Behaviour of Solutions to the Nonlinear Schrödinger Equation in the Presence of a Resonance

by

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Abstract

The present thesis is split in two parts. The first deals with the focusing Nonlinear Schrödinger Equation in one dimension with pure-power nonlinearity near cubic. We consider the spectrum of the linearized operator about the soliton solution. When the nonlinearity is exactly cubic, the linearized operator has resonances at the edges of the essential spectrum. We establish the degenerate bifurcation of these resonances to eigenvalues as the nonlinearity deviates from cubic. The leading-order expression for these eigenvalues is consistent with previous numerical computations.

The second considers the perturbed energy critical focusing Nonlinear Schrödinger Equation in three dimensions. We construct solitary wave solutions for focusing subcritical perturbations as well as defocusing supercritical perturbations. The construction relies on the resolvent expansion, which is singular due to the presence of a resonance. Specializing to pure power focusing subcritical perturbations we demonstrate, via variational arguments, and for a certain range of powers, the existence of a ground state soliton, which is then shown to be the previously constructed solution. Finally, we present a dynamical theorem which characterizes the fate of radially-symmetric solutions whose initial data are below the action of the ground state. Such solutions will either scatter or blow-up in finite time depending on the sign of a certain function of their initial data.

Lay Summary

We conduct a mathematically motivated study to understand qualitative aspects of the nonlinear Schrödinger equation. For this summary, however, we imagine our equation as describing the positions of many cold quantum particles in a cloud. A group of particles may cluster together and persist in this configuration for all time; this structure being called a *soliton.* One aspect we are interested in is the stability of the soliton. It may be stable - a small disruption of the system will be weathered and the soliton will remain, or unstable - a perturbation will cause the particles to break up, destroying the soliton. The soliton also impacts the overall dynamics of the equation. If the particles do not have enough mass or energy to form a soliton, they spread out and *scatter*. On the other hand, with too much mass or energy, they may *blow-up*, coming together to form a singularity.

Preface

A version of Chapter 2 has been published in [19]. I conducted much of the analysis, all of the numerics, and wrote most of the manuscript.

A version of Chapter 3 has, at the time of this writing, been submitted for publication to an academic journal. The preprint is available here [20]. I conducted much of the analysis and wrote most of the manuscript.

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Chapter 1

Introduction

1.1 The Nonlinear Schrödinger Equation

The nonlinear Schrödinger equation (NLS) is the partial differential equation (PDE)

$$i\partial_t u = -\Delta u \pm |u|^{p-1}u. \tag{1.1}$$

It attracts interest from both pure and applied mathematicians, with applications including quantum mechanics, water waves, and optics [35], [74]. Before discussing the applications we establish some terminology and notation. Here $u = u(x,t)$ is a complex-valued function, x is often thought of a spacial variable (in n real dimensions, $x \in \mathbb{R}^n$), and t is typically the temporal variable ($t \in \mathbb{R}$). The operator ∂_t is the partial derivative in time and $\Delta = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2$ is the laplacian. We have chosen the pure power nonlinearity, $\pm|u|^{p-1}u$, with power $p \in (1,\infty)$, but could also replace this nonlinearity with a more complicated function of u. The nonlinearity with the negative sign, $-|u|^{p-1}u$, will be referred to as the *focusing* or *attractive* case while the positive, $+|u|^{p-1}u$, is the *defocusing* or *repulsive* nonlinearity. In applications the most common nonlinearities are the cubic ($p = 3$) and the quintic ($p = 5$). Also common is the cubic-quintic nonlinearity: the sum or difference of cubic and quintic terms.

Let us now briefly describe some of the contexts in which the NLS can be applied. Firstly, in quantum physics the nonlinear Schrödinger equation, and the closely related Gross-Pitaevskii equation (which is NLS subject to an external potential), describes the so called Bose-Einstein condensate. A Bose-Einstein condensate is a cloud of cold bosonic particles all of which are in their lowest energy configuration. In this way, the mass of particles can be described by one wave function; that is $u(x,t)$. In this context the absolute value $|u(x,t)|^2$ describes the probability density to find particles within the cloud at point x in space, and at time t. The inter-particle forces may be either attractive or repulsive, which, corresponds to focusing or defocusing nonlinearities, respectively. See [74] or the review [14] for more information.

The NLS has also applications in water waves, a pursuit that dates back to [91]. For a thorough description see, for example, Chapter 11 of [74] (and references therein). The water-wave problem concerns the dynamics of a wave train propagating at the surface of a liquid. In deep water, the solution $u(x,t)$ to the 1D cubic ($p = 3$) NLS describes an envelope which captures the behaviour of a wave which is modulated only in the direction in which it propagates.

In nonlinear optics, the function u describes a wave propagating in a weakly nonlinear dielectric (see again [74], Chapter 1). In this case Δ is the Laplacian transverse to the propagation and the time variable t is replaced by the spacial variable along the direction of propagation. For example in 3D where $x = (x_1, x_2, x_3)$ the NLS may take the form

$$i\partial_{x_3} u = -(\partial_{x_1}^2 + \partial_{x_2}^2)u \pm |u|^2 u$$

where x_3 is in the direction of propagation.

On the pure mathematical side, the NLS is interesting as a general model of dispersive and nonlinear wave phenomena. The nonlinear Schrödinger equation provides an arena to develop techniques which can be applied to other nonlinear dispersive equations. The NLS is often technically simpler than other equations [15], such as the Korteweg-de Vries equation (KdV), the nonlinear wave equation (NLW), Zakharov system, Boussinesq equation, and various other water-wave models.

1.2 Conserved Quantities, Scaling, and Criticality

The facts we review here are standard and can be found in, for example, the books [15, 35, 74, 77].

The NLS (1.1) has the following conserved quantities

$$\mathcal{M}(u) = \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 \, dx, \quad \mathcal{E}(u) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p+1}|u|^{p+1} \, dx \tag{1.2}$$

often called *mass* and *energy*, respectively. These quantities are conserved in time, that is: $\mathcal{M}(u(x,t)) = \mathcal{M}(u(x,0))$ and $\mathcal{E}(u(x,t)) = \mathcal{E}(u(x,0))$ for sufficiently smooth solutions.

The scaling

$$u(x,t) \mapsto u^\lambda(x,t) := \lambda^{-2/(p-1)} u(\lambda^{-1}x, \lambda^{-2}t) \tag{1.3}$$

preserves the equation. That is if $u(x,t)$ is a solution to (1.1) then $u^\lambda(x,t)$ is also a solution.

When the power p and dimension n are chosen according to the following relation

$$p = \frac{4}{n} + 1$$

our scaling (1.3) also preserves the mass, that is $\mathcal{M}(u^\lambda(\cdot,t)) = \mathcal{M}(u(\cdot,\lambda^{-2}t))$. Such an equation is called *mass critical*. For example, in one dimension ($n = 1$) the quintic ($p = 5$) NLS is mass critical. For values of $p < 4/n + 1$ we call the equation *mass sub-critical* and for $p > 4/n + 1$ we say *mass super-critical*.

Similarly, if

$$p = 1 + \frac{4}{n-2}, \quad n \geq 3$$

then (1.3) preserves the energy, so $\mathcal{E}(u^\lambda(\cdot,t)) = \mathcal{E}(u(\cdot,\lambda^{-2}t))$. We call this equation an *energy critical* equation. For example, in three dimensions ($n = 3$) the quintic ($p = 5$) NLS is energy critical. Again we refer to $p < 1 + 4/(n-2)$ as *energy sub-critical* and $p > 1 + 4/(n-2)$ as *energy super-critical*. Note that for dimensions one and two all equations with $p < \infty$ are energy sub-critical.

1.3 Scattering and Blow-Up

The overall dynamics of the equation are affected by the power p's relation to the mass and energy critical values. By overall dynamics we mean the long time behaviour of the solution subject to an initial condition, ie. the Cauchy problem

$$\begin{cases} i\partial_t u = -\Delta u \pm |u|^{p-1}u \\ u(x,0) = u_0(x) \end{cases} . \tag{1.4}$$

We seek theorems which characterize the solution's eventual behaviour for a large class of initial conditions u_0.

One possibility is that the solution will *scatter* (Chapter 7 of [15], Chapter 3.3 of [74]). This means that the solution $u(x,t)$ will eventually look like a solution to the linear Schrödinger equation. That is, after sufficient time, all nonlinear behaviour has disappeared and only linear behaviour remains.

Solutions to the linear Schrödinger equation (Chapter 2 of [15], Chapter 3.1 of [74])

$$\begin{cases} i\partial_t u = -\Delta u \\ u(x,0) = u_0 \end{cases}$$

evolve in a predictable way. The mass, or L^2 norm, is preserved while the L^∞ norm (supremum) decays according to

$$\|u(x,t)\|_{L^\infty(\mathbb{R}^n)} \lesssim t^{-n/2}\|u_0\|_{L^1(\mathbb{R}^n)}. \tag{1.5}$$

The above norms are defined as

$$\|u(x,t)\|_{L^\infty(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} |u(x,t)| \quad \text{and} \quad \|u(x,t)\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u(x,t)|\; dx$$

and when we write $f(t) \lesssim g(t)$ we mean that there is a constant C, independent of t, such that $f(t) \leq Cg(t)$.

We may think of the linear Schrödinger equation as modeling a free quantum particle. The particle has a tendency to "spread out" due to momentum uncertainty, even if it is well localized to start.

A scattering theorem will then have the following form: for all $u_0 \in X$ (or some subset of X) we have

$$\|u(x,t) - u_{lin}\|_X \to 0, \quad \text{as} \quad t \to \infty.$$

Here u_{lin} is a solution to the linear Schrödinger equation (chosen according to the particular u_0), X is an appropriate function space, and $\|\cdot\|_X$ is a norm on this space.

Scattering is, to some extent, expected in the defocusing (repulsive) case. We think about the Laplacian as being a force for dispersion, as evidenced by the behaviour of the linear Schrödinger equation, and we think of the defocusing nonlinearity as a repellent to the gathering of mass. We single out the result [38], a scattering theorem in the *energy space* $X = H^1$ for defocusing NLS with power p between mass critical and energy critical,

$$1 + \frac{4}{n} < p < 1 + \frac{4}{n-2},$$

in dimensions $n \geq 3$. The space H^1 consists of functions u whose H^1 norm,

$$\begin{aligned}\|u(x,t)\|^2_{H^1(\mathbb{R}^n)} &= \|u(x,t)\|^2_{L^2(\mathbb{R}^n)} + \|u(x,t)\|^2_{\dot{H}^1(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} |u(x,t)|^2\; dx + \int_{\mathbb{R}^n} |\nabla u(x,t)|^2\; dx,\end{aligned}$$

is finite.

For the focusing equation, with $n \geq 3$, $X = H^1$, $1 + 4/n < p < 1 + 4/(n-2)$, we have the scattering theory [73]. The key difference now in the focusing case is the assumption that the energy (H^1) norm of the initial condition, u_0, be small. Here we see that if we do not have enough mass or energy to begin with, the force of dispersion will win out over the focusing nonlinearity's tendency to attract mass together.

In the focusing equation, however, there are other possible fates for solutions besides scattering. If too much mass is assembled, the attractive nonlinearity may cause the solution to break down in finite time; that is, some norm of the solution will *blow-up*. For example, (Chapter 5 of [74]) consider the focusing NLS with $p \geq 1 + 4/n$. There exist initial conditions in $u_0 \in H^1$ such that there exists a time $t_* < \infty$ such that

$$\lim_{t \to t_*} \|\nabla u(x,t)\|_{L^2} = \infty.$$

Important for the discussion of mass super-critical and energy sub-critical equations above, scattering in the defocusing case and blow-up in the focusing case, is the local existence theory. The H^1 local theory ensures an initial data $u_0 \in H^1$ will generate a solution $u(x,t)$ with continuous in time H^1 norm up to some time T. Here T is a non-increasing function of $\|u_0\|_{H^1}$. Repeatedly applying the local theory together with an a priori H^1 bound (such as in the defocusing case) then yields global existence, ie. the solution exists for all time (the maximal time of existence $T_{\max} = \infty$). In the absence of an a priori H^1 bound (focusing case) it's possible that the maximal time of existence for our initial data is finite. Indeed, the local theory provides the following *blow-up criterion*: if $T_{\max} < \infty$ then $\lim_{t \to T_{\max}^-} \|u(\cdot,t)\|_{H^1} = \infty$.

The energy critical case, $p = 1 + 4/(n-2)$, is more challenging since the blow-up criterion generated by the corresponding local theory is more complicated. Nevertheless, in the defocusing energy critical case the global existence and scattering theory is more or less complete. First we have scattering theorems [10] [11] in dimensions $n = 2, 3$, also [40] in dimension $n = 3$, and then [76] in dimensions $n \geq 5$, all where the initial data u_0 was assumed to be radial. The radial assumption was removed in [21] for $n = 3$ and later in [67] for $n = 4$ and finally in [83] for $n \geq 5$. Scattering in the defocusing energy super-critical case remains a substantial open problem.

We discuss scattering and blow-up for the focusing energy critical equation (a study which was initiated in [54]) in Section 1.7 and Chapter 3.

1.4 Solitary Waves and Stability

In the focusing NLS,

$$i\partial_t u = -\Delta u - |u|^{p-1} u,$$

the dispersive force of the Laplacian may be balanced by the attractive nonlinearity leading to solutions that neither scatter nor blow-up. Our NLS may admit solutions of the form

$$u(x,t) = e^{i\omega t}Q(x) \tag{1.6}$$

often called *solitary wave* solutions or *solitons*. We think of solitary waves as bound states or equilibrium solutions: the time dependence is confined to the phase and the absolute value, $|u|$, is preserved. We further classify solitons as *ground states* if they minimize the *action*

$$\mathcal{S}_\omega = \mathcal{E} + \omega\mathcal{M}$$

among all non-zero solutions of the form (1.6).

The time independent function $Q(x)$ is the solitary wave profile, which, satisfies the following elliptic partial differential equation

$$-\Delta Q - |Q|^{p-1}Q + \omega Q = 0. \tag{1.7}$$

The above elliptic equation (1.7) is well studied, with results going back to [72] and [9]. Much of the present thesis concerns solitary waves. In particular, their existence, stability, and impact on the overall dynamics of the equation.

Ground state solitary waves are stable for mass sub-critical powers of p [16] and unstable for mass critical [88] and mass super-critical powers [7]. We call solitary waves (*orbitally*) *stable* if initial conditions close to the soliton produce solutions which remain close to the soliton (modulo symmetries of spatial translation and phase rotation) for all time. More precisely, let $\varphi(x)$ be the spatial profile of the ground state, $u_0(x) \in H^1$ an initial condition, and $u(x,t)$ the solution generated by the initial condition. We say $e^{i\omega t}\varphi$ is (orbitally) stable if for every $\varepsilon > 0$ there exits a $\delta(\varepsilon) > 0$ such that if $\|\phi - u_0\|_{H^1} \leq \delta(\varepsilon)$ then

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^n} \|u(\cdot,t) - e^{i\theta}\phi(\cdot - y)\|_{H^1} < \varepsilon.$$

Otheriwise, we say a soliton is *unstable*. In fact, both [88] and [7] demonstrate that some initial condition close to the soliton blows up in finite time.

To study stability one can employ variational methods, as in [16], [88], [7] or else study the linearized operator around the soliton (see [39], [41], [89]). Understanding the spectrum of the linearized operator is often necessary to establish asymptotic stability results such as those obtained in [6, 13, 22, 23, 26, 36, 65, 68]. A soliton is *asymptotically stable* if initial conditions close to the soliton produce solutions which converge to a (nearby) soliton as $t \to \infty$. The study we initiate in Section 1.6 and Chapter 2 concerns the linearized operator about 1D solitons.

1.5 Resonance

By *resonance*, or *resonance eigenvalue*, we mean a 'would be' eigenvalue, usually at the edge of the continuous spectrum. The resonance is not a true eigenvalue because its resonance eigenfunction does not have sufficient decay to be square integrable. The appearance of a resonance, in the linearized operator about a soliton for example, is a non-generic occurrence. While it is non-generic, however, it does appear in a few key equations; in particular those NLS that we study in Chapter 2 and Chapter 3. Such a resonance may complicate analysis, for example by making singular the resolvent expansion and slowing the time-decay of perturbations to a soliton.

For example, let us consider the linear Schrödinger operator

$$H := -\Delta + V \tag{1.8}$$

in n dimensions where $V = V(x)$ is a potential. We may have a resonance eigenvalue λ with resonance eigenfunction ξ such that

$$H\xi = \lambda\xi$$

but $\xi \notin L^2(\mathbb{R}^n)$.

In 3D we require the resonance ξ be in $L^3_w(\mathbb{R}^3)$, the weak L^3 space. Therefore, ξ may decay like $1/|x|$ at infinity. The original paper [47] computes several terms in the resolvent expansion in the presence of a resonance in 3D. The resolvent is singular as $\lambda \to 0$ and takes the following form (assuming we have no edge-eigenvalue):

$$(H + \lambda^2)^{-1} = O\left(\frac{1}{\lambda}\right) \tag{1.9}$$

as an operator on suitable spaces. Moreover, the time-decay estimate (1.5) is retarded and decays in time like $t^{-1/2}$ instead of $t^{-3/2}$. A restated version of these results are crucial in the analysis of Chapter 3.

In 1D a resonance ξ may not decay at all. If $H\xi = \lambda\xi$ and $\xi \notin L^p$ for any $p < \infty$ but $\xi \in L^\infty$ then we regard ξ as a resonance. The more recent work [48] provides a unified approach to resolvent expansions across all dimensions, and so in 1D in particular, which, we rely on in Chapter 2. The resolvent expansion itself in 1D appears similar to (1.9).

Interestingly, resonances in the Schrödinger operator only appear in dimensions 1-4 [46–48] not in dimensions $n \geq 5$ [45].

1.6 The 1D Linearized NLS

Consider now the following focusing NLS in 1 space dimension

$$i\partial_t u = -\partial_x^2 u - |u|^{p-1}u. \qquad (\mathrm{NLS}_p)$$

Chapter 2 deals with the above equation and so we supply here some background, motivation, and connection to previous works.

The above (NLS_p) is known to exhibit *solitary waves.* Indeed, since we are in 1D the solitons are available in the following explicit form

$$u(x,t) = Q_p(x)e^{it}$$

where

$$Q_p^{p-1}(x) = \left(\frac{p+1}{2}\right)\operatorname{sech}^2\left(\frac{p-1}{2}x\right).$$

One naturally asks about the *stability* of these waves, which leads immediately to an investigation of the spectrum of the *linearized operator* governing the dynamics close to the solitary wave solution.

The linearized operator is obtained by considering a perturbation of the solitary wave,

$$u(x,t) = (Q_p(x) + h(x,t))e^{it},$$

and neglecting all but the leading order in the resulting system. After we complexify, ie. letting $\vec{h} = \begin{pmatrix} h & \bar{h} \end{pmatrix}^T$, we obtain the linear system

$$i\partial_t \vec{h} = \mathcal{L}_p \vec{h}$$

where

$$\mathcal{L}_p = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\left(\begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & -\partial_x^2 + 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p+1 & p-1 \\ p-1 & p+1 \end{pmatrix} Q_p^{p-1}\right)$$

is the linearized operator. See section 2.1. Systematic spectral analysis of the linearized operator has a long history (eg. [39, 89], and for more recent studies [17, 30, 85, 86]).

The principle motivation for Chapter 2 comes from [17] where *resonance* eigenvalues (with explicit resonance eigenfunctions) were observed to sit at the edges (or *thresholds*) of the spectrum for the 1D linearized NLS problem with focusing cubic nonlinearity. Numerically, it was observed that the same problem with power nonlinearity close to $p = 3$ (on both sides) has a true eigenvalue close to the threshold. We establish analytically the observed qualitative behaviour. Stated roughly, the main result of Chapter 2 is:

for $p \approx 3$, $p \neq 3$, *the linearization of the 1D* (NLS_p) *about its soliton has purely imaginary eigenvalues, bifurcating from resonances at the edges of the essential spectrum of linearized* (NLS_3), *whose distance from the thresholds is of order* $(p-3)^4$.

The exact statement is given as Theorem 2.3.1 in Section 2.3, and includes the precise leading order behaviour of the eigenvalues.

The eigenvalues obtained here, being on the imaginary axis, correspond to *stable* behaviour at the linear level. A further motivation for obtaining detailed information about the spectra of linearized operators is that such information is a key ingredient in studying the *asymptotic stability* of solitary waves: see [6, 13, 22, 23, 26, 36, 65, 68] for some results of this type. Such results typically assume the absence of threshold eigenvalues or resonances. The presence of a resonance is an exceptional case which complicates the stability analysis by retarding the time-decay of perturbations. Nevertheless, the asymptotic stability of solitons in the 1D cubic focusing NLS was recently proved in [29]. The proof relies on integrable systems technology and so is only available for the cubic equation. The solitons are known to be stable in the (weaker) orbital sense for all $p < 5$ (the so-called mass subcritical range) while for $p \geq 5$ they are unstable [41, 90], but the question of asymptotic stability for $p < 5$ and $p \neq 3$ seems to be open. The existence (and location) of eigenvalues on the imaginary axis, which is shown here, should play a role in any attempt on this problem.

The generic bifurcation of resonances and eigenvalues from the edge of the essential spectrum was studied by [28] and [84] in three dimensions. Edge bifurcations have also been studied in one dimensional systems using the Evans function in [52] and [53] as well as in the earlier works [50], [51] and [64]. We do not follow that route, but rather adopt the approach of [28, 84] (going back also to [48], and in turn to the classical work [47]), using a *Birman-Schwinger* formulation, resolvent expansion, and *Lyapunov-Schmidt reduction.*

Our work is distinct from [28, 84] due to the unique challenges of working in one dimension, in particular the strong singularity of the free resolvent at zero energy, which among other things increases by one the dimension of the range of the projection involved in the Lyapunov-Schmidt reduction procedure.

Moreover, our work is distinct from all of [28, 52, 53, 84] in that we study the particular (and as it turns out non-generic) resonance and perturbation corresponding to the near-cubic pure-power NLS problem. Generically, a resonance is associated with the birth or death of an eigenvalue, and such is the picture obtained in [28, 52, 53, 84]: an eigenvalue approaches the essential spectrum, becomes a resonance on the threshold and then disappears. In our setting, the eigenvalue approaches the essential spectrum, sits on the threshold as a resonance, then returns as an eigenvalue. The bifurcation is degenerate in the sense that the expansion of the eigenvalue begins at higher order, and the analysis we develop to locate this eigenvalue is thus considerably more delicate.

1.7 A Perturbation of the 3D Energy Critical NLS

In Chapter 3 we consider Nonlinear Schrödinger equations in three space dimensions, of the form

$$i\partial_t u = -\Delta u - |u|^4 u - \varepsilon g(|u|^2)u, \tag{1.10}$$

where ε is a small, real parameter. Equation (1.10) is a perturbed version of the focusing energy critical NLS. This section is devoted to introducing the above equation, providing some background on the unperturbed critical equation, and stating the main theorems of Chapter 3.

The mass and energy of (1.10) are

$$\mathcal{M}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 \, dx, \qquad \mathcal{E}_\varepsilon(u) = \int_{\mathbb{R}^3} \left\{ \frac{1}{2}|\nabla u|^2 - \frac{1}{6}|u|^6 - \frac{\varepsilon}{2}G(|u|^2) \right\} \, dx$$

where $G' = g$. We are particularly interested in the existence (and dynamical implications) of solitary wave solutions of the form

$$u(x,t) = Q(x)e^{i\omega t}$$

of (1.10). We will consider only real-valued solitary wave profiles, $Q(x) \in \mathbb{R}$, for which the corresponding stationary problem is

$$-\Delta Q - Q^5 - \varepsilon f(Q) + \omega Q = 0, \qquad f(Q) = g(Q^2)Q. \tag{1.11}$$

Since the perturbed solitary wave equation (1.11) is the Euler-Lagrange equation for the *action*

$$\mathcal{S}_{\varepsilon,\omega}(u) := \mathcal{E}_\varepsilon(u) + \omega \mathcal{M}(u) \ ,$$

the standard *Pohozaev relations* [34] give necessary conditions for existence of finite-action solutions of (1.11):

$$\begin{aligned} 0 = \mathcal{K}_\varepsilon(u) &:= \frac{d}{d\mu} \mathcal{S}_{\varepsilon,\omega}(T_\mu u)\bigg|_{\mu=1} \\ &= \int |\nabla Q|^2 - \int Q^6 + \varepsilon \int \left(3F(Q) - \frac{3}{2} Q f(Q)) \right) \\ 0 = \mathcal{K}^{(0)}_{\varepsilon,\omega}(u) &:= \frac{d}{d\mu} \mathcal{S}_{\varepsilon,\omega}(S_\mu u)\bigg|_{\mu=1} = \varepsilon \int \left(3F(Q) - \frac{1}{2} Q f(Q) \right) - \omega \int Q^2 \end{aligned} \tag{1.12}$$

where

$$(T_\mu u)(x) := \mu^{\frac{3}{2}} u(\mu x), \qquad (S_\mu u)(x) := \mu^{\frac{1}{2}} u(\mu x)$$

are the scaling operators preserving, respectively, the L^2 norm and the L^6 (and $\dot{H}^1$) norm, and $F' = f$ (so $F(Q) = \frac{1}{2} G(Q^2)$).

The corresponding unperturbed ($\varepsilon = 0$) problem, the 3D quintic equation

$$i \partial_t u = -\Delta u - |u|^4 u, \tag{1.13}$$

is energy critical ie. the scaling

$$u(x,t) \mapsto u_\lambda(x,t) := \lambda^{1/2} u(\lambda x, \lambda^2 t)$$

which preserves (1.13), also leaves invariant its energy

$$\mathcal{E}_0(u) = \int_{\mathbb{R}^3} \left\{ \frac{1}{2} |\nabla u|^2 - \frac{1}{6} |u|^6 \right\} \, dx, \qquad \mathcal{E}_0(u_\lambda(\cdot,t)) = \mathcal{E}_0(u(\cdot, \lambda^2 t)).$$

One implication of energy criticality is that (1.13) fails to admit solitary waves with $\omega \neq 0$ – as can be seen from (1.12) – but instead admits the *Aubin-Talenti static* solution

$$W(x) = \left(1 + \frac{|x|^2}{3} \right)^{-1/2}, \qquad \Delta W + W^5 = 0, \tag{1.14}$$

whose slow spatial decay means it fails to lie in $L^2(\mathbb{R}^3)$, though it does fall in the *energy space*

$$W \notin L^2(\mathbb{R}^3), \quad W \in \dot{H}^1(\mathbb{R}^3) = \{ u \in L^6(\mathbb{R}^3) \mid \|u\|_{\dot{H}^1} := \|\nabla u\|_{L^2} < \infty \}.$$

By scaling invariance, $W_\mu := S_\mu W = \mu^{1/2} W(\mu x)$, for $\mu > 0$, also satisfy (1.14), as do their negatives and spatial translates $\pm W_\mu(\cdot + a)$ $(a \in \mathbb{R}^3)$. These functions (and their multiples) are well-known to be the only functions realizing the best constant appearing in the Sobolev inequality [4, 75]

$$\int_{\mathbb{R}^3} |u|^6 \le C_3 \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^3, \quad C_3 = \frac{\int_{\mathbb{R}^3} W^6}{\left(\int_{\mathbb{R}^3} |\nabla W|^2 \right)^3} = \frac{1}{\left(\int_{\mathbb{R}^3} W^6 \right)^2},$$

where the last equality used $\int |\nabla W|^2 = \int W^6$ (as follows from (1.12)). A closely related statement is that W, together with its scalings, negatives and spatial translates, are the only minimizers of the energy under the Pohozaev constraint (1.12) with $\varepsilon = \omega = 0$:

$$\begin{aligned} &\min\{\mathcal{E}_0(u) \mid 0 \neq u \in \dot{H}^1(\mathbb{R}^3),\ \mathcal{K}_0(u) = 0\} = \mathcal{E}_0(W) = \mathcal{E}_0(\pm W_\mu(\cdot + a)), \\ &\mathcal{K}_0(u) = \int_{\mathbb{R}^3} \left\{ |\nabla u|^2 - |u|^6 \right\}. \end{aligned} \tag{1.15}$$

It follows that for solutions of (1.13) lying energetically ‘below’ W, $\mathcal{E}_0(u) < \mathcal{E}_0(W)$, the sets where $\mathcal{K}_0(u) > 0$ and where $\mathcal{K}_0(u) < 0$ are invariant for (1.13). The celebrated result [54] showed that radially symmetric solutions in the first set scatter to 0, while those in the second set become singular in finite time (in dimensions 3, 4, 5). In this way, W plays a central role in classifying solutions of (1.13), and it is natural to think of W (together with its scalings and spatial translates) as the *ground states* of (1.13). The assumption in [54] that solutions be radially symmetric was removed in [57] for dimensions $n \ge 5$ and then for $n = 4$ in [32]. Removing the radial symmetry assumption appears still open for $n = 3$. A characterization of the dynamics for initial data at the threshold $\mathcal{E}_0(u_0) = \mathcal{E}_0(W)$ appears in [33], and a classification of global dynamics based on initial data slightly above the ground state is given in [60].

Just as the main interest in studying (1.13) is in exploring the implications of critical scaling, the main interest in studying (1.10) and (1.11) here is the effect of *perturbing* the critical scaling, in particular: the emergence of ground state solitary waves from the static solution W, the resulting energy landscape, and its implications for the dynamics.

A natural analogue for (1.11) of the ground state variational problem (1.15) is

$$\min\{\mathcal{S}_{\varepsilon,\omega}(u) \mid u \in H^1 \setminus \{0\}, \mathcal{K}_\varepsilon(u) = 0\}. \tag{1.16}$$

For a study of similar minimization problems see [7] and [8] as well as [3], which treats a large class of critical problems and establishes the existence of ground state solutions. In space dimensions 4 and 5, [1, 2] showed the existence of minimizers for (the analogue of) (1.16), hence of ground state solitary waves, for each $\omega > 0$ and $\varepsilon g(|u|^2)u$ sufficiently small and subcritical; moreover, a blow-up/scattering dichotomy 'below' the ground states in the spirit of [54] holds. Our intention is to establish the existence of ground states, and the blow-up/scattering dichotomy, in the 3-dimensional setting. In dimension 3, the question of the existence of minimizers for (1.16) is more subtle, and we proceed via a perturbative construction, rather than a direct variational method.

A key role in the analysis is played by the linearization of (1.14) around W, in particular the linearized operator

$$H := -\Delta + V := -\Delta - 5W^4, \tag{1.17}$$

which as a consequence of scaling invariance has the following resonance:

$$H \, \Lambda W = 0, \quad \Lambda W := \frac{d}{d\mu} S_\mu W|_{\mu=0} = \left(\frac{1}{2} + x \cdot \nabla\right) W \; \notin L^2(\mathbb{R}^3). \tag{1.18}$$

Indeed $\Lambda W = W^3 - \frac{1}{2}W$ decays like $|x|^{-1}$, and so

$$W, \, \Lambda W \; \in L^r(\mathbb{R}^3) \cap \dot{H}^1(\mathbb{R}^3), \;\; 3 < r \leq \infty.$$

Our first goal is to find solutions to (1.11) where $\omega = \omega(\varepsilon) > 0$ is small and $Q(x) \in \mathbb{R}$ is a perturbation of W in some appropriate sense. One obstacle is that $W \notin L^2$ is a slowly decaying function, whereas solutions of (1.11) satisfy $Q \in L^2$, and indeed are exponentially decaying.

Assumption 1.7.1. *Take $f : \mathbb{R} \to \mathbb{R} \in C^1$ such that $f(0) = 0$ and*

$$|f'(s)| \lesssim |s|^{p_1 - 1} + |s|^{p_2 - 1}$$

with $2 < p_1 \leq p_2 < \infty$. Further assume that

$$\langle \Lambda W, f(W) \rangle < 0.$$

Theorem 1.7.2. *There exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$, there is $\omega = \omega(\varepsilon) > 0$, and smooth, real-valued, radially symmetric $Q = Q_\varepsilon \in H^1(\mathbb{R}^3)$ satisfying (1.11) with*

$$\omega = \omega_1 \varepsilon^2 + \tilde{\omega} \tag{1.19}$$

$$Q(x) = W(x) + \eta(x) \tag{1.20}$$

where

$$\omega_1 = \left(\frac{-\langle \Lambda W, f(W) \rangle}{6\pi} \right)^2,$$

$\tilde{\omega} = O(\varepsilon^{2+\delta_1})$ *for any* $\delta_1 < \min(1, p_1 - 2)$*,* $\|\eta\|_{L^r} \lesssim \varepsilon^{1-3/r}$ *for all* $3 < r \leq \infty$*, and* $\|\eta\|_{\dot{H}^1} \lesssim \varepsilon^{1/2}$*. In particular,* $Q \to W$ *in* $L^r \cap \dot{H}^1$ *as* $\varepsilon \to 0$*.*

Remark 1.7.3. *We have a further decomposition of* η *but the leading order term depends on whether we measure it in* L^r *with* $r = \infty$ *or* $3 < r < \infty$*. See Lemmas 3.1.9 and 3.1.10.*

Remark 1.7.4. *Note that allowable* f *include* $f(Q) = |Q|^{p-1}Q$ *with* $2 < p < 5$*, the subcritical, pure-power, focusing nonlinearities, as well as* $f(Q) = -|Q|^{p-1}Q$ *with* $5 < p < \infty$*, the supercritical, pure power, defocusing nonlinearities. Observe*

$$\begin{aligned} \langle \Lambda W, W^p \rangle &= \int \left(\frac{1}{2} W^{p+1} + W^p (x \cdot \nabla) W \right) \\ &= \int \left(\frac{1}{2} W^{p+1} + \frac{1}{p+1} (x \cdot \nabla) W^{p+1} \right) \\ &= \int \left(\frac{1}{2} - \frac{3}{p+1} \right) W^{p+1} \end{aligned}$$

which is negative when $2 < p < 5$ *and positive when* $p > 5$*.*

Remark 1.7.5. *Since* $Q_\varepsilon \to W$ *in* L^r *for* $r \in (3, \infty]$*, the Pohozaev identity* (1.12)*, together with the divergence theorem, implies that for any such family of solutions, a necessary condition is*

$$\begin{aligned} \langle \Lambda W, f(W) \rangle = \int \left(\frac{1}{2} W f(W) - 3F(W) \right) &= \lim_{\varepsilon \to 0} \int \left(\frac{1}{2} Q_\varepsilon f(Q_\varepsilon) - 3F(Q_\varepsilon) \right) \\ &\leq 0. \end{aligned}$$

Remark 1.7.6. *Note that* $Q \in L^r \cap \dot{H}^1$ $(3 < r \leq \infty)$ *satisfying* (1.11) *lies automatically in* L^2 *(and hence* H^1*): by the Pohozaev relations* (1.12)*:*

$$0 = \int |\nabla Q|^2 - \int Q^6 - \varepsilon \int f(Q) Q + \omega \int Q^2. \tag{1.21}$$

The first two integrals are then finite. We can also bound the third

$$\left| \int f(Q) Q \right| \leq \int |f(Q)||Q| \lesssim \int |Q|^{p_1+1} + \int |Q|^{p_2+1} < \infty$$

since $p_2 + 1 \geq p_1 + 1 > 3$. In this way $\int Q^2$ must be finite. Moreover, since $Q \in L^r$ with $r > 6$, a standard elliptic regularity argument implies that Q is in fact a smooth function. Therefore it suffices to find a solution $Q \in L^r \cap \dot{H}^1$.

The paper [31] considers an elliptic problem similar to (1.11):

$$-\Delta Q + Q - Q^p - \lambda Q^q = 0$$

with $1 < q < 3$, $\lambda > 0$ large and fixed, and $p < 5$ but $p \to 5$. They demonstrate the existence of three positive solutions, one of which approaches W (1.14) as $p \to 5$. The follow up [18] established a similar result with $p \to 5$ but $p > 5$ and $3 < q < 5$. While [31] and [18] are perturbative in nature, their method of construction differs from ours.

The proof of Theorem 1.7.2 is presented in Section 3.1. As the statement suggests, the argument is perturbative – the solitary wave profiles Q are constructed as small (in L^r) corrections to W. The set-up is given in Section 3.1.1. The equation for the correction η involves the resolvent of the linearized operator H. A Lyapunov-Schmidt-type procedure is used to recover uniform boundedness of this resolvent in the presence of the resonance ΛW – see Section 3.1.2 for the relevant estimates – and to determine the frequency ω, see Section 3.1.3. Finally, the correction η is determined by a fixed point argument in Section 3.1.4.

The next question is if the solution Q is a *ground state* in a suitable sense. For this question, we will specialize to pure, subcritical powers $f(Q) = |Q|^{p-1}Q$, $3 < p < 5$, for which the 'ground state' variational problem (1.16) reads

$$\begin{aligned} &\min\{\mathcal{S}_{\varepsilon,\omega}(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{K}_\varepsilon(u) = 0\}, \\ &\mathcal{S}_{\varepsilon,\omega}(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{6}\|u\|_{L^6}^6 - \frac{1}{(p+1)}\varepsilon\|u\|_{L^{p+1}}^{p+1} + \frac{1}{2}\omega\|u\|_{L^2}^2, \\ &\mathcal{K}_\varepsilon(u) = \|\nabla u\|_{L^2}^2 - \|u\|_{L^6}^6 - \frac{3(p-1)}{2(p+1)}\varepsilon\|u\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{1.22}$$

Theorem 1.7.7. *Let $f(Q) = |Q|^{p-1}Q$ with $3 < p < 5$. There exists ε_0 such that for each $0 < \varepsilon \leq \varepsilon_0$ and $\omega = \omega(\varepsilon) > 0$ furnished by Theorem 1.7.2, the solitary wave profile Q_ε constructed in Theorem 1.7.2 is a minimizer of problem* (1.22)*. Moreover, Q_ε is the unique positive, radially-symmetric minimizer.*

Remark 1.7.8. *It follows from Theorem 1.7.7 that the solitary wave profiles are positive: $Q_\varepsilon(x) > 0$.*

Remark 1.7.9. *(see Corollary 3.2.12). By scaling, for each $\varepsilon > 0$ there is an interval $[\underline{\omega}, \infty) \ni \omega(\varepsilon)$, such that for $\omega \in [\underline{\omega}, \infty)$,*

$$Q(x) := \left(\frac{\varepsilon}{\hat{\varepsilon}}\right)^{\frac{1}{5-p}} Q_{\hat{\varepsilon}}\left(\left(\frac{\varepsilon}{\hat{\varepsilon}}\right)^{\frac{2}{5-p}} x\right),$$

where $0 < \hat{\varepsilon} \leq \varepsilon_0$ satisfies $(\omega(\hat{\varepsilon})/\omega) = (\hat{\varepsilon}/\varepsilon)^{4/(5-p)}$, solves the corresponding minimization problem (1.22)*. Here the function $Q_{\hat{\varepsilon}}$ is the solution constructed by Theorem 1.7.2 with $\hat{\varepsilon}$ and $\omega(\hat{\varepsilon})$.*

The proof of Theorem 1.7.7 is presented in Section 3.2. It is somewhat indirect. We first use the $Q = Q_\varepsilon$ constructed in Theorem 1.7.2 simply as test functions to verify

$$\mathcal{S}_{\varepsilon,\omega(\varepsilon)}(Q_\varepsilon) < \mathcal{E}_0(W)$$

and so confirm, by standard methods, that the variational problems (1.22) indeed admit minimizers. By exploiting the unperturbed variational problem (1.15), we show these minimizers approach (up to rescaling) W as $\varepsilon \to 0$. Then the local uniqueness provided by the fixed-point argument from Theorem 1.7.2 implies that the minimizers agree with Q_ε.

Finally, as in [1, 2], we use the variational problem (1.22) to characterize the dynamics of radially-symmetric solutions of the perturbed critical Nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u = -\Delta u - |u|^4 u - \varepsilon |u|^{p-1} u \\ u(x,0) = u_0(x) \in H^1(\mathbb{R}^3) \end{cases} \tag{1.23}$$

'below the ground state', in the spirit of [54]. By standard local existence theory (details in Section 3.3), the Cauchy problem (2.1) admits a unique solution $u \in C([0, T_{max}); H^1(\mathbb{R}^3))$ on a maximal time interval, and a central question is whether the solution *blows-up in finite time* ($T_{max} < \infty$) or is *global* ($T_{max} = \infty$), and if global, how it behaves as $t \to \infty$. We have:

Theorem 1.7.10. *Let $3 < p < 5$ and $0 < \varepsilon < \varepsilon_0$, let $u_0 \in H^1(\mathbb{R}^3)$ be radially-symmetric, and satisfy*

$$S_{\varepsilon,\omega(\varepsilon)}(u_0) < S_{\varepsilon,\omega(\varepsilon)}(Q_\varepsilon),$$

and let u be the corresponding solution to (2.1)*:*

1. *If $K_\varepsilon(u_0) \geq 0$, u is global, and scatters to 0 as $t \to \infty$;*

2. *if $K_\varepsilon(u_0) < 0$, u blows-up in finite time .*

Note that the conclusion is sharp in the sense that Q_ε itself is a global but *non-scattering* solution. Below the action of the ground state the sets where $K_\varepsilon(u) > 0$ and $K_\varepsilon(u) < 0$ are invariant under the equation (1.10). Despite the fact that $K_\varepsilon(u_0) > 0$ gives an a priori bound on the H^1 norm of the solution, the local existence theory is insufficient (since we have the energy critical power) to give global existence/scattering, and so we employ concentration compactness machinery.

The blow-up argument is classical, while the proof of the scattering result rests on that of [54] for the unperturbed problem, with adaptations to handle the scaling-breaking perturbation coming from [1, 2] (higher-dimensional case) and [56] (defocusing case). This is given in Section 3.3.

Chapter 2

A Degenerate Edge Bifurcation in the 1D Linearized NLS

In this chapter, we state and prove the theorem alluded to in Section 1.6. The problem is set up in Section 2.1. In Section 2.2 we collect some results about the relevant operators that are necessary for the bifurcation analysis. Section 2.3 is devoted to the statement and proof of the main result of this chapter: Theorem 2.3.1. The positivity of a certain (explicit) coefficient, which is crucial to the proof, is verified numerically; details of this computation are given in Section 2.4.

2.1 Setup of the Birman-Schwinger Problem

We consider the focusing, pure power (NLS) in one space dimension:

$$i\partial_t u = -\partial_x^2 u - |u|^{p-1}u. \tag{2.1}$$

Here $u = u(x,t) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ with $1 < p < \infty$. The NLS (2.1) admits solutions of the form

$$u(x,t) = Q_p(x)e^{it} \tag{2.2}$$

where $Q_p(x) > 0$ satisfies

$$-Q_p'' - Q_p^p + Q_p = 0. \tag{2.3}$$

In one dimension the explicit solutions

$$Q_p^{p-1}(x) = \left(\frac{p+1}{2}\right) \operatorname{sech}^2\left(\frac{p-1}{2}x\right) \tag{2.4}$$

of (2.3) for each $p \in (1,\infty)$ are classically known to be the unique H^1 solutions of (2.3) up to spatial translation and phase rotation (see e.g. [15]).

In what follows we study the linearized NLS problem. That is, linearize (2.1) about the *solitary wave* solutions (2.2) by considering solutions of the form

$$u(x,t) = (Q_p(x) + h(x,t))\, e^{it}.$$

Then h solves, to leading order (i.e. neglecting terms nonlinear in h)

$$i\partial_t h = (-\partial_x^2 + 1)h - Q_p^{p-1} h - (p-1)Q_p^{p-1}\mathrm{Re}(h).$$

We write the above as a matrix equation

$$\partial_t \vec{h} = J\hat{H}\vec{h}$$

with

$$\vec{h} := \begin{pmatrix} \mathrm{Re}(h) \\ \mathrm{Im}(h) \end{pmatrix} \qquad J^{-1} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\hat{H} := \begin{pmatrix} -\partial_x^2 + 1 - pQ_p^{p-1} & 0 \\ 0 & -\partial_x^2 + 1 - Q_p^{p-1} \end{pmatrix}.$$

The above $J\hat{H}$ is the linearized operator as it appears in [17]. We now consider the system rotated

$$i\partial_t \vec{h} = iJ\hat{H}\vec{h}$$

and find U unitary so that, $UiJ\hat{H}U^* = \sigma_3 H$, where σ_3 is one of the Pauli matrices and with H self-adjoint:

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix},$$

$$H = \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & -\partial_x^2 + 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} p+1 & p-1 \\ p-1 & p+1 \end{pmatrix} Q_p^{p-1} =: \tilde{H} - V^{(p)}.$$

In this way we are consistent with the formulation of [28, 84]. We can also arrive at this system, $i\partial_t \vec{h} = \sigma_3 H \vec{h}$, by letting $\vec{h} = \begin{pmatrix} h & \bar{h} \end{pmatrix}^T$ from the start.

Thus we are interested in the spectrum of

$$\mathcal{L}_p := \sigma_3 H$$

and so in what follows we consider the eigenvalue problem

$$\mathcal{L}_p u = zu, \qquad z \in \mathbb{C}, \qquad u \in L^2(\mathbb{R}, \mathbb{C}^2). \tag{2.5}$$

That the essential spectrum of $\mathcal{L}_p$ is

$$\sigma_{ess}(\mathcal{L}_p) = (-\infty, -1] \cup [1, \infty)$$

and 0 is an eigenvalue of $\mathcal{L}_p$ are standard facts [17].

When $p = 3$ we have the following *resonance* at the threshold $z = 1$ [17]

$$u_0 = \begin{pmatrix} 2 - Q_3^2 \\ -Q_3^2 \end{pmatrix} = 2 \begin{pmatrix} \tanh^2 x \\ -\operatorname{sech}^2 x \end{pmatrix} \tag{2.6}$$

in the sense that

$$\mathcal{L}_3 u_0 = u_0, \qquad u_0 \in L^\infty, \qquad u_0 \notin L^q, \text{ for } q < \infty. \tag{2.7}$$

Our main interest is how this resonance bifurcates when $p \neq 3$ but $|p - 3|$ is small. We now seek an eigenvalue of (2.5) in the following form

$$z = 1 - \alpha^2, \qquad \alpha > 0. \tag{2.8}$$

We note that the spectrum of $\mathcal{L}_p$ for the soliton (2.4) may only be located on the Real or Imaginary axes [17], and so any eigenvalues in the neighbourhood of $z = 1$ must be real. There is also a resonance at $z = -1$ which we do not mention further; symmetry of the spectrum of $\mathcal{L}_p$ ensures the two resonances bifurcate in the same way.

We now recast the problem in accordance with the *Birman-Schwinger formulation* (pp. 85 of [43]), as in [28, 84]. For (2.8), (2.5) becomes

$$(\sigma_3 \tilde{H} - 1 + \alpha^2) u = \sigma_3 V^{(p)} u.$$

The constant-coefficient operator on the left is now invertible so we can write

$$u = (\sigma_3 \tilde{H} - 1 + \alpha^2)^{-1} \sigma_3 V^{(p)} u =: R^{(\alpha)} V^{(p)} u.$$

After noting that $V^{(p)}$ is positive we set

$$w := V_0^{1/2} u, \quad V_0 := V^{(p=3)}$$

and apply $V_0^{1/2}$ to arrive at the problem

$$w = -K_{\alpha,p} w, \qquad K_{\alpha,p} := -V_0^{1/2} R^{(\alpha)} V^{(p)} V_0^{-1/2} \tag{2.9}$$

with

$$R^{(\alpha)} = \begin{pmatrix} (-\partial_x^2 + \alpha^2)^{-1} & 0 \\ 0 & (-\partial_x^2 + 2 - \alpha^2)^{-1} \end{pmatrix}. \tag{2.10}$$

We now seek solutions (α, w) of (2.9) which correspond to eigenvalues $1-\alpha^2$ and eigenfunctions $V_0^{-1/2} w$ of (2.5). The decay of the potential $V^{(p)}$ and hence $V_0^{\frac{1}{2}}$ now allows us to work in the space $L^2 = L^2(\mathbb{R}, \mathbb{C}^2)$, whose standard inner product we denote by $\langle \cdot, \cdot \rangle$.

The resolvent $R^{(\alpha)}$ has integral kernel

$$R^{(\alpha)}(x,y) = \begin{pmatrix} \frac{1}{2\alpha} e^{-\alpha|x-y|} & 0 \\ 0 & \frac{1}{2\sqrt{2-\alpha^2}} e^{-\sqrt{2-\alpha^2}|x-y|} \end{pmatrix}$$

for $\alpha > 0$. We expand $R^{(\alpha)}$ as

$$R^{(\alpha)} = \frac{1}{\alpha} R_{-1} + R_0 + \alpha R_1 + \alpha^2 R_R. \tag{2.11}$$

These operators have the following integral kernels

$$R_{-1}(x,y) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \qquad R_0(x,y) = \begin{pmatrix} -\frac{|x-y|}{2} & 0 \\ 0 & \frac{e^{-\sqrt{2}|x-y|}}{2\sqrt{2}} \end{pmatrix}$$

$$R_1(x,y) = \begin{pmatrix} \frac{|x-y|^2}{4} & 0 \\ 0 & 0 \end{pmatrix}$$

and for $\alpha > 0$ the remainder term R_R is continuous in α and uniformly bounded as an operator from a weighted L^2 space (with sufficiently strong polynomial weight) to its dual. Moreover, since the entries of the full integral kernel $R^{(\alpha)}(x,y)$ are bounded functions of $|x-y|$, we see that the entries of

$$R_R(x,y) = \frac{1}{\alpha^2} \left(R^{(\alpha)}(x,y) - \left(\frac{1}{\alpha} R_{-1}(x,y) + R_0(x,y) + \alpha R_1(x,y)\right) \right)$$

grow at most quadratically in $|x-y|$ as $|x-y| \to \infty$. We also expand the potential $V^{(p)}$ in ε where $\varepsilon := p-3$

$$V^{(p)} = V_0 + \varepsilon V_1 + \varepsilon^2 V_2 + \varepsilon^3 V_R, \qquad \varepsilon := p - 3 \tag{2.12}$$

and

$$V_0 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} Q_3^2 \qquad V_1 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} Q_3^2 + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} q_1$$
$$V_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} q_1 + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} q_2 \qquad V_R = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} q_2 + \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} q_R$$
$$V_0^{1/2} = \frac{1}{2}\begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix} Q_3.$$

Here we have expanded

$$Q_p^{p-1}(x) = Q_3^2(x) + \varepsilon q_1(x) + \varepsilon^2 q_2(x) + \varepsilon^3 q_R(x)$$

and the computation gives

$$Q_3^2(x) = 2\operatorname{sech}^2 x, \qquad q_1(x) = \operatorname{sech}^2 x \left(\frac{1}{2} - 2x \tanh x\right)$$
$$q_2(x) = \frac{1}{2}\left(2x^2 \tanh^2 x \operatorname{sech}^2 x - x^2 \operatorname{sech}^4 x - x \tanh x \operatorname{sech}^2 x\right).$$

By Taylor's theorem, the remainder term $q_R(x)$ satisfies an estimate of the form $|q_R(x)| \leq C(1+|x|^3)\operatorname{sech}^2(x/2)$ for some constant C which is uniform in x and $\varepsilon \in (-1,1)$. We will henceforth write

$$Q \text{ for } Q_3 \quad \text{and} \quad K_{\alpha,\varepsilon} \text{ for } K_{\alpha,p}.$$

2.2 The Perturbed and Unperturbed Operators

We study (2.9), that is:

$$(K_{\alpha,\varepsilon} + 1)w = 0. \tag{2.13}$$

Using the expansions (2.11) and (2.12) for $R^{(\alpha)}$ and $V^{(p)}$ we make the following expansion

$$\begin{aligned} K_{\alpha,\varepsilon} = &\frac{1}{\alpha}\left(K_{-10} + \varepsilon K_{-11} + \varepsilon^2 K_{-12} + \varepsilon^3 K_{R1}\right) \\ &+ K_{00} + \varepsilon K_{01} + \varepsilon^2 K_{02} + \varepsilon^3 K_{R2} \\ &+ \alpha K_{10} + \alpha\varepsilon K_{R3} \\ &+ \alpha^2 K_{R4} \end{aligned} \tag{2.14}$$

where K_{R4} is uniformly bounded and continuous in $\alpha > 0$ and ε in a neighbourhood of 0, as an operator on $L^2(\mathbb{R}, \mathbb{C}^2)$.

Before stating the main theorem we assemble some necessary facts about the above operators.

Lemma 2.2.1. *Each operator appearing in the expansion* (2.14) *for* $K_{\alpha,\varepsilon}$ *is a Hilbert-Schmidt (so in particular bounded and compact) operator from* $L^2(\mathbb{R},\mathbb{C}^2)$ *to itself.*

Proof. This is a straightforward consequence of the spatial decay of the weights which surround the resolvent. The facts that $\|V_0^{-1/2}\| \leq Ce^{|x|}$, and that $\|V_0^{1/2}\| \leq Ce^{-|x|}$, while each of $\|V_0\|$, $\|V_1\|$, $\|V_2\|$ and $\|V_R\|$ can be bounded by $Ce^{-3|x|/2}$ (say if we restrict to $|\varepsilon| < \frac{1}{2}$) imply easily that these operators all have square integrable integral kernels. □

Remark 2.2.2. *The same decay estimates for the potentials used in the proof of Lemma 2.2.1 show that for* $\alpha > 0$ *and* $w \in L^2$ *solving* (2.9) *the corresponding eigenfunction of* (2.5) $u = V_0^{-1/2}w$ *lies in* L^2 *and so the eigenvalue* $z = 1-\alpha^2$ *is in fact a true eigenvalue. Indeed* $w \in L^2 \implies V^{(p)}V_0^{-1/2}w \in L^2$ *and so* $u = -R^{(\alpha)}V^{(p)}V_0^{-1/2}w \in L^2$, *since the free resolvent* $R^{(\alpha)}$ *preserves* L^2 *for* $\alpha > 0$.

We will also need the projections P and $\overline{P}$ which are defined as follows: for $f \in L^2$ let

$$Pf := \frac{\langle v, f\rangle v}{\|v\|^2}, \quad v := V_0^{1/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

as well as the complementary $\overline{P} := 1 - P$. A direct computation shows that for any $f \in L^2$ we have

$$K_{-10}f = -4Pf. \tag{2.15}$$

Note that all operators in the expansion containing R_{-1} return outputs in the direction of v.

Lemma 2.2.3. *The operator* $\overline{P}(K_{00} + 1)\overline{P}$ *has a one dimensional kernel spanned by*

$$w_0 := V_0^{1/2}u_0$$

as an operator from $\mathrm{Ran}(\overline{P})$ *to* $\mathrm{Ran}(\overline{P})$.

Proof. First note that by (2.7)

$$-V_0u_0 = \sigma_3u_0 - \tilde{H}u_0, \qquad [-V_0u_0]_1 = [u_0]_1'' \tag{2.16}$$

from which it follows that

$$Pw_0 = 0, \qquad \text{i.e. } w_0 \in \operatorname{Ran}(\overline{P}).$$

Then a direct computation using (2.16), the expansion (2.14), the expression for R_0, and integration by parts, shows that

$$(K_{00} + 1)w_0 = 2v$$

and so indeed $\overline{P}(K_{00} + 1)\overline{P}w_0 = 0$.

Theorem 5.2 in [48] shows that the kernel of the analogous scalar operator can be at most one dimensional. We will use this argument, adapted to the vector structure, to show that any two non-zero elements of the kernel must be multiples of each other. Take $w \in L^2$ with $\langle w, v\rangle = 0$ and $\overline{P}(K_{00}+1)w = 0$. That is $(K_{00} + 1)w = cv$ for some constant c. This means

$$-V_0^{1/2} R_0 V_0 V_0^{-1/2} w + w = cV_0^{1/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $w = V_0^{1/2}u$ where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. We then obtain, after rearranging and expanding

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c - \frac{1}{2}\int_{\mathbb{R}} |x-y| Q^2(y)\left(2u_1(y) + u_2(y)\right) dy \\ \frac{1}{2\sqrt{2}} \int_{\mathbb{R}} \exp\left(-\sqrt{2}|x-y|\right) Q^2(y)(u_1(y) + 2u_2(y))dy \end{pmatrix}.$$

We now rearrange the first component. Expand

$$\begin{aligned} -\frac{1}{2}\int_{\mathbb{R}} &|x-y|Q^2(y)(2u_1(y) + u_2(y))dy \\ &= -\frac{1}{2}\int_{-\infty}^{x} (x-y)Q^2(y)(2u_1(y) + u_2(y))dy \\ &\quad -\frac{1}{2}\int_{x}^{\infty} (y-x)Q^2(y)(2u_1(y) + u_2(y))dy \end{aligned}$$

and rewrite the first term as

$$\begin{aligned} &-\frac{x}{2}\int_{-\infty}^{x} Q^2(y)(2u_1(y) + u_2(y))dy + \frac{1}{2}\int_{-\infty}^{x} yQ^2(y)(2u_1(y) + u_2(y))dy \\ &= \frac{x}{2}\int_{x}^{\infty} Q^2(y)(2u_1(y) + u_2(y))dy + b - \frac{1}{2}\int_{x}^{\infty} yQ^2(y)(2u_1(y) + u_2(y))dy \end{aligned}$$

where

$$b := \frac{1}{2}\int_{\mathbb{R}} yQ^2(y)(2u_1(y) + u_2(y))dy$$

and where we used $\int_{\mathbb{R}} 2Q^2u_1 + Q^2u_2 = 0$ since $\langle w, v\rangle = 0$. So putting everything back together we see

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c + b + \int_x^\infty (x-y)Q^2(y)\left(2u_1(y) + u_2(y)\right)dy \\ \frac{1}{2\sqrt{2}}\int_{\mathbb{R}} \exp\left(-\sqrt{2}|x-y|\right)Q^2(y)(u_1(y) + 2u_2(y))dy \end{pmatrix}. \tag{2.17}$$

We claim that as $x \to \infty$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} c+b \\ 0 \end{pmatrix}.$$

Observe

$$\begin{aligned} \left|\int_x^\infty (x-y)Q^2(y)\left(2u_1(y)+u_2(y)\right)dy\right| & \\ &\leq \int_x^\infty |y-x|Q^2(y)|2u_1(y)+u_2(y)|dy \\ &\leq \int_x^\infty |y|Q^2(y)|2u_1(y)+u_2(y)|dy \\ &\to 0 \end{aligned}$$

as $x \to \infty$. Here we have used the fact that $w \in L^2$ implies $Q|2u_1 + u_2| \in L^2$ and that $|y|Q \in L^2$. As well, in the second component

$$\begin{aligned} \int_{\mathbb{R}} e^{-\sqrt{2}|x-y|}Q^2(y)(u_1(y)+2u_2(y))dy & \\ =& e^{-\sqrt{2}x}\int_{-\infty}^x e^{\sqrt{2}y}Q^2(y)(u_1(y)+2u_2(y))dy \\ &+ e^{\sqrt{2}x}\int_x^\infty e^{-\sqrt{2}y}Q^2(y)(u_1(y)+2u_2(y))dy \end{aligned}$$

and

$$\begin{aligned}
&\left| e^{-\sqrt{2}x} \int_{-\infty}^{x} e^{\sqrt{2}y} Q^2(y)(u_1(y) + 2u_2(y))dy \right| \\
&\quad \le e^{-\sqrt{2}x} \int_{-\infty}^{x} e^{\sqrt{2}y} Q^2(y)|u_1(y) + 2u_2(y)|dy \\
&\quad \le e^{-\sqrt{2}x} \left(\int_{-\infty}^{x} e^{2\sqrt{2}y} Q^2(y)dy \right)^{1/2} \left(\int_{-\infty}^{x} Q^2(y)|u_1(y) + 2u_2(y)|^2 dy \right)^{1/2} \\
&\quad \le Ce^{-\sqrt{2}x} \left(\int_{-\infty}^{x} e^{2\sqrt{2}y} Q^2(y)dy \right)^{1/2} \\
&\quad \le Ce^{-\sqrt{2}x} \left(\int_{-\infty}^{x} e^{2\sqrt{2}y} e^{-2y} dy \right)^{1/2} \\
&\quad \le Ce^{-\sqrt{2}x} \left(e^{-2\sqrt{2}x} e^{-2x} \right)^{1/2} \le Ce^{-x} \to 0, \quad x \to \infty
\end{aligned}$$

where we again used $Q|u_1 + 2u_2| \in L^2$. Similarly,

$$\left| e^{\sqrt{2}x} \int_{x}^{\infty} e^{-\sqrt{2}y} Q^2(y)(u_1(y) + 2u_2(y))dy \right| \to 0$$

as $x \to \infty$ which addresses the claim.

Next we claim that if $c + b = 0$ in (2.17) then $u \equiv 0$. To address the claim we first note that if $c + b = 0$ then $u \equiv 0$ for all $x \ge X$ for some X, by estimates similar to those just done. Finally, we appeal to ODE theory. Differentiating (2.17) in x twice returns the system

$$u_1'' = -2Q^2 u_1 - Q^2 u_2 \tag{2.18}$$

$$u_2'' - 2u_2 = -Q^2 u_1 - 2Q^2 u_2. \tag{2.19}$$

Any solution u to the above with $u \equiv 0$ for all large enough x must be identically zero.

With the claim in hand we finish the argument. Given two non-zero elements of the kernel, say u and $\tilde{u}$ with limits as $x \to \infty$ (written as above) $c + b$ and $\tilde{c} + \tilde{b}$ respectively, the combination

$$u^* = u - \frac{c + b}{\tilde{c} + \tilde{b}} \tilde{u}$$

satisfies (2.17) but with $u^*(x) \to 0$ as $x \to \infty$, and so $u^* \equiv 0$. Therefore, u and $\tilde{u}$ are linearly dependent, as required. $\square$

Note that K_{00}, and hence $\overline{P}(K_{00}+1)\overline{P}$, is self-adjoint. Indeed

$$\begin{aligned} K_{00} &= -V_0^{1/2} R_0 V_0 V_0^{-1/2} \\ &= -V_0^{-1/2} V_0 R_0 V_0^{1/2} \\ &= (K_{00})^*. \end{aligned}$$

As we have seen above in Lemma 2.2.1, thanks to the decay of the potential, $\overline{P}K_{00}\overline{P}$ is a compact operator. Therefore, the simple eigenvalue -1 of $\overline{P}K_{00}\overline{P}$ is isolated and so

$$(\overline{P}(K_{00}+1)\overline{P})^{-1} : \{v,\ w_0\}^{\perp} \to \{v,\ w_0\}^{\perp} \tag{2.20}$$

exists and is bounded.

With the above preliminary facts assembled, we proceed to the bifurcation analysis.

2.3 Bifurcation Analysis

This section is devoted to the proof of the main result of Chapter 2:

Theorem 2.3.1. *There exists $\varepsilon_0 > 0$ such that for $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$ the eigenvalue problem* (2.13) *has a solution* (α, w) *of the form*

$$\begin{aligned} w &= w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \tilde{w} \\ \alpha &= \varepsilon^2 \alpha_2 + \tilde{\alpha} \end{aligned} \tag{2.21}$$

where $\alpha_2 > 0$, w_0, w_1, w_2 are known (given below), and $|\tilde{\alpha}| < C|\varepsilon|^3$ and $\|\tilde{w}\|_{L^2} < C|\varepsilon|^3$ for some $C > 0$.

Remark 2.3.2. *This theorem confirms the behaviour observed numerically in [17]: for $p \neq 3$ but close to 3, the linearized operator $J\hat{H}$ (which is unitarily equivalent to $i\mathcal{L}_p$) has true, purely imaginary eigenvalues in the gap between the branches of essential spectrum, which approach the thresholds as $p \to 3$. Note Remark 2.2.2 to see that $u = V_0^{-1/2} w$ is a true L^2 eigenfunction of* (2.5)*. In addition, the eigenfunction approaches the resonance eigenfunction in some weighted L^2 space. Furthermore, we have found that α^2, the distance of the eigenvalues from the thresholds, is to leading order proportional to $(p-3)^4$. Finally, note that $\alpha = \varepsilon^2\alpha_2 + O(\varepsilon^3)$ with $\alpha_2 > 0$ gives $\alpha > 0$ for both $\varepsilon > 0$ and $\varepsilon < 0$, ensuring the eigenvalues appear on both sides of $p = 3$.*

The quantities in (2.21) are defined as follows:

$$w_0 := V_0^{1/2} u_0$$
$$Pw_1 := \frac{1}{4} K_{-11} w_0$$
$$\overline{P}w_1 := -\big(\overline{P}(K_{00}+1)\overline{P}\big)^{-1}\left(\frac{1}{4}\overline{P}K_{00}K_{-11}w_0 + \overline{P}K_{01}w_0\right)$$
$$Pw_2 := \frac{1}{4}\big(K_{-11}w_1 + K_{-12}w_0 + \alpha_2(K_{00}+1)w_0\big)$$
$$\begin{aligned}\overline{P}w_2 := -\big(\overline{P}(K_{00}+1)\overline{P}\big)^{-1}\Bigg(&\frac{1}{4}\overline{P}K_{00}K_{-11}w_1 + \frac{1}{4}\overline{P}K_{00}K_{-12}w_0 \\ &+ \frac{\alpha_2}{4}\overline{P}K_{00}(K_{00}+1)w_0 + \overline{P}K_{01}w_1 + \overline{P}K_{02}w_0 + \alpha_2\overline{P}K_{10}w_0\Bigg)\end{aligned}$$
$$\alpha_2 := \frac{-\frac{1}{4}\langle w_0, K_{00}K_{-11}w_1\rangle - \frac{1}{4}\langle w_0, K_{00}K_{-12}w_0\rangle - \langle w_0, K_{01}w_1\rangle - \langle w_0, K_{02}w_0\rangle}{\langle w_0, K_{10}w_0\rangle + \frac{1}{4}\langle w_0, K_{00}(K_{00}+1)w_0\rangle}.$$

Remark 2.3.3. *A numerical computation shows*

$$\alpha_2 \approx 2.52/8 > 0.$$

Since the positivity of α_2 is crucial to the main result, details of this computation are described in Section 2.4.

Note that the functions on which $\overline{P}(K_{00}+1)\overline{P}$ is being inverted in the expressions for $\overline{P}w_1$ and $\overline{P}w_2$ are orthogonal to both w_0 and v, and so these quantities are well-defined by (2.20). The projections to v are zero by the presence of $\overline{P}$. As for the projections to w_0, the identity

$$\langle w_0, \frac{1}{4}K_{00}K_{-11}w_0 + K_{01}w_0\rangle = 0 \tag{2.22}$$

has been verified analytically. It is because of this identity that the $O(\varepsilon)$ term is absent in the expansion of α in (2.21). The fact that

$$\begin{aligned}0 = \langle w_0, \frac{1}{4}K_{00}K_{-11}w_1 + \frac{1}{4}K_{00}K_{-12}w_0 + \frac{\alpha_2}{4}K_{00}(K_{00}+1)w_0 + K_{01}w_1 \\ + K_{02}w_0 + \alpha_2 K_{10}w_0\rangle\end{aligned}$$

comes from our definition of α_2.

The above definitions, along with (2.15), imply the relationships below

$$0 = K_{-10}w_0 \tag{2.23}$$
$$0 = K_{-11}w_0 + K_{-10}w_1 \tag{2.24}$$
$$0 = K_{-10}w_2 + K_{-11}w_1 + K_{-12}w_0 + \alpha_2(K_{00}+1)w_0 \tag{2.25}$$
$$0 = \overline{P}(K_{00}+1)w_1 + \overline{P}K_{01}w_0 \tag{2.26}$$
$$0 = \overline{P}(K_{00}+1)w_2 + \overline{P}K_{01}w_1 + \overline{P}K_{02}w_0 + \alpha_2\overline{P}K_{10}w_0 \tag{2.27}$$

which we will use in what follows.

Using the expression for α in (3.1.1), our expansion (2.14) for $K_{\alpha,\varepsilon}$ now takes the form

$$\begin{aligned} K_{\alpha,\varepsilon} &= \frac{1}{\alpha}\left(K_{-10} + \varepsilon K_{-11} + \varepsilon^2 K_{-12} + \varepsilon^3 K_{R1}\right) \\ &\quad + K_{00} + \varepsilon K_{01} + \varepsilon^2 K_{02} + \varepsilon^3 K_{R2} \\ &\quad + (\alpha_2\varepsilon^2 + \tilde{\alpha})K_{10} + (\alpha_2\varepsilon^2 + \tilde{\alpha})\varepsilon K_{R3} + (\alpha_2\varepsilon^2 + \tilde{\alpha})^2 K_{R4} \\ &=: \frac{1}{\alpha}\left(K_{-10} + \varepsilon K_{-11} + \varepsilon^2 K_{-12} + \varepsilon^3 K_{R1}\right) + K_{00} + \varepsilon\overline{K}_1 + \tilde{\alpha}\overline{K}_2 \end{aligned}$$

where $\overline{K}_1$ is a bounded (uniformly in ε) operator depending on ε but not $\tilde{\alpha}$, while $\overline{K}_2$ is a bounded (uniformly in ε and $\tilde{\alpha}$) operator depending on both ε and $\tilde{\alpha}$.

Further decomposing

$$\tilde{w} = \beta v + W, \qquad \langle W, v\rangle = 0,$$

we aim to show existence of a solution with the remainder terms $\tilde{\alpha}$, β and W small. We do so via a Lyapunov-Schmidt reduction.

First substitute (2.21) to (2.13) and apply the projection $\overline{P}$ to obtain

$$\begin{aligned} 0 &= \overline{P}(K_{\alpha,\varepsilon}+1)w \\ &= \overline{P}(K_{\alpha,\varepsilon}+1)(w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \beta v + W) \\ &= \overline{P}(K_{00}+1)w_0 + \varepsilon\overline{P}(K_{00}+1)w_1 + \varepsilon\overline{P}K_{01}w_0 \\ &\quad + \varepsilon^2\overline{P}(K_{00}+1)w_2 + \varepsilon^2\overline{P}K_{01}w_1 + \varepsilon^2\overline{P}K_{02}w_0 + \varepsilon^2\alpha_2\overline{P}K_{10}w_0 \\ &\quad + \overline{P}(K_{00}+1)(\beta v + W) + \tilde{\alpha}\overline{P}K_{10}w_0 + \overline{P}\left(\varepsilon\overline{K}_1 + \tilde{\alpha}\overline{K}_2\right)(\beta v + W) \\ &\quad + \varepsilon^3\overline{P}\left(K_{R2}w_0 + K_{02}w_1 + K_{01}w_2 + \varepsilon K_{02}w_2 + \varepsilon K_{R2}w_1 + \varepsilon^2 K_{R2}w_2\right) \\ &\quad + (\alpha_2\varepsilon^2 + \tilde{\alpha})\overline{P}K_{10}(\varepsilon w_1 + \varepsilon^2 w_2) + (\alpha_2\varepsilon^2 + \tilde{\alpha})\varepsilon\overline{P}K_{R3}(w_0 + \varepsilon w_1 + \varepsilon^2 w_2) \\ &\quad + (\alpha_2\varepsilon^2 + \tilde{\alpha})^2\overline{P}K_{R4}(w_0 + \varepsilon w_1 + \varepsilon^2 w_2). \end{aligned} \tag{2.28}$$

Making some cancellations coming from Lemma 2.2.3, (2.26) and (2.27) leads to

$$\begin{aligned}
&-\overline{P}(K_{00}+1)\overline{P}W = \\
&\quad \beta\overline{P}K_{00}v + \tilde{\alpha}\overline{P}K_{10}w_0 + \overline{P}\left(\varepsilon\overline{K}_1 + \tilde{\alpha}\overline{K}_2\right)(\beta v + W) \\
&\qquad + \varepsilon^3\overline{P}\left(K_{R2}w_0 + K_{02}w_1 + K_{01}w_2 + \varepsilon K_{02}w_2 + \varepsilon K_{R2}w_1 + \varepsilon^2 K_{R2}w_2\right) \\
&\qquad + (\alpha_2\varepsilon^2 + \tilde{\alpha})\overline{P}K_{10}(\varepsilon w_1 + \varepsilon^2 w_2) + (\alpha_2\varepsilon^2 + \tilde{\alpha})\varepsilon\overline{P}K_{R3}(w_0 + \varepsilon w_1 + \varepsilon^2 w_2) \\
&\qquad + (\alpha_2\varepsilon^2 + \tilde{\alpha})^2\overline{P}K_{R4}(w_0 + \varepsilon w_1 + \varepsilon^2 w_2) \\
&\quad =: \mathcal{F}(W;\varepsilon,\tilde{\alpha},\beta).
\end{aligned}$$

According to (2.20), inversion of $\overline{P}(K_{00}+1)\overline{P}$ on $\mathcal{F}$ requires the solvability condition

$$P_0\mathcal{F} = 0, \qquad P_0 := \frac{1}{\|w_0\|_2^2}\langle w_0,\ \cdot\rangle w_0, \quad \overline{P}_0 := 1 - P_0 \tag{2.29}$$

which we solve together with the fixed point problem

$$W = \left(-\overline{P}(K_{00}+1)\overline{P}\right)^{-1}\overline{P}_0\mathcal{F}(W;\varepsilon,\tilde{\alpha},\beta) =: \mathcal{G}(W;\varepsilon,\tilde{\alpha},\beta) \tag{2.30}$$

in order to solve (2.28).

Write

$$\mathcal{F} := \overline{P}\left(\beta K_{00}v + \tilde{\alpha}K_{10}w_0 + \left(\varepsilon\overline{K}_1 + \tilde{\alpha}\overline{K}_2\right)(\beta v + W) + \varepsilon^3 f_1 + \varepsilon\tilde{\alpha}f_2 + \tilde{\alpha}^2 h_1\right)$$

where f_1 and f_2 denote functions depending on (and L^2 bounded uniformly in) ε but not $\tilde{\alpha}$, while h_1 denotes an L^2 function depending on (and uniformly L^2 bounded in) both ε and $\tilde{\alpha}$.

Lemma 2.3.4. *For any $M > 0$ there exists $\varepsilon_0 > 0$ and $R > 0$ such that for all $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$ and for all $\tilde{\alpha}$ and β with $|\tilde{\alpha}| \le M|\varepsilon|^3$ and $|\beta| \le M|\varepsilon|^3$ there exists a unique solution $W \in L^2 \cap \{v,\ w_0\}^\perp$ of* (2.30) *satisfying $\|W\|_{L^2} \le R|\varepsilon|^3$.*

Proof. We prove this by means of Banach Fixed Point Theorem. We must show that $\mathcal{G}(W)$ maps the closed ball of radius $R|\varepsilon|^3$ into itself and that $\mathcal{G}(W)$ is a contraction mapping. Taking $W \in L^2$ orthogonal to v and w_0 such that $\|W\|_{L^2} \le R|\varepsilon|^3$ and given $M > 0$ where $|\tilde{\alpha}| \le M|\varepsilon|^3$ and $|\beta| \le M|\varepsilon|^3$, we have, using the boundedness of $\left(-\overline{P}(K_{00}+1)\overline{P}\right)^{-1}\overline{P}_0$,

$$\begin{aligned}
&\|\mathcal{G}\|_{L^2}\\
&\leq C|\beta|\|\overline{P}K_{00}v+\overline{P}\left(\varepsilon\overline{K}_1+\tilde{\alpha}\overline{K}_2\right)v\|_{L^2}+C|\tilde{\alpha}|\|\overline{P}\left(K_{10}w_0+\varepsilon f_2+\tilde{\alpha}h_1\right)\|_{L^2}\\
&\quad+C\|\overline{P}\left(\varepsilon\overline{K}_1+\tilde{\alpha}\overline{K}_2\right)W\|_{L^2}+|\varepsilon|^3C\|\overline{P}f_1\|_{L^2}\\
&\leq CM|\varepsilon|^3+CM|\varepsilon|^3+C|\varepsilon|\|W\|_{L^2}+C|\tilde{\alpha}|\|W\|_{L^2}+C|\varepsilon|^3\\
&\leq C|\varepsilon|^3+CR|\varepsilon|^4\\
&\leq R|\varepsilon|^3
\end{aligned}$$

for some appropriately chosen R with $|\varepsilon|$ small enough. Here C is a positive, finite constant whose value changes at each appearance. Next consider

$$\begin{aligned}
&\|\mathcal{G}(W_1)-\mathcal{G}(W_2)\|_{L^2}\\
&\quad\leq C\|\overline{P}\left(\varepsilon\overline{K}_1+\tilde{\alpha}\overline{K}_2\right)\|_{L^2\to L^2}\|W_1-W_2\|_{L^2}\\
&\quad\leq C|\varepsilon|\|\overline{P}\ \overline{K}_1\|_{L^2\to L^2}\|W_1-W_2\|_{L^2}+C|\tilde{\alpha}|\|\overline{P}\ \overline{K}_2\|_{L^2\to L^2}\|W_1-W_2\|_{L^2}\\
&\quad\leq C|\varepsilon|\|W_1-W_2\|_{L^2}\leq\kappa\|W_1-W_2\|_{L^2}
\end{aligned}$$

with $0<\kappa<1$ by taking $|\varepsilon|$ sufficiently small. Hence $\mathcal{G}(W)$ is a contraction, and we obtain the desired result. □

Lemma 2.3.4 provides W as a function of $\tilde{\alpha}$ and β, which we may then substitute into (2.29) to get

$$\begin{aligned}
0&=\langle w_0,\mathcal{F}\rangle\\
&=\beta\langle w_0,K_{00}v\rangle+\tilde{\alpha}\langle w_0,K_{10}w_0\rangle+\varepsilon\beta\langle w_0,\overline{K}_1v\rangle+\tilde{\alpha}\beta\langle w_0,\overline{K}_2v\rangle\\
&\quad+\varepsilon^3\langle w_0,f_1\rangle+\varepsilon\tilde{\alpha}\langle w_0,f_2\rangle+\tilde{\alpha}^2\langle w_0,h_1\rangle+\varepsilon\langle w_0,\overline{K}_1W\rangle+\tilde{\alpha}\langle w_0,\overline{K}_2W\rangle\\
&=:\beta\langle w_0,K_{00}v\rangle+\tilde{\alpha}\langle w_0,K_{10}w_0\rangle+\mathcal{F}_1 \qquad (2.31)
\end{aligned}$$

which is the first of two equations relating $\tilde{\alpha}$ and β.

The second equation is the complementary one to (2.28): substitute

(2.21) to (2.13) but this time multiply by α and take projection P to see

$$\begin{aligned}
0 &= \alpha P(K_{\alpha,\varepsilon}+1)w \\
&= K_{-10}w_0 + \varepsilon(K_{-11}w_0 + K_{-10}w_1) \\
&+ \varepsilon^2(K_{-10}w_2 + K_{-11}w_1 + K_{-12}w_0) + \varepsilon^2\alpha_2(K_{00}+1)w_0 \\
&+ \varepsilon^3(K_{-11}w_2 + K_{-12}w_1 + K_{R1}w_0 + \varepsilon K_{-12}w_2 + \varepsilon K_{R1}w_1 + \varepsilon^2 K_{R1}w_2) \\
&+ \beta K_{-10}v + K_{-10}W + \varepsilon(K_{-11} + \varepsilon K_{-12} + \varepsilon^2 K_{R1})(\beta v + W) \\
&+ \tilde{\alpha}(K_{00}+1)w_0 + \varepsilon^3\alpha_2 P(K_{00}+1)(w_1+\varepsilon w_2) + \varepsilon\tilde{\alpha}P(K_{00}+1)(w_1+\varepsilon w_2) \\
&+ \varepsilon^2\alpha_2 P(K_{00}+1)(\beta v + W) + \tilde{\alpha}P(K_{00}+1)(\beta v + W) \\
&+ \alpha P(\varepsilon K_{01} + \varepsilon^2 K_{02} + \varepsilon^3 K_{R2} + \alpha K_{10} + \alpha\varepsilon K_{R3} + \alpha^2 K_{R4}) \\
&\qquad \times (w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \beta v + W).
\end{aligned} \tag{2.32}$$

After using known information about w_0, w_1, w_2, α_2 coming from (2.23), (2.24), (2.25) and noting that $K_{-10}W = -4PW = 0$ from (2.15) we have

$$\begin{aligned}
0 = &\beta K_{-10}v + \tilde{\alpha}(K_{00}+1)w_0 \\
&+ \varepsilon^3(K_{-11}w_2 + K_{-12}w_1 + K_{R1}w_0 + \varepsilon K_{-12}w_2 + \varepsilon K_{R1}w_1 + \varepsilon^2 K_{R1}w_2) \\
&+ \varepsilon(K_{-11} + \varepsilon K_{-12} + \varepsilon^2 K_{R1})(\beta v + W) \\
&+ \varepsilon^3\alpha_2 P(K_{00}+1)(w_1+\varepsilon w_2) + \varepsilon\tilde{\alpha}P(K_{00}+1)(w_1+\varepsilon w_2) \\
&+ \varepsilon^2\alpha_2 P(K_{00}+1)(\beta v + W) + \tilde{\alpha}P(K_{00}+1)(\beta v + W) \\
&+ \alpha P(\varepsilon K_{01} + \varepsilon^2 K_{02} + \varepsilon^3 K_{R2} + \alpha K_{10} + \alpha\varepsilon K_{R3} + \alpha^2 K_{R4}) \\
&\qquad \times (w_0 + \varepsilon w_1 + \varepsilon^2 w_2 + \beta v + W).
\end{aligned}$$

Written more compactly, this is

$$\begin{aligned}
0 = &\beta K_{-10}v + \tilde{\alpha}(K_{00}+1)w_0 \\
&+ \varepsilon^3 f_4 + \varepsilon\overline{K}_3(\beta v + W) + \tilde{\alpha}\varepsilon f_5 + \tilde{\alpha}\overline{K}_4(\beta v + W) + \tilde{\alpha}^2 h_2
\end{aligned}$$

where $\overline{K}_3$ is a bounded (uniformly in ε) operator containing ε but not $\tilde{\alpha}$, while $\overline{K}_4$ is a bounded (uniformly in ε and $\tilde{\alpha}$) operator containing both ε and $\tilde{\alpha}$. Functions f_4 and f_5 depend on ε (and are uniformly L^2-bounded) but not $\tilde{\alpha}$, while the function h_2 depends on both ε and $\tilde{\alpha}$ (and is uniformly L^2-bounded). To make the relationship between $\tilde{\alpha}$ and β more explicit we take inner product with v

$$\begin{aligned} 0 &= \beta\langle v, K_{-10}v\rangle + \tilde{\alpha}\langle v, (K_{00}+1)w_0\rangle + \varepsilon^3\langle v, f_4\rangle \\ &\quad + \varepsilon\langle v, \overline{K}_3(\beta v + W)\rangle + \tilde{\alpha}\varepsilon\langle v, f_5\rangle + \tilde{\alpha}\langle v, \overline{K}_4(\beta v + W)\rangle + \tilde{\alpha}^2\langle v, h_2\rangle \\ &=: \beta\langle v, K_{-10}v\rangle + \tilde{\alpha}\langle v, (K_{00}+1)w_0\rangle + \mathcal{F}_2. \end{aligned} \tag{2.33}$$

Now let

$$\vec{\zeta} = \begin{pmatrix} \tilde{\alpha} \\ \beta \end{pmatrix}$$

and rewrite (2.31) and (2.33) in the following way

$$A\vec{\zeta} := \begin{pmatrix} \langle w_0, K_{10}w_0\rangle & \langle w_0, K_{00}v\rangle \\ \langle v, (K_{00}+1)w_0\rangle & \langle v, K_{-10}v\rangle \end{pmatrix} \begin{pmatrix} \tilde{\alpha} \\ \beta \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix}$$

which we recast as a fixed point problem

$$\vec{\zeta} = A^{-1}\begin{pmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \end{pmatrix} =: \vec{F}(\tilde{\alpha}, \beta; \varepsilon). \tag{2.34}$$

We have computed

$$A = \begin{pmatrix} 0 & 16 \\ 16 & -32 \end{pmatrix}$$

so in particular, A is invertible. We wish to show there is a solution $(\tilde{\alpha}, \beta)$ of (2.34) of the appropriate size. We establish this fact in the following Lemmas. Lemmas 2.3.5 and 2.3.6 are accessory to Lemma 2.3.7.

Lemma 2.3.5. *The operators and functions $\overline{K}_2$, $\overline{K}_4$ and h_1, h_2 are continuous in $\tilde{\alpha} > 0$.*

Proof. The operators and function in question are compositions of continuous functions of $\tilde{\alpha}$. □

Lemma 2.3.6. *The W given by Lemma 2.3.4 is continuous in $\vec{\zeta}$ for sufficiently small $|\varepsilon|$.*

Proof. Let $(\tilde{\alpha}_1, \beta_1)$ give rise to W_1 and let $(\tilde{\alpha}_2, \beta_2)$ give rise to W_2 via Lemma 2.3.4. Take $|\tilde{\alpha}_1 - \tilde{\alpha}_2| < \delta$ and $|\beta_1 - \beta_2| < \delta$. We show that $\|W_1 - W_2\|_{L^2} < C\delta$

for some constant $C > 0$. Observing $\overline{K}_2$ depends on $\tilde{\alpha}$, we see

$$
\begin{aligned}
&\|W_1 - W_2\|_{L^2} = \\
&\quad \| \left(\overline{P}(K_{00}+1)\overline{P}\right)^{-1} \overline{P}_0\|_{L^2 \to L^2} \|\mathcal{F}(W_1, \vec{\zeta}_1; \varepsilon) - \mathcal{F}(W_2, \vec{\zeta}_2; \varepsilon)\|_{L^2} \\
&\quad \le C \Big\| (\beta_1 - \beta_2) K_{00} v + (\tilde{\alpha}_1 - \tilde{\alpha}_2) K_{10} w_0 + \varepsilon(\beta_1 - \beta_2)\overline{K}_1 v \\
&\qquad + \varepsilon \overline{K}_1 (W_1 - W_2) + \tilde{\alpha}_1 \beta_1 \overline{K}_2(\tilde{\alpha}_1) v - \tilde{\alpha}_2 \beta_2 \overline{K}_2(\tilde{\alpha}_2) v + \tilde{\alpha}_1 \overline{K}_2(\tilde{\alpha}_1) W_1 \\
&\qquad\qquad - \tilde{\alpha}_2 \overline{K}_2(\tilde{\alpha}_2) W_2 + \varepsilon(\tilde{\alpha}_1 - \tilde{\alpha}_2) f_2 + \tilde{\alpha}_1^2 h_1(\tilde{\alpha}_1) - \tilde{\alpha}_2^2 h_1(\tilde{\alpha}_2) \Big\|_{L^2} \\
&\quad \le C\delta + C|\varepsilon| \|W_1 - W_2\|_{L^2} \\
&\qquad + \|\tilde{\alpha}_1 \overline{K}_2(\tilde{\alpha}_1)(W_1 - W_2) + \left(\tilde{\alpha}_1 \overline{K}_2(\tilde{\alpha}_1) - \tilde{\alpha}_2 \overline{K}_2(\tilde{\alpha}_2)\right) W_2\|_{L^2} \\
&\quad \le C\delta + C|\varepsilon| \|W_1 - W_2\|_{L^2}
\end{aligned}
$$

noting that $|\tilde{\alpha}_1| \le M|\varepsilon|^3$. Rearranging the above gives

$$\|W_1 - W_2\|_{L^2} < C\delta$$

for small enough $|\varepsilon|$. □

Lemma 2.3.7. *There exists $\varepsilon_0 > 0$ such that for all $-\varepsilon_0 \le \varepsilon \le \varepsilon_0$ the equation* (2.34) *has a fixed point with $|\tilde{\alpha}|, |\beta| \le M|\varepsilon|^3$ for some $M > 0$.*

Proof. We prove this by means of the Brouwer Fixed Point Theorem. We show that $\vec{F}$ maps a closed square into itself and that $\vec{F}$ is a continuous function. Take $|\tilde{\alpha}|, |\beta| \le M|\varepsilon|^3$ and and so by Lemma 2.3.4 we have $\|W\|_{L^2} \le |\varepsilon|^3 R$ for some $R > 0$. Consider now

$$
\begin{aligned}
&\|A^{-1}\| \, |\mathcal{F}_1| \\
&\quad \le \|A^{-1}\| \Bigg(|\varepsilon||\beta||\langle w_0, \overline{K}_1 v\rangle| + |\tilde{\alpha}||\beta||\langle w_0, \overline{K}_2 v\rangle| + |\varepsilon|^3 |\langle w_0, f_1\rangle| \\
&\qquad + |\varepsilon||\tilde{\alpha}||\langle w_0, f_2\rangle| + |\tilde{\alpha}|^2 |\langle w_0, h_1\rangle| + |\varepsilon||\langle w_0, \overline{K}_1 W\rangle| + |\tilde{\alpha}||\langle w_0, \overline{K}_2 W\rangle| \Bigg) \\
&\quad \le CM|\varepsilon|^4 + CM^2|\varepsilon|^6 + C|\varepsilon|^3 + CM|\varepsilon|^4 + CM^2|\varepsilon|^6 + CR|\varepsilon|^4 \\
&\quad \le C|\varepsilon|^3 + CM|\varepsilon|^4 \le M|\varepsilon|^3
\end{aligned}
$$

and

$$\begin{aligned}
&\|A^{-1}\|\,|\mathcal{F}_2| \\
&\leq \|A^{-1}\| \Bigg(|\varepsilon|^3 |\langle v, f_4\rangle| + |\varepsilon||\langle v, \overline{K}_3(\beta v + W)\rangle| + |\tilde{\alpha}||\varepsilon||\langle v, f_5\rangle| \\
&\qquad\qquad + |\tilde{\alpha}||\langle v, \overline{K}_4(\beta v + W)\rangle| + |\tilde{\alpha}|^2 |\langle v, h_2\rangle| \Bigg) \\
&\leq C|\varepsilon|^3 + CM|\varepsilon|^4 + CR|\varepsilon|^4 + CM|\varepsilon|^4 + CM^2|\varepsilon|^6 + CMR|\varepsilon|^6 + CM^2|\varepsilon|^6 \\
&\leq C|\varepsilon|^3 + CM|\varepsilon|^4 \leq M|\varepsilon|^3
\end{aligned}$$

for some choice of $M > 0$ and sufficiently small $|\varepsilon| > 0$. Here $C > 0$ is a constant that is different at each instant. So $\vec{F}$ maps the closed square to itself.

It is left to show that $\vec{F}$ is continuous. Given $\eta > 0$ take $|\tilde{\alpha}_1 - \tilde{\alpha}_2| < \delta$ and $|\beta_1 - \beta_2| < \delta$. Let $(\tilde{\alpha}_1, \beta_1)$ give rise to W_1 and let $(\tilde{\alpha}_2, \beta_2)$ give rise to W_2 via Lemma 2.3.4. We will also use Lemma 2.3.5 and Lemma 2.3.6. Now consider

$$\begin{aligned}
&|\mathcal{F}_1(\tilde{\alpha}_1, \beta_1) - \mathcal{F}_1(\tilde{\alpha}_2, \beta_2)| \\
&= \Big| \varepsilon(\beta_1 - \beta_2)\langle w_0, \overline{K}_1 v\rangle + \tilde{\alpha}_1\beta_1\langle w_0, \overline{K}_2(\tilde{\alpha}_1)v\rangle - \tilde{\alpha}_2\beta_2\langle w_0, \overline{K}_2(\tilde{\alpha}_2)v\rangle \\
&\qquad + \varepsilon(\tilde{\alpha}_1 - \tilde{\alpha}_2)\langle w_0, f_2\rangle + \tilde{\alpha}_1^2\langle w_0, h_1(\tilde{\alpha}_1)\rangle - \tilde{\alpha}_2^2\langle w_0, h_1(\tilde{\alpha}_2)\rangle \\
&\qquad + \varepsilon\langle w_0, \overline{K}_1(W_1 - W_2)\rangle + \tilde{\alpha}_1\langle w_0, \overline{K}_2(\tilde{\alpha}_1)W_1\rangle - \tilde{\alpha}_2\langle w_0, \overline{K}_2(\tilde{\alpha}_2)W_2\rangle \Big| \\
&\leq C\delta + C\|h_1(\tilde{\alpha}_1) - h_1(\tilde{\alpha}_2)\|_{L^2} \\
&\qquad + C\|W_1 - W_2\|_{L^2} + C\|\overline{K}_2(\tilde{\alpha}_1) - \overline{K}_2(\tilde{\alpha}_2)\|_{L^2 \to L^2} \\
&\leq C\delta < \frac{\eta}{\|A^{-1}\|\sqrt{2}}
\end{aligned}$$

for small enough δ. Similarly we can show

$$|\mathcal{F}_2(\tilde{\alpha}_1, \beta_1) - \mathcal{F}_2(\tilde{\alpha}_2, \beta_2)| \leq C\delta < \frac{\eta}{\|A^{-1}\|\sqrt{2}}$$

for δ small enough. Putting everything together gives $|\vec{F}(\vec{\zeta}_1) - \vec{F}(\vec{\zeta}_2)| < \eta$ as required. Hence $\vec{F}$ is continuous. □

So finally we have solved both (2.28) and (2.32), and hence (2.13), and so have proved Theorem 2.3.1.

2.4 Comments on the Computations

Analytical and numerical computations were used in the above to compute inner products such as the ones appearing in the definition of α_2 (2.21). It was critical to establish that $\alpha_2 > 0$ since the expansion of the resolvent $R^{(\alpha)}$ (2.10) requires $\alpha > 0$. Inner products containing w_0 but not w_1 can be written as an explicit single integral and then evaluated analytically or numerically with good accuracy. For example

$$
\begin{aligned}
&\langle w_0, K_{02} w_0\rangle + \frac{1}{4}\langle w_0, K_{00}K_{-12} w_0\rangle \\
=& -\frac{1}{2}\int_{\mathbb{R}^2} |x-y|\big(4Q^2(x) - 3Q^4(x)\big) \\
&\qquad \times \big(Q^2(y)q_1(y) - q_1(y) + 3Q^2(y)q_2(y) - 4q_2(y) - \frac{c_2}{2}Q^2(y)\big)dydx \\
&+ \frac{1}{2\sqrt{2}}\int_{\mathbb{R}^2} e^{-\sqrt{2}|x-y|}\big(2Q^2(x) - 3Q^4(x)\big) \\
&\qquad \times \big(Q^2(y)q_1(y) - q_1(y) + 3Q^2(y)q_2(y) - 2q_2(y) - \frac{c_2}{4}Q^2(y)\big)dydx \\
=& -\int_{\mathbb{R}} Q^2(y)\big(Q^2(y)q_1(y) - q_1(y) + 3Q^2(y)q_2(y) - 4q_2(y) - \frac{c_2}{2}Q^2(y)\big)dy \\
&- \int_{\mathbb{R}} Q^2(y)\big(Q^2(y)q_1(y) - q_1(y) + 3Q^2(y)q_2(y) - 2q_2(y) - \frac{c_2}{4}Q^2(y)\big)dy \\
\approx& -2.9369
\end{aligned}
$$

where

$$
c_2 = \frac{1}{2}\int_{\mathbb{R}} Q^2 q_1 - q_1 + 3Q^2 q_2 - 4q_2.
$$

To reduce the double integral to a single integral we recall some facts about the integral kernels. Let

$$
h(y) = -\frac{1}{2}\int_{\mathbb{R}} |x-y|\big(4Q^2(x) - 3Q^4(x)\big)dx.
$$

Then h solves the equation

$$
h'' = -4Q^2 + 3Q^4.
$$

Notice that $-4Q^2 + 3Q^4 = -2Q^2u_1 - Q^2u_2$ where u_1 and u_2 are the components of the resonance u_0 (2.6). Observing the equation (2.18) we see that $h = u_1 + c = 2 - Q^2 + c$ for some constant c. We can directly compute

$h(0) = -2$ to find $c = -2$ and so $h = -Q^2$. A similar argument involving (2.19) gives

$$\frac{1}{2\sqrt{2}} \int_{\mathbb{R}} e^{-\sqrt{2}|x-y|} \big(2Q^2(x) - 3Q^4(x)\big) dx = u_2(y) = -Q^2(y).$$

Many of the inner products can be computed analytically. These include the identity (2.22), the entires in the matrix A in (2.34) and the denominator appearing in the expression for α_2. As an example we evaluate the denominator of α_2:

$$\begin{aligned}
&\langle w_0, K_{10} w_0\rangle + \frac{1}{4}\langle w_0, K_{00}(K_{00}+1)w_0\rangle \\
&= -\int_{\mathbb{R}^2} \big(3Q^4(x) - 4Q^2(x)\big) \frac{(x-y)^2}{4} \big(3Q^4(y) - 4Q^2(y)\big) dydx \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^2} \big(4Q^2(x) - 3Q^4(x)\big) |x-y| Q^2(y) dydx \\
&\quad - \frac{1}{4\sqrt{2}} \int_{\mathbb{R}^2} \big(4Q^2(x) - 3Q^4(x)\big) e^{-\sqrt{2}|x-y|} Q^2(y) dydx \\
&= \frac{3}{2}\int_{\mathbb{R}} Q^4(y) dy \\
&= 8
\end{aligned}$$

where the first integral is zero by a direct computation and the remaining double integrals are converted to single integrals as above.

Computing inner products containing w_1 is harder. We have an explicit expression for Pw_1 but lack an explicit expression for $\overline{P}w_1$. Therefore we approximate $\overline{P}w_1$ by numerically inverting $\overline{P}(K_{00}+1)\overline{P}$ in

$$\overline{P}(K_{00}+1)\overline{P}w_1 = -\left(\frac{1}{4}\overline{P}K_{00}K_{-11}w_0 + \overline{P}K_{01}w_0\right) =: g.$$

Note that $\langle g, v\rangle = \langle g, w_0\rangle = 0$. We represent $\overline{P}(K_{00}+1)\overline{P}$ as a matrix with respect to a basis $\{\phi_j\}_{j=1}^N$. The basis is formed by taking terms from the typical Fourier basis and projecting out the components of each function in the direction of v and w_0. Some basis functions were removed to ensure linear independence of the basis. Let $\overline{P}w_1 = \sum_{j=1}^N a_j \phi_j$. Then

$$B\vec{a} = \vec{b}$$

where $B_{j,k} = \langle \phi_j, (K_{00}+1)\phi_k\rangle$ and $b_j = \langle \phi_j, g\rangle$. So we can solve for $\vec{a}$ by inverting the matrix B. Once we have an approximation for $\overline{P}w_1$ we can

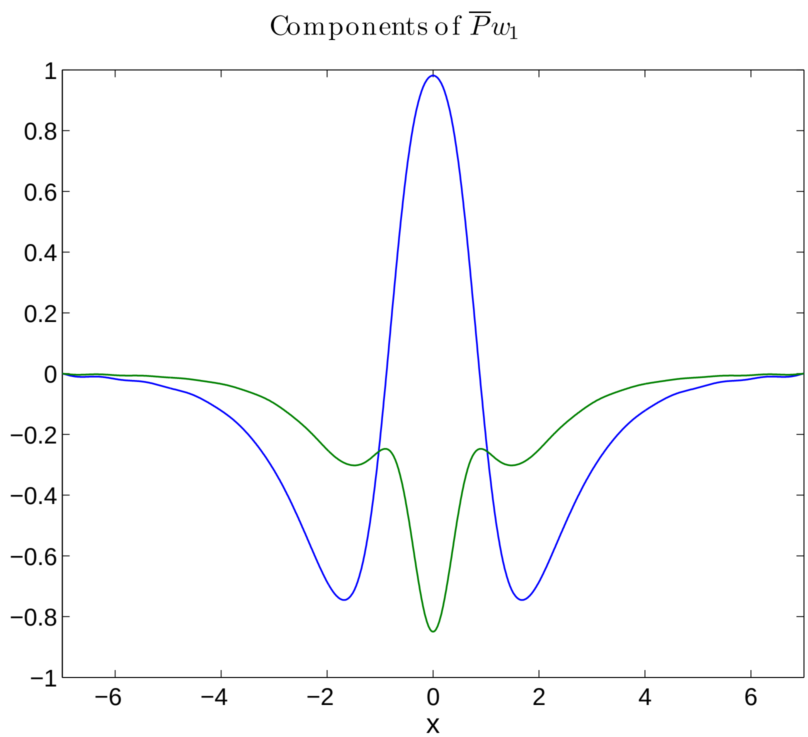

Figure 2.1: The two components of $\overline{P}w_1$ computed numerically with 32 basis terms.

compute $\overline{P}(K_{00}+1)\overline{P}w_1$ directly to observe agreement with the function g. With this agreement we are confident in our numerical algorithm and that our numerical approximation for $\overline{P}w_1$ is accurate. In Figure 2.1 we show the two components of $\overline{P}w_1$ as computed numerically. Figure 2.2 shows the components of the function g with the computed $\overline{P}(K_{00}+1)\overline{P}w_1$ on top.

With an approximation for $\overline{P}w_1$ in hand we can combine it with our explicit expression for Pw_1 and compute inner products containing w_1 in the same way as the previous inner product containing w_0. In this way we establish that $\alpha_2 > 0$. We list computed values for the numerator of α_2 against the number of basis terms used in Table 2.1.

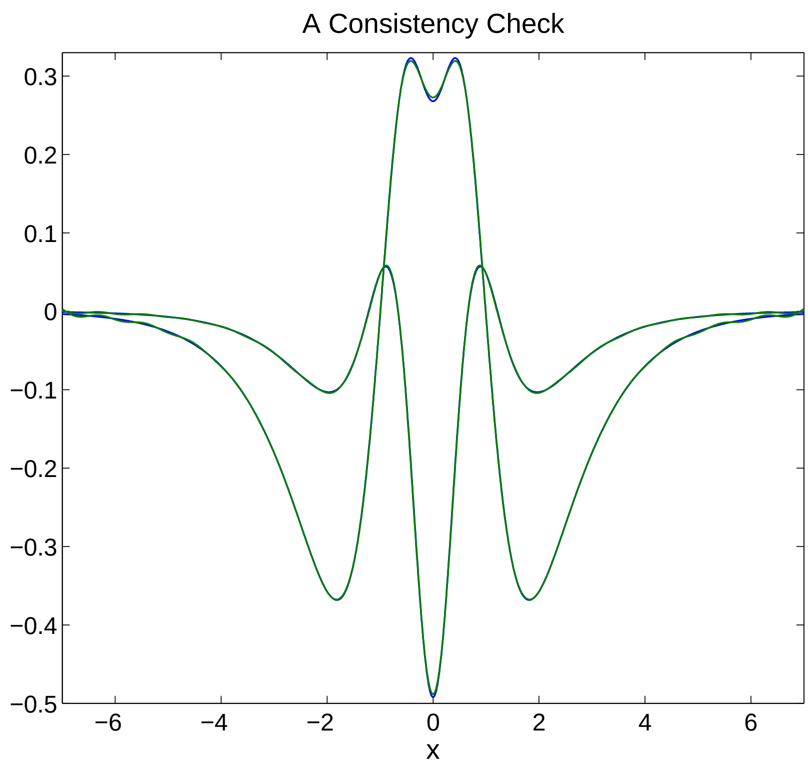

Figure 2.2: The two components of function g with the computed $\overline{P}(K_{00}+1)\overline{P}w_1$ on top. Again 32 basis terms were used in this computation. At this scale the difference can only be seen around zero and at the endpoints.

Number of Basis Terms	$8\alpha_2$
20	2.4992
24	2.5137
28	2.5189
30	2.5201
32	2.5207

Table 2.1: Numerical values for $8\alpha_2$ for the number of basis terms used in the computation.

Chapter 3

Perturbations of the 3D Energy Critical NLS

In this chapter we prove Theorems 1.7.2, 1.7.7, and 1.7.10. The construction of the solitary wave profiles appears in Section 3.1, variational arguments which establish the solitary waves as ground states appear in Section 3.2, and the dynamical (scattering/blow-up) theory appears in Section 3.3.

3.1 Construction of Solitary Wave Profiles

This section is devoted to the proof of Theorem 1.7.2, constructing solitary wave profiles for the perturbed NLS via perturbation from the unperturbed static solution W.

3.1.1 Mathematical Setup

Let $\lambda^2 = \omega$ with $\lambda \geq 0$. Now substitute (1.20) to (1.11) to see

$$(-\Delta - 5W^4 + \lambda^2)\eta = -\lambda^2 W + \varepsilon f(W) + N(\eta)$$

where

$$N(\eta) = (W+\eta)^5 - W^5 - 5W^4\eta + \varepsilon\left(f(W+\eta) - f(W)\right)$$

collects the higher order terms. We can rewrite the above as

$$(H + \lambda^2)\eta = \mathcal{F}, \qquad H = -\Delta + V, \qquad V = -5W^4 \tag{3.1}$$

where

$$\mathcal{F} = \mathcal{F}(\varepsilon, \lambda, \eta) = -\lambda^2 W + \varepsilon f(W) + N(\eta).$$

To understand the resolvent $(H+\lambda^2)^{-1}$ for small λ, we follow [47]. Use the resolvent identity to write

$$(H+\lambda^2)^{-1} = (1 + R_0(-\lambda^2)V)^{-1} R_0(-\lambda^2)$$

where

$$R_0(\zeta) = (-\Delta - \zeta)^{-1}$$

is the free resolvent, and apply Lemma 4.3 of [47] to obtain the expansion

$$(1 + R_0(-\lambda^2)V)^{-1} = -\frac{1}{\lambda}\langle V\psi, \cdot\rangle\psi + O(1) \tag{3.2}$$

where ψ is the normalized resonance eigenfunction (1.18):

$$\psi(x) = \frac{1}{\sqrt{3\pi}}\Lambda W(x), \qquad \int_{\mathbb{R}^3} V\psi = \sqrt{4\pi}.$$

The above expansion is understood in [47] in weighted Sobolev spaces. We choose instead to work in higher L^p spaces. Precise statements are found in the following Section 3.1.2.

To eliminate the singular behaviour as $\lambda \to 0$ we require

$$0 = \langle R_0(-\lambda^2)V\psi, \mathcal{F}(\varepsilon, \lambda, \eta)\rangle. \tag{3.3}$$

Satisfying this condition determines $\lambda = \lambda(\varepsilon, \eta)$. This is done in Section 3.1.3. With this condition met, we can invert (3.1) to see

$$\eta = (H + \lambda^2)^{-1}\mathcal{F} = (H + \lambda^2(\varepsilon, \eta))^{-1}\mathcal{F}(\varepsilon, \lambda(\varepsilon, \eta), \eta) =: \mathcal{G}(\eta, \varepsilon), \tag{3.4}$$

which can be solved for η via a fixed point argument. This is done in Section 3.1.4.

3.1.2 Resolvent Estimates

We collect here some estimates that are necessary for the proof of Theorem 1.7.2.

In order to apply Lemma 4.3 of [47] and so to use the expansion (3.2) in what follows (Lemmas 3.1.1 and 3.1.4) we must have that the operator H has no zero eigenvalue. However, it is true that $H(\partial W/\partial x_j) = 0$ for each $j = 1, 2, 3$. To this end, we restrict ourselves to considering only radial functions. In this way H has no zero eigenvalues and only the one resonance, ΛW (see [33]).

The free resolvent operator $R_0(-\lambda^2)$ for $\lambda > 0$ has integral kernel

$$R_0(-\lambda^2)(x) = \frac{e^{-\lambda|x|}}{4\pi|x|}. \tag{3.5}$$

An application of Young's inequality/generalized Young's inequality gives the bounds

$$\|R_0(-\lambda^2)\|_{L^q\to L^r} \lesssim \lambda^{3(1/q-1/r)-2}, \quad 1\le q\le r\le\infty \tag{3.6}$$

$$\|R_0(-\lambda^2)\|_{L^q_w\to L^r} \lesssim \lambda^{3(1/q-1/r)-2}, \quad 1< q\le r<\infty \tag{3.7}$$

with $3(1/q-1/r)<2$, as well as

$$\|R_0(-\lambda^2)\|_{L^q\to L^r} \lesssim 1 \tag{3.8}$$

where $1<q<3/2$ and $3(1/q-1/r)=2$ (so $3<r<\infty$). We will also need the additional bound

$$\|R_0(-\lambda^2)\|_{L^{\frac{3}{2}-}\cap L^{\frac{3}{2}+}\to L^\infty} \lesssim 1, \tag{3.9}$$

where the $+/-$ means the bound holds for any exponent greater/less than $3/2$, to replace the fact that we do not have (3.8) for $r=\infty$ and $q=3/2$.

Observe also that $R_0(0)=G_0$ has integral kernel

$$G_0(x)=\frac{1}{4\pi|x|}$$

and is formally $(-\Delta)^{-1}$.

We need also some facts about the operator $(1+R_0(-\lambda^2)V)^{-1}$. The idea is that we can think of the full resolvent $(1+R_0(-\lambda^2)V)^{-1}R_0(-\lambda^2)$ as behaving like the free resolvent $R_0(-\lambda^2)$ providing we have a suitable orthogonality condition. Otherwise we lose a power of λ due to the non-invertibility of $(1+G_0V)$: indeed,

$$\psi\in\ker(1+G_0V), \quad V\psi\in\ker\left((1+G_0V)^*=1+VG_0\right). \tag{3.10}$$

First we recall some results of [47]:

Lemma 3.1.1. (Lemmas 2.2 and 4.3 from [47]) *Let s satisfy $3/2<s<5/2$ and denote $\mathcal{B}=B(H^1_{-s},H^1_{-s})$ where H^1_{-s} is the weighted Sobolev space with norm*

$$\|u\|_{H^1_{-s}}=\|(1+|x|^2)^{-s/2}u\|_{H^1}.$$

Then for ζ with $\mathrm{Im}\zeta\ge 0$ we have the expansions

$$1+R_0(\zeta)V=1+G_0V+i\zeta^{1/2}G_1V+o(\zeta^{1/2})$$

$$(1+R_0(\zeta)V)^{-1}=-i\zeta^{-1/2}\langle\cdot,V\psi\rangle\psi+C_0^1+o(1)$$

in $\mathcal{B}$ with $|\zeta| \to 0$. Here C_0^1 is an explicit operator and G_0 and G_1 are convolution with the kernels

$$G_0(x) = \frac{1}{4\pi|x|}, \quad G_1(x) = \frac{1}{4\pi}.$$

Remark 3.1.2. *The expansion is also valid in $B(L^2_{-s}, L^2_{-s})$ where L^2_{-s} is the weighted L^2 space with norm*

$$\|u\|_{L^2_{-s}} = \|(1+|x|^2)^{-s/2}u\|_{L^2}.$$

Remark 3.1.3. *Since our potential only has decay $|V(x)| \lesssim \langle x\rangle^{-4}$ our expansion has one less term than in* [47] *and we use $3/2 < s < 5/2$ rather than $5/2 < s < 7/2$.*

The following is a reformulation of Lemma 3.1.1 but using higher L^p spaces rather than weighted spaces. This reformulation was also used in [44].

Lemma 3.1.4. *Take $3 < r \leq \infty$ and $\lambda > 0$ small. Then*

$$\|(1+R_0(-\lambda^2)V)^{-1}f\|_{L^r} \lesssim \frac{1}{\lambda}\|f\|_{L^r}.$$

If we also have $\langle V\psi, f\rangle = 0$ then

$$\|(1+R_0(-\lambda^2)V)^{-1}f\|_{L^r} \lesssim \|f\|_{L^r}$$

and

$$\begin{aligned}\|(1+R_0(-\lambda^2)V)^{-1}f - \bar{Q}(1+G_0V)^{-1}\bar{P}f\|_{L^r}& \\ \lesssim \left\{\begin{array}{ll}\lambda^{1-3/r}, & 3 < r < \infty \\ \lambda\log(1/\lambda), & r = \infty\end{array}\right\}\|f\|_{L^r}&\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}P &:= \frac{1}{\int V\psi^2}\langle V\psi, \cdot\rangle\psi, \quad \bar{P} = 1 - P \\ Q &:= \frac{1}{\int V\psi}\langle V, \cdot\rangle\psi, \quad \bar{Q} = 1 - Q\end{aligned} \tag{3.12}$$

Proof. We start with the identity

$$\begin{aligned}g := (1+R_0(-\lambda^2)V)^{-1}f &= f - R_0(-\lambda^2)V(1+R_0(-\lambda^2)V)^{-1}f \\ &= f - R_0(-\lambda^2)Vg\end{aligned}$$

so

$$\|g\|_{L^r} \lesssim \|f\|_{L^r} + \|R_0(-\lambda^2)Vg\|_{L^r}.$$

We treat the above second term in two cases. For $3 < r < \infty$ let $1/q = 1/r + 2/3$ and use (3.8) and for $r = \infty$ use (3.9)

$$\begin{aligned}\|R_0(-\lambda^2)Vg\|_{L^r} &\lesssim \begin{cases} \|Vg\|_{L^q}, & 3 < r < \infty \\ \|Vg\|_{L^{3/2-} \cap L^{3/2+}}, & r = \infty \end{cases} \\ &\lesssim \begin{cases} \|V\langle x\rangle^2\|_{L^m}\|g\|_{L^2_{-2}}, & 3 < r < \infty \\ \|V\langle x\rangle^2\|_{L^{6-}\cap L^{6+}}\|g\|_{L^2_{-2}}, & r = \infty \end{cases} \\ &\lesssim \|g\|_{L^2_{-2}}.\end{aligned}$$

Here we used that $|V(x)| \lesssim \langle x\rangle^{-4}$, and with $1/q = 1/m + 1/2$ we have $(4-2)m > 3$. Finally we appeal to Lemma 3.1.1 and use the fact that $L^r \subset L^2_{-2}$ to see

$$\|R_0(-\lambda^2)Vg\|_{L^r} \lesssim \|(1 + R_0(-\lambda^2)V)^{-1}f\|_{L^2_{-2}} \lesssim \frac{1}{\lambda}\|f\|_{L^2_{-2}} \lesssim \frac{1}{\lambda}\|f\|_{L^r}$$

where we can remove the factor of $1/\lambda$ if our orthogonality condition is satisfied.

In light of (3.10),

$$1 + G_0V : L^r \cap V^\perp \to L^r \cap (V\psi)^\perp$$

is bijective, and so we treat the operator $(1 + G_0V)^{-1}$ as acting

$$(1 + G_0V)^{-1} : L^r \cap (V\psi)^\perp \to L^r \cap V^\perp,$$

which is the meaning of the expression $\bar{Q}(1+G_0V)^{-1}\bar{P}$ involving the projections $\bar{P}$ and $\bar{Q}$. That the range should be taken to be $V^\perp$ is a consequence of estimate (3.14) below.

To prove (3.11), expand

$$\begin{aligned}&R_0(-\lambda^2) = G_0 - \lambda G_1 + \lambda^2\tilde{R}, \\ &\tilde{R} := \frac{1}{\lambda^2}\left(R_0(-\lambda^2) - G_0 + \lambda G_1\right) = \frac{1}{\lambda}\left(\frac{e^{-\lambda|x|} - 1 + \lambda|x|}{4\pi\lambda|x|}\right) *\end{aligned}$$

and consider $f \in (V\psi)^\perp \cap L^r$ with $3 < r \leq \infty$. We first establish the estimates

$$\|h\|_{L^q} \lesssim \begin{cases} 1, & 1 < q < \infty \\ \log(1/\lambda), & q = 1 \end{cases}, \qquad h := V\tilde{R}V\psi, \tag{3.13}$$

$$|\langle V, (1 + R_0(-\lambda^2)V)^{-1} f\rangle| \lesssim \left\{ \begin{array}{cc} \lambda, & 3 < r < \infty \\ \lambda \log(1/\lambda), & r = \infty \end{array} \right\} \|f\|_{L^r}. \tag{3.14}$$

For the purpose of these estimates we may make the following replacements: $V\psi \to \langle x\rangle^{-5}$, $V \to \langle x\rangle^{-4}$, and $\tilde{R}(x) \to \min(|x|, 1/\lambda)$. To establish (3.13) we must therefore estimate

$$\langle x\rangle^{-4} \int_{\mathbb{R}^3} \min(|y|, 1/\lambda)\langle y - x\rangle^{-5} dy,$$

and we proceed in two parts:

- Take $|y| \leq 2|x|$. Then

$$\begin{aligned} \langle x\rangle^{-4} \int_{|y|\leq 2|x|} & \min(|y|, 1/\lambda)\langle y - x\rangle^{-5} dy \\ & \lesssim \langle x\rangle^{-4} \min(|x|, 1/\lambda) \int \langle y - x\rangle^{-5} dy \\ & \lesssim \langle x\rangle^{-4} \min(|x|, 1/\lambda) \end{aligned}$$

  and

$$\begin{aligned} \|\langle x\rangle^{-4} & \min(|x|, 1/\lambda)\|_{L^q}^q \\ & \lesssim \int_0^1 r^{q+2} dr + \int_1^{1/\lambda} r^{-3q+2} dr + \frac{1}{\lambda} \int_{1/\lambda}^\infty r^{-4q+2} dr \\ & \lesssim 1 + \left\{ \begin{array}{cc} 1, & q > 1 \\ \log(1/\lambda), & q = 1 \end{array} \right\} + \lambda^{4(q-1)} \\ & \lesssim \begin{cases} 1, & q > 1 \\ \log(1/\lambda), & q = 1 \end{cases}. \end{aligned}$$

- Take $|y| \geq 2|x|$. Then

$$\begin{aligned} \langle x\rangle^{-4} \int_{|y|\geq 2|x|} \min(|y|, 1/\lambda)\langle y - x\rangle^{-5} dy & \lesssim \langle x\rangle^{-4} \int |y|\langle y\rangle^{-5} dy \\ & \lesssim \langle x\rangle^{-4} \end{aligned}$$

  and

$$\|\langle x\rangle^{-4}\|_{L^q} \lesssim 1.$$

With (3.13) established we now prove (3.14). Let $g = (1+R_0(-\lambda^2)V)^{-1}f$ and observe

$$\begin{aligned} 0 &= \frac{1}{\lambda}\langle V\psi, f\rangle \\ &= \frac{1}{\lambda}\langle V\psi, (1+R_0(-\lambda^2)V)g\rangle \\ &= \frac{1}{\lambda}\langle (1+VR_0(-\lambda^2))(V\psi), g\rangle \\ &= \frac{1}{\lambda}\langle (1+V(G_0-\lambda G_1+\lambda^2\tilde{R}))(V\psi), g\rangle \\ &= \langle (-VG_1+\lambda V\tilde{R})(V\psi), g\rangle \\ &= -\frac{1}{\sqrt{4\pi}}\langle V, g\rangle + \lambda\langle h, g\rangle \end{aligned}$$

noting that $(1+VG_0)(V\psi) = 0$. Now

$$|\langle V, g\rangle| \lesssim \lambda \|h\|_{L^{r'}}\|g\|_{L^r} \lesssim \lambda \begin{Bmatrix} 1, & 3 < r < \infty \\ \log(1/\lambda), & r = \infty \end{Bmatrix} \|f\|_{L^r}$$

applying (3.13).

With (3.14) in place we finish the argument. For $f \in L^r \cap (V\psi)^\perp$ we write

$$g = (1+R_0(-\lambda^2)V)^{-1}f \quad \text{and} \quad g_0 = (1+G_0V)^{-1}f.$$

We have

$$0 = (1+R_0(-\lambda^2)V)g - (1+G_0V)g_0$$

and so

$$(1+G_0V)(g-g_0) = -\hat{R}Vg$$

where $\hat{R} = R_0(-\lambda^2) - G_0$. The above also implies $\hat{R}Vg \perp V\psi$. We invert to see

$$g - g_0 = -(1+G_0V)^{-1}\hat{R}Vg + \alpha\psi$$

noting that $\psi \in \ker(1+G_0V)$. Take now inner product with V to see

$$\alpha\langle V, \psi\rangle = \langle V, g\rangle$$

and so

$$|\alpha| \lesssim |\langle V, g\rangle| \lesssim \left\{ \begin{array}{cc} \lambda, & 3 < r < \infty \\ \lambda \log(1/\lambda), & r = \infty \end{array} \right\} \|f\|_{L^r}$$

observing (3.14). It remains to estimate $(1 + G_0 V)^{-1} \hat{R} V g$. We note that

$$\hat{R} = \left(\frac{e^{-\lambda|x|} - 1}{4\pi|x|} \right) *$$

and so for estimates we may replace $\hat{R}(x)$ with $\min(\lambda, 1/|x|)$. There follows by Young's inequality

$$\begin{aligned} \|(1 + G_0 V)^{-1} \hat{R} V g\|_{L^r} &\lesssim \|\hat{R} V g\|_{L^r} \\ &\lesssim \|\min(\lambda, 1/|x|)\|_{L^r} \|V g\|_{L^1} \\ &\lesssim \|\min(\lambda, 1/|x|)\|_{L^r} \|g\|_{L^r} \\ &\lesssim \lambda^{1-3/r} \|f\|_{L^r}. \end{aligned}$$

And so after putting everything together we obtain (3.11). □

We end this section by recording pointwise estimates of the nonlinear terms

$$N(\eta) = (W + \eta)^5 - W^5 - 5W^4\eta + \varepsilon\left(f(W + \eta) - f(W)\right).$$

Bound the first three terms as follows:

$$|(W + \eta)^5 - W^5 - 5W^4\eta| \lesssim W^3\eta^2 + |\eta|^5.$$

For the other term we use the Fundamental Theorem of Calculus and Assumption 1.7.1 to see

$$\begin{aligned} |f(W + \eta) - f(W)| &= \left| \int_0^1 \partial_\delta f(W + \delta\eta) d\delta \right| \\ &= \left| \int_0^1 f'(W + \delta\eta)\eta d\delta \right| \\ &\lesssim |\eta| \sup_{0<\delta<1} \left(|W + \delta\eta|^{p_1 - 1} + |W + \delta\eta|^{p_2 - 1} \right) \\ &\lesssim |\eta| \left(W^{p_1 - 1} + |\eta|^{p_1 - 1} + W^{p_2 - 1} + |\eta|^{p_2 - 1} \right) \\ &\lesssim |\eta| \left(W^{p_1 - 1} + W^{p_2 - 1} \right) + |\eta|^{p_1} + |\eta|^{p_2} \end{aligned}$$

and so together we have

$$|N(\eta)| \lesssim W^3\eta^2 + |\eta|^5 + \varepsilon|\eta|\left(W^{p_1-1} + W^{p_2-1}\right) + \varepsilon|\eta|^{p_1} + \varepsilon|\eta|^{p_2}. \quad (3.15)$$

Similarly

$$\begin{aligned}&|f(W+\eta_1) - f(W+\eta_2)| \\ &\quad \lesssim |\eta_1 - \eta_2|\left(W^{p_1-1} + |\eta_1|^{p_1-1} + |\eta_2|^{p_1-1} + W^{p_2-1} + |\eta_1|^{p_2-1} + |\eta_2|^{p_2-1}\right)\end{aligned}$$

and so

$$\begin{aligned}|N(\eta_1) - N(\eta_2)| \lesssim\ &|\eta_1-\eta_2|(|\eta_1|+|\eta_2|)W^3 + |\eta_1-\eta_2|(|\eta_1|^4+|\eta_2|^4) \\ &+ \varepsilon|\eta_1-\eta_2|\left(W^{p_1-1}+W^{p_2-1}\right) \\ &+ \varepsilon|\eta_1-\eta_2|\left(|\eta_1|^{p_1-1}+|\eta_2|^{p_1-1}+|\eta_1|^{p_2-1}+|\eta_2|^{p_2-1}\right).\end{aligned} \quad (3.16)$$

3.1.3 Solving for the Frequency

We are now in a position to construct solutions to (1.11) and so prove Theorem 1.7.2. The proof proceeds in two steps. In the present section, we will solve for λ in (3.3) for a given small η. Then in the following Section 3.4, we will treat λ as a function of η and solve (3.4). Both steps involve fixed point arguments.

We begin by computing the inner product (3.3). Write

$$0 = \langle R_0(-\lambda^2)V\psi, \mathcal{F}\rangle = \langle R_0(-\lambda^2)V\psi, -\lambda^2 W + \varepsilon f(W) + N(\eta)\rangle$$

so that

$$\lambda \cdot \lambda\langle R_0(-\lambda^2)V\psi, W\rangle = \varepsilon\langle R_0(-\lambda^2)V\psi, f(W)\rangle + \langle R_0(-\lambda^2)V\psi, N(\eta)\rangle. \quad (3.17)$$

It is our intention to find a solution λ of (3.17) of the appropriate size. This is done in Lemma 3.1.6 but we first make some estimates on the leading order inner products appearing above.

Lemma 3.1.5. *We have the estimates*

$$\langle R_0(-\lambda^2)V\psi, f(W)\rangle = -\langle \psi, f(W)\rangle + O(\lambda^{\delta_1}) \quad (3.18)$$

$$\lambda\langle R_0(-\lambda^2)V\psi, W\rangle = 2\sqrt{3\pi} + O(\lambda) \quad (3.19)$$

where δ_1 is defined in the statement of Theorem 1.7.2.

Proof. Firstly

$$\langle R_0(-\lambda^2)V\psi, f(W)\rangle = \langle G_0 V\psi, f(W)\rangle + \langle (R_0(-\lambda^2) - R_0(0))V\psi, f(W)\rangle$$

First note that since $H\psi = 0$ we have $V\psi = -(-\Delta\psi)$ so

$$\langle G_0 V\psi, f(W)\rangle = \langle -(-\Delta)^{-1}(-\Delta\psi), f(W)\rangle = -\langle \psi, f(W)\rangle.$$

Note that this inner product is finite. For the other term use the resolvent identity $R_0(-\lambda^2) - R_0(0) = -\lambda^2 R_0(-\lambda^2)R_0(0)$ to see

$$\langle (R_0(-\lambda^2) - R_0(0))V\psi, f(W)\rangle = \lambda^2\langle R_0(-\lambda^2)\psi, f(W)\rangle.$$

Observe now that

$$\lambda^2|\langle R_0(-\lambda^2)\psi, f(W)\rangle| \leq \lambda^2\|R_0(-\lambda^2)\psi\|_{L^r}\|f(W)\|_{L^{r^*}}$$

where $1/r + 1/r^* = 1$. Choose an $r^* > 1$ with $3/p_1 < r^* < 3/2$. In this way $f(W) \in L^{r^*}$ observing Assumption 1.7.1. We now apply (3.7) with $q = 3$ noting that $3 < r < \infty$. Hence

$$\begin{aligned} \lambda^2|\langle R_0(-\lambda^2)\psi, f(W)\rangle| &\lesssim \lambda^2 \cdot \lambda^{3(1/3-1/r)-2}\|\psi\|_{L^3_w}\|f(W)\|_{L^{r^*}} \\ &\lesssim \lambda^{1-3/r}. \end{aligned}$$

If $p_1 \geq 3$ we can take r as large as we like. Otherwise we must take $3 < r < 3/(3-p_1)$ and so $1-3/r$ can be made close to $p_1 - 2$ (from below). We now see (3.18).

Next on to (3.19). Note that this computation is taken from [44]. First we isolate the troublesome part of W and write

$$W = \frac{\sqrt{3}}{|x|} + \tilde{W}.$$

There is no problem with the second term since $\tilde{W} \in L^{6/5}$ and $V\psi \in L^{6/5}$ so we can use (3.8) with $q = 6/5$ and $r = 6$ to see

$$\begin{aligned} \lambda|\langle R_0(-\lambda^2)V\psi, \tilde{W}\rangle| &\lesssim \lambda\|R_0(-\lambda^2)V\psi\|_{L^6}\|\tilde{W}\|_{L^{6/5}} \\ &\lesssim \lambda\|V\psi\|_{L^{6/5}}\|\tilde{W}\|_{L^{6/5}} \qquad (3.20) \\ &\lesssim \lambda. \qquad (3.21) \end{aligned}$$

Set $g := V\psi$ and concentrate on

$$\lambda\sqrt{3}\left\langle R_0(-\lambda^2)g, \frac{1}{|x|}\right\rangle = \sqrt{\frac{6}{\pi}}\lambda\left\langle \frac{\hat{g}(\xi)}{|\xi|^2+\lambda^2}, \frac{1}{|\xi|^2}\right\rangle$$

where we work on the Fourier Transform side, using Plancherel's theorem. So

$$\begin{aligned}
&\sqrt{\frac{6}{\pi}}\lambda\left\langle\frac{\hat{g}(\xi)}{|\xi|^2+\lambda^2},\frac{1}{|\xi|^2}\right\rangle \\
&\qquad = \sqrt{\frac{6}{\pi}}\lambda\,\hat{g}(0)\left\langle\frac{1}{|\xi|^2+\lambda^2},\frac{1}{|\xi|^2}\right\rangle+\sqrt{\frac{6}{\pi}}\lambda\left\langle\frac{\hat{g}(\xi)-\hat{g}(0)}{|\xi|^2+\lambda^2},\frac{1}{|\xi|^2}\right\rangle
\end{aligned}$$

where the first term is the leading order. We invert the Fourier Transform and note that $\hat{g}(0)=(2\pi)^{-\frac{3}{2}}\int g$ to see

$$\begin{aligned}
\sqrt{\frac{6}{\pi}}\lambda\,\hat{g}(0)\left\langle\frac{1}{|\xi|^2+\lambda^2},\frac{1}{|\xi|^2}\right\rangle &= \sqrt{3}\left(\int g\right)\lambda\left\langle\frac{e^{-\lambda|x|}}{4\pi|x|},\frac{1}{|x|}\right\rangle \\
&= \sqrt{3}\int g = 2\sqrt{3\pi}.
\end{aligned}$$

We now must bound the remainder term. It is easy for the high frequencies

$$\int_{|\xi|\geq 1}\frac{|\hat{g}(\xi)-\hat{g}(0)|}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi \lesssim \|\hat{g}\|_{L^\infty}\int_{|\xi|\geq 1}\frac{d\xi}{|\xi|^4}\lesssim \|g\|_{L^1}\lesssim 1.$$

For the low frequencies note that since $|x|g\in L^1$ we have that $\nabla\hat{g}$ is continuous and bounded. In light of this set

$$h(\xi):=\phi(\xi)\left(\hat{g}(\xi)-\hat{g}(0)-\nabla\hat{g}(0)\cdot\xi\right)$$

where ϕ is a smooth, compactly supported cutoff function with $\phi=1$ on $|\xi|\leq 1$. Now since

$$\int_{|\xi|\leq 1}\frac{\xi}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi = 0$$

we have

$$\int_{|\xi|\leq 1}\frac{\hat{g}(\xi)-\hat{g}(0)}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi = \int_{|\xi|\leq 1}\frac{h(\xi)}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi$$

and so bound this integral instead. If we recall the form of g we see $|g|\lesssim \langle x\rangle^{-5}$ and so $(1+|x|^{1+\alpha})g\in L^1$ for some $\alpha>0$. Therefore $(1+|x|^{1+\alpha})\check{h}\in L^1$ and noting also that $\nabla h(0)=0$ we see $|\nabla h(\xi)|\lesssim \min(1,|\xi|^\alpha)$. The Mean

Value Theorem along with $h(0) = 0$ then gives $|h(\xi)| \lesssim \min(1, |\xi|^{1+\alpha})$. With this bound established we consider two regions of the integral

$$\int_{|\xi|\leq\lambda} \frac{|h(\xi)|}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi \lesssim \int_{|\xi|\leq\lambda} \frac{|\xi|}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi \lesssim \int_{|\zeta|\leq 1} \frac{1}{|\zeta|(|\zeta|^2+1)}d\zeta \lesssim 1$$

and

$$\int_{\lambda\leq|\xi|\leq 1} \frac{|h(\xi)|}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi \lesssim \int_{\lambda\leq|\xi|\leq 1} \frac{|\xi|^{1+\alpha}}{|\xi|^2(|\xi|^2+\lambda^2)}d\xi \lesssim \lambda^\alpha \int_{1\leq|\zeta|\leq 1/\lambda} \frac{|\zeta|^{\alpha-1}}{|\zeta|^2+1}d\zeta \lesssim \lambda^\alpha\cdot\lambda^{-\alpha} \lesssim 1.$$

Putting everything together gives (3.19). □

With the above estimates in hand we turn our attention to solving (3.17).

Lemma 3.1.6. *For any $R > 0$ there exists $\varepsilon_0 = \varepsilon_0(R) > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ and given a fixed $\eta \in L^\infty$ with $\|\eta\|_{L^\infty} \leq R\varepsilon$ the equation* (3.17) *has a unique solution $\lambda = \lambda(\varepsilon, \eta)$ satisfying $\varepsilon\lambda^{(1)}/2 \leq \lambda \leq 3\varepsilon\lambda^{(1)}/2$ where*

$$\lambda^{(1)} = \frac{-\langle \Lambda W, f(W)\rangle}{6\pi} > 0. \tag{3.22}$$

Moreover, we have the expansion

$$\lambda = \lambda^{(1)}\varepsilon + \tilde{\lambda}, \qquad \tilde{\lambda} = O(\varepsilon^{1+\delta_1}). \tag{3.23}$$

Remark 3.1.7. *Writing the resolvent as* (3.5)*, and thus the subsequent estimates* (3.6)*-*(3.8)*, require $\lambda > 0$ and so it is essential that we have established $\lambda^{(1)} > 0$. This is the source of the sign condition in Assumption 1.7.1.*

Proof. We first estimate the remainder term. Take $\varepsilon\lambda^{(1)}/2 \leq \lambda \leq 3\varepsilon\lambda^{(1)}/2$ and η with $\|\eta\|_{L^\infty} \leq R\varepsilon$. We establish the estimate

$$|\langle R_0(-\lambda^2)V\psi, N(\eta)\rangle| \lesssim \varepsilon^{1+\delta_1}. \tag{3.24}$$

We deal with each term in (3.15). Take $j = 1, 2$. We frequently apply (3.6), (3.8) and Hölder:

- $$\begin{aligned}|\langle R_0(-\lambda^2)V\psi, W^3\eta^2\rangle| &\lesssim \|R_0(-\lambda^2)V\psi\|_{L^6}\|W^3\eta^2\|_{L^{6/5}}\\ &\lesssim \|V\psi\|_{L^{6/5}}\|\eta\|_{L^\infty}^2\|W^3\|_{L^{6/5}}\\ &\lesssim \varepsilon^2\end{aligned}$$
- $$\begin{aligned}|\langle R_0(-\lambda^2)V\psi, \eta^5\rangle| &\lesssim \|R_0(-\lambda^2)V\psi\|_{L^1}\|\eta^5\|_{L^\infty}\\ &\lesssim \lambda^{-2}\|V\psi\|_{L^1}\|\eta\|_{L^\infty}^5\\ &\lesssim \varepsilon^3\end{aligned}$$
- $$\begin{aligned}\varepsilon|\langle R_0(-\lambda^2)V\psi, \eta^{p_j}\rangle| &\lesssim \varepsilon\|R_0(-\lambda^2)V\psi\|_{L^1}\|\eta^{p_j}\|_{L^\infty}\\ &\lesssim \varepsilon\lambda^{-2}\|V\psi\|_{L^1}\|\eta\|_{L^\infty}^{p_j}\\ &\lesssim \varepsilon\cdot\varepsilon^{p_j-2}\end{aligned}$$

The term that remains requires two cases. First take $p_j > 3$ then

$$\begin{aligned}\varepsilon|\langle R_0(-\lambda^2)V\psi, \eta W^{p_j-1}\rangle| &\lesssim \varepsilon\|R_0(-\lambda^2)V\psi\|_{L^r}\|\eta W^{p_j-1}\|_{L^{r^*}}\\ &\lesssim \varepsilon\|V\psi\|_{L^q}\|\eta\|_{L^\infty}\|W^{p_j-1}\|_{L^{r^*}}\\ &\lesssim \varepsilon^2\end{aligned}$$

where we have used (3.8) for some $r^* < 3/2$ and $r > 3$. Now if instead $2 < p_j \leq 3$ we use (3.6) with $r^* = (3/(p_j-1))^+$ so $1 - 1/r = ((p_j-1)/3)^-$ and

$$\begin{aligned}\varepsilon|\langle R_0(-\lambda^2)V\psi, \eta W^{p_j-1}\rangle| &\lesssim \varepsilon\|R_0(-\lambda^2)V\psi\|_{L^r}\|\eta W^{p_j-1}\|_{L^{r^*}}\\ &\lesssim \varepsilon\lambda^{3(1-1/r)-2}\|V\psi\|_{L^1}\|\eta\|_{L^\infty}\|W^{p_j-1}\|_{L^{r^*}}\\ &\lesssim \varepsilon\cdot\varepsilon^{(p_j-2)^-}\end{aligned}$$

and so we establish (3.24).

With the estimates (3.18), (3.19), (3.24) in hand we show that a solution to (3.17) of the desired size exists. For this write (3.17) as a fixed point problem

$$\lambda = \mathcal{H}(\lambda) := \frac{\varepsilon\langle R_0(-\lambda^2)V\psi, f(W)\rangle + \langle R_0(-\lambda^2)V\psi, N(\eta)\rangle}{\lambda\langle R_0(-\lambda^2)V\psi, W\rangle} \tag{3.25}$$

with the intention of applying Banach Fixed Point Theorem. We show that for a fixed η with $\|\eta\|_{L^\infty} \lesssim \varepsilon$ the function $\mathcal{H}$ maps the interval $\varepsilon\lambda^{(1)}/2 \leq \lambda \leq 3\varepsilon\lambda^{(1)}/2$ to itself and that $\mathcal{H}$ is a contraction.

First note that $-\langle\psi, f(W)\rangle > 0$ by Assumption 1.7.1 and so after observing (3.18), (3.19), (3.24) we see that $\mathcal{H}(\lambda) > 0$. Furthermore for ε small enough we have that $\varepsilon\lambda^{(1)}/2 \le \mathcal{H}(\lambda) \le 3\varepsilon\lambda^{(1)}/2$ and so $\mathcal{H}$ maps this interval to itself.

We next show that $\mathcal{H}$ is a contraction. Take $\varepsilon\lambda^{(1)}/2 \le \lambda_1, \lambda_2 \le 3\varepsilon\lambda^{(1)}/2$ and again keep η fixed with $\|\eta\|_{L^\infty} \le R\varepsilon$. Write

$$\mathcal{H}(\lambda) = \frac{a(\lambda) + b(\lambda)}{c(\lambda)}$$

so that

$$\begin{aligned} |\mathcal{H}(\lambda_1) - \mathcal{H}(\lambda_2)| &\le \frac{|a_1||c_2 - c_1| + |a_1 - a_2||c_1| + |b_1||c_2 - c_1| + |b_1 - b_2||c_1|}{|c_1 c_2|} \\ &\lesssim |a_1 - a_2| + |b_1 - b_2| + \varepsilon|c_1 - c_2| \end{aligned}$$

using (3.18), (3.19), (3.24). We treat each piece in turn.

First

$$\begin{aligned} |a_1 - a_2| &= \varepsilon|\langle\left(R_0(-\lambda_1^2) - R_0(-\lambda_2^2)\right) V\psi, f(W)\rangle| \\ &= \varepsilon|\lambda_1^2 - \lambda_2^2||\langle R_0(-\lambda_1^2) R_0(-\lambda_2^2) V\psi, f(W)\rangle| \end{aligned}$$

by the resolvent identity. Continuing we see

$$|a_1 - a_2| \lesssim \varepsilon^2 |\lambda_1 - \lambda_2| \|R_0(-\lambda_1^2) R_0(-\lambda_2^2) V\psi\|_{L^r} \|f(W)\|_{L^{r^*}}$$

where $1/r + 1/r^* = 1$. Note that by Assumption 1.7.1 we have $f(W) \in L^{r^*}$ for some $1 < r^* < 3/2$ so $3 < r < \infty$. Applying now (3.8) we get

$$|a_1 - a_2| \lesssim \varepsilon^2 |\lambda_1 - \lambda_2| \|R_0(-\lambda_2^2) V\psi\|_{L^q}$$

with $3(1/q - 1/r) = 2$ so $1 < q < 3/2$. Now apply the bound (3.6)

$$\begin{aligned} |a_1 - a_2| &\lesssim \varepsilon^2 |\lambda_1 - \lambda_2| \lambda^{3(1-1/q)-2} \|V\psi\|_{L^1} \\ &\lesssim \varepsilon^{3(1-1/q)} |\lambda_1 - \lambda_2| \end{aligned}$$

and note that $3(1 - 1/q) > 0$.

Next consider

$$|b_1 - b_2| = |\langle R_0(-\lambda_1^2) V\psi, N(\eta)\rangle - \langle R_0(-\lambda_2^2) V\psi, N(\eta)\rangle|.$$

Proceeding as in the previous argument and using (3.6) we see

$$\begin{aligned} |b_1 - b_2| &\lesssim \varepsilon|\lambda_1 - \lambda_2| \|R_0(-\lambda_1^2) R_0(-\lambda_2^2) V\psi\|_{L^r} \|N(\eta)\|_{L^{r^*}} \\ &\lesssim \varepsilon|\lambda_1 - \lambda_2| \lambda_1^{-2} \|R_0(-\lambda_2^2) V\psi\|_{L^r} \|N(\eta)\|_{L^{r^*}} \end{aligned}$$

for $1/r + 1/r^* = 1$. We can estimate this term (using different r and r^* for different portions of $N(\eta)$) using the computations leading to (3.24) to achieve

$$|b_1 - b_1| \lesssim \varepsilon^{-1} \cdot \varepsilon^{1+\delta_1}|\lambda_1 - \lambda_2| = \varepsilon^{\delta_1}|\lambda_1 - \lambda_2|.$$

Lastly consider

$$\varepsilon|c_1 - c_2| = \varepsilon|\lambda_1\langle R_0(-\lambda_1^2)V\psi, W\rangle - \lambda_2\langle R_0(-\lambda_2^2)V\psi, W\rangle|.$$

Again we write $W = \sqrt{3}/|x| + \tilde{W}$ where $\tilde{W} \in L^{6/5}$. The second term is easy. We compute

$$\begin{aligned}
&\varepsilon|\lambda_1\langle R_0(-\lambda_1^2)V\psi, \tilde{W}\rangle - \lambda_2\langle R_0(-\lambda_2^2)V\psi, \tilde{W}\rangle| \\
&\quad\lesssim \varepsilon|\lambda_1 - \lambda_2||\langle R_0(-\lambda_1^2)V\psi, \tilde{W}\rangle| + \varepsilon^3|\lambda_1 - \lambda_2||\langle R_0(-\lambda_1^2)R_0(-\lambda_2^2)V\psi, \tilde{W}\rangle| \\
&\quad\lesssim \varepsilon|\lambda_1 - \lambda_2| + \varepsilon^3\lambda_1^{-2}\lambda_2^{3(1-1/6)-2}|\lambda_1 - \lambda_2|\|V\psi\|_{L^1}\|\tilde{W}\|_{L^{6/5}} \\
&\quad\lesssim \varepsilon|\lambda_1 - \lambda_2|
\end{aligned}$$

where we have used (3.21) once and (3.6) twice. For the harder term we follow the computations which establish (3.19) and so work on the Fourier Transform side

$$\begin{aligned}
&\varepsilon\lambda_1\langle R_0(-\lambda_1^2)V\psi, 1/|x|\rangle - \varepsilon\lambda_2\langle R_0(-\lambda_2^2)V\psi, 1/|x|\rangle \\
&\quad= C\varepsilon\lambda_1\left\langle \frac{\hat{g}(\xi)}{|\xi|^2 + \lambda_1^2}, \frac{1}{|\xi|^2}\right\rangle - C\varepsilon\lambda_2\left\langle \frac{\hat{g}(\xi)}{|\xi|^2 + \lambda_2^2}, \frac{1}{|\xi|^2}\right\rangle \\
&\quad= C\varepsilon\lambda_1\left\langle \frac{\hat{g}(\xi) - \hat{g}(0)}{|\xi|^2 + \lambda_1^2}, \frac{1}{|\xi|^2}\right\rangle - C\varepsilon\lambda_2\left\langle \frac{\hat{g}(\xi) - \hat{g}(0)}{|\xi|^2 + \lambda_2^2}, \frac{1}{|\xi|^2}\right\rangle \\
&\quad= C\varepsilon(\lambda_1 - \lambda_2)\left\langle \frac{\hat{g}(\xi) - \hat{g}(0)}{|\xi|^2 + \lambda_1^2}, \frac{1}{|\xi|^2}\right\rangle \\
&\qquad+ C\varepsilon\lambda_2\left\langle (\hat{g}(\xi) - \hat{g}(0))\left(\frac{1}{|\xi|^2 + \lambda_1^2} - \frac{1}{|\xi|^2 + \lambda_2^2}\right), \frac{1}{|\xi|^2}\right\rangle
\end{aligned}$$

where we have used the fact that

$$\lambda_1\left\langle \frac{\hat{g}(0)}{|\xi|^2 + \lambda_1^2}, \frac{1}{|\xi|^2}\right\rangle = \lambda_2\left\langle \frac{\hat{g}(0)}{|\xi|^2 + \lambda_2^2}, \frac{1}{|\xi|^2}\right\rangle.$$

Continuing as in the computations used to establish (3.19) , we bound

$$\varepsilon|\lambda_1 - \lambda_2|\left|\left\langle \frac{\hat{g}(\xi) - \hat{g}(0)}{|\xi|^2 + \lambda_1^2}, \frac{1}{|\xi|^2}\right\rangle\right| \lesssim \varepsilon|\lambda_1 - \lambda_2|$$

and

$$\begin{aligned}
&\varepsilon\lambda_2 \left| \left\langle (\hat{g}(\xi) - \hat{g}(0)) \left(\frac{1}{|\xi|^2 + \lambda_1^2} - \frac{1}{|\xi|^2 + \lambda_2^2} \right), \frac{1}{|\xi|^2} \right\rangle \right| \\
&\quad \lesssim \varepsilon\lambda_2(\lambda_1 + \lambda_2)|\lambda_1 - \lambda_2| \int \frac{d\xi}{|\xi|(|\xi|^2 + \lambda_1^2)(|\xi|^2 + \lambda_2^2)} \\
&\quad \lesssim \varepsilon|\lambda_1 - \lambda_2| \int \frac{d\zeta}{|\zeta|(|\zeta|^2 + 1)(|\zeta|^2 + \lambda_2^2/\lambda_1^2)} \\
&\quad \lesssim \varepsilon|\lambda_1 - \lambda_2|.
\end{aligned}$$

In this way we finally have

$$\varepsilon|c_1 - c_2| \leq \varepsilon|\lambda_1 - \lambda_2|.$$

So, putting everything together we see that by taking ε sufficiently small,

$$|\mathcal{H}(\lambda_1) - \mathcal{H}(\lambda_2)| < \kappa|\lambda_1 - \lambda_2|$$

for some $0 < \kappa < 1$, and hence $\mathcal{H}$ is a contraction. Therefore (3.25) has a unique fixed point of the desired size.

To find the leading order $\lambda^{(1)}$ let λ take the form in (3.23), substitute to (3.17) use estimates (3.18), (3.19), (3.24) and ignore higher order terms. An inspection of the higher order terms gives the order of $\tilde{\lambda}$. □

In this way we now think of λ as a function of η. We will also need the following Lipshitz condition for what follows in Lemma 3.1.9.

Lemma 3.1.8. *The λ generated via Lemma 3.1.6 is Lipshitz continuous in η in the sense that*

$$|\lambda_1 - \lambda_2| \lesssim \varepsilon^{\delta_1} \|\eta_1 - \eta_2\|_{L^\infty}.$$

Proof. Take η_1 and η_2 with $\|\eta_1\|_{L^\infty}, \|\eta_2\|_{L^\infty} \leq R\varepsilon$. Let η_1 and η_2 give rise to λ_1 and λ_2 respectively through Lemma 3.1.6. Consider now the difference

$$\begin{aligned}
|\lambda_1 - \lambda_2| &= \left| \frac{\varepsilon\langle R_0(-\lambda_1^2)V\psi, f(W)\rangle + \langle R_0(-\lambda_1^2)V\psi, N(\eta_1)\rangle}{\lambda_1\langle R_0(-\lambda_1^2)V\psi, W\rangle} \right. \\
&\qquad \left. - \frac{\varepsilon\langle R_0(-\lambda_2^2)V\psi, f(W)\rangle + \langle R_0(-\lambda_2^2)V\psi, N(\eta_2)\rangle}{\lambda_2\langle R_0(-\lambda_2^2)V\psi, W\rangle} \right| \\
&=: \left| \frac{a(\lambda_1) + b(\lambda_1, \eta_1)}{c(\lambda_1)} - \frac{a(\lambda_2) + b(\lambda_2, \eta_2)}{c(\lambda_2)} \right|
\end{aligned}$$

observing (3.25). Now we estimate

$$
\begin{aligned}
|\lambda_1-\lambda_2| \leq & \left|\frac{b(\lambda_1,\eta_1)-b(\lambda_1,\eta_2)}{c(\lambda_1)}\right| + \left|\frac{a(\lambda_1)+b(\lambda_1,\eta_2)}{c(\lambda_1)} - \frac{a(\lambda_2)+b(\lambda_2,\eta_2)}{c(\lambda_2)}\right| \\
\leq & C|\langle R_0(-\lambda_1^2)V\psi, N(\eta_1)-N(\eta_2)\rangle| + \kappa|\lambda_1-\lambda_2|
\end{aligned}
$$

for some $0<\kappa<1$. The second term has been estimated using the computations of Lemma 3.1.6 and taking ε small enough. Now we estimate the first. Observing the terms in (3.16) we use the same procedure that established (3.24) to obtain

$$|\langle R_0(-\lambda_1^2)V\psi, N(\eta_1)-N(\eta_2)\rangle| \lesssim \varepsilon^{\delta_1}\|\eta_1-\eta_2\|_{L^\infty}.$$

So together we now see

$$(1-\kappa)|\lambda_1-\lambda_2| \lesssim \varepsilon^{\delta_1}\|\eta_1-\eta_2\|_{L^\infty}$$

which gives the desired result. □

3.1.4 Solving for the Correction

We next solve (3.4), given that (3.3) holds. Recall the formulation of (3.4) as the fixed-point equation

$$\eta = \mathcal{G}(\eta,\varepsilon) = (H+\lambda^2)^{-1}\mathcal{F}$$

where in light of Lemma 3.1.6, we take $\lambda=\lambda(\varepsilon,\eta)$ and $\mathcal{F}=\mathcal{F}(\varepsilon,\lambda(\varepsilon,\eta),\eta)$ so that (3.3) holds.

Lemma 3.1.9. *There exists $R_0>0$ such that for any $R\geq R_0$, there is $\varepsilon_1=\varepsilon_1(R)>0$ such that for each $0<\varepsilon\leq\varepsilon_1$, there exists a unique solution $\eta\in L^\infty$ to* (3.4) *with $\|\eta\|_{L^\infty}\leq R\varepsilon$. Moreover, we have the expansion*

$$\eta = \varepsilon\bar{Q}(1+G_0V)^{-1}\bar{P}\left(G_0f(W) - \lambda^{(1)}\sqrt{3}\lambda R_0(-\lambda^2)|x|^{-1}\right) + O_{L^\infty}(\varepsilon^{1+\delta_1})$$

where $\bar{P}$ and $\bar{Q}$ are given in (3.12).

Proof. We proceed by means of Banach Fixed Point Theorem. We show that $\mathcal{G}(\eta)$ maps a ball to itself and is a contraction. In this way we establish a solution to $\eta=\mathcal{G}(\eta,\varepsilon,\lambda(\varepsilon,\eta))$ in (3.4).

Let $R>0$ (to be chosen) and take $\varepsilon<\varepsilon_0(R)$ as in Lemma 3.1.6. In this way given $\eta\in L^\infty$ with $\|\eta\|_{L^\infty}\leq R\varepsilon$ we can generate

$$\lambda=\lambda(\varepsilon,\eta)=\lambda^{(1)}\varepsilon+o(\varepsilon).$$

We aim to take ε smaller still in order to run fixed point in the L^∞ ball of radius $R\varepsilon$.

Consider

$$\begin{aligned}\|\mathcal{G}\|_{L^\infty} &= \|(1+R_0(-\lambda^2)V)^{-1}R_0(-\lambda^2)\mathcal{F}\|_{L^\infty}\\ &\lesssim \|R_0(-\lambda^2)\mathcal{F}\|_{L^\infty}\end{aligned}$$

in light of Lemma 3.1.4 and since we have chosen λ to satisfy (3.3). Continuing with

$$\|\mathcal{G}\|_{L^\infty} \lesssim \|R_0(-\lambda^2)\left(-\lambda^2 W + \varepsilon f(W) + N(\eta)\right)\|_{L^\infty}$$

we treat each term separately. For the first term it is sufficient to replace W with $1/|x|$ (otherwise we simply apply (3.9))

$$\begin{aligned}\lambda^2\|R_0(-\lambda^2)W\|_{L^\infty} &\lesssim \lambda\left\|\lambda R_0(-\lambda^2)\frac{1}{|x|}\right\|_{L^\infty}\\ &\lesssim \lambda\left\|\lambda\int\frac{e^{-\lambda|y|}}{|y|}\frac{1}{|x-y|}dy\right\|_{L^\infty}\\ &\lesssim \lambda\left\|\int\frac{e^{-|z|}}{|z|}\frac{1}{|\lambda x - z|}dz\right\|_{L^\infty}\\ &\lesssim \lambda\left\|\left(\frac{e^{-|x|}}{|x|}*\frac{1}{|x|}\right)(\lambda x)\right\|_{L^\infty}\\ &\lesssim \lambda \lesssim \varepsilon.\end{aligned}$$

Now for the second term use (3.9)

$$\varepsilon\|R_0(-\lambda^2)f(W)\|_{L^\infty} \lesssim \varepsilon\|f(W)\|_{L^{3/2-}\cap L^{3/2+}} \lesssim \varepsilon.$$

And for the higher order terms we employ (3.6) and (3.9)

- $$\begin{aligned}\|R_0(-\lambda^2)(W^3\eta^2)\|_{L^\infty} &\lesssim \|W^3\eta^2\|_{L^{3/2-}\cap L^{3/2+}}\\ &\lesssim \|W^3\|_{L^{3/2-}\cap L^{3/2+}}\|\eta\|_{L^\infty}^2\\ &\lesssim R^2\varepsilon^2\end{aligned}$$
- $\|R_0(-\lambda^2)\eta^5\|_{L^\infty} \lesssim \lambda^{-2}\|\eta^5\|_{L^\infty} \lesssim \lambda^{-2}\|\eta\|_{L^\infty}^5 \lesssim R^5\varepsilon^3$
- $$\begin{aligned}\varepsilon\|R_0(-\lambda^2)\eta^{p_j}\|_{L^\infty} &\lesssim \varepsilon\lambda^{-2}\|\eta^{p_j}\|_{L^\infty}\\ &\lesssim \varepsilon^{-1}\|\eta\|_{L^\infty}^{p_j}\\ &\lesssim R^{p_j}\varepsilon^{p_j-1}\end{aligned}$$

for $j = 1,2$. The remaining remainder term again requires two cases. For $p_j > 3$ we use (3.9) to see

$$\begin{aligned}\varepsilon \left\| R_0(-\lambda^2)\left(\eta W^{p_j-1}\right)\right\|_{L^\infty} &\lesssim \varepsilon \|\eta W^{p_j-1}\|_{L^{3/2-}\cap L^{3/2+}} \\ &\lesssim \varepsilon \|\eta\|_{L^\infty} \|W^{p_j-1}\|_{L^{3/2-}\cap L^{3/2+}} \\ &\lesssim R\varepsilon^2\end{aligned}$$

and for $2 < p_j \leq 3$ we apply (3.6)

$$\begin{aligned}\varepsilon \left\| R_0(-\lambda^2)\left(\eta W^{p_j-1}\right)\right\|_{L^\infty} &\lesssim \varepsilon \lambda^{(p_j-1)^- -2} \|\eta\|_{L^\infty} \|W^{p_j-1}\|_{L^{3/(p_j-1)^+}} \\ &\lesssim R\varepsilon^{1+(p_j-2)^-}\end{aligned}$$

Collecting the above yields

$$\|\mathcal{G}\|_{L^\infty} \leq C\varepsilon \left(1 + R^2\varepsilon + R^5\varepsilon^2 + R^{p_1}\varepsilon^{p_1-2} + R^{p_2}\varepsilon^{p_2-2} + R\varepsilon + R\varepsilon^{(p_1-2)-}\right) \tag{3.26}$$

and so taking $R_0 = 2C$, $R \geq R_0$, and then ε small enough so that $R\varepsilon + R^4\varepsilon^2 + R^{p_1-1}\varepsilon^{p_1-2} + R^{p_2-1}\varepsilon^{p_2-2} + \varepsilon + \varepsilon^{(p_1-2)-} \leq \frac{1}{2C}$, we arrive at

$$\|\mathcal{G}\|_{L^\infty} \leq R\varepsilon.$$

Hence $\mathcal{G}$ maps the ball of radius $R\varepsilon$ in L^∞ to itself.

Now we show that $\mathcal{G}$ is a contraction. Take η_1 and η_2 and let them give rise to λ_1 and λ_2 respectively. Again $\|\eta_j\|_{L^\infty} \leq R\varepsilon$ and denote $\mathcal{F}(\eta_j)$ by $\mathcal{F}_j$, $j = 1,2$. Consider

$$\begin{aligned}&\|\mathcal{G}(\eta_1,\varepsilon) - \mathcal{G}(\eta_2,\varepsilon)\|_{L^\infty} \\ &= \|(1+R_0(-\lambda_1^2)V)^{-1}R_0(-\lambda_1^2)\mathcal{F}_1 - (1+R_0(-\lambda_2^2)V)^{-1}R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty} \\ &\leq \|(1+R_0(-\lambda_1^2)V)^{-1}\left(R_0(-\lambda_1^2)\mathcal{F}_1 - R_0(-\lambda_2^2)\mathcal{F}_2\right)\|_{L^\infty} \\ &\quad + \|\left((1+R_0(-\lambda_1^2)V)^{-1} - (1+R_0(-\lambda_2^2)V)^{-1}\right)R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty} \\ &\leq \|R_0(-\lambda_1^2)\mathcal{F}_1 - R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty} \\ &\quad + \|\left((1+R_0(-\lambda_1^2)V)^{-1} - (1+R_0(-\lambda_2^2)V)^{-1}\right)R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty} \\ &\leq \|R_0(-\lambda_1^2)\left(\mathcal{F}_1 - \mathcal{F}_2\right)\|_{L^\infty} + \|\left(R_0(-\lambda_1^2) - R_0(-\lambda_2^2)\right)\mathcal{F}_2\|_{L^\infty} \\ &\quad + \|\left((1+R_0(-\lambda_1^2)V)^{-1} - (1+R_0(-\lambda_2^2)V)^{-1}\right)R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty} \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}\end{aligned}$$

where we have applied Lemma 3.1.4, observing the orthogonality condition. We treat each part in turn.

Start with I. This computation is similar to those previous. We also apply Lemma 3.1.8:

$$\begin{aligned}\|R_0(-\lambda_1^2)\,(\mathcal{F}_1-\mathcal{F}_2)\,\|_{L^\infty} &= \|R_0(-\lambda_1^2)\left((\lambda_2^2-\lambda_1^2)W+N(\eta_1)-N(\eta_2)\right)\|_{L^\infty}\\ &\lesssim |\lambda_1-\lambda_2|+\varepsilon^{\delta_1}\|\eta_1-\eta_2\|_{L^\infty}\\ &\lesssim \varepsilon^{\delta_1}\|\eta_1-\eta_2\|_{L^\infty}.\end{aligned}$$

Part II is also similar to previous computations:

$$\begin{aligned}\|\left(R_0(-\lambda_1^2)-R_0(-\lambda_2^2)\right)\mathcal{F}_2\|_{L^\infty} &= |\lambda_1^2-\lambda_2^2|\|R_0(-\lambda_1^2)R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty}\\ &\lesssim |\lambda_1+\lambda_2||\lambda_1-\lambda_2|\lambda_1^{-2}\|R_0(-\lambda_2^2)\mathcal{F}_2\|_{L^\infty}\\ &\lesssim \varepsilon^1\cdot\varepsilon^{-2}\cdot\varepsilon|\lambda_1-\lambda_2|\\ &\lesssim |\lambda_1-\lambda_2|\\ &\lesssim \varepsilon^{\delta_1}\|\eta_1-\eta_2\|_{L^\infty}.\end{aligned}$$

Part III is the hardest. First we find a common denominator

$$\begin{aligned}&(1+R_0(-\lambda_1^2)V)^{-1}-(1+R_0(-\lambda_2^2)V)^{-1}\\ &\quad=(1+R_0(-\lambda_1^2)V)^{-1}(1+R_0(-\lambda_2^2)V)(1+R_0(-\lambda_2^2)V)^{-1}\\ &\qquad-(1+R_0(-\lambda_1^2)V)^{-1}(1+R_0(-\lambda_1^2)V)(1+R_0(-\lambda_2^2)V)^{-1}\\ &\quad=(1+R_0(-\lambda_1^2)V)^{-1}\left(R_0(-\lambda_2^2)V-R_0(-\lambda_1^2)V\right)(1+R_0(-\lambda_2^2)V)^{-1}\end{aligned}$$

so that

$$\begin{aligned}&\left((1+R_0(-\lambda_1^2)V)^{-1}-(1+R_0(-\lambda_2^2)V)^{-1}\right)R_0(-\lambda_2^2)\mathcal{F}_2=\\ &(1+R_0(-\lambda_1^2)V)^{-1}(R_0(-\lambda_2^2)V-R_0(-\lambda_1^2)V)(1+R_0(-\lambda_2^2)V)^{-1}R_0(-\lambda_2^2)\mathcal{F}_2\\ &=(1+R_0(-\lambda_1^2)V)^{-1}\left(R_0(-\lambda_2^2)V-R_0(-\lambda_1^2)V\right)\mathcal{G}(\eta_2).\end{aligned}$$

Now

$$\mathrm{III}=\|(1+R_0(-\lambda_1^2)V)^{-1}\left(R_0(-\lambda_2^2)V-R_0(-\lambda_1^2)V\right)\mathcal{G}(\eta_2)\|_{L^\infty}$$

and here we just suffer the loss of one λ (Lemma 3.1.4) to achieve

$$\begin{aligned}\mathrm{III} &\lesssim \lambda_1^{-1}\|\left(R_0(-\lambda_2^2)V-R_0(-\lambda_1^2)V\right)\mathcal{G}(\eta_2)\|_{L^\infty}\\ &\lesssim \lambda_1^{-1}|\lambda_2^2-\lambda_1^2|\|R_0(-\lambda_2^2)R_0(-\lambda_1^2)V\mathcal{G}(\eta_2)\|_{L^\infty}\\ &\lesssim \lambda_1^{-1}|\lambda_2+\lambda_1||\lambda_2-\lambda_1|\lambda_2^{-1/2}\|R_0(-\lambda_1^2)V\mathcal{G}(\eta_2)\|_{L^2}\\ &\lesssim \varepsilon^{-1}\cdot\varepsilon^1|\lambda_2-\lambda_1|\lambda_2^{-1/2}\lambda_1^{-1/2}\|V\mathcal{G}(\eta_2)\|_{L^1}\\ &\lesssim \varepsilon^{-1}|\lambda_2-\lambda_1|\|V\|_{L^1}\|\mathcal{G}(\eta_2)\|_{L^\infty}\end{aligned}$$

and using Lemma 3.1.8 and (3.26) we see

$$\mathrm{III} \lesssim |\lambda_1 - \lambda_2| \lesssim \varepsilon^{\delta_1} \|\eta_1 - \eta_2\|_{L^\infty}.$$

Hence, by taking ε smaller still if needed, we have

$$\|\mathcal{G}(\eta_1, \varepsilon) - \mathcal{G}(\eta_2, \varepsilon)\|_{L^\infty} \leq \kappa \|\eta_1 - \eta_2\|_{L^\infty}$$

for some $0 < \kappa < 1$ and so $\mathcal{G}$ is a contraction. Therefore, invoking the Banach fixed-point theorem, we have established the existence of a unique η, with $\|\eta\|_{L^\infty} \leq R\varepsilon$, satisfying (3.4).

To see the leading order observe the order of the terms appearing in the previous computations as well as the following. First if $p_1 \geq 3$ then

$$\begin{aligned}
\varepsilon \| \left(R_0(-\lambda^2) - G_0\right) f(W) \|_{L^\infty} &\lesssim \varepsilon \lambda^2 \|R_0(-\lambda^2) G_0 f(W)\|_{L^\infty} \\
&\lesssim \varepsilon \lambda^2 \cdot \lambda^{-1^-} \|G_0 f(W)\|_{L^{3+}} \\
&\lesssim \varepsilon \lambda^{1^-} \|f(W)\|_{L^{1+}} \\
&\lesssim \varepsilon^{2^-}
\end{aligned}$$

and if instead $2 < p_1 < 3$ then take $3/q = (p_1 - 2)^-$ and

$$\begin{aligned}
\varepsilon \| \left(R_0(-\lambda^2) - G_0\right) f(W) \|_{L^\infty} &\lesssim \varepsilon \lambda^2 \|R_0(-\lambda^2) G_0 f(W)\|_{L^\infty} \\
&\lesssim \varepsilon \lambda^2 \cdot \lambda^{3/q-2} \|G_0 f(W)\|_{L^q} \\
&\lesssim \varepsilon \lambda^{3/q} \|f(W)\|_{L^{(3/p_1)+}} \\
&\lesssim \varepsilon^{1+(p_1-2)^-}.
\end{aligned}$$

The lemma is now proved. □

With the existence of η established we can improve the space in which η lives.

Lemma 3.1.10. *The η established in Lemma 3.1.9 is in $L^r \cap \dot{H}^1$ for any $3 < r \leq \infty$. The function η also enjoys the bounds*

$$\begin{aligned}
\|\eta\|_{L^r} &\lesssim \varepsilon^{1-3/r} \\
\|\eta\|_{\dot{H}^1} &\lesssim \varepsilon^{1/2}
\end{aligned}$$

for all $3 < r \leq \infty$. Furthermore we have the expansion

$$\eta = \bar{Q}(1 + G_0 V)^{-1} \bar{P} R_0(-\lambda^2)(-\lambda^2 \sqrt{3}|x|^{-1}) + \tilde{\eta}$$

with

$$\|\tilde{\eta}\|_{L^r} \lesssim \max\left\{\begin{cases} \varepsilon^{1-}, & \text{if } 2 < p_1 < 3 \text{ and } r = 3/(p_1-2) \\ \varepsilon, & \text{else} \end{cases}, \varepsilon^{p_1-2+1-3/r}, \varepsilon^{2(1-3/r)}\right\}$$

for $3 < r < \infty$ *and where* $\bar{P}$ *and* $\bar{Q}$ *are given in* (3.12).

Proof. The computations which produce (3.26) are sufficient to establish the result with $r = \infty$. Take $3 < r < \infty$ and consider:

$$\|\eta\|_{L^r} \lesssim \lambda^2\|R_0(-\lambda^2)W\|_{L^r} + \varepsilon\|R_0(-\lambda^2)f(W)\|_{L^r} + \|R_0(-\lambda^2)N(\eta)\|_{L^r}.$$

For the first term use (3.7)

$$\lambda^2\|R_0(-\lambda^2)W\|_{L^r} \lesssim \lambda^2 \cdot \lambda^{3(1/3-1/r)-2}\|W\|_{L^3_w} \lesssim \varepsilon^{1-3/r}$$

to see the leading order contribution.

While the second term contributed to the leading order in Lemma 3.1.9 it is inferior to the first term when measured in L^r. We do however need several cases. Suppose that $3 \le p_1 < 5$ or $r > 3/(p_1-2)$ and apply (3.8) with $1/q = 1/r + 2/3$

$$\varepsilon\|R_0(-\lambda^2)f(W)\|_{L^r} \lesssim \varepsilon\|f(W)\|_{L^q} \lesssim \varepsilon.$$

Note that under these conditions $f(W) \in L^q$. Now suppose that $2 < p_1 < 3$ and $r = 3/(p_1-2)$ and apply (3.6) with $q = (3/p_1)^+$

$$\varepsilon\|R_0(-\lambda^2)f(W)\|_{L^r} \lesssim \varepsilon\lambda^{3(1/q-(p_1-2)/3)-2}\|f(W)\|_{L^q} \lesssim \varepsilon^{1^-}.$$

And if $2 < p_1 < 3$ and $3 < r < 3/(p_1-2)$ apply (3.7) with $q = 3/p_1$

$$\varepsilon\|R_0(-\lambda^2)f(W)\|_{L^r} \lesssim \varepsilon\lambda^{3(p_1/3-1/r)-2}\|f(W)\|_{L^q_w} \lesssim \varepsilon^{1-3/r+p_1-2}.$$

And thirdly the remaining terms. First use (3.8) where $1/q = 1/r + 2/3$ to see

$$\|R_0(-\lambda^2)(W^3\eta^2)\|_{L^r} \lesssim \|W^3\eta^2\|_{L^q} \lesssim \|W^3\|_{L^{3/2}}\|\eta\|_{L^r}\|\eta\|_{L^\infty} \lesssim \varepsilon\|\eta\|_{L^r}$$

and now use (3.6) with $1/q = 1/r$ to obtain

$$\|R_0(-\lambda^2)\eta^5\|_{L^r} \lesssim \lambda^{-2}\|\eta^5\|_{L^r} \lesssim \lambda^{-2}\|\eta\|_{L^r}\|\eta\|^4_{L^\infty} \lesssim \varepsilon^2\|\eta\|_{L^r}.$$

and similarly for $j = 1, 2$

$$\varepsilon \|R_0(-\lambda^2)\eta^{p_j}\|_{L^r} \lesssim \varepsilon\lambda^{-2}\|\eta^{p_j}\|_{L^r} \lesssim \varepsilon^{-1}\|\eta\|_{L^\infty}^{p_j-1}\|\eta\|_{L^r} \lesssim \varepsilon^{p_j-2}\|\eta\|_{L^r}$$

noting that $p_2 - 2 \geq p_1 - 2 > 0$. For the last remainder term we have two cases. If $p_j > 3$ then use (3.8) with $1/q = 1/r + 2/3$

$$\varepsilon\|R_0(-\lambda^2)\left(\eta W^{p_j-1}\right)\|_{L^r} \lesssim \varepsilon\|\eta W^{p_j-1}\|_{L^q} \lesssim \varepsilon\|\eta\|_{L^r}\|W^{p_j-1}\|_{L^{3/2}} \lesssim \varepsilon\|\eta\|_{L^r}.$$

If instead $2 < p_j \leq 3$ then we need (3.7) with $1/q = (p_j - 1)/3$ so that

$$\begin{aligned}
\varepsilon\|R_0(-\lambda^2)\left(\eta W^{p_j-1}\right)\|_{L^r} &\lesssim \varepsilon\lambda^{3(1/q-1/r)-2}\|\eta W^{p_j-1}\|_{L^q_w} \\
&\lesssim \varepsilon\lambda^{p_1-1-3/r-2}\|\eta\|_{L^\infty}\|W^{p_j-1}\|_{L^q_w} \\
&\lesssim \varepsilon^{p_j-2}\varepsilon^{1-3/r}
\end{aligned}$$

So together we have

$$\|\eta\|_{L^r} \leq C\varepsilon^{1-3/r} + \kappa\|\eta\|_{L^r}$$

where κ may be chosen sufficiently small to yield the desired L^r bound for $3 < r < \infty$. An inspection of the higher order terms gives the size of $\tilde{\eta}$. We also must note Lemma 3.1.4. There are several competing terms which determine the size of $\tilde{\eta}$ depending on p_1 and r.

On to the $\dot{H}^1$ norm. We need the identity

$$\begin{aligned}
\eta &= (1 + R_0(-\lambda^2)V)^{-1}R_0(-\lambda^2)\mathcal{F} \\
&= R_0(-\lambda^2)\mathcal{F} - R_0(-\lambda^2)V(1 + R_0(-\lambda^2)V)^{-1}R_0(-\lambda^2)\mathcal{F} \\
&= R_0(-\lambda^2)\mathcal{F} - R_0(-\lambda^2)V\eta
\end{aligned}$$

so we have two parts

$$\|\eta\|_{\dot{H}^1} \leq \|R_0(-\lambda^2)\mathcal{F}\|_{\dot{H}^1} + \|R_0(-\lambda^2)V\eta\|_{\dot{H}^1}.$$

For the first

$$\begin{aligned}
\|R_0(-\lambda^2)\mathcal{F}\|_{\dot{H}^1} &\lesssim \lambda^2\|R_0(-\lambda^2)W\|_{\dot{H}^1} + \varepsilon\|R_0(-\lambda^2)f(W+\eta)\|_{\dot{H}^1} \\
&\quad + \|R_0(-\lambda^2)\left(W^3\eta^2 + W^2\eta^3 + W\eta^4 + \eta^5\right)\|_{\dot{H}^1}
\end{aligned}$$

and

$$\begin{aligned}
\lambda^2\|R_0(-\lambda^2)W\|_{\dot{H}^1} &\lesssim \lambda^2\|R_0(-\lambda^2)\nabla W\|_{L^2} \\
&\lesssim \lambda^2 \cdot \lambda^{1/2-2}\|\nabla W\|_{L^{3/2}_w} \\
&\lesssim \varepsilon^{1/2}
\end{aligned}$$

and

$$\begin{aligned}\varepsilon\|R_0(-\lambda^2)f(W+\eta)\|_{\dot{H}^1} &\lesssim \varepsilon\|R_0(-\lambda^2)f'(W+\eta)(\nabla W+\nabla\eta)\|_{L^2}\\ &\lesssim \varepsilon\lambda^{-1/2}\|f'(W+\eta)\nabla W\|_{L^1}\\ &\quad+\varepsilon\lambda^{1^-}\|f'(W+\eta)\nabla\eta\|_{L^{6/5-}}\\ &\lesssim \varepsilon^{1/2}\|f'(W+\eta)\|_{L^{3-}}\|\nabla W\|_{L^{3/2+}}\\ &\quad+\varepsilon^{0^+}\|f'(W+\eta)\|_{L^{3-}}\|\nabla\eta\|_{L^2}\\ &\lesssim \varepsilon^{1/2}+\kappa\|\eta\|_{\dot{H}^1}\end{aligned}$$

with κ small and

$$\begin{aligned}&\|R_0(-\lambda^2)\left(W^3\eta^2+W^2\eta^3+W\eta^4+\eta^5\right)\|_{\dot{H}^1}\\ &\qquad\lesssim \|R_0(-\lambda^2)\eta(\nabla W f_1+\nabla\eta f_2)\|_{L^2}\end{aligned}$$

where f_1 and f_2 are in L^2 so

$$\begin{aligned}&\|R_0(-\lambda^2)\left(W^3\eta^2+W^2\eta^3+W\eta^4+\eta^5\right)\|_{\dot{H}^1}\\ &\quad\lesssim \lambda^{-1/2}\|\eta\|_{L^\infty}\left(\|\nabla W\|_{L^2}\|f_1\|_{L^2}+\|\nabla\eta\|_{L^2}\|f_2\|_{L^2}\right)\\ &\quad\lesssim \varepsilon^{1/2}+\kappa\|\eta\|_{\dot{H}^1}.\end{aligned}$$

For the second

$$\|R_0(-\lambda^2)V\eta\|_{\dot{H}^1}=\left\|\left(\nabla\frac{e^{-\lambda|x|}}{|x|}\right)*(V\eta)\right\|_{L^2}=\left\|\left(\lambda^2 g(\lambda x)\right)*(V\eta)\right\|_{L^2}$$

where $g\in L_w^{3/2}$. So using weak Young's we obtain

$$\begin{aligned}\|R_0(-\lambda^2)V\eta\|_{\dot{H}^1} &\lesssim \lambda^2\|g(\lambda x)\|_{L_w^{3/2}}\|V\eta\|_{L^{6/5}}\\ &\lesssim \lambda^2\cdot\lambda^{-2}\|V\|_{L^{3/2}}\|\eta\|_{L^6}\\ &\lesssim \|\eta\|_{L^6}\\ &\lesssim \varepsilon^{1/2}.\end{aligned}$$

So putting everything together gives

$$\|\eta\|_{\dot{H}^1}\le C\left(\varepsilon^{1/2}+\kappa\|\eta\|_{\dot{H}^1}\right)$$

which gives the desired bound by taking κ sufficiently small. □

Combining Lemmas 3.1.6, 3.1.9, 3.1.10 and Remark 1.7.6 completes the proof of Theorem 1.7.2.

At this point we demonstrate the following monotonicity result which will be used in Section 3.2.

Lemma 3.1.11. *Suppose that $f(W) = W^p$ with $3 < p < 5$. Take ε_1 and ε_2 with $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$. Let ε_1 give rise to λ_1 and η_1 and let ε_2 give rise to λ_2 and η_2 via Theorem 1.7.2. We have*

$$|(\lambda_2 - \lambda_1) - \lambda^{(1)}(\varepsilon_2 - \varepsilon_1)| \lesssim o(1)|\varepsilon_2 - \varepsilon_1|. \tag{3.27}$$

Proof. We first establish the estimate

$$|\lambda_2 - \lambda_1| \leq \left(\lambda^{(1)} + o(1)\right)|\varepsilon_2 - \varepsilon_1| \tag{3.28}$$

We write, as in Lemma 3.1.6 and Lemma 3.1.8

$$\begin{aligned}\lambda_2 - \lambda_1 &= \frac{a(\varepsilon_2, \lambda_2) + b(\varepsilon_2, \lambda_2, \eta_2)}{c(\lambda_2)} - \frac{a(\varepsilon_1, \lambda_1) + b(\varepsilon_1, \lambda_1, \eta_1)}{c(\lambda_1)} \\ &= \frac{a(\varepsilon_2, \lambda_2) - a(\varepsilon_1, \lambda_2) + b(\varepsilon_2, \lambda_2, \eta_2) - b(\varepsilon_1, \lambda_2, \eta_2)}{c(\lambda_2)} \\ &\quad + \frac{a(\varepsilon_1, \lambda_2) + b(\varepsilon_1, \lambda_2, \eta_2)}{c(\lambda_2)} - \frac{a(\varepsilon_1, \lambda_1) + b(\varepsilon_1, \lambda_1, \eta_1)}{c(\lambda_1)}.\end{aligned}$$

The second line, containing only ε_1 and not ε_2, has been dealt with in the proof of Lemma 3.1.8 and so there follows

$$\begin{aligned}|\lambda_2 - \lambda_1| &\leq \left|\frac{a(\varepsilon_2, \lambda_2) - a(\varepsilon_1, \lambda_2) + b(\varepsilon_2, \lambda_2, \eta_2) - b(\varepsilon_1, \lambda_2, \eta_2)}{c(\lambda_2)}\right| \\ &\quad + o(1)\|\eta_2 - \eta_1\|_{L^\infty} + o(1)|\lambda_2 - \lambda_1| \\ &\leq |\varepsilon_2 - \varepsilon_1|\left(\lambda^{(1)} + o(1)\right) + o(1)\|\eta_2 - \eta_1\|_{L^\infty} + o(1)|\lambda_2 - \lambda_1|.\end{aligned}$$

For the η's we estimate

$$\|\eta_2 - \eta_1\|_{L^\infty} \lesssim o(1)\|\eta_2 - \eta_1\|_{L^\infty} + |\lambda_2 - \lambda_1| + |\varepsilon_2 - \varepsilon_1|$$

appealing to Lemma 3.1.9. So putting everything together we have

$$|\lambda_2 - \lambda_1| \leq \left(\lambda^{(1)} + o(1)\right)|\varepsilon_2 - \varepsilon_1|$$

establishing (3.28).

Now we proceed to the more refined (3.27). Observing the computations leading to (3.28) we have

$$|\lambda_2 - \lambda_1 - (\varepsilon_2 - \varepsilon_1)\lambda^{(1)}| \leq \left| \frac{a(\varepsilon_2, \lambda_2) - a(\varepsilon_1, \lambda_2)}{c(\lambda_2)} - (\varepsilon_2 - \varepsilon_1)\lambda^{(1)} \right| \\ + o(1)\|\eta_2 - \eta_1\|_{L^\infty} + o(1)|\lambda_2 - \lambda_1|.$$

By (3.28) the last two terms are of the correct size and so we focus on the first. We have

$$\begin{aligned} &\left| \frac{a(\varepsilon_2, \lambda_2) - a(\varepsilon_1, \lambda_2)}{c(\lambda_2)} - (\varepsilon_2 - \varepsilon_1)\lambda^{(1)} \right| \\ &\qquad = \left| (\varepsilon_2 - \varepsilon_1) \left(\frac{\langle R_0(-\lambda_2^2)V\psi, W^p \rangle}{\lambda_2 \langle R_0(-\lambda_2^2)V\psi, W \rangle} - \lambda^{(1)} \right) \right| \\ &\qquad = o(1)|\varepsilon_2 - \varepsilon_1| \end{aligned}$$

noting (3.18) and (3.19). And so, putting everything together we achieve

$$|\lambda_2 - \lambda_1 - (\varepsilon_2 - \varepsilon_1)\lambda^{(1)}| \lesssim o(1)|\varepsilon_2 - \varepsilon_1|$$

as desired. □

3.2 Variational Characterization

It is not clear from the construction that the solution Q is in any sense a *ground state* solution. It is also not clear that the solution is positive. In this section we first establish the existence of a ground state solution; one that minimizes the action subject to a constraint. We then demonstrate that this minimizer must be our constructed solution. In this way we prove Theorem 1.7.7.

In this section we restrict our nonlinearity and take only $f(Q) = |Q|^{p-1}Q$ with $3 < p < 5$. Then the action is

$$\mathcal{S}_{\varepsilon,\omega}(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{6}\|u\|_{L^6}^6 - \frac{\varepsilon}{p+1}\|u\|_{L^{p+1}}^{p+1} + \frac{\omega}{2}\|u\|_{L^2}^2. \tag{3.29}$$

We are interested in the constrained minimization problem

$$m_{\varepsilon,\omega} := \inf\{\mathcal{S}_{\varepsilon,\omega}(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\},\ \mathcal{K}_\varepsilon(u) = 0\} \tag{3.30}$$

where

$$\mathcal{K}_\varepsilon(u) = \frac{d}{d\mu}\mathcal{S}_{\varepsilon,\omega}(T_\mu u)\bigg|_{\mu=1} = \|\nabla u\|_{L^2}^2 - \|u\|_{L^6}^6 - \frac{3(p-1)}{2(p+1)}\varepsilon\|u\|_{L^{p+1}}^{p+1}$$

and $(T_\mu u)(x) = \mu^{3/2} u(\mu x)$ is the L^2 scaling operator. Note that for $Q_\varepsilon = W + \eta$ as constructed in Theorem 1.7.2 we have $\mathcal{K}_\varepsilon(Q_\varepsilon) = 0$ since any solution to (1.11) will satisfy $\mathcal{K}_\varepsilon(Q) = 0$.

Before addressing the minimization problem we investigate the implications of our generated solution Q_ε with specified ε and corresponding $\omega = \omega(\varepsilon)$. In particular there is a scaling that generates for us additional solutions to the equation

$$-\Delta Q - Q^5 - \varepsilon|Q|^{p-1}Q + \omega Q = 0 \tag{3.31}$$

with $3 < p < 5$.

Remark 3.2.1. *For any* $0 < \tilde{\varepsilon} \leq \varepsilon_0$*, we have solutions to* (3.31) *given by*

$$Q^\mu = \mu^{1/2} Q_{\tilde{\varepsilon}}(\mu \cdot)$$

with $\varepsilon = \mu^{(5-p)/2}\tilde{\varepsilon}$ *and* $\omega = \mu^2 \omega(\tilde{\varepsilon})$*. So for any* $\varepsilon > 0$*, we obtain the family of solutions*

$$\{\ Q^\mu \mid \mu = \left(\frac{\varepsilon}{\tilde{\varepsilon}}\right)^{\frac{2}{5-p}},\ \tilde{\varepsilon} \in (0, \varepsilon_0]\ \}$$

with

$$\omega = \left(\frac{\varepsilon}{\tilde{\varepsilon}}\right)^{\frac{4}{5-p}} \omega(\tilde{\varepsilon})\ \in\ \left[\left(\frac{\varepsilon_0}{\tilde{\varepsilon_0}}\right)^{\frac{4}{5-p}} \omega(\tilde{\varepsilon_0}),\ \infty\right)$$

since as $\tilde{\varepsilon} \downarrow 0$, $\left(\frac{\varepsilon}{\tilde{\varepsilon}}\right)^{\frac{4}{5-p}} \omega(\tilde{\varepsilon})\ \sim\ \tilde{\varepsilon}^{\frac{2(3-p)}{5-p}} \to \infty$*.*

We now address the minimization problem by first addressing the existence of a minimizer.

Lemma 3.2.2. *Take* $3 < p < 5$*. Let* $Q = Q_\varepsilon$ *solving* (3.31) *with* $\omega = \omega(\varepsilon)$ *be as constructed in Theorem 1.7.2. There exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$ *we have*

$$\mathcal{S}_{\varepsilon,\omega(\varepsilon)}(Q_\varepsilon) < \frac{1}{3}\|W\|_{L^6}^6 = \mathcal{S}_{0,0}(W).$$

It follows, see Proposition 2.1 of [1], which is in turn based on the earlier [12], that the variational problem (3.30) *with* $\omega = \omega(\varepsilon)$ *admits a non-negative, radially-symmetric minimizer, which moreover solves* (3.31)*.*

Proof. We compute directly, ignoring higher order contributions. Using (1.21) we write the action as

$$\begin{aligned}\mathcal{S}_{\varepsilon,\omega}(Q) &= \frac{1}{3}\int Q^6 + \frac{p-1}{2(p+1)}\varepsilon\int |Q|^{p+1}\\ &= \frac{1}{3}\int (W+\eta)^6 + \frac{p-1}{2(p+1)}\varepsilon\int |W+\eta|^{p+1}.\end{aligned}$$

Rearranging we have

$$\mathcal{S}_{\varepsilon,\omega}(Q) - \frac{1}{3}\int W^6 = 2\int W^5\eta + \frac{p-1}{2(p+1)}\varepsilon\int W^{p+1} + O(\varepsilon^2)$$

where the higher order terms are controlled for $3 < p < 5$:

- $\|W^4\eta^2\|_{L^1} \lesssim \|W^4\|_{L^1}\|\eta\|_{L^\infty}^2 \lesssim \varepsilon^2$
- $\|\eta^6\|_{L^1} \lesssim \|\eta\|_{L^6}^6 \lesssim \varepsilon^3$
- $\varepsilon\|W^p\eta\|_{L^1} \lesssim \varepsilon\|W^p\|_{L^1}\|\eta\|_{L^\infty} \lesssim \varepsilon^2$
- $\varepsilon\|\eta^{p+1}\|_{L^1} \lesssim \varepsilon\|\eta\|_{L^{p+1}}^{p+1} \lesssim \varepsilon^{p-1}$.

We now compute

$$\begin{aligned}2\int W^5\eta &= 2\left\langle W^5, (H+\lambda^2)^{-1}\left(\varepsilon W^p - \lambda^2 W + N(\eta)\right)\right\rangle\\ &= 2\left\langle W^5, (1+R_0(-\lambda^2)V)^{-1}\bar{P}R_0(-\lambda^2)\left(\varepsilon W^p - \lambda^2 W + N(\eta)\right)\right\rangle\end{aligned}$$

where we have inserted the definition of η from (3.4) and so identify the two leading order terms. There is no problem to also insert the projection $\bar{P}$ from (3.12) since we have the orthogonality condition (3.3) by the way we defined ε, λ, η.

We approximate in turn writing only R_0 for $R_0(-\lambda^2)$. In what follows we use the operators $(1+VG_0)^{-1}$ and $(1+VR_0)^{-1}$. The former as acts on the spaces

$$(1+VG_0)^{-1} : L^1 \cap (\Lambda W)^{\perp} \to L^1 \cap (1)^{\perp}$$

and the later has the expansion

$$(1+VR_0)^{-1} = \frac{1}{\lambda}\langle \Lambda W, \cdot\rangle V\Lambda W + O(1)$$

in L^1. We record here also the adjoint of $\bar{P}$:

$$\bar{P}^* = 1 - P^*, \qquad P^* = \frac{\langle \Lambda W, \cdot \rangle}{\int V(\Lambda W)^2} V \Lambda W.$$

To estimate the first term write

$$\begin{aligned} 2\varepsilon \langle W^5, (1 + R_0 V)^{-1} \bar{P} R_0 W^p \rangle &= 2\varepsilon \langle (1 + V R_0)^{-1} W^5, \bar{P} R_0 W^p \rangle \\ &= 2\varepsilon \langle (1 + V G_0)^{-1} W^5, \bar{P} R_0 W^p \rangle + O(\varepsilon^2). \end{aligned}$$

The error is controlled with a resolvent identity:

$$\begin{aligned} &\varepsilon \left| \left\langle \left((1 + V R_0)^{-1} - (1 + V G_0)^{-1} \right) W^5, \bar{P} R_0 W^p \right\rangle \right| \\ &\quad = \varepsilon \left| \left\langle (1 + V R_0)^{-1} V (G_0 - R_0)(1 + V G_0)^{-1} W^5, \bar{P} R_0 W^p \right\rangle \right| \\ &\quad = \varepsilon \left| \left\langle \bar{P}^* (1 + V R_0)^{-1} V (G_0 - R_0) \left(-W^5/4 + V \Lambda W/2 \right), R_0 W^p \right\rangle \right| \\ &\quad \lesssim \varepsilon \left\| \bar{P}^* (1 + V R_0)^{-1} V (G_0 - R_0) \left(-W^5/4 + V \Lambda W/2 \right) \right\|_{L^1} \| R_0 W^p \|_{L^\infty} \\ &\quad \lesssim \varepsilon \left\| V \bar{R} \left(-W^5/4 + V \Lambda W/2 \right) \right\|_{L^1} \| W^p \|_{L^{3/2-} \cap L^{3/2+}} \\ &\quad \lesssim \varepsilon \lambda \\ &\quad \lesssim \varepsilon^2 \end{aligned}$$

where we have substituted $G_0 - R_0 = \lambda G_1 + \bar{R}$, note that $G_1(-W^5/4 + V\Lambda W/2) = 0$ since $(-W^5/4 + V\Lambda W/2) \perp 1$, and have computed

$$\left\| V \bar{R} \left(-W^5/4 + V \Lambda W/2 \right) \right\|_{L^1} \lesssim \int \langle x \rangle^{-1} dx \int \lambda \frac{|\lambda y|}{\langle \lambda y \rangle} \langle x - y \rangle^{-5} dy \lesssim \lambda.$$

Continuing, we have

$$\begin{aligned} 2\varepsilon \langle W^5, (1 + R_0 V)^{-1} \bar{P} R_0 W^p \rangle &= 2\varepsilon \langle \bar{P}^* (1 + V G_0)^{-1} W^5, R_0 W^p \rangle + O(\varepsilon^2) \\ &= -\frac{1}{2} \varepsilon \langle W^5, R_0 W^p \rangle + O(\varepsilon^2) \\ &= -\frac{1}{2} \varepsilon \langle R_0 W^5, W^p \rangle + O(\varepsilon^2) \\ &= -\frac{1}{2} \varepsilon \langle G_0 W^5, W^p \rangle + O(\varepsilon^2) \\ &= -\frac{1}{2} \varepsilon \langle W, W^p \rangle + O(\varepsilon^2) \\ &= -\frac{1}{2} \varepsilon \int W^{p+1} + O(\varepsilon^2) \end{aligned}$$

where the other error term is bounded:

$$\varepsilon \left|\langle (R_0 - G_0)W^5, W^p\rangle\right| \lesssim \varepsilon\lambda^2 \left|\langle R_0 G_0 W^5, W^p\rangle\right| \lesssim \varepsilon\lambda^2 \left|\langle R_0 W, W^p\rangle\right| \lesssim \varepsilon^2$$

observing the computations that produce (3.19).

For the second term we proceed in a similar manner

$$\begin{aligned}
-2\lambda^2 \langle W^5, (1+R_0V)^{-1}\bar{P}R_0 W\rangle &= \frac{1}{2}\lambda^2\langle W^5, R_0 W\rangle + O(\varepsilon^{2^-}) \\
&= \frac{1}{2}\lambda\sqrt{3}\int W^5 + O(\varepsilon^{2^-}) \\
&= 6\pi\lambda + O(\varepsilon^{2^-}) \\
&= -\varepsilon\langle \Lambda W, W^p\rangle + O(\varepsilon^{2^-}) \\
&= \varepsilon\left(\frac{3}{p+1} - \frac{1}{2}\right)\int W^{p+1} + O(\varepsilon^{2^-})
\end{aligned}$$

again referring to (3.19) and also Remark 1.7.4. The error term coming from the difference of the resolvents is similar. Note

$$\begin{aligned}
&\lambda^2 \left|\left\langle \left((1+VR_0)^{-1} - (1+VG_0)^{-1}\right) W^5, \bar{P}R_0 W\right\rangle\right| \\
&\quad\lesssim \lambda^2 \left\|\bar{P}^*(1+VR_0)^{-1}V(G_0 - R_0)\left(-W^5/4 + V\Lambda W/2\right)\right\|_{L^1} \|R_0 W\|_{L^\infty} \\
&\quad\lesssim \lambda^3 \|R_0 W\|_{L^\infty} \\
&\quad\lesssim \lambda^3\lambda^{-1^-} \|W\|_{L^{3+}} \\
&\quad\lesssim \lambda^{2^-} \\
&\quad\lesssim \varepsilon^{2^-}.
\end{aligned}$$

The term coming from $N(\eta)$ is controlled similarly, and so, all together we have

$$\begin{aligned}
\mathcal{S}_{\varepsilon,\omega}(Q) - \frac{1}{3}\int W^6 &= \left(\frac{3}{p+1} - \frac{1}{2} - \frac{1}{2} + \frac{p-1}{2(p+1)}\right)\varepsilon\int W^{p+1} + O(\varepsilon^{2^-}) \\
&= -\frac{p-3}{2(p+1)}\varepsilon\int W^{p+1} + O(\varepsilon^{2^-})
\end{aligned}$$

which is negative for $3 < p < 5$ and $\varepsilon > 0$ and small. We note that when $p = 3$, this leading order term vanishes. □

Lemma 3.2.3. *Take* $3 < p < 5$*. Denote by* $V = V_\varepsilon$ *a non-negative, radially-symmetric minimizer for* (3.30) *with* $\omega = \omega(\varepsilon)$ *(as established in Lemma*

3.2.2). Then for any $\varepsilon_j \to 0$, V_{ε_j} is a minimizing sequence for the (unperturbed) variational problem

$$\mathcal{S}_{0,0}(W) = \min\{\mathcal{S}_{0,0}(u) \mid u \in \dot{H}^1 \setminus \{0\},\ \mathcal{K}_0(u) = 0\} \tag{3.32}$$

in the sense that

$$\mathcal{K}_0(V_{\varepsilon_j}) \to 0, \qquad \limsup_{\varepsilon \to 0} \mathcal{S}_{0,0}(V_{\varepsilon_j}) \le \mathcal{S}_{0,0}(W).$$

Proof. Since

$$0 = K_\varepsilon(V) = K_0(V) - \frac{3(p-1)}{2(p+1)}\varepsilon \int V^{p+1},$$

and by Lemma 3.2.2,

$$S_{0,0}(W) > m_{\varepsilon,\omega(\varepsilon)} = S_{\varepsilon,\omega(\varepsilon)}(V) = S_{0,0}(V) - \frac{1}{p+1}\varepsilon \int V^{p+1} + \frac{1}{2}\omega \int V^2, \tag{3.33}$$

the lemma will be implied by the claim:

$$\varepsilon \int V^{p+1} \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.34}$$

To address the claim, first introduce the functional

$$\begin{aligned}\mathcal{I}_{\varepsilon,\omega}(u) &:= S_{\varepsilon,\omega}(u) - \frac{1}{3}K_\varepsilon(u) \\ &= \frac{1}{6}\int |\nabla u|^2 + \frac{1}{6}\int |u|^6 + \frac{p-3}{2(p+1)}\varepsilon \int |u|^{p+1} + \frac{1}{2}\omega \int |u|^2\end{aligned}$$

and observe that since $\mathcal{K}_\varepsilon(V) = 0$,

$$\mathcal{I}_{\varepsilon,\omega(\varepsilon)}(V) = \mathcal{S}_{\varepsilon,\omega}(V) < \mathcal{S}_{0,0}(W)$$

and so the following quantities are all bounded uniformly in ε:

$$\int |\nabla V|^2, \ \int V^6, \ \varepsilon \int V^{p+1}, \ \omega \int V^2 \ \lesssim 1.$$

By interpolation

$$\begin{aligned}\varepsilon \int V^{p+1} &\le \varepsilon \|V\|_{L^2}^{(5-p)/2} \|V\|_{L^6}^{3(p-1)/2} \\ &\lesssim \varepsilon \omega^{-(5-p)/4} \left(\omega \int V^2\right)^{(5-p)/4}.\end{aligned}$$

So (3.34) holds, provided that $\varepsilon^{4/(5-p)} \ll \omega$. Since $\omega \sim \varepsilon^2$, this indeed holds for $3 < p < 5$.

With the claim in hand we can finish the argument. The fact that $\mathcal{K}_0(V) \to 0$ now follows from $\mathcal{K}_\varepsilon(V) = 0$. Also, from Lemma 3.2.2 we know that for $\varepsilon \geq 0$

$$\mathcal{S}_{0,0}(V) - \frac{\varepsilon}{p+1} \int V^{p+1} \leq \mathcal{S}_{\varepsilon,\omega}(V) \leq \mathcal{S}_{0,0}(W)$$

and so $\limsup_{\varepsilon\to 0} \mathcal{S}_{0,0}(V) \leq \mathcal{S}_{0,0}(W)$. □

Lemma 3.2.4. *For a sequence $\varepsilon_j \downarrow 0$, let $V = V_{\varepsilon_j}$ be corresponding non-negative, radially-symmetric minimizers of* (3.30) *with $\omega = \omega(\varepsilon_j)$. There is a subsequence ε_{j_k} and a scaling $\mu = \mu_k$ such that along the subsequence,*

$$V^\mu = \mu^{1/2} V(\mu\cdot) \to \nu W$$

in $\dot{H}^1$ with $\nu = 1$.

Proof. The result with $\nu = 1$ or $\nu = 0$ follows from the bubble decomposition of Gérard [37] (see eg. the notes of Killip and Vişan [58], in particular Theorem 4.7 and the proof of Theorem 4.4). Therefore we need only eliminate the possibility that $\nu = 0$.

If $\nu = 0$ then $\int |\nabla V_\varepsilon|^2 \to 0$ (along the given subsequence). Then by the Sobolev inequality,

$$0 = \mathcal{K}_\varepsilon(V_\varepsilon) = (1 + o(1)) \int |\nabla V_\varepsilon|^2 - \frac{3(p-1)}{2(p+1)} \varepsilon \int V_\varepsilon^{p+1},$$

and so

$$\int |\nabla V_\varepsilon|^2 \lesssim \varepsilon \int V_\varepsilon^{p+1}.$$

However, we have already seen

$$\int |\nabla V_\varepsilon|^2 \lesssim \varepsilon \int V_\varepsilon^{p+1} \lesssim \varepsilon \omega^{-(5-p)/4} \left(\omega \int V_\varepsilon^2\right)^{(5-p)/4} \left(\int |\nabla V_\varepsilon|^2\right)^{3(p-1)/4}$$

via interpolation and so

$$\left(\int |\nabla V_\varepsilon|^2\right)^{(7-3p)/4} \lesssim \varepsilon \omega^{-(5-p)/4} \left(\omega \int V_\varepsilon^2\right)^{(5-p)/4} \to 0$$

as above. Note that $(7-3p)/4 = -3(p-7/3)/4 < 0$. Hence $\nu = 0$ is impossible and so we conclude that $\nu = 1$. The result follows. □

Remark 3.2.5. *This lemma implies in particular that for* $V = V_\varepsilon$, $\omega = \omega(\varepsilon)$, $S_{0,0}(V) = S_{0,0}(V^\mu) \to S_{0,0}(W)$, *and so by* (3.33) *and* (3.34),

$$\omega \int V^2 \ \to 0.$$

Remark 3.2.6. *Note that* V^μ *is a minimizer of the minimization problem* (3.30), *and a solution to* (3.31), *with* ε *and* ω *replaced with*

$$\tilde{\varepsilon} = \mu^{\frac{5-p}{2}}\varepsilon, \qquad \tilde{\omega} = \mu^2\omega.$$

Under this scaling the following properties are preserved:

$$\begin{aligned} \tilde{\varepsilon}^{\frac{4}{5-p}} &= \mu^2 \varepsilon^{\frac{4}{5-p}} \ll \mu^2\omega = \tilde{\omega} \\ \tilde{\varepsilon} \int (V^\mu)^{p+1} &= \varepsilon \int V^{p+1} \ \to 0 \\ \tilde{\omega} \int (V^\mu)^2 &= \omega \int V^2 \ \to 0. \end{aligned}$$

Moreover,

$$\tilde{\varepsilon} \to 0, \quad \tilde{\omega} \to 0,$$

the latter since otherwise $\|V^\mu\|_{L^2} \to 0$ *along some subsequence, contradicting* $V^\mu \to W \notin L^2$ *in* $\dot{H}^1$, *and then the former by the first relation above.*

Lemma 3.2.7. *Let*

$$V^\mu = W + \tilde{\eta}, \quad \|\tilde{\eta}\|_{\dot{H}^1} \to 0, \quad \tilde{\varepsilon} \to 0$$

be a sequence as provided by Lemma 3.2.4. There is a further scaling

$$\nu = \nu_{\tilde{\varepsilon}} = 1 + o(1)$$

so that

$$(V^\mu)^\nu = W^\nu + \tilde{\eta}^\nu =: W + \hat{\eta}$$

retains $\|\hat{\eta}\|_{\dot{H}^1} \to 0$, *but also satisfies the orthogonality condition*

$$0 = \langle R_0(-\hat{\omega})V\psi, \mathcal{F}(\hat{\varepsilon}, \hat{\omega}, \hat{\eta})\rangle \tag{3.35}$$

with the corresponding $\hat{\varepsilon} = \nu^{(5-p)/2}\tilde{\varepsilon}$ *and* $\hat{\omega} = \nu^2\tilde{\omega}$.

Proof. We may rewrite the above inner-product as

$$\langle R_0(-\hat{\omega})V\psi, \mathcal{F}(\hat{\varepsilon}, \hat{\omega}, \hat{\eta})\rangle = \frac{-5}{\sqrt{3}}\langle \hat{\eta}, (H+\hat{\omega})R_0(-\hat{\omega})W^4\Lambda W\rangle$$

and observe from the resonance equation (1.18)

$$5R_0(-\hat{\omega})W^4\Lambda W = \Lambda W - \hat{\omega}R_0(-\hat{\omega})\Lambda W$$

and so

$$\begin{aligned}(H+\hat{\omega})R_0(-\hat{\omega})W^4\Lambda W &= (-\Delta + \hat{\omega} - 5W^4)R_0(-\hat{\omega})W^4\Lambda W \\ &= (1 - 5W^4R_0(-\hat{\omega}))W^4\Lambda W = \hat{\omega}W^4R_0(-\hat{\omega})\Lambda W\end{aligned}$$

so the desired orthogonality condition reads

$$0 = \frac{1}{\sqrt{\hat{\omega}}}\langle \hat{\eta}, (H+\hat{\omega})R_0(-\hat{\omega})W^4\Lambda W\rangle = \sqrt{\hat{\omega}}\langle W^\nu - W + \tilde{\eta}^\nu, W^4R_0(-\hat{\omega})\Lambda W\rangle.$$

Now since $\Lambda W = \frac{d}{d\mu}W^\mu|_{\mu=1}$, by Taylor expansion

$$\|W^\nu - W - (\nu-1)\Lambda W\|_{L^6} \lesssim (\nu-1)^2,$$

and using (3.7)

$$\|W^4R_0(-\hat{\omega})\Lambda W\|_{L^{\frac{6}{5}}} \lesssim \|R_0(-\hat{\omega})\Lambda W\|_{L^\infty} \lesssim \frac{1}{\sqrt{\hat{\omega}}},$$

we arrive at

$$0 = (\nu-1)\left(\sqrt{\hat{\omega}}\langle \Lambda W,\ W^4R_0(-\hat{\omega})\Lambda W\rangle\right) + O((\nu-1)^2) + O(\|\tilde{\eta}^\nu\|_{L^6})$$

Computations exactly as for (3.19) lead to

$$\sqrt{\hat{\omega}}\langle \Lambda W,\ W^4R_0(-\hat{\omega})\Lambda W\rangle = \frac{6\sqrt{3}\pi}{5} + O(\sqrt{\hat{\omega}}),$$

and so the desired orthogonality condition reads

$$0 = (\nu-1)(1+o(1)) + O((\nu-1)^2) + O(\|\tilde{\eta}^\nu\|_{L^6})$$

which can therefore be solved for $\nu = 1 + o(1)$ using $\|\tilde{\eta}^\nu\|_{L^6} = o(1)$. □

The functions

$$W_{\hat{\varepsilon}} := (V^{\mu})^{\nu} = W + \hat{\eta}$$

produced by Lemma 3.2.7 solve the minimization problem (3.30), and the PDE (3.31), with ε and ω replaced (respectively) by $\hat{\varepsilon} \to 0$ and $\hat{\omega}$. Since $\nu_{\hat{\varepsilon}} = 1 + o(1)$, the properties

$$\hat{\varepsilon}^{\frac{4}{5-p}} \ll \hat{\omega} \to 0, \quad \hat{\varepsilon} \int W_{\hat{\varepsilon}}^{p+1} \to 0, \quad \hat{\omega} \int W_{\hat{\varepsilon}}^2 \to 0$$

persist.

It remains to show that that $V_{\hat{\varepsilon}}$ agrees with $Q_{\hat{\varepsilon}}$ constructed in Theorem 1.7.2. First:

Lemma 3.2.8. *For $3 < r \leq \infty$, and $\hat{\varepsilon}$ sufficiently small,*

$$\|\hat{\eta}\|_{L^r} \lesssim \hat{\varepsilon} + \sqrt{\hat{\omega}}^{1-\frac{3}{r}}.$$

Proof. Since $W_{\hat{\varepsilon}}$ is a solution of (1.11), the remainder $\hat{\eta}$ must satisfy (3.4). So

$$\begin{aligned} \|\hat{\eta}\|_{L^r} &= \|(H + \hat{\omega})^{-1} \left(-\hat{\omega} W + \hat{\varepsilon} f(W) + N(\hat{\eta})\right) \|_{L^r} \\ &\lesssim \hat{\varepsilon} + \sqrt{\hat{\omega}}^{1-\frac{3}{r}} + \|R_0(-\hat{\omega}) N(\hat{\eta})\|_{L^r} \end{aligned}$$

using (3.35) and after observing the computations of Lemma 3.1.9. We now establish the required bounds on the remainder, beginning with $3 < r < \infty$. Let $q \in (1, \frac{3}{2})$ satisfy $\frac{1}{q} - \frac{1}{r} = \frac{2}{3}$:

- $\|R_0(-\hat{\omega}) \hat{\varepsilon} W^{p-1} \hat{\eta}\|_{L^r} \lesssim \hat{\varepsilon} \|W^{p-1} \hat{\eta}\|_{L^q} \lesssim \hat{\varepsilon} \|W\|_{L^{\frac{3}{2}(p-1)}}^{p-1} \|\hat{\eta}\|_{L^r} \lesssim \hat{\varepsilon} \|\hat{\eta}\|_{L^r}$
- $\|R_0(-\hat{\omega}) \hat{\varepsilon} \hat{\eta}^p\|_{L^r} \lesssim \hat{\varepsilon} \hat{\omega}^{-\frac{5-p}{4}} \|\hat{\eta}^p\|_{L^{\frac{6r}{6+(p-1)r}}} \lesssim o(1) \|\hat{\eta}\|_{L^6}^{p-1} \|\hat{\eta}\|_{L^r}$
$\lesssim o(1) \|\hat{\eta}\|_{L^r}$
- $\|R_0(-\hat{\omega}) W^3 \hat{\eta}^5\|_{L^r} \lesssim \|W^3 \hat{\eta}^2\|_{L^q} \lesssim \|W\|_{L^6}^3 \|\hat{\eta}\|_{L^6} \|\hat{\eta}\|_{L^r} \lesssim o(1) \|\hat{\eta}\|_{L^r}$
- $\|R_0(-\hat{\omega}) \hat{\eta}^5\|_{L^r} \lesssim \|\hat{\eta}^5\|_{L^q} \lesssim \|\hat{\eta}\|_{L^6}^4 \|\hat{\eta}\|_{L^r} \lesssim o(1) \|\hat{\eta}\|_{L^r}$

where in the second inequality we used $\hat{\omega} \gg \hat{\varepsilon}^{4/(5-p)}$. Combining, we have achieved

$$\|\hat{\eta}\|_{L^r} \lesssim \hat{\varepsilon} + \sqrt{\hat{\omega}}^{1-\frac{3}{r}} + o(1) \|\hat{\eta}\|_{L^r}$$

and so obtain the desired estimate for $3 < r < \infty$. It remains to deal with $r = \infty$. The first three estimates proceed similarly, while the last one uses the now-established L^r estimate:

- $\|R_0(-\hat\omega)\hat\varepsilon W^{p-1}\hat\eta\|_{L^\infty} \lesssim \hat\varepsilon\|W^{p-1}\hat\eta\|_{L^{\frac{3}{2}-}\cap L^{\frac{3}{2}+}}$
$\lesssim \hat\varepsilon\|W\|^{p-1}_{L^{\frac{3}{2}(p-1)-}\cap L^{\frac{3}{2}(p-1)+}}\|\hat\eta\|_{L^\infty} \lesssim \hat\varepsilon\|\hat\eta\|_{L^\infty}$
- $\|R_0(-\hat\omega)\hat\varepsilon\hat\eta^p\|_{L^\infty} \lesssim \hat\varepsilon\hat\omega^{-\frac{5-p}{4}}\|\hat\eta^p\|_{L^{\frac{6}{p-1}}} \lesssim o(1)\|\hat\eta\|^{p-1}_{L^6}\|\hat\eta\|_{L^\infty}$
$\lesssim o(1)\|\hat\eta\|_{L^\infty}$
- $\|R_0(-\hat\omega)W^3\hat\eta^5\|_{L^\infty} \lesssim \|W^3\hat\eta^2\|_{L^{\frac{3}{2}-}\cap L^{\frac{3}{2}+}} \lesssim \|W\|^3_{L^{6-}\cap L^{6+}}\|\hat\eta\|_{L^6}\|\hat\eta\|_{L^\infty}$
$\lesssim o(1)\|\hat\eta\|_{L^\infty}$
- $\|R_0(-\hat\omega)\hat\eta^5\|_{L^\infty} \lesssim \|\hat\eta^5\|_{L^{\frac{3}{2}-}\cap L^{\frac{3}{2}+}} \lesssim \|\hat\eta\|^4_{L^{6-}\cap L^{6+}}\|\hat\eta\|_{L^\infty}$
$\lesssim (\hat\varepsilon + \hat\omega^{\frac{1}{4}-})^4\|\hat\eta\|_{L^\infty} \lesssim o(1)\|\hat\eta\|_{L^\infty}$

which, combined, establish the desired estimate with $r = \infty$. Strictly speaking, these are *a priori* estimates, since we do not know $\hat\eta \in L^r$ for $r > 6$ to begin with. However, the typical argument of performing the estimates on a series of smooth functions that approximate η remedies this after passing to the limit. □

Lemma 3.2.9. *Write $\hat\omega = \hat\lambda^2$. For $\hat\varepsilon$ sufficiently small, $\|\hat\eta\|_{L^\infty} \lesssim \hat\varepsilon$, and $\hat\lambda = \lambda(\hat\varepsilon, \hat\eta)$ as given in Lemma 3.1.6. Moreover, $W_{\hat\varepsilon} = W + \hat\eta = Q_{\hat\varepsilon}$.*

Proof. From the orthogonality equation (3.35),

$$\begin{aligned} 0 &= \langle R_0(-\hat\lambda^2)V\psi, -\hat\lambda^2 W + \hat\varepsilon W^p + N(\hat\eta)\rangle \\ &= -\hat\lambda\cdot\hat\lambda\langle R_0(-\hat\lambda^2)V\psi, W\rangle + \hat\varepsilon\langle R_0(-\hat\lambda^2)V\psi, W^p\rangle + \langle R_0(-\hat\lambda^2)V\psi, N(\hat\eta)\rangle. \end{aligned}$$

Now re-using estimates (3.18) and (3.19), as well as

- $|\langle R_0(-\hat\lambda^2)V\psi, W^3\hat\eta^2\rangle| \lesssim \|R_0(-\hat\lambda^2)V\psi\|_{L^6}\|W^3\hat\eta^2\|_{L^{6/5}}$
$\lesssim \|V\psi\|_{L^{6/5}}\|\hat\eta\|^2_{L^\infty}\|W^3\|_{L^{6/5}} \lesssim \|\hat\eta\|^2_{L^\infty}$
- $|\langle R_0(-\hat\lambda^2)V\psi, \hat\eta^5\rangle| \lesssim \|R_0(-\hat\lambda^2)V\psi\|_{L^6}\|\hat\eta^5\|_{L^{\frac{6}{5}}}$
$\lesssim \|V\psi\|_{L^{\frac{6}{5}}}\|\hat\eta\|^5_{L^6} \lesssim \|\hat\eta\|^5_{L^6}$
- $|\langle R_0(-\hat\lambda^2)V\psi, \hat\varepsilon\hat\eta^p\rangle| \lesssim \hat\varepsilon\|R_0(-\hat\lambda^2)V\psi\|_{L^6}\|\hat\eta^p\|_{L^{\frac{6}{5}}}$
$\lesssim \hat\varepsilon\|V\psi\|_{L^{\frac{6}{5}}}\|\hat\eta\|^p_{L^{\frac{6}{5}p}} \lesssim \hat\varepsilon\cdot\|\hat\eta\|^p_{L^{\frac{6}{5}p}}$
- $|\langle R_0(-\hat\lambda^2)V\psi, \hat\varepsilon W^{p-1}\hat\eta\rangle| \lesssim \hat\varepsilon\|R_0(-\hat\lambda^2)V\psi\|_{L^6}\|W^{p-1}\hat\eta\|_{L^{\frac{6}{5}}}$
$\lesssim \hat\varepsilon\|V\psi\|_{L^{\frac{6}{5}}}\|W^{p-1}\|_{L^{\frac{3}{2}}}\|\hat\eta\|_{L^6} \lesssim \hat\varepsilon\|\hat\eta\|_{L^6}$,

combined with Lemma 3.2.8, yields

$$(\hat{\lambda} - \lambda^{(1)}\hat{\varepsilon})(1 + O(\hat{\lambda}^{1-})) = O(\hat{\lambda}^2 + \hat{\varepsilon}^2 + \hat{\varepsilon}\hat{\lambda}^{\frac{1}{2}})$$

from which follows

$$\hat{\lambda} - \lambda^{(1)}\hat{\varepsilon} \ll \hat{\varepsilon},$$

and then by Lemma 3.2.8 again,

$$\|\hat{\eta}\|_{L^r} \lesssim \hat{\varepsilon}^{1-\frac{3}{r}}, \quad 3 < r \leq \infty.$$

It now follows from Lemma 3.1.6 that $\hat{\lambda} = \lambda(\hat{\varepsilon}, \hat{\eta})$ for $\hat{\varepsilon}$ small enough.

Finally, the uniqueness of the fixed-point in the L^∞-ball of radius $R\hat{\varepsilon}$ from Lemma 3.1.9 implies that $W_{\hat{\varepsilon}} = Q_{\hat{\varepsilon}}$, where $Q_{\hat{\varepsilon}}$ is the solution constructed in Theorem 1.7.2. □

We have so far established that, up to subsequence, and rescaling, a sequence of minimizers V_{ε_j} eventually coincides with a solution Q_ε as constructed in Theorem 1.7.2: $\xi_j^{1/2} V_{\varepsilon_j}(\xi_j \cdot) = Q_{\hat{\varepsilon}_j}$ (here $\xi_j = \nu_j \mu_j$). It remains to remove the scaling ξ_j and establish that $\hat{\varepsilon}_j = \varepsilon_j$:

Lemma 3.2.10. *Suppose $V(x) = \xi^{-\frac{1}{2}} Q_{\hat{\varepsilon}}(x/\xi)$ solves (3.31) with $\omega = \omega(\varepsilon)$ (as given in Theorem 1.7.2), where $\hat{\varepsilon} = \xi^{(5-p)/2}\varepsilon$, $\hat{\omega} = \xi^2\omega$, and $\hat{\omega} = \omega(\hat{\varepsilon})$. Then $\xi = 1$ and $\hat{\varepsilon} = \varepsilon$, and so $V = Q_\varepsilon$.*

Proof. By assumption $\hat{\omega} = \xi^2\omega(\varepsilon) = \omega(\hat{\varepsilon})$, so

$$\omega(\varepsilon) = \Omega_\varepsilon(\hat{\varepsilon}), \qquad \Omega_\varepsilon(\hat{\varepsilon}) := \left(\frac{\varepsilon}{\hat{\varepsilon}}\right)^{4/(5-p)} \omega(\hat{\varepsilon}).$$

This relation is satisfied if $\hat{\varepsilon} = \varepsilon$ ($\xi = 1$), and our goal is to show it is not satisfied for any other value of $\hat{\varepsilon}$. Thus we will be done if we can show that Ω_ε is monotone in $\hat{\varepsilon}$. Take ε_1 and ε_2 with $0 < \varepsilon_1 < \varepsilon_2 \leq \varepsilon_0$. Let $\alpha = 4/(5-p) > 2$ and assume that $0 < \varepsilon_2 - \varepsilon_1 \ll \varepsilon_1$. Denoting $\omega(\varepsilon_j) = \lambda_j^2$, we estimate:

$$\begin{aligned}
\varepsilon^{-\alpha}\left(\Omega_\varepsilon(\varepsilon_2) - \Omega_\varepsilon(\varepsilon_1)\right) &= \varepsilon_2^{-\alpha}\lambda_2^2 - \varepsilon_1^{-\alpha}\lambda_1^2 \\
&= \varepsilon_2^{-\alpha}(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1) + \lambda_1^2\left(\varepsilon_2^{-\alpha} - \varepsilon_1^{-\alpha}\right) \\
&\approx \varepsilon_1^{-\alpha}\lambda^{(1)}(\varepsilon_2 - \varepsilon_1) \cdot 2\lambda^{(1)}\varepsilon_1 \\
&\quad + \varepsilon_1^2(\lambda^{(1)})^2\varepsilon_1^{-\alpha}\left(-\frac{\alpha}{\varepsilon_1}(\varepsilon_2 - \varepsilon_1) + O\left(\left(\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_1}\right)^2\right)\right) \\
&\approx \varepsilon_1^{1-\alpha}(\lambda^{(1)})^2(\varepsilon_2 - \varepsilon_1)\,(2 - \alpha) \\
&< 0
\end{aligned}$$

where we have used Lemma 3.27. With the monotonicity argument complete we conclude that $\varepsilon = \hat{\varepsilon}$ and $\xi = 1$ so there follows $V = Q_\varepsilon$. □

The remaining lemma completes the proof of Theorem 1.7.7:

Lemma 3.2.11. *There is $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$ and $\omega = \omega(\varepsilon)$, the solution Q_ε of* (3.31) *constructed in Theorem 1.7.2 is the unique positive, radially symmetric solution of the minimization problem* (3.30).

Proof. This is the culmination of the previous series of Lemmas. We know that minimizers $V = V_\varepsilon$ exist by Lemma 3.2.2. Arguing by contradiction, if the statement is false, there is a sequence V_{ε_j}, $\varepsilon_j \to 0$, of such minimizers, for which $V_{\varepsilon_j} \neq Q_{\varepsilon_j}$. We apply Lemmas 3.2.3, 3.2.4, 3.2.7, 3.2.8 and 3.2.10 in succession to this sequence, to conclude that along a subsequence, V_{ε_j} and Q_{ε_j} eventually agree, a contradiction. □

Finally, for a given ε, we establish a range of ω for which a minimizer exists and is, up to scaling, a constructed solution. This addresses Remark 1.7.9.

Corollary 3.2.12. *Fix $\varepsilon > 0$ and take $\omega \in [\underline{\omega}, \infty)$ where*

$$\underline{\omega} = \varepsilon^{4/(5-p)} \varepsilon_0^{-4/(5-p)} \omega(\varepsilon_0) \le \omega(\varepsilon).$$

The minimization problem (3.30) *with ε and ω has a solution Q given by*

$$Q(x) = \mu^{1/2} Q_{\hat{\varepsilon}}(\mu x)$$

where $Q_{\hat{\varepsilon}}$ is a constructed solution with $0 < \hat{\varepsilon} \le \varepsilon_0$ and corresponding $\omega(\hat{\varepsilon})$. The scaling factor, μ, satisfies the relationships

$$\varepsilon = \hat{\varepsilon}\mu^{(5-p)/2}, \qquad \omega = \omega(\hat{\varepsilon})\mu^2.$$

Proof. Fix $\varepsilon > 0$. Take any $0 < \hat{\varepsilon} \le \varepsilon_0$ and corresponding constructed $\omega(\hat{\varepsilon})$ and constructed solution $Q_{\hat{\varepsilon}}$. Then, for scaling $\mu = (\varepsilon/\hat{\varepsilon})^{2/(5-p)}$ the function

$$Q(x) = \mu^{1/2} Q_{\hat{\varepsilon}}(\mu x)$$

is a solution to the elliptic problem (3.31) with ε and $\omega = \omega(\hat{\varepsilon})\mu^2$. Recall from Lemma 3.2.10 that $\omega(\hat{\varepsilon})\mu^2$ is monotone in $\hat{\varepsilon}$. Taking $\hat{\varepsilon} \downarrow 0$ yields $\omega \to \infty$. Setting $\hat{\varepsilon} = \varepsilon_0$ yields $\omega = \underline{\omega}$.

In other words if we fix ε and $\omega \in [\underline{\omega}, \infty)$ from the start we determine an $\hat{\varepsilon}$ and μ that generate the desired Q. We claim that the function $Q(x)$

is a minimizer of the problem (3.30) with ε and ω. Suppose not. That is, suppose there exists a function $0 \neq v \in H^1$ with $\mathcal{K}_\varepsilon(v) = 0$ such that $\mathcal{S}_{\varepsilon,\omega}(v) < \mathcal{S}_{\varepsilon,\omega}(Q)$. Set $w(x) = \mu^{-1/2} v(\mu^{-1}x)$ and note that $0 = \mathcal{K}_\varepsilon(v) = \mathcal{K}_{\hat{\varepsilon}}(w)$. We now see

$$\mathcal{S}_{\hat{\varepsilon},\omega(\hat{\varepsilon})}(w) = \mathcal{S}_{\varepsilon,\omega}(v) < \mathcal{S}_{\varepsilon,\omega}(Q) = \mathcal{S}_{\hat{\varepsilon},\omega(\hat{\varepsilon})}(Q_{\hat{\varepsilon}})$$

which contradicts the fact that $Q_{\hat{\varepsilon}}$ is a minimizer of the problem (3.30) with $\hat{\varepsilon}$ and $\omega(\hat{\varepsilon})$. Therefore, $Q(x)$ is a minimizer of (3.30) with ε and ω, which concludes the proof. □

3.3 Dynamics Below the Ground States

In this final section we establish Theorem 1.7.10, the scattering/blow-up dichotomy for the perturbed critical NLS (2.1).

We begin by summarizing the local existence theory for (2.1). This is based on the classical *Strichartz estimates* for the solutions of the homogeneous linear Schrödinger equation

$$i\partial_t u = -\Delta u, \;\; u|_{t=0} = \phi \in L^2(\mathbb{R}^3) \;\; \Longrightarrow \;\; u(x,t) = e^{it\Delta}\phi \; \in C(\mathbb{R}, L^2(\mathbb{R}^3))$$

and the inhomogeneous linear Schrödinger equation (with zero initial data)

$$i\partial_t u = -\Delta u + f(x,t), \;\; u|_{t=0} = 0 \;\; \Longrightarrow \;\; u(x,t) = -i\int_0^t e^{i(t-s)\Delta} f(\cdot,s)ds \;\; :$$

$$\|e^{it\Delta}\phi\|_{S(\mathbb{R})} \le C\|\phi\|_{L^2(\mathbb{R}^3)}, \quad \left\| \int_0^t e^{i(t-s)\Delta} f(\cdot,s)ds \right\|_{S(I)} \le C\|f\|_{N(I)}, \tag{3.36}$$

where we have introduced certain Lebesgue norms for space-time functions $f(x,t)$ on a time interval $t \in I \subset \mathbb{R}$:

$$\|f\|_{L^r_t L^q_x(I)} = \left\| \|f(\cdot,t)\|_{L^q(\mathbb{R}^3)} \right\|_{L^r(I)},$$
$$\|f\|_{S(I)} := \|f\|_{L^\infty_t L^2_x(I) \cap L^2_t L^6_x(I)}, \quad \|f\|_{N(I)} := \|f\|_{L^1_t L^2_x(I) + L^2_t L^{\frac{6}{5}}_x(I)},$$

together with the integral (Duhamel) reformulation of the Cauchy problem (2.1):

$$u(x,t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-s)\Delta}(|u|^4 u + \varepsilon|u|^{p-1}u)ds$$

which in particular gives the sense in which we consider $u(x,t)$ to be a *solution* of (2.1). This lemma summarizing the local theory is standard (see, for example [15, 56]):

Lemma 3.3.1. *Let $3 \leq p < 5$, $\varepsilon > 0$. Given $u_0 \in H^1(\mathbb{R}^3)$, there is a unique solution $u \in C((-T_{min}, T_{max}); H^1(\mathbb{R}^3))$ of (2.1) on a maximal time interval $I_{max} = (-T_{min}, T_{max}) \ni 0$. Moreover:*

1. *space-time norms: $u, \nabla u \in S(I)$ for each compact time interval $I \subset I_{max}$;*
2. *blow-up criterion: if $T_{max} < \infty$, then $\|u\|_{L_t^{10} L_x^{10}([0,T_{max}))} = \infty$ (with similar statement for T_{min});*
3. *scattering: if $T_{max} = \infty$ and $\|u\|_{L_t^{10} L_x^{10}([0,\infty))} < \infty$, then u scatters (forward in time) to 0 in H^1:*

$$\exists\, \phi_+ \in H^1(\mathbb{R}^3) \text{ s.t. } \|u(\cdot,t) - e^{it\Delta}\phi_+\|_{H^1} \to 0 \text{ as } t \to \infty$$

   (with similar statement for T_{min});

4. *small data scattering: for $\|u_0\|_{H^1}$ sufficiently small, $I_{max} = \mathbb{R}$, $\|u\|_{L_t^{10} L_x^{10}(\mathbb{R})} \lesssim \|\nabla u_0\|_{L^2}$, and u scatters (in both time directions).*

Remark 3.3.2. *The appearance here of the $L_t^{10} L_x^{10}$ space-time norm is natural in light of the Strichartz estimates* (3.36)*. Indeed, interpolation between $L_t^\infty L_x^2$ and $L_t^2 L^6$ shows that*

$$\|e^{it\Delta}\phi\|_{L_t^r L_x^q(\mathbb{R})} \lesssim \|\phi\|_{L^2}, \quad \frac{2}{r} + \frac{3}{q} = \frac{3}{2}, \quad 2 \leq r \leq \infty$$

(such an exponent pair (r,q) is called admissible), so then if $\nabla\phi \in L^2$, by a Sobolev inequality,

$$\|e^{it\Delta}\phi\|_{L_x^{10}} \lesssim \|\nabla e^{it\Delta}\phi\|_{L_x^{\frac{30}{13}}} \in L_t^{10},$$

since $(r = 10, q = \frac{30}{13})$ is admissible.

The next lemma is a standard extension of the local theory called a *perturbation* or *stability* result, which shows that any 'approximate solution' has an actual solution remaining close to it. In our setting (see [56, 78]):

Lemma 3.3.3. *Let $\tilde{u} : \mathbb{R}^3 \times I \to \mathbb{C}$ be defined on time interval $0 \in I \subset \mathbb{R}$ with*

$$\|\tilde{u}\|_{L_t^\infty H_x^1(I) \cap L_t^{10} L_x^{10}(I)} \leq M,$$

and suppose $u_0 \in H^1(\mathbb{R}^3)$ satisfies $\|u_0\|_{L^2} \leq M$. There exists $\delta_0 = \delta_0(M) > 0$ such that if for any $0 < \delta < \delta_0$, $\tilde{u}$ is an approximate solution of (2.1) *in the sense*

$$\|\nabla e\|_{L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}(I)} \leq \delta, \qquad e := i\partial_t \tilde{u} + \Delta \tilde{u} + |\tilde{u}|^4 \tilde{u} + \varepsilon |\tilde{u}|^{p-1} \tilde{u},$$

with initial data close to u_0 in the sense

$$\|\nabla\left(\tilde{u}(\cdot, 0) - u_0\right)\|_{L^2} \leq \delta,$$

then the solution u of (2.1) *with initial data u_0 has $I_{max} \supset I$, and*

$$\|\nabla\left(u - \tilde{u}\right)\|_{S(I)} \leq C(M)\delta.$$

Remark 3.3.4. *The space-time norm $\nabla e \in L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}$ in which the error is measured is natural in light of the Strichartz estimates* (3.36)*, since $L_t^{\frac{10}{7}} L_x^{\frac{10}{7}}$ is the dual space of $L_t^{\frac{10}{3}} L_x^{\frac{10}{3}}$, and $(\frac{10}{3}, \frac{10}{3})$ is an admissible exponent pair.*

Given a local existence theory as above, an obvious next problem is to determine if the solutions from particular initial data u_0 are *global* ($I_{max} = \mathbb{R}$), or exhibit *finite-time blow-up* ($T_{max} < \infty$ and/or $T_{min} < \infty$). Theorem 1.7.10 solves this problem for radially-symmetric initial data lying 'below the ground state' level of the action: for any $\varepsilon > 0$, $\omega > 0$, set

$$m_{\varepsilon,\omega} := \inf\{\mathcal{S}_{\varepsilon,\omega}(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{K}_\varepsilon(u) = 0\} \tag{3.37}$$

(see (1.22) for expressions for the functionals $\mathcal{S}_{\varepsilon,\omega}$ and $\mathcal{K}_\varepsilon$), and note that for $\varepsilon \ll 1$ and $\omega = \omega(\varepsilon)$, by Theorem 1.7.7 we have $m_{\varepsilon,\omega} = S_{\varepsilon,\omega}(Q_\varepsilon)$. From here on, we fix a choice of

$$\varepsilon > 0, \quad \omega > 0, \quad p \in (3, 5)$$

(though some results discussed below extend to $p \in (\frac{7}{3}, 5)$):

Theorem 3.3.5. *Let $u_0 \in H^1(\mathbb{R}^3)$ be radially-symmetric and satisfy*

$$\mathcal{S}_{\varepsilon,\omega}(u_0) < m_{\varepsilon,\omega},$$

and let u be the corresponding solution to (2.1)*:*

1. *If* $\mathcal{K}_\varepsilon(u_0) \geq 0$*, u is global, and scatters to* 0 *as* $t \to \pm\infty$*;*

2. *if* $\mathcal{K}_\varepsilon(u_0) < 0$*, u blows-up in finite time (in both time directions).*

Remark 3.3.6. *The argument which gives the finite-time blow-up (the second statement) is classical, going back to [61, 62]. It rests on the following ingredients: conservation of mass and energy imply* $\mathcal{S}_{\varepsilon,\omega}(u) \equiv \mathcal{S}_{\varepsilon,\omega}(u_0) < m_{\varepsilon,\omega}$*, so that the condition* $\mathcal{K}_\varepsilon(u) < 0$ *is preserved (by definition of* $m_{\varepsilon,\omega}$*); a spatially cut-off version of the formal variance identity for (NLS)*

$$\frac{d^2}{dt^2}\frac{1}{2}\int |x|^2|u(x,t)|^2 dx = \frac{d}{dt}\int x \cdot \Im\left(\bar{u}\nabla u\right) dx = 2\mathcal{K}_\varepsilon(u)\ ; \tag{3.38}$$

and exploitation of radial symmetry to control the errors introduced by the cut-off. In fact, a complete argument in exactly our setting is given as the proof of Theorem 1.3 in [1] (it is stated there for dimensions ≥ 4 *but in fact the proof covers dimension* 3 *as well). So we will focus here only on the proof of the first (scattering) statement.*

The concentration-compactness approach of Kenig-Merle [54] to proving the scattering statement is by now standard. In particular, [2] provides a complete proof for the analogous problem in dimensions ≥ 5. In fact, the proof there is more complicated for two reasons: there is no radial symmetry restriction; and in dimension n, the corresponding nonlinearity includes the term $|u|^{p-1}u$ with $p > 1 + \frac{4}{n}$ loses smoothness, creating extra technical difficulties. We will therefore provide just a sketch of the (simpler) argument for our case, closely following [56], where this approach is implemented for the *defocusing* quintic NLS perturbed by a cubic term, and taking the additional variational arguments we need here from [1, 2], highlighting points where modifications are needed.

In the next lemma we recall some standard variational estimates for functions with action below the ground state level $m_{\varepsilon,\omega}$. The idea goes back to [63], but proofs in this setting are found in [1, 2]. Recall the 'unperturbed' ground state level is attained by the Aubin-Talenti function W:

$$m_{0,0} := \mathcal{E}_0(W) = \inf\{\mathcal{E}_0(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{K}_0(u) = 0\},$$

and introduce the auxilliary functional

$$\begin{aligned}\mathcal{I}_\omega(u) &:= \mathcal{S}_{\varepsilon,\omega}(u) - \frac{2}{3(p-1)}\mathcal{K}_\varepsilon(u) \\ &= \frac{p-\frac{7}{3}}{2(p-1)}\int |\nabla u|^2 + \frac{5-p}{6(p-1)}\int |u|^6 + \frac{1}{2}\omega\int |u|^2\end{aligned}$$

which is useful since all its terms are positive, and note

$$\mathcal{K}_\varepsilon(u) \geq 0 \implies \|u\|_{H^1}^2 \lesssim \mathcal{I}_\omega(u) \leq \mathcal{S}_{\varepsilon,\omega}(u). \tag{3.39}$$

Define, for $0 < m^* < m_{\varepsilon,\omega}$, the set

$$\mathcal{A}_{m^*} := \{u \in H^1(\mathbb{R}^3) \mid \mathcal{S}_{\varepsilon,\omega}(u) \leq m^*,\ K_\varepsilon(u) > 0\}$$

and note that it is is preserved by (2.1):

$$u_0 \in \mathcal{A}_{m^*} \implies u(\cdot,t) \in \mathcal{A}_{m^*} \text{ for all } t \in I_{max}.$$

Indeed, by conservation of mass and energy $\mathcal{S}_{\varepsilon,\omega}(u(\cdot,t)) = \mathcal{S}_{\varepsilon,\omega}(u_0) \leq m^*$. Moreover if for some $t_0 \in I_{max}$, $\mathcal{K}_\varepsilon(u(\cdot,t_0)) \leq 0$, then by H^1 continuity of $u(\cdot,t)$ and of $\mathcal{K}_\varepsilon$, we must have $\mathcal{K}_\varepsilon(u(\cdot,t_1)) = 0$ for some $t_1 \in I_{max}$, contradicting $m^* < m_{\varepsilon,\omega}$.

Lemma 3.3.7. *1. $m_{\varepsilon,\omega} \leq m_{0,0}$, and* (3.37) *admits a minimizer if $m_{\varepsilon,\omega} < m_{0,0}$;*

2. we have

$$m_{\varepsilon,\omega} = \inf\{\mathcal{I}_\omega(u) \mid u \in H^1(\mathbb{R}^3) \setminus \{0\}, \mathcal{K}_\varepsilon(u) \leq 0\}, \tag{3.40}$$

and a minimizer for this problem is a minimizer for (3.37)*, and vice versa;*

3. given $0 < m^* < m_{\varepsilon,\omega}$, there is $\kappa(m^*) > 0$ such that

$$u \in \mathcal{A}_{m^*} \implies \mathcal{K}_\varepsilon(u) \geq \kappa(m^*) > 0. \tag{3.41}$$

After the local theory, and in particular the perturbation Lemma 3.3.3, the key analytical ingredient is a *profile decomposition*, introduced into the analysis of critical nonlinear dispersive PDE by [5, 55]. This version, taken from [56] (and simplified to the radially-symmetric setting), can be thought of as making precise the lack of compactness in the Strichartz estimates for $\dot{H}^1(\mathbb{R}^3)$ data, when the data is bounded in $H^1(\mathbb{R}^3)$:

Lemma 3.3.8. *([56], Theorem 7.5) Let $\{f_n\}_{n=1}^\infty$ be a sequence of radially symmetric functions, bounded in $H^1(\mathbb{R}^3)$. Possibly passing to a subsequence, there is $J^* \in \{0,1,2,\ldots\} \cup \{\infty\}$ such that for each finite $1 \leq j \leq J^*$ there exist (radially symmetric) 'profiles' $\phi^j \in \dot{H}^1 \setminus \{0\}$, 'scales' $\{\lambda_n^j\}_{n=1}^\infty \subset (0,1]$, and 'times' $\{t_n^j\}_{n=1}^\infty \subset \mathbb{R}$ satisfying, as $n \to \infty$,*

$$\lambda_n^j \equiv 1 \ \text{ or } \ \lambda_n^j \to 0, \qquad t_n^j \equiv 0 \ \text{ or } \ t_n^j \to \pm\infty.$$

If $\lambda_n^j \equiv 1$ *then additionally* $\phi^j \in L^2(\mathbb{R}^3)$. *For some* $0 < \theta < 1$, *define*

$$\phi_n^j(x) := \begin{cases} \left[e^{it_n^j\Delta}\phi^j\right](x) & \lambda_n^j \equiv 1 \\ (\lambda_n^j)^{-\frac{1}{2}}\left[e^{it_n^j\Delta}P_{\geq(\lambda_n^j)^\theta}\phi^j\right]\left(\frac{x}{\lambda_n^j}\right) & \lambda_n^j \to 0, \end{cases}$$

where $P_{\geq N}$ *denotes a standard smooth Fourier multiplier operator (Littlewood-Paley projector) which removes the Fourier frequencies* $\leq N$. *Then for each finite* $1 \leq J \leq J^*$ *we have the decomposition*

$$f_n = \sum_{j=1}^{J} \phi_n^j + w_n^J$$

with:

- *small remainder:* $\lim_{J\to J^*} \limsup_{n\to\infty} \|e^{it\Delta}w_n^J\|_{L_t^{10}L_x^{10}(\mathbb{R})} = 0$
- *decoupling: for each* J, $\lim_{n\to\infty}\left[\mathcal{M}(f_n) - \sum_{j=1}^{J}\mathcal{M}(\phi_n^j) - \mathcal{M}(w_n^J)\right] = 0$, *and the same statement for the functionals* $\mathcal{E}_\varepsilon$, $\mathcal{K}_\varepsilon$, $\mathcal{S}_{\omega,\varepsilon}$ *and* $\mathcal{I}_\omega$;
- *orthogonality:* $\lim_{n\to\infty}\left[\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|t_n^j(\lambda_n^j)^2 - t_n^k(\lambda_n^k)^2|}{\lambda_n^j\lambda_n^k}\right] = \infty$ *for* $j \neq k$.

The global existence and scattering statement 1 of Theorem 1.7.10 is established by a contradiction argument. For $0 < m < m_{\varepsilon,\omega}$, set

$$\tau(m) := \sup\left\{\|u\|_{L_t^{10}L_x^{10}(I_{max})} \mid \mathcal{S}_{\varepsilon,\omega}(u_0) \leq m,\ \mathcal{K}_\varepsilon(u_0) > 0\right\}$$

where the supremum is taken over all radially-symmetric solutions of (2.1) whose data u_0 satisfies the given conditions. It follows from the local theory above that τ is non-decreasing, continuous function of m into $[0,\infty]$, and that $\tau(m) < \infty$ for sufficiently small m (by part 4 of Lemma 3.3.1). By parts 2-3 of Lemma 3.3.1, if $\tau(m) < \infty$ for all $m < m_{\varepsilon,\omega}$, the first statement of Theorem 1.7.10 follows. So we suppose this is *not* the case, and that in fact

$$m^* := \sup\{m \mid 0 < m < m_{\varepsilon,\omega},\ \tau(m) < \infty\} < m_{\varepsilon,\omega}.$$

By continuity, $\tau(m^*) = \infty$, and so there exists a sequence $u_n(x,t)$ of *global*, radially-symmetric solutions of (2.1) satisfying

$$\mathcal{S}_{\varepsilon,\omega}(u_n) \leq m^*, \qquad \mathcal{K}_\varepsilon(u_n(\cdot,0)) > 0, \tag{3.42}$$

and

$$\lim_{n\to\infty} \|u_n\|_{L_t^{10}L_x^{10}([0,\infty))} = \lim_{n\to\infty} \|u_n\|_{L_t^{10}L_x^{10}((-\infty,0])} = \infty \tag{3.43}$$

(the last condition can be arranged by time shifting, if needed). The idea is to pass to a limit in this sequence in order to obtain a solution sitting at the threshold action m^*.

Lemma 3.3.9. *There is a subsequence (still labelled u_n) such that $u_n(x,0)$ converges in $H^1(\mathbb{R}^3)$.*

Proof. This is essentially Proposition 9.1 of [56], with slight modifications to incorporate the variational structure. We give a brief sketch. The sequence $u_n(\cdot,0)$ is bounded in H^1 by (3.39), so we may apply the profile decomposition Lemma 3.3.8: up to subsequence,

$$u_n(\cdot,0) = \sum_{j=1}^{J} \phi_n^j + w_n^J.$$

If we can show there is only one profile ($J^* = 1$), that $\lambda_n^1 \equiv 1$, $t_n^1 \equiv 0$, and that $w_n^1 \to 0$ in H^1, we have proved the lemma. By (3.42) and the decoupling,

$$\begin{aligned} m^* - \frac{2}{3(p-1)}\kappa(m^*) &\geq \mathcal{S}_{\varepsilon,\omega}(u_n(\cdot,0)) - \frac{2}{3(p-1)}\mathcal{K}_\varepsilon(u_n(\cdot,0)) \\ &= \mathcal{I}_\omega(u_n(\cdot,0)) = \sum_{j=1}^{J} \mathcal{I}_\omega(\phi_n^j) + \mathcal{I}_\omega(w_n^J) + o(1), \end{aligned}$$

and since $\mathcal{I}_\omega$ is non-negative, we have, for n large enough, $\mathcal{I}_\omega(\phi_n^j) < m^*$ for each j and $\mathcal{I}_\omega(w_n^J) < m^*$. Since $m^* < m_{\varepsilon,\omega}$, it follows from (3.40) that $\mathcal{K}_\varepsilon(\phi_n^j) > 0$ and $\mathcal{K}_\varepsilon(w_n^J) \geq 0$, so also $\mathcal{S}_{\varepsilon,\omega}(\phi_n^j) > 0$ and $\mathcal{S}_{\varepsilon,\omega}(w_n^J) \geq 0$. Hence if there is more than one profile, by the decoupling

$$m^* \geq \mathcal{S}_{\varepsilon,\omega}(u_n(\cdot,0)) = \sum_{j=1}^{J} \mathcal{S}_{\varepsilon,\omega}(\phi_n^j) + \mathcal{S}_{\varepsilon,\omega}(w_n^J) + o(1),$$

we have, for each j, and n large enough, for some $\eta > 0$,

$$\mathcal{S}_{\varepsilon,\omega}(\phi_n^j) \leq m^* - \eta, \quad \mathcal{K}_\varepsilon(\phi_n^j) > 0. \tag{3.44}$$

Following [56], we introduce *nonlinear profiles* v_n^j associated to each ϕ_n^j.

First, suppose $\lambda_n^j \equiv 1$. If $t_n^j \equiv 0$, then $v_n^j = v^j$ is defined to be the solution to (2.1) with initial data ϕ^j. If $t_n^j \to \pm\infty$, v^j is defined to be the solution scattering (in H^1) to $e^{it\Delta}\phi^j$ as $t \to \pm\infty$, and $v_n^j(x,t) := v^j(t + t_n^j)$. In both cases, it follows from (3.44) that v_n^j is a global solution, with $\|v_n^j\|_{L_t^{10}L_x^{10}(\mathbb{R})} \leq \tau(m^* - \eta) < \infty$.

For the case $\lambda_n^j \to 0$, we simply let v_n^j be the solution of (2.1) with initial data ϕ_n^j. As in [56] Proposition 8.3, v_n^j is approximated by the solution $\tilde{u}_n^j$ of the *unperturbed* critical NLS (1.13) (since the profile is concentrating, the sub-critical perturbation 'scales away') with data ϕ_n^j (or by a scattering procedure in case $t_j^n \to \pm\infty$). The key additional point here is that by (3.44), and since $m^* < m_{\varepsilon,\omega} \leq m_{0,0}$, it follows that for n large enough

$$\mathcal{E}_0(v_n^j) \leq m^* < m_{0,0}, \quad \mathcal{K}_0(v_n^j) > 0,$$

and so by [54], $\tilde{u}_n^j$ is a *global* solution of (1.13), with $\|\tilde{u}_n^j\|_{L_t^{10}L_x^{10}(\mathbb{R})} \leq C(m^*) < \infty$. It then follows from Lemma 3.3.3 that the same is true of v_n^j.

These nonlinear profiles are used to construct what are shown in [56] to be increasingly accurate (for sufficiently large J and n) approximate solutions in the sense of Lemma 3.3.3,

$$u_n^J(x,t) := \sum_{j=1}^{J} v_n^j(x,t) + e^{it\Delta}w_n^J$$

which are moreover global with uniform space-time bounds. This contradicts (3.43).

Hence there is only one profile: $J^* = 1$, and the decoupling also implies $\|w_n^1\|_{H^1} \to 0$. Finally, the possibilities $t_n^1 \to \pm\infty$ or $\lambda_n^1 \to 0$ are excluded just as in [56], completing the argument. $\square$

Given this lemma, let $u_0 \in H^1(\mathbb{R}^3)$ be the H^1 limit of (a subsequence) of $u_n(x,0)$, and let $u(x,t)$ be the corresponding solution of (2.1) on its maximal existence interval $I_{max} \ni 0$. We see $\mathcal{S}_{\varepsilon,\omega}(u) = \mathcal{S}_{\varepsilon,\omega}(u_0) \leq m^*$. Whether u is global or not, it follows from Lemma 3.3.1 (part 2), (3.43) and Lemma 3.3.3, that

$$\|u\|_{L_t^{10}L_x^{10}(I_{max})} = \infty, \quad \text{hence also } \mathcal{S}_{\varepsilon,\omega}(u) = m^*.$$

It follows also that

$$\{u(\cdot,t) \mid t \in I_{max}\} \text{ is a pre-compact set in } H^1(\mathbb{R}^3).$$

To see this, let $\{t_n\}_{n=1}^{\infty} \subset I_{max}$, and note that since

$$\|u\|_{L_t^{10}L_x^{10}((-T_{min},t_n])} = \|u\|_{L_t^{10}L_x^{10}([t_n,T_{max}))} = \infty,$$

and so (the proof of) Lemma 3.3.9 applied to the sequence $u(x, t-t_n)$ implies that $\{u(x,t_n)\}$ has a convergent subsequence in H^1.

The final step is to show that this ‘would-be’ solution u with these special properties, sometimes called a *critical element* cannot exist. For this, first note that u must be global: $I_{max} = \mathbb{R}$. This is because if, say, $T_{max} < \infty$, then for any $t_n \to T_{max}-$, $u(\cdot, t_n) \to \tilde{u}_0 \in H^1(\mathbb{R}^3)$ (up to subsequence) in H^1, by the pre-compactness. Then by comparing u with the solution $\tilde{u}$ of (2.1) with initial data $\tilde{u}_0$ at $t = t_n$ using Lemma (3.3.3), we conclude that u exists for times beyond T_{max}, a contradiction.

Finally, the possible existence of (the now global) solution u is ruled out via a suitably cut-off version of the virial identity (3.38), using (3.41), and the compactness to control the errors introduced by the cut-off, exactly as in [56] (Proposition 10, and what follows it). □

Chapter 4

Directions for Future Study

We collect here some suggestions (and speculations) for future problems relating to the results of Chapters 2 and 3.

4.1 Improvements and Extensions of the Current Work

As mentioned in Section 1.6 the asymptotic stability for the 1D mass subcritical ($p \neq 3$) NLS is still open. This problem presents serious challenges. A successful attempt on this problem may involve detailed knowledge of the spectrum of the linearized operator as obtained in Chapter 2.

The results of Chapter 3 apply only to dimension 3. The works [1, 2] addressed the dynamics (scattering/blow-up) for the perturbed energy critical NLS (below the threshold) in dimensions $n \geq 4$. Nevertheless it would be interesting to see if the construction of Section 3.1 can be achieved in 4D. One would need to redo the resolvent estimates in Section 3.1.2 using [46] in place of [47], as well as perform many of the computations in 4D since we have used the particular 3D free resolvent expansion (3.5) and particular 3D Young's inequality (3.6), (3.7) and so on. If the same methods produce constructed solutions in 4D it's possible we could complete the analysis of Sections 3.2 and 3.3 to achieve a dynamical theorem. While this theorem already exists [1, 2] some comparison may be interesting.

In dimensions $n \geq 5$ the linearized operator (1.17) no longer has a resonance (as a resonance appearing in dimensions 5D and higher is impossible [45]) but does have an edge-eigenvalue. Replacing the role of [47] with [45] we may be able to use our methods to achieve the results of Chapter 3 in 5D and higher.

One important case we are missing from Sections 3.2 and 3.3 is the case when $p = 3$. Of course, the cubic-quintic NLS is important for applications and seems as well to present interesting mathematical difficulties. While we are able to construct solitary wave solutions in Section 3.1 we cannot demonstrate these solutions to be ground states in Section 3.2. In Lemma

3.2.2 the leading order in the computation of $\mathcal{S}_{\varepsilon,\omega}(Q)$ is zero when $p = 3$ and the next to leading order is difficult to resolve. If we were able to at least demonstrate the existence of a ground state we could run the arguments of Section 3.3 to achieve our dynamical theorem for $p = 3$. (although a ground state in this case may simply not exist). Supposing that a ground state solution does exist for $p = 3$ we would still have difficulty to demonstrate our constructed solutions as the ground state (it also simply may not be) as the series of Lemmas 3.2.3, 3.2.4, 3.2.7, 3.2.8 and 3.2.10 in Section 3.2 all require $3 < p < 5$. One could look for additional boundary case arguments to add or else try to demonstrate that our constructed solution is not a ground state.

4.2 Small Solutions to the Gross-Pitaevskii Equation

A distinct but related problem is to consider solutions to the Gross-Pitaevskii equation (the nonlinear Schrödinger equation with a potential)

$$i\partial_t \psi = (-\Delta + V)\psi \pm |\psi|^{p-1}\psi \tag{4.1}$$

with small initial data $\psi(x, 0) = \psi_0 \in H^1$ in n dimensions. Function $V(x)$ is a potential which we can think of as a Schwartz (fast decaying) function. The power p should be taken mass super-critical but energy sub-critical to ensure solutions do not blow-up in finite time. Global well-posedness is a result of conservation of mass and energy and the smallness of initial data (see [15] Chapter 6). We think mainly about H^1 results but there are also results where further localizing assumptions ($\psi_0 \in L^1$) on the initial data are made such as [66] [69] [70] [87].

The simplest result is when $-\Delta + V$ has no bound states nor a resonance. In this case all solutions of (4.1) will scatter; a result due to [49] and [73]. The result with a single simple bound state was obtained for a class of nonlinearities in [42] in 3D. This result was extended to 1D with supercritical power nonlinearity in [59]. In this situation the bound state of $-\Delta + V$ generates nonlinear bound states of (4.1). These nonlinear bound states give rise to stable solitary waves. Both [42] and [59] establish that any small solution can be decomposed into a piece approaching a nonlinear bound state and a piece that scatters. The state of the art is [27] where they treat the 3D cubic defocusing equation with many simple eigenvalues. There are some conditions on the relative values of the eigenvalues but the treatable cases are quite generic. While the result obtained is analogous to [42] and

[59] the methods used in this paper differ from [42] and [59]. The paper [27] uses instead variational methods. By means of a Birkhoff normal forms argument they find an effective Hamiltonian which gives rise to a nonlinear Fermi Golden Rule. The connection between the Hamiltonian structure and a Fermi Golden Rule was originally introduced in [24]. See also [25] for the intuition behind the argument. For us the special case of two eigenvalues in [27] will be most relevant. There are also some other results for two bound states [71] [79] [80] [81] [82] which impose stronger restrictions on the initial data and placement of the eigenvalues.

Our questions revolve around the case when operator $-\Delta + V$ has a resonance. That is when $(-\Delta + V)\phi_1 = 0$ for some $\phi_1 \notin L^2$ but $\phi_1 \in L^q$ for some $2 < q \leq \infty$ where q depends on the dimension n. For the statement of questions bellow we also assume that $-\Delta + V$ also has a simple bound state, that is $(-\Delta + V)\phi_0 = e_0\phi_0$ with $e_0 < 0$. We could, however, ask analogous questions if $-\Delta + V$ has no bound state and just the resonance or if instead of a resonance we have a simple threshold eigenvalue.

Firstly, we may suspect that ϕ_1 will generate nonlinear bound states. When $H := -\Delta + V$ has a bound sate the nonlinear problem (4.1) admits a family of nonlinear bound states $Q_0[z]$ parametrized by $z = (Q_0, \phi_0)$ with eigenvalue $E_0 = E_0[|z|]$. This follows from standard bifurcation theory (see the Appendix of [42]). The existence of nonlinear bound states coming from the resonance eigenfunction does not follow immediately from the true eigenvalue result. If for example we write

$$Q_1(x) = z\phi_1(x) + q(x) \tag{4.2}$$

then to have $Q_1 \in L^2$ we cannot have $q \in L^2$. The Birman-Schwinger trick used in Chapter 2 does not work due to the presence of the nonlinear term in (4.1) but we can instead proceed as in Chapter 3. A formal and preliminary computation suggests that generically in the defocusing case the resonance will not yield nonlinear bound states while in the focusing case we will have nonlinear bound states with corresponding eigenvalues close to the threshold. The idea is to follow the analysis of Section 3.1 and substitute (4.2) to the nonlinear eigenvalue equation

$$(H + \lambda^2)Q_1 \pm |Q_1|^{p-1}Q_1 = 0$$

and write the resulting equation as a fixed point problem for q

$$q = (H + \lambda^2)^{-1}f.$$

Here $f = f(\lambda, z, q)$. From [47] and [48] we have the resolvent expansion

$$(H+\lambda^2)^{-1} = \frac{1}{\lambda}\langle R_0(\lambda)V\phi_1, \cdot\rangle\phi_1$$

where $R_0(\lambda) = (-\Delta+\lambda^2)^{-1}$ is the free resolvent. The idea now is to solve

$$0 = \langle R_0(\lambda)V\phi_1, f\rangle$$
$$q = (H+\lambda^2)^{-1}f$$

for $\lambda = \lambda(z)$ via fixed point arguments where q is in a higher L^p space. Adjustments of the fixed point arguments of Section 3.1 may prove fruitful in this setting.

Secondly, we may consider the asymptotic stability of the ground state family $e^{-iE_0t}Q_0$. One approach would be to proceed as in [42] but with weaker decay estimates. Taking the power in the nonlinearity higher may help in this direction. An alternative, and perhaps more favourable, approach to this problem would be to understand the spectrum of the linearized operator around Q_0 in the presence of the resonance ϕ_1. We comment on this direction. Since we are interested in the stability of the family of ground state solitary waves we consider a solution to (4.1) of the form

$$\psi(x,t) = e^{-iE_0t}\big(Q_0(x)+\xi(x,t)\big)$$

where ξ is a small perturbation of Q_0. Since Q_0 is the ground state we can take it positive and real. Substituting the above to (4.1) and removing known information about Q_0 yields

$$i\partial_t\xi = (H-E_0)\xi \pm (p-1)Q_0^{p-1}\mathrm{Re}\xi + Q_0^{p-1}\xi + N(\xi)$$

where $N(\xi)$ is nonlinear in ξ. The above with N removed is the linearized equation. We complexify by letting

$$\vec{\xi} := \begin{pmatrix}\xi\\ \bar{\xi}\end{pmatrix}$$

to see

$$i\partial_t\vec{\xi} = \mathcal{L}\vec{\xi}$$

where

$$\mathcal{L} := \begin{pmatrix} H-E_0 \pm \frac{p+1}{2}Q_0^{p-1} & \pm\frac{p+1}{2}Q_0^{p-1} \\ \mp\frac{p+1}{2}Q_0^{p-1} & -\Big(H-E_0\pm\frac{p+1}{2}Q_0^{p-1}\Big)\end{pmatrix}.$$

The unperturbed $\mathcal{L}_0$ for $z = 0$ then has a resonance at the threshold. Indeed

$$\mathcal{L}_0 = \begin{pmatrix} H - e_0 & 0 \\ 0 & -(H - e_0) \end{pmatrix}$$

has essential spectrum $(-\infty, e_0] \cup [-e_0, \infty)$, a double eigenvalue at 0 and a resonance on each threshold. The question is then about the spectrum of $\mathcal{L}$. The full operator in question has essential spectrum $(-\infty, E_0] \cup [-E_0, \infty)$ as well as the double eigenvalue at 0 but the fate of the resonance has yet to be seen. Again, a formal preliminary computation suggests that generically in the focusing case we have a true eigenvalue close to the threshold and that in the defocusing case the resonance disappears. In 3D we may be able to apply [28] which treats general perturbations of the linearized operator. In 1D the experience gained from Chapter 2 may be of use. Once the spectrum of the operator $\mathcal{L}$ is understood we may be able to proceed as in [79–82] to obtain asymptotic stability results for the ground state as well as the excited state, in any case where it may exist. A difficulty in this series of papers was the eigenvalues of H being close together or, after complexification, $\mathcal{L}$ having eigenvalues close to zero. For us any non-zero eigenvalues of $\mathcal{L}$ will be close to the threshold and so far from zero.

Finally, we may wish to understand the long time behaviour of all small solutions of (4.1). That is obtain an asymptotic stability and completeness theorem in the spirit of [42], [59], [27]. This is the most substantial and challenging question present. The goal is to prove a theorem resembling the following. For any small solution ψ of (4.1) we have the unique decomposition

$$\psi(t) = Q_0[z_0(t)] + Q_1[z_1(t)] + \eta(t)$$

in the presence of nonlinear bound states connected to the resonance or

$$\psi(t) = Q_0[z_0(t)] + \eta(t)$$

in the absence of nonlinear bound states connected to the resonance. In the above z_j and η must also enjoy some smallness properties. Additionally the z_j should converge in some sense as $t \to \infty$ where only one z_j converges to a nonzero value. The function η should scatter and converges to a solution of the free linear Schrödinger equation. While we would like to, after understanding the existence on nonlinear bound states and the linearized operator, proceed as in [42], [59], [27] we draw attention to the fact that eigenvalues arbitrarily close to the threshold will adversely affect the necessary decay estimates. The convergence of ψ (if at all) to a nonlinear bound state will be slow and achieving such an asymptotic stability and completeness theorem (if true) may be beyond current mathematical technology.

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