Inégalites quantitatives et convexité

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Thèse de doctorat

Discipline : Mathématiques

présentée par

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Inégalités quantitatives et Convexité

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Soutenue le 7 juillet 2017 devant le jury composé de :

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Résumé

Cette thèse est divisée en trois parties. Les deux premieres sont constituées chacune d'articles soumis disponibles sur arXiv, respectivement *More on functional and quantitative versions of the isoperimetric inequality* et *Dimensional transport inequalities and Brascamp-Lieb inequalities* alors que la dernière est constituée de remarques sur l'isopérimétrie.

Nous nous intéressons dans un premier temps à une version fonctionnelle de l'inégalité isopérimétrique généralisant les versions ensemblistes et fonctionnelles classiques. Dans ce même article, nous donnons une version quantitative de l'inégalité isopérimétrique avec un reste faisant intervenir la distance de Wasserstein.

Puis, nous étudions dans *Dimensional transport inequalities and Brascamp-Lieb inequalities* des inégalités de transport pour les mesures convexes. La linéarisation de ces inégalités de transport redonnent les inégalités de Brascamp-Lieb dimensionnelles. Nous en donnons aussi une forme quantitative.

Enfin, dans un troisième temps, nous étudions les inégalités isopérimétriques avec une fonction poids pour les mesures convexes. Nous traitons le cas de la dimension 1 en montrant qu'une constante de Cheeger existe et nous en donnons une estimation.

Mots clés: Transport de mesures, mesure convexe, inégalité de transport, inégalités du type Brascamp-Lieb, inégalité isopérimétrique, inégalités quantitatives, entropie, inégalités de variance dimensionelles.

Abstract

This thesis is divided in three parts. The two first are constituted by submitted papers available in arXiv, respectively *More on functional and quantitative versions of the isoperimetric inequality* and *Dimensional transport inequalities and Brascamp-Lieb inequalities* whereas the last chapter is dedicated to remarks on isoperimetry.

In the first paper, we are interested in a functional version of the isoperimetric inequality which generalizes the version for sets and the classical functional ones. We also give a quantitative version of the isoperimetric inequality with a remainder term involving Wasserstein's distance.

In the second one, we study transport inequalities for convex measures. Linearization of our transport inequalities retrieve the dimensional forms of Brascamp-Lieb inequalities. We also give a quantitative forms of these inequalities.

Finally, we investigate weighted isoperimetric inequalities for convex measures. We treat the case of dimension 1. We note that the associated Cheeger constant exists et we give an estimation of this constant.

Keywords: mass transportation, convex measure, transport inequality, Brascamp-Lieb type inequalities, isoperimetric inequality, quantitative inequalities, entropy, dimensional variance inequalities.

1 Introduction

1.1 Généralités

1.1.1 Transport de mesure et inégalités quantitatives

La décennie 1996-2006 a confirmé le succès des méthodes de transport optimal pour traiter les inégalités fonctionnelles importantes en géométrie, probabilités et analyse harmonique. Avant d'énoncer de tels résultats, énonçons quelques résultats relatifs à la notion de transport optimal. Cette notion est apparue dans [Mo] en 1781 et a connu un regain d'intérêt pendant la Seconde Guerre Mondiale avec Leonid Kantorovich dans [Ka]. Avant d'exposer les différentes formulations dûes à Monge et Kantorovich, faisons un rappel sur les mesures images. Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces mesurables munis de mesures de probabilités μ et ν respectivement. Soit $T : \mathcal{X} \to \mathcal{Y}$ une application mesurable. On dit que ν est l'image de μ par T et on note $T_{\#}\mu = \nu$ si pour toute partie mesurable $A \subseteq \mathcal{Y}$,

$$\nu(A) = \mu\left(T^{-1}(A)\right).$$

Nous avons par conséquent la formule suivante : pour toute fonction $b : \mathcal{Y} \to \mathbb{R}$ mesurable, bornée

$$\int_{\mathcal{Y}} b\, d\nu = \int_{\mathcal{X}} b \circ T\, d\mu.$$

Nous pouvons énoncer la formulation du problème de Monge:

Problème 1.1. *Étant donné deux espaces de probabilités $(\mathcal{X}, \mu)$ et $(\mathcal{Y}, \nu)$, et une fonction mesurable (appelée "coût") $c : \mathcal{X} \times \mathcal{Y} \to [0, +\infty]$, on considère le problème de minimation suivant :*

$$\inf \int_{\mathcal{X}} c(x, T(x))\, d\mu(x), \tag{1}$$

où l'infimum est pris sur toutes les applications mesurables $T : \mathcal{X} \to \mathcal{Y}$ telles que $T_{\#}\mu = \nu$.

La formulation du problème selon Monge peut être mal posée: si μ est une mesure de Dirac et si ν n'en est pas une, on ne peut trouver d'application mesurable T telle que $T_{\#}\mu = \nu$. On peut améliorer la formulation en la suivante, relaxée, dite formulation de Kantorovich.

Problème 1.2. *Soient $(\mathcal{X}, \mu)$ et $(\mathcal{Y}, \nu)$ deux espaces de probabilités et une fonction mesurable $c : \mathcal{X} \times \mathcal{Y} \to [0, +\infty]$, on considère le problème de minimisation suivant*

$$\mathcal{W}_c(\mu, \nu) := \inf \int_{\mathcal{X} \times \mathcal{Y}} c(x, y)\, d\pi(x, y), \tag{2}$$

*où l'infimum est pris sur les mesures de probabilités π sur $\mathcal{X} \times \mathcal{Y}$ ayant μ et ν pour marginales. De telles mesures sont appelées **plan de transport optimal**.*

Lorsque $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$ et $c(x,y) = |x-y|^p$ avec $p \geq 1$, on définit la distance de Wasserstein par

$$W_p^p(\mu,\nu) := \mathcal{W}_c(\mu,\nu).$$

Le problème de Kantorovich admet une formulation duale, dualité dite de Kantorovich :

$$\mathcal{W}_c(\mu,\nu) = \sup\left(\int_{\mathcal{X}} \phi(x)\, d\mu(x) + \int_{\mathcal{Y}} \psi(y)\, d\nu(y)\right), \tag{3}$$

où le supremum est pris sur toutes les fonctions $\phi : \mathcal{X} \to \mathbb{R}$ et $\psi : \mathcal{Y} \to \mathbb{R}$ mesurables et telles que $\phi(x) + \psi(y) \leq c(x,y)$ avec $x \in \mathcal{X}$ et $y \in \mathcal{Y}$. Bien sûr, mis sous cette formulation générale, le problème de Monge-Kantorovich admettra (ou non) des solutions dépendant de la fonction c et des espaces considérés. On peut obtenir des résulats non triviaux dans le cas où c est semi-continue inférieurement et $\mathcal{X}, \mathcal{Y}$ sont des espaces polonais (espace métrisable et séparable). Avant de donner un tel résultat, nous définissons la notion de c-convexité.

Définition 1.1. *Soient $\mathcal{X}$ et $\mathcal{Y}$ deux ensembles et soit $c : \mathcal{X} \times \mathcal{Y} \to (-\infty, +\infty]$. On dit que $\psi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ est c-convexe si elle n'est pas identiquement égale à $+\infty$ et s'il existe $\phi : \mathcal{Y} \to \mathbb{R} \cup \{\pm\infty\}$ telle que*

$$\psi(x) = \sup_{y \in \mathcal{Y}} \{\phi(y) - c(x,y)\}, \qquad \forall x \in \mathcal{X}.$$

On définit alors la c-transformée ψ^c par

$$\psi^c(y) = \inf_{x \in \mathcal{X}} \{\psi(x) + c(x,y)\}, \qquad \forall y \in \mathcal{Y},$$

et son c-sous gradient

$$\partial_c \psi = \{(x,y) \in \mathcal{X} \times \mathcal{Y}, \psi^c(y) - \psi(x) = c(x,y)\}.$$

Soit enfin le c-sous gradient au point x $\partial_c \psi(x)$ que l'on peut définir par

$$\partial_c \psi(x) = \{y \in \mathcal{Y}, (x,y) \in \partial_c \psi\}.$$

La définition prédente donne lieu à la remarque suivante :

Remarque 1. Lorsque $\mathcal{X} = \mathcal{Y} = \mathbb{R}^n$ et $c(x,y) = -x \cdot y$ où $x \cdot y$ est le produit scalaire usuel de $\mathbb{R}^n$, la c-transformée s'apparente à la transformée de Legendre $V^\star$ d'une fonction V définie par

$$V^\star(y) = \sup_{x \in \mathbb{R}^n} \{x \cdot y - V(x)\}, \qquad \forall y \in \mathbb{R}^n.$$

Cette définition de la transformée de Legendre a comme conséquence l'inégalité suivante, dite inégalité de Young :

$$\forall x, y \in \mathbb{R}^n, \qquad x \cdot y \leq V(x) + V^\star(y).$$

Voici un résultat général d'existence et d'unicité de plan de transport optimal, pour lequel nous renvoyons à [Vi].

Théorème 1.1. *Soient $\mathcal{X}$ et $\mathcal{Y}$ deux espaces métriques, séparables. Soit c : $\mathcal{X} \times \mathcal{Y} \to \mathbb{R}_+$ une fonction coût semi-continue inférieurement. Soient μ et ν deux mesures de probabilités sur $\mathcal{X}$ et $\mathcal{Y}$ respectivement. Si $\mathcal{W}_c(\mu, \nu) < +\infty$ et si pour toute fonction c-convexe $\psi : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ l'ensemble $\{x \in \mathcal{X}$ tel que $\partial_c \psi(x) \neq \emptyset\}$ est négligeable pour μ alors il existe un unique couple (en loi) (μ, ν) satisfaisant le problème de minimisation de Kantorovich.*

On ne peut parler de transport optimal sans invoquer les résultats de Brenier [Br], améliorés par McCann [Mc1, Mc2].

Théorème 1.2 (Brenier-McCann)**.** *Si μ et ν sont deux mesures de probabilité sur $\mathbb{R}^n$ avec μ absolument continue par rapport à la mesure de Lebesgue alors il existe une fonction convexe φ définie sur $\mathbb{R}^n$ telle que $T = \nabla\varphi$ vérifie $T_\# \mu = \nu$. De plus, $\nabla\varphi$ est unique μ-presque partout. Nous appellerons cette application le transport de Brenier.*

Ramarquons que la convexité de φ assure qu'elle est différentiable presque-partout sur son support et par construction elle est la solution du problème de Kantorovich pour la fonction "coût" $c(x, y) = |x - y|^2/2$. Nous renvoyons à [Mc1] pour les détails de la preuve. Si nous supposons que les mesures μ et ν possèdent des densités F et G respectivement par rapport à la mesure de Lebesgue, alors

$$\int_{\mathbb{R}^n} b(y)\, G(y)\, dy = \int_{\mathbb{R}^n} b(\nabla\varphi(x))\, F(x)\, dx,$$

pour toute fonction $b : \mathbb{R}^n \to \mathbb{R}_+$ mesurable, bornée. Si φ est C^2, le changement de variable $y = \nabla\varphi(x)$ montre que φ est solution de l'équation de Monge-Ampère

$$F(x) = G(\nabla\varphi(x)) \det D^2\varphi(x). \tag{4}$$

Ici $D^2\varphi(x)$ est la matrice hessienne de φ au point x. Les résultats de régularité de Caffarelli nous renseignent sur des conditions suffisantes pour que (4) soit vraie au sens classique.

Théorème 1.3 (Caffarelli [Ca1, Ca2])**.** *Soient E et K deux ouverts non vides et bornés de $\mathbb{R}^n$. On suppose K convexe. Soient f et g deux fonctions boréliennes strictement positives, bornées et telles que $1/f$ et $1/g$ soient bornées sur respectivement E et K. On suppose en outre que les mesures $d\mu(x) = f(x)\, dx$*

et $d\nu(y) = g(y)\,dy$ *sont deux mesures de probabilité. Soit* $T = \nabla\varphi$ *le transport de Brenier entre* μ *et* ν*. Sous ces hypothèses,* φ *est strictement convexe et* $\varphi \in C^{1,\beta}(E)$ *. Si l'on suppose de plus* f *et* g *continues alors* $\varphi \in W^{2,p}_{loc}(E)$ *. Et si,* f *et* g *sont* α*-hölderiennes alors* $\varphi \in C^{2,a}(E)$ *pour tout* $0 < a < \alpha$*.*

Si les conditions énoncées par Caffarelli assurent l'existence de (4) au sens fort, McCann remarqua dans [Mc2] que (4) reste vraie au sens faible sans hypothèse supplémentaire sur F et G, la hessienne $D^2\varphi(x)$ étant à comprendre au sens d'Alexandrov, c'est-à-dire comme la partie absolument continue de la hessienne distribution de φ. Une autre possibilité (équivalente à la précédente presque-partout) est de remarquer qu'une fonction convexe admet presque partout un développement de Taylor de la forme

$$\varphi(x+h) = \varphi(x) + \nabla\varphi(x)\cdot h + \frac{1}{2}D^2\varphi(x)(h)\cdot h + |h|^2\,\epsilon(h),$$

avec $\lim_{h\to 0}\epsilon(h) = 0$. Même si en pratique il est difficile voire impossible de calculer le transport de Brenier, cette méthode n'est pas dénuée d'applications. Une des plus célèbres est sûrement l'inégalité de Brunn-Minkowski.

Théorème 1.4 (Inégalité de Brunn-Minkowski)**.** *Soient* A *et* B *deux compacts non vides de* $\mathbb{R}^n$ *alors*

$$|A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}, \tag{5}$$

où $|\cdot|$ *désigne la mesure de Lesbesgue dans* $\mathbb{R}^n$ *et* $A+B$ *est la somme de Minkowski* $\{a+b, a\in A, b\in B\}$ *.*

Remarque 2. Nous énonçons l'inégalité de Brunn-Minkowski pour A et B compacts pour s'assurer la mesurabilité de $A+B$.

Nous écrivons le schéma de la preuve ci-après (en omettant la discussion sur la régularité de φ) pour montrer l'élégance de la méthode. On se ramène au cas où A et B sont de même mesure, disons 1 pour simplifier. Soient $d\mu(x) = 1_A(x)\,dx$ et $d\nu(y) = 1_B(y)\,dy$ où 1_A et 1_B sont les fonctions indicatrices des ensembles A et B respectivement. Soit $T = \nabla\varphi$ le transport de Brenier entre μ et ν. Soit enfin $S = Id + T$. On a donc:

$$\begin{aligned} |A+B| &= \int_{|A+B|} dx \\ &\geq \int_{S(A)} dx \\ &\underbrace{=}_{u=s(x)} \int_A \det \nabla S \\ &= \int_A \det \left(Id + D^2\varphi \right). \end{aligned}$$

Ainsi en utilisant l'inégalité de Jensen et la concavité de $\det^{1/n}$ sur l'ensemble des matrices positives (i.e. $\det^{1/n}(H_1+H_2) \geq \det^{1/n}(H_1) + \det^{1/n}(H_2)$), on récupère

$$|A+B|^{1/n} \geq \int_A \left(\det (Id)^{1/n} + \det (\nabla\varphi)^{1/n} \right).$$

L'équation de Monge-Ampère donne dans ce cas $\det \left(D^2\varphi \right) = 1$, ainsi $|A+B|^{1/n} \geq 1+1 = |A|^{1/n} + |B|^{1/n}$. Citons comme autre application du transport optimal l'inégalité de Prékopa-Leindler [Pr1, Pr2].

Théorème 1.5 (Prékopa-Leindler)**.** *Soient $f, g : \mathbb{R}^n \to \mathbb{R}_+$ deux fonctions mesurables et vérifiant $\int f = \int g = 1$. Soit $t \in (0,1)$ et soit h une fonction définie sur $\mathbb{R}^n$ vérifiant*

$$h(z) \geq \sup_{z=(1-t)x+ty} f(x)^{1-t} g(y)^t, \qquad \forall z \in \mathbb{R}^n.$$

Alors, on a

$$\int h \geq \left(\int f \right)^{1-t} \left(\int g \right)^t.$$

On peut, en posant $h^*(z) = \sup_{z=(1-t)x+ty} f(x)^{1-t} g(y)^t$, énoncer l'inégalité de Prékopa-Leindler sous la forme suivante

$$\int h^* \geq \left(\int f \right)^{1-t} \left(\int g \right)^t.$$

Nous devons quand même faire attention au problème de mesurabilité portant sur h^*.

On peut encore citer les inégalités de Brascamp-Lieb inverses de Barthe [Ba] ou bien les inégalités étudiées Sobolev de Cordero-Erausquin-Nazaret-Villani dans [CE-Na-Vi]. La liste est longue et on se contentera de renvoyer à [Vi] pour plus d'informations.

A priori, pour toute inégalité dans laquelle les cas d'égalité sont connus se pose dans la question de son amélioration quantitative, c'est-à-dire de la quantification de la proximité à un cas d'égalité lorsque l'inégalité est presque saturée. Comme résultat allant en ce sens, on peut citer celui de Alzer [Al] donnant une forme quantitative de l'inégalité arithmético-géométrique.

Proposition 1.1 (Alzer). *Soient $x_1, \cdots, x_n$ des réels strictement positifs et soient $p_1, \cdots p_n$ des réels positifs tels que $\sum_{i=1}^n p_i = 1$. On note $A_n = \sum_{i=1}^n p_i x_i$ et $G_n = \prod_{i=1}^n x_i^{p_i}$. Alors, on a la version quantitative suivante de l'inégalité arithmético-géométrique*

$$A_n - G_n \geq \frac{1}{2} \frac{1}{\sup_{1 \leq i \leq n} x_i} \sum_{i=1}^n p_i \left(x_i - G_n\right)^2.$$

Les travaux récents concernent, par exemple, les inégalités de Sobolev [Ci-Fu-Ma-Pr], les inégalités isopérimétriques [Fi-Ma-Pr1] et de type Brunn-Minkowski [Ba-Bö, El-Kl, Fi-Ma-Pr2] et les inégalités de Gagliardo-Nirenberg-Sobolev [Ca-Fi]. Énonçons par exemple une forme quantitative de l'inégalité de Brunn-Minkowski dûe à Figalli-Maggi-Pratelli [Fi-Ma-Pr2] (la meilleure estimation de $C(n)$ est dûe à Segal [Se])

Théorème 1.6 (Figalli-Maggi-Pratelli). *Soient A et B deux convexes de $\mathbb{R}^n$ avec, pour simplifier, $|A| = |B| = 1$, alors*

$$\left|\frac{A+B}{2}\right| \geq |A| \left(1 + C(n) \inf_{z \in \mathbb{R}^n} \left|A \Delta (B+z)\right|^2\right), \tag{6}$$

où Δ désigne la différence symétrique et $C(n)$, obtenue par les auteurs, est de l'ordre de n^{-7}.

Pour démontrer cela, les auteurs commencent par établir une inégalité de type Cheeger à trace: pour tout corps convexe $E \subset \mathbb{R}^n$ tel que $B(0,r) \subseteq E \subseteq B(0,R)$ et pour toute fonction $f : \mathbb{R}^n \to \mathbb{R}$ assez régulière, on a:

$$\frac{n\sqrt{2}}{\log 2} \frac{R}{r} \int_E |\nabla f| \geq \inf_{c \in \mathbb{R}} \int_{\partial E} |f - c| \, d\mathcal{H}^{n-1}. \tag{7}$$

Grâce à l'équation de Monge-Ampère (4) et (7), on obtient la version quantitative suivante de l'inégalité isopérimétrique

$$\frac{p_K(E)}{n |K|^{\frac{1}{n}} |E|^{\frac{n-1}{n}}} - 1 \geq \frac{C}{n^5} \inf_{x \in \mathbb{R}^n} \left|E \Delta (x+K)\right|^2, \tag{8}$$

où E est compact, K convexe, et

$$p_K(E) = \liminf_{\epsilon \to 0} \frac{|E + \epsilon K| - |E|}{\epsilon}$$

est la mesure de bord de E. À partir de cette version quantitative de l'isopérimétrie, voici la fin de l'argument de Figalli-Maggi-Pratelli [Fi-Ma-Pr2] permettant de d'obtenir (6). L'inégalité (8) donne les deux lignes suivantes

$$p_E(E+K) \geq n|E|^{1/n}|E+K|^{1/n'}\left(1 + \frac{\inf_{x\in\mathbb{R}^n}|(E+K)\,\Delta\,(x+E)|}{Cn^5}\right),$$

et

$$p_K(E+K) \geq n|K|^{1/n}|E+K|^{1/n'}\left(1 + \frac{\inf_{x\in\mathbb{R}^n}|(E+K)\,\Delta\,(x+K)|}{Cn^5}\right).$$

En remarquant que $n|E+K| = p_{E+K}(E+K) = p_E(E+K) + p_K(E+K)$ et le fait que $|E\Delta K| \leq |E\Delta G| + |G\Delta K|$ pour $E, K, G \subseteq \mathbb{R}^n$, on obtient (6). On peut néanmoins signaler qu'il n'y a, pour l'instant, pas de résultat vraiment définitif, quelle que soit l'inégalité en question. En effet, on sait, dans presque tous les cas étudiés que les inégalités quantitatives obtenues ne sont pas optimales à cause du comportement des constantes en la dimension.

1.1.2 La classification de Borell des mesures convexes

On peut montrer que l'inégalité de Brunn-Minkowski est équivalente à la forme adimensionnelle suivante (que l'on peut retrouver en utilisant l'inégalité de Prékopa-Leindler): pour tout A et B compacts non vides,

$$|(1-t)A + tB| \geq |A|^{1-t}|B|^t, \qquad \forall t \in (0,1).$$

Cela signifie que la mesure de Lebesgue est log-concave ou bien, pour reprendre la terminologie de Borell, elle est 0-concave. Les mesures log-concaves généralisent les ensembles convexes dans le sens où si E est convexe alors la mesure $1_E(x)\,dx$ est log-concave. Rappelons la terminologie introduite par Borell sur les mesures convexes qui généralisent les mesures log-concaves. Soit $\alpha \in [-\infty, +\infty]$ et soit μ une mesure de Radon sur $\mathbb{R}^n$. On dit que μ est α-concave si

$$\mu((1-t)A + tB) \geq \left((1-t)\mu(A)^\alpha + t\mu(B)^\alpha\right)^{1/\alpha}, \tag{9}$$

pour tout $t \in (0,1)$ et pour tous compacts $A, B \subset \mathbb{R}^n$. Lorsque $\alpha = 0$, le terme de droite est à comprendre au sens de $\mu(A)^{1-t}\mu(B)^t$, lorsque $\alpha = -\infty$ le terme de droite doit s'interpréter comme $\min\{\mu(A), \mu(B)\}$ et comme $\max\{\mu(A), \mu(B)\}$

lorsque $\alpha = +\infty$. On remarque que l'inégalité (9) devient de plus en plus forte lorsque α croît. Ainsi, le cas $\alpha = -\infty$ décrit le plus grand ensemble de mesures que l'on appelle mesures convexes. Profitons en pour définir la notion de fonction γ-concave. Soit f une fonction continue sur $\mathbb{R}^n$ (ou bien sur un ouvert convexe $\Omega \subseteq \mathbb{R}^n$), on dit que f est γ-concave si

$$f\left((1-t)\,x + ty\right) \geq \left((1-t)\,f(x)^{\gamma} + tf(y)^{\gamma}\right)^{1/\gamma},$$

pour tous $x, y \in \mathbb{R}^n$, avec les mêmes définitions que ci-dessus lorsque $\gamma = 0, -\infty$ ou $+\infty$. Citons un résultat de Borell dans [Bor1] caractérisant les mesures convexes.

Théorème 1.7 (Borell)**.** *Soit μ une mesure de Radon sur $\mathbb{R}^n$. Soit F le plus petit sous espace affine contenant le support de μ. Notons d la dimension de F et m_d la mesure de Lebesgue sur F. Alors pour tout $-\infty \leq s \leq 1/d$, μ est α-concave si et seulement s'il existe une fonction γ-concave $\psi \in L^1_{loc}(F, m_d)$ telle que $\psi \geq 0$ et $d\mu = \psi\, dm_d$ avec $\gamma = \frac{\alpha}{1-d\alpha} \in [-1/d, +\infty]$.*

Ce résultat permet de simplifier l'écriture des mesures convexes. Nous distinguons alors deux cas:

Cas 1. *Ce cas correspond au cas où $\alpha \leq 0$. On pose (en gardant les mêmes notations que ci-dessus), $\beta = -1/\gamma = d - 1/\alpha \geq d$. Nous travaillons donc avec des densités de la forme $\rho_{\beta,W}(x) = W(x)^{-\beta}$ où W est convexe sur $\mathbb{R}^n$ ou sur un ouvert convexe Ω de $\mathbb{R}^n$.*

Et,

Cas 2. *Ce cas correspond au cas où $0 < \alpha \leq 1/d$. On pose alors $\beta = 1/\gamma \geq 0$. Nous travaillons alors avec des densités de la forme $\rho_{\beta,W}(x) = W(x)^{\beta}$ où W est concave sur un support compact convexe $\Omega \subset \mathbb{R}^n$.*

L'ensemble des mesures log-concaves (pouvant s'écrire $d\mu(x) = \exp(-V(x))\, dx$ avec V convexe sur $\mathbb{R}^n$) forme l'ensemble le plus remarquable parmi les mesures convexes car il contient la célèbre distribution gaussienne $d\gamma_n(x) = \frac{1}{(2\pi)^{n/2}} \exp\left(-|x|^2/2\right) dx$.

1.1.3 Inégalités isopérimétriques, spectrales et de Brascamp-Lieb pour les mesures de probabilité convexes

Citons un résultat majeur concernant les mesures log-concaves: l'inégalité de Brascamp-Lieb [Br-Li]. Pour tout mesure de probabilité log-concave sur $\mathbb{R}^n$ de la forme $d\mu = e^{-V} dm_n$ (m_n étant la mesure de Lebesgue sur $\mathbb{R}^n$) et pour toute fonction $u : \mathbb{R}^n \to \mathbb{R}$ localement Lipschitz telle que $\int u\, d\mu = 0$,

$$\int u^2 d\mu \leq \int \left(D^2 V\right)^{-1} \nabla u \cdot \nabla u \, d\mu.$$

Une question naturelle se pose: peut-on énoncer une inégalité Brascamp-Lieb pour des mesures convexes générales ? La question est en particulier intéressante si l'on suppose $D^2 V \geq \lambda I$ avec $\lambda > 0$, puisqu'alors on obtient l'inégalité suivante dite de Poincaré

$$\int u^2 d\mu \leq \frac{1}{\lambda} \int |\nabla u|^2 \, d\mu.$$

Rappelons la terminologie relative aux inégalités de Poincaré. Soient $p, q \in [1, +\infty]$. Une mesure μ définie sur $\mathbb{R}^n$ satisfait l'inégalité de Poincaré (p, q) , s'il existe $c > 0$ tel que pour toute fonction $f \in W_{loc}^{1,1} \cap L^p(\mu)$,

$$\left(\int |\nabla f|^q \, d\mu\right)^{1/q} \geq c \left(\int \left|f - m_f(\mu)\right|^p d\mu\right)^{1/p}, \tag{10}$$

où $m_f(\mu)$ est une valeur médiane de f pour la mesure μ (c'est-à-dire que $m_f(\mu)$ vérifie $\mu\{f > t\} \leq 1/2$ pour tout $t \geq m_f(\mu)$ et $\mu\{f > t\} > 1/2$ pour tout $t < m_f(\mu)$). On notera par $h_{p,q}(\mu) > 0$ la meilleure constante (i.e. la plus grande) vérifiant (10). Lorsque $p = q = 2$, on retrouve la notion d'inégalité de Poincaré. Un autre cas est remarquable, c'est lorsque $p = q = 1$. La constante $h_{1,1}(\mu)$ (si elle existe), elle liée au problème d'isopérimétrie pour la mesure μ, c'est-à-dire à l'existence d'une constante $D_{Che}(\mu) > 0$, appelée constante de Cheeger, telle que pour tout $A \subseteq \mathbb{R}^n$ mesurable

$$\mu^+(A) \geq D_{Che}(\mu) \min\{\mu(A), 1 - \mu(A)\},$$

où $\mu^+(A)$ est le périmètre de A et est définie par

$$\mu^+(A) = \liminf_{\epsilon \to 0} \frac{\mu(A + \epsilon B_2^n) - \mu(A)}{\epsilon} = \int_A \rho_\mu(x) \, d\mathcal{H}^{n-1}(x), \tag{11}$$

où ρ_μ est la densité de μ et $\mathcal{H}^{n-1}$ est la mesure de Hausdorff $(n-1)$-dimensionnelle. Avant de parler du problème de l'isopérimétrie, terminons la partie dédiée aux constantes de Poincaré en faisant le lien entre ces dernières et la première valeur propre non nulle du Laplacien. Soit $K \subset \mathbb{R}^n$ un corps convexe de volume 1. Soit $d\mu(x) = 1_K(x) \, dx$. On s'intéresse au Laplacien $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ sur le domaine suivant:

$$H = \left\{f \in C^2(K) \, ; \nabla f \cdot n = 0 \text{ sur } \partial K\right\},$$

où n est la normale extérieure à K de sorte que l'on ait la formule d'intégration par parties suivante:

$$\langle g, \Delta f\rangle = \int g\Delta f = -\int \nabla f \cdot \nabla g, \qquad \forall f \in H, g \in C^1(K).$$

Comme $-\Delta$ est autoadjoint et positif, la théorie classique des opérateurs assure que $-\Delta$ admet une suite de valeurs propres $0 = \lambda_0(\mu) < \lambda_1(\mu) < \lambda_2(\mu) < \cdots$. La première valeur propre non nulle, $\lambda_1(\mu)$ est liée à $h_{2,2}(\mu)$ par les lignes suivantes (notons que le noyau de $-\Delta$ est réduit aux fonctions constantes sur K):

$$\begin{aligned}
\lambda_1(K) &= \inf_{f \perp \text{fonctions constantes}} \frac{\langle f, -\Delta f\rangle}{\langle f, f\rangle} \\
&= \inf_{f\in H, \int f\, d\mu = 0} \frac{\int |\nabla f|^2 \, d\mu}{\int f^2 d\mu} \\
&= \inf_{f\in C^1(K), \int f\, d\mu = 0} \frac{\int |\nabla f|^2 \, d\mu}{\int f^2 d\mu}.
\end{aligned}$$

Ainsi $h_{2,2}(\mu) = \sqrt{\lambda_1(\mu)}$. Un résultat analogue existe pour les mesures log-concaves $d\mu = e^{-V} dm_n$ avec V convexe et m_n la mesure de Lebesgue sur $\mathbb{R}^n$ mais avec $\Delta_V f := \Delta f - \nabla V \cdot \nabla f$.

Le problème de l'isopérimétrie est le suivant: étant données une mesure de probabilité μ et une mesure de bord μ^+ définie comme ci-dessus (11), peut-on trouver une classe de boréliens $\mathcal{C}$ telle que pour tout borélien $A \subseteq \mathbb{R}^n$, il existe $B \in \mathcal{C}$ tel que $\mu(A) = \mu(B)$ et $\mu^+(A) \geq \mu^+(B)$? Et si l'on note I_μ le fonction isopérimétrique définie sur $(0,1)$ par $I_\mu(t) = \inf_{A\subseteq\mathbb{R}^n, \mu(A)=t} \mu^+(A)$, peut-on trouver $D_{Che}(\mu) > 0$ tel que $I_\mu(t) \geq D_{Che}(\mu) \min\{t, 1-t\}$ (notons en effet que I_μ est symétrique par rapport à $1/2$). Par exemple dans le cas de la mesure gaussienne, la question a été résolue dans les années 70 par Borell [Bor3] et Sudakov-Tsirelson (voir [Bo1] et [Eh] pour d'autres approches dont une par symétrisation).

Théorème 1.8. *Soit γ_n la mesure de probabilité gaussienne standard et centrée dans $\mathbb{R}^n$. Soit aussi*

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt.$$

Alors la fonction isopérimétrique I_{γ_n} de γ_n est donnée par $I_{\gamma_n} = \psi' \circ \psi^{-1}$. De manière équivalente a une caractérisation des ensembles extrémaux $\mathcal{C}$: ce sont les demi-espaces. On en déduit également que γ_n admet une constante de Cheeger $D_{Che}(\gamma_n) > 0$.

Revenons aux constantes de Poincaré. Dans [Mi1], E. Milman prouva, sous certaines hypothèses, que les constantes de Poincaré sont équivalentes, plus précisément:

Théorème 1.9 (E. Milman). *Il existe une constante universelle $C > 0$, tel que pour toute mesure log-concave μ définie sur $\mathbb{R}^n$ et pour $p, q, p', q' \in \mathbb{R}$ tels que $1 \leq p \leq q \leq +\infty$ et $1 \leq p' \leq q' \leq +\infty$, tel que*

$$h_{p,q}(\mu) \leq Cp' h_{p',q'}(\mu).$$

D'autres résultats vont dans ce sens et nous renvoyons à Ledoux [Le] ou encore Cheeger [Ch].

Le problème de l'estimation de ces constantes de Poincaré est difficile, même dans le cas L^2, c'est-à-dire lorsque $p = q = 2$. Les premières estimations dans le cas de mesures uniformes sur des convexes sont dûes à Payne et Weinberger et font intervenir le diamètre du convexe. Dans certain cas, par exemple dans le cas d'un intervalle $I = (-a, a) \subset \mathbb{R}$ avec $a > 0$, on a $h_{2,2}\left(\frac{1}{2a} 1_I(x)\, dx\right) = \frac{\pi}{a}$. On peut généraliser cela en des estimations pour les pavés de $\mathbb{R}^n$ grâce aux résultats de [Bo-Ho1]. Citons enfin le remarquable résultat de Kannan, Lovász et Simonovits dans [Ka-Lo-Si] dans le cas des mesures uniformes.

Théorème 1.10 (Kannan-Lovász-Simonovits). *Pour tout corps convexe K, de mesure 1 pour simplifier, on a*

$$h_{1,1}(1_K(x)\, dx) \geq \frac{\ln 2}{M_1(K)},$$

où $M_1(K) = \int_K |x|\, dx$.

Cela améliore les estimations faisant intervenir le diamètre car $M_1(K)$ pouvant être bien plus petit que le diamètre. Les auteurs utilisent la méthode de localisation développée plus tôt par dans [Lo-Si] que l'on peut énoncer sous la forme élégante suivante:

Théorème 1.11 (Localisation). *Soient f et g deux fonctions semi-continues inférieurement et intégrables sur $\mathbb{R}^n$. On suppose en outre que*

$$\int_{\mathbb{R}^n} f(x)\, dx > 0 \text{ et } \int_{\mathbb{R}^n} g(x)\, dx > 0.$$

Alors il existe $a, b \in \mathbb{R}^n$ et une fonction affine $l : [0,1] \to \mathbb{R}_+$ tels que

$$\int_0^1 f((1-t)a + tb)\, l(t)^{n-1}\, dt > 0 \text{ et } \int_0^1 g((1-t)a + tb)\, l(t)^{n-1}\, dt > 0.$$

Voir aussi [Fr-Gu] et [Bo-Le2] pour d'autres approches et des exemples d'application de cette méthode de localisation. Fermons l'apparté sur les inégalités de Poincaré en parlant de la conjecture que Kannan, Lovász et Simonovits énonçent dans [Ka-Lo-Si]. On dit qu'un corps convexe $K \subset \mathbb{R}^n$ est en position isotropique s'il est de volume un et s'il existe $\alpha > 0$ tel que

$$\int_K |x \cdot y|^2 \, dx = \alpha \, |y|^2 \, , \qquad \forall y \in \mathbb{R}^n.$$

Dans ce cas, on note $L_K := \frac{1}{n} \int_K |x|^2 \, dx$ la constante d'isotropie de K. La conjecture de KLS est la suivante:

Conjecture 1.1. *Pour tout corps convexe K en position isotropique, on a*

$$h_{1,1}(K) \geq c$$

où $c > 0$ est une constante universelle.

Revenons aux inégalités Brascamp-Lieb pour les mesures convexes. Nous distinguons selon que l'on soit dans le **Cas 1** ou le **Cas 2**. En gardant les notations du **Cas 1**, en posant $g = \phi f$, on a pour toute fonction $f : \Omega \to \mathbb{R}$ assez regulière,

$$(\beta + 1) \, Var_{\mu_\beta}(f) \leq \int \frac{\left\langle \left(D^2 \phi\right)^{-1} \nabla g, \nabla g \right\rangle}{\phi} d\mu_\beta + \frac{n}{\beta - n} \left(\int f d\mu_\beta \right)^2 ,$$

Alors qu'en se plaçant dans le **Cas 2**, on a

$$(\beta - 1) \, Var_{\mu_\beta}(f) \leq \int \frac{\left\langle \left(-D^2 \phi\right)^{-1} \nabla g, \nabla g \right\rangle}{\phi} d\mu_\beta + \frac{n}{n + \beta} \left(\int f d\mu_\beta \right)^2 .$$

Ces inégalités établies par Bobkov et Ledoux [Bo-Le3] par linéarisation de l'inégalité de Borell-Brascamp-Lieb sont connues sous le nom d'inégalités de Brascamp-Lieb dimensionnelles et elles permettent de retrouver l'inégalité de Brascamp-Lieb [Br-Li]. Nous renvoyons à Bobkov-Ledoux [Bo-Le3], et à Nguyen [Ng] pour diverses discussions autour de ces inégalités et pour une approche L^2 de celles-i.

1.2 Résultats

1.2.1 Chapitre 2

Ce chapitre reprend notre article *More on functional and quantitative versions of the isoperimetric inequality.* Dans ce papier, nous donnons une généralisation de l'inégalité isopérimétrique à l'aide de la transformée de Legendre d'une fonction, la version classique étant retrouvée en prenant des fonctions indicatrices d'ensembles :

Théorème 1.12. *Étant donné une fonction convexe* $V : \mathbb{R}^n \to \mathbb{R}_+$ *telle que* $Z_V := \int_{\mathbb{R}^n} \left(1 + \frac{V}{n-1}\right)^{-n} < +\infty$ *et* $d\mu_V = \frac{1}{Z_V}\left(1 + \frac{1}{n-1} V(x)\right)^{-n} dx$.

Alors pour toute fonction f *positive localement-lipschitz sur* $\mathbb{R}^n$

$$p_V(f) \geq \left[n Z_V^{1/n} \int_{\mathbb{R}^n} \left(1 + \frac{1}{n-1} V\right) d\mu_V \right] \|f\|_{L^{n'}(\mathbb{R}^n)}, \tag{12}$$

où

$$p_V(f) := \int_{\mathbb{R}^n} V^\star \left(\frac{-\nabla f}{f^{\frac{n}{n-1}}} \right) f^{\frac{n}{n-1}} + \left(\int_{\mathbb{R}^n} V d\mu_V \right) \left(\int_{\mathbb{R}^n} f^{n'} \right)$$

et $1/n + 1/n' = 1$.

Lorsque l'on prend dans (12) $f = 1_E$ et pour $V = 1_K^\infty$ avec 1_K^∞ la fonction "indicatrix" de K définie par

$$1_K^\infty(x) = \begin{cases} 0, & \text{si } x \in K \\ +\infty & \text{si } x \notin K, \end{cases}$$

on obtient l'inégalité isopérimétrique, à savoir:

$$p_K(E) \geq n |K|^{\frac{1}{n}} |E|^{\frac{n}{n-1}}.$$

Dans ce même papier, nous montrons aussi la forme quantitative suivante de l'inégalité isopérimétrique avec un reste faisant intervenir la distance de Wasserstein.

Théorème 1.13. *Soient* K *un corps convexe et* E *un borélien de même mesure, disons* 1 *pour simplifier. On suppose aussi que* E *et* K *ont le même barycentre. Notons*

$$p_K(E) = \liminf_{\epsilon \to 0} \frac{|E + \epsilon K| - |E|}{\epsilon},$$

le périmètre de E *relativement à* K*. Alors, on a l'inégalité suivante*

$$R(E, K) := \frac{p_K(E)}{n |K|^{\frac{1}{n}} |E|^{\frac{n}{n-1}}} - 1 \geq \frac{c}{n} \mathcal{W}_{c_{\lambda_E}}(\lambda_E, \lambda_K), \tag{13}$$

où λ_E, λ_K *désignent respectivement les mesures de probabilité uniformes sur* E *et* K*, et* c_{λ_E} *est la fonction coût définie par*

$$c_{\lambda_E}(x, y) = \mathcal{F}(D_{Che}(\lambda_E) |y - x|)$$

avec $\mathcal{F}(t) := t - \log(1 + t)$ *et* $D_{Che}(\lambda_E)$ *la constante de Cheeger de la mesure de probabilité* λ_E.

Nous mettons notre résultat (13) en compétition avec la version quantitative de l'inégalité isopérimétrique de [Fi-Ma-Pr1], à savoir:

$$R\left(E,K\right)\geq\frac{p_{K}\left(E\right)}{\left|K\right|^{\frac{1}{n}}\left|E\right|^{\frac{n}{n-1}}}-1\geq C\,\frac{\inf_{z\in\mathbb{R}^{n}}\left|E\Delta\left(z+K\right)\right|}{n^{5}},\tag{14}$$

où $C>0$ est une constante universelle.

Pour cela, nous nous plaçons dans des exemples importants de la géométrie des corps convexes. Nous exhibons aussi des exemples pour lesquels le reste dans (13) est arbitrairement grand alors que celui dans (14) reste borné par 1. Les résultats de Maggi-Figalli-Pratelli permettent néanmoins d'obtenir une version quantitative de l'inégalité de Brunn-Minkowski. Nous en donnons une mais cette dernière semble difficilement exploitable.

1.2.2 Chapitre 3

Ce chapitre reprend notre article *Dimensional transport inequalities and Brascamp-Lieb inequalities.* Dans cet article, nous retrouvons les inégalités de Borell-Brascamp-Lieb dimensionelles grâce à une inégalité de transport. On dit que μ satisfait une inégalité de transport si:

$$\alpha\left(\mathcal{W}_{c}\left(\mu,\cdot\right)\right)\leq H\left(\cdot\|\mu\right),$$

où α est une fonction croissante sur $[0,+\infty)$ vérifiant $\alpha\left(0\right)=0$ et $H\left(\cdot\|\mu\right)$ est une entropie relative à μ. L'entropie qui apparaît le plus fréquemment s'écrit sous la forme

$$H\left(f\right)=-\int f\log\left(f\right).$$

Cette forme est bien adaptée pour des mesures log-concaves, ce qui n'est pas le cas dans ce qui nous intéresse ici. Nous ne détaillerons pas l'entropie ici. La linéarisation de ces inégalités de transport (c'est-à-dire que l'on étudie le développement limité de $H\left(\left(1+\epsilon g\right)\rho\|\rho\right)$) redonne les inégalités de Brascamp-Lieb dimensionnelles [Bo-Le3]. Nous en donnons aussi des formes quantitatives. Nous avons dû distinguer les cas, selon que l'on soit dans le **Cas 1** ou **Cas 2**. Nous énonçons ici seulement l'inégalité quantitative dans le **Cas 1**.

Theorem 1.1. *En reprenant les notations du **Cas 1**, et en notant $\rho_{\beta,W}\left(x\right)=\frac{W(x)^{\beta}}{\int W^{\beta}}$ avec $\beta\geq0$ et $W\geq0$ concave sur son support compact Ω, alors on a*

$$\int\left(-D^{2}W+\frac{c}{\beta+1}h_{W}\left(\rho_{\beta,W}\right)I\right)^{-1}\nabla f\cdot\nabla f\rho_{\beta,W}\geq\beta\int g^{2}W\rho_{\beta,W},$$

avec $\int g\,\rho_{\beta,W}=0, \int xg\,(x)\,\rho_{\beta,W}\,(x)=0, f=gW$ *et* $h_W\,(\rho_{\beta,W})$ *une constante de Cheeger à poids.*

La linéarisation est un procédé standard et il est bien connu que la linéarisation des inégalités de transport donne des inégalités de type Poincaré. La procédure pour linéariser les distances de Wasserstein est aussi connue [Ot-Vi] et il est important de noter que seul le comportement local de la fonction coût importe, voir aussi [Go-Lé].

1.2.3 Chapitre 4

Dans ce court chapitre, nous nous intéressons aux inégalités de Poincaré pour les mesures convexes. Peut-on, grâce aux inégalités de Borell-Brascamp-Lieb dimensionnelles, obtenir une inégalité de Cheeger (ou isopérimétrique) pour ces mesures, c'est-à-dire une inégalité du type

$$\mu^+\,(A)\geq D_{Che}\,(\mu)\min\left\{\mu\,(A)\,,1-\mu\,(A)\right\}.$$

La réponse fut donnée par Bobkov dans [Bo5], que pour une mesure α-concave avec $\alpha<0$, on a

$$\mu^+\,(A)\geq\frac{C_\alpha}{\|\,|x|\,\|_{L^1(\mu)}}\min\left\{\mu\,(A)\,,1-\mu\,(A)\right\}^{1-\alpha},$$

où $C_\alpha>0$ dépend continûment de α.

Pour essayer de palier à ce problème, deux approches sont possibles. La première est d'affaiblir les normes utilisées, c'est-à-dire ne plus utiliser les normes L^p comme dans l'inégalité (10). Une seconde approche, que nous suivons, est d'introduire des fonctions poids, c'est-à-dire de travailler avec une mesure de bord de la forme

$$\mu_w^+\,(A)=\int_A w\,(x)\,d\mathcal{H}^{n-1}\,(x)\,.$$

Avant de parler de nos résultats, citons le résulat de E. Milman dans [Mi2] donnant une réponse pour la première approche.

Théorème 1.14 (E. Milman)**.** *Soit μ une mesure α-concave avec $\alpha<0$, on pose, suivant les notations introduites dans* ***Cas 2****, $\beta=n-1/\alpha>n$ en posant $N=n-\beta<0$, on a pour toute fonction f localement Lipschitz telle que $m_f\,(\mu)=0$,*

$$\|\,|\nabla f|\,\|_{L^{q,p\frac{N-1}{N}}(\mu)}\geq C_{p,q}\|f\|_{L^{p,p\frac{N-1}{N}}(\mu)},\tag{15}$$

avec p et q vérifiant $\frac{N}{N-1}\leq p\leq -N$ et $1/q=1/p+1/N$ et $\|f\|_{L^{\alpha,r}}\,(\mu)$ est définie, pour $r\in(0,+\infty]$ et $\alpha\in(0,+\infty)$ par

$$\|f\|_{L^{\alpha,r}(\mu)} := \left(r \int_0^{+\infty} t^r \mu \{|f| \geq t\}^{r/\alpha} \frac{dt}{t} \right)^{1/r}.$$

Comme indiqué ci-dessus, nous avons décidé de nous nous intéresser aux inégalités de Cheeger avec un poids. Étant donné une mesure de probabilité μ α-concave avec $\alpha < 0$ de la forme $d\mu(x) = \phi(x)^{-\beta} dx$ (voir la description du **Cas 1**) avec ϕ convexe, on pose la mesure de bord

$$\nu^+(A) = \int_A \phi(x) \phi(x)^{-\beta} d\mathcal{H}^{n-1}(x). \tag{16}$$

Nous nous intéressons à la résolution du problème isopérimétrique suivant (Cheeger avec un poids):

Problème 1.3. *Peut-on décrire une classe d'ensembles disons $\mathcal{C}$, tel que pour tout borélien $A \subseteq \mathbb{R}^n$, il existe $B \in \mathcal{C}$ vérifiant: $\mu(A) = \mu(B)$ et $\nu^+(A) \geq \nu^+(B)$. Pour ces ensembles $B \in \mathcal{C}$, (et donc pour tous les boréliens) peut-on trouver une constante $c > 0$ tel que pour tout borélien $B \subseteq \mathbb{R}^n$ (si une telle constante existe, nous noterons $Che(\mu)$ la meilleure (i.e. la plus grande))*

$$\nu^+(B) \geq c \min\{\mu(B), 1 - \mu(B)\}? \tag{17}$$

Peut-on décrire les cas d'égalité dans (17)*?*

Des inégalités à poids avaient déjà été étudiées. Par exemple dans [Bo-Le2, Bo-Le3], Bobkov et Ledoux établissent une inégalité de type Cheeger à poids pour les mesures de Cauchy, puis ils en déduissent une inégalité de type Poincaré pour les mesures α-concave avec $\alpha \leq 0$.

Théorème 1.15 (Bobkov-Ledoux)**.** *Soit μ une mesure de probabilité α-concave avec $\alpha \leq 0$. Alors pour toute fonction f localement Lipschitz définie sur $\mathbb{R}^n$ telle que $\int |x|^2 |f(x)| d\mu(x) < +\infty$, alors*

$$Var_\mu(f) \leq C_\alpha \int |\nabla f(x)|^2 \left(m_0^2 + \alpha^2 |x|^2\right) d\mu(x),$$

avec $m_0 = \exp \int \log |x| d\mu(x)$.

Nous répondons à ces questions dans $\mathbb{R}$. Nous suivons le papier de Bobkov [Bo4] dans lequel il donne une estimation de la constante de Cheeger pour les mesures log-concaves réelles. Pour cela, nous nous plaçons dans le **Cas 2**, c'est-à-dire que nous travaillons avec des mesures de probalité convexes de la forme $d\mu(x) = W(x)^{-\beta} dx$ avec $\beta > n$ (nous supposons, pour simplifier $\int W^{-\beta} = 1$). Soit $\nu^+(A) = \int_A W(x) W(x)^{-\beta} d\mathcal{H}^{n-1}(x)$. Nous montrons qu'il existe $Che_W(\mu_{\beta,W}) > 0$ tel que pour tout borélien $A \subseteq \mathbb{R}$

$$\nu^{+}(A) \geq Che_W(\mu_{\beta,W}) \min\{\mu(A), 1-\mu(A)\}. \tag{18}$$

Comme Bobkov [Bo4] donnant une estimation de la constante de Cheeger, nous donnons aussi une estimation de $Che_W(\mu_{\beta,W})$ à l'aide de $\int |x|\, d\mu_{\beta,W}(x)$. Nous pensons que l'inégalité (18) se généralise à $\mathbb{R}^n$.

2 More on functional and quantitative versions of the isoperimetric inequality

2.1 Introduction

We shall work on the Euclidean space $(\mathbb{R}^n, \cdot, |\cdot|)$. The sharp (anisotropic) isoperimetric inequality can be stated as follow: given a convex body $K \subset \mathbb{R}^n$ (having zero in its interior), if we denote by

$$n' = \frac{n}{n-1}$$

the Lebesgue conjugate to n, we have for every Borel set $E \subset \mathbb{R}^n$ that

$$p_K(E) \geq n\, |K|^{\frac{1}{n}}\, |E|^{\frac{1}{n'}}, \tag{19}$$

with equality if $E = \lambda K + a$ for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Here

$$p_K(E) = \liminf_{\epsilon \to 0} \frac{|E + \epsilon K| - |E|}{\epsilon}.$$

Equivalently, if E has a regular enough boundary ∂E, then

$$p_K(E) = \int_{\partial E} h_K(-\nu(x))\, d\mathcal{H}^{n-1}(x),$$

where

$$h_K(z) := \sup_{y \in K} y \cdot z, \qquad \forall z \in \mathbb{R}^n.$$

is the support function of the body K, $\nu(x)$ is the outer unit normal to ∂E at $x \in \partial E$, and $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure on ∂E. The classical Euclidean isoperimetric inequality corresponds to the case when $K = B_2^n = \{|\cdot| \leq 1\}$, the Euclidean unit ball.

We want to analyse functional versions of (19). Replacing E by a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}_+$ is standard. We have decided to work with nonnegative functions recalling that for f with values in $\mathbb{R}$ we can apply the result to $|f|$ and use the fact that for $f \in W^{1,1}_{loc}(\mathbb{R}^n)$, we have, almost-everywhere, $\nabla |f| = \pm \nabla f$. The functional inequality takes the same form

$$p_K(f) \geq n\, |K|^{\frac{1}{n}} \left(\int_{\mathbb{R}^n} f^{n'} \right)^{1/n'}, \tag{20}$$

with equality if $f = 1_E$ (provided the gradient term below is understood as a capacity of the bounded variation function 1_E). Here,

$$p_K(f) = \int_{\mathbb{R}^n} h_K(-\nabla f(x))\, dx.$$

The inequality (19) can be proven directly using a mass transportation method, as observed by Gromov, see the appendix of [Mi-Sc]. In the case of the Euclidean ball, $K = B_2^n$, we recover

$$p_{B_2^n}(f) = \left\| |\nabla f| \right\|_{L^1(\mathbb{R}^n)}.$$

Extending the convex body K to a (convex) function or measure is less obvious. First, one needs to have a proper extension of the notion of support function h_K for a convex function V. Actually, the integral term $\int h_K(-\nabla f)$ needs a proper interpretation, so that non only a convex function will enter the game, but also some "convex measure" (in the terminology of Borell [Bor1] and [Bor2]) associated to it. This has been studied recently in several papers. In particular, in [Kl1] corresponding extensions of the isoperimetric inequality (19) have been proposed. See also [Co-Fr] and [Mi-Ro]. Here we will establish a new inequality that has the advantage to contain the geometric versions (19) and (20). We will do that by picking a good category of convex measures. First, we need to introduce some notation. Let $V : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a nonnegative convex function such that $Z_V := \int_{\mathbb{R}^n} \left(1 + \frac{1}{n-1} V(x)\right)^{-n} dx < +\infty$. We associate to V the probability measure

$$d\mu_V(x) = \frac{1}{Z_V} \left(1 + \frac{1}{n-1} V(x)\right)^{-n} dx. \tag{21}$$

Our generalization for $p_K(f)$ is as follows, for $f : \mathbb{R}^n \to \mathbb{R}_+$ locally Lipschitz we put

$$p_V(f) := \int_{\mathbb{R}^n} V^* \left(\frac{-\nabla f}{f^{\frac{n}{n-1}}} \right) f^{\frac{n}{n-1}} + \left(\int_{\mathbb{R}^n} V d\mu_V \right) \left(\int_{\mathbb{R}^n} f^{n'} \right)$$

where V^* is the Legendre's transform of V,

$$V^*(y) = \sup_{x \in \mathbb{R}^n} x \cdot y - V(x), \qquad \forall y \in \mathbb{R}^n.$$

In particular, we have following inequality (known as Young's inequality):

$$\forall x, y \in \mathbb{R}^n, \qquad x \cdot y \le V(x) + V^*(y), \tag{22}$$

with equality when $y = \nabla V(x)$. Note that when

$$V = 1_K^\infty := \begin{cases} 0 & \text{on } K \\ +\infty & \text{outside } K \end{cases}$$

is the "indicatrix" of a convex set K, then $p_V(f) = \int_{\mathbb{R}^n} h_K(-\nabla f) = p_K(f)$ since $V = 0$ μ_V almost-everywhere and $V^* = h_K$. The general isoperimetric-Sobolev inequality, we can get is then as follow.

Theorem 2.1. *Let V be a nonnegative convex function with $Z_V = \int_{\mathbb{R}^n}(1+\frac{V}{n-1})^{-n} < +\infty$ and μ_V the associated probability measure* (21)*. Then, for every nonnegative locally Lipschitz function f on $\mathbb{R}^n$ we have*

$$p_V(f) \geq \left[n\, Z_V^{\frac{1}{n}} \int_{\mathbb{R}^n} (1+\frac{1}{n-1}V)d\mu_V\right] \|f\|_{L^{n'}(\mathbb{R}^n)}, \tag{23}$$

and, when V is finite, with equality when $f(x) = \left(1+\frac{1}{n-1}V(x-a)\right)^{-(n-1)}$ with $a \in \mathbb{R}^n$.

Thanks to the remark prior to the theorem, we see that when $V = 1_K^\infty$, inequality (23) becomes exactly (20).

The second topic of the present paper is mainly independent of what we discussed so far, although based again on mass transport methods. We aim at presenting some quantitative forms of the geometric isoperimetric inequality (19) that involve a Kantorovich-Rubinstein (or Wassertein) distance cost to an extremizer. For $u : \mathbb{R}^n \to \mathbb{R}^n$ a Borel map and μ a measure in $\mathbb{R}^n$, we write $u_\sharp\mu$ for the measure defined by

$$u_\sharp\mu(M) := \mu\left(u^{-1}(M)\right)$$

for all Borel sets $M \subseteq \mathbb{R}^n$. It is called the push-forward of μ through u.

For μ and ν two probability measures in $\mathbb{R}^n$, and a (cost) function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$, we define the Kantorovich-Rubinstein or Wasserstein transportation cost $\mathcal{W}_c(\mu,\nu)$ by

$$\begin{aligned}\mathcal{W}_c(\mu,\nu) &= \inf_{T:\mathbb{R}^n\to\mathbb{R}^n:T_\sharp\mu=\nu} \int_{\mathbb{R}^n} c(x,T(x))d\mu(x) \\ &= \inf_\pi \iint_{\mathbb{R}^n\times\mathbb{R}^n} c(x,y)d\pi(x,y)\end{aligned}$$

where the infimum is taken over probability measures π on $\mathbb{R}^n \times \mathbb{R}^n$ that have μ and ν as marginals, respectively. For $1 \leq p \leq +\infty$ and $c(x,y) = |y-x|^p$, the p-th power classical p-Kantorovich-Rubinstein distance, W_p^p, is recovered. We refer to [Vi] for details.

Our measures will be uniform measures on the sets E and K. Given a Borel set E, we will denote by λ_E the Lebesgue measure restricted to E and normalized to be a probability measure,

$$d\lambda_E(x) = \frac{1_E(x)}{|E|}dx$$

where 1_E is the indicator function of the set E. Given a Borel set E, we will denote by $\widetilde{E}$ the homothetic of volume one of the set E, namely

$$\widetilde{E} = \frac{1}{|E|^{\frac{1}{n}}}E.$$

Note that for $u \in GL_n(\mathbb{R})$ we have $\lambda_{u(E)} = u_\sharp \lambda_E$. And with some abuse of notation, we denote for $t > 0$ by $t_\sharp \mu$ the image of μ under the dilation by t, we have

$$t_\sharp \lambda_E = \lambda_{tE} \quad \text{and} \quad \lambda_{\widetilde{E}} = \frac{1}{|E|^{\frac{1}{n}}} \sharp \lambda_E.$$

Our cost function will depend on the set E. Recall that for a probability measure μ on $\mathbb{R}^n$, its *Cheeger constant* $D_{Che}(\mu)$ is the best (i.e. largest) constant such that the following inequality holds for all Borel sets A:

$$\mu^+(A) \geq D_{Che}(\mu) \min\{\mu(A), 1 - \mu(A)\}$$

where μ^+ denotes the measure of the perimeter (or Minkowski content) associated to μ. It can be defined by:

$$\mu^+(A) := \liminf_{\epsilon \to 0} \frac{\mu(A_\epsilon) - \mu(A)}{\epsilon},$$

where $A_\epsilon = \{x \in \mathbb{R}^n : \operatorname{dist}(x, A) < \epsilon\}$. Equivalently, if we denote by $h_{p,q}(\mu)$ the best nonnegative constant for which the inequality

$$\left(\int |\nabla f(x)|^q \, d\mu(x)\right)^{\frac{1}{q}} \geq h_{p,q}(\mu) \left(\int \left| f(x) - \int f \, d\mu \right|^p d\mu(x)\right)^{\frac{1}{p}}$$

holds for all $f \in W^{1,1}_{\text{loc}} \cap L^p(\mu)$, then $h_{1,1}(\mu) \leq D_{Che}(\mu) \leq 2h_{1,1}(\mu)$. Let $\mathcal{F}$ be the convex, increasing function defined on $\mathbb{R}_+$ by

$$\mathcal{F}(t) := t - \log(1 + t).$$

The function $\mathcal{F}$ behaves like t^2 for t small and like t for t large, and

$$\min\left\{t^2, t\right\} \leq \mathcal{F}(t) \leq 2 \min\left\{t^2, t\right\}, \qquad \forall t \geq 0.$$

This function appears in several mass transport proofs to give a remainder term, to instance in [Fi-Ma-Pr2], [Ba-Ko], [CE-Go].

Given a probability measure μ, we will use the following cost $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ that is also used in [CE]:

$$c_\mu(x, y) := \mathcal{F}\left(D_{\text{Che}}(\mu) \, |y - x|\right) \tag{24}$$

which behaves like $D_{Che}(\mu)^2 \, |y - x|^2$ for small distances, and like $D_{Che}(\mu) \, |y - x|$ for large ones.

Our main result is the following extension of the isoperimetric inequality.

Theorem 2.2. *Let K be a convex body on $\mathbb{R}^n$. Given a Borel set $E \subset \mathbb{R}^n$ with locally Lipschitz boundary and $\int_{\widetilde{E}} x\,dx = \int_{\widetilde{K}} x\,dx$, we have*

$$R(E,K) := \frac{p_K(E)}{n\,|K|^{\frac{1}{n}}\,|E|^{\frac{n}{n-1}}} - 1 \geq \frac{c}{n}\mathcal{W}_{c_{\lambda_{\widetilde{E}}}}\left(\lambda_{\widetilde{E}}, \lambda_{\widetilde{K}}\right), \tag{25}$$

for some universal constant $c > 0$, and as a consequence

$$R(E,K) \geq \frac{c}{n}\mathcal{F}\left(D_{Che}(\lambda_E) W_1(\lambda_E, \lambda_K)\right) \tag{26}$$

We emphasize here a weakness of this result: the remainder term depends on E (on the Cheeger constant of $\widetilde{E}$, precisely). But in some geometric problems, $\widetilde{E}$ will not be too wild : it will belong to a family of sets for which we have a good control on $D_{Che}(\lambda_{\widetilde{E}})$, as we will see later. Since the condition $\int_{\widetilde{E}} xdx = \int_{\widetilde{K}} xdx$ can always be achieved by translating E, we can drop this assumption provided the transportation term is replaced by $\inf_{v\in\mathbb{R}^n} \mathcal{W}_{c_{\lambda_{\widetilde{E}}}}(\tau_v \lambda_{\widetilde{E}}, \lambda_{\widetilde{K}})$ where $\tau_v\nu$ is the image of the measure ν by the translation by v in $\mathbb{R}^n$. Note that we still have equality if $E = \lambda K$ for some $\lambda > 0$.

When E is convex, it is known that $D_{Che}(\lambda_E) > 0$. In this case, we also know that up to numerical constants, $D_{Che}(\lambda_E)$ is the same as $h_{2,2}(\lambda_E)$, the Poincaré constant associated to E (or the inverse of the spectral gap).

Let us compare our results to existing quantitative Sobolev and isoperimetric inequalities, obtained by Figalli-Maggi-Pratelli. In [Fi-Ma-Pr2], there is a quantitative isoperimetric inequality (the numerical constant we use are the improved ones obtained by Segal [Se]):

$$p_K(E) \geq n\,|K|^{\frac{1}{n}}\,|E|^{\frac{1}{n'}}\left(1 + \frac{C}{n^7}A_K(E)^2\right), \tag{27}$$

where $A_K(E) := \inf\left\{\frac{|E\Delta(x_0+rK)|}{|E|} : x_0 \in \mathbb{R}^n, r^n\,|K| = |E|\right\}$ and C is a numerical constant.

This result of Figalli-Maggi-Pratelli is much deeper and in general stronger than ours, since it is universal (the bound does not depend on geometry of E, as in our case). We can note however that the quantity $\frac{C}{n^7}A_K(E)^2$ decreases to 0 when the dimension n goes to $+\infty$.

Actually, there are some particular cases in which our result might give a better bound, both in fixed dimension and when the dimension grows. The reason is that the transportation cost term can be rather large. For instance, we will give examples where our remainder, $\mathcal{F}\left(D_{Che}(\lambda_E) W_1(\lambda_E, \lambda_K)\right)$ decreases slower than $\frac{1}{n^7}$ of the inequality (27).

The rest of the paper is organized as follows. In the next section, we collect some results on optimal transportation theory. Then, we will prove our two Theorems above. In a final section, we will compute our reminder term in several situation of interest arising in convex geometry.

I would like to thank my Professor Dario Cordero-Erausquin for his encouragements, his careful reviews and his many useful discussions.

2.2 Proof of Theorem 2.1

We first give some background about optimal transportation.

2.2.1 Background on optimal transportation

The following Theorem, due to Brenier [Br] and refined then by McCann [Mc1], is the main result in optimal transportation.

Theorem 2.3. *If μ and ν are two probability measures on $\mathbb{R}^n$ and μ absolutely continuous with respect to Lebesgue measure, then there exists a convex function ϕ such that $T = \nabla\phi$ transports μ onto ν. Moreover, T is uniquely determined μ almost-everywhere.*

That means that for every nonnegative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$,

$$\int_{\mathbb{R}^n} b(y)\, d\nu(y) = \int_{\mathbb{R}^n} b(T(x))\, d\mu(x)\,. \tag{28}$$

If μ and ν have densities, say F and G, (28) becomes

$$\int_{\mathbb{R}^n} b(y)\, G(y)\, dy = \int_{\mathbb{R}^n} b(\nabla\phi(x))\, F(x)\, dx. \tag{29}$$

If ϕ is C^2 the change of variables $y = \nabla\phi(x)$ in (29) gives the Monge-Ampère equation, for $F(x)\,dx$ almost-every $x \in \mathbb{R}^n$:

$$F(x) = G(\nabla\phi(x)) \det\left(D^2\phi(x)\right), \tag{30}$$

where $D^2\phi$ is the hessian matrix of ϕ.

Remark 1. When T is the Brenier map between λ_E and λ_K with E and K two convex bodies with same volume, (30) is simpler:

$$\det\left(D^2\phi(x)\right) = 1,$$

for λ_E almost-every $x \in E$.

The question of regularity of ϕ can be asked because, in the previous equality (30), ϕ seemed to be required C^2. In fact, this is not the case, as it was established by McCann [Mc2] that we can give an almost-everywhere sense to (30) by rather standard arguments from measure theory. This almost-everywhere theory is sufficient for most applications, including the one in the present paper but it requires some further arguments that will be discussed later.

2.2.2 Proof of Theorem 2.1: the inequality

Let us recall the frame. Let $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ a nonnegative convex function such that $Z_V = \int_{\mathbb{R}^n} \left(1 + \frac{1}{n-1} V(x)\right)^{-n} dx < +\infty$. So we define the probability measure μ_V by

$$d\mu_V(x) = \frac{1}{Z_V}\left(1 + \frac{1}{n-1} V(x)\right)^{-n} dx.$$

Let $f : \mathbb{R}^n \to \mathbb{R}_+$ a Borel function such that $0 < \int_{\mathbb{R}^n} f^{\frac{n}{n-1}} < +\infty$. So we can define the probability measure μ by

$$d\mu(x) = \frac{f^{\frac{n}{n-1}}(x)}{\int_{\mathbb{R}^n} f^{\frac{n}{n-1}}} dx.$$

Let $T = \nabla\varphi$ the Brenier between μ and μ_V and we start by studying the regularity of φ. It is sufficient to prove the Theorem for measures μ and μ_V whose support is $\mathbb{R}^n$. We also can assume that f is the convolution of a function compactly supported and a mollifier, so that f is smooth and converges rapidly to 0 at $+\infty$. Then, it is known that prove that $\varphi \in W^{2,1}_{loc}(\mathbb{R}^n)$ and the following equality

$$\int_{\mathbb{R}^n} f\,\Delta\varphi = -\int_{\mathbb{R}^n} \nabla f \cdot \nabla\varphi \tag{31}$$

is valid. Let us prove now the first part of Theorem 2.1.

Proof. We first need the following Fact.

Fact 2.1. *[CE-Na-Vi] Let $d\mu(x) = F(x)\,dx$ and $d\nu(y) = G(y)\,dy$ two probability measures on $\mathbb{R}^n$. Let $T = \nabla\varphi$ the Brenier map between μ and ν. Then, the following inequality holds:*

$$\int_{\mathbb{R}^n} G^{1-\frac{1}{n}} \leq \frac{1}{n}\int_{\mathbb{R}^n} F^{1-\frac{1}{n}}\Delta\varphi.$$

Let us give the proof this Fact for completeness.

Proof. We start with Monge-Ampère equation, for μ almost-every $x \in \mathbb{R}^n$, we have:

$$F(x) = G(\nabla\varphi(x)) \det\left(D^2\varphi(x)\right).$$

Then, for μ almost-every $x \in \mathbb{R}^n$ and thanks to arithmetic-geometric inequality:

$$G^{-\frac{1}{n}}(\nabla\varphi(x)) \leq F^{-\frac{1}{n}}(x)\,\frac{\Delta\varphi(x)}{n}. \tag{32}$$

An integration with respect to $d\mu(x) = F(x)\,dx$ gives:

$$\begin{aligned}
\frac{1}{n}\int_{\mathbb{R}^n} F^{1-\frac{1}{n}}(x)\,\Delta\varphi(x)\,dx \quad &\geq \quad \int_{\mathbb{R}^n} G^{-\frac{1}{n}}(\nabla\varphi(x))\,dx \\
&\underbrace{=}_{(29)} \quad \int_{\mathbb{R}^n} G^{1-\frac{1}{n}}(x)\,dx.
\end{aligned}$$

□

If we apply this Fact to our situation, it gives:

$$\int_{\mathbb{R}^n} \frac{\left(1+\frac{1}{n-1}V\right)^{-n+1}}{Z_V^{\frac{1}{n'}}} \leq \frac{1}{n}\int_{\mathbb{R}^n} \frac{f}{\|f\|_{L^{n'}(\mathbb{R}^n)}}\Delta\varphi,$$

and

$$n\,\|f\|_{L^{n'}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\left(1+\frac{1}{n-1}V\right)^{-n+1}}{Z_V^{\frac{1}{n'}}} \leq \int_{\mathbb{R}^n} f\Delta\varphi.$$

As $n\,\|f\|_{L^{n'}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \frac{\left(1+\frac{1}{n-1}V\right)^{-n+1}}{Z_V^{\frac{1}{n'}}} = n\,\|f\|_{L^{n'}(\mathbb{R}^n)} Z_V^{\frac{1}{n}} \int_{\mathbb{R}^n}\left(1+\frac{1}{n-1}V\right)d\mu_V$, we now have the following lines:

$$
\begin{aligned}
n\|f\|_{L^{n'}(\mathbb{R}^n)} Z_V^{\frac{1}{n}} \int_{\mathbb{R}^n} \left(1+\frac{1}{n-1}V\right) d\mu_V &\leq \int_{\mathbb{R}^n} f\,\Delta\varphi \\
&\underbrace{=}_{(31)} \int_{\mathbb{R}^n} (-\nabla f)\cdot\nabla\varphi \\
&= \int_{\mathbb{R}^n} \left(\frac{-\nabla f}{f^{\frac{n}{n-1}}}\cdot\nabla\varphi\right) f^{\frac{n}{n-1}} \\
&\underbrace{\leq}_{(22)} \int_{\mathbb{R}^n} V^*\left(\frac{-\nabla f}{f^{\frac{n}{n-1}}}\right) f^{\frac{n}{n-1}} + \int_{\mathbb{R}^n} V(\nabla\varphi)\, f^{\frac{n}{n-1}} \\
&= \int_{\mathbb{R}^n} V^*\left(\frac{-\nabla f}{f^{\frac{n}{n-1}}}\right) f^{\frac{n}{n-1}} + \left(\int_{\mathbb{R}^n} V\circ T d\mu\right)\left(\int_{\mathbb{R}^n} f^{n'}\right) \\
&= \int_{\mathbb{R}^n} V^*\left(\frac{-\nabla f}{f^{\frac{n}{n-1}}}\right) f^{\frac{n}{n-1}} + \left(\int_{\mathbb{R}^n} V d\mu_V\right)\left(\int_{\mathbb{R}^n} f^{n'}\right).
\end{aligned}
$$

□

2.2.3 Case of equality

In this subsection, we establish that the inequality (23) becomes an equality when $f(x) = \left(1+\frac{1}{n-1}V(x)\right)^{-(n-1)}$ with $V:\mathbb{R}^n \to \mathbb{R}$ a finite convex function. Note that in this case the Brenier map $T=\nabla\varphi$ is $T(x)=x$ so $D^2\varphi = I$ and Monge-Ampère equation is, for μ almost-every $x\in\mathbb{R}^n$,

$$
f^{\frac{n}{n-1}}(x) = \left(1+\frac{1}{n-1}V(T(x))\right)^{-n} \underbrace{\det\left(D^2\varphi(x)\right)}_{=1}. \tag{33}
$$

If we come back to the proof of the inequality (23), we remark that we use only two inequalities: the inequality in Fact 2.1 (which is an arithmetic-geometric inequality) and Young's inequality.

We note that in our case case, the inequality in Fact 2.1 in an equality since $D^2\varphi = I$ so $\det\left(D^2\varphi(x)\right) = \frac{\Delta\varphi(x)}{n}$.
Let us treat now Young's inequality. We have an equality in Young's inequality if (and only if) for μ almost-every $x\in\mathbb{R}^n$

$$
\frac{-\nabla f(x)}{f^{\frac{n}{n-1}}(x)} = \nabla V(T(x)) = \nabla V(x). \tag{34}
$$

To get this equality, let us take the $-\frac{1}{n}$ power in (33) to get

$$f^{\frac{1}{n-1}}(x) = 1 + \frac{1}{n-1} V(T(x)) = 1 + \frac{1}{n-1} V(x).$$

If we compute the gradient of the previous line, we find (34).

Remark 2. One can prove that (23) is an equality *if and only if* $f(x) = \left(1 + \frac{1}{n-1} V(x-a)\right)^{-(n-1)}$ with $a \in \mathbb{R}^n$. We decided not to prove this because it is technical. Let us speak about this. If we have an equality in (23), we have an equality in the inequality in Fact 2.1 and an equality in Young's inequality.

An equality in Fact 2.1 means that $\det\left(D^2\varphi(x)\right) = \frac{\Delta\varphi(x)}{n}$ so the matrix $D^2\varphi(x)$ has only one eigenvalue, say $\lambda(x)$ and $D^2\varphi(x) = \lambda(x) I$. The main difficulty is to show that the function λ is constant, which is the case when φ is C^2 smooth (the details are analyzed in [CE-Na-Vi]). If we assume that, it is easy to conclude that, up to translations, $\mu = \mu_V$.

2.3 Proof of Theorem 2.2

Here we establish Theorem 2.2. It was noted by Figalli, Maggi and Pratelli [Fi-Ma-Pr2] (and Segal [Se]) that for this kind of result, the general situation follows from the case the two bodies have same volume, equal to one. For completeness, let us recall the argument. Let E Borel set and K a convex body in $\mathbb{R}^n$. The following Lemma establishes a link between $R(E,K) = \frac{p_K(E)}{n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}}} - 1$ and $R\left(\widetilde{E}, \widetilde{K}\right) = \frac{p_{\widetilde{K}}\left(\widetilde{E}\right)}{n\left|\widetilde{K}\right|^{\frac{1}{n}}\left|\widetilde{E}\right|^{\frac{1}{n'}}} - 1$ where $\widetilde{E}$ and $\widetilde{K}$ are respectively $\frac{E}{|E|^{\frac{1}{n}}}$ et $\frac{K}{|K|^{\frac{1}{n}}}$.

Lemma 2.1. *With the previous notations,*

$$R(E,K) = \frac{p_K(E)}{n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}}} - 1 = \frac{p_{\widetilde{K}}\left(\widetilde{E}\right)}{n\left|\widetilde{K}\right|^{\frac{1}{n}}\left|\widetilde{E}\right|^{\frac{1}{n'}}} - 1 = R\left(\widetilde{E}, \widetilde{K}\right).$$

Proof. Let us note that for all $\epsilon > 0$, we have

$$\frac{|E + \epsilon K| - |E|}{\epsilon} = |E|^{\frac{1}{n}} |K|^{\frac{1}{n'}} \frac{\left|\widetilde{E} + \epsilon \frac{|K|^{\frac{1}{n}}}{|E|^{\frac{1}{n}}} \widetilde{K}\right| - \left|\widetilde{E}\right|}{\epsilon \frac{|K|^{\frac{1}{n}}}{|E|^{\frac{1}{n}}}}.$$

By taking the limit, we get the equality. □

Therefore, if we have established Theorem 2.2 for two sets of volume one, we have the general statement by applying it to $\widetilde{E}$ and $\widetilde{K}$. So in the rest of this section, E is a Borel set with smooth boundary and K a convex body, both with volume one, $|E| = |K| = 1$.

As in [Fi-Ma-Pr2] and [Se], the argument to establish Theorem 2.2 starts with optimal transportation. The following Lemma gives a first minimization for the deficit $R(E,K) = \frac{p_K(E)}{n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}}} - 1$.

Lemma 2.2. *[Fi-Ma-Pr2] Let E and K two convex bodies in $\mathbb{R}^n$ with same measure 1. Let $T = \nabla\phi$ the Brenier map between the measures λ_E and λ_K. We note by $0 < \lambda_1 \leq \cdots \leq \lambda_n$ the eigenvalues of $D^2\phi$. Then, we have the following inequality*

$$R(E,K) \geq \int_{\mathbb{R}^n} (\lambda_A - \lambda_G)\, d\lambda_E, \tag{35}$$

where $\lambda_A = \frac{\lambda_1 + \cdots + \lambda_n}{n}$ and $\lambda_G = \Pi_{i=1}^n \lambda_i^{\frac{1}{n}}$.

Before briefly recalling the proof of this Lemma, let us speak about the regularity of the optimal transport. It is known, see [Ca1, Ca2], that when $T = \nabla\phi$ is the Brenier map between $d\mu(x) = f(x)\,dx$ and $d\nu(y) = g(y)\,dy$ two probability measures supported on two open bounded sets, respectively E and K, with f and g are α-Hölder, bounded and with $\frac{1}{f}$ and $\frac{1}{g}$ bounded too, then $\phi \in C^{2,\beta}(E)$ for all $0 < \beta < \alpha$.

Proof. As $|E| = |K| = 1$, we can write

$$n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}} = \int_{\mathbb{R}^n} n\left(\det\left(D^2\phi\right)\right)^{\frac{1}{n}} d\lambda_E = \int_E n\left(\det\left(D^2\phi\right)\right)^{\frac{1}{n}},$$

because $\det\left(D^2\phi\right) = 1$ thanks to the Remark 1. The arithmetic-geometric inequality gives

$$n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}} \leq \int_E \operatorname{div} T(x)\,dx.$$

The divergence theorem provides

$$\int_E \operatorname{div} T(x)\,dx = \int_{\partial E} T(x)\cdot\nu_E(x)\,d\mathcal{H}^{n-1}(x),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Haussdorf measure. By definition of the support function h_K, of K, and since $T(E) \subseteq K$ we therefore get

$$n|K|^{\frac{1}{n}}|E|^{\frac{1}{n'}} \leq \int_E \operatorname{div}T(x)\,dx \leq \int_{\partial E} h_K(\nu_E(x))\,d\mathcal{H}^{n-1}(x) = p_K(E).$$

Thus

$$\frac{p_K(E)}{n|K|^{\frac{1}{n'}}|E|^{\frac{1}{n}}} - 1 \geq \int_{\mathbb{R}^n} \left(\frac{\text{div}T}{n} - 1\right) d\lambda_E \geq \int_{\mathbb{R}^n} (\lambda_A - \lambda_G)\, d\lambda_E.$$

□

To go on, we need a quantitative version of the arithmetic-geometric inequality. The following result is due to Alzer [Al].

Lemma 2.3. *[Al] Let* $0 < \lambda_1 \leq \cdots \leq \lambda_n$. *Let* $\lambda_A = \frac{\lambda_1 + \cdots + \lambda_n}{n}$ *and* $\lambda_G = \Pi_{i=1}^n \lambda_i^{\frac{1}{n}}$. *We have*

$$\sum_{i=1}^n (\lambda_i - \lambda_G)^2 \leq 2n\lambda_n (\lambda_A - \lambda_G). \tag{36}$$

We can now complete the proof of Theorem . If $T = \nabla\varphi$ is the Brenier map between λ_E and λ_K for two bodies E and K of volume 1. With the previous notations, if we use (35) and (36), we get

$$(\lambda_A - \lambda_G) \geq \frac{1}{2n}\frac{\|D^2\phi - \text{Id}\|_{\text{HS}}^2}{\lambda_n} \geq \frac{1}{2n}\frac{\|D^2\phi - \text{Id}\|_{\text{HS}}^2}{1 + \|D^2\phi - \text{Id}\|_{\text{HS}}} \geq \frac{c}{n}\text{tr}\left(\mathcal{F}\left(\|D^2\theta\|_{\text{HS}}\right)\right),$$

where $\theta(x) = \phi(x) - \frac{|x|^2}{2}$ and $\|\cdot\|_{\text{HS}}$ refers to the Hilbert-Schmidt norm of a $n \times n$ matrix. Let us remark that $\lambda_G = 1$, thanks to, once again, Remark 1. So we have

$$R(E,K) \geq \frac{c}{n}\int_{\mathbb{R}^n} \text{tr}\left(\mathcal{F}\left(\|D^2\theta\|_{\text{HS}}\right)\right). \tag{37}$$

The treatment of this term is stated in the next Lemma and we refer to [CE].

Lemma 2.4. *[CE] Let* μ *a probability measure on* $\mathbb{R}^n$ *absolutely continuous with respect to the Lebesgue measure and* $\theta \in W^{2,1}_{loc}(\mathbb{R}^n)$ *with* $D^2\theta + Id \geq 0$ *almost-everywhere. We assume* $|\nabla\theta| \in L^1(\mu)$ *and* $\int_{\mathbb{R}^n} \nabla\theta\, d\mu = 0$. *Then,*

$$\int_{\mathbb{R}^n} \text{tr}\left(\mathcal{F}\left(D^2\theta\right)\right) d\mu \geq c\int_{\mathbb{R}^n} \mathcal{F}\left(D_{Che}(\mu)\,|\nabla\theta|\right)\, d\mu,$$

for some numerical constant $c > 0$.

Note that our assumption $\int_E x\,dx = \int_K x\,dx$ rewrites as $\int_E \nabla\theta = 0$, so if we use the previous Lemma with $\mu = \lambda_E$, in (37) we find

$$\begin{aligned} R(E,K) &\geq \frac{c}{n}\int_{\mathbb{R}^n} \mathcal{F}\left(D_{Che}|\nabla\theta|\right) d\lambda_E \\ &\geq \frac{c}{n}\mathcal{W}_{c_{\lambda_E}}(\lambda_E, \lambda_K). \end{aligned}$$

2.4 Some examples

Here we give some examples where our result (i.e. Theorem 2.2) gives good bounds for the remainder term, better than the one in [Fi-Ma-Pr2]. We will give an example in dimension 2 where our remainder term, depending on a parameter, can be as large as we want and an example in dimension n. We recall that the remainder term in (27) is bounded by 1 when E and K have for measure 1 and decreases to 0 with $\frac{1}{n^7}$ when the dimension n grows.

2.4.1 In dimension 2

In this section, we give a toy example in dimension 2. Let, for $\alpha > 0$, $E_\alpha = \left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right] \times \left[-\frac{1}{2\alpha}, \frac{1}{2\alpha}\right]$ and $K_\alpha = \left[-\frac{\alpha^2}{2}, \frac{\alpha^2}{2}\right] \times \left[-\frac{1}{2\alpha^2}, \frac{1}{2\alpha^2}\right]$. We will prove the following:

Proposition 2.1. *With the previous notations, we have:*

$$\lim_{\alpha \to +\infty} D_{Che}\left(\lambda_{E_\alpha}\right) W_1\left(\lambda_{E_\alpha}, \lambda_{K_\alpha}\right) = +\infty.$$

Proof. As $W_1\left(\lambda_{E_\alpha}, \lambda_{K_\alpha}\right) = W_1\left(\lambda_{K_\alpha}, \lambda_{E_\alpha}\right)$, we give an estimation of the last term. Let $T = \nabla\phi$ the Brenier map which transports the measure λ_{K_α} onto the measure λ_{E_α} (by Monge-Ampère equation, it verifies det $D^2\phi = 1$, in particular it preserves the volume). Let $K'_\alpha = \left[\frac{\alpha^2}{4}, \frac{\alpha^2}{2}\right] \times \left[-\frac{1}{2\alpha^2}, \frac{1}{2\alpha^2}\right]$. Then, we have:

$$W_1\left(\lambda_{E_\alpha}, \lambda_{K_\alpha}\right) = \int_{\mathbb{R}^n} |T(x) - x|\, d\lambda_{K_\alpha}(x) \geq \int_{K'_\alpha} |T(x) - x|\, dx \geq \left|K'_\alpha\right| \operatorname{dist}\left(K'_\alpha, E_\alpha\right).$$

So, we have

$$W_1\left(\lambda_{E_\alpha}, \lambda_{K_\alpha}\right) \geq \frac{1}{4}\left(\frac{\alpha^2}{4} - \frac{\alpha}{2}\right).$$

□

We need an estimation of $D_{Che}\left(\lambda_{E_\alpha}\right)$. This constant could be computed explicitly but it is rather easier to compare this constant to the Poincaré constant $h_{2,2}\left(\lambda_{E_\alpha}\right)$. Indeed, it is known that, up to numerical constants, that $D_{Che}\left(\lambda_{E_\alpha}\right)$ is the same as $h_{2,2}\left(\lambda_{E_\alpha}\right)$, see [Le] and [Mi1]. As $\lambda_{E_\alpha} = \lambda_{\left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]} \otimes \lambda_{\left[-\frac{1}{2\alpha}, \frac{1}{2\alpha}\right]}$, then $h_{2,2}\left(E_\alpha\right) \geq \min\left\{h_{2,2}\left(\lambda_{\left[-\frac{\alpha}{2}, \frac{\alpha}{2}\right]}\right), h_{2,2}\left(\lambda_{\left[-\frac{1}{2\alpha}, \frac{1}{2\alpha}\right]}\right)\right\}$, see [Bo-Ho2] and [Bo7]. Since $h_{2,2}\left(\lambda_{[-a,a]}\right) = \frac{\pi}{a}$ for $a > 0$, it follows with Theorem 2.2 that:

$$R\left(E_{\alpha}, K_{\alpha}\right) \geq \frac{1}{4}\mathcal{F}\left(\frac{c\pi}{2\alpha}\left(\frac{\alpha^2}{4}-\frac{\lambda}{2}\right)\right),$$

for some numerical constant $c > 0$. In particular the remainder term $R\left(E_{\alpha}, K_{\alpha}\right)$ is not bounded when α grows to $+\infty$ whereas the remainder in (27) remains bounded.

2.4.2 Estimation of $W_1\left(\lambda_K, \lambda_L\right)$ for K and L isotropic convex bodies

Here K and L are two convex bodies with measure 1. We say that a convex body K is in isotropic position if $|K| = 1$, it is centered and there exists $\alpha > 0$, such that

$$\int_K |x \cdot y|^2 \, dx = \alpha \, |y|^2, \qquad \forall y \in \mathbb{R}^n.$$

For an isotropic convex body K, we define its isotropic constant $L_K (= \sqrt{\alpha})$ by:

$$L_K^2 = \frac{1}{n}\int_K |x|^2 \, dx.$$

We also define for any convex isotropic body K

$$M\left(K\right) = \frac{1}{\sqrt{n}}\int_K |x| \, dx.$$

Using Hölder inequality and Borell deviation inequality [Bor1, Bor2], we have:

$$cL\left(K\right) \leq M\left(K\right) \leq L\left(K\right),$$

for some numerical constant $c > 0$. For backgrounds, we refer to [Br-Gi-Va-Vr]. Our goal is here is to prove the following Proposition.

Proposition 2.2. *Let K and L two convex bodies of volume 1 in isotropic position. Then, the following estimation for $W_1\left(\lambda_K, \lambda_L\right)$ holds:*

$$\sqrt{n}\left|M\left(K\right) - M\left(L\right)\right| \leq W_1\left(\lambda_K, \lambda_L\right) \leq c\left(L_K + L_L\right)\sqrt{n} + 8, \tag{38}$$

for some numerical constant $c > 0$. In connection with some isoperimetric estimates, we are mainly interested with lower bounds.

We are mainly interested in the lower bound provided by this Proposition. The upper bound (which is far from sharp when K are L are closed to each other) is stated only to emphasize that the generic expected order of magnitude is $\sqrt{n}$ on both sides.

Proof. We first work on the left hand-side of (38). Let us recall the dual of $W_1(\lambda_K, \lambda_L)$, known as Kantorovich's duality:

$$W_1(\mu, \nu) = \sup_{\phi\, 1-\mathrm{Lip}} \left\{ \int_{\mathbb{R}^n} \phi\, d\mu - \int_{\mathbb{R}^n} \phi\, d\nu \right\}. \tag{39}$$

If we take in (39), $\phi(x) = |x|$ or $-|x|$, we have:

$$\begin{aligned} W_1(\lambda_K, \lambda_L) &\geq \left| \int_{\mathbb{R}^n} |x|\, d\lambda_K(x) - \int_{\mathbb{R}^n} |x|\, d\lambda_L(x) \right| \\ &= |M(K) - M(L)|. \end{aligned}$$

Let us treat now the right hand-side of (38). Let T be a transport map (in particular, it verifies $\det(\nabla T) = 1$) which transports the measure λ_K onto the measure λ_L, so $W_1(\lambda_K, \lambda_L) \leq \int_{\mathbb{R}^n} |T(x) - x|\, d\lambda_K(x) = \int_K |T(x) - x|\, dx$. Let us recall a deep result of Paouris, see [Pa].

Theorem 2.4. *[Pa] There exists a numerical constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\left| \left\{ x \in K : |x| \geq c\sqrt{n} L_K t \right\} \right| \leq \exp\left(-\sqrt{n} t\right), \qquad \forall t \geq 1. \tag{40}$$

Let $t \geq 1$ such that $\exp(-\sqrt{n}t) \leq \frac{1}{n}$ (note that $t = 1$ works, we will set this value for t) so the following sets $K_1 = \{x \in K : |x| < c\sqrt{n} L_K\}$ and $L_1 = \{x \in L : |x| < c\sqrt{n} L_L\}$ have their volumes bigger than $1 - \frac{1}{n}$. Finally, let $K_2 = T^{-1}(L_1)$. Since $\det(\nabla T) = 1$, we have $|K_2| = |L_1|$. We can now conclude thanks to the following inequality:

$$W_1(\lambda_K, \lambda_L) \leq \int_K |T(x) - x|\, dx = \int_{K_1 \cap K_2} |T(x) - x|\, dx + \int_{K \setminus (K_1 \cap K_2)} |T(x) - x|\, dx. \tag{41}$$

Let us estimate the two remaining integrals. Thanks to the definitions of K_1 and K_2, we have: $\int_{K_1 \cap K_2} |T(x) - x|\, dx \leq c(L_K + L_L)\sqrt{n}$. For the second, we need the fact that $K, L \subseteq B\left(0, \sqrt{n(n+2)}\right) \subseteq B(0, 2n)$, see [Ka-Lo-Si]. Then, for $x \in K \setminus (K_1 \cap K_2)$, we have $|T(x) - x| \leq 4n$ and $|K \setminus (K_1 \cap K_2)| \leq \frac{2}{n}$, that gives $\int_{K \setminus (K_1 \cap K_2)} |T(x) - x|\, dx \leq 8$. Going back to (41), we finally have:

$$W_1(\lambda_K, \lambda_L) \leq \int_K |T(x) - x|\, dx \leq c(L_K + L_L)\sqrt{n} + 8.$$

□

3 Dimensional transport inequalities and Brascamp-Lieb inequalities

3.1 Introduction

We shall begin by recalling Borell's terminology [Bor1, Bor2] about convex measures. Although we will not use explicitly Borell's results, it allows to explain the values and internal relations between the parameters appearing in our study.

Let $\alpha \in [-\infty, +\infty]$. A Radon probability measure μ on $\mathbb{R}^n$ (or on an open convex set $\Omega \subseteq \mathbb{R}^n$) is called α-concave, if it satisfies

$$\mu\left(tA + (1-t)B\right) \geq \left(t\mu\left(A\right)^{\alpha} + (1-t)\,\mu\left(B\right)^{\alpha}\right)^{\frac{1}{\alpha}}, \tag{42}$$

for all $t \in (0,1)$ and for all Borel sets $A, B \subset \mathbb{R}^n$. When $\alpha = 0$, the right-hand side of (42) is understood as $\mu\left(A\right)^{t}\mu\left(B\right)^{1-t}$: μ is a log-concave measure. When $\alpha = -\infty$, the right-hand side is understood as $\min\left\{\mu\left(A\right), \mu\left(B\right)\right\}$ and when $\alpha = +\infty$ as $\max\left\{\mu\left(A\right), \mu\left(B\right)\right\}$. We remark that the inequality (42) is getting stronger when α increases, so the case $\alpha = -\infty$ describes the largest class whose members are called convex or hyperbolic probability measures. In [Bor1, Bor2], Borell proved that a measure μ on $\mathbb{R}^n$ absolutely continuous with respect to the Lebesgue measure is α-concave (and verifies (42)) if and only if $\alpha \leq \frac{1}{n}$ and μ is supported on some open convex subset $\Omega \subseteq \mathbb{R}^n$ where it has a nonnegative density p which satisfies, for all $t \in (0,1)$,

$$p\left(tx + (1-t)\,y\right) \geq \left(tp\left(x\right)^{\alpha_n} + (1-t)\,p\left(y\right)^{\alpha_n}\right)^{\frac{1}{\alpha_n}}, \qquad \forall x, y \in \Omega, \tag{43}$$

where $\alpha_n := \frac{\alpha}{1-n\alpha} \in \left[-\frac{1}{n}, +\infty\right]$. Note that this amounts to the concavity of $\alpha_n p^{\alpha_n}$. In particular, μ is log-concave if and only if it has a log-concave density ($\alpha = \alpha_n = 0$).

We shall focus on the densities rather than on the measures, so let us reverse the perspective. We are given $\kappa > -1$, or more precisely,

$$\kappa \in \left[-\frac{1}{n}, +\infty\right] = \left[-\frac{1}{n}, 0\right[\cup \{0\} \cup \left]0, +\infty\right[, \tag{44}$$

and a probability density ρ on $\mathbb{R}^n$ (by this we mean a nonnegative Borel function with $\int \rho = 1$), with the property that $\kappa\rho^{\kappa}$ is concave on its support. Borell's result then tells us that the (probability) measure with density ρ is $\kappa(n) := \frac{\kappa}{1+n\kappa}$-concave measure on $\mathbb{R}^n$. This suggests two different behaviors depending on the sign of κ since $\rho^{\kappa(n)}$ is convex or concave. Let us describe them.

Case 1
This corresponds to $\kappa \in]0,+\infty]$ (that is, for measures, $0 < \kappa(n) \le \frac{1}{n}$). We set $\beta := \frac{1}{\kappa} \in [0,+\infty)$ and we work with densities of the form $\rho_\beta(x) = \frac{W(x)^\beta}{\int_\Omega W^\beta}$ where $W : \mathbb{R}^n \to \mathbb{R}_+$ is concave on its support. Note that the measure is supported on $\Omega = \{W > 0\} \subset \mathbb{R}^n$, which is an open bounded convex set. The typical examples are the measures defined by

$$d\tau_{\sigma,\beta}(x) = \frac{1}{C_{\sigma,\beta}} \left(\sigma^2 - |x|^2\right)_+^\beta dx, \qquad \beta > 0, \sigma > 0,$$

where $C_{\sigma,\beta} = \int_{\mathbb{R}^n} \left(\sigma^2 - |x|^2\right)_+^\beta dx = \sigma^{2\beta+n}\pi^{\frac{n}{2}} \frac{\Gamma(\beta+1)}{\Gamma\left(\beta+\frac{n}{2}+1\right)}$ is a normalizing constant.
Case 2
This corresponds to $\kappa \in [-\frac{1}{n}, 0[$ (that is, for measures, $\kappa(n) \le 0$). We set $\beta := -\frac{1}{\kappa} = n - \frac{1}{\kappa(n)} \ge n$ and we work with densities of the form $\rho_\beta(x) = \frac{W(x)^{-\beta}}{\int_\Omega W^{-\beta}}$ where $W : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is a convex function. Note that the support of the measure is given by the convex set $\{W < +\infty\}$. The typical examples are the (generalized) Cauchy probability measures defined by

$$d\mu_\beta(x) = \frac{1}{C_\beta} \left(1 + |x|^2\right)^{-\beta} dx, \qquad \beta > \frac{n}{2},$$

where $C_\beta = \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{-\beta} dx = \pi^{\frac{n}{2}} \frac{\Gamma\left(\beta-\frac{n}{2}\right)}{\Gamma(\beta)}$ is a normalizing constant.

In the sequel, we shall adopt the following unified notation. Given κ as in (44), we consider a nonnegative function $W : \mathbb{R}^n \to \mathbb{R}^+$ with the convention that

$$\begin{cases} \text{when } \kappa > 0, & W \text{ is concave on the bounded open convex set } \{W > 0\} \\ \text{when } \kappa < 0, & W \text{ is convex on } \mathbb{R}^n, \end{cases}$$

with the property that

$$\int W^{1/\kappa} < +\infty;$$

we then define the density

$$\rho_{\kappa,W}(x) = \frac{1}{\int W^{1/\kappa}} W^{1/\kappa}(x). \tag{45}$$

Our first goal is to study generalized transport inequalities for these probability measures (which we identify with the density).

Let μ a probability measure on $\mathbb{R}^n$, we recall that a transport inequality is an inequality of the form

$$\alpha\left(\mathcal{W}_c(\mu,\cdot)\right) \le H(\cdot\|\mu),$$

where α is an increasing function on $[0,+\infty)$ with $\alpha(0)=0$, $\mathcal{W}_c(\mu,\cdot)$ is the Kantorovich distance from μ and $H(\cdot\|\mu)$ a relative entropy with respect to μ. Let us recall that given a cost function $c:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$, the Kantorovich distance $\mathcal{W}_c(\mu,\nu)$ between two probability measures μ and ν on $\mathbb{R}^n$ is defined by

$$\mathcal{W}_c(\mu,\nu)=\inf_\pi\iint_{\mathbb{R}^n\times\mathbb{R}^n}c(x,y)\,d\pi(x,y)$$

where the infimum is taken over all probability measures π on $\mathbb{R}^n\times\mathbb{R}^n$ projecting on μ and ν respectively. In case where $c(x,y)=|x-y|^p$, with $p\geq 1$, we note

$$\mathcal{W}_c(\mu,\nu)=W_p^p(\mu,\nu).$$

The relative entropy is defined as follows.

Definition 3.1 (Entropy)**.** *Let κ, W and $\rho_{\kappa,W}$ be given as in the paragraph before* (45)*. Given a probability density ρ on $\mathbb{R}^n$ we introduce the (κ,W)-entropy*

$$H_{\kappa,W}(\rho):=\frac{1}{\kappa}\int\left(\rho^{1+\kappa}-\rho\right)+\frac{\kappa+1}{-\kappa}\int\rho W$$

provided the integrands are integrable (we set $H_{\kappa,W}(\rho)=+\infty$ otherwise). The relative entropy is then defined by

$$\begin{aligned}H_{\kappa,W}(\rho||\rho_{\kappa,W}) &:= H_{\kappa,W}(\rho)-H_{\kappa,W}(\rho_{\kappa,W})\\ &= \frac{1}{\kappa}\int\left(\rho^{\kappa+1}-(\kappa+1)\rho W\right)+\int W^{1+1/\kappa}.\end{aligned}$$

The reader can convince himself that the functional $\rho\to H_{\kappa,W}(\rho)$ is convex in ρ (note the role played by the sign of $\kappa>-1$) and that $H_{\kappa,W}(\rho||\rho_{\kappa,W})\geq 0$. Let us emphasize that the log-concave case corresponds to the case $\kappa\to 0$. We can approximate it from above or from below. For instance, given a convex function V with $V\to+\infty$ at infinity, if we set, for $\kappa<0$ close to zero with $W(x)=W_\kappa(x)=(1-\kappa V(x))_+$ then, as $\kappa\to 0^-$,

$$\rho_{\kappa,W_\kappa}\to\rho_V:=\frac{1}{\int e^{-V}}\,e^{-V}$$

and $H_{\kappa,W_\kappa}(\rho||\rho_{\kappa,W_\kappa})\to\int\log\left(\frac{\rho}{\rho_V}\right)\rho$, the classical relative entropy of ρ with respect to ρ_V.

Generalized transport inequalities have been studied in [CE-G-H] in order to study quasilinear parabolic-elliptic equations, under some uniform convexity assumption. The following result can be seen as a dimensional form of the transport inequality for log-concave measures stated in [CE] that goes back to earlier work by Bobkov and Ledoux. The cost is defined by

$$c_{\kappa,W}(x,y) = \frac{\kappa+1}{-\kappa}\Big[W(y) - W(x) - \nabla W(x)\cdot(y-x)\Big]. \tag{46}$$

According to the context recalled before (45), note that $c_{\kappa,W}(x,y) \geq 0$, with $c_{\kappa,W}(x,x) = 0$. This cost is actually mainly independent of κ, which is there only to distinguish between convex ($\kappa < 0$) and concave ($\kappa > 0$) situations. The general transport inequality is as follows.

Theorem 3.1. *Let κ, W and $\rho_{\kappa,W}$ be given as in the paragraph before* (45)*. Then we have the following transport inequality, for the entropy and cost defined above: for any probability density ρ on $\mathbb{R}^n$,*

$$W_{c_{\kappa,W}}(\rho_{\kappa,W},\rho) \leq H_{\kappa,W}(\rho||\rho_{\kappa,W}) \tag{47}$$

According to the discussion above, when $W(x) = W_\kappa(x) = (1+\kappa V(x))_+$ and $\kappa \to 0^-$, the transport inequality recalled in [CE] for $\rho_V := e^{-V}$, namely

$$W_{c_V}(\rho_V,\rho) \leq H_V(\rho||\rho_V),$$

is recovered, for the cost $c_V(x,y) = V(y) - V(x) - \nabla V(y)\cdot(y-x)$ and the relative entropy $H_V(\rho||\rho_V) = \int \log\left(\frac{\rho}{\rho_V}\right)\rho$.

The previous inequality is therefore not surprising, and it requires only a minor work to extract it from [CE-G-H].

Interestingly enough, we will show that the previous inequality allows to reproduce, by a linearization procedure, the dimensional Brascamp-Lieb inequalities obtained by Bobkov and Ledoux [Bo-Le1] and Nguyen [Ng] thus providing a mass transport approach to them.

Our second goal is to obtain quantitative versions of the transport inequality above. Before announcing our results, we need some notation. Let the function $\mathcal{F}$ be defined on $\mathbb{R}_+$ by

$$\mathcal{F}(t) := t - \log(1+t), \qquad \forall t \geq 0.$$

The function $\mathcal{F}$ is an increasing, convex function on $\mathbb{R}_+$ and it behaves like t^2 when t is small and like t when t is large, more precisely:

$$\frac{1}{4}\min\left\{t,t^2\right\} \leq \mathcal{F}(t) \leq \min\left\{t,t^2\right\}, \qquad \forall t \geq 0.$$

We introduce next a (weighted) isoperimetric type constant. Given a probability measure μ, we denote by $h_W(\mu)$ the best nonnegative constant such that the following inequality

$$\int \mathcal{F}(|\nabla f|)\, W d\mu \geq \int \mathcal{F}\left(h_W(\mu)\left|f - m_f\right|\right) d\mu \tag{48}$$

holds for every smooth enough function $f \in L^1(\mu)$. One may hope that $h_W(\rho_{\kappa,W}) > 0$. We shall briefly discuss this in the last section.

It may be convenient to change the notation and focus rather on the parameter

$$\beta = \pm\frac{1}{\kappa}$$

according to the **Case 1** and **Case 2** detailed previously. With some abuse of notation, we will denote $\rho_{\beta,W}$, $H_{\beta,W}$ and $c_{\beta,W}$ the corresponding quantities.

So, more explicitly, in **Case 1**, which corresponds to $\kappa > 0$, we are given a $\beta \in [0, +\infty[$ and function $W : \mathbb{R}^n \to \mathbb{R}^+$ concave on its support such that

$$\rho_{\beta,W}(x) := \frac{W(x)^\beta}{\int W^\beta}$$

is a probability density. In this **Case 1** the cost is $c_{\beta,W}(x,y) = (\beta+1)\Big(W(x) - W(y) + \nabla W(x)\cdot(y-x)\Big)$ and the relative entropy $H_{\beta,W}(\rho\|\rho_{\beta,W}) := \int\Big(\beta\rho^{1+1/\beta} - (\beta+1)\rho W\Big) + \int W^{\beta+1}$.

Theorem 3.2. *Under the notation of **Case 1** recalled above, introduce the costs*

$$\widetilde{c}(x,y) = c\mathcal{F}\left(h_W(\rho_{\beta,W})\ |y-x|\right)$$

where $c > 0$ is some fixed numerical constant, and

$$c(x,y) = c_{\beta,W}(x,y) + \widetilde{c}(x,y).$$

Then, if ρ and $\rho_{\beta,W}$ have the same center of mass, we have

$$H_{\beta,W}(\rho||\rho_{\beta,W}) \geq \mathcal{W}_c(\rho_{\beta,W}, \rho). \tag{49}$$

Remark 3. Since the inequality $\mathcal{W}_c(\rho, \rho_{\beta,W}) = \mathcal{W}_{c_{\beta,W}+\widetilde{c}}(\rho, \rho_{\beta,W}) \geq \mathcal{W}_{c_{\beta,W}}(\rho, \rho_{\beta,W}) + \mathcal{W}_{\widetilde{c}}(\rho, \rho_{\beta,W})$ holds, the second transport inequality gives a remainder term for the first inequality:

$$H_{\beta,W}(\rho\|\rho_{\beta,W}) - \mathcal{W}_{c_{\beta,W}}(\rho, \rho_{\beta,W}) \geq \mathcal{W}_{\widetilde{c}}(\rho, \rho_{\beta,W}). \tag{50}$$

In **Case 2**, which corresponds to $\kappa \in \left[-\frac{1}{n}, 0\right[$, we are given a $\beta \geq n$ and function $W : \mathbb{R}^n \to \mathbb{R}^+$ convex such that

$$\rho_{\beta,W}(x) := \frac{W(x)^{-\beta}}{\int W^{-\beta}}$$

is a probability density. In this **Case 2** the cost is $c_{\beta,W}(x,y) = (\beta-1)\Big(W(y) - W(x) - \nabla W(x)\cdot(y-x)\Big)$ and the relative entropy $H_{\beta,W}(\rho\|\rho_{\beta,W}) := \int \Big((\beta-1)\,\rho W - \beta\rho^{1-1/\beta}\Big) + \int W^{1-\beta}$.

Theorem 3.3. *Under the notation of **Case 2** recalled above, introduce the costs*

$$\widetilde{c}(x,y) = \frac{c}{\beta}\left(1-\frac{n}{\beta}\right)^2 \mathcal{F}\left(h_W(\rho_{\beta,W})\ |y-x|\right)$$

where $c>0$ is some fixed numerical constant, and

$$c(x,y) = c_{\beta,W}(x,y) + \widetilde{c}(x,y).$$

Then, if ρ and $\rho_{\beta,W}$ have the same center of mass, we have

$$H_{\beta,W}(\rho||\rho_{\beta,W}) \geq \mathcal{W}_c(\rho_{\beta,W}, \rho). \tag{51}$$

Remark 4. As in the **Case 1**, this gives a remainder term for the first transport inequality:

$$H_\beta(\rho) - \mathcal{W}_{c_{\beta,W}}(\rho, \rho_{\beta,W}) \geq \mathcal{W}_{\widetilde{c}}(\rho, \rho_{\beta,W}). \tag{52}$$

The idea of the proof is to transport the densities ρ onto the measure $\rho_{\beta,W}$. Cordero in [CE] uses optimal transportation to obtain a transport inequality for log-concave measures. We recall some backgrounds about mass transportation at the beginning of the following section but we refer to [Vi] for a detailed approach.

In a second section, we will use transport inequalities to retrieve some dimensional versions of Brascamp-Lieb inequalities. Such inequalities had already been studied by Bobkov and Ledoux in [Bo-Le1] where they use a Prékopa-Leindler type inequality. More recently, Nguyen in [Ng] retrieve these inequalities with a L^2- Hörmander method. Our approach is different. From transport inequalities (Theorems 3.1 and 3.2), we will use a linearization procedure to retrieve these inequalities.

I would like to thank my Professor Dario Cordero-Erausquin for his encouragements, his careful reviews and his many useful discussions.

3.2 Proof of Theorem 3.1

In this part, we do not use the notation β because there it is useless to separate the proof between **Case 1** and **Case 2**. We can assume that $\int W^{1/\kappa} = 1$. The proof is based on optimal transportation. Let us recall briefly what it is about. Let two probability measures μ and ν on $\mathbb{R}^n$. We say a map $T : \mathbb{R}^n \to \mathbb{R}^n$ transports the measure μ onto the measure ν if:

$$\nu(B) = \mu\left(T^{-1}(B)\right), \qquad \text{for all borelian sets } B \subseteq \mathbb{R}^n.$$

This gives a transport equation: for all nonnegative Borel function $b : \mathbb{R}^n \to \mathbb{R}_+$,

$$\int_{\mathbb{R}^n} b(y)\, d\nu(y) = \int_{\mathbb{R}^n} b(T(x))\, d\mu(x). \tag{53}$$

When μ and ν have densities with respect to Lebesgue measure (it will be always the case in this paper), say F and G, (53) becomes:

$$\int_{\mathbb{R}^n} b(y)\, G(y)\, dy = \int_{\mathbb{R}^n} b(T(x))\, F(x)\, dx. \tag{54}$$

The existence of a such map T is resolved by the following Theorem of Brenier [Br] and refined by McCann [Mc1].

Theorem 3.4. *If μ and ν are two probability measures on $\mathbb{R}^n$ and μ is absolutely continuous with respect to Lebesgue measure, then there exists a convex function φ defined on $\mathbb{R}^n$ such that $\nabla\varphi$ transports μ onto ν. Furthermore, $\nabla\varphi$ is uniquely determined μ almost-everywhere.*

As φ is convex on its domain, it is differentiable μ almost-everywhere. If we assume φ of class C^2, the change of variables $y = \nabla\varphi(x)$ in (54) shows that φ satisfies the Monge-Ampère equation, for μ almost-every $x \in \mathbb{R}^n$:

$$F(x) = G(\nabla\varphi(x)) \det D^2\varphi(x). \tag{55}$$

Here $D^2\varphi(x)$ stands for the Hessian matrix of φ at the point x. Cafarelli's Theorems [Ca1] and [Ca2] asserts the validity of (55) in classical sense when the functions F and G are Hölder-continuous and strictly positive on their respective supports. Generally speaking, the matrix $D^2\varphi(x)$ can be defined with the Taylor expansion of φ (μ almost-everywhere)

$$\varphi(x+h) =_{h\to 0} \varphi(x) + \nabla\varphi(x)\cdot h + \frac{1}{2}D^2\varphi(x)(h)\cdot h + o\left(|h|^2\right).$$

In our case we are given a probability density ρ on $\mathbb{R}^n$, which we can assume to be, by approximation, continuous and strictly positive. Let $T = \nabla\varphi$ the Brenier map between $\rho_{\kappa,W}$ and ρ. Because $\rho_{\kappa,W}$ has a convex support, and is continuous on its support, we know that $\varphi \in W^{2,1}_{loc}$. Then the following integration by parts formula

$$\int f \, \Delta\varphi = -\int \nabla\varphi \cdot \nabla f$$

is valid for any smooth enough function $f : \Omega \to \mathbb{R}$. We begin by writing Monge-Ampère equation:

$$\rho_{\kappa,V}(x) = \rho(T(x)) \det D^2\varphi. \tag{56}$$

It follows that for

$$\rho(T(x))^{\kappa} = \rho_{\kappa,W}(x)^{\kappa} \left(\det D^2\varphi\right)^{-\kappa} = W(x) \left(\det D^2\varphi\right)^{-\kappa} \tag{57}$$

Recall that for $\kappa \in [-\frac{1}{n}, +\infty]$, the functional

$$M \to \frac{1}{\kappa}\det{}^{-\kappa}(M)$$

is concave on the set of nonnegative symmetric $n \times n$ matrices. If we consider the tangent at the identity matrix I we find that

$$\frac{1}{\kappa}\det{}^{-\kappa}(M) \geq \frac{1}{\kappa} - \operatorname{tr}(M - I).$$

Actually, for future use, let us introduce

$$\mathcal{G}_{\kappa}(M) := \frac{1}{\kappa}\det{}^{-\kappa}(M) - \frac{1}{\kappa} + \operatorname{tr}(M - I) \geq 0. \tag{58}$$

So if we introduce the displacement function function $\theta(x) = \varphi(x) - |x|^2/2$ so that $T(x) = \nabla\varphi(x) = x + \nabla\theta(x)$ we have

$$\frac{1}{\kappa}\rho(T(x))^{\kappa} \geq W(x)\left(\frac{1}{\kappa} - \Delta\theta(x)\right) + W(x)\mathcal{G}_{\kappa}(D^2\theta(x))$$

Integrating with respect to $\rho_{\kappa,W} = W^{1/\kappa}$ and performing an integration by parts (note that $W^{1+\frac{1}{\kappa}} \to 0$ at infinity) we find

$$\begin{aligned}
\frac{1}{\kappa}\int \rho^{1+\kappa} &\geq \frac{1}{\kappa}\int \rho_{\kappa,W}^{1+\kappa} - \int W^{1+\frac{1}{\kappa}}\Delta\theta + \int W\mathcal{G}_{\kappa}(D^2\theta(x))\rho_{\kappa,W} \\
&= \frac{1}{\kappa}\int \rho_{\kappa,W}^{1+\kappa} + \frac{1+\kappa}{\kappa}\int W^{\frac{1}{\kappa}}\nabla W \cdot \nabla\theta + \int W\mathcal{G}_{\kappa}(D^2\theta(x))\rho_{\kappa,W}
\end{aligned}$$

By definition of mass transport we have

$$\frac{1+\kappa}{\kappa}\int W(y)\,\rho(y)\,dy = \frac{1+\kappa}{\kappa}\int W(T(x))W^{\frac{1}{\kappa}}(x)\,dx$$

so adding the left-hand expression to the left and the right-hand expression to the right we find (adding also the required cosmetic constant) that

$$H_{\kappa,W}(\rho) = H_{\kappa,W}(\rho_{\kappa,W}) + \int c_{\kappa,W}(x,T(x))\,\rho_{\kappa,W} + \int W\mathcal{G}_\kappa(D^2\theta(x))\rho_{\kappa,W},$$

or equivalently

$$H_{\kappa,W}(\rho||\rho_{\kappa,W}) \geq \int c_{\kappa,W}(x,T(x))\,\rho_{\kappa,W} + \int W\mathcal{G}_\kappa(D^2\theta(x))\rho_{\kappa,W}. \tag{59}$$

In particular, since $\mathcal{G}_\kappa \geq 0$, we find, by the definition of the transportation cost, the inequality stated in Theorem 3.1.

3.3 Remainder terms (Theorems 3.2 and 3.3)

The main step is to obtain a quantitative form of the inequality (58) and the approach is not the same whether we are in **Case 1** or in **Case 2**. The rest of the proof is exactly the same.

3.3.1 Case 1

We start from (59) and try to exploit the last term in order to get an improved inequality. The following Lemma gives a quantitative form of the inequality (58).

Lemma 3.1. *Under the notation of **Case 1**, for any symmetric $n \times n$ matrix M, we have*

$$\mathcal{G}_\kappa(M) \geq c\sum_{i=1}^{n} \min\left\{\mu_i^2, |\mu_i|\right\}, \tag{60}$$

where $\mu_1, \cdots, \mu_n$ are the eigenvalues of $M - I$ and for some numerical constant $c > 0$.

Proof. The main point is the following inequality valid for $t \geq -1$,

$$\log(1+t) \leq t - c\min\left\{t^2, |t|\right\},$$

where $c > 0$ is a numerical constant (for instance $c = \frac{3}{10}$ works). Then, applying it with μ_i and after summing, this gives

$$\sum_{i=1}^{n} \log(1+\mu_i) \leq \sum_{i=1}^{n} \mu_i - c\sum_{i=1}^{n} \min\left\{|\mu_i|, \mu_i^2\right\},$$

and

$$\begin{aligned}\prod_{i=1}^{n}\left(1+\mu_i\right)^{-1/\beta} &\geq \exp\left(-\frac{1}{\beta}\sum_{i=1}^{n}\mu_i+c\frac{1}{\beta}\sum_{i=1}^{n}\min\left\{\left|\mu_i\right|,\mu_i^2\right\}\right)\\ &\geq 1-\frac{1}{\beta}\sum_{i=1}^{n}\mu_i+c\frac{1}{\beta}\sum_{i=1}^{n}\min\left\{\left|\mu_i\right|,\mu_i^2\right\}.\end{aligned}$$

Since $\prod_{i=1}^{n}\left(1+\mu_i\right)^{-1/\beta}=\det^{-1/\beta}\left(M\right)$ and $\sum_{i=1}^{n}\mu_i=\operatorname{tr}\left(M-I\right)$ dividing by $\frac{1}{\beta}>0$ ends the proof.

□

Let us prove now Theorem 3.2.

Proof. We go back to (59) and we use the previous Lemma to minimize $\int W\mathcal{G}_\kappa\left(D^2\theta\left(x\right)\right)\rho_{\beta,W}$:

$$\int W\mathcal{G}_\kappa\left(D^2\theta\left(x\right)\right)\rho_{\beta,W}\geq c\int \operatorname{tr}\left(\mathcal{F}\left(D^2\theta\left(x\right)\right)\right)W\rho_{\beta,W}$$

Now, we follow the approach of Cordero-Erausquin in [CE].

Lemma 3.2. *[CE] For any $n\times n$ symmetric matrix M with eigenvalues larger than -1, we have:*

$$\operatorname{tr}\left(\mathcal{F}\left(M\right)\right)\geq\frac{1}{8}\int_{\mathbb{S}^{n-1}}\mathcal{F}\left(\sqrt{n}\left|Mu\right|\right)d\sigma\left(u\right).$$

This gives

$$\int W\mathcal{G}_\kappa\left(D^2\theta\left(x\right)\right)\rho_{\beta,W}\geq c\int_{\mathbb{S}^{n-1}}\left(\int\mathcal{F}\left(\sqrt{n}\left|D^2\theta\left(x\right)u\right|\right)W\rho_{\beta,W}\right)d\sigma\left(u\right)\quad(61)$$

Since $D^2\theta\left(x\right)u=\nabla\left(\nabla\theta\left(x\right)\cdot u\right)$, and using (48) in (61), we find

$$\int W\mathcal{G}_\kappa\left(D^2\theta\left(x\right)\right)\rho_{\beta,W}\geq c\int_{\mathbb{S}^{n-1}}\int\mathcal{F}\left(h_W\left(\rho_{\beta,W}\right)\sqrt{n}\left|\nabla\theta\cdot u\right|\right)\rho_{\beta,W}d\sigma\left(u\right)\quad(62)$$

Note that since ρ and $\rho_{\beta,W}$ have the same center of mass, we have

$$\int\nabla\theta\left(x\right)\cdot u\,\rho_{\beta,W}=0.$$

Before going on, let us use the following Fact.

Fact 3.1. *There exists* $c_n > 0$*, such that for all* $x \in \mathbb{R}^n$*, we have*

$$\int_{\mathbb{S}^{n-1}} |x \cdot u| \, d\sigma(u) = c_n |x| .$$

Moreover, one can prove that there exists two positive numerical constants, say c *and* C*, such that* $c \le c_n \sqrt{n} \le C$*.*

Proof. It is easy to see that $N(x) := \int_{\mathbb{S}^{n-1}} |x \cdot u| \, d\sigma(u)$ is a norm invariant with rotations, then it is a multiple of the Euclidean norm. It is classical, see [Bor1, Bor2], that $c_n \simeq \sqrt{\int_{\mathbb{S}^{n-1}} |x \cdot u|^2 \, d\sigma(u)}$ (i.e. up to numerical constants). Then, one can prove, using concentration of measures, that

$$\sqrt{\int_{\mathbb{S}^{n-1}} |x \cdot u|^2 \, d\sigma(u)} \simeq \frac{1}{\sqrt{n}}.$$

□

Using Fubini's theorem, Jensen's inequality ($\mathcal{F}$ is convex) and Fact 1 in (62), we find

$$\begin{aligned}
\int W \mathcal{G}_\kappa \left(D^2 \theta(x)\right) \rho_{\beta,W} &\ge c \int \mathcal{F}\left(h_W(\rho_{\beta,W}) \sqrt{n} \int_{\mathbb{S}^{n-1}} |\nabla \theta(x) \cdot u| \, d\sigma(u)\right) \rho_{\beta,W} \\
&\ge C \int \mathcal{F}\left(h_W(\rho_{\kappa,W}) |\nabla \theta(x)|\right) \rho_{\beta,W} \\
&= C \int \tilde{c}(x, T(x)) \rho_{\beta,W}
\end{aligned}$$

Replacing this inequality in (59) finishes the proof of Theorem 3.2.

□

3.3.2 Case 2

As we say at the beginning of this part, the proof of Theorem 3.3 is very similar as the one for Theorem 3.2, the only difference is proof of the quantitative form of (58). That is the goal of the following Lemma.

Lemma 3.3. *Under the notation of* ***Case 2****, for any nonnegative, symmetric* $n \times n$ *matrix* M *and for all* $\beta \ge n$*, we have:*

$$\mathcal{G}_\kappa(M) \ge \frac{3}{64\beta} \left(1 - \frac{n}{\beta}\right)^2 \mathcal{F}\left(\|M - I\|_{\mathrm{HS}}\right). \tag{63}$$

Proof. We introduce the probability measure μ defined on $\mathbb{R}$ by $d\mu = \frac{1}{\beta}\delta_1 + \cdots + \frac{1}{\beta}\delta_n + \left(1 - \frac{n}{\beta}\right)\delta_{n+1}$, the function ϕ defined on $[-1, +\infty)$ by $\phi(x) = \log(1+x)$ and the function f defined on $\mathbb{R}$ by:

$$f(x) = \begin{cases} \mu_i \, \text{if } x = i \, \text{with} \, i \in \{1, \cdots, n\}, \\ 0 \, \text{else.} \end{cases}$$

Let us note that $\phi\left(\int_{\mathbb{R}} f d\mu\right) = \log\left(1 + \frac{1}{\beta}\sum_{i=1}^{n} \mu_i\right)$ and $\int_{\mathbb{R}} \phi(f)\, d\mu = \log\left(\prod_{i=1}^{n} (1+\mu_i)^{\frac{1}{\beta}}\right)$.
We start with this inequality: for all $s, t \in \mathbb{R}_+$,

$$\log(s) \leq \log(t) + \frac{s-t}{t} - \frac{(s-t)^2}{2\max\{s,t\}^2}. \tag{64}$$

In (64), taking $s = 1 + f$ and $t = 1 + m = 1 + \int_{\mathbb{R}} f d\mu$, then integrating with respect to the measure μ, it gives:

$$\int_{\mathbb{R}} \phi(f)\, d\mu \leq \phi(m) - \frac{1}{2}\int_{\mathbb{R}} \frac{(f-m)^2}{\max\{1+m, 1+f\}^2} d\mu.$$

Then, we have:

$$\begin{aligned} \phi(m) - \int_{\mathbb{R}} \phi(f)\, d\mu &\geq \frac{1}{2}\int_{\mathbb{R}} \frac{(f-m)^2}{\max\{1+m, 1+f\}^2} d\mu \\ &\geq \frac{1}{4\left(1+\mu_{\max}^2\right)} \int_{\mathbb{R}} (f-m)^2\, d\mu. \end{aligned}$$

Let us compute $\int_{\mathbb{R}} (f-m)^2\, d\mu$.

$$\begin{aligned} \int_{\mathbb{R}} (f-m)^2\, d\mu &= \int_{\mathbb{R}} f^2 d\mu - \left(\int_{\mathbb{R}} f d\mu\right)^2 \\ &= \frac{1}{\beta}\sum_{i=1}^{n} \mu_i^2 - \frac{1}{\beta^2}\left(\sum_{i=1}^{n} \mu_i\right)^2 \\ &\underbrace{\geq}_{\text{Cauchy-Schwarz inequality}} \frac{1}{\beta}\left(1 - \frac{n}{\beta}\right)\sum_{i=1}^{n} \mu_i^2 \\ &= \frac{1}{\beta}\left(1 - \frac{n}{\beta}\right)\|M - I\|_{\text{HS}}^2. \end{aligned}$$

So, we have:

$$\phi\left(m\right)-\int_{\mathbb{R}}\phi\left(f\right)d\mu\geq\frac{1}{4\beta}\left(1-\frac{n}{\beta}\right)\frac{\|M-I\|_{\mathrm{HS}}^{2}}{1+\mu_{\max}^{2}}=:z.$$

Taking the exponential, this yields $e^{\int_{\mathbb{R}}\phi(f)d\mu}\leq e^{-z}e^{\phi(m)}$ then $e^{\phi(m)}-e^{\int_{\mathbb{R}}\phi(f)d\mu}\geq\left(1-e^{-z}\right)e^{\phi(m)}$. It is easy to see that $z\in\left[0,\frac{1}{2}\right]$, so the inequality $1-e^{-z}\geq\frac{3}{4}z$ holds. Finally, we have established the following inequality:

$$1+\mathrm{tr}\left(M-I\right)-\det\left(M\right)^{\frac{1}{\beta}}\geq\frac{3}{16\beta}\left(1-\frac{n}{\beta}\right)\frac{\|M-I\|_{\mathrm{HS}}^{2}}{1+\mu_{\max}^{2}}\left(1+\frac{1}{\beta}\mathrm{tr}\left(M-I\right)\right).$$

To conclude, we discuss whether $\mu_{\max}$ is bigger than 1 or not.
We assume $\mu_{\max}\leq 1$. In this case, we have:

$$1+\mathrm{tr}\left(M-I\right)-\det\left(M\right)^{\frac{1}{\beta}}\geq\frac{3}{32\beta}\left(1-\frac{n}{\beta}\right)^{2}\|M-I\|_{\mathrm{HS}}^{2}.$$

We assume $\mu_{\max}\geq 1$. First, we work on $1+\frac{1}{\beta}\mathrm{tr}\left(M-I\right)$. This yields the following lines:

$$\begin{aligned}\beta\left(1+\frac{1}{\beta}\mathrm{tr}\left(M-I\right)\right) &= \beta+\sum_{i=1}^{n}\mu_{i}\\ &\geq \left(\beta-\left(n-1\right)\right)+\mu_{\max}\\ &\geq \mu_{\max}\\ &\geq \frac{1}{n}\sum_{i=1}^{n}|\mu_{i}|\\ &\geq \frac{1}{n}\sqrt{\sum_{i=1}^{n}\mu_{i}^{2}}\\ &= \frac{1}{n}\|M-I\|_{\mathrm{HS}}.\end{aligned}$$

Consequently,

$$1+\mathrm{tr}\left(M-I\right)-\det\left(M\right)^{\frac{1}{\beta}}\geq\frac{3}{16n\beta^{2}}\left(1-\frac{n}{\beta}\right)\|M-I\|_{\mathrm{HS}}\frac{\|M-I\|_{\mathrm{HS}}^{2}}{1+\mu_{\max}^{2}}.$$

We finally conclude thanks to

$$\frac{\|M-I\|_{\mathrm{HS}}^{2}}{1+\mu_{\max}^{2}}\geq\frac{n\mu_{\max}^{2}}{1+\mu_{\max}^{2}}\geq\frac{1}{2}n$$

and

$$1+\mathrm{tr}\,(M-I)-\det\,(M)^{\frac{1}{\beta}} \geq \frac{3}{32\beta^2}\left(1-\frac{n}{\beta}\right)\|M-I\|_{\mathrm{HS}}.$$

In the two cases, we have at the same time the inequality:

$$\begin{aligned} 1+\mathrm{tr}\,(M-I)-\det\,(M)^{\frac{1}{\beta}} &\geq \frac{3}{32\beta^2}\left(1-\frac{n}{\beta}\right)^2 \min\left\{\|M-I\|_{\mathrm{HS}}, \|M-I\|_{\mathrm{HS}}^2\right\} \\ &\geq \frac{3}{32\beta^2}\left(1-\frac{n}{\beta}\right)^2 \mathcal{F}\left(\|M-I\|_{\mathrm{HS}}\right). \end{aligned}$$

Multiplying by $\beta>0$, this concludes the proof of the Lemma.

□

Let us end the proof of Theorem 3.3.

Proof. Let us plug (63) in (59), we obtain,

$$\int W\mathcal{G}_\kappa(D^2\theta(x))\rho_{\beta,W} \geq \frac{3}{64\beta}\left(1-\frac{n}{\beta}\right)^2\int \mathcal{F}\left(\|M-I\|_{\mathrm{HS}}\right)W\rho_{\beta,W}.$$

The rest of the proof is the same as the one for Theorem 3.2.

□

3.4 Linearization and dimensional Brascamp-Lieb inequalities

3.4.1 Dimensional Brascamp-Lieb inequalities

The goal of this part is to recover dimensional Brascamp-Lieb inequalities. For that, we linearize our transport inequality we established in Theorem 3.1. Let us cite the result we need for that.

Lemma 3.4. *[CE]*

Let $c:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}_+$ a function such that $c(y,y)=0$ and $c(x,y)\geq \delta_0|x-y|^2$ for all $x,y\in\mathbb{R}^n$ and for some $\delta_0>0$. We assume that for every $y\in\mathbb{R}^n$, there exists a nonnegative, symmetric matrix $n\times n$, say H_y, such that

$$c(y,y+h) =_{h\to 0} \frac{1}{2}H_y h\cdot h+|h|^2 o(1).$$

Then, if μ is a probability measure on $\mathbb{R}^n$ and g is C^1 compactly supported with $\int_{\mathbb{R}^n} g\, d\mu = 0$, we have

$$\liminf_{\epsilon\to 0} \frac{1}{\epsilon^2}\mathcal{W}_c\left(\mu,\left(1+\epsilon g\right)\mu\right) \geq \frac{1}{2}\frac{\left(\int_{\mathbb{R}^n} g\, f\, d\mu\right)^2}{\int_{\mathbb{R}^n} H_y^{-1}\nabla f\cdot\nabla f d\mu},$$

for any function C^1 compactly supported f.

Using this Lemma to linearize inequality (47) gives

Theorem 3.5. *With the notation of Theorem 3.1 and assuming $\int W^{1/\kappa} = 1$, we have the following inequality*

$$-\kappa\int\left(D^2W\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\kappa,W} \geq \int g^2\,W\,\rho_{\kappa,W}, \tag{65}$$

with $\int g\,\rho_{\kappa,W} = 0$ and $f = g\,W$.

As we said in Introduction, we can now retrieve dimensional Brascamp-Lieb inequalities. For example, in **Case 1**, (65) becomes

$$\int\left(-D^2W\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\beta,W} \geq \beta\int g^2\,W\,\rho_{\beta,W},$$

whereas in **Case 2**

$$\int\left(D^2W\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\beta,W} \geq \beta\int g^2\,W\,\rho_{\beta,W}.$$

Proof. Let us remark first that, when $h\to 0$,

$$c\left(y,y+h\right) = \frac{1}{2}\left(\frac{\kappa+1}{-\kappa}D^2W\left(y\right)\right)\left(h\right)\cdot h + \left|h\right|^2 o\left(1\right). \tag{66}$$

Let us compute, for g verifying $\int g\,\rho_{\kappa,W} = 0$

$$\liminf_{\epsilon\to 0}\frac{1}{\epsilon^2}H_{\kappa,W}\left(\left(1+\epsilon g\right)\rho_{\kappa,W}\|\rho_{\kappa,W}\right).$$

Thanks to Theorem 1, it we will have a maximization of

$$\liminf_{\epsilon\to 0}\frac{1}{\epsilon^2}W_{c_{\kappa,W}}\left(\rho_{\kappa,W},\left(1+\epsilon g\right)\rho_{\kappa,W}\right).$$

Using the definition of the entropy, we have

$$H_{\kappa,W}\left(\left(1+\epsilon g\right)\rho_{\kappa,W}\|\rho_{\kappa,W}\right) = \frac{\kappa+1}{2}\epsilon^2\int g^2\,\rho_{\kappa,W}^{1+\kappa} + o\left(\epsilon^2\right). \tag{67}$$

Putting together (66) and (67) thanks to the above Lemma gives the following inequality, for all function f C^1 compactly supported

$$\frac{\kappa+1}{2}\int g^2\,W\rho_{\kappa,W}\geq\frac{1}{2}\frac{\left(\int g\,f\,\rho_{\kappa,W}\right)^2}{\int\left(\frac{\kappa+1}{-\kappa}D^2W\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\kappa,W}}.$$

Taking $f=g\,W$ concludes the proof of the Theorem.

□

3.5 Quantitative forms

In this section, we are interested by giving some quantitative forms of the inequalities stated in (49) and (51). The main argument is, once again, Lemma 3.4: we use it with the costs we introduced in Theorems 3.2 and 3.3. We separate our result whether we are in **Case 1** or in **Case 2**.

Theorem 3.6. *Under the notation of **Case 1**, we have the following inequality*

$$\int\left(-D^2W+\frac{c}{\beta+1}h_W\left(\rho_{\beta,W}\right)I\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\beta,W}\geq\beta\int g^2\,W\,\rho_{\beta,W},$$

for some numerical constant $c>0$ and with $\int g\,\rho_{\beta,W}=0$, $\int xg\left(x\right)\rho_{\beta,W}\left(x\right)=0$ and $f=g\,W$.

And

Theorem 3.7. *Under the notation of **Case 2**, we have the following inequality*

$$\int\left(D^2W+\frac{c}{\beta\left(\beta-1\right)}\left(1-\frac{n}{\beta}\right)^2h_W\left(\rho_{\beta,W}\right)I\right)^{-1}\nabla f\cdot\nabla f\,\rho_{\beta,W}\geq\beta\int g^2\,W\,\rho_{\beta,W},$$

for some numerical constant $c>0$ and with $\int g\,\rho_{\beta,W}=,0$ $\int xg\left(x\right)\rho_{\beta,W}\left(x\right)=0$ and $f=g\,W$.

The proofs are very similar as the one for Theorem 3.5. Anyway, let us proof Theorem 3.6 (the proof of Theorem 3.7 is the same).

Proof. We keep the notation of Theorem 3.2. As $\int g\,\rho_{\beta,W}=0$ and $\int xg\left(x\right)\rho_{\beta,W}\left(x\right)=0$, the measures $\rho_{\beta,W}$ and $\left(1+\epsilon g\right)\rho_{\beta,W}$ are both probability measures with the same center of mass. Thanks to Theorem 3.2, it is enough to give as estimation of the relative entropy instead of $\mathcal{W}_c\left(\rho_{\beta,W},\left(1+\epsilon g\right)\rho_{\beta,W}\right)$. Proof of Theorem 3.5 gives for the relative entropy:

$$\begin{aligned} H_{\kappa,W}\left(\left(1+\epsilon g\right)\rho_{\kappa,W}\|\rho_{\kappa,W}\right) &= \frac{\kappa+1}{2}\epsilon^2\int g^2\,\rho_{\kappa,W}^{1+\kappa}+o\left(\epsilon^2\right) \\ &= \frac{\beta+1}{2\beta}\epsilon^2\int g^2\,\rho_{\kappa,W}^{1+\kappa}+o\left(\epsilon^2\right). \end{aligned}$$

Thanks to the definition of $\mathcal{F}$, one have

$$\lim_{h\to 0} c\left(y,y+h\right)=\frac{1}{2}\left(-\left(\beta+1\right)D^2W\left(y\right)+ch_W\left(\rho_{\beta,W}\right)I\right),$$

for some numerical constant $c>0$. Using Lemma 3.4 with $f=g\,W$ permits to conclude the proof.

□

3.6 Further remarks on weighted Poincaré inequalities

3.6.1 Generality on weighted Poincaré inequalities

In (48), we introduced $h_W\left(\mu\right)$ as the best nonnegative constant such that the inequality

$$\int\mathcal{F}\left(\left|\nabla f\right|\right)W\,d\mu\geq\int\mathcal{F}\left(h_W\left(\mu\right)\left|f-m_f\right|\right)d\mu$$

holds for every smooth enough $f\in L^1\left(\mu\right)$. Nevertheless, we are convinced that the following definition for weighted Poincaré inequality (note that the weight has not the same place):

$$\int\mathcal{F}\left(\frac{1}{h_W\left(\mu\right)}W\left|\nabla f\right|\right)d\mu\geq\int\mathcal{F}\left(\left|f-m_f\right|\right)d\mu, \tag{68}$$

is more natural. The next Proposition goes in this way.

Proposition 3.1. *Let μ a probability measure with a support $\Omega\subseteq\mathbb{R}^n$ and let $\omega:\Omega\to\mathbb{R}_+$ a function. If we assume that there exists $h\left(\mu\right)>0$ such that*

$$\int_\Omega\left|\frac{1}{h\left(\mu\right)}\nabla f\right|\,\omega\,d\mu\geq\int_\Omega\left|f-m_f\right|d\mu \tag{69}$$

for every smooth enough function $f\in L^1\left(\mu\right)$ then the following inequality

$$\int_\Omega\mathcal{F}\left(\frac{1}{h\left(\mu\right)}\omega\left|\nabla f\right|\right)d\mu\geq\int_\Omega\mathcal{F}\left(\left|f-m_f\right|\right)d\mu$$

holds.

The proof is identical to the one of Bobkov-Houdré [Bo-Ho2] (see [CE]). In the next section, we give an example where inequality (48) is fulfilled.

3.6.2 Example of weighted Poincaré inequality

Let us recall the result we will use.

Theorem 3.8. *[Bo-Le3]*

*Let $\kappa \in (-\infty, 0]$ and let μ a κ-concave measure defined on $\mathbb{R}^n$ (i.e. with the notation introduced in Introduction, we are in **Case 2**). Let $m = \exp\left(\int_{\mathbb{R}^n} \log(|x|)\, d\mu(x)\right)$ (note that m is finite). Then, for any non-empty Borel sets A and B in $\mathbb{R}^n$ located at distance $\epsilon = \operatorname{dist}(A, B) > 0$*

$$\mu(A)\,\mu(B) \leq \frac{C_\kappa}{2\epsilon} \int_{\mathbb{R}^n \backslash (A \cup B)} (m - \kappa |x|)\, d\mu(x) \tag{70}$$

with C_κ depending continuously in κ in the indicated range.

Thanks to (70), let us deduce a weighted Poincaré inequality. Let us take K a non-empty compact set with smooth enough boundary and if set $A = K \backslash S$ and $B = (\mathbb{R}^n \backslash K) \backslash S$ where S is the closure of the $\frac{\epsilon}{2}$-neighborhood of ∂K in (70) and letting $\epsilon \to 0$, we have:

$$\mu(K)\,(1 - \mu(K)) \leq \frac{C_\kappa}{2} \mu_\omega^+(K), \tag{71}$$

where $\omega(x) = m - \kappa |x|$ and $m = \exp\left(\int_{\mathbb{R}^n} \log(|x|)\, d\mu(x)\right)$. If we use the coarea formula in (71), we finally have

$$\int_{\mathbb{R}^n} \left| f(x) - m_f \right| d\mu(x) \leq \frac{C_\kappa}{2} \int_{\mathbb{R}^n} |\nabla f(x)|\, \omega(x)\, d\mu(x). \tag{72}$$

Now, we give an example of density ρ_β such that the measure $\rho_\beta(x)\,dx$ $\kappa(n)$-concave which satisfy (48). For this, let for $\beta > n$, $\rho_\beta(x) = \frac{1}{Z_\beta}\left(1 + |x|^2\right)^{-\beta} = \frac{1}{Z_\beta} W(x)^{-\beta}$ where $Z_\beta = \int_{\mathbb{R}^n} \left(1 + |x|^2\right)^{-\beta} dx$. Recalling the notation we used in the introduction, we have $\beta = -\frac{1}{\kappa} = \frac{n\kappa(n) - 1}{\kappa(n)}$ then $\kappa(n) = \frac{1}{n - \beta}$. Then, the measure $\rho_\beta(x)\,dx$ is κ-concave with $\kappa = \frac{1}{n-\beta}$. If $m = \exp\left(\int \log(|x|)\, \rho_\beta\right)$, using (72), we can write, for all function $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough (noting C_β instead of C_κ)

$$\begin{aligned} \int \left| f(x) - m_f \right| \rho_\beta &\leq \frac{C_\beta}{2} \int |\nabla f(x)| \left(m + \frac{1}{\beta - n} |x| \right) \rho_\beta \\ &\leq \frac{C_\beta}{2} \max\left\{ m, \frac{1}{\beta - n} \right\} \int |\nabla f(x)|\, (1 + |x|)\, \rho_\beta. \end{aligned}$$

Proposition 3.1 gives

$$\int \mathcal{F}\left(\left|f(x)-m_f\right|\right)\rho_\beta \leq \int \mathcal{F}\left(\frac{C_\beta}{2}\max\left\{m,\frac{1}{\beta-n}\right\}|\nabla f|\left(1+|x|\right)\right)\rho_\beta.$$

Remarking that $\mathcal{F}(ab) \leq \max\left\{a,a^2\right\}\mathcal{F}(b)$ and $(1+|x|)^2 \leq 3\left(1+|x|^2\right)$, this gives

$$\frac{1}{3}\int \mathcal{F}\left(\left|f(x)-m_f\right|\right)\rho_\beta \leq \int \mathcal{F}\left(\frac{C_\beta}{2}\max\left\{m,\frac{1}{\beta-n}\right\}|\nabla f|\right)\left(1+|x|^2\right)\rho_\beta.$$

Since $fort \geq 0$, $\mathcal{F}(t/12) \leq \frac{1}{3}\mathcal{F}(t)$, we have

$$\int \mathcal{F}\left(\frac{1}{12}\left|f(x)-m_f\right|\right)\rho_{\beta,W} \leq \int \mathcal{F}\left(\frac{C_\beta}{2}\max\left\{m,\frac{1}{\beta-n}\right\}|\nabla f|\right)W\rho_\beta,$$

or equivalently

$$\int \mathcal{F}\left(\frac{1}{6C_\beta\max\left\{m,\frac{1}{\beta-n}\right\}}\left|f-m_f\right|\right)\rho_\beta \leq \int \mathcal{F}\left(|\nabla f|\right)W\rho_\beta. \tag{73}$$

Thus (73) provides an example where (48) is satisfied with the constant $h_W\left(\rho_{\beta,W}\right) = \frac{1}{6C_\beta\max\left\{m,\frac{1}{\beta-n}\right\}} > 0$. To conclude properly, one can give an estimation of m. If we note, for $q \geq 0$

$$m_q = \left(\int_{\mathbb{R}^n}|x|^q\rho_\beta(x)\,dx\right)^{1/q},$$

then we have the following lines

$$\begin{aligned} m &\leq m_1 \\ &= \frac{\int_{\mathbb{R}^n}\frac{|x|}{\left(1+|x|^2\right)^\beta}dx}{\int_{\mathbb{R}^n}\frac{1}{\left(1+|x|^2\right)^\beta}dx} \\ &= \frac{\int_0^{+\infty}\frac{r^n}{(1+r^2)^\beta}dr}{\int_0^{+\infty}\frac{r^{n-1}}{(1+r^2)^\beta}dr}. \end{aligned}$$

It remains to give an estimation of

$$\frac{\int_0^{+\infty} \frac{r^n}{(1+r^2)^\beta} dr}{\int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^\beta} dr}.$$

If we note

$$I_n(\beta) = \int_0^{+\infty} \frac{r^n}{(1+r^2)^\beta} dr = \int_0^{+\infty} r^n e^{-\beta \log\left(1+r^2\right)} dr,$$

one can have, thanks to Laplace's method (see [Di])

$$I_n(\beta) \underset{\beta \to +\infty}{\sim} \frac{1}{2} \Gamma\left(\frac{n+1}{2}\right) \beta^{-\frac{n+1}{2}}$$

and

$$\frac{\int_0^{+\infty} \frac{r^n}{(1+r^2)^\beta} dr}{\int_0^{+\infty} \frac{r^{n-1}}{(1+r^2)^\beta} dr} \underset{\beta \to +\infty}{\sim} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \sqrt{\beta}.$$

Since $\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \underset{n \to +\infty}{\sim} \sqrt{\frac{n}{2}}$, we have $m \leq C\sqrt{n\beta}$ for some numerical constant C and for all $\beta \geq n$, this finally gives

$$h_W(\rho_{\beta,W}) \geq \frac{1}{6C_\beta \max\left\{C\sqrt{n\beta}, \frac{1}{\beta-n}\right\}}.$$

4 Remarks on weighted isoperimetric estimates for convex measures in dimension one

4.1 Introduction

In this note, we are interested in Cheeger type inequalities for convex measures (we recall below what it is) in order to obtain an isoperimetric inequality for these measures. In [Bo-Ho2], Bobkob and Houdré give an example of a convex measure, defined by $d\mu(x) = \frac{1}{(|x|+2)^2}$, for which the isoperimetric function $I_\mu(t) := \inf_{A,\,\mu(A)=t} \mu^+(A) = \frac{1}{2}\min\left\{t^2, (1-t)^2\right\}$, where $\mu^+(A)$ is the perimeter of A with respect to μ. Thus, a such measure cannot verify a linear isoperimetric inequality (in the sense $I_\mu(t) \geq c\min\{t, 1-t\}$ for some constant $c > 0$ and for all $t \in (0,1)$). Two different approaches can solve this problem. The first, developed by E. Milman in [Mi1] is to establish Sobolev inequalities for Lorentz quasi-norms. Here, our approach is to work with weighted Cheeger inequalities. Before announcing our results, let us recall some backgrounds.

Borell's results [Bor1, Bor2] assert that a convex probability measure $\mu_{\beta,W}$ on $\mathbb{R}$ can be defined as follow:

$$d\mu_{\beta,W}(x) = \frac{1}{Z_{\beta,W}} W(x)^{-\beta}\, dx,$$

where W is a convex function on $\mathbb{R}$ such that $\lim_{x\to\pm\infty} W(x) = +\infty$ and $\beta > 1$. $Z_{\beta,W} = \int_{\mathbb{R}} W(x)^{-\beta}\, dx < +\infty$ is a normalizing constant. Let $\phi_{\beta,W} : \mathbb{R} \to (0,1)$ defined by

$$\phi_{\beta,W}(x) = \int_{-\infty}^{x} d\mu_{\beta,W}(t).$$

Let us finally define the measure $\mu^+_{\beta,W}$, which is the W-weighted "surface" measure associated to $\mu_{\beta,W}$, by

$$\mu^+_{\beta,W}(A) := \int_A \frac{W(x)}{Z_{\beta,W}^{-1/\beta}} W(x)^{-\beta} \frac{d\mathcal{H}^0(x)}{Z_{\beta,W}}, \qquad \forall A \subseteq \mathbb{R}\text{ borelian set.}$$

We insist that that here the notation $\mu^+_{\beta,W}$ refers to a weighted (by a multiple of W) area measure associated to $\mu_{\beta,W}$ and not the usual μ^+-area. Up to a constant, it rather corresponds to $(\mu_{W,\beta-1})^+$ in the classical notation. But adding an extra parameter W in the notation would have been too heavy.

We can define on $(0,1)$ the weighted isoperimetric function $I_{\mu_{\beta,W},W}$ by

$$I_{\mu_{\beta,W},W}(t) = \inf \mu^+_{\beta,W}(A), \tag{74}$$

where the infimum is taken all over Borel sets A with $\mu_{\beta,W}(A) = t$ $(t \in (0,1))$. Note that $I_{\mu_{\beta,W},W}$ is symmetric with respect to $\frac{1}{2}$. The following Proposition tells that it is enough to compute (74) with some union of disjoint intervals with disjoint boundaries. For the proof, we refer at the proof of Proposition 13.1 in [Bo-Ho2].

Proposition 4.1 (Bobkov-Houdré)**.** *For any* $t \in (0,1)$, *we have*

$$I_{\mu_{\beta,W},W}(t) = \inf\left\{\sum_{i=1}^{n} \mu^{+}_{\beta,W}((a_i,b_i))\right\}, \tag{75}$$

where the infimum is taken all over $A = \bigcup_{i=1}^{n}(a_i,b_i)$ *union of disjoint intervals with disjoint boundaries and with* $\mu_{\beta,W}(A) = t$.

The following Proposition refined the previous result. The infimum in (74) is attained on a half-line.

Proposition 4.2. *With the above notation, the infimum in* (74) *is attained at an interval of the form* $(-\infty,a)$ *or* $(b,+\infty)$.

This Proposition allows us to state an isoperimetric inequality:

Corollary 4.1. *With the above notation, there exists* $c_{\beta,W} > 0$, *for any* $t \in (0,1)$

$$I_{\mu_{\beta,W},W}(t) \geq c_{\beta,W} \min\{t, 1-t\}. \tag{76}$$

We will note $Che_W(\mu_{\beta,W})$ *the best (i.e. the largest) constant* $c_{\beta,W}$ *which satisfies* (76) *for all* $t \in (0,1)$. *Note, thanks to Proposition 4.2, that*

$$Che_W(\mu_{\beta,W}) = \inf_{0<t<1} \frac{\min\left\{\mu^{+}_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t)\right)\right), \mu^{+}_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t), +\infty\right)\right)\right\}}{\min\{t, 1-t\}}. \tag{77}$$

In a last section, we give an estimation of $Che_W(\mu_{\beta,W})$ depending on the variance of $\mu_{\beta,W}$ and a functional form of (76).

4.2 Proofs of Proposition 4.2 and Corollary 4.1

We begin by proving Proposition 4.2.

Proof. Let A a borelian of measure t such that $I_{\mu_{\beta,W},W}(t) = \mu^{+}_{\beta,W}(A)$. Thanks to Proposition 4.1, we can assume that A is an union of disjoint intervals, say $A = \bigcup_{i=1}^{r}(a_i,b_i)$. Since W is a convex function, $W^{-\beta}$ is non-decreasing on an interval $(-\infty, x_0)$ and non-increasing on $(x_0,+\infty)$. Take all intervals (a_i,b_i), $i \in U \subseteq \{1,\cdots n\}$, which are situated on the left of x_0, then

$$\mu^+_{\beta,W}\left(\bigcup_{i\in U}(a_i,b_i)\right)\geq \mu^+_{\beta,W}\left(\left(-\infty,\phi^{-1}_{\beta,W}(t_1)\right)\right),$$

where $t_1=\mu_{\beta,W}\left(\bigcup_{i\in U}(a_i,b_i)\right)$. In the same way, we define $V\subseteq\{1,\cdots,n\}$ such that if $i\in V$ then (a_i,b_i) is situated on the right of x_0. If we note $t_2=\mu_{\beta,W}\left(\bigcup_{i\in V}(a_i,b_i)\right)$, then

$$\mu^+_{\beta,W}\left(\bigcup_{i\in V}(a_i,b_i)\right)\geq \mu^+_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t_2),+\infty\right)\right).$$

We have to treat now an interval, say (b,c) such that $x_0\in(b,c)$. We write $B=(-\infty,a)\cup(b,c)\cup(d,+\infty)$ with $(-\infty,a)\subset(-\infty,x_0)$ and $(d,+\infty)\subset(x_0,+\infty)$. Let us consider $C=(a,b)\cup(c,d)$. We have $\mu_{\beta,W}(C)=1-t$ and $\mu^+_{\beta,W}(C)=\mu^+_{\beta,W}(B)$. Then, let $D=(-\infty,e)\cup(f,+\infty)$ with $e\leq x_0\leq f$. We have $\mu_{\beta,W}((-\infty,e))=\mu_{\beta,W}((a,b))$ and $\mu_{\beta,W}((f,+\infty))=\mu_{\beta,W}((c,d))$, so $\mu_{\beta,W}(D)=\mu_{\beta,W}(C)$. It is easy to see that $\mu^+_{\beta,W}(C)\geq\mu^+_{\beta,W}(D)$. Finally, let $E=(e,f)$. We have $\mu_{\beta,W}(E)=t$ and $\mu^+_{\beta,W}(A)\geq\mu^+_{\beta,W}(E)$. For an interval (a,b) of measure t, let the function h defined on $(-\infty,\phi_{\beta,W}(t))$ defined by

$$\begin{aligned}h(a)&=\mu^+_{\beta,W}((a,b))\\&=\frac{1}{Z^{1-1/\beta}_{\beta,W}}\left(W(a)^{1-\beta}+W(b)^{1-\beta}\right).\end{aligned}$$

Since $t=\phi_{\beta,W}(b)-\phi_{\beta,W}(a)$, we have $b=\phi^{-1}_{\beta,W}(t+\phi_{\beta,W}(a))$ and then $b'=\frac{W(a)^{-\beta}}{W(b)^{-\beta}}$. This gives:

$$h'(a)=\frac{(1-\beta)}{Z^{1-1/\beta}_{\beta,W}}W(a)^{-\beta}\left(W'(a)+W'(b)\right).$$

As $a\to W'(a)+W'(b)$ is a non-decreasing function (recall that W is convex), there are only three possibilities:

• $a\to W'(a)+W'(b)\geq 0$ when $a\in\left(-\infty,\phi^{-1}_{\beta,W}(1-t)\right)$, then $h'(a)\leq 0$. So, if we note $F=\left(\phi^{-1}_{\beta,W}(1-t),+\infty\right)$, we have $\mu_{\beta,W}(F)=\mu_{\beta,W}(E)=t$ and $\mu^+_{\beta,W}(E)\geq\mu^+_{\beta,W}(F)$.

• $a\to W'(a)+W'(b)\leq 0$ when $a\in\left(-\infty,\phi^{-1}_{\beta,W}(1-t)\right)$, then $h'(a)\geq 0$. So, if we note $F=\left(-\infty,\phi^{-1}_{\beta,W}(t)\right)$, we have $\mu_{\beta,W}(F)=\mu_{\beta,W}(E)=t$ and $\mu^+_{\beta,W}(E)\geq\mu^+_{\beta,W}(F)$.

• $a \to W'(a) + W'(b)$ changes its sign in $\left(-\infty, \phi_{\beta,W}^{-1}(1-t)\right)$: there exists $a_0 \in \left(-\infty, \phi_{\beta,W}^{-1}(1-t)\right)$ such that $h'(a)$ is non-negative on $(-\infty, a_0)$ and non-positive on $(a_0, \phi_{\beta,W}(1-t))$. So, if we note $F = \left(-\infty, \phi_{\beta,W}^{-1}(t)\right)$ if $e \leq a_0$ and $F = \left(\phi_{\beta,W}^{-1}(1-t), +\infty\right)$ if $e \geq a_0$, we have in both cases $\mu_{\beta,W}(F) = \mu_{\beta,W}(E) = t$ and $\mu_{\beta,W}^{+}(E) \geq \mu_{\beta,W}^{+}(F)$.

□

We can now prove Corollary 4.1 using Proposition 4.2.

Proof. We keep the notation above. We note $\alpha = \min\left\{\mu_{\beta,W}((-\infty, x_0)), \mu_{\beta,W}((x_0, +\infty))\right\}$ (recall that W is non-increasing on $(-\infty, x_0)$ and non-decreasing on $(x_0, +\infty)$). Let A a borelian set and we note $t = \mu_{\beta,W}(A)$. Using the above Proposition, there exists B an interval of the form $(-\infty, a)$ or $(b, +\infty)$ such that $\mu_{\beta,W}(A) = \mu_{\beta,W}(B)$ and $\mu_{\beta,W}^{+}(A) \geq \mu_{\beta,W}^{+}(B)$. So it is enough to prove Corollary 4.1 for $B = (-\infty, a)$ or $B = (b, +\infty)$. We assume $B = (-\infty, a)$, the other case can be treated with the same method and $\mu_{\beta,W}(B) \leq \frac{\alpha}{2}$. We note $t = \mu_{\beta,W}(B)$. Let the function ψ defined on $\left(0, \frac{\alpha}{2}\right)$ by

$$\psi(t) = \mu_{\beta,W}^{+}((-\infty, a)) = \frac{1}{Z_{\beta,W}^{1-1/\beta}} W^{1-\beta}\left(\phi_{\beta,W}^{-1}(t)\right),$$

since $a = \phi_{\beta,W}^{-1}(t)$. The computation of ψ gives

$$\psi'(t) = \frac{1-\beta}{Z_{\beta,W}^{-1/\beta}} W'\left(\phi_{\beta,W}^{-1}(t)\right).$$

As ψ' is non-negative on $\left(0, \frac{\alpha}{2}\right)$ and W is convex on $\mathbb{R}$, we can write

$$\psi(t) = \psi(t_1) - \psi(0) = \psi'(c)\, t \geq \frac{1-\beta}{Z_{\beta,W}^{-1/\beta}} W'\left(\phi_{\beta,W}^{-1}\left(\frac{\alpha}{2}\right)\right) = c_{1,\beta,W}\, t, \tag{78}$$

with $c_{1,\beta,W} = (1-\beta) W'\left(\phi_{\beta,W}^{-1}\left(\frac{\alpha}{2}\right)\right) > 0$ and with $c \in (0, t) \subseteq \left(0, \frac{\alpha}{2}\right)$. When $B = \mu_{\beta,W}((b, +\infty))$ with $\mu_{\beta,W}(B) \leq t$, one can define $c_{2,\beta,W} > 0$ such that

$$\theta(t) := \mu_{\beta,W}((b, +\infty)) \geq c_{2,\beta,W}\, t. \tag{79}$$

Noting $c_{\beta,W} = \min\left\{c_{1,\beta,W}, c_{2,W,\beta}\right\}$, we have

$$\mu_{\beta,W}^{+}(B) \geq c_{\beta,W}\mu_{\beta,W}(B). \tag{80}$$

Now, we need to generalize (80) without the assumption $\mu_{\beta,W}(B) \leq \frac{\alpha}{2}$. For that, let us write for $t \in \left(0, \frac{1}{2}\right)$

$$\begin{aligned} I_{\mu_{\beta,W},W}(t) &\geq I_{\mu_{\beta,W},W}\left(\frac{t}{1/2\alpha}\right) \\ &\geq c_{\beta,W}\frac{t}{1/2\alpha} \\ &= (2\alpha c_{\beta,W})\, t. \end{aligned}$$

□

Remark 5. Proposition 4.2 and Corollary 4.1 remain valid when the measure $\mu_{\beta,W}$ has a compact support.

4.3 Estimation of $Che_W(\mu_{\beta,W})$

The proof of Corollary 4.1 does not give a good estimation of $Che_W(\mu_{\beta,W})$. In this section, we give an estimation of $Che_W(\mu_{\beta,W})$ depending on

$$\widetilde{M_1(\mu_{\beta,W})} := \int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|\, d\mu_{\beta,W}(x)\; d\mu_{\beta,W}(y).$$

More precisely, we have:

Proposition 4.3. *With the above notation and assuming that $Var(\xi)$ exists, then, we have the following inequality*

$$\frac{2\left(\frac{5}{3}\right)^{\frac{\beta-1}{\beta}}}{\widetilde{M_1(\mu_{\beta,W})}^{\frac{\beta-1}{\beta}}} \leq Che_W(\mu_{\beta,W}) \leq \frac{I_\beta^{\frac{\beta-1}{\beta}}}{\widetilde{M_1(\mu_{\beta,W})}^{\frac{\beta-1}{\beta}}}, \tag{81}$$

where $I_\beta = \int_0^1\int_0^1 \left(\int_x^y \frac{1}{\min\{t,1-t\}^{\frac{\beta}{\beta-1}}} dt\right) dx\, dy.$

We first give two Lemmas we will use in the proof.

Lemma 4.1. *Let, for $t \in (0,1)$,*

$$J_{\mu_{\beta,W},W}(t) = \mu^+_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t)\right)\right) = \frac{1}{Z^{1-1/\beta}_{\beta,W}} W^{1-\beta}\left(\phi^{-1}_{\beta,W}(t)\right)$$

and

$$\widetilde{Che_W(\mu_{\beta,W})} = \inf_{0<t<1} \frac{J_{\mu_{\beta,W},W}(t)}{\min\{t, 1-t\}}. \tag{82}$$

Then, $\widetilde{Che_W(\mu_{\beta,W})} = Che_W(\mu_{\beta,W})$.

And

Lemma 4.2. *The function* $t \to J_{\mu_{\beta,W},W}(t)$ *is concave on* $(0,1)$. *Therefore,*

$$Che_W(\mu_{\beta,W}) = 2J_{\mu_{\beta,W},W}\left(\frac{1}{2}\right).$$

We postpone the easy and classical proofs of these Lemmas and we prove first Proposition 4.3.

Proof. We start by writing, as in Bobkov's argument [Bo4],

$$\begin{aligned}
\widetilde{M_1(\mu_{\beta,W})} &= \int_{\mathbb{R}}\int_{\mathbb{R}} |p-q|\, d\mu_{\beta,W}(p)\, d\mu_{\beta,W}(q) \\
&= \int_0^1\int_0^1 \left|\phi_{\beta,W}^{-1}(x) - \phi_{\beta,W}^{-1}(y)\right| dx\, dy \\
&= \int_0^1\int_0^1 \left|\int_x^y \frac{Z_{\beta,W}}{W\left(\phi_{\beta,W}^{-1}(t)\right)^{-\beta}} dt\right| dx\, dy \\
&= \int_0^1\int_0^1 \left|\int_x^y \frac{1}{J_{\mu_{\beta,W},W}(t)^{\frac{\beta}{\beta-1}}} dt\right| dx\, dy.
\end{aligned}$$

We begin by proving the right-hand side of (81). Thanks to Lemma 4.1, we can write $J_{\mu_{\beta,W},W}(t)^{\frac{\beta}{\beta-1}} \geq Che_W(\mu_{\beta,W})^{\frac{\beta}{\beta-1}} \min\{t, 1-t\}^{\frac{\beta}{\beta-1}}$. This gives

$$\widetilde{M_1(\mu_{\beta,W})} \leq \frac{1}{Che_W(\mu_{\beta,W})^{\frac{\beta}{\beta-1}}} I_\beta$$

and concludes for the left-hand side. Let us proof now the left-hand side of the inequality (81). We follow the proof of the Proposition 4.1 in [Bo4]. Since $J_{\mu_{\beta,W},W}$ is concave (Lemma 4.2) on $(0,1)$, one can write, for some θ

$$J_{\mu_{\beta,W},W}(t) \leq J_{\mu_{\beta,W},W}\left(\frac{1}{2}\right) + \theta\left(t - \frac{1}{2}\right) := J_\theta(t), \qquad 0 < p < 1.$$

We have $|\theta| \leq 2J_{\mu_{\beta,W},W}\left(\frac{1}{2}\right)$ since $J_{\mu_{\beta,W},W}$ remains non-negative on $(0,1)$. This yields:

$$\begin{aligned}
\widetilde{M_1(\mu_{\beta,W})} &= \int_0^1\int_0^1 \left|\int_x^y \frac{1}{J_{\mu_{\beta,W},W}(t)^{\frac{\beta}{\beta-1}}} dt\right| dx\, dy \\
&\geq \frac{1}{2}\int_0^1\int_0^1\int_{[x,y]} \frac{1}{J_\theta(t)^{\frac{\beta}{\beta-1}}} dt dx\, dy := u(\theta).
\end{aligned}$$

The function $\theta \to 1/J_\theta(t)^{\frac{\beta}{\beta-1}}$ is convex, so is u. As u is symmetric around 0, we have $u(\theta) \geq u(0)$ for all θ. And finally,

$$\begin{aligned}\widetilde{M_1(\mu_{\beta,W})} &\geq \frac{1}{J_{\mu_{\beta,W},W}(1/2)^{\frac{\beta}{\beta-1}}} \int_0^1 \int_0^1 |x-y|\, dx\, dy \\ &= \frac{5}{3J_{\mu_{\beta,W},W}(1/2)^{\frac{\beta}{\beta-1}}}.\end{aligned}$$

Using once again Lemma 4.2, we finally have

$$\frac{1}{2} Che_W(\mu_{\beta,W}) \geq \frac{\left(\frac{5}{3}\right)^{\frac{\beta-1}{\beta}}}{\widetilde{M_1(\mu_{\beta,W})}^{\frac{\beta-1}{\beta}}}.$$

□

Let us proof now the two Lemmas. We begin by proving Lemma 4.1.

Proof. Since it is clear that $Che_W(\mu_{\beta,W}) \leq \widetilde{Che_W(\mu_{\beta,W})}$, it remains to prove that $Che_W(\mu_{\beta,W}) \geq \widetilde{Che_W(\mu_{\beta,W})}$. Let t_0 which realizes the infimum in (77). There are two possibilities:

• If $\min\left\{\mu^+_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t_0)\right)\right), \mu^+_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t_0), +\infty\right)\right)\right\} = \mu^+_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t_0)\right)\right)$, then one can write

$$\widetilde{Che_W(\mu_{\beta,W})} \leq \frac{\mu^+_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t_0)\right)\right)}{\min\{t_0, 1-t_0\}} = Che_W(\mu_{\beta,W}).$$

• Else, if

$$\min\left\{\mu^+_{\beta,W}\left(\left(-\infty, \phi^{-1}_{\beta,W}(t_0)\right)\right), \mu^+_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t_0), +\infty\right)\right)\right\} = \mu^+_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t_0), +\infty\right)\right),$$

one can write

$$\widetilde{Che_W(\mu_{\beta,W})} \leq \frac{\mu^+_{\beta,W}\left(\left(\phi^{-1}_{\beta,W}(1-t_0), +\infty\right)\right)}{\min\{t_0, 1-t_0\}} = Che_W(\mu_{\beta,W}).$$

□

And for Lemma 4.2:

Proof. An easy computation gives

$$J'_{\mu_{\beta,W},W}\left(t\right)=\left(1-\beta\right)W'\left(\phi_{\beta,W}^{-1}\left(t\right)\right).$$

Since W is convex, it follows that $J'_{\mu_{\beta,W},W}$ is a non-increasing function so $J_{\mu_{\beta,W},W}$ is convex on $\left(0,1\right)$. As $J_{\mu_{\beta,W},W}$ is concave, we have for all $t\in\left(0,1\right)$

$$J_{\mu_{\beta,W},W}\left(t\right)\geq 2J_{\mu_{\beta,W},W}\left(\frac{1}{2}\right)\min\left\{t,1-t\right\},$$

with equality when $t=1/2$. $\square$

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