# Manifestly Gauge Invariant Transition Amplitudes and Thermal Influence Functionals in QED and Linearized Gravity

by

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# Abstract

Einstein's theory of General Relativity tells us that gravity is not a force but rather it is the curvature of spacetime itself. Spacetime is a dynamical object evolving and interacting similar to any other system in nature. The equivalence principle requires **everything** to couple to gravity in the same way. Consequently, as a matter of principle it is impossible to truly isolate a system—it will always be interacting with the dynamical spacetime in which it resides. This may be detrimental for large mass quantum systems since interaction with an environment can decohere a quantum system, rendering it effectively classical. To understand the effect of a 'spacetime environment', we compute the Feynman-Vernon influence functional (IF), a useful tool for studying decoherence. We compute the IF for both the electromagnetic and linearized gravitational fields at finite temperature in a manifestly gauge invariant way. Gauge invariance is maintained by using a modification of the Faddeev-Popov technique which results in the integration over all gauge equivalent configurations of the system. As an intermediate step we evaluate the gauge invariant transition amplitude for the gauge fields in the presence of sources. When used as an evolution kernel the transition amplitude projects initial data onto a physical (gauge-invariant) subspace of the Hilbert space and time-evolves the states within that physical subspace. The states in this physical subspace satisfy precisely the same constraint equations which one implements in the constrained quantization method of Dirac. Thus we find that our approach is the path-integral equivalent of Dirac's. In the gauge invariant computation it is clear that for gauge theories the appropriate separation between system and environment is not a) matter and gauge field, but rather b) matter (dressed with a coherent field) and radiation field. This implies that only the state of the radiation field can be traced out to obtain a reduced description of the matter. We stress the importance of gauge invariance and the implementation of constraints because it resolves the disagreement between results in reported recent literature in which influence functionals were computed in different gauges without consideration of constraints.

# Lay Summary

The macroscopic world is described well by classical physics. At a moment of time you say i) where things are, ii) how fast they are moving, and iii) how hard are they being pushed or pulled. This is all you need to say where everything will be at a later time and how fast it will all be moving. This description works well until you look at very small things like electrons. We've seen experimentally that small things are better described by quantum mechanics. In quantum mechanics things are more strange. For example, an object can be in more than one place at a time. Since we don't see macroscopic objects behaving this way, we expect there to be some sort of cross-over. In this thesis we build some mathematical tools which will help to understand how fluctuations in the gravitational field may be responsible for this cross-over.

# Preface

The idea to compute a gravitational influence functional was provided by the author's supervisor Dr. Philip Stamp. Aside from this initial idea, the work done in this thesis was performed independently by the author.

# Table of Contents

**Abstract** . . . . . ii

**Lay Summary** . . . . . iii

**Preface** . . . . . iv

**Table of Contents** . . . . . v

**Acknowledgements** . . . . . vii

**Dedication** . . . . . viii

**1 Introduction** . . . . . 1
- 1.1 Previous Approaches to Gravitational Decoherence . . . . . 4
  - 1.1.1 Decoherence in Modified Quantum Theories . . . . . 5
  - 1.1.2 Decoherence in Conventional Quantum Theory . . . . 6
- 1.2 Quantum Gravity as a Low Energy Effective Field Theory . 10
- 1.3 Feynman-Vernon Influence Functional . . . . . 12
- 1.4 Difficulty Defining an Influence Functional in a Gauge Theory 16
- 1.5 Quantum Gravity Path Integral - Boundary Terms . . . . . 18
  - 1.5.1 Gibbons-Hawking-York Boundary Term . . . . . 18
  - 1.5.2 Gauge Transformation Boundary Term . . . . . 20

**2 Transition Amplitudes in Gauge Theories** . . . . . 23
- 2.1 Quantum Electrodynamics (QED) . . . . . 24
  - 2.1.1 Free U(1) Theory . . . . . 24
  - 2.1.2 Interacting QED . . . . . 29
- 2.2 Linearized Quantum Gravity . . . . . 34

**3 Influence Functional in QED and linearized Gravity** . . . . 43
- 3.1 Thermal Photon Bath Influence Functional . . . . . 43
- 3.2 Thermal Graviton Bath Influence Functional . . . . . 45
- 3.3 Importance of Gauge Invariance . . . . . 48

**4 Conclusion** . . . . . 52

**Bibliography** . . . . . 55

## Appendices

**A The Classical Action in QED and Linear Gravity** . . . . . 63

**B Instantaneous Interaction Terms** . . . . . 66

**C Derivation of the Master Equation from the Path Integral** 69

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# Dedication

To my mother,

For the unconditional support she has provided which has allowed me to pursue my academic dreams, and for teaching me that excellence only comes through unfailing dedication and hard work.

# Chapter 1

# Introduction

Quantum systems are fragile and easily disrupted by interactions with environmental disturbances. Decoherence can occur when an environment disturbs a quantum system even if energy is not exchanged [1]. It is the same thing that makes quantum systems so rich that makes them so fragile, they can build extraordinarily complex correlations. Counterintuitive as it may be, in general an environment decoheres a quantum system not by physically destroying it but by developing intricate correlations with the state of the system. For a quantum system strongly correlated with its environment, a demonstration of its quantum behaviour requires the careful control of the state of the environment in addition to the system. Thus if the state of the environment is not carefully tracked then the system alone may not exhibit quantum behaviour (interference for example). Quantum correlation, referred to as quantum *entanglement*, was initially thought of as a demonstration of the incompleteness of quantum theory [2] but is now understood as such a fundamental feature of nature that there is an entire field based around the idea that it might be the glue which holds spacetime together [3].

To study quantum systems experimentally we try to isolate quantum systems to avoid environmental interactions, but as a matter of principle no system can be entirely isolated. Einstein's theory of General Relativity (GR) has taught us that gravity should not be thought of as a force between objects but rather as a manifestation of the curvature of spacetime itself. Spacetime is no longer a fixed background on which objects interact, it too is a dynamical object. Since it provides the casual structure for all other objects in nature, spacetime is necessarily interacting with **everything**; nothing can be shielded from gravitational interaction. Spacetime itself is an ever present environment which may cause the decoherence of quantum systems.

Gravitational decoherence is a hard problem to properly define let alone study and solve. In the typical discussion of decoherence one has two quantum systems, commonly referred to as the central system and the environment. A measurement/experiment involving only the central system is

insensitive to not only the state of the environment but also the correlations that exist between the environment and central system. Operationally this ignorance is claimed by summing (or tracing) over all possible states of the environment. Any information stored as correlations between the central system and environment is lost, *i.e.* the entropy of the central system is increased. It is then said that the central system has undergone decoherence. In quantum mechanics (QM) alone this leads to a number of unresolved questions related to the interpretation of the theory. For example, what defines a measurement? Does the wavefunction/density matrix simply represent the information we have, or is it something physical? Typically such questions are set aside because we can still use QM to predict probabilities of and correlations between measurement outcomes without thinking about the ontology of measurement operators and state vectors. This is the FAPP ('for all practical purposes') approach, which has remained popular because of how accurate quantum mechanical predictions can be regardless of the chosen interpretation. Note, for example, the agreement between theory and experiement of the anomalous magnetic moment of the electron to better than 1 part in $10^{11}$ [4]. However once gravitation is included the foundational questions which could once be ignored FAPP now become essential problems which make it unclear what/how computations can be done in quantum gravity. For example, if spacetime is treated as a quantum mechanical object then what constitutes an observer/measuring apparatus? Even if spacetime is treated classically we can ask, if a central system interacts with an environment that later gets thrown into a black hole, are the correlations permanently lost? The latter question seeds the well know black hole information paradox [5]. Questions of this type make it difficult to formulate meaningful questions involving QM and gravity. The problem is that we simply do not yet have a complete accepted theory incorporating both quantum mechanics and general relativity.

There is not yet a unification of GR and QM, however there are a number of different approaches to this unification. In the most popular approaches, string theory and loop quantum gravity, QM holds at all scales however at short distances the classical notion of spacetime is drastically modified, *i.e.* general relativity breaks down. There are a number of alternative approaches in which QM suffers a breakdown at macroscopic scales as a result of gravitation. These theories predict new mechanisms which can look like intrinsic decoherence of quantum states (as opposed to "environmental decoherence" caused by interaction with environmental degrees of freedom) [6–14]. For compatibility with previous experimental observations the predicted decoherence rates must be negligible at microscopic scales,

but they are expected to become appreciable around mesoscopic scales and to completely suppress quantum effects on macroscopic scales. Recently there has been much effort towards the development of "table-top" quantum gravity experiments which aim to study quantum theory at mesoscopic scales right where intrinsic decoherence mechanisms are expected to be seen. To understand the results of such experiments it is essential to understand how environmental decoherence may occur in conventional quantum gravity (theories without modifications to QM). We can see the extreme sensitivity to decoherence as providing an useful probe of quantum gravity. In this thesis, we develop tools which will be useful for the study of decoherence in the effective field theory description of conventional quantum gravity. Specifically we compute the Feynman-Vernon influence functional for a bath of gravitons at finite temperature. As a warm-up for quantum gravity we first study Quantum Electrodynamics (QED)[1]. As we will show, in the low energy limit which we are interested in the treatments of the two theories are quite similar.

This thesis is organized as follows. This chapter provides an introduction to the study of gravitational decoherence. In Section 1.1 we provide a brief discussion of some previous work on gravitational decoherence in: classical gravity, gravitationally induced intrinsic decoherence theories, and conventional quantum gravity theories. In Section 1.2 we discuss how although we lack a full theory of quantum gravity we can still make quantitative predictions in a low energy limit. In Section 1.3 we provide a general introduction to the Feynman-Vernon influence functional, the primary tool for understanding decoherence in the path-integral formalism. In Section 1.4 we discuss the additional subtleties in formulating an influence functional for gauge theories. In Section 1.5 we discuss some subtleties regarding boundary terms in quantum gravitational path integrals. In Chapter 2 we introduce the gauge theories which we will be studying. En route to computing the influence functional, we compute the gauge invariant propagator for these gauge theories using a modification of the FP trick applied to transition kernels. It must be understood that we are using the word propagator to mean the amplitude for the system to evolve from one configuration to another configuration in a given time, *i.e.* the Feynman transition kernel. The word propagator in field theory has become synonomous with correlator, but it should be clear that we are not discussing correlation functions. For

[1]By QED we are not specifically referring to the theory involving Dirac fermions coupled to a $U(1)$ gauge field but rather a quantum theory of a generic not-yet-specified matter which is minimally coupled to a $U(1)$ gauge field. Of course this includes Dirac fermions but also *e.g* charged scalar fields and point particles.

instructive purposes we'll first show how such a computation is done for free U(1) theory, then for QED, and ultimately for linearized gravity with matter. In Chapter 3 we compute the influence functional for both QED and linearized gravity at finite temperature. We also discuss the result one would obtain if the gauge invariance of these theories is not properly treated and show that this matches a result reported in recent literature [15]. All of these discussions and results are summarized in the conclusion, Chapter 4. Supplementary materials are presented in the appendices.

Throughout we will use units in which ($\hbar = c = k_{\rm B} = 1$), and a mostly positive metric convention ($-+++$). We will use the shorthand notation $\int_{t_i}^{t_f} d^4x = \int_{t_i}^{t_f} dt \int d^3x$. The letters $i, f$ will be reserved to label initial and final quantities. They will never be used as indices. We will use Greek letters for spacetime indices and Latin letters in the middle of the alphabet (except $i, f$) for spatial indices *e.g.* $(j, k, l)$. Latin letters at the start of the alphabet *e.g.* $(a, b, c)$ will be used to label miscellaneous discrete quantities. This being said, we will often suppress spacetime indices as well as the arguments of functions to avoid cluttered expressions. It should be clear from the context which objects are vectors and tensors.

## 1.1 Previous Approaches to Gravitational Decoherence

Gravitational decoherence is a massive topic which receives both theoretical and experimental interest from communities ranging from quantum information to opto-mechanics to quantum cosmology. The idea that gravitation may be responsible for the quantum to classical transition was present and discussed in the early Sixties. Comments can be found in the Feynman lectures on gravitation [16] as well as in the work of Rosen [17]. The first actual model was soon after proposed by Karolyhazy [6]. Of course, since we lack a full theory of quantum gravity, our understanding of gravitational decoherence is still incomplete and it remains an actively studied topic.

In this section we will attempt to provide a bit of an overview of the different ways in which gravitational decoherence has been studied. Our aim is not to provide a comprehensive review but rather to help situate our work in the appropriate historical context. More detailed summaries of previous works are provided in [14, 18–20].

### 1.1.1 Decoherence in Modified Quantum Theories

As previously mentioned, theories incorporating both quantum mechanics and gravity can separated into two groups: those which hold QM sacred, and those willing to violate QM at macroscopic scales. Of the theories which violate QM, there are different approaches distinguished by how gravitation enters the theory.

The earliest ideas of this kind, as well as those popularized by Penrose involve taking the Heisenberg uncertainty relations for quatum systems living on a spacetime manifold and understanding this as leading to an inherent fuzziness to spacetime. This fuzziness leads to the loss of phase relations in the different branches of the matter wavefunction and thus a pure state evolves into a mixed state [6, 7]. In these theories decoherence is an intrinsic process in nature.

Another class of theories assume that gravity is inherently classical but QM is necessarily modified. In particular Mielnik [21] emphasized that, 'either the gravitation is not classical or quantum mechanics is not orthodox'. These theories view QM as a linear approximation to a more complicated non-linear theory where the non-linearities arise from gravitation [8–10]. This leads to a modification to the Einstein field equation where the RHS of the equation is replaced by the expectation value of the stress-energy tensor operator. The non-linear nature of these theories leads to a dynamical collapse of the wavefunction which proceeds most rapidly for macroscopic objects. While this is not necessarily decoherence, the inability to sustain macroscopic superpositions is a feature common to both decoherence models and collapse models. Without proper attention to detail an observation of dynamical collapse could be misinterpreted as environmental decoherence.

In another approach the graviational field is treated as a classical fluctuating stochastic variable. This is an example of a stochastic collapse theory. Stochastic collapse theories propose that some new universal mechanism is responsible for suppressing macroscropic quantum fluctuations [22]. They have been well studied outside of a gravitational context as a potential means for understanding macrorealism and the collapse of the wavefunction (see [19] and references therein). The idea that the underlying mechanism may be related to gravitation was popularized by Diósi [11, 12]. In these theories the corresponding evolution equation for the matter density matrix is no longer the unitary von Neumann equation, but instead it is a Markovian master equation describing the decoherence of pure states. Diósi's approach is limited for a number of reasons, the most obvious of which being that it is not relativistic. In addition the theory has been criticized by Ghirardi *et*

*al* who claim it is not internally consistent [13].

The final intrinsic decoherence theory we'll describe is that of correlated worldlines (CWL theory) [14]. In CWL theory the linearity of QM is broken in the path integral. In a conventional path integral computation one sums up all paths (independent of one another) which a system can take to evolve from one configuration to a later configuration. In CWL theory it is assumed that the conventional path integral is the lowest order term in an infinite series of multiple path integrals. The second order term includes a sum over all pairs of paths, the third order goes over all triplets, and the n-th order over all n-tuplets of paths. What distinguishes CWL theory from the conventional path integral is that the paths are no longer independent from one another, they are correlated. Arguments based on the equivalence principle and indistinguishability in QM suggest that the correlations are generated by gravitation. This leads to "path-bunching" where paths "gravitate" towards each other. This suppresses quantum fluctuations for macroscopic objects and thus predicts a quantum-to-classical crossover. This too is a theory without decoherence, but again the experimental signature of this crossover may be misattributed to environmental decoherence if we do not take care to understand both effects well.

It is clear then that we can no longer naively attribute decoherence in experiments to dirt/noise. Experimental signatures of the above theories may already be lying in our data, but we don't yet understand conventional gravitational decoherence well enough to interpret the data and place bounds on these theories. Since the Planck energy scale is far beyond present the day energy frontier we cannot afford to dismiss potential experimental data which probes theories of quantum gravity. In the next section we will review some efforts to understand and model decoherence in conventional quantum gravity.

### 1.1.2 Decoherence in Conventional Quantum Theory

Without a full theory of quantum gravity, the various approaches to decoherence in conventional quantum gravity have been rather *ad hoc*. Assumptions must be made about how to correctly describe spacetime, and many different choices have been made. In most of the approaches we will discuss the environment consists of small fluctuations about a classical background spacetime metric (typically Minkowski, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$). Without fixing a classical background it is unclear how to even describe the central system—how can we possible describe a particle in a superposition of two locations in a coordinate-independent way [7]? It has been argued that diffeomor-

phism invariance is so strict that there are no local observables in quantum gravity; the only observables are defined on the boundary of the spacetime manifold [23, 24]. This is exemplifed by the ADM mass being a boundary integral [25]. Indeed this is the spirit of the holographic principle [26, 27], and is realized by the AdS/CFT correspondence [28–30]. Until we further develop our understanding of quantum gravity we must assume a classical background on which the central system and gravitational fluctuations can live. This being said, since the gravitational fields created in any man-made experiment will certainly be in the weak-field regime we can hypothesize that this assumption is reasonable for all practical purposes.

Before we discuss the many approaches which are similar to ours (considering only small spacetime fluctuations) its worth mentioning some more speculative ideas which do not involve a static classical background metric. It was hypothesized that even conventional quantum gravity is inherently non-unitary. This was an idea which recieved a lot of attention from Hawking as a potential resolution to the black hole information paradox. Some of his ideas included: thermalization due to the formation and evaporation of black holes [5], metric fluctuations destroying global hyperbolicity [31], and the branching off of "baby universes" connected to ours by wormholes [32–34]. It is notable that the wormhole computations suggest a crossover mass scale above which macroscopic objects would rapidly decohere, but this calculation was not performed within a controlled approximation and thus its legitimacy is questionable.

Recent attention has been given to the idea that even static classical gravitational fields may cause quantum decoherence. This idea was proposed by [35] and has since been a hotly debated topic [36–40] (see also [20] and refs. therein). The basic setup for this idea is a single object consisting of many microscopic constituents (for example a large molecule). The degrees of freedom of the system decompose into center-of-mass motion and internal excitations. The claim in [35] is that the center-of-mass motion becomes coupled to the internal degrees of freedom due to the gravitational time dilation felt by the internal modes based on the trajectory of the center-of-mass. The internal modes act as "internal clocks" and it is argued that if these clocks register different proper time durations then they can no longer constructively interfere. The gravitational field does not act as the environment but rather the internal modes constitute the environment while the center-of-mass degree of freedom is the central system. This is an interesting idea, but since it is still hotly debated (and a different flavour than the gravitational decoherence we are interested in) we will not have more to comment on it.

In the more common approaches a classical metric is fixed (typically Minkowski) and the effect of linearized metric fluctuations on a central system are studied. Such fluctuations occur in classical gravity where one can assume that astrophysical or cosmological processes produce a stochastic background of classical gravitational waves [41, 42]. More interesting to the quantum gravity community is the effect of quantum fluctuations associated with the zero-point motion of the metric field. The quantum fluctuations have been modelled in a number of different ways. Many have treated the metric perturbation as a stochastic variable with a gaussian probability distribution. This was done for a non-relativistic particle coupled to fluctuations of the conformal factor in [43], *i.e.* the scalar $\phi$ in the metric expansion $h_{\mu\nu} = (2\phi + \phi^2)\eta_{\mu\nu}$. Such a metric ansatz does not satisfy Einstein's equations, but this was soon after remedied by including shearing modes [44]. Isotropic pertubations were well studied, and it was shown that a perturbation of the form $h_{ij} = \xi(t)\delta_{ij}$ with mean $\langle\xi(t)\rangle = 0$, $\langle\xi(t)\xi(t')\rangle = M_P^{-1}\delta(t-t')$ would lead to decoherence into the energy basis with a rate $\Gamma \sim (\Delta E)^2/M_P$ [45, 46]. Here $\Delta E$ is the difference in energy of the two states under consideration in the off-diagonal element of the density matrix. In all of these models the correlation functions for the noise were not derived but simply assumed. Their models contained free parameters which were identified with the Planck constants of appropriate units (mass, time, etc.). In the papers of Ford [47, 48], care was taken to perform a proper QFT computation of the transverse-traceless graviton thermal corrleation function for use in a stochastic model. The novelty in this model is that the quantum particle was not coupled directly to the gravitons but rather the walls confining the particle were. The background fluctuations then led to fluctuations in the size of the container causing decoherence for the particle inside. Ford predicts a decoherence rate arising from interaction with a graviton bath at temperature $T$, $\Gamma \sim T(\Delta E/M_P)$. This depends only linearly on the energy difference and, in contrast to the previous computations, it vanishes at zero temperature.

The above models attempted to describe the quantum fluctuations of the metric by treating it as a classical fluctuating variable. This can provide some intuition but to truly directly understand quantum gravitational effects we need to treat the metric perturbation quantum mechanically. This was done in [15, 49–53]. In each of these references a decoherence rate for a sample matter system was either computed or estimated. It is difficult to directly compare the rates though because the different authors considered quite different matter systems, *e.g.* massive scalar coherent states, point particles in one-dimension, a point particle in an interferometer, and photon

wavepackets. We will later see that our computation is sufficiently general that all of these examples could in principle be studied.

In each reference the computation was done differently and not all results agree quantitatively or even qualitatively. We'll have more to say in Chapter 3 about specifically how the results disagree and why this arises. In all cases the Einstein-Hilbert action is linearized and the metric perturbation is quantized either using the path-integral or canonical quantization. The initial state of the metric pertubation and matter is assumed to be uncorrelated. In all but [49] (where the initial state is vacuum) the metric perturbation is assumed to be thermal. In references [15, 49, 50, 52] the path-integral is used and the effect of the environment is captured in an influence functional. The central system in [49, 50] is restricted to be a collection of non-relativistic point particles. In [52] the generalization is made to general matter coupling to the metric perturbation via the stress-energy tensor $\mathcal{L}_{\text{int}} \propto h_{\mu\nu}T^{\mu\nu}$. In each of [49, 50, 52] transverse-traceless (TT) gauge is assumed altough this is a gauge choice which cannot be consistently made in the presence of matter [54]. Blencowe [15] chose an initially thermal state and used the harmonic gauge fixing term (a valid choice), however there was a drastic overcounting of the gravitational degrees of freedom. Although the TT gauge choice cannot be consistently made in the presence of matter it still remains true that the two TT polarizations are the only independent degrees of freedom in the metric perturbation; the remaining components of $h_{\mu\nu}$ are constrained variables. We will have much more to say about this in the upcoming chapters, but for now it suffices to say that there has not yet been a satisfactory quantum mechanical derivation of the influence functional describing a thermal bath of gravitons. Certainly at high temperatures this should be well approximated by the classical results, but a proper QM calculation would allow one to interpolate between quantum and classical regimes.

In the other two references [51, 53] the metric perturbation is quantized canonically. They both make use of the ADM $(3+1)$ decomposition to describe the time evolution of a spatial 3-metric $h_{ij}$ [25]. In the ADM formalism (even at the linear level) one must explicitly deal with the constraints of general relativity. In [51] a gauge is fixed and the constraints are imposed on the field operators, whereas in [53] they use Dirac constrained system formalism [55, 56] to implement the constraints on the states and describe the dynamics in a manifestly gauge invariant way. In both cases the constraints are properly treated and as a result only the two independent graviton degrees of freedom of the metric act as an environment. They both assume an initially thermal state for the gravitons, and they both compute

a master equation describing the evolution of the reduced density matrix for the matter. It was not clear whether the results of [51] were gauge invariant, but because the Dirac formalism is manifestly gauge invariant there is no concern about the results derived in [53].

From the proper treatment of the constraints of the theory one learns that it is not enough for a decoherence model to be relativistic. To quote [51], since the Hamiltonian and momentum constraints of GR generate gauge transformations which correspond to temporal and spatial reparameterizations, "*Any postulate of dynamical or stochastic fluctuations that correspond to space and time reparameterizations conflicts with the fundamental symmetries of GR*". This comment has direct bearing on all of the modified QM theories, the convential theories with fluctuations modelled as stochastic variables, as well as the fully quantum theories. In particular this statement indicates that the results of Blencowe [15] cannot be correct since their result was obtained by integrating over fluctuations of all components of the metric perturbation $h_{\mu\nu}$. In Section 3.3 we explicity show how Blencowe's result is obtained if the constraints/gauge-invariance of theory are not properly accounted for.

It then seems as if the problem is solved and all that needs to be done is to analyze the master equation derived in [53]. In principle this may be correct but their master equation cannot be solved in general and it is not immediately obvious how to implement different approximations. This is why an influence functional is powerful—the path integral approach offers new approximation schemes (semiclassical, eikonal, etc.). This is why we are interested in computing the gravitational influence functional in a way which satisfies the constraints of the theory and is manifestly gauge invariant.

## 1.2 Quantum Gravity as a Low Energy Effective Field Theory

As previously mentioned, both string and loop approaches predict modifications to GR at high energies. If we assume quantum theory holds at all scales then there is a simple argument to suggest a breakdown of GR at high energies. Einstein's theory of general relativity is described by the Einstein-Hilbert action[2]

$$S_{EH} = \frac{M_P^2}{2} \int d^4x \sqrt{|g|}\, R, \tag{1.1}$$

[2]This excludes boundary terms. Such terms do not modify the current discussion but we will later discuss how they are essential for a number of other reasons.

where $M_P = (8\pi G)^{-1/2}$ is the Planck mass, $g$ is the determinant of the metric tensor, and $R$ is the Ricci scalar. This action is extremized by a spacetime metric satisfying Einstein's equation. Classical general relativity has been extraodinarily well tested (see [57–59] and references therein). Indeed the recent discovery of gravitational waves emitted from a black hole merger event has verified a long standing prediction of classical general relativity [60] and has opened up a new paradigm in strong-field testing [61]. One can choose to describe the metric in terms of a deviation from the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_P$, and expand the Einstein-Hilbert action as an infinite series in $h_{\mu\nu}/M_P$. The result is the gauge theory of a massless spin-2 particle (graviton) with an infinite number of interaction terms [62]. Every term with $n$ factors of $h$ will be multiplied by $M_P^{2-n}$. The lowest order terms in the expansion $n = 2$ have no powers of the Planck mass and simply look like standard kinetic terms for a relativistic field $\mathcal{L}_0 \sim \partial h \partial h$. In principle one can take this theory and try to perform standard perturbative quantum field theory (QFT) computations using Feynman diagrammatics [63–65]. This is straightforward until loop diagrams are considered and one must contend with the fact that the theory is non-renormalizable.

As in perturbative treatments of other quantum field theories the various ultraviolet divergences that arise from loop diagrams are absorbed into renormalized coupling constants and physical observables can be computed in terms of these renormalized couplings. Of course the value of the renormalized couplings must still be measured experimentally. In a renormalizable theory all ultraviolet divergences can be absorbed into a finite number of renormalized coupling constants, and only a finite number of experiments need to be performed before the theory can make unambiguous predictions. Perturbative quantum GR is a non-renormalizable theory though; to account for all ultraviolet diverences an infinite number of counterterms must be added, and thus infinitely many experiments must be performed to determine all of the the renormalized couplings [66, 67]. These counterterms extend beyond the Einstein-Hilbert action, in fact they include higher powers of the Riemann tensor. Naively this spells disaster for the theory, suggesting it has no predictive power.

One resolution to this apparent problem is quite simple; we should understand Einstein gravity as an effective field theory which is a low energy approximation of an unknown microscopic "UV complete" theory. The appropriate microscopic variables may be strings or loops for example. Recall that higher order interaction terms in the Lagrangian were multiplied by larger and larger powers of $M_P^{-1}$. At low energies $E \ll M_P$ an $n$th order interaction term $\mathcal{L}_{\text{int}} \sim h^n$ is suppressed by a factor $(E/M_P)^{n-2}$. From di-

mensional grounds the same should be true of the higher order $\mathcal{O}(R^n)$ terms parameterized by renormalized coulping constants. The renormalized coupling constants should have the form of a dimensionless constant multiplying an appropriate power of $M_P^{-1}$ where the power is determined by the number of derivatives appearing in the term. Since the derivatives quantify the energy scale of a given process, we see that the higher order terms are supressed by higher and higher order powers of $E/M_P$. Although quantum GR is perturbatively non-renormalizable, the infinite number of undetermined renormalized couplings are all coefficients of interaction terms which are highly suppressed at low energies. There has been significant work done studying quantum GR as an effective field theory [68–72]. In principle one can make predictions accurate to any finite order $\mathcal{O}\left((E/M_P)^n\right)$ and only need to measure a finite number of coupling constants. Since $M_P \sim 10^{18}\,\text{GeV}$ is an extraordinarily high energy in particle physics contexts these corrections are indeed very small. Thus as a low energy effective theory perturbative quantum gravity regains its predictive power. This approach is ignorant of the details of the underlying high energy theory, which appear only in the values of the renormalized couplings. Of course one can immediately see that at the Planck scale $E \sim M_P$ every interaction term becomes relevant and the theory loses predictive power. This is the regime in which string theory or loop quantum gravity may become the appropriate description of nature.

A UV complete theory would be necessary to describe situations in which the classical theory predicts extreme curvature, *i.e.* near black hole or cosmological singularities. Such extreme scales are however infeasible for man-made experiments. As a result, if QM holds at all scales then the previously mentioned table-top quantum gravity experiments should be very well described by the effective field theory of quantum gravity. The lowest order effective field theory predictions, involving no additional renormalized couplings should be universal and independent of the UV completion of the theory[3].

## 1.3 Feynman-Vernon Influence Functional

The primary tool for studying decoherence in the path-integral formalism is the Feynman-Vernon influence functional (IF) [73]. In this section we will review the derivation of a general IF. In the next section we'll discuss why

[3] At this order the cosmological constant is also relevant but we will assume it is negligibly small.

this simple derivation may be complicated in a gauge theory such as QED or gravity. The primary result of this thesis is a resolution to these difficulties.

In what follows we will necessarily use density matrices rather than wavefunctions. When describing density matrices $\rho(\phi;\phi')$ the number of variables doubles but the expressions often contain similar/identical factors for the primed and unprimed variables. It is convenient to use the following condensed notation coming from the Schwinger-Keldysh formalism [74, 75]. For a generic functional $f$ we write $f[\underline{\phi}] \equiv f[\phi;\phi']$. The action will always have the special form $S[\phi;\phi'] = S[\phi] - S[\phi']$. For path integrals we write

$$\int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi} \equiv \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{\phi'_i}^{\phi'_f} \mathcal{D}\phi'. \tag{1.2}$$

Consider a bipartite quantum system. The two interacting quantum subsystems will referred to as the "central system" and the "environment". The central system will have states labelled by $\phi$ and the environment by $X$. These are collective variables written in compact notation, both the system and environment can in principle have many degrees of freedom. The state of the central system and environment at time $t_i$ is described by the density matrix

$$\rho(\underline{\phi}_i, \underline{X}_i). \tag{1.3}$$

Given the initial data, the density matrix at a later time $t_f \equiv t_i + T$ can be determined,

$$\rho(\underline{\phi}_f, \underline{X}_f) = \sum_{\underline{\phi}_i \underline{X}_i} \mathcal{K}(\underline{\phi}_f, \underline{X}_f; \underline{\phi}_i, \underline{X}_i)\rho(\underline{\phi}_i, \underline{X}_i), \tag{1.4}$$

where the density matrix propagator is

$$\mathcal{K}(\underline{\phi}_f, \underline{X}_f; \underline{\phi}_i, \underline{X}_i) \equiv K(\phi_f, X_f; \phi_i, X_i)K^*(\phi'_f, X'_f; \phi'_i, X'_i). \tag{1.5}$$

The kernel $K$ is just the usual propagator for the system and environment, i.e.

$$K(\phi_f, X_f; \phi_i, X_i) = \langle \phi_f, X_f; t_f | \phi_i, X_i; t_i \rangle \tag{1.6}$$

Supposing only the central system was of observational interest and the environment represented some unobserved degrees of freedom, the central system can be fully described by the reduced density matrix obtained by tracing over the environmental degrees of freedom

$$\rho_\phi(\underline{\phi}_f) \equiv \sum_{X_f} \rho(\phi_f, X_f; \phi'_f, X_f). \tag{1.7}$$

If the central system and environment are initially uncorrelated the total density matrix factorizes,

$$\rho(\underline{\phi}_i, \underline{X}_i) = \rho_\phi(\underline{\phi}_i)\rho_X(\underline{X}_i). \tag{1.8}$$

As time proceeds the interactions between the two will generally lead to correlations. The evolution of the system's reduced density matrix is then generally non-unitary since information describing the correlations with the environment is lost when the trace is performed. We can describe this non-unitary evolution of the system's reduced density matrix using an effective reduced density matrix propagator. The reduced density matrix evolves in the usual linear way

$$\rho_\phi(\underline{\phi}_f) = \sum_{\underline{\phi}_i} \underline{\mathcal{K}}_\phi(\underline{\phi}_f; \underline{\phi}_i)\rho_\phi(\underline{\phi}_i), \tag{1.9}$$

and the reduced density matrix propagator is

$$\underline{\mathcal{K}}_\phi(\underline{\phi}_f; \underline{\phi}_i) \equiv \sum_{\underline{X}_f \underline{X}_i} \delta(X_f - X'_f)\mathcal{K}(\underline{\phi}_f, \underline{X}_f; \underline{\phi}_i, \underline{X}_i)\rho_X(\underline{X}_i). \tag{1.10}$$

Using the path integral representation of the propagator we can proceed to write a path integral representation for the reduced density matrix propagator. We will assume the system and environment can be described by an action of the form

$$S[\phi, X] = S[\phi] + S[X] + S_{\rm int}[\phi, X]. \tag{1.11}$$

The propagator for the system is then

$$K(\phi_f, X_f; \phi_i, X_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{iS[\phi]} \int_{X_i}^{X_f} \mathcal{D}X\, e^{iS[X]+iS_{\rm int}[\phi,X]}. \tag{1.12}$$

The reduced density matrix propagator can then be written as

$$\underline{\mathcal{K}}_\phi(\underline{\phi}_f; \underline{\phi}_i) = \int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi}\, e^{iS[\underline{\phi}]}\mathcal{F}[\underline{\phi}], \tag{1.13}$$

where the entire effect of the environment on the system is contained within the functional $\mathcal{F}[\underline{\phi}]$ called the influence functional [73]. The influence functional has the path-integral expression

$$\mathcal{F}[\underline{\phi}] \equiv \sum_{\underline{X}_f \underline{X}_i} \delta(X_f - X'_f)\, \rho_X(\underline{X}_i) \int_{\underline{X}_i}^{\underline{X}_f} \mathcal{D}\underline{X}\, e^{iS[\underline{X}]+iS_{\rm int}[\underline{\phi},\underline{X}]}. \tag{1.14}$$

Note that the double path integral is just a product of propagators for the environment subject to a "frozen" background configuration of the central system. Without specifying the system dynamics $S[\phi]$ one can still evaluate the influence functional so long as the interaction $S_{\rm int}[\phi, X]$ and the environmental dynamics $S[X]$ are known.

A initially thermal environment is commonly used, and is in fact the one we will use in the next section. It will be useful to have an explicit expression for the thermal density matrix. A thermal density matrix corresponding to a canonical ensemble at temperature $\beta^{-1}$ has the eigenfunction expansion

$$\rho_\beta(\underline{X}) = \sum_n e^{-\beta E_n}\psi_n(X)\psi_n^*(X'), \tag{1.15}$$

where the $\psi_n(X)$ are energy eigenfunctions of the Hamiltonian for the environment alone. Of course this should be divided by the partition function to ensure the density matrix is properly normalized. To keep the notation compact we will not explicitly write overall normalizations. Ultimately the normalization of the influence functional is determined by the fact that it must equal 1 in the limit that the coupling vanishes. Compare the above thermal density matrix with the eigenfunction expansion of the propagator for the environment

$$K(\underline{X}) = \sum_n e^{-iE_nT}\psi_n(X)\psi_n^*(X'). \tag{1.16}$$

The thermal density matrix can then be seen as the analytic continuation of the propagator to imaginary times if we make the identification $T = -i\beta$. Analytically continuing the path integral representation of the propagator to imaginary times we can then write the path integral representation for the thermal density matrix

$$\rho_\beta(\underline{X}) = \int_{X'}^{X} \mathcal{D}X''\, e^{-S_E[X'']}, \tag{1.17}$$

where $S_E = -iS|_{T=-i\beta}$ is the Euclidean action. The integration is now over paths from $X' \to X$ in imaginary time $T = -i\beta$. The influence functional for an initially thermal environment can then be written as the multiple path integral

$$\mathcal{F}[\underline{\phi}] = \sum_{\underline{X}_f\underline{X}_i} \delta(X_f - X_f') \int_{\underline{X}_i}^{\underline{X}_f} \mathcal{D}\underline{X} \int_{X_i'}^{X_i} \mathcal{D}X'' e^{iS[\underline{X}]+iS_{\rm int}[\underline{\phi},\underline{X}]-S_E[X'']}. \tag{1.18}$$

In many cases of interest the path integrals cannot be evaluated exactly and approximate techniques must be used. There is one case of particular relevance for which the influence functional can be evaluated exactly. This is the case that the environment is described by a free action $S[X]$ which is quadratic in $X$ and an interaction which is linear in $X$, $\mathcal{L}_{\rm int}[\phi, X] = Xg[\phi]$, for some functional of the system $g[\phi]$. Assuming an initial state which is gaussian, *e.g.* a vacuum or thermal state, both the path integrals and the boundary integrals are gaussian and the influence functional can be evaluated exactly. For a system minimally coupled to a $U(1)$ gauge field, thermal photons provide an environment precisely fitting the above criteria. An analogous computation will show that the same is true in linearized Einstein gravity.

## 1.4 Difficulty Defining an Influence Functional in a Gauge Theory

Einstein gravity is a beautifully geometric theory. As motivated by the equivalence principle, the theory is coordinate independent. Indeed the Einstein-Hilbert action Eq. (1.1) is invariant under diffeomorphisms[4] and thus there is a redundancy in our description of the spacetime manifold. Einstein gravity is then a gauge theory since two metrics which are related by a coordinate transformation are considered physically equivalent. In this section we will discuss how the above discussion of the influence functional becomes complicated in gauge theories.

The above general discussion of the influence functional required a clear divide between system $\phi$, and environment $X$. In gauge theories, this divide is not always straightforward because gauge theories are constrained theories. One says a theory is constrained if its Lagrangian $L(\phi, \dot{\phi})$ is singular, *i.e.* the generalized velocities $\dot{\phi}^a$ cannot be expressed in terms of the canonical momenta $\pi_a = \frac{\partial L(\phi,\dot{\phi})}{\partial \dot{\phi}^a}$ [55]. Classically this implies the existence of a set of primary constraint equations relating the canonical variables,

$$\psi_m(\phi^a, \pi_a) = 0, \tag{1.19}$$

$m = 1, 2, ..., N$. The number $N$ of primary constraints depends on the theory. Consistent time evolution, $\dot{\psi}_m = 0$, may require that additional "secondary" constraints be appended to the original set of primary constraints. The secondary constraints are treated on the same footing as the primary,

[4] This is only true up to a boundary term which will be further discussed in Section 1.5.

so this process may be repeated and further tertiary constraints may be required to ensure the secondary constraints are held throughout time evolution. This process is to be repeated until a complete set of consistent constraints is obtained.

The existence of constraints implies that not all of the canonical variables are independent. Thus in a constrained theory we cannot always naively partition the system into a central system and environment where some of the generalized coordinates describe the central system and the remaining coordinates describe the environment. A valid partition must respect the constraints. A basic example of this is a particle constrained to move on a surface defined by $z = f(x, y)$. Obviously we could not divide the coordinates $x, y, z$ of the particle into a central system $x, y$ and environment $z$ because knowledge of the state of the central system would uniquely determine the state of the environment. Even if our measurement apparatus could only see the shadow of the particle on the $x, y$ plane and not its height $z$, we could uniquely determine $z$. Although the Lagrangian may contain the generalized coordinate variables $x, y, z$ and $\lambda$ (the Lagrange multiplier) it would be completely incorrect to treat them all as independent quantum operators unless a restriction is placed on the Hilbert space. Of course this is a trivial example but it is a useful reminder of this essential idea.

In a theory of a gauge field interacting with matter, the constraints are such that the state of the gauge field is fundamentally correlated with the state of the matter. We will see how this comes about shortly. For now we can see that in order to define an influence functional in a gauge theory one must first identify the independent variables. Only then can a partition be made between matter and environment and a partial trace performed.

The above considerations were general for constrained theories however gauge theories are a particular subset of constrained theories. In a gauge theory there is the additional complication that there is a redundancy in the description of the system. The degrees of freedom which are invariant under gauge transformation are called "physical" whereas the remaining degrees of freedom are considered "unphysical". The two approaches to studying gauge theories are

1. Work in the full (extended) phase space but understand that observables must be gauge invariant quantities.

2. Impose a gauge condition which restricts the system to a reduced phase space in which every degree of freedom is physical.

We will choose option 1, and work in the extended phase space with

manifestly gauge invariant objects. This approach is beneficial because it prevents any doubts about the gauge invariance of the results.

## 1.5 Quantum Gravity Path Integral - Boundary Terms

### 1.5.1 Gibbons-Hawking-York Boundary Term

Before we can proceed to the quantize the linearized metric perturbations there is a technical note which must be made about the Einstein-Hilbert action Eq. (1.1); the Ricci scalar contains second derivatives of the metric. This causes a problem which may seem purely academic at the classical level, but will be very disruptive quantum mechanically.

To see how the problem arises classically consider a Lagrangian containing only first derivatives $\mathcal{L}(\phi, \partial_\mu \phi)$. After integration by parts the variation of the action is

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \oint_\Sigma dS^\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi, \tag{1.20}$$

where $\Sigma$ is the boundary of the system. The boundary term which is generated by the necessary integration by parts depends linearly on the variation at the boundary. One can assume the variation vanishes along the boundary (Dirichlet conditions) and obtain a well defined functional derivative $\delta S/\delta\phi =$ (equation of motion). Now consider a Lagrangian depending on second derivatives $\mathcal{L}(\phi, \partial^2\phi)$. After integration by parts the variation of the action is

$$\delta S = \int d^4x \left( \frac{\partial \mathcal{L}}{\partial \phi} + \partial^2 \frac{\partial \mathcal{L}}{\partial(\partial^2 \phi)} \right) \delta\phi - \oint_\Sigma dS^\mu \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial^2\phi)} \delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial^2\phi)} \partial_\mu \delta\phi \right). \tag{1.21}$$

Now although the bulk term depends only on the the variation $\delta\phi$, the boundary term depends both the variation as well as the derivatives of the variation normal to the boundary surface. To fix both $\delta\phi$ and $\partial_\mu\delta\phi$ equal to zero on the boundary would be equivalent to imposing both Dirichlet and Neumann conditions. This cannot be consistently done if the equation of motion is a second order differential equation. Thus the action is not functionally differentiable, *i.e.* $\delta S/\delta\phi \neq$ (equation of motion). Classically this feels like a purely academic problem because the equations of motion are unaffected by the addition of a total derivative term to the Lagrangian. A Lagrangian linear in $\partial^2\phi$ can be made into a Lagrangian quadratic in $\partial_\mu\phi$

by the addition of a suitable total derivative term. For example, there is usually no concern when one writes the free scalar Lagrangian $\mathcal{L} = \frac{1}{2}\phi\partial^2\phi$ since it is obviously equal to $\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ up to the addition of a total derivative $\frac{1}{2}\partial_\mu(\phi\partial^\mu\phi)$.

At the quantum level, one is not free to simply add total derivatives to the Lagrangian without affecting the results of the computation. Suppose we had two Lagrangians $\mathcal{L}_1$ and $\mathcal{L}_2$, where $\mathcal{L}_1$ depended on first derivatives, $\mathcal{L}_2$ on second derivatives, and the two are related by a total derivative $\mathcal{L}_2 - \mathcal{L}_1 = \partial_\mu B^\mu$. Classically one would not distinguish between the two since if one simply ignores boundary terms they produce the same equation of motion. However in quantum mechanics one obtains different results depending on which action they choose. Suppose one started with a classical theory defined by the Lagrangian $\mathcal{L}_2$. Naively quantizing the theory one would define the propagator as

$$K(\phi_f;\phi_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{i\int_\mathcal{V}\mathcal{L}_2} = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{i\oint_{\partial\mathcal{V}} dS_\mu B^\mu}\, e^{i\int_\mathcal{V}\mathcal{L}_1}. \tag{1.22}$$

Although the bulk integrations are unchanged in the choice between $\mathcal{L}_1$ and $\mathcal{L}_2$, *i.e.* the stationary phase path is the same in both cases, the phase accumulated along the same path will depend on which Lagrangian was chosen.

How then are we to decide which of the two Lagrangians is the appropriate one to use to define the propagator for the quantized theory[5]? This question was addressed in quantum gravity by York as well as Hawking and Gibbons [76–78]. The linearity of quantum theory implies that propagators have the convolution property $K(c;a) = \int dbK(c;b)K(b;a)$. If the action along a path between $\phi_a$ and $\phi_b$ is denoted $S[b,a]$, then this convolution property holds if and only if

$$S[c,a] = S[c,b] + S[b,a]. \tag{1.23}$$

If the action depends on the normal derivative on the boundary then it does not satisfy this condition. Taking for example the constant time surfaces $t_{a,b,c}$. If $\phi_{ba}(t)$ is a path between $t_a$ and $t_b$, then we can impose the condition that both $\phi_{ba}(t_b) = \phi_b$ and $\phi_{cb}(t_b) = \phi_b$, however for general paths the time derivatives will not agree at $t_b$. If the the boundary term in the action depended only on the value of $\phi$ then it is clear that (1.23) would hold whereas if the action depended on the time derivative of $\phi$ on the boundaries

[5]Furthermore, in general if we have a classical theory defined by a class of equivalent Lagrangians, how are we to decide which is the correct one to use in the propagator?

it would not. As a result, in order to maintain the convolution property of the propagator we must use classical actions containing only first derivatives.

Returning to the problem of interest, since the Ricci scalar contains both first and second derivatives of the metric one must add an appropriate total derivative to the EH Lagrangian to remove the second derivative terms. The appropriate term to add is called the Gibbons-Hawking-York (GHY) term

$$S_{GHY} = \epsilon M_P^2 \oint_\Sigma d^3y \sqrt{|h|}\, K, \tag{1.24}$$

where $\Sigma$ is the boundary hypersurface, $h$ is the determinant of the induced metric on $\Sigma$, $K$ is the trace of the extrinsic curvature $K_{ab}$ of $\Sigma$, and $\epsilon = \pm 1$ depending on whether $\Sigma$ is timelike or spacelike. The correct (containing only first derivatives) action to include in the path-integral for quantum GR is thus

$$S_g = S_{EH} + S_{GHY} = \frac{M_P^2}{2} \int d^4x \sqrt{|g|}\, R + \epsilon M_P^2 \oint_\Sigma d^3y \sqrt{|h|}\, K. \tag{1.25}$$

In the presence of matter one simply adds the covariant matter action[6]

$$S_M = \int d^4x \sqrt{|g|}\, \mathcal{L}_M(\phi, g_{\mu\nu}). \tag{1.26}$$

### 1.5.2 Gauge Transformation Boundary Term

We are interested in making predictions for table-top quantum gravity experiments. Such experiments will be well within the regime of weak curvature (the length scales involved are far larger than the Schwarzschild radius of the system). In this case we can treat the metric as a perturbation about the Minkowsi metric $g_{\mu\nu} = \eta_{\mu\nu} + \frac{2}{M_P} h_{\mu\nu}$, where $h_{\mu\nu}/M_P$ is assumed much smaller than 1. The factor of 2 is a matter of preference. We'll be interested in the time evolution of the system, which in quantum mechanics corresponds to the amplitude to make a transition from an initial state defined on an initial time slice to a final state defined on a later time slice. It is assumed that all fields vanish sufficiently fast at spatial infinity so that we can integrate by parts freely on spatial derivatives without picking up surface terms. The relevant boundary $\Sigma = \Sigma_i \cup \Sigma_f$ then consists of two hypersurfaces of constant time. To lowest order in $h_{\mu\nu}/M_P$ the above action can be

[6] This excludes fermionic matter which couples directly to the connection, as well as non-minimally coupled matter, *e.g.* scalar fields coupling directly to the curvature through terms like $\phi R$.

written as the sum of free terms and an interaction term,

$$S = \oint_{\Sigma} d^3x\, h^{ij}\pi^{(1)}_{ij} - \int_{t_i}^{t_f} d^4x \left( h^{\mu\nu}G^{(1)}_{\mu\nu} - \mathcal{L}_M(\phi, \eta_{\mu\nu}) - \frac{1}{M_P}h^{\mu\nu}T_{\mu\nu} \right), \tag{1.27}$$

where

$$\pi^{(1)}_{ij} \equiv K^{(1)}_{ij} - \delta_{ij}K^{(1)} \tag{1.28}$$

is the linearized conjugate momentum to $h_{ij}$,

$$K^{(1)}_{ij} = \frac{1}{2}(\partial_0 h_{ij} - \partial_i h_{0j} - \partial_j h_{0i}) \tag{1.29}$$

is the linearized extrinsic curvature, and

$$G^{(1)}_{\mu\nu} = \frac{1}{2}(-\partial^2 h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial^\rho\partial_\mu h_{\rho\nu} + \partial^\rho\partial_\nu h_{\rho\mu} - \eta_{\mu\nu}\partial^\sigma\partial^\rho h_{\sigma\rho} + \eta_{\mu\nu}\partial^2 h) \tag{1.30}$$

is the linearized Einstein tensor. Indices are now raised and lowered with the Minkowsi metric, and we use the shorthand notation for the trace $h = h^\mu{}_\mu$. The superscript '(1)' is used to emphasize that these quantities are first-order in $h_{\mu\nu}$. In what follows we will drop the superscript '(1)' since all geometric objects are linearized. The stress-energy tensor $T_{\mu\nu}$ is defined as the right-hand side of Einstein's equation

$$T_{\mu\nu} = -2\frac{\partial\mathcal{L}_M(\phi, g_{\mu\nu})}{\partial g^{\mu\nu}}\bigg|_{g=\eta} + \eta_{\mu\nu}\mathcal{L}_M. \tag{1.31}$$

It must be mentioned that to the same order in $h/M_P$ there is a graviton self-interaction term of the form $\frac{h}{M_P}(\partial h)^2$. This three-graviton vertex term is the lowest order contribution of the infinitely many non-linear graviton self-interaction terms. Loosely speaking we can think of $(\partial h)^2$ as graviton stress-energy and the three graviton interaction term as the metric perturbation coupling to its own stress-energy (as it should according to the equivalence principle). By neglecting this term we are assuming that stress-energy carried by the gravitons is negligible compared to that of the matter. It is not yet clear if this is a valid approximation for the following reasons. This term is responsible for a long-range interaction between gravitons and matter whereas the $hT$ term which we retain is a local interaction. It is the long-range interaction which leads to a pole in the forward direction of $2 \to 2$ graviton-matter scattering amplitude analogous to the pole in Rutherford scattering [79]. Unlike Rutherford scattering where this pole can be ignored due to screening effects at long distances, there are no screening effects in

gravity. It will be the topic of future work to assess the validity of ignoring the 3-graviton vertex in this context. For now we will operate under the hypothesis that for sufficiently small graviton energies this term is negligible.

The full gravitational action is invariant under diffeomorphisms $\xi : x^\mu \to \xi^\mu(x)$, $g_{\mu\nu} \to g^\xi_{\mu\nu}$ which leave the boundaries unchanged [80]. Here $g^\xi$ is defined by

$$g_{\mu\nu}(x) = \frac{\partial \xi^\rho(x)}{\partial x^\mu} \frac{\partial \xi^\sigma(x)}{\partial x^\nu} g^\xi_{\rho\sigma}(\xi(x)). \tag{1.32}$$

This symmetry still holds in the linearized theory so long as $h_{\mu\nu}/M_P \ll 1$ is preserved. These transformations are of the form $x^\mu \to x^\mu + \frac{2}{M_P}\xi^\mu$, $h_{\mu\nu} \to h^\xi_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$ where $\xi$ is of the same order as $h_{\mu\nu}$. Under a transformation of the above form which does not vanish on the constant time surfaces the linearized action changes by a boundary term

$$S \to S - 2 \oint_\Sigma d^3x \, \xi_0 \left( 2G^{00} + M_P^{-1} \mathcal{L}_M \right). \tag{1.33}$$

Classically these terms are irrelevant but as we've previously discussed, boundary terms cannot generally be discarded in quantum theory. Had we not included the GHY term in the gravitational action we would not have obtained this boundary term. We will soon see that this boundary term is essential for the implementation of constraints in the path integral formulation of the theory.

# Chapter 2

# Transition Amplitudes in Gauge Theories

As discussed in the previous section, to define and compute an influence functional for a gauge theory we will need to take care of two additional subtleties. Firstly, because the theory is constrained we must take care distinguishing the central system from the environment so that a partial trace can be performed. Secondly, we must be sure to that the results of our computations are gauge invariant. In what follows we will see that both of these points will be addressed naturally using the path integral representation of the transition kernel.

Ultimately we are looking for an effective propagator for the reduced density matrix of the central system, and it is clear from equations (1.9) and (1.10) that one first needs the propagator for the joint system. One advantage of the path integral formulation is that it uses the Lagrangian rather than Hamiltonian. In the usual analyses of constrained systems one starts with a singular Lagrangian, computes the Hamiltonian, determines the constraints, and then proceeds to quantize the system. However all of the information was contained in the Lagrangian from the start, so there should be an equivalent formulation of the quantum theory involving only the Lagrangian which still correctly handles the constraints.

The first system we will consider is a theory of matter $\phi$, coupled to a $U(1)$ gauge field $A_\mu$. We are using $\phi$ as shorthand notation for the matter variables. If the matter were a complex scalar field $\phi$ would represent both the field and its complex conjugate, if the matter were a Dirac fermion field $\phi$ would represent a Dirac spinor, etc. The second system we will consider is a theory of matter $\phi$ coupled to linearized Einstein gravity (1.27). Although we are ultimately interested in gravity, the calculations will look similar for the two theories so it will be instructive to study QED first.

## 2.1 Quantum Electrodynamics (QED)

The action for a matter system minimally coupled to a $U(1)$ gauge field can be written as

$$S[\phi, A_\mu] = \int_{t_i}^{t_f} d^4x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + \mathcal{L}_M \right), \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, $J^\mu$ is a global $U(1)$ current, and $\mathcal{L}_M$ depends only on the matter variables. The amplitude for the system to evolve from a configuration $(\phi_i, A_i^\mu(\mathbf{x}))$ at time $t_i$ to configuration $(\phi_f, A_f^\mu(\mathbf{x}))$ at time $t_f \equiv t_i + T$ is given by the propagator

$$K(A_f, \phi_f; A_i, \phi_i) = \int_{\phi_1}^{\phi_f} \mathcal{D}\phi \, e^{iS_M[\phi]} \int_{A_i}^{A_f} \mathcal{D}A_\mu \, e^{iS_0[A] + i\int_{t_i}^{t_f} d^4x A_\mu J^\mu}. \tag{2.2}$$

At first glance it is not clear whether this propagator is even well defined. One possible objection is that in the Hamiltonian formalism one sees that $A^0(\mathbf{x})$ is not a canonical variable and thus does not serve as a label for a quantum state. Furthermore, the above path integral is formally infinite for any choice of boundary data. We will see shortly that both of these issues are handled naturally in the path integral formalism when gauge invariance is treated carefully.

### 2.1.1 Free U(1) Theory

Before we compute the propagator for the full system including matter, we will familiarize ourselves with the simpler case of free electrodynamics. In this case the propagator for the free Maxwell field is

$$K(A_f; A_i) = \int_{A_i}^{A_f} \mathcal{D}A_\mu \, e^{iS_0[A]}. \tag{2.3}$$

The propagator is manifestly gauge invariant because the action and measure are gauge invariant. To verify this, consider independent transformations of the boundary data, $A_{i,f} \to A^\Lambda_{i,f} = A_{i,f} + \partial\Lambda_{i,f}$. The transformed propagator is

$$K(A_f^{\Lambda_f}; A_i^{\Lambda_i}) = \int_{A_i + \partial\Lambda_i}^{A_f + \partial\Lambda_f} \mathcal{D}A_\mu \, e^{iS_0[A]}. \tag{2.4}$$

Now simply change integration variables, $A' = A + \partial\Lambda$, where $\Lambda(x)$ satisfies $\Lambda(\mathbf{x}, t_{i,f}) = \Lambda_{i,f}(\mathbf{x})$. In terms of the primed variable the propagator is

$$K(A_f^{\Lambda_f}; A_i^{\Lambda_i}) = \int_{A_i}^{A_f} \mathcal{D}A'_\mu \, e^{iS_0[A'-\partial\Lambda]} = \int_{A_i}^{A_f} \mathcal{D}A'_\mu \, e^{iS_0[A']} = K(A_f, A_i), \tag{2.5}$$

and the propagator is thus invariant under independent gauge transformations on the boundary data.

We can use this gauge invariance to make contact with the canonical formalism. Following [81] we can perform a spectral decomposition of the propagator into energy eigenfunctions

$$K(A_f; A_i) = \sum_n e^{-iE_n T} \Psi_n[A_f] \Psi_n^*[A_i]. \tag{2.6}$$

The wavefunctionals $\Psi[A]$ which comprise $K$ will be called physical states. Since the propagator is invariant under independent gauge transformations of its boundary data, the physical states are gauge invariant $\Psi[A] = \Psi[A^\Lambda]$. We can take this simple equation, and functionally differentiate both sides with respect to $\Lambda$ to obtain

$$0 = -i \left. \frac{\delta \Psi[A^\Lambda]}{\delta \Lambda(x)} \right|_{\Lambda=0} = i\partial_j \frac{\delta \Psi[A]}{\delta A_j(x)} = \partial_j \hat{E}^j(x) \Psi[A]. \tag{2.7}$$

Thus physical states are wavefunctionals in the extended configuration space which satisfy Gauss' law as an eigenvalue equation. Actually this equation was derived by only considering gauge transformations of the spatial components, in the next section we will proceed more carefully and see that the invariance of the physical state under timelike gauge transformations implies the additional constraint $\hat{E}^0 \Psi[A] = 0$. To obtain these equations we used the operator representation of the canonical momentum

$$\hat{\pi}^\mu = \frac{\partial \hat{\mathcal{L}}}{\partial(\partial_0 A_\mu)} \equiv -i \frac{\delta}{\delta A_\mu} \tag{2.8}$$

and the fact that the electric field $E^\mu$ is the (negative of the) conjugate momentum to $A_\mu$. This functional derivative representation can be discussed entirely within the path-integral formalism without passing to a canonical approach (see [82] or Appendix C for details). These physical states are precisely the states which are considered in Dirac quantization [55, 56]. We've thus written down the path-integral representation of the propagator between Dirac's physical states. It is easy to check that this propagator

projects arbitrary sates onto the space of physical states. As a consequence, when evolving an initial state using the transition kernel we can always first project the initial state onto the physical subspace. Thus we can work exclusively with physical states without loss of generality.

The action is quadratic in the field, so the integral should be naturally evaluated by shifting variables $A_\mu = \hat{A}_\mu + \chi_\mu$ where $\hat{A}_\mu$ is a path which extremizes the action while subject to the boundary conditions $\hat{A}^\mu(\mathbf{x}, t_{i,f}) = A^\mu_{i,f}(\mathbf{x})$. This approach does not work though because there is no unique solution to the classical equation of motion subject to these boundary conditions. If $\hat{A}_\mu$ is a solution then $\hat{A}_\mu + \partial_\mu \Lambda$ is also a solution satisfying the boundary conditions so long as $\partial_\mu \Lambda \to 0$ as $t \to t_{i,f}$. As a result the integral is infinite. As identified by Faddeev and Popov (FP) this infinite gauge group volume can be factored out as a constant overall normalization [83]. We will ignore such overall normalizations because the normalization of the influence function will be fixed at the end of the computation anyways. The FP trick is typically used in path integrals which do not have fixed boundary data. A modification of the FP trick can be used if the boundary data is properly treated, *i.e.* integrated over all gauge equivalent configurations, and in fact this procedure implements the constraints which one would find in a Hamiltonian framework. This idea was introduced and used in [80, 81, 84] and we will generalize their approach in two ways. Firstly, we generalize from their particular gauge choice (temporal gauge) to arbitrary gauge fixing functions. Secondly, in the upcoming sections we generalize their results to include gauge fields which are coupled to matter.

We start by multiplying the propagator by

$$1 = \int \mathcal{D}\Lambda \, \Delta(A^\Lambda) \delta(\mathcal{G}(A^\Lambda)), \tag{2.9}$$

where $\Lambda(x)$ is a smooth function vanishing at spatial infinity, $\mathcal{G}(A^\Lambda)$ imposes a gauge condition, and $\Delta(A^\Lambda) = \det \left| \frac{\delta \mathcal{G}(A^\Lambda)}{\delta \Lambda} \right|$ is the associated FP determinant. We can then change variables $A^\Lambda = A'$ write the path integral as

$$K(A_f; A_i) = \int \mathcal{D}\Lambda \int_{A_i^{\Lambda_i}}^{A_f^{\Lambda_f}} \mathcal{D}A_\mu \, \Delta(A) \delta(\mathcal{G}(A)) \, e^{iS_0[A]}. \tag{2.10}$$

Note that we had to use the gauge invariance of the action and measure to obtain this expression. In the standard application of the FP trick (applied to integrals without fixed boundary data) the gauge group volume integral factors out as an overall normalization. In our case the boundary data

ends up depending on the gauge group elements so we cannot immediately factor out this volume. When the propagator is written in this form its gauge invariance is obvious. The prescription for computing the propagator involves first fixing a gauge and evaluating a gauge dependent integral, then integrating the result over all gauge equivalent boundary data.

To proceed it is useful to look at the path integral as the limit of discrete integrals on time slices

$$K(A_f;A_i) = \lim_{N\to\infty} \prod_{n=1}^{N+1} \int d\Lambda(t_n) \int_{A_i^j+\partial^j\Lambda(t_1)}^{A_f^j+\partial^j\Lambda(t_N)} dA^j(t_n) \tag{2.11}$$
$$\times \int_{A_i^0-\frac{1}{\epsilon}(\Lambda(t_2)-\Lambda(t_1))}^{A_f^0-\frac{1}{\epsilon}(\Lambda(t_{N+1})-\Lambda(t_N))} dA^0(t_n)\Delta(A)\delta(\mathcal{G}(A))e^{iS[A]},$$

where $\epsilon \equiv= \frac{t_f-t_i}{N}$. Note that it was necessary to split the vector field into spacelike and timelike components because the gauge transform of the timelike component $A^0$ depends on the time derivative of $\Lambda$. In a path integral a time derivative is defined as the difference of a quantity evaluated on consecutive time slices whereas spatial derivatives are evaluated on a single time slice. We can now see that the gauge group integrations for $n \neq 1, 2, N, N+1$ can be factored out immediately as an overall normalization since the integrand is independent of $\Lambda$ for intermediate times. The only dependence on $\Lambda(t_{N+1})$ and $\Lambda(t_2)$ is in the boundary data for the $A^0$ integral. The integrals over $\Lambda(t_{N+1})$ and $\Lambda(t_2)$ can then be understood as integrals over the boundary data for $A^0$. The two remaining integrals are over $\Lambda(t_N) \equiv \Lambda_f$ and $\Lambda(t_1) \equiv \Lambda_i$. The propagator can then be written as

$$K(A_f;A_i) = \int d\Lambda_f d\Lambda_i \int \mathcal{D}A^0 \int_{A_i^j+\partial^j\Lambda_i}^{A_f^j+\partial^j\Lambda_f} \mathcal{D}A^j\, \Delta(A)\delta(\mathcal{G}(A))e^{iS[A]}. \tag{2.12}$$

We now see that just as a Hamiltonian formalism would suggest, the propagator is independent of $A^0$. It serves as no more than a Lagrange multiplier.

We can take this expression further by decomposing the spatial vector field into longitudinal and transverse components,

$$\begin{aligned} A_j^L &= \frac{\partial_j\partial_k}{\nabla^2}A^k, \\ A_j^T &= \left(\delta_{jk} - \frac{\partial_j\partial_k}{\nabla^2}\right)A^k. \end{aligned} \tag{2.13}$$

We are using the shorthand notation for the Green's function of the Laplacian $\nabla^{-2}(x)$. In this notation it is an integral operator,

$$\nabla^{-2}(x)f(x) = -\int d^3y \frac{f(y)}{4\pi|x-y|}. \tag{2.14}$$

It is obvious that the transverse components are invariant under gauge transformation. The integral over gauge equivalent boundary data is thus an integral over the longitudinal part of the boundary data. We then arrive at the expression for the propagator

$$K(A_f; A_i) = \int \mathcal{D}A_0 \mathcal{D}A^L \int_{A_i^T}^{A_f^T} \mathcal{D}A^T \, \Delta(A)\delta(\mathcal{G}(A))e^{iS[A]}. \tag{2.15}$$

This is precisely the Faddeev formula applied to free $U(1)$ gauge theory [85]. Faddeev derived his formula however by passing first to a canonical Hamiltonian framework and then later constructing the path integral. In our derivation the Lagrangian was used start to finish and the physical degrees of freedom emerged naturally when we integrated over gauge equivalent boundary data. For the specific choice of temporal gauge $\mathcal{G}(A) = A_0$ this formula was obtained in a similar manner by [81].

It is convenient to choose the Coulomb gauge $\mathcal{G}(A) = \partial_j A^j$. In this case both the timelike and longitudinal integrals can be done and their results along with the FP determinant can be factored out as overall constants. The kernel then takes the simple form in terms of only the transverse field components (*i.e.* the radiation),

$$K(A_f; A_i) = \int_{A_i^T}^{A_f^T} \mathcal{D}A^T \, e^{iS[A^T]}, \tag{2.16}$$

where the action for the transverse components is

$$S[A^T] = -\frac{1}{2}\int_{t_i}^{t_f} d^4x (\partial_\mu A_j^T)(\partial^\mu A^{j\,T}) = -\frac{1}{2}\int_{t_i}^{t_f} d^4x \, P^{\mu\nu}\partial_\sigma A_\mu \partial^\sigma A_\nu. \tag{2.17}$$

This is written in terms of the transverse projector $P^{\mu\nu} = \sum_a \epsilon_a^\mu \epsilon_a^\nu$, here written in terms of the two orthogonal transverse polarization vectors. Explicitly, $P^{0\mu} = 0$, $P^{ij} = \left(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2}\right)$. Clearly this is not Lorentz invariant but that is expected since Lorentz invariance was broken when we identified preferred spacelike slices $\Sigma_{i,f}$.

The remaining integral is easily evaluated by shifting the integration variable by the classical solution. The fluctuation determinant factors out

as an overall constant and we're left with the result given in terms of the classical action for the transverse field evolving between $A_i^T$ and $A_f^T$ in time $T = t_f - t_i$,

$$K(A_f; A_i) = e^{iS_{cl}[A_i^T \to A_f^T]}. \tag{2.18}$$

The expression for the classical action is presented in Appendix A.

Regardless of the longitudinal or timelike data we fix at the endpoints, the gauge invariance of the action has made the propagator depend only on the boundary data which lies in the gauge invariant subspace of the full configuration space (*i.e.* the transverse components).

### 2.1.2 Interacting QED

Now that we have seen the simple example of free $U(1)$ theory let's see how the computation of the propagator is changed when we include charged matter. Of course we cannot perform the integral over the gauge field and the matter variables but for the purposes of computing an influence functional we only need to perform the gauge field integral. The QED propagator is

$$K(A_f, \phi_f; A_i, \phi_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{A_i}^{A_f} \mathcal{D}A_\mu \, e^{iS[A,\phi]}. \tag{2.19}$$

The action is invariant under the $U(1)$ gauge transformation $\phi \to \phi^\Lambda = e^{i\Lambda}\phi$, $A_\mu \to A_\mu^\Lambda = A_\mu + \partial_\mu \Lambda$. As a result the propagator is invariant under independent gauge transformations of the boundary data,

$$\begin{aligned} \int_{\phi_i^{\Lambda_i}}^{\phi_f^{\Lambda_f}} \mathcal{D}\phi \int_{A_i^{\Lambda_i}}^{A_f^{\Lambda_f}} \mathcal{D}A_\mu \, e^{iS[A,\phi]} &= \int_{\phi_i}^{\phi_f} \mathcal{D}\phi' \int_{A_i}^{A_f} \mathcal{D}A'_\mu \, e^{iS[(A')^{-\Lambda},(\phi')^{-\Lambda}]} \\ &= \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{A_i}^{A_f} \mathcal{D}A_\mu \, e^{iS[A,\phi]}. \end{aligned} \tag{2.20}$$

The first equality was merely a change of variables, $\phi' = \phi^\Lambda, A'_\mu = A_\mu^\Lambda$, where the gauge transformation $\Lambda(x)$ matches the gauge transformations on the boundaries, $\Lambda(\mathbf{x}, t_{i,f}) = \Lambda_{i,f}(\mathbf{x})$. The second equality used the gauge invariance of the action. Thus the propagator is indeed invariant under independent gauge transformations of its boundary data,

$$K(A_f^{\Lambda_f}, \phi_f^{\Lambda_f}; A_i^{\Lambda_i}, \phi_i^{\Lambda_i}) = K(A_f, \phi_f; A_i, \phi_i). \tag{2.21}$$

This invariance implies that the physical states which comprise the propagator are also gauge invariant,

$$\Psi[A^\Lambda, \phi^\Lambda] = \Psi[A, \phi]. \tag{2.22}$$

This expression can be written in a more illuminating way if we use the following functional identities,

$$f[\phi + g] = e^{\int d^3x g \frac{\delta}{\delta\phi}} f[\phi], \tag{2.23}$$

$$f[e^{i\Lambda}\phi] = e^{i\int d^3x \Lambda\phi\frac{\delta}{\delta\phi}} f[\phi]. \tag{2.24}$$

The first identity is a straightforward application of a linear shift operator, while the second is a slight generalization from the group of translations to the group of dilatations [86]. The second identity can be rewritten by noting that the $U(1)$ Noether charge is defined as

$$\mathcal{J}^0 = -i\phi\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = -i\phi\Pi, \tag{2.25}$$

where $\Pi$ is the conjugate momentum to the field $\phi$. Writing the conjugate momentum in its operator representation we can then see that the $U(1)$ transformation on a functional is generated by the charge density

$$f[e^{i\Lambda}\phi] = e^{-i\int d^3x \Lambda\hat{\mathcal{J}}^0} f[\phi]. \tag{2.26}$$

With these identities we can write Eq. (2.22) as

$$\exp\left[\int d^3x\, \partial_t\Lambda\frac{\delta}{\delta A_0} + \Lambda\left(-\partial_j\frac{\delta}{\delta A_j} + i\phi\frac{\delta}{\delta\phi}\right)\right]\Psi[A,\phi] = \Psi[A,\phi]. \tag{2.27}$$

or equivalently

$$\left[\int d^3x\, \partial_t\Lambda\frac{\delta}{\delta A_0} + \Lambda\left(-\partial_j\frac{\delta}{\delta A_j} + i\phi\frac{\delta}{\delta\phi}\right)\right]\Psi[A,\phi] = 0 \tag{2.28}$$

Since $\partial_t\Lambda$ is a variable independent of $\Lambda$ and this equation holds for arbitrary values of these parameters, the coefficient of $\partial_t\Lambda$ and the coefficient of $\Lambda$ must vanish independently. Rewriting the functional derivatives in terms as the operators which they represent we obtain two equations which physical states satisfy

$$\hat{E}^0\Psi[A,\phi] = 0 \tag{2.29}$$

and

$$\left(\partial_j\hat{E}^j - \hat{\mathcal{J}}^0\right)\Psi[A,\phi] = 0. \tag{2.30}$$

These are precisely the constraint equations one would impose in the canonical Dirac quantization of this system. As a consequence of gauge invariance, physical quantum states still satisfy Gauss' law as an eigenvalue equation.

As in free $U(1)$ theory, the path integral is formally divergent and again the gauge group volume must be factored out using the modified FP trick. Multiplying the above equation by (2.9) and changing variables as we did previously we obtain

$$K(A_f,\phi_f;A_i,\phi_i)=\int\mathcal{D}\Lambda\int_{\phi_i^{\Lambda_i}}^{\phi_f^{\Lambda_f}}\mathcal{D}\phi\int_{A_i^{\Lambda_i}}^{A_f^{\Lambda_f}}\mathcal{D}A_\mu\,\Delta(A)\delta(\mathcal{G}(A))e^{iS[A,\phi]}. \tag{2.31}$$

The gauge transformation on the matter boundary data can be rewritten in an illuminating way if we use identity (2.24). The propagator can then be written as

$$\begin{aligned}K(A_f,\phi_f;A_i,\phi_i)&= \\ &=\int\mathcal{D}\Lambda\,\hat{U}_\Lambda(t_f)\left(\int_{\phi_i}^{\phi_f}\mathcal{D}\phi\int_{A_i^{\Lambda_i}}^{A_f^{\Lambda_f}}\mathcal{D}A_\mu\,\Delta(A)\delta(\mathcal{G}(A))e^{iS[A,\phi]}\right)\hat{U}_\Lambda^\dagger(t_i),\end{aligned} \tag{2.32}$$

where the operator which effects gauge transformations on the matter variables is

$$\hat{U}_\Lambda(t)=e^{-i\int d^3x\Lambda(x,t)\hat{\mathcal{J}}^0(x,t)}. \tag{2.33}$$

As we did in the above example of free $U(1)$ theory, we will re-express this path integral as the limit of discrete integrals on constant time slices,

$$\begin{aligned}K(A_f,\phi_f;A_i,\phi_i)&=\lim_{N\to\infty}\prod_{n=1}^{N+1}\int d\Lambda(t_n) \\ &\times\hat{U}_\Lambda(t_N)\left(\int_{\phi_i}^{\phi_f}d\phi(t_n)\int_{A_i^{\Lambda_i}}^{A_f^{\Lambda_f}}dA_\mu(t_n)\,\Delta(A)\delta(\mathcal{G}(A))e^{iS[A,\phi]}\right)\hat{U}_\Lambda^\dagger(t_1).\end{aligned} \tag{2.34}$$

Again, the gauge transformations of the boundary data are of the form $A^\mu_{i,f}+\partial^\mu\Lambda_{i,f}$. The spatial gradients are defined on a single time-slice, but the time derivatives are are a difference between variables on consecutive time-slices, $\partial_0\Lambda(t_1)=\frac{1}{\epsilon}(\Lambda(t_2)-\Lambda(t_1))$, and $\partial_0\Lambda(t_N)=\frac{1}{\epsilon}(\Lambda(t_{N+1})-\Lambda(t_N))$. Since the integral depends only on $\Lambda(t_n)$ for $n=1,2,N,N+1$ the gauge group volume can be factored out for intermediate times. As well since the only dependence on $\Lambda(t_{2,N+1})$ is in the gauge transform of $A^0_{i,f}$, the integrals over $\Lambda(t_{2,N+1})$ act as integrals over the boundary data for $A^0$. Only two

integrals remain, and the propagator can be written as

$$K(A_f,\phi_f;A_i,\phi_i) = \int d\Lambda_i d\Lambda_f \tag{2.35}$$
$$\times\, \hat{U}_\Lambda(t_f)\left(\int_{\phi_i}^{\phi_f}\mathcal{D}\phi\int\mathcal{D}A_0\int_{A_i^j+\partial^j\Lambda_i}^{A_f^j+\partial^j\Lambda_f}\mathcal{D}A_j\,\Delta(A)\delta(\mathcal{G}(A))e^{iS[A,\phi]}\right)\hat{U}_\Lambda^\dagger(t_i).$$

The simplest choice of gauge fixing function is Coulomb gauge $\mathcal{G}(A)=\partial_j A^j$. In this case the FP determinant factors out as an overall constant. For intermediate times this gauge choice fixes the longitudinal part of the field to be zero. The gauge fixing delta functions for the boundary times $\delta(\mathcal{G}(A_{i,f}))$ then impose the condition $\partial_j A^j_{i,f} = -\nabla^2\Lambda_{i,f}$ which can be solved for $\Lambda_{i,f}$. The integrals over $\Lambda_{i,f}$ can then be performed using the Coulomb gauge fixing delta functions and the resulting propagator is

$$K(A_f,\phi_f;A_i,\phi_i) = \hat{U}_C(t_f)\left(\int_{\phi_i}^{\phi_f}\mathcal{D}\phi\int\mathcal{D}A_0\int_{A_i^T}^{A_f^T}\mathcal{D}A^T\,e^{iS[A^0,A^T,\phi]}\right)\hat{U}_C^\dagger(t_i). \tag{2.36}$$

Here we've introduced the notation

$$\hat{U}_C(t) = \exp\left(-i\int d^3x A^j(x,t)\hat{C}_j(x,t)\right), \tag{2.37}$$

where $\hat{C}_j(x,t) = -\frac{\partial}{\partial x^j}\int d^3y \frac{1}{4\pi|x-y|}\hat{\mathcal{J}}^0(y,t)$ is the Coulomb electric field created by charge density $\hat{\mathcal{J}}^0$. The operator $\hat{U}_C$ acts in a very intuitive way. If we consider a state $\Psi[\phi]$ describing the matter then the state $\hat{U}_C\Psi[\phi]$ satisfies

$$\hat{E}^j(x)\hat{U}_C\Psi[\phi] = i\frac{\delta}{\delta A_j(x)}\exp\left(-i\int d^3y A^j(y,t)\hat{C}_j(y,t)\right)\Psi[\phi] \tag{2.38}$$
$$= \left(-\frac{\partial}{\partial x_j}\int d^3y\frac{\hat{\mathcal{J}}^0(y)}{4\pi|x-y|}\right)\hat{U}_C\Psi[\phi].$$

Thus the action of the operator $\hat{U}_C$ is to create a coherent Coulomb electric field around the matter. The Coulomb electric field is the solution to the Gauss law constraint equation, so $\hat{U}_C$ makes a state satisfy Gauss' law and thus be gauge-invariant. As a result of Eqs. (2.29) and (2.30) physical states are of the form

$$\Psi[A,\phi] = \hat{U}_C\,\psi[A^T,\phi]. \tag{2.39}$$

Physical states consist of transverse photons and matter with its accompanying Coulomb field. All of the dependence on the longitudinal part of the field is in $\hat{U}_C$ and the state does not depend on $A_0$. The unconstrained part of the wavefunctional $\psi[A^T, \phi]$ can take on any form.

This is reminiscent of Dirac's approach to gauge invariant QED [87]. He constructed gauge invariant fermion field operators which created an a electron with an accompanying coherent gauge field. In his case it was ambiguous what the accompanying field should be since any transverse field could be added to the Coulomb field and the resulting field operator would still be gauge invariant. We do not have this ambiguity since all of the dependence on the transverse field lies in $\psi[A^T, \phi]$. It should be emphasized that our results are indeed different. We are describing general dressed states not dressed field operators.

Returning to the evaluation of the path integral, in Coulomb gauge the action is

$$S[A^0, A^T, \phi] = S_M[\phi] + S_\gamma[A^T, \phi] + \int_{t_i}^{t_f} d^4x \left( \frac{1}{2} A_0 \partial_j \partial^j A^0 + A_0 J^0 \right), \quad (2.40)$$

where the action for transverse photons is

$$S_\gamma[A^T, \phi] = \int_{t_i}^{t_f} d^4x \, P^{\mu\nu} \left( -\frac{1}{2} \partial_\sigma A_\mu \partial^\sigma A_\nu + A_\mu \mathcal{J}_\nu \right). \quad (2.41)$$

The $A_0$ integral can be immediately done and it merely adds a new interaction term to the matter action corresponding to the Coulomb force between the charge densities,

$$\int \mathcal{D}A_0 \, e^{iS[A^0, A^T, \phi]} = e^{iS[A^T, \phi] + iS_C[\phi]} \quad (2.42)$$

where the instantaneous Coulomb interaction term is

$$S_C[\phi] = -\frac{1}{2} \int_{t_i}^{t_f} dt \int d^3x d^3y \frac{J^0(x,t) J^0(y,t)}{4\pi |x-y|}. \quad (2.43)$$

With the timelike and longitudinal integrals done the final expression for the propagator is written in terms of only the independent data

$$K(A_f, \phi_f; A_i, \phi_i) = \hat{U}_C(t_f) \left( \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{A_i^T}^{A_f^T} \mathcal{D}A^T \, e^{iS[A^T, \phi] + iS_C[\phi]} \right) \hat{U}_C^\dagger(t_i), \quad (2.44)$$

or equivalently

$$K(A_f,\phi_f;A_i,\phi_i) = \hat{U}_C(t_f)\left(\int_{\phi_i}^{\phi_f}\mathcal{D}\phi\, e^{iS_M[\phi]+iS_C[\phi]}\right)\hat{U}_C^\dagger(t_i) \times \int_{A_i^T}^{A_f^T}\mathcal{D}A^T\, e^{iS_\gamma[A^T,\phi]}. \tag{2.45}$$

The latter form is convenient because the remaining path integral on the gauge field has been separated from the matter integral and can be evaluated as the integral for a transverse electromagnetic radiation field on a frozen background source field.

The path integral for the transverse field can again be evaluated by simply shifting the integration variable by the classical solution. The fluctuation determinant again factors out as an overall constant and the result is just the classical action for the transverse field on a frozen background source field,

$$K(A_f,\phi_f;A_i,\phi_i) = \hat{U}_C(t_f)\left(\int_{\phi_i}^{\phi_f}\mathcal{D}\phi\, e^{iS_M[\phi]+iS_C[\phi]}\right)\hat{U}_C^\dagger(t_i)\, e^{iS_{cl}[A_i^T\to A_f^T,\phi]}, \tag{2.46}$$

where $S_{cl}[A_i^T \to A_f^T,\phi]$ is a straightforward to compute but lengthy expression which we present in Appendix A.

As a result of gauge invariance there are constraints implemented on the system enforcing that i) the kernel is independent of $A_0$ and ii) the dependence of the kernel on $A^L$ is determined entirely by the charge density of the matter (Gauss' law). This implies that physical states are of the form $\hat{U}_C\,\psi[A^T,\phi]$. Indeed the only components of the electromagnetic field which are independent of the matter are the transverse components. Comparing Eq. (2.46) to Eq. (1.12) we see that the natural partition into central system and environment is not matter and gauge field, but rather matter (with its coherent Coulomb field) and radiation field. A partial trace which results in a physical state must then be only over the transverse photon degrees of freedom since the longitudinal degrees of freedom are constrained by Gauss' law. This will be used in Section 3 to compute the influence functional.

## 2.2 Linearized Quantum Gravity

Now that we have familiarized ourselves with the computation of a gauge invariant propagation kernel in a simple gauge theory lets proceed to quantum gravity. As mentioned in the introduction we will use the effective field

theory approach to quantum gravity in the low energy linearized approximation.

The action for matter coupled to linearized Einstein gravity (including the GHY term) is given in Eq. (1.27),

$$S[h,\phi] = \oint_\Sigma d^3x\, h^{ij}\pi^{(1)}_{ij} - \int_{t_i}^{t_f} d^4x \left( h^{\mu\nu}G^{(1)}_{\mu\nu} - \mathcal{L}_M(\phi,\eta_{\mu\nu}) - \frac{1}{M_P}h^{\mu\nu}T_{\mu\nu}\right). \tag{2.47}$$

The amplitude to evolve from an initial configuration $(\phi_i, h_i^{\mu\nu})$ to a later configuration $(\phi_f, h_f^{\mu\nu})$ is given by the path integral representation of the propagator

$$K(h_f,\phi_f;h_i,\phi_i) = \int_{\phi_i}^{\phi_f}\mathcal{D}\phi\int_{h_i}^{h_f}\mathcal{D}h_{\mu\nu}\, e^{iS[h,\phi]}. \tag{2.48}$$

Again, as with Eq. (2.2) one may take objection to this expression since we know from the canonical formalism that $h_{0\nu}$ are not canonical variables, as well as the path integral being formally divergent. We've seen already in Sections 2.1.1 and 2.1.2 that these issues were resolved naturally by carefully factoring out the gauge redundancy from the integral. The same is true of this gravitational path integral.

As emphasized in Section 1.5.2 this action is not invariant under gauge transformations but rather it changes by a boundary term (1.33). Lets see what this implies for the propagator. A gauge transformation of the boundary data causes a change in the propagator

$$\begin{aligned} &K(h_f^{\xi_f},\phi_f^{\xi_f};h_i^{\xi_i},\phi_i^{\xi_i}) = \\ &\quad = e^{-2i\int_{\Sigma_f}\xi_0\left(2\hat{G}^{00}+\frac{1}{M_P}\hat{\mathcal{L}}_M\right)}K(h_f,\phi_f;h_i,\phi_i)e^{2i\int_{\Sigma_i}\xi_0\left(2\hat{G}^{00}+\frac{1}{M_P}\hat{\mathcal{L}}_M\right)}, \end{aligned} \tag{2.49}$$

where $\phi^\xi$ denotes the matter variables under the transformation $x^\mu \to x^\mu + \frac{2}{M_P}\xi^\mu$. The propagator is then invariant under (small) spatial diffeomorphisms but not under diffeomorphisms changing the initial and final time coordinates. The physical states which comprise the propagator then transform as

$$\Psi[h^\xi,\phi^\xi] = e^{-2i\int d^3x\,\xi_0\left(2\hat{G}^{00}+\frac{1}{M_P}\hat{\mathcal{L}}_M\right)}\Psi[h,\phi]. \tag{2.50}$$

This expression can be rewritten in a more physically illuminating way. Since $\xi/M_P \ll 1$ we can write a functional $f[\phi]$ of the transformed variable using a linear shift operator

$$f[\phi^\xi] = f\left[\phi + \frac{2}{M_P}\xi^\mu\partial_\mu\phi\right] = e^{\frac{2}{M_P}\int d^3x\,\xi^\mu(\partial_\mu\phi)\frac{\delta}{\delta\phi}}f[\phi]. \tag{2.51}$$

The metric perturbation also transforms by a linear shift, so we can write

$$f[h^{\xi}_{\mu\nu}] = e^{2\int d^3x(\partial_\mu\xi_\nu)\frac{\delta}{\delta h_{\mu\nu}}} f[h_{\mu\nu}]. \tag{2.52}$$

We now rewrite the functional derivatives in terms of the operators which they represent, $-i\frac{\delta}{\delta\phi} = \hat{\Pi}$ and $-i\frac{\delta}{\delta h_{\mu\nu}} = \hat{\pi}^{\mu\nu}$ where $\Pi = \frac{\partial\mathcal{L}_M}{\partial(\partial_0\phi)}$ and $\pi^{\mu\nu} = \frac{\partial\mathcal{L}_g}{\partial(\partial_0 h_{\mu\nu})}$. The definition of the stress-tensor as the Noether current associated with spacetime translations is

$$T^{\mu\nu} = -(\partial^\nu\phi)\frac{\partial\mathcal{L}_M}{\partial(\partial_\mu\phi)} + \eta^{\mu\nu}\mathcal{L}_M, \tag{2.53}$$

so as an operator the 4-momentum density can be written

$$\hat{T}^{0\nu} = i(\partial^\nu\phi)\frac{\delta}{\delta\phi} + \eta^{0\nu}\hat{\mathcal{L}}_M. \tag{2.54}$$

The functional identities are then expressible as,

$$f[\phi^{\xi}] = e^{-i\frac{2}{M_P}\int d^3x\,\xi_\mu\left(\hat{T}^{\mu 0} - \eta^{\mu 0}\hat{\mathcal{L}}_M\right)} f[\phi], \tag{2.55}$$

$$f[h^{\xi}_{\mu\nu}] = e^{2i\int d^3x\,(\partial_\nu\xi_\mu)\hat{\pi}^{\mu\nu}} \tag{2.56}$$

Another useful formula is the relation between the conjugate momentum to the metric perturbation and the Einstein tensor, $-\partial_j\pi^{jk} = 2G^{0k}$. Using these relations we can rewrite Eq. (2.50) as

$$\exp\left[2i\int d^3x\,\left((\partial_0\xi_\nu)\hat{\pi}^{0\nu} + \xi_\nu\left(2\hat{G}^{0\nu} - M_P^{-1}\hat{T}^{0\nu}\right)\right)\right]\Psi[h,\phi] = \Psi[h,\phi], \tag{2.57}$$

or equivalently

$$\left[\int d^3x\,\left((\partial_0\xi_\nu)\hat{\pi}^{0\nu} + \xi_\nu\left(2\hat{G}^{0\nu} - M_P^{-1}\hat{T}^{0\nu}\right)\right)\right]\Psi[h,\phi] = 0 \tag{2.58}$$

Since this equation holds for arbitrary $\xi^\mu$ and $\partial_0\xi^\mu$, and they are independent parameters, their coefficients must independently vanish. We then obtain two constraint equations which the physical states satisfy,

$$\hat{\pi}^{0\nu}\Psi[h,\phi] = 0, \tag{2.59}$$

$$\left(\hat{G}^{0\nu} - \frac{1}{2M_P}\hat{T}^{0\nu}\right)\Psi[h,\phi] = 0. \tag{2.60}$$

The first equation says that physical states do not depend on the time-like components $h_{0\nu}$. The second equation can be identified as the (linearized) Hamiltonian and momentum constraint equations of general relativity. These are precisely the constraint equations one would impose on physical states in the Dirac canonical formalism [55, 56]. Just as we did in QED, here we obtain these constraint equations as a consequence of the gauge invariance of the propagator.

Now that we've seen how the gauge invariance of the propagator implies that physical states satisfy certain constraint equations, we can proceed to evaluate the path integral (for the metric variables). This proceeds completely analogously to the QED example. To factor out the gauge redundancy we multiply by

$$1 = \int \mathcal{D}\xi\, \Delta(h^\xi)\delta(\mathcal{G}(h^\xi)). \tag{2.61}$$

Note that the integral is over $\xi^\mu$ including those which change the boundaries. We multiply by the above FP factor and change variables to obtain

$$\begin{aligned} K(h_f,\phi_f;h_i,\phi_i) = \int \mathcal{D}\xi^\mu \int_{\phi_i^{\xi_i}}^{\phi_f^{\xi_f}} \mathcal{D}\phi \int_{h_i^{\xi_i}}^{h_f^{\xi_f}} \mathcal{D}h_{\mu\nu} \\ \times \Delta(h)\delta(\mathcal{G}(h))\, e^{iS[h,\phi]+2i\oint_\Sigma d^3x\,\xi_0\left(2G^{00}+M_P^{-1}\mathcal{L}_M\right)}. \end{aligned} \tag{2.62}$$

Using the identities Eq. (2.55) and Eq. (2.56) we can rewrite the propagator in the convenient form

$$\begin{aligned} &K(h_f,\phi_f;h_i,\phi_i) = \\ &\quad = \int \mathcal{D}\xi^\mu\, \hat{U}_{\xi^\mu}(t_f) \left( \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{h_i^{\xi_i}}^{h_f^{\xi_f}} \mathcal{D}h_{\mu\nu}\, \Delta(h)\delta(\mathcal{G}(h))\, e^{iS[h,\phi]} \right) \hat{U}^\dagger_{\xi^\mu}(t_i), \end{aligned} \tag{2.63}$$

where we've defined the operator

$$\hat{U}_{\xi^\mu}(t) = e^{2i\int d^3x\,\xi_0\left(2G^{00}-M_P^{-1}\hat{T}^{00}\right)-M_P^{-1}\xi_j\hat{T}^{0j}}. \tag{2.64}$$

The next step is to cut the path integral into discrete time slices. Recall how the gauge transform of $A_0$ depended on the time derivative $\partial_0\Lambda$ and this allowed us to use rewrite the integrals $\int d\Lambda(t_2)\int d\Lambda(t_{N+1})$ as integrals over the boundary data for $A_0$. The same situation occurs here for $h_{0\nu}$. The gauge transform of the spatial components $h^\xi_{jk}$ depends only only spatial

derivatives and thus only on $\xi^j$ at times $t_{1,N} \equiv t_{i,f}$. The gauge transform of the timelike components is

$$h^{\xi}_{0\nu}(t_f) = h_{0\mu}(t_f) + \partial_\nu \xi_0(t_N) + \frac{1}{\epsilon}\left(\xi_\nu(t_{N+1}) - \xi_\nu(t_N)\right) \tag{2.65}$$
$$h^{\xi}_{0\nu}(t_i) = h_{0\mu}(t_i) + \partial_\nu \xi_0(t_1) + \frac{1}{\epsilon}\left(\xi_\nu(t_2) - \xi_\nu(t_1)\right)$$

A simple change of variables allows us to write the integrals over $\xi_\nu(t_{N+1})$ and $\xi_\nu(t_2)$ as integrals over the boundary data for $h_{0\nu}$. Once this is done the boundary data is independent of $\xi_0$. In fact the only dependence on $\xi_0$ is in the boundary operator (2.64). The integrand in Eq. (2.63) doesn't depend on $\xi_\mu$ for intermediate times and thus the gauge group volume can be factored out as an overall normalization. The resulting path integral is

$$K(h_f, \phi_f; h_i, \phi_i) = \int d\xi^j_f \int d\xi^j_i \tag{2.66}$$
$$\times\, \delta(\hat{\mathcal{H}})\hat{U}_{\xi^j}\left(\int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int \mathcal{D}h_{0\nu} \int_{h_i^{\xi_i}}^{h_f^{\xi_f}} \mathcal{D}h_{jk}\, \Delta(h)\delta(\mathcal{G}(h))\, e^{iS[h,\phi]}\right) \hat{U}^\dagger_{\xi^j}\delta^\dagger(\hat{\mathcal{H}}).$$

The operator

$$\delta(\hat{\mathcal{H}}) \equiv \int d\xi_0\, e^{4i\int d^3x\, \xi_0 \left(G^{00} - \frac{1}{2M_P}\hat{T}^{00}\right)} \tag{2.67}$$

is the projector onto the kernel of the operator $\hat{G}^{00} - \frac{1}{2M_P}\hat{T}^{00}$. That is, it projects onto the subspace of the Hilbert space satisfying the Hamiltonian constraint $\hat{\mathcal{H}}\Psi = 0$. Since the overall normalization is irrelevant, we can assume that $\delta(\hat{\mathcal{H}})$ returns 1 when acting on a state which satisfies the Hamiltonian constraint and returns zero otherwise.

To evaluate the path integral it is convenient to further decompose the metric perturbation. Similar to the transverse and longitudinal decomposition of a vector field, a symmetric tensor field can be decomposed into longitudinal, transverse-trace, and transverse-traceless (TT) parts

$$h_{jk} = h^L_{jk} + h^T_{jk} + h^{TT}_{jk} \tag{2.68}$$

which satisfy $\partial^j h^T_{jk} = 0$, $\partial^j h^{TT}_{jk} = 0$, and $\delta^{jk} h^{TT}_{jk} = 0$. Explicit expressions are obtained using the transverse projector $P_{ij}$,

$$h^L_{jk} = \left(\delta^a_j \delta^b_k - P^a_j P^b_k\right) h_{ab}, \tag{2.69}$$

$$h^T_{jk} = \frac{1}{2} P_{jk} P^{ab} h_{ab}, \tag{2.70}$$

$$h_{jk}^{TT} = \left( P_j^a P_k^b - \frac{1}{2} P_{jk} P^{ab} \right) h_{ab}. \tag{2.71}$$

It is easy to check that only the longitudinal part transforms under gauge transformation.

Along with this decomposition we must choose a gauge fixing function. The most convenient choice is transverse gauge $\mathcal{G}(h) = \partial_j h^{j\nu} = 0$. With this choice the FP determinant factors out as an overall constant. This gauge choice sets the longitudinal part of the field to zero for intermediate times. For times $t_{i,f}$ it enforces $\partial^j h_{jk}^L + \nabla^2 \xi_k + \partial^j \partial_k \xi_j = 0$. This can be solved for $\xi^j(t_{i,f})$ to find

$$\xi^j(t_{i,f}) = -\frac{1}{\nabla^2} \left( \delta_a^j - \frac{\partial^j \partial_a}{2\nabla^2} \right) \partial_b h_{i,f}^{ab}. \tag{2.72}$$

We are now able to evaluate the integrals over all of the different components of the metric perturbation $h_{00}$, $h_{0j}$, $h_{jk}^T$, $h_{jk}^L$, and $h_{jk}^{TT}$. The $h_{00}$ integral can be done immediately since $h_{00}$ appears only linearly in the action. It simply produces a delta function enforcing the Hamiltonian constraint for every intermediate time in the path integral. The Hamiltonian constraint fixes the transverse-trace part of the metric perturbation in terms of the longitudinal part and the matter stress tensor. Using the Hamiltonian constraint the transverse-trace part can then be immediately integrated. Next the $h_{0j}$ integral can be performed and it is a straightforward gaussian integral. The result is an instantaneous gravitational interaction term added to the matter action analogous to the $A_0$ integration in QED generating the Coulomb interaction term. The last remaining integral is the trivial integral is over the longitudinal part.

The resulting path integral expression for the gravitational propagator is

$$\begin{aligned} K(h_f, \phi_f; h_i, \phi_i) =& \delta(\hat{\mathcal{H}})\, \hat{U}_G \left( \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{iS_M[\phi] + iS_{SG}[\phi]} \right) \hat{U}_G^\dagger \delta^\dagger(\hat{\mathcal{H}}) \\ & \times \int_{h_i^{TT}}^{h_f^{TT}} \mathcal{D}h_{jk}^{TT}\, e^{iS_g[h^{TT}, \phi]}. \end{aligned} \tag{2.73}$$

We have defined the instantaneous gravitational self-interaction term

$$\begin{aligned} S_{SG}[\phi] = & -\frac{1}{4M_P^2} \int_{t_i}^{t_f} d^4x \\ & \times \frac{1}{\nabla^2} \left( T^{00} T^{00} - 4 T^{0j} P_{jk} T^{0k} + 2 T^{00} P_{jk} T^{jk} + \frac{\partial_0 T^{00} \partial_0 T^{00}}{\nabla^2} \right). \end{aligned} \tag{2.74}$$

The first term is simply the Newton potential while the rest of the terms are purely relativistic. This expression may seem unfamiliar but it is indeed the correct gravitational analogue of the Coulomb interaction in QED. This is demonstrated in Appendix B. The transverse-traceless action is

$$S_g[h^{TT},\phi] = \int_{t_i}^{t_f} d^4x\, \Pi^{\mu\nu\alpha\beta}\left(-\frac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h_{\alpha\beta} + \frac{1}{M_P}h_{\mu\nu}T_{\alpha\beta}\right), \tag{2.75}$$

written in terms of the TT projector $\Pi^{\mu\nu\alpha\beta} = \frac{1}{2}\left(P^{\mu\alpha}P^{\nu\beta} + P^{\mu\beta}P^{\nu\alpha} - P^{\mu\nu}P^{\alpha\beta}\right)$. The operators $\hat{U}_G$ are defined as

$$\hat{U}_G = \exp\left(i\frac{1}{M_P}\int d^3x\, h^{jk}\hat{B}_{jk}\right), \tag{2.76}$$

where

$$\hat{B}_{jk} = -\frac{1}{\nabla^2}\left(\delta_{jl}\partial_k + \delta_{kl}\partial_j - \frac{\partial_j\partial_k\partial_l}{\nabla^2}\right)\hat{T}^{0l}. \tag{2.77}$$

To see how $\hat{U}_G$ acts consider a state $\Psi[\phi]$ describing only the matter. The state $\hat{U}_G\Psi[\phi]$ then satisfies

$$\partial_j\left(i\frac{\delta}{\delta h_{jk}}\right)\hat{U}_G\Psi[\phi] = -\frac{1}{M_P}\partial_j\hat{B}^{jk}\hat{U}_G\Psi[\phi] = \hat{T}^{0k}\hat{U}_G\Psi[\phi]. \tag{2.78}$$

This can be rewritten in terms of the conjugate momentum $\hat{\pi}^{jk}$ and we can see that the state $\hat{U}_G\Psi[\phi]$ satisfies the momentum constraint

$$\hat{\mathcal{P}}^k\hat{U}_G\Psi[\phi] \equiv \left(\hat{G}^{0k} - \frac{1}{2M_P}T^{0k}\right)\hat{U}_G\Psi[\phi] = 0. \tag{2.79}$$

In the same way the operator $\hat{U}_C$ created a coherent Coulomb electric field in QED (ensuring the that physical states satisfy Gauss' law), the operator $\hat{U}_G$ creates a coherent gravitational field which ensures that physical states satisfy the momentum constraint and are thus invariant under spatial gauge transformations. As a result of the constraints Eq. (2.59) and Eq. (2.60), physical states are of the form $\Psi[h,\phi] = \delta(\hat{\mathcal{H}})\,\hat{U}_G\,\psi[h^{TT},\phi]$. They are indepedent of the timelike components $h_{0\nu}$, and the dependence on the longitudinal field is entirely in $\hat{U}_G$. The unconstrained part of the wavefunctional $\psi[h^{TT},\phi]$ can take on any form, since $h^{TT}$ and $\phi$ are the true independent degrees of freedom. The only dependence on the transverse-trace part of the field is through the projector $\delta(\hat{\mathcal{H}})$. This projector ensures physical wavefunctions are non-zero only if the transverse-trace part of the field takes

on a value such that the Hamiltonian constraint is satisfied. This is quite unlike the momentum constraint which is satisfied by dressing matter states with coherent fields using the operator $\hat{U}_G$. The reason for this difference is that the momentum constraint is a relation between the coordinates and momenta $(h_{jk}, \pi^{jk})$ whereas the Hamiltonian constraint is a constraint only on the coordinates, $G^{00} = \frac{1}{2}(\partial_j\partial_k - \delta_{jk}\nabla^2)h^{jk}$. The Hamiltonian constraint then restricts physical states to be wavefunctionals which only have support on the kernel of $\hat{\mathcal{H}}$ rather than wavefunctionals with support on the entire space of three-metric configurations. This is a point argued by Kuchar [88] which was imposed by hand in [89] and derived in a way very similar to us in [80].

Finally, the path integral for the TT parts of the metric perturbation can be done by shifting the integration variable by the classical solution. The fluctuation determinant factors out as an overall constant and the result is the classical action for the TT field on a frozen background source. The final expression for the propagator, having evaluated the gravitational path integral is

$$K(h_f, \phi_f; h_i, \phi_i) = \delta(\hat{\mathcal{H}})\, \hat{U}_G \left( \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{iS_M[\phi] + iS_{SG}[\phi]} \right) \hat{U}_G^\dagger\, \delta^\dagger(\hat{\mathcal{H}})\, e^{iS_{cl}[h^{TT}_{i\to f}, \phi]} \tag{2.80}$$

The classical action $S_{cl}[h^{TT}_{i\to f}, \phi]$ is presented in Appendix A.

As a result of gauge invariance, there are constraints implemented on the system enforcing that i) the propagator is independent of $h_{0\nu}$ ii) the dependence of the propagator on $h^L$ is determined by the momentum density of the matter (due to the momentum constraint) and iii) the dependence of the propagator on the trace $\delta^{jk}h_{jk}$ is constrained by energy density of the matter (due to the Hamiltonian constraint). As a result the only independent degrees of freedom are the matter degrees of freedom and the transverse-traceless graviton degrees of freedom. Physical states can then be written as $\Psi[h, \phi] = \delta(\hat{\mathcal{H}})\, \hat{U}_G\, \psi[h^{TT}, \phi]$. The appropriate partition into central system and environment is then matter (dressed by appropriate coherent gravitational field) and transverse-traceless gravitons. A partial trace resulting in a state which is still physical must then be only over the transverse-traceless graviton degrees of freedom.

We've now seen three examples of how the gauge invariance of the propagator leads to the idea of gauge invariant physical states. These physical states which comprise the propagator are annihilated by operators which generate gauge transformation. Such equations are precisely the constraint equations which one imposes in Dirac's canonical quantization of constrained

systems. We computed the path integrals for the gauge fields using a modification of the FP trick. The result is a manifestly gauge invariant propagator which is equivalent to the propagator one would obtain in the canonical Dirac formalism (see [53]). As a result of our computations it is clear how to divide the central system from the environment. In QED the environment is the transverse (photon) part of the gauge field, while the rest of the gauge field is constrained by the state of the matter. In linear quantum gravity the environment is the transverse-traceless (graviton) part of the gauge field, while the rest of the gauge field is constrained by the state of the matter. The physical picture in QED is that of dressed charges (charges surrounded by their associated Coulomb electric field) interacting with a field of transverse photons. The picture in gravity is similar, there is matter dressed by a gravitational field and the matter interacts with TT gravitons. In the next chapter we will take these results and compute the influence functional describing the interaction of these central systems with the radiation environment in each theory.

# Chapter 3

# Influence Functional in QED and linearized Gravity

## 3.1 Thermal Photon Bath Influence Functional

In section 2.1.2 we showed that as a result of gauge invariance the propagator is composed of physical states $\Psi[A,\phi]$ which have the form $\Psi[A,\phi] = \hat{U}_C\,\psi[A^T,\phi]$, which makes it clear that the only independent degrees of freedom are that of the matter and transverse field components. It was also mentioned that because the propagator projects onto the space of physical states, we can work exclusively with physical states without loss of generality. Physical density matrices are of the form

$$\rho[\underline{A},\underline{\phi}] = \hat{U}_C\,\tilde{\rho}[\underline{A}^T,\underline{\phi}]\,\hat{U}_C^\dagger. \tag{3.1}$$

We will use tilde to denote density matrices which only depend on matter and/or transverse field components. The tilded density matrices describe unphysical states but the appropriate physical state can be obtained simply by conjugating $\tilde{\rho}$ with the operator $\hat{U}_C$. For brevity we can work with the density matrices $\tilde{\rho}$.

The transverse field components can be traced out to obtain the reduced density matrix for the matter

$$\tilde{\rho}_\phi[\underline{\phi}] = \int d\underline{A}^T\,\delta(A^T - A^{T\prime})\tilde{\rho}[\underline{A}^T,\underline{\phi}]. \tag{3.2}$$

For physical states it is possible that the transverse degrees of freedom are uncorrelated with the matter, but obviously the longitudinal part of the field must be correlated with the matter. A state in which the transverse field components are uncorrelated with the matter is written as a product state $\tilde{\rho}[\underline{A}^T,\underline{\phi}] = \rho_\gamma[\underline{A}^T]\tilde{\rho}_\phi[\underline{\phi}]$.

Now we can return ourselves to the overarching question related to decoherence. Suppose the environment, *i.e* the transverse field components, is initially uncorrelated with the matter. We can evolve the state forward in

time and trace out the environment to obtain the final (physical) state of the matter. The reduced density matrix for the matter can then be written as

$$\rho_\phi[\underline{\phi}_f] = \hat{U}_C \left( \int d\underline{\phi}_i \, \tilde{\rho}_\phi[\underline{\phi}_i] \int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi} \, e^{iS_M[\underline{\phi}]+iS_C[\underline{\phi}]} \mathcal{F}[\underline{\phi}] \right) \hat{U}_C^\dagger, \tag{3.3}$$

and the influence functional is given by

$$\mathcal{F}[\underline{\phi}] = \int d\underline{A}_f^T \int d\underline{A}_i^T \, \delta(A_f^T - A_f^{T\prime}) \rho[\underline{A}_i^T] \int_{\underline{A}_i^T}^{\underline{A}_f^T} \mathcal{D}\underline{A}^T \, e^{iS[\underline{A}^T,\underline{\phi}]}. \tag{3.4}$$

Equations (3.3) and (3.4) are of precisely the same form as Eqs. (1.13) and (1.14) for an environment consisting of transverse photons. The upshot of this manifestly gauge invariant computation is that it has become clear that only the transverse photons can act as an environment in QED while the timelike and longitudinal parts of the gauge field act to produce the Coulomb interaction in the matter action and to dress the matter with a coherent Coulomb electric field.

At this point it is straightforward to compute the influence functional for an initially thermal photon bath. A state describing a thermal gauge field is of the form

$$\tilde{\rho}[\underline{A}^T, \underline{\phi}] = \tilde{\rho}_\phi[\underline{\phi}] \int_{A^{T\prime}}^{A^T} \mathcal{D}A^{T\prime\prime} \, e^{-S_E[A^{T\prime\prime}]}. \tag{3.5}$$

The influence functional for a bath of thermal photons is then given by the functional integral

$$\mathcal{F}[\underline{\phi}] = \int d\underline{A}_f^T \, \delta(A_f^T - A_f^{T\prime}) \int d\underline{A}_i^T \int_{\underline{A}_i^T}^{\underline{A}_f^T} \mathcal{D}\underline{A}^T \int_{A_i^{T\prime}}^{A_i^T} \mathcal{D}A^{T\prime\prime} \, e^{iS[\underline{A}^T,\underline{\phi}]-S_E[A^{T\prime\prime}]}. \tag{3.6}$$

Each of the functional integrals is gaussian and can be evaluated immediately by shifting the integration about the classical path. The remaining integrals are gaussian as well and are done by direct integration. At this stage the environment has been reduced to a collection of independent oscillators, a well studied case [73, 90]. For a theory in which the $U(1)$ current is independent of $A_\mu$ (this excludes scalar QED for example) and the coupling term is $\mathcal{L}_{\text{int}} = eA_\mu \mathcal{J}^\mu$ the result is expressed conveniently in terms of an influence phase,

$$\mathcal{F}[\underline{\phi}] = e^{i\Phi[\underline{\phi}]}. \tag{3.7}$$

The influence phase is given by the expression

$$i\Phi[\underline{\phi}] = i\int_{t_i}^{t_f} d^4x \int_{t_i}^{x^0} d^4\tilde{x}\left[\mathcal{J}_\mu(x) - \mathcal{J}'_\mu(x)\right] \times \left(\left[\mathcal{J}_\nu(\tilde{x}) + \mathcal{J}'_\nu(\tilde{x})\right]\gamma^{\mu\nu}(x-\tilde{x}) + i\left[\mathcal{J}_\nu(\tilde{x}) - \mathcal{J}'_\nu(\tilde{x})\right]\eta^{\mu\nu}(x-\tilde{x})\right). \tag{3.8}$$

The dissipation and noise kernels are respectively given by

$$\gamma^{\mu\nu}(x) = \frac{e^2}{2}\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\, P^{\mu\nu}(p)\frac{\sin px^0}{p}, \tag{3.9}$$

$$\eta^{\mu\nu}(x) = \frac{e^2}{2}\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \coth\left(\frac{\beta p}{2}\right) P^{\mu\nu}(p)\frac{\cos px^0}{p}, \tag{3.10}$$

where $\beta$ is the inverse temperature.

We can now finally write down the physical (gauge-invariant) reduced density matrix for matter interacting with a thermal bath of photons,

$$\rho_\phi[\underline{A}^L_f, \underline{\phi}_f] = \int d\underline{\phi}_i\, \tilde{\rho}_\phi[\underline{\phi}_i]\left(\hat{U}_C \int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi}\, e^{iS_M[\underline{\phi}]+iS_C[\underline{\phi}]+i\Phi[\underline{\phi}]}\,\hat{U}_C^\dagger\right). \tag{3.11}$$

It should be noted that the influence functional Eq. (3.8) is not a new result, see [91]. The manifestly gauge invariant computation is new however. Regardless, the QED computation was considered a warm-up for the quantum gravity computation which has had more debate in the literature.

## 3.2 Thermal Graviton Bath Influence Functional

The discussion for linear gravity is nearly identical to the above for QED. We've seen in section 2.2 that as result of gauge invariance physical states are of the form $\Psi[h,\phi] = \delta(\hat{\mathcal{H}})\hat{U}_G\psi[h^{TT},\phi]$. The only independent degrees of freedom are the TT gravitons and the matter. We can write physical density matrices in the form

$$\rho[\underline{h},\underline{\phi}] = \delta(\hat{\mathcal{H}})\,\hat{U}_G\,\tilde{\rho}[\underline{h}^{TT},\underline{\phi}]\,\hat{U}_G^\dagger\,\delta^\dagger(\hat{\mathcal{H}}). \tag{3.12}$$

All of the dynamics is described by the evolution of the unphysical tilded density matrix, and the physical state can be obtained at the end of the computation by replacing the operators $\delta(\hat{\mathcal{H}})\,\hat{U}_G$.

A state in which the environment (TT gravitons) is initially uncorrelated with matter as well as being thermal can be written as

$$\tilde{\rho}[\underline{h}^{TT}, \underline{\phi}] = \tilde{\rho}_{\phi}[\underline{\phi}] \int_{h^{TT\prime}}^{h^{TT}} \mathcal{D}h^{TT\prime\prime}\, e^{-S_E[h^{TT\prime\prime}]}. \tag{3.13}$$

Assuming an initial condition of this form is convenient from a calculational perspective since it allows for an exact evaluation of the influence functional integrals, but we must ask if it is physically reasonable. As we've previously emphasized, gravity is unique because it cannot be screened. This implies that the central system and graviton environment will **always** be correlated to some degree. In our computation we will assume the above simple form for the initial state, but this must be understood as an approximation, the accuracy of which we cannot yet quantify since we do not have an answer to the question, "How close to a product state can one prepare a bipartite system composed of some type of matter and a collection of gravitons?" This is a question of interest for future work.

The influence functional describing a bath of initially thermal gravitons is then given by the functional integral

$$\begin{aligned}\mathcal{F}[\underline{\phi}] = \int d\underline{h}_f^{TT} \int d\underline{h}_i^{TT}\, \delta(h_f^{TT} - h_f^{TT\prime}) \\ \times \int_{\underline{h}_i^{TT}}^{\underline{h}_f^{TT}} \mathcal{D}\underline{h}^{TT} \int_{h_i^{TT\prime}}^{h_i^{TT}} \mathcal{D}h^{TT\prime\prime}\, e^{iS_g[\underline{h}^{TT}, \underline{\phi}] - S_E[h^{TT}]}.\end{aligned} \tag{3.14}$$

The integration can be done straightforwardly as again all integrals are gaussian. The resulting influence phase $i\Phi[\underline{\phi}]$ for linear gravity is

$$\begin{aligned}i\Phi[\underline{\phi}] = i \int_{t_i}^{t_f} d^4x \int_{t_i}^{x^0} d^4\tilde{x}\, \left[T_{\mu\nu}(x) - T'_{\mu\nu}(x)\right] \\ \times \left( \left[T_{\sigma\rho}(\tilde{x}) + T'_{\sigma\rho}(\tilde{x})\right] \gamma^{\mu\nu\sigma\rho}(x - \tilde{x}) + i \left[T_{\sigma\rho}(\tilde{x}) - T'_{\sigma\rho}(\tilde{x})\right] \eta^{\mu\nu\sigma\rho}(x - \tilde{x}) \right).\end{aligned} \tag{3.15}$$

The dissipation and noise kernels are

$$\gamma^{\mu\nu\sigma\rho}(x) = \frac{1}{2M_P^2} \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\, \Pi^{\mu\nu\sigma\rho}(p) \frac{\sin px^0}{p}, \tag{3.16}$$

$$\eta^{\mu\nu\sigma\rho}(x) = \frac{1}{2M_P^2} \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \coth\left(\frac{\beta p}{2}\right) \Pi^{\mu\nu\sigma\rho}(p) \frac{\cos px^0}{p}. \tag{3.17}$$

The physical (gauge-invariant) reduced density matrix for matter interacting with a thermal bath of gravitons is then

$$\rho_\phi[\underline{h}^L_f, \underline{h}^T_f, \underline{\phi}_f] = \int d\underline{\phi}_i \, \tilde{\rho}_\phi[\underline{\phi}_i] \times \left( \delta(\hat{\mathcal{H}}) \, \hat{U}_G \int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi} \, e^{iS_M[\underline{\phi}]+iS_{SG}[\underline{\phi}]+i\Phi[\underline{\phi}]} \, \hat{U}^\dagger_G \, \delta^\dagger(\hat{\mathcal{H}}) \right). \tag{3.18}$$

These influence functionals, Eqs. (3.8) and Eq. (3.15) fully describe the interaction between the environment and the central system, and thus form the basis for a discussion of decoherence. With the IF one could derive a master equation, or a quantum Langevin equation, or attempt to approximate the functional integral itself. These next steps will be the subject of future work when we choose a specific matter system and compute its decoherence rate.

It must be noted that the above influence functional has been previously derived in [52] however they incorrectly imposed TT gauge in the path-integral. This is a gauge choice which can only be consistently made in vacuum and not in the presence of matter. Moreover, as we will discuss further in the following section, it is incorrect to simply impose a gauge condition on the path integral without following the FP procedure as we have and integrating over gauge equivalent boundary data. Naively imposing a gauge condition leads to a gauge dependent result as evidenced by the disagreement between [15, 49, 50, 52]. Mistakes aside, the influence functional reported in [52] is indeed in agreement with ours (up to a factor of 2). To understand why they obtain the correct influence functional while making an incorrect gauge-choice we need to return to the evaluation of the path integral expression for the propagator (2.63). The modified FP trick we used forced us to integrate over gauge-equivalent boundary data. Given a field $h_{\mu\nu}$ the only gauge invariant parts are the TT components[7]. Since the only boundary data which is unaffected is that for the TT components, it is the TT components which end up constituting an environment, and thus it is the TT components which get traced over and it is the TT components which contribute to the influence functional. Integrating over gauge equivalent boundary data then implied that we integrate over the boundary data for $h_{\mu 0}$ and $h^L_{jk}$. Without fixed boundary data these variables were acting simply as Lagrange multipliers in the action which generated the "gauge

[7]The transverse-trace part is also gauge invariant, however it is constrained by the Hamiltonian constraint and thus cannot be considered as independent variable

invariant dressing" operators and the self-gravity interaction terms. By incorrectly enforcing TT-gauge the authors discarded the terms necessary to make the matter state gauge invariant (*i.e.* the dressing operators) and also discarded the terms which describe the self-gravitational interaction for the matter, but they retained the essential terms which described the TT radiation *i.e* the environment.

In the limit that the matter is a collection of point particles, Eq. (3.15) has been reported in [49, 50] but in both of these situations it was assumed that the metric perturbation was purely TT. As a result they missed out on the part of the field constrained by the matter but still obtained the correct influence functional. In neither reference are the gauge invariance and constraints discussed. It is the manifestly gauge invariant computation that we have done which makes it clear how this computation is to be correctly done.

The constraints of the theory have indeed been properly treated in [51, 53], however both computations were done using the canonical linearized ADM formalism rather than using the functional integral approach. The computation done in [51] was in a fixed gauge, while the computation done in [53] was manifestly gauge invariant. They ensured gauge invariance by using the Dirac formalism. Since the path integral formalism developed here has been demonstrated as equivalent to the Dirac formalism, we can regard our result as the path integral equivalent of [53]. The strength of our approach is that functional integrals make certain approximation schemes much more convenient, *e.g.* semi-classical and eikonal techniques may be employed.

## 3.3 Importance of Gauge Invariance

The requirement of gauge invariance led us to correctly identify the correct independent degrees of freedom and separate them from the variables which were constrained. This was essential because in a gauge fixed computation one could mistakenly treat all components of the gauge field as the environment and arrive at a qualitatively different influence functional. A result recently reported in the literature suffers from this error [15]. In this section we will intentionally make this error and show that we reproduce the result of [15].

Recall how gauge invariance was maintained in the path integral. When we used the FP trick the integral over the gauge group for intermediate times factored out as a constant. However it was an essential point that there

were remaining integrals over all gauge equivalent boundary data. This is a distinct difference from the way the FP trick is used in the computation of a partition function or generating functional since in both cases there is no fixed boundary data. In these situations, if the FP determinant also factors out the result of the FP trick is to simply insert a gauge fixing delta function in the path integral and this delta function is often rewritten as a gauge fixing term in the action.

In [15] the author made an essential mistake which led to a prediction that the influence functional depended on every component of the matter stress tensor rather than just the TT components. They handled the gauge invariance of the action by merely inserting the harmonic gauge fixing term in the action. As a result they obtained a qualitatively different result from ours. The addition of the harmonic gauge fixing term broke the gauge invariance of the theory making all components of $h_{\mu\nu}$ seemingly independent.

If we intentionally make this mistake we can reproduce their results. The gravitational action Eq. (1.27) can be rewritten

$$S = \int_{t_i}^{t_f} d^4x \left( -\frac{1}{2}\partial^\sigma h^{\mu\nu}\partial_\sigma \overline{h}_{\mu\nu} + \partial_\mu \overline{h}^{\mu\nu}\partial^\sigma \overline{h}_{\sigma\nu} + \mathcal{L}_M + \frac{1}{M_P} h^{\mu\nu} T_{\mu\nu} \right), \tag{3.19}$$

where the overline denotes trace reversal $\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$. The addition of the harmonic gauge fixing term

$$S_{gf} = -\int_{t_i}^{t_f} d^4x \partial_\mu \overline{h}^{\mu\nu}\partial^\sigma \overline{h}_{\sigma\nu} \tag{3.20}$$

cancels the second term in Eq. (3.19). In this gauge fixed theory the propagator is

$$K(h_f, \phi_f; h_i, \phi_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi \int_{h_i}^{h_f} \mathcal{D}h_{\mu\nu}\, e^{iS_M + i\int_{t_i}^{t_f} d^4x \left(-\frac{1}{2}\partial^\sigma h^{\mu\nu}\partial_\sigma \overline{h}_{\mu\nu} + \frac{1}{M_P} h^{\mu\nu}T_{\mu\nu}\right)}. \tag{3.21}$$

The path integral over $h_{\mu\nu}$ can be immediately evaluated,

$$K(h_f, \phi_f; h_i, \phi_i) = \int_{\phi_i}^{\phi_f} \mathcal{D}\phi\, e^{iS_M + i(S+S_{gf})[h_i^{\mu\nu} \to h_f^{\mu\nu}, \phi]}. \tag{3.22}$$

Unlike the gauge invariant propagator Eq. (2.80) this propagator treats all components of $h_{\mu\nu}$ on equal footing. Clearly this propagator maps between states in the full configuration space $\psi[h_{\mu\nu}, \phi]$ rather than projecting onto the physical subspace. In the full configuration space all components of the

gauge field are independent variables and a state in which the matter is uncorrelated with the gauge field would be written as a product $\rho[\underline{h},\underline{\phi}] = \rho_g[\underline{h}_{\mu\nu}]\rho_\phi[\underline{\phi}]$. Considering the matter as the central system the full gauge field would then seem to be the environment, and a partial trace would be over all of $h_{\mu\nu}$. An uncorrelated state with a thermal gauge field would then be of the form

$$\rho[\underline{h},\underline{\phi}] = \rho_\phi[\underline{\phi}] \int_{h'}^{h} \mathcal{D}h''_{\mu\nu}\, e^{-(S+S_{gf})_E[h''_{\mu\nu}]}. \tag{3.23}$$

For such an initial state the reduced density matrix at a later time is given by the path integral

$$\rho_\phi[\underline{\phi}_f] = \int d\underline{\phi}_i\, \rho[\underline{\phi}_i] \int_{\underline{\phi}_i}^{\underline{\phi}_f} \mathcal{D}\underline{\phi}\, e^{iS_M[\underline{\phi}]+i\Phi[\underline{\phi}]}, \tag{3.24}$$

where the influence phase for the gauge fixed theory is of precisely the same form as in the gauge invariant theory Eq. (3.15) but with the essential difference that in the dissipation and noise kernels Eq. (3.16) and (3.17) the TT projection operator is replaced by the trace-reversal operator $P^{\mu\nu\sigma\rho} = \frac{1}{2}(\eta^{\mu\sigma}\eta^{\mu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})$. The noise and dissipation kernels can then be written in terms of the scalar kernels, $\gamma_{gf}^{\mu\nu\sigma\rho} = 2P^{\mu\nu\sigma\rho}D$ and $\eta_{gf}^{\mu\nu\sigma\rho} = 2P^{\mu\nu\sigma\rho}N$, where

$$D(x) = \frac{1}{4M_P^2}\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\, \frac{\sin px^0}{p}, \tag{3.25}$$

$$N(x) = \frac{1}{4M_P^2}\int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}}\, \coth\left(\frac{\beta p}{2}\right)\, \frac{\cos px^0}{p}. \tag{3.26}$$

To make contact with reference [15] we can take the path integral representation of the reduced density matrix evolution and determine the master equation. The process of passing from a path integral to a master equation in this situation is computationally no different than the tetbook derivation of the Schrödinger equation from the path integral in ordinary quantum mechanics [82]. See Appendix C for the derivation. To lowest order in $M_P^{-1}$, the master equation satisfied by the reduced density matrix in Eq. (3.24) is

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= -i[\hat{H}_M,\rho(t)] \\ &\quad - \int_{t_i}^{t} d\tilde{x}^0 \int d^3x d^3\tilde{x}\, \Big(\eta^{\mu\nu\sigma\rho}(x-\tilde{x})[\hat{T}_{\mu\nu}(x),[\hat{T}_{\sigma\rho}(\tilde{x}),\rho(t)]] \\ &\qquad\qquad -i\gamma^{\mu\nu\sigma\rho}(x-\tilde{x})[\hat{T}_{\mu\nu}(x),\{\hat{T}_{\mu\nu}(\tilde{x}),\rho(t)\}]\Big)\,.\end{aligned} \tag{3.27}$$

If we choose the kernels from the gauge invariant theory, we reproduce equation (19) of [53] (up to a relative sign between the dissipation and noise terms). However if instead we choose the kernels from the naively gauge fixed theory the above master equation exactly reproduces equation (17) of [15].

Rather than depending only on the TT components of the stress-tensor the influence functional (and therefore the master equation) in this naively gauge fixed theory depends on all of the components of the stress tensor. The naively gauge fixed theory has thus made a qualitatively different prediction from our manifestly gauge invariant theory. In the limit that the matter is moving non-relativistically the (00) component of the stress tensor dominates the other components. This component is essentially the mass density of the system, and an influence functional which depends on $T^{00}$ would suggest that systems with sufficiently large mass would decohere regardless of the dynamics and shape of the object. This is the conclusion drawn in [15] which we now claim to be incorrect due to the mistreatment of the gauge invariance/constraints of the theory. The correct computation of the decoherence rates of a number of example systems will be the topic of future works. For now, a general statement can be made as a correction of this mistaken conclusion. Since the correct gauge invariant influence functional depends on the TT part of the stress tensor rather than the (00) components, it should be the change in the mass quadrupole moment not the mass monopole moment which is most important in quantum gravitational decoherence. Such a statement should not be surprising as the time derivative of the mass quadrupole moment is a quantity which of central importance in the classical theory of gravitational wave emission.

# Chapter 4

# Conclusion

In this thesis we developed a tool which will be useful to study decoherence in quantum gravity. We used an effective field theory approach which is expected to capture the universal low every behaviour of whatever the UV completion of quantum gravity may be. In particular, we computed the Feynman-Vernon influence functional for a matter system interacting with a bath of thermal gravitons.

There have been a number of calculations of the influence functional (or the related master equation) in linearized quantum gravity in the literature. Of all these references only [51, 53] include an appropriate discussion of the fact that quantum gravity is a constrained theory. Furthermore only [53] has a manifestly gauge invariant computation. They use the canonical formalism to compute a master equation for the reduced density matrix for matter interacting with a thermal bath of gravitons. The discussion of constraints and gauge invariance is of essential importance because different gauge-fixed results have been reported in the literature and there are qualitative disagreements between them depending on the gauge choice. Until now it has not been clear which result to trust and why the different results don't agree. Although the master equation presented in [53] is gauge invariant their approach is limited to the canonical quantization formalism which lacks clear approximation schemes. In this paper we studied quantum gravitational decoherence using the functional integral formalism, which invites future computations to use functional integral approximation techniques such as diagrammatic, semi-classical, and eikonal expansions.

We presented the first manifestly gauge invariant computation of the reduced density matrix propagator for matter interacting with a thermal bath of gravitons in a path integral representation. En route we developed a manifestly gauge invariant representation of the transition amplitude between gauge field configurations in the presence of matter. We also demonstrated how the first-class constraints in both QED and linearized gravity emerge as natural consequences of gauge invariance in the path integral representation of the propagator. The entire approach was within the functional integral framework, and we demonstrated that our approach is the path integral

equivalent of the Dirac quantization of first-class constrained systems. Our result verifies the validity of the influence functional and master equation which depend only on the TT parts of the matter's stress-energy tensor. After obtaining this main result we explained the mistake made in [15] which led the author to derive a master equation primarily depending on the mass density $T^{00}$.

Our manifestly gauge invariant computation led to the following formal statements which when actually applied to some specific examples provide an illuminating physical picture. The theories we studied are gauge theories so the physical states must be invariant under certain gauge transformations. To be invariant under gauge transformations it is sufficient for the states to be annihilated by the generators of gauge transformation. This condition provides us with functional differential equations that the states must satisfy which can be seen as the constraint equations of the canonical framework. Thus the wavefunctionals for physical states cannot be arbitrary functionals on the configuration space. Some of the gauge field degrees of freedom are independent but there are other (constrained) degrees of freedom for which the form of the wavefunctional is determined entirely by the constraint equation, *i.e.* the requirement of gauge invariance.

In the QED example, the transverse part of the field is gauge invariant and thus an independent degree of freedom whereas the dependence of physical wavefunctionals on the longitudinal part of the field is determined entirely by the Gauss law constraint. In order for physical states to satisfy Gauss' law, the charged matter is always dressed by a coherent state of the electric field corresponding the appropriate Coulomb field. The physical picture is not matter interacting with a vector boson field, but rather it is matter with its accompanying Coulomb dressing that interacts with a field consisting of two types of transverse polarized photons.

The gravity example is quite similar. The analogue of Gauss' law (the momentum constraint) constrains the dependence on the longitudinal part of the field. This constraint similarly requires the matter to be dressed by a coherent gravitational field. In gravity there is also the Hamiltonian constraint which constrains the configuration space variables, effectively eliminating the trace-part of the field. The remaining TT degrees of freedom are independent. The physical picture is analogous to QED; we should not think of matter interacting with a symmetric rank-2 tensor field but rather as matter (dressed by its appropriate coherent gravitational field) interacting with two fields corresponding to the two TT polarizations of the graviton.

In future works we plan to apply these results to problems of quantum decoherence by considering specific physical systems and studying the dy-

namics of their reduced density matrices and computing decoherence rates. Interesting applications include the harmonic oscillator, an extended body which consists of particles coupled via harmonic potentials, a uniformly accelerated particle, as well as corrections to the $2 \to 2$ scattering of matter excitations. Another further application of the influence functional is to the resurging field of infrared gauge theory physics and its potential relation to the black hole information paradox. It has recently been demonstrated that the emission of soft gauge bosons from a scattering process can lead to an outgoing state which is almost completely diagonal in the momentum basis [92]. This is inline with the idea that soft bosons may skirt the no-hair theorem and resolve the black hole information paradox by storing information holographically [93]. To obtain the influence functional we traced over all outgoing states of the photon/graviton fields, including infinitely long wavelength modes. The influence functional may then be used as a tool for understanding information loss due to soft bremsstrahlung in addition to the information loss expected from environmental fluctuations.

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# Appendix A

# The Classical Action in QED and Linear Gravity

In Chapter 2 we evaluated path integrals over the gauge fields $A_\mu$ and $h_{\mu\nu}$. A major point of this thesis was that not all variables are independent and as a result of the gauge invariance of the action we ended up integrating over gauge equivalent boundary data. The only dynamical path integrals which we had to evaluate were over the transverse components $A_j^T$ and TT components $h_j^{TT}$. The evaluation of these integrals is identical for the two theories so here we will just evaluate the QED integral. The relevant integral we are interested in evaluating is found in Eq. (2.45)

$$\int_{A_i^T}^{A_f^T} \mathcal{D}A^T \, e^{iS[A^T,\phi]}. \tag{A.1}$$

The integrals for the free theory in both real time $t_f - t_i$ and Euclidean time $i\beta$ are special cases of the above.

As mentioned when this integral was first seen, it can be evaluated by shifting the integration variable by a function which extremizes the action. That is, the solution to the classical equation of motion subject to the boundary condition that at times $t_i$, $t_f$ the classical solution matches the boundary data of the path integral. The integral over fluctuations is then independent of the boundary data. Since the action is quadratic in the fields the integral over fluctuations simply returns a functional determinant which is an overall constant irrelevant for our purposes. The result of the integral is then just exp $iS_{cl}$ where $S_{cl}$ is the action evaluated for the classical path.

The action for the transverse photons is Eq. (2.41)

$$S[A^T,\phi] = \int_{t_i}^{t_f} d^4x \, P^{\mu\nu} \left( -\frac{1}{2} \partial_\sigma A_\mu \partial^\sigma A_\nu + A_\mu \mathcal{J}_\nu \right). \tag{A.2}$$

The transverse projector can be written as a sum over orthonormal transverse polarization basis vectors, $P_{\mu\nu} = \sum_{\lambda=1}^{2} \epsilon_\mu^\lambda \epsilon_\nu^\lambda$. For a 4-vector $v^\mu$ we

define the transverse part $v^\lambda = \epsilon^\lambda_\mu v^\mu$. The action for transverse photons can then be written as

$$S[A^T, \phi] = \sum_{\lambda=1}^{2} \int_{t_i}^{t_f} d^4x \left( -\frac{1}{2} \partial_\sigma A^\lambda \partial^\sigma A^\lambda + A^\lambda \mathcal{J}^\lambda \right). \tag{A.3}$$

The vector components are decoupled and the action is just the sum of actions for two independent massless scalar fields $A^\lambda$ coupled to their respective sources $\mathcal{J}^\lambda$. Performing a Fourier decomposition on the spatial variables the action can be written

$$S[A^T, \phi] = \sum_{\lambda=1}^{2} \int_{t_i}^{t_f} \int \frac{d^3p}{(2\pi)^3} \left( \frac{1}{2} |\partial_t A^\lambda|^2 - \frac{1}{2} \mathbf{p}^2 |A^\lambda|^2 + A^\lambda(p) \mathcal{J}^\lambda(-p) \right). \tag{A.4}$$

This is precisely the sum of actions for a continuum of decoupled harmonic oscillators $A^\lambda(p)$ with unit mass, frequencies $\omega^\lambda(p) = |p|$, each coupled to a force $\mathcal{J}^\lambda(p)$. The evaluation of the action and thus the path integral for a harmonic oscillator is a textbook exercise but for completeness we will briefly review it [82].

The classical equation of motion for a given mode is

$$(\partial_t^2 + |p|^2) A^\lambda = \mathcal{J}^\lambda. \tag{A.5}$$

The general solution to this equation is the sum of a homogeneous solution and an inhomogenous solution obtained by integrating $\mathcal{J}^\lambda$ with the retarded Green's function. The coefficients of integration are fixed by requiring the solution to satisfy the boundary conditions $A^\lambda(t_{i,f}) = A^\lambda_{i,f}$. The solution satisfying these boundary conditions is

$$\begin{aligned} A^\lambda_{cl}(p) =& \frac{A^\lambda_f}{\sin pT} \sin p(t - t_i) + \frac{A^\lambda_i}{\sin pT} \sin p(t_f - t) \\ &+ \int_{t_i}^{t} ds\, \mathcal{J}^\lambda(s) \frac{\sin p(t-s)}{p} - \frac{\sin p(t - t_i)}{\sin pT} \int_{t_i}^{t_f} ds\, \mathcal{J}^\lambda(s) \frac{\cos p(t-s)}{p}, \end{aligned} \tag{A.6}$$

where $T = t_f - t_i$. Substituting this solution into the action and summing

over modes we obtain the expression for the classical action

$$S_{cl}[A_i^T \to A_f^T, \phi] = \sum_{\lambda=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{p}{2\sin pT} \Bigg( (|A_i^\lambda|^2 + |A_f^\lambda|^2)\cos pT - 2A_f^{\lambda *} A_i^\lambda \tag{A.7}$$
$$+ \frac{2A_i^{\lambda *}}{p} \int_{t_i}^{t_f} dt\, \mathcal{J}^\lambda \sin p(t_f - t) + \frac{2A_f^{\lambda *}}{p} \int_{t_i}^{t_f} dt\, \mathcal{J}^\lambda \sin p(t - t_i)$$
$$- \frac{2}{p^2} \int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \mathcal{J}^{\lambda *}(t) \mathcal{J}^\lambda(s) \sin p(t_f - t) \sin p(s - t_i) \Bigg).$$

The generalization to gravity is obvious. The TT projector can be written as a sum over orthonormal TT basis tensors $\Pi^{\mu\nu\sigma\rho} = \sum_{\lambda=1}^{2} \epsilon_{\mu\nu}^\lambda \epsilon_{\sigma\rho}^\lambda$. Defining the TT part $h^\lambda = \epsilon_{\mu\nu}^\lambda h^{\mu\nu}$, the gravitational action decouples into that of two scalar fields and the classical action is precisely the same as above if $(A, \mathcal{J})$ are replaced by $(h, T)$. The action for the free field is the $\mathcal{J} = 0$ case of the above action and the Euclidean action is obtained from the real time action by $S_E = -iS|_{T=-i\beta}$.

# Appendix B

# Instantaneous Interaction Terms

In the evaluation of the functional integral over the metric variables we found that some of the integrals generated a new interaction term for the matter Eq. (2.74) which is instantaneous. Such a term arose in a way completely analogous to the Coulomb interaction term in QED. In what follows we will show that this term is indeed the correct interaction potential analogous to the Coulomb potential.

Consider the generating functional for the photon field

$$Z[J] = \int \mathcal{D}A_\mu \, e^{iS_0[A] + i\int d^4x \, A_\mu J^\mu}, \tag{B.1}$$

where $J^\mu$ is a conserved classical current. The functional integral can be performed and the result computes the vacuum energy

$$E_0\left(\int dt\right) = i \log Z[J] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} J^\mu D_{\mu\nu} J^\nu, \tag{B.2}$$

where $D_{\mu\nu}(p) = -\eta_{\mu\nu}/p^2$ is the time-ordered Green's function for the electromagnetic field. For compactness we won't write the pole prescription explicitly but this does not affect the results. Now we will follow a procedure done in [94].

We want to introduce a convenient orthonormal basis of vectors where one is purely timelike $n^\mu = (1,0,0,0)$, one is longitudinal $\epsilon_p^\mu = (0, \mathbf{p}/|\mathbf{p}|)$, and the other two are transverse. We will call the transverse vectors $\epsilon_r^\mu$ $(r = 1,2)$. An explicit expression for $\epsilon_p^\mu$ is

$$\epsilon_p^\mu = \frac{p^\mu + (p\cdot n)n^\mu}{(p^2 + (p\cdot n)^2)^{\frac{1}{2}}}. \tag{B.3}$$

It can be checked that $p_\mu \epsilon_p^\mu = |p|$ and $p_\mu \epsilon_r^\mu = 0$. The Minkowski metric can then be written in terms of this orthonormal basis $\eta^{\mu\nu} = -n^\mu n^\nu + \epsilon_p^\mu \epsilon_p^\nu + \sum_r \epsilon_r^\mu \epsilon_r^\mu$. Since we are interested in the contraction of $\eta_{\mu\nu}$ with the conserved

current $p_\mu J^\mu = 0$ we can expand out these basis vectors and drop any terms linear in $p^\mu$. Doing so, the Green's function can be written as

$$D^{\mu\nu} = -\frac{\eta^{\mu\nu}}{p^2} = -\frac{\sum_r \epsilon_r^\mu \epsilon_r^\mu}{p^2} + \frac{n^\mu n^\nu}{p^2 + (p\cdot n)^2}. \tag{B.4}$$

Now we recognize the sum over transverse polarization basis vectors as the transverse projector, and note that $p^2 + (p\cdot n)^2 = |p|^2$. With the Green's function rewritten in this way the log of the generating functional can be written in the nice form

$$\begin{aligned} E_0\left(\int dt\right) &= -\frac{1}{2}\int d^4p \frac{J^\mu P_{\mu\nu} J^\nu}{p^2} + \frac{1}{2}\int dt \int \frac{d^3p}{(2\pi)^3}\frac{J^0 J^0}{|p|} \\ &= -\frac{1}{2}\int d^4p \frac{J^\mu P_{\mu\nu} J^\nu}{p^2} + \frac{1}{2}\int dt \int d^3x d^3y \frac{J^0(x) J^0(y)}{4\pi|x-y|}. \end{aligned} \tag{B.5}$$

What we've done is separate the contribution from transverse photons from the interaction energy of the source. The first term describes transverse photons and the second term can be identified as $-S_C$, the Coulomb interaction potential. For a static source we simply find that the energy of the configuration is given by the Coulomb potential energy.

We can repeat the same exercise for linearized gravity. The generating functional for linearized metric perturbation is given by

$$i\log Z[T] = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4} T^{\mu\nu} D_{\mu\nu\sigma\rho} T^{\sigma\rho}, \tag{B.6}$$

where $T^{\mu\nu}$ is a conserved classical current and $D_{\mu\nu\sigma\rho} = -P_{\mu\nu\alpha\beta}/p^2$ is the time-ordered Green's function for the metric perturbation. The index structure $P_{\mu\nu\sigma\rho} = \frac{1}{2}(\eta^{\mu\sigma}\eta^{\mu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\sigma\rho})$ corresponds to the trace-reversal operator. We can use the above decomposition again for each factor of the Minkowski metric that appears in the trace-reversal operator, and again discard all terms linear in $p^\mu$ because $p_\mu T^{\mu\nu} = 0$. Of course this still leaves many terms. When contracted with the stress tensor many of these terms can are seen to be duplicates of each other. Finally, the generating functional may be written down in this orthonormal basis,

$$\begin{aligned} i\log Z[T] = &-\frac{1}{2M_P^2}\int \frac{d^4p}{(2\pi)^4}\frac{T^{\mu\nu}\Pi_{\mu\nu\sigma\rho}T^{\sigma\rho}}{p^2} \\ &+ \frac{1}{4M_P^2}\int dt \int d^3x \frac{1}{\nabla^2}\Big(T^{00}T^{00} - 4T^{0j}P_{jk}T^{0k} \\ &\qquad +2T^{00}P_{jk}T^{jk} + \frac{\partial_0 T^{00}\partial_0 T^{00}}{\nabla^2}\Big). \end{aligned} \tag{B.7}$$

Having separated off the contribution from TT gravitons we found a static interaction potential. This term can readily be identified as $-S_{SG}$, Eq. (2.74). We then see that indeed the instantaneous gravitational interaction term is precisely analogous to the Coulomb interaction in QED.

# Appendix C

# Derivation of the Master Equation from the Path Integral

In this section we will review how one derives the Schrödinger equation from a path integral representation of the propagator. We will then proceed to make only a slight generalization to show how a master equation for a reduced density matrix can be derived from a reduced density matrix propagator.

To derive the Schrödinger equation we will need to quickly establish some textbook results. Firstly, the derivative of the propagator with respect to the final coordinate is

$$\partial_{x_f} K = \int_{x_i}^{x_f} \mathcal{D}x \left( i \frac{\partial S}{\partial x_f} \right) e^{iS}. \tag{C.1}$$

Since we are taking the quantum average of $\partial_{x_f} S$ Ehrenfest's theorem allows us to use the equation from classical physics $\partial_{x_f} S = p_f$, where $p = \frac{\partial L}{\partial \dot{x}}$ is the canonical momentum. Thus, we can write

$$-i\partial_{x_f} K = \int_{x_i}^{x_f} \mathcal{D}x \, p_f \, e^{iS}. \tag{C.2}$$

Secondly, the time derivative of the propagator is

$$\partial_{t_f} K = \int_{x_i}^{x_f} \mathcal{D}x \left( i \frac{\partial S}{\partial t_f} \right) e^{iS}, \tag{C.3}$$

which can be rewritten using the classical equation $\partial_{t_f} S = -H(x_f, p_f)$, where $H = \dot{x}p - L$ is the Hamiltonian of the system

$$\partial_{t_f} K = -i \int_{x_i}^{x_f} \mathcal{D}x \, H(x_f, p_f) \, e^{iS}. \tag{C.4}$$

Using the momentum relation to re-express $p_f$ as a derivative we obtain the differential equation satisfied by the propagator

$$\partial_{t_f} K = -iH(x_f, -i\partial_{x_f})K, \tag{C.5}$$

which of course is precisely the Schrödinger equation. If the propagator satisfies this equation then upon integrating with initial data we find that a wavefunction at time $t_f$ satisfies the same equation

$$\frac{\partial \psi(x,t)}{\partial t} = -iH(x, -i\partial_x)\psi(x,t). \tag{C.6}$$

Rather than writing the state in position basis we could write the Schrödinger equation as basis independent equation by introducing the abstract Hamiltonian operator. This gives the general expression of the Schrödinger equation

$$\partial_t |\psi(t)\rangle = -i\hat{H}(\hat{x}, \hat{p})|\psi(t)\rangle. \tag{C.7}$$

Now suppose we had two distinct particles with coordinate labels $x^1$ and $x^2$, each having its own free action with corresponding free Hamiltonian $H_j(x^j, p_j)$, and let the particles be coupled via a retarded interaction. The propagator for such a system is

$$\begin{aligned} &K(x^1_f, x^2_f, t_f; x^1_i, x^2_i, t_i) = \\ &\quad = \int_{x^1_i}^{x^1_f} \mathcal{D}x^1 \int_{x^2_i}^{x^2_f} \mathcal{D}x^2 \, e^{iS_1[x^1] + iS_2[x^2] + i\int_{t_i}^{t_f} dt \int_{t_i}^{t} ds \Phi(x^1, \dot{x}^1, x^2, \dot{x}^2; t, s)}. \end{aligned} \tag{C.8}$$

The above derivation implies that the Schrödinger equation for such a system is then

$$\begin{aligned} \frac{\partial \psi(x^1, x^2, t)}{\partial t} = &- i\left(H_1(x^1, \hat{p}_1) + H_2(x^2, \hat{p}_2)\right)\psi(x^1, x^2, t) \\ &- \left(\int_{t_i}^{t} ds\, \Phi(x^1, \hat{\dot{x}}^1, x^2, \hat{\dot{x}}^2; t, s)\right)\psi(x^1, x^2, t), \end{aligned} \tag{C.9}$$

where the velocity operators are to be interpreted as functions of $x^j$ and $\hat{p}_j$. If $\Phi$ is a function only of coordinates and not velocities then then the situation is simple and the canonical momenta are just that of the free Hamiltonian and the velocities can be solved for as usual. There is a difficulty however if $\Phi$ is a function of velocities. In that case $p_j$ depends on the derivatives of $\Phi$, which in turn depends on $p_j$ through the velocities. The expression of the velocities in terms of the momenta is then an iterative Born series.

If the interaction is assumed small we can make a Born approximation and truncate the iterative equation at lowest order. In this approximation the canonical momenta are simply given by that of the free Hamiltonian. We'll call the Born approximated operator $\Phi_0(x^1, \hat{p}^1, x^2, \hat{p}^2; t, s)$.

To bring us closer to an expression looking like a master equation lets assume that the interaction has the form

$$\begin{aligned}\Phi_0(x^1, \hat{p}^1, x^2, \hat{p}^2; t, s) = \big(\hat{f}_1(t) - \hat{f}_2(t)\big) D(t, s) \big(\hat{f}_1(s) + \hat{f}_2(s)\big) \\ + i\big(\hat{f}_1(t) - \hat{f}_2(t)\big) N(t, s) \big(\hat{f}_1(s) - \hat{f}_2(s)\big)\end{aligned} \tag{C.10}$$

where $D(t, s)$ and $N(t, s)$ are just functions of time and $\hat{f}_j(t) = f(\hat{x}^j, \hat{p}_j, t)$. If we also write $\hat{H}_j = H(\hat{x}^j, \hat{p}_j)$ then we can expand out the interaction and write the Schrödinger equation as

$$\begin{aligned}\frac{\partial \psi(x^1, x^2, t)}{\partial t} = -i\Bigg(\hat{H}_1 + \hat{H}_2 - \int_{t_i}^{t} ds\, D(t, s) \big(\hat{f}_1(t) - \hat{f}_2(t)\big) \big(\hat{f}_1(s) + \hat{f}_2(s)\big) \\ - i \int_{t_i}^{t} ds\, N(t, s) \big(\hat{f}_1(t) - \hat{f}_2(t)\big) \big(\hat{f}_1(s) - \hat{f}_2(s)\big)\Bigg) \psi(x^1, x^2, t)\end{aligned} \tag{C.11}$$

Rather than working in position space this expression can be made more compact if we invent some notation and write a basis independent expression. Let's define the operators $\rho$ and $\hat{f}(t)$ such that the position space matrix elements are $\hat{f}_1(t)\psi(x^1, x^2, t) = \langle x_1 | \hat{f}(t) \rho | x_2 \rangle$ and $\hat{f}_2(t)\psi(x^1, x^2, t) = \langle x_1 | \rho \hat{f}(t) | x_2 \rangle$. Depending on which side $\hat{f}$ acts we can describe either particle. With this step, and the final assumption that $H_2 = -H_1$ we can write the Schrödinger equation for this strange system in the basis independent form

$$\frac{\partial \rho}{\partial t} = i[\hat{H}(t), \rho] - \int_{t_i}^{t} ds \big(N(t, s)[\hat{f}(t), [\hat{f}(s), \rho(t)]] - iD(t, s)[\hat{f}(t), \{\hat{f}(s), \rho(t)\}]\big) \tag{C.12}$$

Now, it would take a rather strange system of two particles to be described by a Hamiltonian which has i) an interaction of this form as well as ii) $H_2 = -H_1$. Luckily with this computation we were never really interested in a system of two real particles. Instead we were interested in the equation of motion satisfied by the reduced density matrix. In this case the two coordinates $x^1, x^2$ do not label different particles, they label the *ket* and *bra* part of the density matrix respectively. For a full density matrix there are no terms which "couple" the two paths taken by these coordinates (*i.e.* the

forward and backward in time paths). We've seen though that a generic form of the influence functional is $\mathcal{F}[\underline{\phi}] = e^{i\Phi[\underline{\phi}]}$, and it certainly couples the paths. The strange retarded interaction (C.10) is then nothing other than the influence phase. We've thus seen that a reduced density matrix which evolves according to a propagator of the form Eq. (C.8), will satisfy the master equation Eq. (C.12). The generalization from a single coordinate $x$ to a generic set of coordinates $\phi$ is immediate.