphi}(0) = -\partial_{xx} f(0, 0)/\partial_y f(0, 0) = 2\gamma \neq 0$. Moreover, φ is as smooth as f. The Taylor expansion of φ with an integral remainder at 0 is

$$y = \varphi(x) = x^2 r(x), \quad r(x) = \int_0^1 (1 - s)\, \ddot{\varphi}(sx) ds \tag{3.56}$$

where $r \in C^1(\mathbb{R}, \mathbb{R})$ and $r(0) = \ddot{\varphi}(0)/2 = \gamma \neq 0$. The expression of x with respect to y depends on the sign of $\ddot{\varphi}(0)$ (or the sign of γ) as follows:

If $\gamma > 0$ then $r(0) > 0$ and by the continuity of r, it follows that $r(x) > 0$ in some neighborhood of 0. On that neighborhood, define the continuously differentiable function $\phi(x) = x\sqrt{r(x)}$: at 0, ϕ has a nonzero derivative since $r(0) \neq 0$. Through the inverse function theorem, there exists an inverse function ϕ^{-1} in some neighborhood I of $\phi(0) = 0$. From $y = \phi^2(x)$, it is required that $y \geq 0$. It entails I_γ is simply a right neighborhood of 0. Hence, for $y \in I_\gamma$, $x = \phi^{-1}(\sqrt{y})$ or $x = \phi^{-1}(-\sqrt{y})$.

If $\gamma < 0$ then $y = -\phi^2(x)$ with $\phi(x) = x\sqrt{-r(x)}$. A similar proof holds for y belongs to I_γ which is now a left neighborhood of 0. The conclusion is obtained by replacing $\sqrt{y}$ by $\sqrt{-y}$ and the function ψ satisfies $\dot{\psi}(0) = \dot{\phi}^{-1}(0) = 1/\sqrt{-\gamma}$.

In both cases, by denoting $\psi = \phi^{-1}$, we have shown that there exists a function ψ defined in a neighborhood I_γ of 0 satisfying $\psi(0) = 0$ and $\dot{\psi}(0) = \dot{\phi}^{-1}(0) = 1/\sqrt{|\gamma|}$ such that $x = \psi(\sqrt{s_\gamma y})$ or $x = \psi(-\sqrt{s_\gamma y})$. This concludes the proof. □

Remark 3.25. Another way to express the square-root singularity of x is $x = \sqrt{\beta(y)}$. However, with the present expression, say $x = \lambda(\sqrt{y})$, we have a better regularity of λ since λ is smooth but β is not always smooth. For example, if $\lambda(y) = y + y^2$, $y \in \mathbb{R}^+$, then $\lambda \in C^\infty(\mathbb{R}^+)$. However,

$\beta(y) = \lambda^2(\sqrt{y}) = (\sqrt{y} + y)^2 = y + 2y\sqrt{y} + y^2$ is not so smooth.

In $m + 2$ dimensions($m \geq 1$)

Consider a general function f of $m + 2$ variables with a non-zero gradient, the square-root singularity is shown.

Theorem 3.26 — Square-root singularity in $m + 2$ dimensions. Consider the smooth function $f(x, y, \mathbf{z}) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \longrightarrow \mathbb{R}$ such that at point $\mathbf{X}_0 = (x_0, y_0, \mathbf{z}_0)$, it satisfies the following conditions:

1. $f(\mathbf{X}_0) = 0$,
2. $\partial_x f(\mathbf{X}_0) = 0$,
3. $\partial_{xx} f(\mathbf{X}_0) \neq 0$,
4. $\partial_y f(\mathbf{X}_0) \neq 0$.

Denote $2\gamma = -\partial_{xx} f(\mathbf{X}_0)/\partial_y f(\mathbf{X}_0) \neq 0$ and $s_\gamma = \operatorname{sign} \gamma$. Then, there exist three neighborhoods V_{x_0}, V_{y_0}, and $V_{\mathbf{z}_0}$ of x_0, y_0 and $\mathbf{z}_0$ respectively, and two smooth scalar functions $\eta : V_{\mathbf{z}_0} \longrightarrow V_{x_0}$ and $\alpha : V_{\mathbf{z}_0} \longrightarrow V_{y_0}$ satisfying $\eta(\mathbf{z}_0) = x_0$ and $\alpha(\mathbf{z}_0) = y_0$ such that the set

$$S_c = \{(x, y, \mathbf{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m,\ f(x, y, \mathbf{z}) = 0 \text{ and } \partial_x f(x, y, \mathbf{z}) = 0\} \tag{3.57}$$

is parameterized by $S_c \cap \Omega = \{(\eta(\mathbf{z}),\ \alpha(\mathbf{z}),\ \mathbf{z}),\ \mathbf{z} \in V_{\mathbf{z}_0}\}$ where $\Omega = V_{x_0} \times V_{y_0} \times V_{\mathbf{z}_0}$. Let B_γ be a

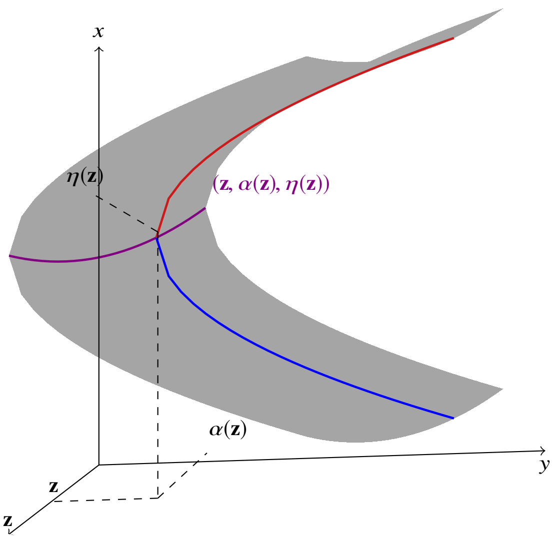


FIGURE 3.2 – Square-root singularity in dimension 3.

subset of $V_{y_0} \times V_{\mathbf{z}_0}$ where

$$B_\gamma = \{(y, \mathbf{z}) \in V_{y_0} \times V_{\mathbf{z}_0},\ s_\gamma(y - \alpha(\mathbf{z})) \geq 0\}. \tag{3.58}$$

There exists a smooth scalar function ψ defined on B_γ such that there are two graphs:

$$x = \eta(\mathbf{z}) + \psi(\sqrt{s_\gamma(y - \alpha(\mathbf{z}))}, \mathbf{z}), \quad (y, \mathbf{z}) \in B_\gamma, \tag{3.59}$$

and

$$x = \eta(\mathbf{z}) + \psi(-\sqrt{s_\gamma(y - \alpha(\mathbf{z}))}, \mathbf{z}), \quad (y, \mathbf{z}) \in B_\gamma. \tag{3.60}$$

In particular, $\psi(0, \mathbf{z}_0) = 0$ and $\partial_y \psi(0, \mathbf{z}_0) = 1/\sqrt{|\gamma|}$.

An illustration of the square-root singularity in 3 dimensions is depicted in Figure 3.2.

The function ψ is not defined on the whole neighborhood $V_{y_0} \times V_{\mathbf{z}_0}$, but only in one part of it, which is B_γ. The subset B_γ defined in the theorem gives the one-sided condition from which the square-root singularity arises. When the sign of γ is known, B_γ is simply a region above or below and including the hypersurface $y = \alpha(\mathbf{z})$, if γ is positive or negative, respectively. A more general result when the condition (3) above does not hold is discussed in Remark 3.27.

Proof. Let S_c be the manifold of m dimensions which is the intersection of the two hypersurfaces of dimension $m + 1$ near $(x_0, y_0, \mathbf{z}_0)$, $S_c = S^0 \cap S^1$ where $S^0 = \{(x, y, \mathbf{z}) \in \Omega, f(x, y, \mathbf{z}) = 0\}$ and $S^1 = \{(x, y, \mathbf{z}) \in \Omega, G(x, y, \mathbf{z}) = 0\}$. Moreover, it can be parameterized in a neighborhood of $\mathbf{z}$ by two parametric functions η and α as follows.

From (1) and (4), by the implicit function theorem, there exist two neighborhoods V of $(x_0, \mathbf{z}_0)$ and W of y_0, and a function $\varphi : V \to W$, $(x, \mathbf{z}) \mapsto y = \varphi(x, \mathbf{z})$ such that $f(x, \varphi(x, \mathbf{z}), \mathbf{z}) = 0$ for $(x, \mathbf{z}) \in V$, $\varphi(x_0, \mathbf{z}_0) = y_0$. In particular, $\partial_x \varphi(x_0, \mathbf{z}_0) = 0$ and $\partial_{xx} \varphi(x_0, \mathbf{z}_0) = -\partial_{xx} f(x_0, y_0, \mathbf{z}_0)/\partial_y f(x_0, y_0, \mathbf{z}_0) = 2\gamma \neq 0$. Moreover, φ has the same smoothness as f.

For $(x, \mathbf{z}) \in V$, once again we apply the implicit function theorem for the function $G(x, \mathbf{z}) = \partial_x f(x, \varphi(x, \mathbf{z}), \mathbf{z})$ which satisfies:

$$G(x_0, \mathbf{z}_0) = \partial_x f(x_0, y_0, \mathbf{z}_0) = 0, \tag{3.61}$$

$$\partial_x G(x_0, \mathbf{z}_0) = \partial_{xx} f(x_0, y_0, \mathbf{z}_0) + \partial_{yx} f(x_0, y_0, \mathbf{z}_0) \partial_x \varphi(x_0, \mathbf{z}_0) = \partial_{xx} f(x_0, a_0, \mathbf{z}_0) \neq 0. \tag{3.62}$$

This gives the existence of the neighborhoods $B_{\mathbf{z}_0}$ of $\mathbf{z}_0$ and B_{x_0} of x_0 such that there is a smooth function $\eta : B_{\mathbf{z}_0} \to B_{x_0}$, $\mathbf{z} \mapsto x = \eta(\mathbf{z})$ satisfying $x_0 = \eta(\mathbf{z}_0)$, $G(\eta(\mathbf{z}), \mathbf{z}) = 0$ for all $\mathbf{z} \in B_{\mathbf{z}_0}$. It follows that $\partial_x f(\eta(\mathbf{z}), \alpha(\mathbf{z}), \mathbf{z}) = 0$ for all $\mathbf{z} \in B_{\mathbf{z}_0}$ where $\alpha(\mathbf{z}) := \varphi(\eta(\mathbf{z}), \mathbf{z})$ belongs to a neighborhood of y_0, denoted V_{y_0}. In particular, $\alpha(\mathbf{z}_0) = \varphi(x_0, \mathbf{z}_0) = y_0$.

As a consequence, there exists a local parameterization of S_c:

$$S_c = \{(x, y, \mathbf{z}) = (\eta(\mathbf{z}),\ \alpha(\mathbf{z}),\ \mathbf{z}),\ \mathbf{z} \in B_{\mathbf{z}_0}\}. \tag{3.63}$$

This parameterization can be seen directly by using the implicit function theorem for the vector function $H = (f, \partial_x f)$ with the invertible matrix

$$D_{x,y}H(\mathbf{X}_0) = \begin{bmatrix} \partial_x f & \partial_y f \\ \partial_{xx} f & \partial_{yx} f \end{bmatrix} (\mathbf{X}_0) = \begin{bmatrix} 0 & \neq 0 \\ \neq 0 & \partial_{yx} f(\mathbf{X}_0) \end{bmatrix}. \tag{3.64}$$

We will now show that the square-root singularity arises along the hypersurface $y = \alpha(\mathbf{z})$. For each fixed $\mathbf{z} \in B_{\mathbf{z}_0}$, the Taylor expansion with an integral remainder of φ with respect to x near $\eta(\mathbf{z})$ reads

$$\varphi(x, \mathbf{z}) = \varphi(\eta(\mathbf{z}), \mathbf{z}) + \partial_x \varphi(\eta(\mathbf{z}), \mathbf{z})\big(x - \eta(\mathbf{z})\big) \tag{3.65}$$

$$+ (x - \eta(\mathbf{z}))^2 \int_0^1 (1 - s) \partial_{xx} \varphi \left(s(x - \eta(\mathbf{z})) + \eta(\mathbf{z}), \mathbf{z}\right) ds. \tag{3.66}$$

Since $y = \varphi(x, \mathbf{z})$, $\partial_x \varphi(\eta(\mathbf{z}), \mathbf{z}) = 0$, $\varphi(\eta(\mathbf{z}), \mathbf{z}) = \alpha(\mathbf{z})$, this yields

$$y - \alpha(\mathbf{z}) = (x - \eta(\mathbf{z}))^2 r(x - \eta(\mathbf{z}), \mathbf{z}), \tag{3.67}$$

where $r(x - \eta(\mathbf{z}), \mathbf{z})$ is a smooth function with respect to x and $\mathbf{z}$, satisfying $r(0, \mathbf{z}) = \partial_{xx} \varphi(\eta(\mathbf{z}), \mathbf{z})/2$.

The remainder $r(x - \eta(\mathbf{z}), \mathbf{z})$ has the sign of γ for all $\mathbf{z}$ in a neighborhood of $\mathbf{z}_0$. Consider the function $\gamma(\mathbf{z}) = r(0, \mathbf{z})$, $\gamma(\mathbf{z})$: it is as smooth as r and satisfies $\gamma(\mathbf{z}_0) = \partial_{xx} \varphi(x_0, \mathbf{z}_0)/2 = \gamma \neq 0$. Without loss of generality, assume that $\gamma > 0$. By continuity, $\gamma(\mathbf{z}) > 0$ in some neighborhood of $\mathbf{z}_0$, say $V_{\mathbf{z}_0}$. It follows that $r(0, \mathbf{z}) > 0$ for $\mathbf{z} \in V_{\mathbf{z}_0}$. Similarly, by the continuity of r with respect to x, $r(x, \mathbf{z}) > 0$ for all x in a neighborhood of $\eta(\mathbf{z})$.

Let $B_\gamma \subset V_{y_0} \times V_{\mathbf{z}_0}$ be the region adjacent to and including the hypersurface $y = \alpha(\mathbf{z})$ such that

$$B_\gamma = \{(y, \mathbf{z}) \in V_{y_0} \times V_{\mathbf{z}_0},\ \gamma(y - \alpha(\mathbf{z})) \geq 0\}. \tag{3.68}$$

Denote by ϕ the function $\phi(x - \eta(\mathbf{z}), \mathbf{z}) = (x - \eta(\mathbf{z}))\sqrt{r(x - \eta(\mathbf{z}), \mathbf{z})}$, (3.67) then becomes

$$y - \alpha(\mathbf{z}) = (\phi(x - \eta(\mathbf{z}), \mathbf{z}))^2. \tag{3.69}$$

This follows that $y - \alpha(\mathbf{z}) \geq 0$. The subset B_γ is then a region above and including the hypersurface $y = \alpha(\mathbf{z})$. Since ϕ satisfies $\phi(0, \mathbf{z}) = 0$, $\partial_x \phi(0, \mathbf{z}) = \sqrt{r(0, \mathbf{z})} = \sqrt{\gamma(\mathbf{z})} \neq 0$, there exists an inverse function ϕ^{-1} in some neighborhood of 0. The inverse function theorem is used uniformly with respect to the parameter $\mathbf{z}$ and eventually reduces the neighborood of $\mathbf{z}_0$ and x_0. Therefore, for $(y, \mathbf{z}) \in B_\gamma$, (3.69) yields $\phi(x - \eta(\mathbf{z}), \mathbf{z}) = \sqrt{y - \alpha(\mathbf{z})}$ or $\phi(x - \eta(\mathbf{z}), \mathbf{z}) = -\sqrt{y - \alpha(\mathbf{z})}$. This gives $x = \eta(\mathbf{z}) + \phi^{-1}(\sqrt{y - \alpha(\mathbf{z})}, \mathbf{z})$ or $x = \eta(\mathbf{z}) + \phi^{-1}(-\sqrt{y - \alpha(\mathbf{z})}, \mathbf{z})$. In particular, $\phi^{-1}(0, \mathbf{z}) = 0$ and $\partial_y \phi^{-1}(0, \mathbf{z}) = 1/\sqrt{\gamma(\mathbf{z})}$.

Similarly, for the case $\gamma < 0$, $r(x - \eta(\mathbf{z}), \mathbf{z}) < 0$ for x in some neighborhood of $\eta(\mathbf{z})$, it follows that $y - \alpha(\mathbf{z}) = -\phi^2(x - \eta(\mathbf{z}), \mathbf{z})$, where $\phi(x - \eta(\mathbf{z}), \mathbf{z}) = (x - \eta(\mathbf{z}))\sqrt{-r(x - \eta(\mathbf{z}), \mathbf{z})}$. The subset B_γ is now the region below and including the hypersurface $y = \alpha(\mathbf{z})$. The same conclusions hold with a change of $\sqrt{y - \alpha(\mathbf{z})}$ in $\sqrt{\alpha(\mathbf{z}) - y}$ and $\dot{\phi}^{-1}(0, \mathbf{z}) = 1/\sqrt{-\gamma(\mathbf{z})}$.

Consequently, by denoting $\psi = \phi^{-1}$, we have shown that ψ is defined in B_γ such that

$$x = \eta(\mathbf{z}) + \psi(\sqrt{s_\gamma(y - \alpha(\mathbf{z}))}, \mathbf{z}) \quad \text{or} \quad x = \eta(\mathbf{z}) + \psi(-\sqrt{s_\gamma(y - \alpha(\mathbf{z}))}, \mathbf{z}). \tag{3.70}$$

In particular, $\psi(0, \mathbf{z}_0) = 0$, $\partial_y \psi(0, \mathbf{z}_0) = 1/\sqrt{\gamma(\mathbf{z}_0)} = 1/\sqrt{|\gamma|}$. This ends the proof. □

3.4.2 Power-root singularity

The square-root singularity shown in Theorem 3.26 relies on the non-vanishing second derivative $\partial_{xx} f(x_0, y_0, \mathbf{z}_0)$. If it vanishes to zero, then higher order derivatives of f must be considered. The following remark gives analogous conditions from which we define power-root singularity.

Remark 3.27. Let $n \le m + 1$ and $f(x, y, \mathbf{z}) : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ be a smooth function, such that at point $\mathbf{X}_0 = (x_0, y_0, \mathbf{z}_0)$, it satisfies the following conditions:

1. $f(\mathbf{X}_0) = \partial_x f(\mathbf{X}_0) = \partial_{xx} f(\mathbf{X}_0) = \ldots = \partial^{n-1}_{x^{n-1}} f(\mathbf{X}_0) = 0$, $\partial^n_{x^n} f(\mathbf{X}_0) \neq 0$, $n \ge 3$,
2. $\partial_y f(\mathbf{X}_0) \neq 0$,
3. linear independance of vectors $\nabla_{(x,y,\mathbf{z})} \partial^\ell_x f(x_0, y_0, \mathbf{z}_0)$, $0 \le \ell < n$,

then a n^{th}-root singularity emerges.

Let us explain this remark with more details. Denote $\gamma = -f^{(n)}(\mathbf{X}_0)/n! \partial_y f(\mathbf{X}_0) \neq 0$ and $s_\gamma = \operatorname{sign} \gamma$. There exist three neighborhoods V_{x_0}, V_{y_0}, and $V_{\mathbf{z}_0}$ of x_0, y_0 and $\mathbf{z}_0$ respectively, and smooth scalar functions $\eta : V_{\mathbf{z}_0} \to V_{x_0}$, $\alpha : V_{\mathbf{z}_0} \to V_{y_0}$ such that the critical set

$$S_c = \{(x, y, \mathbf{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^m, \ \partial^\ell_{x^\ell} f(x, y, \mathbf{z}) = 0, \forall \ell = 0, \ldots, n-1\} \tag{3.71}$$

can be parametrized under the classical assumption (3). The set S_c has dimension $m + 2 - n$. Therefore the functions η and α do not depend on all components of $\mathbf{z}$. Two cases arise when solving the equation $f(x, y, \mathbf{z}) = 0$ on a small box $\Omega = V_{x_0} \times V_{y_0} \times V_{\mathbf{z}_0}$:

1. if $n = 2\ell$, $\ell \in \mathbb{N}$: let B_γ be subset of $V_{y_0} \times V_{\mathbf{z}_0}$ defined by

$$B_\gamma = \{(y, \mathbf{z}) \in V_{y_0} \times V_{\mathbf{z}_0}, \ \gamma(y - \alpha(\mathbf{z})) \ge 0\}. \tag{3.72}$$

Then there exist a smooth real function ψ such that $\forall \mathbf{z} \in B_\gamma$ there are only two branches in Ω:

$$x = \eta(\mathbf{z}) + \psi(\pm(s_\gamma(y - \alpha(\mathbf{z})))^{\frac{1}{n}}, \mathbf{z}), \tag{3.73}$$

2. Otherwise, if $n = 2\ell + 1$, $\ell \in \mathbb{N}$, then there exist an interval I containing $\alpha(\mathbf{z})$, and a smooth function ψ defined on $I \times V_{\mathbf{z}_0}$ such that,

$$x = \eta(\mathbf{z}) + \psi\big(\big(y - \alpha(\mathbf{z})\big)^{\frac{1}{n}}, \mathbf{z}\big). \tag{3.74}$$

In both cases, ψ is not degenerate at $(0, \mathbf{z}_0)$ with respect to the first variable: $\psi(0, \mathbf{z}_0) = 0$ and $\partial_y \psi(0, \mathbf{z}_0) = 1/\gamma^{\frac{1}{n}} \neq 0$.

The existence of a cube-root or fourth-root singularity is studied in [10]. In the present work, the so-called power-root singularity is defined for any positive integer n.

Definition 3.28 — Power-root singularity. A function F defined in a subset of $\mathbb{R}^m$ is said to have an n^{th}-root singularity at 0 if there exists $n > 0$ such that, by a change of variable if needed, F can be written as

$$F(\mathbf{X}) = f(X_1^{\frac{1}{n}}, X_2, \ldots, X_m) \tag{3.75}$$

where f is a smooth function.

Using this definition, Theorem 3.27 states the conditions to have a power-root singularity. This singularity is associated to the multiplicity of a root of a function whose definition is recalled below.

Definition 3.29 — Multiplicity. Given a smooth real function f, a positive integer n is said to be the multiplicity of a root r of f, denoted by $\text{mult}(f)(r) = n$, if it satisfies

$$f(r) = \dot{f}(r) = \cdots = f^{(n-1)}(r) = 0 \quad \text{and} \quad f^{(n)}(r) \neq 0. \tag{3.76}$$

By convention, we denote $\text{mult}(f)(r) = 0$ when $f(r) \neq 0$. Before stating the main results of this section, recall that $\underline{\mathbf{M}} = \mathbf{diag}(m_j)_{j=1}^{N-1}$ and $\underline{\mathbf{K}} = (k_{ij})_{i,j=1}^{N-1}$ are the matrices obtained from $\mathbf{M}$ and $\mathbf{K}$ by removing their last row and column, with $\mathbf{l}_N = \mathbf{e}_N^\top \mathbf{K}$ as the last row of $\mathbf{K}$ and $\underline{\mathbf{l}}_N$ is the last row of $\mathbf{K}$ removing the last entry k_{NN}. Lemma 3.32 requires the following assumptions on the matrix $\underline{\mathbf{D}} = \underline{\mathbf{M}}^{-1}\underline{\mathbf{K}}$ and the matrix $\mathbf{P}$ of eigenvectors of $\mathbf{M}^{-1}\mathbf{K}$ (see Section 3.1).

Assumption 3.30. Assume that rank $\left(\underline{\mathbf{l}}_N, \underline{\mathbf{l}}_N\underline{\mathbf{D}}, \ldots, \underline{\mathbf{l}}_N\underline{\mathbf{D}}^{N-2}\right) = N - 1$ is maximal.

This assumption states that the vectors $\underline{\mathbf{l}}_N\underline{\mathbf{D}}^i$ are linearly independent and therefore constitute an invertible matrix. Such an assumption is well-known in controllability, for instance, the so-called Kalman's criterion to ensure controllability of a linear system. Another application is refered to Krylov subspace which is used in modern iterative methods for finding one (or a few) eigenvalues of large sparse matrices or solving large systems of linear equations. This assumption is also related to the Frobenius decomposition with the block companion matrix [14].

Assumption 3.31. Assume that all the eigenvalues of $\mathbf{K}$ are distinct, and none of the last components of the eigenvectors $\mathbf{Pe}_k$ shall vanish, i.e. $P_{Nk} \neq 0$, for all $k = 1, \ldots, N$.

This assumption is natural to have N linear grazing orbits [25]. The following lemma shows that for a non-zero solution to a linear differential system to exist, the multiplicity of the last entry of the solution must be bounded and related to the dimension of the solution.

Lemma 3.32. Assume that either Assumption 3.30 or Assumption 3.31 holds. Let $\mathbf{x}(t) \in \mathbb{R}^N$ be a solution to the linear differential system

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}. \tag{3.77}$$

Then, either $\mathrm{mult}(x_N)(0) \leq 2N - 1$ or $\mathbf{x} \equiv \mathbf{0}$.

In other words, for a non-zero solution to (3.77) to exist, $\mathrm{mult}(x_N)(t) < 2N$, for all t. We first prove this Lemma by using Assumption 3.30; the proof under Assumption 3.31 is provided after.

Proof. **With Assumption 3.30:** The statement of Lemma 3.32 is equivalent to saying that if $\mathrm{mult}(x_N)(0) > 2N - 1$, that is

$$x_N(0) = \dot{x}_N(0) = \ldots = x_N^{(2N-1)}(0) = 0, \tag{3.78}$$

then $\mathbf{x} \equiv \mathbf{0}$. From the N^{th} equation of (3.77), $m_N \ddot{x}_N + \mathbf{l}_N \mathbf{x} = 0$. Differentiating this equation $2N - 3$ times and using (3.78) implies that, for each $k = 1, \ldots, 2N - 3$, the following relation holds

$$(A_k): \qquad \underline{\mathbf{l}}_N\, \underline{\mathbf{x}}^{(k)}(0) = 0. \tag{3.79}$$

The other $N - 1$ equations form a reduced system $\underline{\mathbf{M}}\ddot{\underline{\mathbf{x}}} + \underline{\mathbf{K}}\underline{\mathbf{x}} = \underline{\mathbf{C}}$ where $\underline{\mathbf{x}} = (x_j)_{j=1}^{N-1}$ and $\underline{\mathbf{C}} = -x_N \underline{\mathbf{l}}_N^{\top}$. Differentiating this system with respect to t gives $\underline{\mathbf{M}}\underline{\mathbf{x}}^{(3)}(0) + \underline{\mathbf{K}}\dot{\underline{\mathbf{x}}}(0) = \mathbf{0}$, or $\underline{\mathbf{x}}^{(3)}(0) + \underline{\mathbf{D}}\dot{\underline{\mathbf{x}}}(0) = \mathbf{0}$. Similarly, differentiating this system $2N - 5$ times yields

$$(B_k): \qquad \underline{\mathbf{x}}^{(k+2)}(0) + \underline{\mathbf{D}}\underline{\mathbf{x}}^{(k)}(0) = \mathbf{0}, \text{ for each } k = 1, \ldots, 2N - 5. \tag{3.80}$$

Since $x_N(0) = \dot{x}_N(0) = 0$, it is sufficient to prove that $\underline{\mathbf{x}}(0) = \mathbf{0}$ and $\dot{\underline{\mathbf{x}}}(0) = \mathbf{0}$. To show the former, a linear system with $\underline{\mathbf{x}}(0)$ as the unknown is constructed as follows. The relation (A_1) from (3.79) gives the first equation of the linear system: $\underline{\mathbf{l}}_N\, \underline{\mathbf{x}}(0) = 0$. Multiplying (B_1) with $\underline{\mathbf{l}}_N$, and using the relation (A_3) to eliminate the term $\underline{\mathbf{l}}_N \underline{\mathbf{x}}^{(3)}(0)$ results in the second equation: $\underline{\mathbf{l}}_N \underline{\mathbf{D}}\underline{\mathbf{x}}(0) = 0$. The third equation is obtained after the following steps. First, multiply (B_3) with $\underline{\mathbf{l}}_N$, then use the relation (A_5) to get $\underline{\mathbf{l}}_N \underline{\mathbf{D}}\underline{\mathbf{x}}^{(3)}(0) = 0$. This vanishing term appears when (B_1) is multiplied by $\underline{\mathbf{l}}_N \underline{\mathbf{D}}$, which gives rise to the third equation: $\underline{\mathbf{l}}_N \underline{\mathbf{D}}^2 \underline{\mathbf{x}}(0) = 0$. The same recursive process can be used for each $k = 1, \ldots, N-1$: the use of the relations of odd indices $(A_{2k-1}, B_{2k-3}, B_{2k-5}, \ldots, B_1)$ gives the k^{th} equation of the system

$$\underline{\mathbf{l}}_N \underline{\mathbf{D}}^{k-1} \underline{\mathbf{x}}(0) = 0. \tag{3.81}$$

Combining the $N-1$ equations (3.81), $k = 1, \ldots, N-1$, it follows that $\underline{\mathbf{x}}(0)$ satisfies $\underline{\mathbf{l}}_N \underline{\mathbf{x}}(0) = 0$, $\underline{\mathbf{l}}_N \underline{\mathbf{D}}\underline{\mathbf{x}}(0) = 0$, ..., $\underline{\mathbf{l}}_N \underline{\mathbf{D}}^{N-2} \underline{\mathbf{x}}(0) = 0$. By Assumption 3.30, the unique solution of this linear system is $\underline{\mathbf{x}}(0) = \mathbf{0}$. Together with the hypothesis that $x_N(0) = 0$, then $\mathbf{x}(0) = \mathbf{0}$.

Similarly, $\underline{\dot{\mathbf{x}}}(0) = \mathbf{0}$ is shown by constructing another linear system with $\underline{\dot{\mathbf{x}}}(0)$ as the variable. The k^{th} equation $\underline{\mathbf{l}}_N \underline{\mathbf{D}}^{k-1} \underline{\dot{\mathbf{x}}}(0) = 0$, $k = 1, \ldots, N-1$, of the system can be derived by using a similar recursive process now involving the even-indexed relations $(A_{2k}, B_{2k-2}, B_{2k-4}, \ldots, B_2)$, whence, $\underline{\mathbf{l}}_N \underline{\dot{\mathbf{x}}}(0) = 0$, $\underline{\mathbf{l}}_N \underline{\mathbf{D}}\underline{\dot{\mathbf{x}}}(0) = 0$, $\cdots$, $\underline{\mathbf{l}}_N \underline{\mathbf{D}}^{N-2} \underline{\dot{\mathbf{x}}}(0) = 0$. It then follows that $\underline{\dot{\mathbf{x}}}(0) = \mathbf{0}$, and together with the assumption $\dot{x}_N(0) = 0$, eventually gives $\dot{\mathbf{x}}(0) = \mathbf{0}$.

It was shown that $\mathbf{x}(0) = \mathbf{0}$ and $\dot{\mathbf{x}}(0) = \mathbf{0}$. A solution of (3.77) associated to this initial data is then identically zero: $\mathbf{x}(t) \equiv \mathbf{0}$. This ends the proof by using Assumption 3.30.

With Assumption 3.31: Consider the N^{th} component of $\mathbf{x}(t)$:

$$\begin{aligned} x_N(t) = \mathbf{e}_N^\top \mathbf{x}(t) &= \mathbf{e}_N^\top \left(\mathbf{P} \cos(t\mathbf{\Omega}) \mathbf{P}^{-1} \mathbf{x}(0) + \mathbf{P}\mathbf{\Omega}^{-1} \sin(t\mathbf{\Omega}) \mathbf{P}^{-1} \dot{\mathbf{x}}(0) \right), \\ &= \sum_{k=1}^{N} (\alpha_k \cos(\omega_k t) + \beta_k \sin(\omega_k t))\, \mathbf{e}_N^\top \mathbf{P}\mathbf{e}_k = \sum_{k=1}^{N} (\alpha_k \cos(\omega_k t) + \beta_k \sin(\omega_k t))\, v_k, \end{aligned} \tag{3.82}$$

where α_k and β_k are coefficients depending on the initial data $[\mathbf{x}(0), \dot{\mathbf{x}}(0)]$, $v_k = \mathbf{e}_\mathbf{N}^\top \mathbf{P}\mathbf{e}_k$ are the components of the last row of the matrix $\mathbf{P}$.

Assume that $\text{mult}(x_N)(0) > 2N-1$, using (3.78) for x_N and $N-1$ first even-order derivatives of x_N yields $\sum_{k=1}^N \alpha_k v_k = 0$, $\sum_{k=1}^N \alpha_k \omega_k^2 v_k = 0$, ..., $\sum_{k=1}^N \alpha_k \omega_k^{2N-2} v_k = 0$. Denote $\lambda_k = \omega_k^2$, this can be rewritten as a linear system where α_k is the unknown:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_N \\ & & \ddots & \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \cdots & \lambda_N^{N-1} \end{bmatrix} \begin{bmatrix} v_1 & 0 & \cdots & 0 \\ 0 & v_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & v_N \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.83}$$

By Assumption 3.31, it follows that λ_k, $k = 1, \ldots, N$ are distinct, and $v_k = P_{Nk} \neq 0$ for all $k = 1, \ldots, N$, then there is a unique solution $\alpha_k = 0$, for all $k = 1, \ldots, N$.

Similarly, using (3.78) for N first odd-order derivatives of x_N yields Similarly, using (3.78) for N first odd-order derivatives of x_N yields

$$\sum_{k=1}^{N} \beta_k \omega_k v_k = 0, \quad \sum_{k=1}^{N} \beta_k \omega_k^3 v_k = 0, \quad \ldots, \quad \sum_{k=1}^{N} \beta_k \omega_k^{2N-1} v_k = 0. \tag{3.84}$$

This can be rewritten as a linear system where the β_k are the unknowns:

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_N \\ & & \ddots & \\ \lambda_1^{N-1} & \lambda_2^{N-1} & \dots & \lambda_N^{N-1} \end{bmatrix} \begin{bmatrix} \omega_1 v_1 & 0 & \dots & 0 \\ 0 & \omega_2 v_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \omega_N v_N \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.85}$$

With the same argument, the unique solution of this sytem is $\beta_k = 0$, for all $k = 1, \ldots, N$. As a consequence, all the components of $\mathbf{x}(t)$ are zero, hence, $\mathbf{x}(t) \equiv \mathbf{0}$ for all t. Lemma 3.32 is then proven. □

Remark 3.33. The main idea of the proof using Assumption 3.31 is to show that the set of functions $\{\cos(\omega_k t), \sin(\omega_k t)\}_{k=1,\ldots,N}$ is linearly independent. This is a very classical proof in linear algebra.

This result can be proven to be true for all component x_k of $\mathbf{x}$, $k = 1, \ldots, N$ with suitable Assumption 3.30 or 3.31. Let us apply this general result to the solution to (3.1) in order to show that at the contact time, u_N has at most $2N - 1$ derivatives which vanish. Between the successive closing contacts, the system is linear and hence the solution $\mathbf{u}(t)$ is analytic. Therefore, for the sake of simplicity, assume that all the derivatives are taken on the left of 0, $u_N^{(n)-}$ is denoted for the left n^{th}-derivative of u_N at 0 with respect to t.

Proposition 3.34. Assume that $\mathbf{u}(t)$ is a solution to (3.1) which has a closing contact at $t = 0$, i.e. $u_N(0) = d$. Then, under Assumption 3.30 or 3.31,

$$0 \leq \text{mult}(\dot{u}_N^-)(0) \leq 2N - 1. \tag{3.86}$$

Moreover, if there is a sticking phase after $t = 0$ of duration τ and with one similar assumption as Assumption 3.30 or 3.31 for the $(N-1) \times (N-1)$ sticking system rewritten in a suitable basis where $\underline{\mathbf{l}}_N \underline{\dot{\mathbf{u}}}$ corresponds to the last entry of the solution, then

$$0 \leq \text{mult}(\underline{\mathbf{l}}_N \underline{\dot{\mathbf{u}}})(\tau) \leq 2N - 3. \tag{3.87}$$

Proof. The first part of the proposition is obtained by applying Lemma 3.32 to $\mathbf{x} = \dot{\mathbf{u}}^-$. Notice that the system of motion is linear outside the successive closing contacts, i.e.

$$\mathbf{M}\ddot{\mathbf{u}}^- + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{3.88}$$

Hence, $\dot{\mathbf{u}}$ is the solution to $\mathbf{M}\ddot{\mathbf{u}}^- + \mathbf{K}\dot{\mathbf{u}} = \mathbf{0}$. By Lemma 3.32, it follows that $\text{mult}(\dot{u}_N^-)(0) \leq 2N-1$ or $\dot{\mathbf{u}}(t) \equiv \mathbf{0}$. However, if $\dot{\mathbf{u}}(t) \equiv \mathbf{0}$, then substituting this into the linear system, it follows that $\mathbf{u}(t) \equiv \mathbf{0}$ which is in contradiction with the hypothesis that $u_N(0) = d > 0$.

If there is a sticking phase at $t = 0$, then $\dot{u}_N^-(0) = 0$ and $\dot{u}_N^-(\tau) = 0$ where τ is the end of sticking phase. During the sticking phase from $t = 0$ to $t = \tau$, the N^{th} mass lies on the obstacle,

$$m_N \ddot{u}_N^-(t) + \mathbf{l}_N \mathbf{u}(t) = R(t), \tag{3.89}$$

where $R(t) \leq 0$, and $F(t) = \mathbf{l}_N \mathbf{u}(t)$. The duration of the sticking phase τ is also the end of the sticking phase, it is implicitly presented by

$$F(\tau) = \mathbf{l}_N \mathbf{u}(\tau) = \underline{\mathbf{l}}_N \underline{\mathbf{u}}(\tau) + k_{NN} d = 0, \tag{3.90}$$

$$F(\tau + \delta) > 0, \quad \forall \delta \in]0\,;\delta_0[, \quad \delta_0 > 0, \tag{3.91}$$

where $\underline{\mathbf{u}}$ is the solution of the sticking system $\underline{\mathbf{M}}\ddot{\underline{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = \underline{\mathbf{C}}$, with $\underline{\mathbf{C}} = -d\,\underline{\mathbf{l}}_N^\top$. Differentiating this system with respect to t yields $\underline{\mathbf{M}}\dddot{\underline{\mathbf{u}}} + \underline{\mathbf{K}}\dot{\underline{\mathbf{u}}} = \mathbf{0}$. A change of variables $\dot{\underline{\mathbf{u}}} = \mathbf{Q}\underline{\mathbf{v}}$ using an $\underline{\mathbf{M}}$-orthogonal matrix $\mathbf{Q}$, i.e. $\mathbf{Q}^\top \underline{\mathbf{M}} \mathbf{Q} = \mathbf{I}$, such that $\underline{v}_{N-1} = c\,\underline{\mathbf{l}}_N \dot{\underline{\mathbf{u}}}$, with $c \neq 0$ yields

$$\ddot{\underline{\mathbf{v}}} + \mathbf{Q}^\top \underline{\mathbf{K}} \mathbf{Q} \underline{\mathbf{v}} = \mathbf{0}. \tag{3.92}$$

Invoking Theorem 3.32 for this $(N-1) \times (N-1)$ reduced system, it follows that $\text{mult}(\underline{\mathbf{l}}_N \dot{\underline{\mathbf{u}}})(\tau) \leq 2N-3$. □

In the following proposition, the singularity of the duration of the free-flight and of the duration of sticking phase is shown. These durations have a relationship with the first return time. More precisely, if $\mathsf{W} \in \mathcal{H}^- \cup \mathcal{H}_G^0$, then the first return time of an orbit generating from W coincides with the duration of the free-flight $s(\mathsf{W})$. Otherwise, if $\mathsf{W} \in \mathcal{H}_S^0$, then the first return time involves the duration of sticking phase and the duration of the free-flight: $T(\mathsf{W}) = \tau(\mathsf{W}) + s(\mathsf{W})$. The bounded multiplicity shown in Proposition 3.34 is the main ingredient to prove the singularity.

Proposition 3.35 — Power-root singularity. Suppose that $\mathsf{W}_0 \in \mathcal{H}_{\mathcal{P}}$ generates an orbit $[\mathbf{u}(t), \dot{\mathbf{u}}(t)]$ of (3.1) which has a closing contact at $t = T_0$. Assumption 3.30 or 3.31 and a similar assumption for the sticking system hold.

1. If $\mathsf{W}_0 \in \mathcal{H}^-$ and $\mathsf{W}(T_0) \in \mathcal{H}^0$, then the duration of the free-flight of the orbit generated from a nearby W until the next closing contact, denoted by $s(\mathsf{W})$, has an n^{th}-root singularity at W_0 with

$$n = 1 + \text{mult}(\dot{u}_N^-)(T_0), \quad 1 \leq n \leq 2N. \tag{3.93}$$

2. If $\mathsf{W}_0 \in \mathring{\mathcal{H}}_S^0$, the interior of the set $\mathcal{H}_S^0$ in the topology of $\mathcal{H}^0$, i.e. there is a sticking phase

right after $t = 0$ with the duration $\tau_0 = \tau(\mathsf{W}_0)$, and then for all $\mathsf{W} \in \mathcal{H}^0_S$, W near W_0, the duration of the sticking phase $\tau = \tau(\mathsf{W})$ has an m^{th}-root singularity at W_0 with

$$m = 1 + \text{mult}(\underline{l}_N \underline{\dot{\mathbf{u}}})(\tau(\mathsf{W}_0)), \quad 1 \leq m \leq 2N - 2. \tag{3.94}$$

Remark 3.36. The first return time can involve two power-root singularities: one from the duration of the sticking phase, and one from the duration of the free-flight phase.

Remark 3.37. When $\mathsf{W}_0 \in \mathcal{H}^0_{\mathcal{G}} \cap \bar{\mathcal{H}}^0_S$, the behaviour of the first return time is more complicated. It depends whether W belongs to $\mathcal{H}^0_S$ or not. In the former, there are two power-root singularities while in the latter, there is a single power-root singularity.

Proof. The idea of the proof is to use the implicit function theorem for the duration of the free-flight or the duration of the sticking phase.

1. Consider the duration T_0 of the free-flight between the two closing contacts at $t = 0$ and $t = T_0$. For (s, W) in the neighborhood of (T_0, W_0), consider the function $f(s, \mathsf{W}) = \mathbf{e}_N^\top \mathbf{R}(s)\mathsf{SW}$. It is smooth and defined for all arguments and corresponds to the nonsmooth function (due to the closing contacts) $u_N(s, \mathsf{W})$. A main point is to apply the implicit function theorem to f and then to interpret the result for the nonsmooth function u_N. By Proposition 3.34, it follows that $0 \leq \text{mult}(\dot{u}_N^-)(T_0) \leq 2N - 1$. Using Remark 3.27 for the function f in the neighborhood of (T_0, W_0), it follows that there exists a subset containing W_0 such that s has an n^{th}-root singularity at W_0 with $n = 1 + \text{mult}(\dot{u}_N^-)(T_0)$, and thus, $1 \leq n \leq 2N$.
2. Recall that $\mathbf{U}(t, \mathsf{W}) = [\mathbf{u}(t, \mathsf{W}), \dot{\mathbf{u}}(t, \mathsf{W})]$ is the solution to (3.1) associated to the initial data W. During the sticking phase, $u_N(t, \mathsf{W}) = d$ and $\dot{u}_N(t, \mathsf{W}) = 0$ for all $t \in [0\,; \tau]$. By denoting $F(t, \mathsf{W}) = \mathbf{l}_N \mathbf{u}(t, \mathsf{W}) = \underline{\mathbf{l}}_N \underline{\mathbf{u}}(t, \mathsf{W}) + k_{NN} d$, the last equation of (3.1) yields $m_N \ddot{u}_N(t, \mathsf{W}) = -F(t, \mathsf{W}) + R(t, \mathsf{W})$, where the reaction from the obstacle is $R(t, \mathsf{W}) \leq 0, \forall t \in [0\,; \tau]$. Hence, the sticking phase starts at $t = 0$ when $F(0, \mathsf{W}) = 0$ and $F(\delta, \mathsf{W}) < 0$, $\forall \delta \in\,]0\,; \delta_0[$, $\delta_0 > 0$. The second condition ensures that the total force acting on the N^{th} mass is strictly positive right after $t = 0$ and therefore the mass stays on the obstacle. The sticking phase holds as long as $F(t, \mathsf{W}) \leq 0$, it ends at $t = \tau$ when $F(\tau, \mathsf{W}) = 0$ and $F(\tau + \delta, \mathsf{W}) > 0$, $\forall \delta \in\,]0\,; \delta_0[$, $\delta_0 > 0$. The latter condition makes sure that the total force acting on the N^{th} mass becomes strictly negative right after $t = \tau$ and therefore the mass leaves the obstacle. Similarly, denote by $F^*(t) = \underline{\mathbf{l}}_N \underline{\mathbf{u}}(t) + k_{NN} d$, where $\underline{\mathbf{u}}$ is the solution of the smooth system $\underline{\mathbf{M}}\underline{\ddot{\mathbf{u}}} + \underline{\mathbf{K}}\underline{\mathbf{u}} = \underline{\mathbf{C}}$, with $\underline{\mathbf{C}} = -d\underline{\mathbf{l}}_N^\top$. By using Remark 3.27 for the smooth function F^* in the neighborhood of (τ_0, W_0), it follows that there is a subset of $\mathcal{H}^0_S$ containing W_0 such that τ has an m^{th}-root singularity at W_0 with $m = 1 + \text{mult}(\dot{F})(\tau(\mathsf{W}_0))$. Proposition 3.34 implies that $0 \leq \text{mult}(\underline{\mathbf{l}}_N \underline{\dot{\mathbf{u}}})(\tau(\mathsf{W}_0)) \leq 2N - 3$, therefore $1 \leq m \leq 2N - 2$. □

Remark 3.38. The lower bound of this multiplicity is optimal for a chain, i.e. $1/2N$ is the greatest lower bound of the power-root singularity of the duration s. Let us verify this by showing that, if $u_N(T_0) = d$, $\dot{u}_N(T_0) = \ldots = u_N^{(2N-1)}(T_0) = 0$ then there is a unique data of (3.1) corresponding to the solution with the maximal $2N$-root singularity of s.

Notice that in this case, the symmetric matrix $\mathbf{K}$ has the form

$$\begin{bmatrix} k_{11} & k_{21} & 0 & \ldots & 0 & 0 \\ k_{21} & k_{22} & k_{32} & \ldots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \ldots & k_{N-1,N-1} & k_{N,N-1} \\ 0 & 0 & 0 & \ldots & k_{N,N-1} & k_{NN} \end{bmatrix} \tag{3.95}$$

with $k_{i+1,i} \neq 0$ for all $i = 1, \ldots, N-1$. Outside the closing contacts, $\mathbf{u}$ is the solution of the linear system

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{3.96}$$

From the last equation of (3.96): $m_N\ddot{u}_N + k_{N,N-1}u_{N-1} + k_{NN}u_N = 0$, together with $k_{N,N-1} \neq 0$, it follows that

$$u_{N-1}(0) = C_{N-1}d \text{ and } \dot{u}_{N-1}(0) = \ldots = u_{N-1}^{(2N-3)}(0) = 0, \quad C_{N-1} = -k_{NN}/k_{N,N-1}. \tag{3.97}$$

Similarly, the recursive process can be used until the $(k+1)^{\text{th}}$ equation, $k = 2, \ldots, N-1$, which together with the assumption $k_{k,k-1} \neq 0$ give

$$u_k(0) = C_k d \text{ and } \dot{u}_k(0) = \ldots = u_k^{2k-1}(0) = 0, \tag{3.98}$$

where $C_k = -(k_{kk}C_k + k_{k+1,k}C_{k+1})/k_{k,k-1}$. As a consequence, an initial data is obtained:

$$\mathbf{u}(0) = [C_1 d, \ldots, C_{N-1}d, d], \; \dot{\mathbf{u}}(0) = \mathbf{0}. \tag{3.99}$$

Hence, there exists a unique solution to (3.1) associated to this initial data.

However, it is not sure that $1/(2N-2)$ is the greatest lower bound of the singularity of τ since it depends on the admissibility condition of the end of the sticking phase which is $\underline{\mathbf{l}}_N \underline{\mathbf{u}}^{(2N-2)}(\tau) > 0$.

Remark 3.39 — Power root singularity for the 2-dof chain. Let us clarify the power-root singularity when $N = 2$. In this case, the duration of the free-flight may have a square-root, cube-root, or at most fourth-root singularity, while the duration of the sticking phase is analytic.

An illustration of these power-root singularities is given in Figure 3.3.

Let $W_0 = [u_1(0), d, \dot{u}_1(0), \dot{u}_2^-(0)] \in \mathcal{H}_{\mathcal{P}}$ be the state of the system at $t = 0$. The interesting criteria happens when the first contact ($u_2(T_0) = d$) is with zero velocity, $\dot{u}_2^-(T_0) = 0$. That means $W(T_0) \in \mathcal{H}^0$.

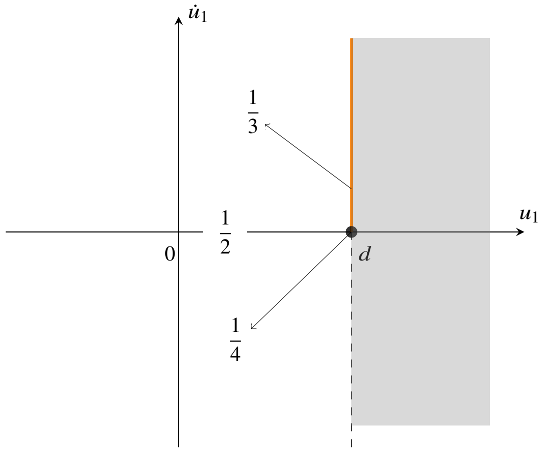


FIGURE 3.3 – Power-root singularity in the plane $(u_1, \dot{u}_1)$ which is isomorphic to the set $\mathcal{H}^0$ since $u_2 = d$ and $\dot{u}_2 = 0$. The left part, $u_1 < d$ or $u_1 = d, \dot{u}_1 = 0$, corresponds to the grazing contact. The orange branch $u_1 = d$ and $\dot{u}_1 > 0$ corresponds to the beginning of the sticking phase. The other parts do not coincide with closing contacts. The gray part $u_1 > d$ corresponds to a state during the sticking phase. The dashed line $u_1 = d$ and $\dot{u}_1 < 0$ corresponds to the end of a sticking phase.

If $W_0 \in \mathcal{H}^- \cup \mathcal{H}^0_{\mathcal{G}}$, i.e there is no sticking phase after $t = 0$, three possibilities are considered:

1. $u_1(T_0) < d$: This gives $m_2\ddot{u}_2^-(T_0) = k_2(u_1(T_0) - u_2(T_0)) < 0$. The duration of the free-flight $s = s(W)$ has a square-root singularity at W_0.
2. $u_1(T_0) = d$ and $\dot{u}_1(T_0) > 0$: this is the beginning of a sticking phase, $\ddot{u}_2(T_0) = 0$ and $m_2\dddot{u}_2^-(T_0) = k_2(\dot{u}_1(T_0) - \dot{u}_2(T_0)) > 0$. Hence, $s = s(W)$ has a cube-root singularity at W_0.
3. $u_1(T_0) = d$ and $\dot{u}_1(T_0) = 0$: this gives $\ddot{u}_2^-(T_0) = \dddot{u}_2^-(T_0) = 0$. However, $m_2 u_2^{(4)-}(T_0) = k_2(\ddot{u}_1(T_0) - \ddot{u}_2^-(T_0)) = k_1 k_2 d > 0$. This case corresponds to a grazing contact and the fourth-root singularity of $s(W)$ arises.

If $W_0 \in \mathcal{H}^0_{\mathcal{S}}$, i.e. there is a sticking phase of duration τ_0 at $t = 0$ and there are two possibilities: $u_1(0) > d$ or ($u_1(0) = d$ and $\dot{u}_1(0) > 0$). During a sticking phase, u_1 is a solution of $m_1\ddot{u}_1 + (k_1 + k_2)u_1 = k_2 d$. The end of the sticking phase is at $t = \tau_0$, when $u_1(\tau_0) = d$ and $\dot{u}_1(\tau_0) < 0$. This shows that $\mathrm{mult}(\dot{u}_1)(\tau_0) \leq 2N - 3 = 1$. This is a particular case where the duration of the sticking phase $\tau(W)$ is analytic for $W \in \mathcal{H}^0_{\mathcal{S}}$.

An example of a cubic-root singularity is depicted in Figure 3.4.

3.4.3 Discontinuous first return time

A consequence of Theorem 3.8 is the discontinuity of the return time near many grazing orbits. The simplest case is stated in the next corollary. This result may gives many consequences: possible

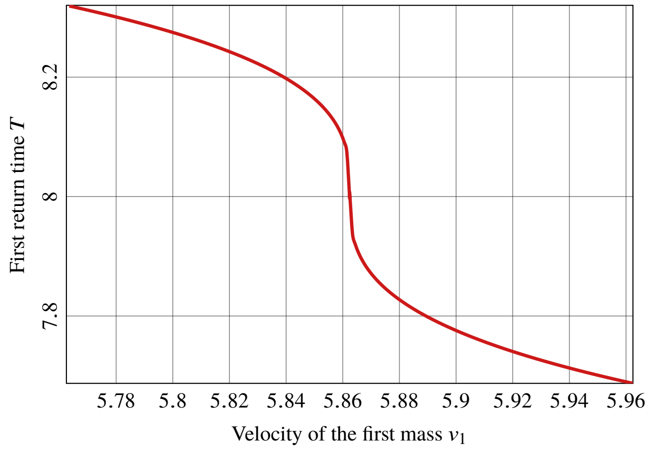


FIGURE 3.4 – First return time $T(v_1)$ with respect to $\dot{u}_1$ (near a periodic solution with one sticking phase per period [28]). A cubic-root singularity appears near $\dot{u}_1(0) = 5.86$.

discontinuous first return map, more important singularity than the square-root singularity, immediate instability, complex patterns... These consequences are postponed to forthcoming works.

Corollary 3.40 — Discontinuous first return time. If $\mathsf{W}_0 \in \mathcal{H}^-$ and $\mathsf{W}(T_0) \in \mathcal{H}^0$ where $T_0 = T(\mathsf{W}_0) > 0$ and $\mathsf{W}(T_0)$ satisfies Assumption 3.7 which says that the grazing contact is not degenerate, then the first return time $T(\mathsf{W})$ is discontinuous at W_0.

Proof. Theorem 3.8 states that $T(\mathsf{W})$ is near $T_0 = T(\mathsf{W}_0)$ only on at most a half neighborhood of W_0: $\mathcal{B}_k = \{\mathsf{W} \in V_{\mathsf{W}_0},\ s_k(W_k - \alpha(\underline{\mathsf{W}})) \geq 0\}$. Thus, $T(\mathsf{W})$ is not near T_0 for $s_k(W_k - \alpha(\underline{\mathsf{W}})) < 0$. □

3.5 The square-root instability

In this section, the so-called square-root instability is introduced for a fixed-point of the map. Loosely speaking, the appearance of the square-root singularity may affect the dynamics and the fixed-point may become unstable.

Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a map defined by $\mathbf{F}(\mathbf{X}) = \mathbf{G}(\sqrt{|x_1|}, x_2, \ldots, x_n)$ where $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ is at least a C^2 function. $\mathbf{F}$ has a fixed-point $\mathbf{0} \in \mathbb{R}^n$. Consider the dynamical system obtained by iterating $\mathbf{F}$:

$$\mathbf{X}^{m+1} = \mathbf{F}(\mathbf{X}^m), \quad \mathbf{X}^m = (x_1, x_2, \ldots, x_n)^m \in \mathbb{R}^n, \quad m = 1, 2, \ldots \tag{3.100}$$

With the presence of the square-root term $\sqrt{x_1}$, a question on the stability of the fixed-point $\mathbf{0}$ then arises. This section provides a generic condition for which the fixed-point $\mathbf{0}$ of the map $\mathbf{F}$ is unstable. Besides, counterexamples show the dynamical complexity of the map in the vicinity of its fixed-point even in small dimensions.

3.5.1 A nonlinear n dimensional map

In the following theorem, it is shown under a specific condition that the square-root term acting on the component x_1 of $\mathbf{X}$ yields the instability of the fixed-point $\mathbf{0}$. Moreover, this instability occurs along the direction of x_1.

Theorem 3.41 — Unstable fixed-point. Suppose $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$ are functions such that $\mathbf{F}(\mathbf{X}) = \mathbf{G}(\underline{\mathbf{X}})$, where $\mathbf{X} = (x_1, \dots, x_n)$ and $\underline{\mathbf{X}} = (\sqrt{|x_1|}, x_2, \dots, x_n)$. If the function $\mathbf{G}$ belongs to C^2, $\mathbf{G}(\mathbf{0}) = \mathbf{0}$, and the Jacobian $\mathbf{DG}(\mathbf{0}) = (a_{ij})_{i,j=1}^n$ satisfies

$$a_{11} \neq 0 \tag{3.101}$$

then $\mathbf{0}$ is an unstable fixed-point of $\mathbf{F}$.

Remark 3.42. More generally, if the square-root is activated on the component x_k, then the instability of the fixed-point $\mathbf{0}$ can be obtained by evaluating the value of $a_{kk} = \partial_{x_k} g_k(\mathbf{0})$.

Proof. Denote $\mathbf{G}(\mathbf{X})$ by $(g_1, g_2, \dots, g_n)(\mathbf{X}) \in \mathbb{R}^n$. Our goal is to show that there is a neighborhood of $\mathbf{0}$ such that many points arbitrarily close to $\mathbf{0}$ will go out of that neighborhood.

The Taylor expansion with an integral remainder of each g_i near $\mathbf{0}$, $i = 1, \dots, n$ is

$$\begin{aligned} g_i(\underline{\mathbf{X}}) = a_{i1}\sqrt{|x_1|} + \sum_{j=2}^{n} a_{ij}x_j + |x_1| r_{i,11}(\underline{\mathbf{X}}) + \\ + 2\sum_{j=2}^{n} \sqrt{|x_1|} x_j\, r_{i,1j}(\underline{\mathbf{X}}) + 2\sum_{n\geq k>l>1} x_k x_l\, r_{i,kl}(\underline{\mathbf{X}}) + \sum_{j=2}^{n} x_j^2\, r_{i,jj}(\underline{\mathbf{X}}), \end{aligned} \tag{3.102}$$

where $r_{i,k\ell}(\underline{\mathbf{X}}) = \int_0^1 (1-s)\partial^2_{k\ell} g_i(s\underline{\mathbf{X}})ds$, for all $k, \ell = 1, \dots, n$. On a compact set which will be chosen later, there exists a constant $M > 0$ such that $|r_{i,k\ell}(\mathbf{X})| \leq M$ for all $k, \ell = 1, \dots, n$ and for each $i = 1, \dots, n$. Hence,

$$|a_{i1}|\sqrt{|x_1|} - \sum_{j=2}^{n} |a_{ij}||x_j| - M\,\theta(\mathbf{X}) \leq |g_i(\underline{\mathbf{X}})| \leq |a_{i1}|\sqrt{|x_1|} + \sum_{j=2}^{n} |a_{ij}||x_j| + M\,\theta(\mathbf{X}), \tag{3.103}$$

where $\theta(\mathbf{X}) = |x_1| + 2\sqrt{|x_1|}\sum_{j=2}^{n}|x_j| + 2\sum_{n\geq k>\ell>1}|x_k||x_\ell| + \sum_{j=2}^{n} x_j^2$. A suitable neighborhood, denoted by D_ϵ, is constructed so that a sequence starting from any point in D_ϵ will eventually go away from $\mathbf{0}$ in the direction of x_1. To define D_ϵ, the following notations are needed. Let α_i, $i = 2, \dots, n$, be:

$$\alpha_i = \begin{cases} \dfrac{|a_{11}|}{2n\,|a_{1i}|} & \text{if } a_{1i} \neq 0, \quad (3.104a) \\ 1 & \text{if } a_{1i} = 0. \quad (3.104b) \end{cases}$$

Through this definition, it follows that $\alpha_j|a_{1j}| \leq |a_{11}|/2n$, the equality occurs if $a_{1j} \neq 0$, otherwise it is a strict inequality. For $\mathbf{X} \in \mathbb{R}^n$ such that $|x_j| \leq \alpha_j\sqrt{|x_1|}$, $\forall j = 2, \ldots, n$, we have $|a_{1j}||x_j| \leq |a_{1j}|\alpha_j\sqrt{|x_1|}$, and hence,

$$\sum_{j=2}^{n} |a_{1j}||x_j| \leq \sum_{j=2}^{n} |a_{1j}|\alpha_j\sqrt{|x_1|} \leq (n-1)\frac{|a_{11}|}{2n}\sqrt{|x_1|}. \tag{3.105}$$

Moreover, the following inequality holds

$$\theta(\mathbf{X}) \leq |x_1|\left(1 + 2\sum_{j=2}^{n} \alpha_j + 2\sum_{n\geq k>l>1} \alpha_k\alpha_l + \sum_{j=2}^{n} \alpha_j^2\right). \tag{3.106}$$

Denote $\alpha = 1+2\sum_{j=2}^{n}\alpha_j+2\sum_{n\geq k>l>1}\alpha_k\alpha_l+\sum_{j=2}^{n}\alpha_j^2$, the above inequality becomes $\theta(\mathbf{X}) \leq \alpha|x_1|$. Denote also $\gamma_0 = a_{11}^2/(2n\alpha M)^2$. Then, for $\mathbf{X} \in \mathbb{R}^n$ such that $|x_1| \leq \gamma_0$, we have

$$M\alpha|x_1| \leq \frac{|a_{11}|}{2n}\sqrt{|x_1|}. \tag{3.107}$$

Since $M\theta(\mathbf{X}) \leq M\alpha|x_1|$, it follows that

$$M\theta(\mathbf{X}) \leq \frac{|a_{11}|}{2n}\sqrt{|x_1|}. \tag{3.108}$$

Let $C > 0$ be a constant such that

$$C \geq \frac{2}{|a_{11}|}\left(|a_{i1}| + \sum_{j=2}^{n} |a_{ij}|\alpha_j + M\alpha\right), \ \forall i = 2, \ldots, n. \tag{3.109}$$

Let $\gamma > 0$ such that $\gamma = \min_{i=2,\ldots,n}\{\gamma_i > 0 \,|\, C\gamma_i \leq \alpha_i\sqrt{\gamma_i}\}$. Therefore, for any $\mathbf{X} \in \mathbb{R}^n$ such that $|x_1| \leq \gamma$, we have $C|x_1| \leq \alpha_i\sqrt{|x_1|}$, for all $i = 2, \ldots, n$. We now can define D_ϵ as follows. Let $\epsilon = \min\{\gamma_0, \gamma, a_{11}^2/8\} > 0$. Consider

$$D = \{\mathbf{X} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ |x_i| \leq C|x_1|, \ \forall i = 2, \ldots, n\}, \tag{3.110}$$

$$D_\epsilon = \{\mathbf{X} = (x_1, \ldots, x_n) \in \mathbb{R}^n, \ |x_1| \leq \epsilon \text{ and } |x_i| \leq C|x_1|, \ \forall i = 2, \ldots, n\}. \tag{3.111}$$

This choice of D_ϵ avoids the criteria in which the instability of $\mathbf{0}$ is hidden by starting at a point near $\mathbf{0}$ but the sequence comes back at $\mathbf{0}$ after one step. See Example 3.44 for more information. An illustration of D_ϵ is provided in Figure 3.5. Any $\mathbf{X} \in D_\epsilon$ satisfies the important inequalities as stated next.

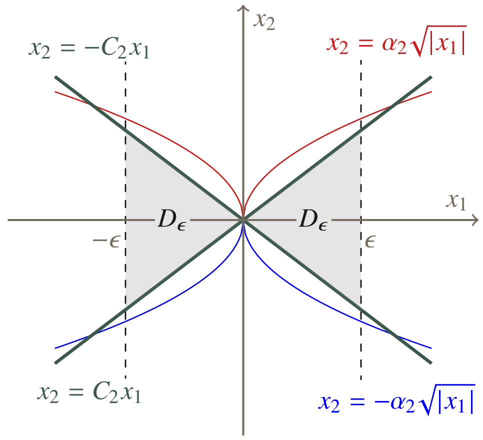


FIGURE 3.5 – Neighborhood D_ϵ

Lemma 3.43. For any $\mathbf{X} \in D_\epsilon$, the following inequalities hold:

$$\frac{|a_{11}|}{2}\sqrt{|x_1|} \le |g_1(\underline{\mathbf{X}})| \le \frac{3|a_{11}|}{2}\sqrt{|x_1|}, \tag{3.112}$$

$$|g_i(\underline{\mathbf{X}})| \le C_i|g_1(\underline{\mathbf{X}})|, \ \forall i = 2,\dots,n. \tag{3.113}$$

Proof. Suppose that $\mathbf{X} \in D_\epsilon$ then $|x_j| \le C|x_1|$ for all $j = 2,\dots,n$, and $|x_1| \le \epsilon \le \gamma$. By the definition of γ, it follows that $C|x_1| \le \alpha_j\sqrt{|x_1|}$. Hence, $|x_j| \le C|x_1| \le \alpha_j\sqrt{|x_1|}$. Thus (3.105) holds. Substituting (3.105) and (3.108) into (3.103) when $i = 1$, the right hand side (RHS) of (3.103) becomes

$$\text{RHS}(3.103) \le |a_{11}|\sqrt{|x_1|} + (n-1)\frac{|a_{11}|}{2n}\sqrt{|x_1|} + \frac{|a_{11}|}{2n}\sqrt{|x_1|} \le \frac{3|a_{11}|}{2}\sqrt{|x_1|}. \tag{3.114}$$

Similarly,

$$\text{LHS}(3.103) \ge |a_{11}|\sqrt{|x_1|} - (n-1)\frac{|a_{11}|}{2n}\sqrt{|x_1|} - \frac{|a_{11}|}{2n}\sqrt{|x_1|} \ge \frac{|a_{11}|}{2}\sqrt{|x_1|}. \tag{3.115}$$

Therefore, $\mathbf{X}$ satisfies (3.112).

Let us prove inequality (3.113). For $i = 2,\dots,n$, expressions (3.103) and (3.108) imply

$$|g_i(\underline{\mathbf{X}})| \le \left(|a_{i1}| + \sum_{j=2}^{n}|a_{ij}|\alpha_j + M\alpha\right)\sqrt{|x_1|}. \tag{3.116}$$

Via inequality (3.112), it is shown that $|a_{11}|\sqrt{|x_1|} \le 2|g_1(\tilde{\mathbf{X}})|$. Hence, to prove (3.113), it is sufficient to show that

$$\frac{2}{|a_{11}|}\left(|a_{i1}| + \sum_{j=2}^{n}|a_{ij}|\alpha_j + M\alpha\right) \le C. \tag{3.117}$$

This is true by the choice of C given in (3.109). Inequality (3.113) is then proven. □

Back to the proof of Theorem 3.41, the idea is to show that the recurrence goes away from $\mathbf{0}$ in the direction of the first component. In other words, the square-root singularity acting on the first component plays an important role via the inequalities (3.112) and (3.113).

Let us show that, for any $0 < \delta < \epsilon$, if the sequence $(\mathbf{X}^m)_{m\geq 1}$ in D_ϵ, where $\mathbf{X}^m = (x_1^m, \dots, x_n^m)$, is defined by

$$\mathbf{X}^0 = (\delta/2, 0, \dots, 0) \in D_\epsilon, \quad \mathbf{X}^{m+1} = \mathbf{F}(\mathbf{X}^m), \quad m \geq 1, \tag{3.118}$$

then there exists a $N_0 > 0$ such that $\mathbf{X}^{N_0} \notin D_\epsilon$. From (3.113), $\mathbf{X}^1 = \mathbf{F}(\mathbf{X}^0) = \mathbf{G}(\underline{\mathbf{X}}^0) \in D$. If $|x_1^1| > \epsilon$ then $\mathbf{X}^1 \notin D_\epsilon$. This shows the instability of $\mathbf{0}$. Otherwise, if $|x_1^1| \leq \epsilon$ and $|x_i^1| = |g_i(\mathbf{X}^0)| \leq C|x_1^1|$ by inequality (3.113), then $\mathbf{X}^1 \in D_\epsilon$, and $\mathbf{X}^2$ is considered. If there exists $\mathbf{X}^m \in D_\epsilon$ for all $m \geq 1$ then

$$\frac{a_{11}}{2}\sqrt{|x_1^m|} \leq |x_1^{m+1}| \leq \frac{3a_{11}}{2}\sqrt{|x_1^m|}. \tag{3.119}$$

Consider the sequence z_m defined by $z_{m+1} = a_{11}/2\sqrt{|z_m|}$, $z_0 = x_1^0 = \delta/2 \leq \epsilon \leq a_{11}^2/8$. Then, inequality (3.119) yields $|x_1^m| \geq z_m$. It is known that z_m increasingly converges to $a_{11}^2/4$ in an interval $(0, a_{11}^2/4)$. Hence, there exists $N_0 > 0$ such that $|z_{N_0}| > a_{11}^2/8 \geq \epsilon$, and thus $|x_1^{N_0}| \geq |z_{N_0}| > \epsilon$. That means, $\mathbf{X}^{N_0} \notin D_\epsilon$ and hence $\mathbf{0}$ is unstable. □

Example 3.44. Consider $\mathbf{F}(x, y) = \mathbf{G}(\sqrt{|x|}, y)$ when $\mathbf{G}$ is a linear map

$$\mathbf{G}(\mathbf{X}) = \mathbf{G}(x, y) = \mathbf{A}\mathbf{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \; a \neq 0. \tag{3.120}$$

Figure 3.6(a) shows that the fixed-point $(0, 0)$ of $\mathbf{F}$ is unstable and the recurrence goes away from this point along the line $y = cx/a$. An interesting case is when $c = \alpha a$, $d = \alpha b$ with $\alpha \neq 0$. The instability of $(0, 0)$ is hidden if the sequence starts at a point belonging to the curve $(C) : y = -a\sqrt{|x|}/b$ because the sequence stays at $(0, 0)$ as soon as the second step (see Figure 3.6(b)). To show the instability, it is required to start at a point (x_0, y_0) which does not lie on C. This is the reason why the set D_ϵ is chosen as specified in the previous proof.

3.5.2 Two-dimensional maps with critical instability

As proven in Theorem 3.41, the square-root instability of the fixed-point appears when $a_{11} \neq 0$. Otherwise, when a_{11} vanishes, the square-root singularity may appear under additional assumptions on the other components of $\mathbf{A}$.

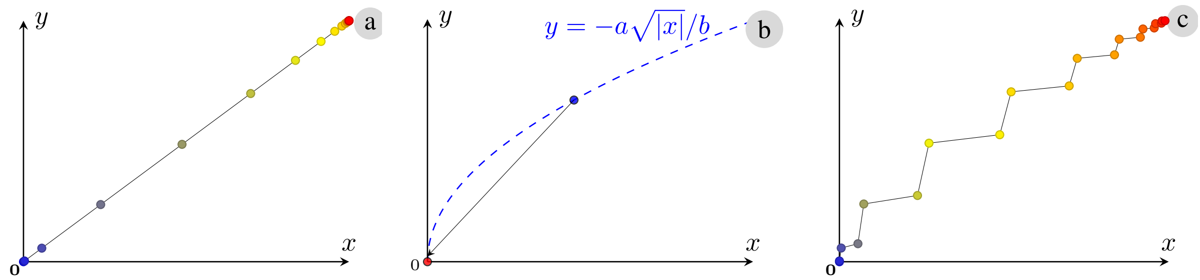


FIGURE 3.6 – Instability of the fixed-point $(0,0)$: (a) when $a = 1$, $c = 2$, and $b = d = 0$, the recurrence goes away from $(0,0)$ along the line $y = cx/a$; (b) when $c = \alpha a$ and $d = \alpha b$, the instability is hidden if starting at a point on the curve $C : y = -a\sqrt{|x|}/b$; (c) when $b = c = 1$ and $a = d = 0$. The gradient color scale [ ] shows initial iterates in blue to final iterates in red irrespective of the magnitude.

Proposition 3.45. Suppose $\mathbf{F}, \mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{F}(x, y) = \mathbf{G}(\sqrt{|x|}, y)$ where $\mathbf{G}$ is a linear map satisfying $\mathbf{G}(x, y) = (by, cx + dy)$ with $bc \neq 0$. Then, $(0,0)$ is an unstable fixed-point of $\mathbf{F}$.

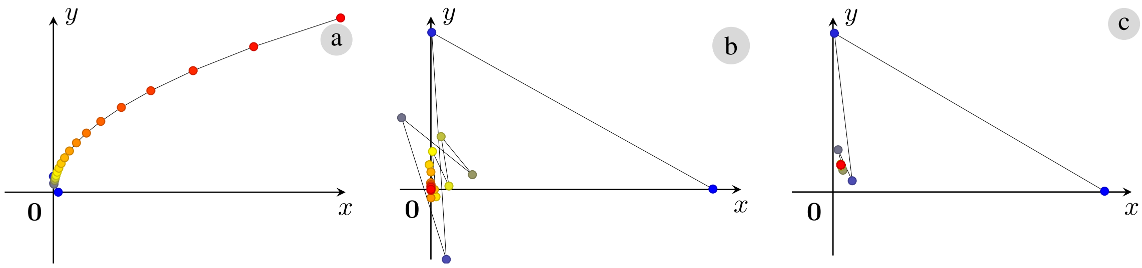


FIGURE 3.7 – Instability of the fixed-point $(0,0)$: (a) when $0 < d = 0.5 < 1$ and $c = 1$; (b) when $-1 < d = -0.5 < 0$ and $c = 1$; (c) when $d = 0$ and $c = 1$. The gradient color scale [ ] shows initial iterates in blue to final iterates in red irrespective of the magnitude.

Proof. By considering the map $\mathbf{F}^2(x, y) = \mathbf{F}(\mathbf{F}(x, y)) = (bc\sqrt{|x|} + bd\,y,\ cd\sqrt{|x|} + d^2\,y + c\sqrt{|by|})$, it is seen that $a_{11} = bc \neq 0$, then the same technique as in the proof of Theorem 3.41 is used to show the instability of the fixed-point $(0,0)$ of $\mathbf{F}^2$. □

Denote

$$\mathbf{DG}(x, y) = \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}, \tag{3.121}$$

Then, in the case of a linear map $\mathbf{G}$, the fixed-point $(0,0)$ of $\mathbf{F}$ is stable if and only if $bc = 0$ and $|d| < 1$. The square-root term in x disappears by the condition $bc = 0$, the stability of the 2-dimensional map becomes the one of 1-dimensional linear map, and hence $|d| < 1$ is needed to have the stability.

Example 3.46. A simple example of this instability is shown in Fig. 3.6(c) where $b = c = 1$ and $a = d = 0$. It is seen that, starting at a point closed to $(0, 0)$, the recurrence goes away from this point.

Example 3.47. There are also the cases where $(0, 0)$ is stable for the linear map $\mathbf{G}$ but unstable for the nonlinear map $\mathbf{G}$. Suppose $\mathbf{F}(x, y) = \mathbf{A}X = (0, \sqrt{|x|} + dy)$ with

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & d \end{bmatrix}, \tag{3.122}$$

where $|d| < 1$. Thus, $(0, 0)$ is a stable fixed-point of $\mathbf{F}$. However, by adding a nonlinear term $\sqrt{|x|}y$ into $\mathbf{F}$ to have $\underline{\mathbf{F}}(x, y) = (\sqrt{|x|}y, \sqrt{|x|} + dy)$, then $(0, 0)$ becomes unstable numerically.

More precise, there are three cases:

1. for $0 < d < 1$, the recurrence oscillates for several steps then goes away from $(0, 0)$ along a curve like $y = k\sqrt{|x|}$ (see Figure 3.7a).
2. for $-1 < d < 0$, the dynamics is different, it seems that the ocurrence always oscillates around $(0, 0)$. This case is harder to see if it is stable or unstable (see Figure 3.7b).
3. For $d = 0$, the fixed-point $(0, 0)$ of the linear map $\mathbf{F}$ is asymptotically stable. It is not asymptotically stable if there is the nonlinear term $\sqrt{|x|}y$, but it might be still stable. In some cases the recurrence may tend to another fixed-point (see Figure 3.7c).

3.6 Dynamics in the vicinity of the grazing orbits

The aim of this section is first to prove the square-root singularity near a grazing contact: Theorems 3.8 and 3.10 and then to study the possible square-root dynamics near periodic solutions with one grazing contact per period. Such periodic solutions of the N-dof system are limited to N linear grazing orbits and there is no other solutions. Periodic solutions with sticking contacts are excluded in this current work. Take note that $\mathcal{H}_{\mathcal{P}}$ ensures that the first return time exists and finite. A challenging point in building the FRT is to check the nonlocal admissibility condition: $u_N(t) < d$ before the closing contact. Moreover, data may lead to the first return time not in the vicinity of T_0 but instead in the vicinity of $2T_0$, $3T_0$ and so on. We pay attention to studying the class of the initial data near W_0 in $\mathcal{H}^-$ or $\mathcal{H}^0_{\mathcal{G}}$ which lead to some orbits having their FRT near T_0.

3.6.1 The first return time

This section deals with the proof of Theorem 3.8. It will be shown that, in a class of initial data, the FRT has a particular form containing a square-root term. First, a lemma is stated, then a long proof is proposed. Recall that $\Phi(t, \mathsf{W}) = \mathbf{e}_N^\top \mathbf{R}(t)\mathsf{SW}$ is smooth. It coincides with u_N in the neighborhood of (T_0, W_0) as long as $u_N(t) \leq d$.

Lemma 3.48. The set $\mathcal{K}$ of indices $\left\{i \in \{N+1, \ldots, 2N\},\ \partial_{W_i}\Phi(T_0, \mathsf{W}_0) \neq 0\right\}$ is not empty.

In other words, there exists at least one non-vanishing partial derivative of Φ with respect to at least one initial velocity. This means also that a component W_k stated in Theorem 3.8 always exits.

Proof. This is proved by contradiction. The smooth function $\Phi(t, \mathsf{W})$ can be written as

$$\Phi(t, \mathsf{W}) = \mathbf{e}_N^\top \mathbf{R}(t)\mathsf{SW} = \mathbf{e}_N^\top \left(\mathbf{P}\cos(t\boldsymbol{\Omega})\mathbf{P}^{-1}\mathbf{u} + \mathbf{P}\boldsymbol{\Omega}^{-1}\sin(t\boldsymbol{\Omega})\mathbf{P}^{-1}\mathbf{v}\right). \tag{3.123}$$

If $\mathcal{K} = \emptyset$ then $\partial_{W_i}\Phi(T_0, \mathsf{W}_0) = 0$ for all $i = N+1, \ldots, 2N$. It follows that

$$\mathbf{e}_N^\top\, \mathbf{P}\, \boldsymbol{\Omega}^{-1}\, \mathbf{sin}(T_0\boldsymbol{\Omega})\, \mathbf{P}^{-1}\, \mathbf{e}_i = 0 \text{ for all } i = 1, \ldots, N. \tag{3.124}$$

In other words, the N^{th} row of the matrix $\mathbf{A} = \mathbf{P}\, \boldsymbol{\Omega}^{-1}\, \mathbf{sin}(T_0\boldsymbol{\Omega})\, \mathbf{P}^{-1}$ is zero. Therefore, $\text{rank}(\mathbf{A}) < N$, this is impossible. Hence $\mathcal{K} \neq \emptyset$. □

Now, the implicit equation $\Phi(T, \mathsf{W}) = d$ is solved with a function $T = \theta(\mathsf{W})$ on a half neighborhood of W_0. At the end, it is proven that $\theta(\mathsf{W})$ is the first return time $T(\mathsf{W})$.

Proof. Consider the initial condition $\mathsf{W}_0 \in \mathcal{H}^- \cup \mathcal{H}^0_{\mathcal{G}}$ at initial instant $t = 0$, the associated orbit has the first grazing contact at $T_0 = T(\mathsf{W}_0)$. The first return time T is implicitly given by the equation $u_N(T, \mathsf{W}) = d$. The nonsmooth function u_N is replaced by the smooth function Φ to can apply the implicit function theorem to the equation:

$$\Phi(T, \mathsf{W}) = d, \tag{3.125}$$

where $\Phi(T_0, \mathsf{W}_0) = u_N(T_0, \mathsf{W}_0) = d$, $\partial_t\Phi(T_0, \mathsf{W}_0) = \dot{u}_N^-(T_0, \mathsf{W}_0) = 0$. In order to apply Theorem 3.26, the following lemma is needed. As a consequence of Lemma 3.48, there exists $k \in \mathcal{K}$ where $\partial_{W_k}\Phi(T_0, \mathsf{W}_0) \neq 0$. Denote $\underline{\mathsf{W}}$ is the reduced vector obtained from W by removing W_k. Together with Assumption 3.7, function $\Phi(t, \mathsf{W})$ can be seen as $\Phi(t, W_k, \underline{\mathsf{W}})$ where W_0 corresponds to $(W_{0k}, \underline{\mathsf{W}}_0)$, and f satisfies the following conditions: $\Phi(T_0, W_{0k}, \underline{\mathsf{W}}_0) = d$, $\partial_t\Phi(T_0, W_{0k}, \underline{\mathsf{W}}_0) = 0$, $\partial_{W_k}\Phi(T_0, W_{0k}, \underline{\mathsf{W}}_0) \neq 0$, $\partial_{tt}\Phi(T_0, W_{0k}, \underline{\mathsf{W}}_0) \neq 0$. By applying Theorem 3.26, there exist neighborhoods $V_{\underline{\mathsf{W}}_0}$, V_{T_0} and $V_{W_{0k}}$ of $\underline{\mathsf{W}}_0$, T_0 and W_{0k} respectively and smooth scalar functions η, α such that

$$\eta : V_{\underline{\mathsf{W}}_0} \to V_{T_0},\ \underline{\mathsf{W}} \mapsto T = \eta(\underline{\mathsf{W}}) \tag{3.126}$$

$$\alpha : V_{\underline{\mathsf{W}}_0} \to V_{W_{0k}},\ \underline{\mathsf{W}} \mapsto W_k = \alpha(\underline{\mathsf{W}}) \tag{3.127}$$

satisfying $\eta(\underline{\mathsf{W}}_0) = T_0$ and $\alpha(\underline{\mathsf{W}}_0) = W_{0k}$. The set S_c defined as the intersection of the two hypersurfaces $\Phi = d$ and $\partial_t\Phi = 0$ can be parameterized as follows

$$S_c = \{(t, W_k, \underline{\mathsf{W}}) \in V_{T_0} \times V_{W_{0k}} \times V_{\underline{\mathsf{W}}_0},\ \Phi(t, W_k, \underline{\mathsf{W}}) = d \text{ and } \partial_t\Phi(t, W_k, \underline{\mathsf{W}}) = 0\}, \quad (3.128)$$

$$= \{(\eta(\underline{\mathsf{W}}), \alpha(\underline{\mathsf{W}}), \underline{\mathsf{W}}),\ \underline{\mathsf{W}} \in V_{\underline{\mathsf{W}}_0}\}. \quad (3.129)$$

Recall that $\mathcal{B}_k = \{\mathsf{W} = (W_k, \underline{\mathsf{W}}) \in V_{W_{0k}} \times V_{W_{0k}},\ s_k(W_k - \alpha(\underline{\mathsf{W}})) \geq 0\}$ where $s_k = \text{sign}(\gamma_k)$. It is the region adjacent to and including the hypersurface $W_k = \alpha(\underline{\mathsf{W}})$. By applying Theorem 3.26, there exists a smooth function ψ such that there are two graphs to solve equation (3.125):

$$T = \theta(\mathsf{W}) = \eta(\underline{\mathsf{W}}) + \psi(\pm\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}))}, \underline{\mathsf{W}}),\ \mathsf{W} \in \mathcal{B}_k, \quad (3.130)$$

where $\psi(0, \underline{\mathsf{W}}_0) = 0$ and $\partial_{W_k}\psi(0, \underline{\mathsf{W}}_0) = |\gamma_k|^{-1/2}$. We choose the branch of $\theta(\mathsf{W})$ corresponding to the admissibility condition for the velocity at the contact, which is

$$\dot{\Phi}(\theta(\mathsf{W}), \mathsf{W}) \geq 0. \quad (3.131)$$

Denote $\mathsf{F}(\mathsf{W}) = \mathbf{R}(\theta(\mathsf{W}))\mathbf{S}\mathsf{W}$. Using the asymptotic expansion up to the first order of $\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}_0))}$, in the direction of the k^{th} component of W, and take note that $\eta(\underline{\mathsf{W}}_0) = T_0$, one has $\theta(\mathsf{W}) = T_0 \pm |\gamma_k|^{-1/2}\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}_0))} + O(h^2) = T_0 + h + O(h^2)$ where $h = \pm|\gamma_k|^{-1/2}\sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}_0))}$. This also follows that

$$W_k = \alpha(\underline{\mathsf{W}}_0) \pm \gamma_k h^2 = W_{0k} + O(h^2),\ \text{since } \alpha(\underline{\mathsf{W}}_0) = W_{0k} \text{ (see Theorem 3.26)}, \quad (3.132)$$

$$\mathsf{W} = \mathsf{W}_0 + h^2\mathbf{e}_k,\ \mathbf{e}_k^\top = [0, \dots, 1, \dots, 0] \in \mathbb{R}^{2N}. \quad (3.133)$$

Similarly,

$$\mathbf{cos}(\theta(\mathsf{W})\,\mathbf{\Omega}) = \mathbf{cos}\left((T_0 + h + O(h^2))\,\mathbf{\Omega}\right) = \mathbf{cos}(T_0\,\mathbf{\Omega}) - h\,\mathbf{\Omega}\,\mathbf{sin}(T_0\,\mathbf{\Omega}) + O(h^2), \quad (3.134)$$

$$\mathbf{sin}(\theta(\mathsf{W})\,\mathbf{\Omega}) = \mathbf{sin}\left((T_0 + h + O(h^2))\,\mathbf{\Omega}\right) = \mathbf{sin}(T_0\,\mathbf{\Omega}) + h\,\mathbf{\Omega}\,\mathbf{cos}(T_0\,\mathbf{\Omega}) + O(h^2). \quad (3.135)$$

Hence, $\mathbf{R}(\theta(\mathsf{W}))$ can be written as $\mathbf{R}(T_0 + h + O(h^2)) = \mathbf{R}(T_0) + h\dot{\mathbf{R}}(T_0) + O(h^2)$ where $\dot{\mathbf{R}}(t)$ denotes the matrix whose elements are the derivatives of the elements of $\mathbf{R}$:

$$\dot{\mathbf{R}}(t) = \begin{bmatrix} -\mathbf{P}\,\mathbf{\Omega}\,\mathbf{sin}(t\,\mathbf{\Omega})\mathbf{P}^{-1} & \mathbf{P}\,\mathbf{cos}(t\,\mathbf{\Omega})\mathbf{P}^{-1} \\ -\mathbf{P}\,\mathbf{\Omega}^2\,\mathbf{cos}(t\,\mathbf{\Omega})\mathbf{P}^{-1} & -\mathbf{P}\,\mathbf{\Omega}\,\mathbf{sin}(t\,\mathbf{\Omega})\mathbf{P}^{-1} \end{bmatrix}. \quad (3.136)$$

Inserting these into F and neglecting the higher order term of h leads to

$$\mathsf{F}(\mathsf{W}) \approx \left(\mathbf{R}(T_0) + h\,\dot{\mathbf{R}}(T_0)\right)\,\mathbf{S}\,\mathsf{W}_0 \approx \begin{bmatrix}\mathbf{u}(T_0,\mathsf{W}_0)\\ \mathbf{v}(T_0,\mathsf{W}_0)\end{bmatrix} + h\begin{bmatrix}\mathbf{v}(T_0,\mathsf{W}_0)\\ -\mathbf{K}\mathbf{u}(T_0,\mathsf{W}_0)\end{bmatrix}. \tag{3.137}$$

Thus, $\dot{\Phi}(\theta(\mathsf{W}),\mathsf{W}) = \mathsf{e}_{2N}^\top P(\mathsf{W}) \approx \mathbf{e}_N^\top\left(\mathbf{v}(T_0,\mathsf{W}_0) - h\,\mathbf{K}\,\mathbf{u}(T_0,\mathsf{W}_0)\right) = -h\,\mathbf{e}_N^\top\mathbf{K}\,\mathbf{u}(T_0,\mathsf{W}_0)$. where e_{2N} is the vector in $\mathbb{R}^{2N}$ such that all coordinates are 0 except the $2N^{\text{th}}$ one equal to 1, similarly for the vector $\mathbf{e}_N$ in $\mathbb{R}^N$. Based on the N^{th} equation of (3.1) taken at $t = T_0$, $m_N\ddot{u}_N^-(T_0) + \mathbf{e}_N^\top\mathbf{K}\,\mathbf{u}(T_0) = 0$ it follows that $m_N\ddot{u}_N^-(T_0,\mathsf{W}_0) = -\mathbf{e}_N^\top\mathbf{K}\,\mathbf{u}(T_0,\mathsf{W}_0)$. Thus,

$$\dot{\Phi}(\theta(\mathsf{W}),\mathsf{W}) \approx h\,m_N\ddot{u}_N^-(T_0,\mathsf{W}_0). \tag{3.138}$$

Therefore, since $\ddot{u}_N^-(T_0,\mathsf{W}_0) < 0$ then, in order to have $\dot{\Phi}(\theta(\mathsf{W})) \geq 0$, h is chosen with the negative sign. Hence, $\theta(\mathsf{W})$ satisfies (3.131). □

The proof of Theorem 3.8 ends by showing that $\theta(\mathsf{W}) = T(\mathsf{W})$. This means that $\Phi(t,\mathsf{W}) < d$ for all $t \in \,]0\,;\theta(\mathsf{W})[$.

Proof. First, $\ddot{\Phi}(T_0,\mathsf{W}_0) < 0$ since it is not zero and $\dot{\Phi}(T_0,\mathsf{W}_0) = 0$, so it is mandatory to have $\Phi(t,\mathsf{W}_0) < d$ for $t < T_0$ and t near T_0. By the smoothness of Φ with respect to (t,W), there exists a neighborhood $[T_-\,;T_+]\times V_1$ of (T_0,W_0) and $\delta_1 > 0$ such that $\ddot{\Phi}(T,\mathsf{W}) < -\delta_1 < 0$ in this neighborhood and $\theta(\mathsf{W}) \in [T_-\,;T_+]$ for $\mathsf{W} \in V_1$. This will be the crucial point to conclude at the end that $\Phi(t,\mathsf{W}) < d$ for all $t \in \,]T_-\,;\theta(\mathsf{W})[$.

Second, $\dot{\Phi}(0,\mathsf{W}_0) < 0$ because W_0 belongs to $\mathcal{H}^-$. Thus there exists a neighborhood $[0\,;T_2]\times V_2$ of $(0,\mathsf{W}_0)$ with $V_2 \subset V_1$ and $\delta_2 > 0$ such that $\dot{\Phi}(T,\mathsf{W}) < -\delta_2 < 0$ in this neighborhood. This implies that $\Phi(t,\mathsf{W}) < d - \delta_2 t < d$ for all $t \in \,]0\,;T_2]$.

Let $\delta_3 = 2^{-1}\delta_2 T_2 > 0$, there exists a neighborhood of W_0 denoted by $V_3 \subset V_2$ such that $|\Phi(t,\mathsf{W}) - \Phi(t,\mathsf{W}_0)| < \delta_3$ for all $(t,\mathsf{W}) \in [T_2\,;T_-]\times V_3$. This yields $\Phi(t,\mathsf{W}) < d - \delta_3 < d$ for all $(t,\mathsf{W}) \in [T_2\,;T_-]\times V_3$.

Finally, on $[T_-\,;\theta(\mathsf{W})]$ for $\mathsf{W} \in V_3$, the function $\Phi(t,\mathsf{W})$ is concave and the velocity at time $\theta(\mathsf{W})$ is nonnegative obtained from (3.131), so $\Phi(t,\mathsf{W}) < d$ on $[T_-\,;\theta(\mathsf{W})[\,\times V_3$ which concludes the proof. □

This ends the proof of main Theorem 3.8.

The proof of Theorem 3.10 is the same except two points: the neighborhood of W_0 is smaller and the proof that $\Phi(t,\mathsf{W}) < d$ for $0 < t$ small enough. The difference is that the initial velocity of the last mass is zero. Hopefully, the acceleration is negative: $\ddot{u}_N^+(0,\mathsf{W}_0) < 0$ which is enough to get $\Phi(t,\mathsf{W}) < d$ for $0 < t$ small enough and conclude the proof of Theorem 3.10.

3.6.2 Square-root dynamics in the vicinity of a linear grazing orbit (LGO)

This subsection explores the possible square-root dynamics near the LGO. One feature of the LGO is that the sticking phase does not occur near such a mode. This property is proven in the coming Proposition. Then, the square-root dynamics coefficients are computed. At least one of them, non-vanishing, will activate the square-root dynamics near the LGOs. If the first return map has a particular expression with the square-root term in a class of initial data then one may expect the instability of the LGOs.

Throughout this section, let us consider the j^{th} LGO (the notation LGO_j is also used below) with period $T_j = T(\mathsf{W}_0)$ [25] associated to the initial data

$$\mathsf{W}_0 = [\mathbf{u}_0^\top, \mathbf{v}_0^\top]^\top, \quad \mathbf{u}_0 = \frac{d}{P_{Nj}}\mathbf{P}\mathbf{e}_j, \quad \mathbf{v}_0 = \mathbf{0}. \tag{3.139}$$

Proposition 3.49. The sticking phase does not occur near the j^{th} LGO, $j = 1, \ldots, N$.

Proof. From the N^{th} equation in (3.1), $m_N \ddot{u}_N(t) + \mathbf{e}_N^\top \mathbf{K}\mathbf{u}(t) = R(t)$, $R(t) \le 0$, the sticking phase does not occur if $u_N(t) = d$ and $F(t) = \mathbf{e}_N^\top \mathbf{K}\mathbf{u}(t) > 0$. By the periodicity of LGO, it is sufficient to show that $F(0) > 0$. The initial data of the LGO yields

$$F(0) = \frac{d}{P_{Nj}}\mathbf{e}_N^\top \mathbf{K}\mathbf{P}\mathbf{e}_j = \frac{d}{P_{Nj}}\mathbf{e}_N^\top \mathbf{M}\mathbf{P}\,\mathbf{\Omega}^2\,\mathbf{P}^{-1}\mathbf{P}\mathbf{e}_j = d m_N \omega_j^2 > 0 \text{ for } d > 0. \tag{3.140}$$

In some cases when $d > 0$ and for some initial data in $\mathcal{B}_k^+$ defined in Theorem 3.10, the associated orbit takes less time to come back to the Poincaré section. This is the consequence of Theorem 3.10 and a feature of the LGO.

Corollary 3.50. Consider a LGO with period T_j and an initial perturbation of its initial data $\mathsf{W} = \mathsf{W}_0 + w\mathbf{e}_k$ with $k \neq 2N$ (or $k = 2N$ and $\sigma > 0$ where $\sigma = sign(\gamma_k)$), then there exists $\delta > 0$ such that $0 < \sigma w < \delta$ and $T(\mathsf{W}) \le T_j$.

Such one-sided condition on the first return time is already known for nonlinear modes with one impact per period near a LGO [25]. It is expected that this inequality is valid in a larger set near W_0 but of course not in the whole neighborhood if $k \neq 2N$.

Proof. To apply Theorem 3.10, Assumption 3.7 must be verified. The second time derivative of u_N at the grazing point is

$$\ddot{u}_N(T_j, \mathsf{W}_0) = -\frac{1}{m_N}\mathbf{e}_N^\top \mathbf{K}\mathbf{u}(T_j) = -\frac{1}{m_N}\mathbf{e}_N^\top \mathbf{K}\mathbf{u}(0) = -d\omega_j^2 < 0. \tag{3.141}$$

Hence, by Theorem 3.8, the first return T takes the form given by (3.17). Moreover, since

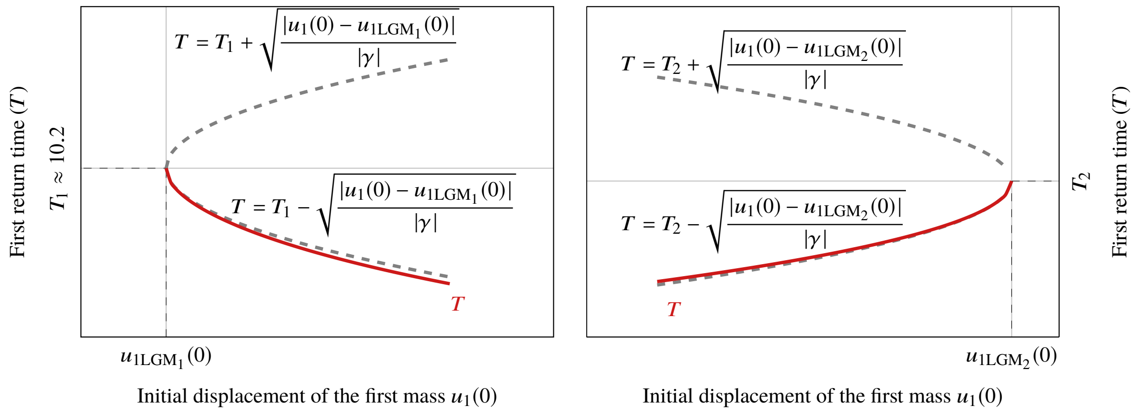


Figure 3.8 – First return time (red lines) with respect to the initial displacement of the first mass: (a) near the first LGO, (b) near the second LGO.

$\ddot{u}_N(T_0, \mathsf{W}_0) < 0$, it follows that in the direction of the k^{th} component of W,

$$T = \eta(\underline{\mathsf{W}}_0) + \psi(-\sqrt{\sigma w}, \underline{\mathsf{W}}_0), \quad \eta(\underline{\mathsf{W}}_0) = T_j, \quad \partial_{W_k}\psi(0, \underline{\mathsf{W}}_0) = 1/\sqrt{|\gamma_k|}. \tag{3.142}$$

Hence, $T = T_j - 1/\sqrt{|\gamma_k|}\sqrt{\sigma w} + O(w) \leq T_j$. □

Let us name $u_{1\text{LGO}_1}(0)$ and $u_{1\text{LGO}_2}(0)$, the initial displacement of the first mass along LGO_1 and LGO_2, respectively. The square-root singularity of the FRT with respect to the initial u_1, when $u_1(0) \gtrsim u_{1\text{LGO}_1}(0)$ or $u_1(0) \lesssim u_{1\text{LGO}_2}(0)$ is shown in Figure 3.8. The red lines are computed numerically, while the dashed lines illustrate the Taylor expansion of ψ. It is seen that in both cases, $T \leq T_j$, $j = 1, 2$.

The following is the proof of Theorem 3.14 which gives the computation of the coefficients C_k and the generic condition that causes the square-root dynamics near the LGO.

Proof. As proven in Proposition 3.49, the sticking phase does not occur near the LGOs, thus the first return map takes the form (3.20a), $\mathcal{F}(\mathsf{W}) = \mathbf{R}(T(\mathsf{W}))\mathbf{S}\mathsf{W}$ where the FRT has the square-root dependence (3.17) in the subset $\mathcal{B}_k^+$. Using the proof in Subsection 3.6.1, the first return map $\mathcal{F}$ can be rewritten as $\mathcal{F}(\mathsf{W}) = \mathbf{G}(\bar{\mathsf{W}}) = (g_1, \ldots, g_{2N})(\bar{\mathsf{W}})$ where $\bar{\mathsf{W}} = [\bar{W}_i]_{i=1}^{2N}$, with a change of variables:

$$\bar{W}_k = \sqrt{s_k(W_k - \alpha(\underline{\mathsf{W}}_0))} \text{ and } \bar{W}_i = \underline{\mathsf{W}}_i, \quad \forall i \neq k. \tag{3.143}$$

Note that $g_N(\bar{\mathsf{W}}) = d$. From the proof in Subsection 3.6.1, it follows that $\bar{W}_k = \sqrt{|\gamma_k|}h$, together with (3.137), $\mathcal{F}$ is written as

$$\mathcal{F}(\mathsf{W}) = \mathbf{G}(W_1, \ldots, \sqrt{|\gamma_k|}h, \ldots, W_{2N}) \approx \begin{bmatrix} \mathbf{u}(T_j, \mathsf{W}_0) \\ \mathbf{v}(T_j, \mathsf{W}_0) \end{bmatrix} + h \begin{bmatrix} \mathbf{v}(T_j, \mathsf{W}_0) \\ -\mathbf{K}\mathbf{u}(T_j, \mathsf{W}_0) \end{bmatrix}. \tag{3.144}$$

For $k \in \{1, \ldots, 2N\}$, $k \neq N$ such that $\partial_{W_k} u_N(T_j, \mathsf{W}_0) \neq 0$, the coefficient associated with the square-root term h is $C_k = a_{kk} = \partial_{\bar{W}_k} g_k(T_j, \mathsf{W}_0) = \sqrt{|\gamma_k|}\partial_h g_k(T_j, \mathsf{W}_0)$. The expression of C_k is then obtained via (3.144),

$$C_k = \begin{cases} \sqrt{|\gamma_k|}\mathbf{e}_k^\top \mathbf{v}(T_j, \mathsf{W}_0) & \text{if } 1 \leq k < N, \quad (3.145a) \\ -\sqrt{|\gamma_k|}\mathbf{e}_{k-N}^\top \mathbf{K}\mathbf{u}(T_j, \mathsf{W}_0) & \text{if } N < k \leq 2N. \quad (3.145b) \end{cases}$$

Since $\mathbf{v}(T_j) = \mathbf{v}_0 = \mathbf{0}$, it follows that $C_k = 0, \forall 1 \leq k < N$. Another way to write C_k is by using the modal coordinates $[\mathbf{q}, \dot{\mathbf{q}}]$ where $\mathbf{u} = \mathbf{P}\mathbf{q}$, as follows:

$$C_k = \begin{cases} \sqrt{|\gamma_k|}\mathbf{e}_k^\top \mathbf{P}\dot{\mathbf{q}}(T_j, \mathsf{W}_0) & \text{if } 1 \leq k < N, \quad (3.146a) \\ -\sqrt{|\gamma_k|}\mathbf{e}_{k-N}^\top \mathbf{M}\mathbf{P}\,\mathbf{\Omega}^2\,\mathbf{q}(T_j, \mathsf{W}_0) & \text{if } N < k \leq 2N. \quad (3.146b) \end{cases}$$

It is shown in Lemma 3.48 that $\mathcal{K} \neq \emptyset$. For each $k \in \mathcal{K}$, the coefficient C_k is

$$C_k = -\sqrt{|\gamma_k|}\mathbf{e}_{k-N}^\top \mathbf{M}\mathbf{P}\,\mathbf{\Omega}^2\,\mathbf{q}(T_j) = -\sqrt{|\gamma_i|}dm_i\omega_j^2 \frac{P_{kj}}{P_{Nj}}. \tag{3.147}$$

By the hypothesis that there exists $i \in \mathcal{K}$ where $P_{ij} \neq 0$, it follows that $C_i \neq 0$. This non-vanishing coefficient then facilitates the square-root dynamics near the LGO. □

The square-root dynamics is however activated near W_0 and in a particular class of initial data. If the orbits stay in that regime of initial data, the dynamics will follows the framework of Section 3.5 and one may be able to determine the instability of the LGOs.

Conclusions

The existence of the solutions to the N-degree-of-freedom vibro-impact system with a unilateral contact condition and without source terms is given in chapter 1. The difficulty while giving a simple proof for the uniqueness of the solution of (1.1) is also discussed.

In chapter 2, a class of periodic solutions involving sticking phases is investigated on a two-degree-of-freedom linear oscillator subjected to a unilateral contact condition. In particular, periodic orbits with one sticking phase per period are considered. The set of such orbits is characterized by only one parameter, the free flight duration, which belongs to a discrete set. Hence, solutions with 1-SPP might or might not exist, and when they do exist, they are isolated. This property can also be seen via the one-to-one correspondence between the solutions with 1-SPP and the ones with 1-IPP. A numerical procedure is given in order to seek for all possible 1-SPP and examples of 1-SPPs are presented. The prestressed structure is also explored; the solutions with 1-SPP with finite duration of sticking phase as well as with infinite duration of sticking phase are found. For N-degree-of-freedom with $N > 2$, the symmetry $\mathbf{u}(t) = \mathbf{u}(-t)$ which is heavily used in the two-degree-of-freedom is unknown, and the sticking system is of $N - 1 > 1$ dimensions. For these reasons, the results on the periodic solutions to N-degree-of-freedom vibro-impact systems involving sticking phases remain to be done.

The First Return Map which is used in seeking for periodic solutions is investigated in order to study the stability or instability of the periodic orbits. This map is determined based on its domain of definition- the Poincaré section and the First Return Time of the orbits to this section. In chapter 3, the well-known square-root singularity of the First Return Time in the vicinity of grazing orbits is carefully studied in a rigorous mathematical framework. In this work, the Poincaré section, that is the domain of definition of the First Return Map, is chosen to be a subset of the impact hyperplane. This critical choice ensures that the FRM is well defined as the orbit comes back to the Poincaré section an infinite number of times. The square-root singularity is shown to appear and it is only defined on a subset of the neighborhood of the initial data leading to a grazing orbit. This implies the discontinuity of the FRT. This result may give rise to a possibly discontinuous FRM as well as immediate instability of the periodic grazing orbits. The square-root dynamics is shown to emerge from the square-root singularity of the FRT if one of the coefficients related to a linear map does not vanish. Under a generic condition of the matrix of eigenvectors, the square-root dynamics near a linear grazing orbit is proven to exist. However, this square-root dynamics is only activated in a particular class of initial data. Hence, the theory on the instability of the fixed point of a map exhibiting the square-root term is not applied directly. Further study on the FRM is needed in order to understand better the global dynamics. The continuity of the FRM remains an open question. When this is clear then the stability of the linear grazing orbits and 1-SPP periodic solutions follow as the next steps. Besides, challenges on the results of the First Return Time arise with the presence of internal resonances. It is however interesting since there are many periodic solutions involving one grazing contact per period other than the N linear grazing modes (LGM).

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Appendix A

Power-root singularity for a mass-spring chain

In the case of a chain, i.e. the stiffness matrix **K** has the form as in (3.95), there is a simpler proof for Proposition 3.34 as stated below.

Lemma A.1. Suppose $\mathbf{u}(t)$ is a solution to (3.1) which models a chain of N masses and has a closing contact at $t = 0$. Under the assumption that $k_{j+1,j} \neq 0$ for all $j = 1, \ldots, N-1$, then $\mathrm{mult}(\dot{u}_N^-)(0) \leq 2N-1$. Moreover, if there is sticking phase starting at $t = 0$, then $\mathrm{mult}(\dot{u}_{N-1})(\tau) \leq 2N-3$.

Proof. The proof includes two parts. The first part shows that $\mathrm{mult}(\dot{u}_N^-)(0) \leq 2N-1$. Otherwise, $\mathrm{mult}(\dot{u}_N^-)(0) \geq 2N$, i.e. $\dot{u}_N^-(0) = \cdots = u_N^{(2N-1)-}(0) = u_N^{(2N)-}(0) = 0$ and it is shown that $\mathbf{u}(0) = \mathbf{0}$ and $\dot{\mathbf{u}}(0) = \mathbf{0}$.

Outside the closing contacts, the system is linear:

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}. \tag{A.1}$$

Let $\mathbf{v} = \dot{\mathbf{u}}$, then outside the closing contacts, $\mathbf{v}$ is the solution of the linear system

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{0}. \tag{A.2}$$

Moreover, $v_N = \dot{u}_N$, from the assumption, v_N satisfies $v_N^{(\ell)}(0) = 0$ for all $1 \leq \ell \leq 2N-1$. From the last equation of (A.2): $m_N\ddot{v}_N + k_{N,N-1}v_{N-1} + k_{NN}v_N = 0$, and since $k_{N,N-1} \neq 0$, it follows that $v_{N-1}(0) = \cdots = v_N^{(2N-3)}(0) = 0$. Similarly, from the $(N-1)^{\text{th}}$ equation and by the assumption that $k_{N-1,N-2} \neq 0$, $v_{N-2}(0) = \cdots = v_{N-2}^{(2N-5)}(0) = 0$. In the end, the second equation and $k_{21} \neq 0$ give $v_1(0) = \dot{v}_1(0) = 0$. As a consequence, we have $\mathbf{v}(0) = \mathbf{0}$, $\dot{\mathbf{v}}(0) = \mathbf{0}$ and the corresponding solution is $\mathbf{v} \equiv \mathbf{0}$, $\dot{\mathbf{v}} \equiv \mathbf{0}$. Thus, $\dot{\mathbf{u}} \equiv \mathbf{0}$ and $\ddot{\mathbf{u}} \equiv \mathbf{0}$. Substitution of these identities into the linear system (A.1) yields $\mathbf{K}\mathbf{u} = \mathbf{0}$. This induces $\mathbf{u} \equiv \mathbf{0}$ since $\det(\mathbf{K}) \neq 0$. However, this contradicts the fact that $u_N(0) = d > 0$, i.e. the solution cannot rest at its equilibrium $\mathbf{0}$.

The next part shows that $\mathrm{mult}(\dot{u}_{N-1})(\tau) \leq 2N-3$ when a sticking phase arises after $t = 0$. Based on the last equation of (A.1), $m_N\ddot{u}_N + k_{N,N-1}u_{N-1} + k_{NN}u_N = 0$, the end of the sticking phase at $t = \tau$ satisfies

$$k_{N,N-1}u_{N-1}(\tau) = -k_{NN}d, \tag{A.3}$$

$$k_{N,N-1}u_{N-1}(\tau + \delta) > -k_{NN}d, \quad \forall\, 0 < \delta < 1. \tag{A.4}$$

The sticking system is then a $(N-1)$ degree-of-freedom non-homogeneous system $\underline{\mathbf{M}\ddot{\mathbf{u}}} + \underline{\mathbf{K}\mathbf{u}} = \underline{\mathbf{C}}$ where $\underline{\mathbf{C}} = [0, \ldots, 0, k_{N-1,N}d]^{\top}$. Now, assume that $\mathrm{mult}(\dot{u}_{N-1})(\tau) \geq 2N-2$, then $\dot{u}_{N-1}(\tau) = \cdots = u_{N-1}^{(2N-2)}(\tau) = 0$. By a similar proof for the reduced $(N-1)\times(N-1)$ system with the last entry of the solution is u_{N-1} instead of u_N, it follows that $\mathrm{mult}(\dot{u}_{N-1})(\tau) \leq 2N-3$. □

Appendix B

One-impact-per-period dynamics with internal resonances near grazing orbits

Summary In this appendix, we discuss the behaviour of periodic solutions with one-impact-per-period (1-IPP) where the period is near linear natural frequencies in the case of internal resonances [25].

Consider the N-dof system (3.1) with the assumption of internal resonances, i.e. $(w_1, w_2, \ldots, w_N)$ are $\mathbb{Z}$-dependent. That is $w_j = k_j w_1$ for $k_j \in \mathbb{Q}^+$, $j = 2, \ldots, N$. Let $\alpha(N)$ be the set of all the non empty subsets of $\{1, 2, \ldots, N\}$. Suppose T_0 belongs to $\cap_{j\in\alpha(N)} T_j\mathbb{N}$.

Remark B.1. For each $T \notin \cup_{j=1}^{N} T_j\mathbb{N}$, it is proven in [25] that there is a unique periodic solution with one impact-per-period with T as the period. This unique periodic solution corresponds to the initial data

$$\mathbf{u}(0;T) = \dot{u}_N^+(0;T)\mathbf{w}(T) \quad \text{and} \quad \dot{\mathbf{u}}^+(0;T) = \dot{u}_N^+(0;T)\mathbf{e}_N, \tag{B.1}$$

where $\dot{u}_N^+(0;T) = d/w_N(T)$. Recall that $w_N(T) = \sum_{j=1}^{N} a_j \Phi_j(T)$ with the coefficients

$$a_j = P_{Nj}P_{jN}^{-1} \quad \text{and} \quad \Phi_j(t) = \frac{\sin(\omega_j t)}{\omega_j(1 - \cos(\omega_j t))}, \quad j = 1, \ldots, N. \tag{B.2}$$

Remark B.2. When $T_0 \in \cap_{j\in\alpha(N)} T_j\mathbb{N}$, a linear grazing periodic solution of (3.1) is the periodic solution with period T_0 associated with the initial data

$$\mathbf{u}(0;T_0) = [u_k(0;T_0)]_{k=1}^{N}, \quad u_k(0;T_0) = d\frac{\sum_{j\in\alpha(N)} a_{kj}\omega_j^{-2}}{\sum_{j\in\alpha(N)} a_j\omega_j^{-2}}, \quad k = 1, \ldots, N, \tag{B.3}$$

$$\dot{\mathbf{u}}^+(0;T_0) = \mathbf{0}, \tag{B.4}$$

where $w_k(T) = \sum_{j=1}^{N} a_{kj}\Phi_j(T)$ and $a_{kj} = P_{kj}P_{jN}^{-1}$.

In particular, when $\alpha(N)$ is a singleton, then T_0 is kT_j, $k \in \mathbb{N}$, we recover the linear grazing modes or linear subharmonics corresponding to the linear natural frequency T_j or the subharmonic frequency kT_j in the non internal resonance case.

Proposition B.3. Suppose $T \notin \cup_{j=1}^{N} T_j\mathbb{N}$, T is near $T_0 \in \cap_{j\in\alpha(N)} T_j\mathbb{N}$. Then the initial data $[\mathbf{u}(0;T), \dot{\mathbf{u}}^+(0;T)]$ in (B.1) converges to the initial data (B.3) of the corresponding grazing orbit.

Proof. If $T_0 \in \cap_{j\in\alpha(N)} T_j\mathbb{N}$ then $T_0 = m_j T_j$, for $j \in \alpha(N)$. Hence, for every $j \in \alpha(N)$, the function Φ_j can be expanded as

$$\Phi_j(T) = \frac{\sin(\omega_j T)}{\omega_j(1-\cos(\omega_j T))} \sim \frac{2}{\omega_j^2(T - m_j T_j)} \sim \frac{2}{\omega_j^2(T - T_0)}. \tag{B.5}$$

Thus

$$w_N(T) = \sum_{j=1}^{N} a_j \Phi_j(T) \sim \sum_{j\in\alpha(N)} a_j \Phi_j(T) \sim \sum_{j\in\alpha(N)} \frac{2a_j}{\omega_j^2} \frac{1}{T - T_0} \quad \text{when} \quad T \to T_0. \tag{B.6}$$

From the condition $u_N(0;T_0) = d = \dot{u}_N^+(0;T) w_N(T)$,

$$\dot{u}_N^+(0;T) \sim \frac{d}{\sum_{j\in\alpha(N)} a_j \Phi_j(T)} \sim \frac{d}{\sum_{j\in\alpha(N)} 2a_j\omega_j^{-2}} (T - T_0). \tag{B.7}$$

It follows that $\dot{u}_N^+(0;T) \to 0$ when $T \to T_0$. The study of the initial displacements is similar. Precisely, the initial displacement is $u_k(0;T) = d w_k(T)/w_N(T)$ where

$$w_k(T) = \sum_{j=1}^{N} a_{kj} \Phi_j(T) \sim \sum_{j\in\alpha(N)} \frac{2a_{kj}}{\omega_j^2} \frac{1}{T - T_0}. \tag{B.8}$$

Hence,

$$u_k(0;T) \to d \frac{\sum_{j\in\alpha(N)} \frac{2a_{kj}}{\omega_j^2} \frac{1}{T - T_0}}{\sum_{j\in\alpha(N)} \frac{2a_j}{\omega_j^2} \frac{1}{T - T_0}} = d \frac{\sum_{j\in\alpha(N)} a_{kj}\omega_j^{-2}}{\sum_{j\in\alpha(N)} a_j\omega_j^{-2}}. \tag{B.9}$$

As a consequence, $u_k^+(0;T) \to u_k(0;T_0)$ when $T \to T_0$. In the end, we obtain that the initial data of the periodic solution with period T converges to the initial data (B.3) of the grazing mode corresponding to T_0 when $T \to T_0$. □

The above proposition shows that there exists a family of 1-IPP orbits with period $T \notin \cup_{j=1}^{N} T_j\mathbb{N}$, T is near a linear period or a subharmonic $T_0 \in \cap_{j\in\alpha(N)} T_j\mathbb{N}$ and these 1-IPP stay near the new linear grazing periodic solutions. Numerical examples for two-dof and three-dof systems are given in Figures (B.1) and (B.2).

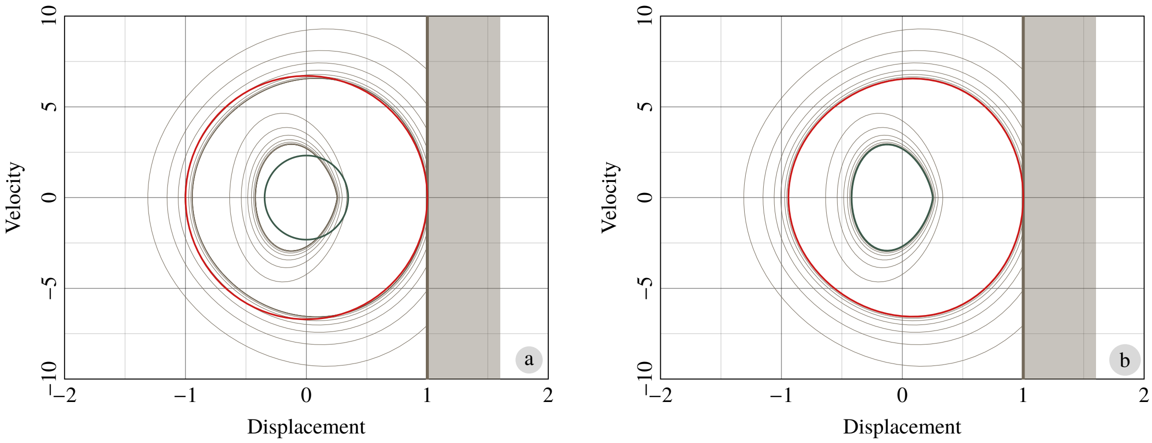

FIGURE B.1 – A 2-dof system when $T_1 = 2T_2$: (a) The 1-IPP solutions are not near to the LGO; (b) they are near to the new periodic solution with grazing.

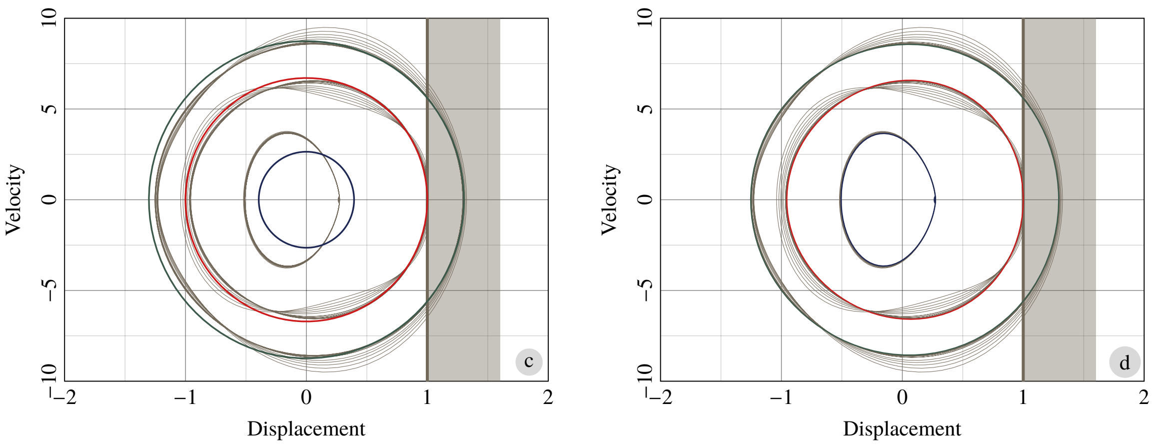

FIGURE B.2 – A 3-dof system when T near T_1: (c) the associated 1-IPPs are not near the usual linear grazing mode with fundamental period T_1; (d) the associated 1-IPPs are near the grazing orbit with initial data (B.3).

Contents

On some periodic solutions of discrete vibro-impact oscillators with a unique unilateral contact condition

Abstract

The purpose of this thesis is to investigate N degree-of-freedom vibro-impact oscillators with an unilateral contact. The dynamics is linear in the absence of contact; it is governed by an impact law otherwise. Trajectories that display a sticking phase are identified. The First Return Map is a fundamental tool to explore such periodic solutions. The Poincaré section is tangent to grazing orbits and thus yields the well-known square-root singularity, as already reported in Mechanics, which is here revisited in a rigorous mathematical framework. Another important singularity is exhibited: the discontinuity of the First Return Time. Finally, the square-root dynamics near the linear grazing orbits which may lead to the instability of these linear grazing orbits is studied. It is found that the square-root dynamics emerges from the square-root singularity of the First Return Time if one of the coefficients related to a linear map does not vanish. Under a generic condition of the matrix of eigenvectors, the square-root dynamics near a linear grazing orbit is proven to exist.

Keywords: nonsmooth analysis, vibro-impact systems, unilateral contact, periodic solutions, sticking phase, linear grazing orbit, poincaré map

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Sur des solutions périodiques d'oscillateurs discrets à vibro-impact avec une unique condition de contact unilatéral

Résumé

Le but de cette thèse est d'étudier des oscillateurs à vibro-impact à N degrés de liberté avec une condition de contact unilatéral. La dynamique est linéaire en l'absence de contact ; elle est régie par une loi d'impact autrement. Des trajectoires présentant une phase de contact collant sont identifiées. L'application de premier retour de Poincaré est un outil fondamental pour étudier la dynamique près de solutions périodiques. La section de Poincaré est tangente aux orbites rasantes et conduit à une singularité en « racine carrée », déjà connue en Mécanique, singularité revisitée dans un cadre mathématique rigoureux. Elle implique la discontinuité du temps de premier retour. Enfin, la dynamique en racine carrée près des modes linéaires rasants, qui peut conduire à l'instabilité de ces modes, est abordée. On constate que la dynamique en racine carrée émerge de la singularité en racine carrée du temps de premier retour si l'un des coefficients liés à une carte linéaire ne s'annule pas. Pour une condition générique de la matrice des vecteurs propres, la dynamique en racine carrée près d'une mode linéaire rasants est prouvée.

Mots clés : analyse non lisse, systèmes discrets à vibro-impact, contact unilatéral, solutions périodiques, mode linéaire rasant, l'application de poincaré

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