Eikonal Analysis of Linearized Quantum Gravity

A Functional Approach

by

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Abstract

The low energy effective theory of quantized gravity is currently our most successful attempt at unifying general relativity and quantum mechanics. It is expected to serve as the universal low energy limit of any future microscopic theory of quantum gravitation, so it is crucial to properly understand its low frequency, long wavelength, "infrared" limit. However, this effective theory suffers from the same kind of infrared divergences as theories like quantum electrodynamics. It is the aim of this work to characterize these divergences and isolate the infrared behavior of quantum gravity using functional methods. We begin with a review of infrared divergences, and how they are treated in QED. This includes a brief overview of the known applications of functional methods to the problem. We then discuss the construction of the effective field theory of quantum gravity in the linearized limit, coupled to scalar matter. Proceeding to the main body of the work, we employ functional techniques to derive the form of the scalar propagator after soft graviton radiation is integrated out. An eikonal form for the generating functional of the theory is then presented. In the final chapter, we use this generating functional to derive the soft graviton theorem and the eikonal form of the two-body scalar scattering amplitude. The result is a concise derivation of multiple known results, as well as a demonstration of the factorization of soft graviton radiation against the eikonal amplitude. We conclude with some comments on how these results can be extended, and we argue that the functional framework is the best candidate for a unified understanding of all relevant infrared features of quantum gravity.

Lay Summary

It is often said that the two most successful theories of modern physics - quantum mechanics and general relativity - are incompatible. This is not strictly true. At low energies, these two theories can be combined to some success, though the approach has its own limits. This work examines how to unify the discussion of various phenomena in quantum gravity under one mathematical framework, in the limit of extremely low energies. It is hoped that such a description will also contain insight into more general properties of quantum gravity and quantum field theory.

Preface

This dissertation is original, unpublished, independent work by the author, C. DeLisle.

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For Cindi.

Chapter 1

Introduction

From Newtonian gravity to electroweak theory, the draw of unifying seemingly unrelated phenomena under one framework has been responsible for much of the current landscape of physical understanding. Tying together quantum mechanics (QM) and general relativity (GR) remains perhaps the most sought-after unification in the field. In many aspects, the two theories seem utterly at odds with one another, and even the question of whether gravity *must* be quantized is still unanswered. However, it is a remarkable fact that assuming gravity is quantized, GR and QM can get along at low energies as an effective field theory. Indeed, in the low energy limit we expect any sensible theory of quantized gravity to reduce to this effective description in this picture. Using this intuition as motivation, we note that the infrared limit of the effective field theory may provide a more accessible way to search for hints of a full unified theory of quantum gravity, as opposed to the more difficult path of quantizing GR at extremely high energies and short distances, as is attempted by theories of strings [6], loops [7], etc.

The problem with this approach is that even our best understood physical theory - quantum electrodynamics - is still suffering from a problem of infinities in its infrared limit. While it is known in a technical sense how to make physical predictions by carefully discarding these divergences [29–31], it should be uncontroversial to say that their origin and cancellation is unsatisfactorily understood on physical grounds. The same can be said for analogous problems that remain in the quantum gravity case.

It is the purpose of this work to apply a powerful functional technique originally developed by E.S. Fradkin [2] to the problem of IR divergences in quantum gravity. We aim to show that this method is well-suited to discussion of these issues in a single language by reproducing known results from the literature. We also offer comments and examples of ways in which these known results can be extended or better understood in the functional picture. This work too is a sort of unification, in that we believe the functional picture of soft gravity provides an easier way to analyze all of the contributions to IR issues in parallel.

The organization of the thesis is as follows. In Chapter 1, we introduce these general issues arising in the study of IR physics in quantum field theory, using QED as an example, and lay out the basic elements of the most simple theory of quantum gravity coupled to matter - Einstein GR minimally coupled to one real scalar field. Chapter 2 then deals with how to analyze this theory in the functional eikonal framework. We derive the eikonal form of the generating functional, as well as the scalar propagator. Finally, chapter 5 shows how to use these results to compute S-matrix elements, as well as demonstrating how the soft graviton theorem emerges naturally in this limit from the nonperturbative calculation. We conclude with some comments on what new results may emerge from this framework.

What is meant by *soft*? For the entirety of this thesis, we define *soft* radiation as that which has momentum q that satisfies

$$q^2 \ll m^2, \tag{1.1}$$

where m is the mass of the scalar particle. For a scalar particle interacting with virtual soft quanta, this means that the scalar loses (gains) negligible momentum to emitted (absorbed) gravitons. This enforces the condition that, throughout the process, the scalar remains nearly on-shell. Calling the momentum of the scalar at some point in the process k, this means

$$k^2 \approx m^2, \tag{1.2}$$

even in situations where the virtual particle momenta are integrated over. Furthermore, the words "soft" and "eikonal" will often be used interchangeably.

Regarding Notation: In this thesis, we use the mostly-minus metric signature,

$$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \tag{1.3}$$

use commas to denote partial derivatives ($A_{,\alpha} \equiv \partial_\alpha A$), and employ the Einstein summation convention, where repeated indices are summed implicitly. We also commonly suppress coordinate-dependence, Lorentz indices, integration measures, and integration variables, where these things should be clear from the context. A few examples:

$$\int fj \equiv \int d^4x\, f(x)j(x), \tag{1.4}$$

$$\int jA \equiv \int d^4x\, j_\mu(x)A^\mu(x), \tag{1.5}$$

$$\int \frac{\delta}{\delta I}\Delta\frac{\delta}{\delta I} \equiv \int d^4x \int d^4y \frac{\delta}{\delta I^{\mu\nu}(x)}\Delta^{\mu\nu\alpha\beta}(x,y)\frac{\delta}{\delta I^{\alpha\beta}(y)}, \tag{1.6}$$

etc. Unless otherwise specified, seeing $\int$ with no measure implies $\int d^4x$ or $\int d^4x \int d^4y$ depending on the context. For momentum integrals, we use the notation

$$\sum_q^{(D)} \equiv \int \frac{d^Dq}{(2\pi)^D}, \tag{1.7}$$

with the special case

$$\sum_q \equiv \sum_q^{(4)} \equiv \int \frac{d^4q}{(2\pi)^4}, \tag{1.8}$$

since we usually work in (3+1) dimensions. Lastly, we choose to use natural units, where $\hbar = c = 1$.

1.1 Infrared Problems in Quantum Field Theory

Quantum field theory - quantum electrodynamics (QED) in particular - is the most well-tested theory in the history of science [25]. It is the foundation of our current understanding of the interaction of matter via all forces except gravity, though as we will see, even gravity can in some limits be accounted for. In spite of its experimental success, the theory is still imperfect. We more or less know how to obtain sensible results from QED calculations despite many integrals resulting in infinite answers, though some of the current methods for dealing with such divergences leave a lot to be desired. Divergences that come from low energy massless excitations are particularly interesting, and in this section we will discuss why that is.

In this introduction, we introduce the topic of IR divergences using results as presented in [29] and chapter 6 of [26], and borrow their conventions as needed. The discussion of IR divergences is first presented from a diagrammatic perspective, followed by some background on the functional methods used to attack the same problems. We assume the reader is familiar with QED and its Feynman rules.

1.1.1 IR Divergences

Consider the amplitude corresponding to Figure 1.1 in QED.

The momentum q can be thought of as an arbitrary external source. Evaluation of this diagram involves an integral over the virtual photon momentum l - an integral (eq. (6.38) of [26]) which is seen to diverge both as $|l| \to \infty$ and as $|l| \to 0$. The first type of divergence, we call a *UV divergence*. These are infinities coming from the naive assumption that the theory of QED is valid up to arbitrarily high energy scales. We will not be concerned with this type of infinity in this work except to refer to textbook discussions of renormalization, in e.g. [5, 26].

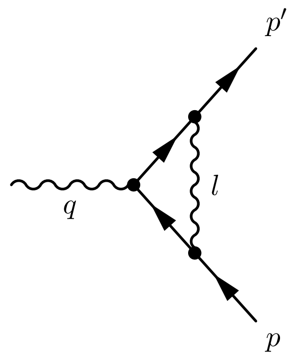


Figure 1.1: Loop correction to the QED vertex.

As the photon momentum goes to zero however, we encounter *IR divergences*. These are caused by the photon's masslessness and we will discuss their effects here. Beyond showing up as an infinity in the calculation of an amplitude, IR divergences appear in computation of observables. The effect of this diagram's IR divergence on the differential cross section in the limit of strong external field ($q^2 \to \infty$, used for computational convenience) turns out to be

$$\frac{d\sigma}{d\Omega}(p \to p') = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) \ln\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right] \tag{1.9}$$

where $\alpha \equiv e^2/4\pi$, $\left(\frac{d\sigma}{d\Omega}\right)_0$ is the cross section without the inclusion of the virtual photon, m is the mass of the electron, and μ is a fictitious photon mass used to regulate the integral. Taking the massless limit, we see that this makes the logarithm diverge. This form was first discovered by Sudakov [27] and is now referred to as the Sudakov double logarithm.

What are we to do with this? Of course observed cross sections do not diverge. The key actually lies in the expression of a seemingly unrelated process - the emission of a single low energy photon. This amplitude really has two contributions, displayed in figure 1.2, as the photon can couple to either the ingoing or outgoing fermion leg.

This amplitude is tree level, and so easy to compute. The addition of the

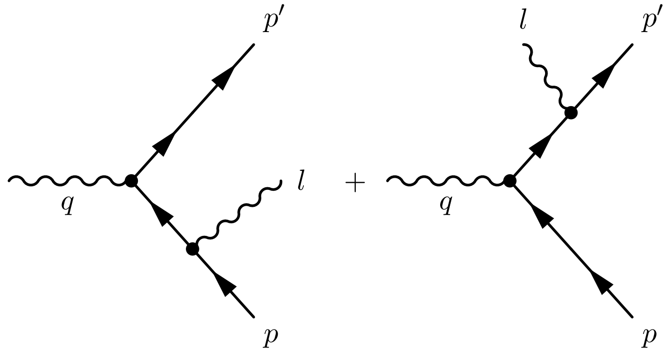


Figure 1.2: Amplitude for scattering and producing a low energy photon.

external photon, as $|l| \to 0$, causes the following amendment to the original cross section:

$$\frac{d\sigma}{d\Omega}(p \to p' + \gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[\frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) \ln\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right] \tag{1.10}$$

Remarkably, this looks identical to the second term in (1.9), with the opposite sign. This means that these processes on their own seem to be unphysical, while their *sum* is perfectly finite. To interpret this result, we must realize that in any real experiment, we cannot measure emitted photons of arbitrarily low energy. We denote the lower end of the momentum scale we can resolve by λ. To compute the inclusive cross section of these two processes, we simply add the two previous results, but integrate the momentum of the emitted photon from 0 to λ, allowing for all photon energies that are indistinguishable to our experiment. The divergent contributions to the physical result give

$$\frac{d\sigma}{d\Omega}_{\text{phys}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \ln\left(\frac{-q^2}{m^2}\right) \ln\left(\frac{-q^2}{\lambda^2}\right) + \mathcal{O}(\alpha^2)\right]. \tag{1.11}$$

This is a result depending only on perfectly well-defined physical parameters. The continuation of this argument to higher orders in perturbation theory is not trivial. Details can be found in [29, 31], but we will give an overview of the results here.

In ref. [27], Sudakov actually goes beyond showing eq. (1.9). He sums all similar diagrams in perturbation theory (vertex corrections involving low energy virtual quanta) and shows that the result exponentiates. This means that the full contribution of the vertex correction, after summing to all orders, is

$$\frac{d\sigma}{d\Omega}(p \to p') = \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right]. \tag{1.12}$$

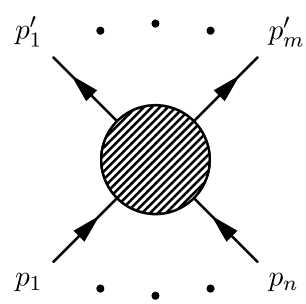
Notice that now

$$\frac{d\sigma}{d\Omega}(p \to p') \xrightarrow{\mu \to 0} 0 \tag{1.13}$$

so that this IR divergence causes the cross section to vanish! What remains is to look at the contribution of processes that involve emission of more than one soft photon. We will go over this in some detail, as we will make contact with this derivation in later chapters.

Consider some *known* amplitude in QED. Let this amplitude be denoted $\mathcal{M}$. Specifically, we consider the amplitude for n fermions to scatter to m fermions, with no incoming or outgoing photons.

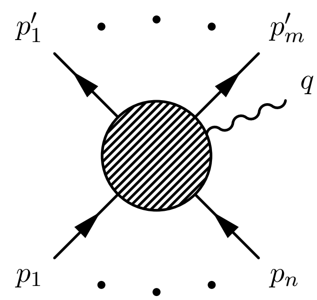
$\mathcal{M} =$



(1.14)

We now ask the question: what is the effect on this amplitude of adding one additional soft outgoing photon line to this amplitude? We take this photon to have momentum q, polarization $\epsilon_\mu(q)$, and assume the limit $|q| \to 0$.

$$\mathcal{M} \to \mathcal{M}' = \tag{1.15}$$



First we note that attaching the line anywhere inside the blob will not lead to any infrared divergences, as the particles in the blob are generically off-shell. However, consider what happens when we attach the line to the i^{th} *outgoing* on-shell matter line with momentum p_i'.

This multiplies $\mathcal{M}$ by a vertex function and a propagator factor. We enforce the momentum shell condition for p_i', and neglect the small q^2 terms, making the total multiplicative factor

$$\frac{e(p_i')_\mu}{p_i' \cdot q - i\delta}, \tag{1.16}$$

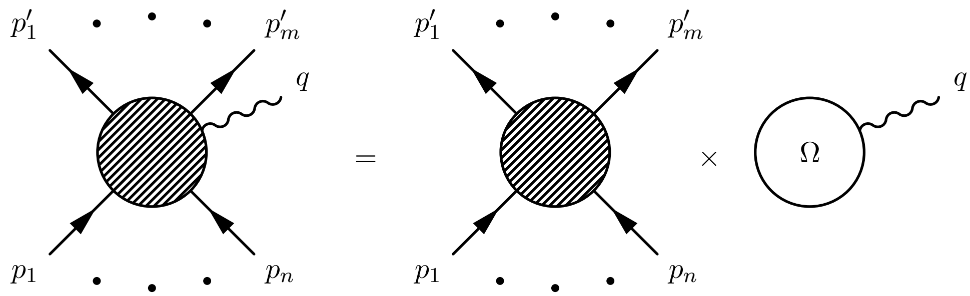
which is clearly divergent as $|q| \to 0$. Similarly, if we had attached the photon line to the j^{th} *incoming* fermion, we would multiply the amplitude by a factor

$$-\frac{e(p_j)_\mu}{p_j \cdot q + i\delta}. \tag{1.17}$$

For $n \to m$ scattering then, the amplitude becomes simply

$$\mathcal{M}' = \mathcal{M} \times e\left[\sum_i^m \frac{(p_i')_\mu}{p_i' \cdot q - i\delta} - \sum_j^n \frac{(p_j)_\mu}{p_j \cdot q + i\delta}\right] \equiv \mathcal{M} \times e\Omega_{\mu\nu}, \tag{1.18}$$

or in pictures:



(1.19)

The important thing here is that the contribution from the soft radiation *factorizes* against the known hard amplitude $\mathcal{M}$. The omegas are factors that are gauge invariant and do not depend on the spin of the hard particles. The Ω notation is simply to have a consistent name for these soft factors throughout this work. It is also shown in [29] that the generalization to N photon emissions is straightforward:

$$\mathcal{M}^{(N)} = \mathcal{M} \times \prod_l^N e \left[\sum_i^m \frac{(p_i')_\mu}{p_i' \cdot q_l - i\delta} - \sum_j^n \frac{(p_j)_\mu}{p_j \cdot q_l + i\delta} \right] \equiv \mathcal{M} \times \prod_l^N e(\Omega_\mu)_l, \tag{1.20}$$

or, attaching the outgoing photon wavefunction and normalization factors to write the full S-matrix element:

$$\mathcal{M} \times \prod_l^N \frac{e}{(2\pi)^{3/2}\sqrt{2|q_l|}} \left[\sum_i^m \frac{\epsilon^* \cdot p_i'}{p_i' \cdot q_l - i\delta} - \sum_j^n \frac{\epsilon^* \cdot p_j}{p_j \cdot q_l + i\delta} \right] \equiv \mathcal{M} \times \prod_l^N e\Omega_l. \tag{1.21}$$

Equation (1.21) is called the *soft photon theorem*, as it describes the change in the amplitude due to the addition of N external soft photon lines.

Let us apply this result to our simple scattering example, generalizing the process in Figure 1.2. We recall that q now means $p' - p$ once again. The correction to the cross section from all possible numbers of emitted photons with unmeasurable momenta (also summed over polarizations) is given by

$$
\begin{aligned}
&\lim_{\mu\to 0}\sum_n \frac{1}{n!}\left[\int_\mu^\lambda \frac{dk^0}{2\pi}\sum_k^{(3)}\frac{-e^2}{2k^0}\left(\frac{p'^\nu}{p'\cdot k}-\frac{p^\nu}{p\cdot k}\right)\left(\frac{p'_\nu}{p'\cdot k}-\frac{p_\nu}{p\cdot k}\right)\right]^n \\
&= \exp\left[\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{\lambda^2}{\mu^2}\right)\right]\xrightarrow{\mu\to 0}\infty.
\end{aligned}
\tag{1.22}
$$

This infinite result seems unphysical, but combining this with the vertex correction (1.12), as we did before, the physical cross section is

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}_{\text{phys}} &= \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{\mu^2}\right)\right]\exp\left[\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{\lambda^2}{\mu^2}\right)\right] \\
&= \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\ln\left(\frac{-q^2}{m^2}\right)\ln\left(\frac{-q^2}{\lambda^2}\right)\right].
\end{aligned}
\tag{1.23}
$$

This result contains information from all orders in perturbation theory, and again only depends on physical parameters. This form can be shown to reproduce semiclassical predictions for the number of photons radiated as a function of the parameters [26].

This sort of cancellation seems to imply that in order to talk about physical transitions, we should speak only of processes including infinite amounts of soft radiation. In fact, with the above results, only these inclusive cross sections are nonzero and finite. That this radiation is inevitable was already discussed by Bloch and Nordsieck [30] long before the modern derivations of the above results in e.g., [29, 31]. While our interpretation in terms of unmeasurable soft modes is a practical one, formally it leaves much to be desired. This discontent can be seen in the works of many authors who attempt to redefine asymptotic states or the S-matrix in order to write down a theory that never includes IR divergences in the first place [51–57]. We note here that application of the functional formalism discussed next to these sorts of "dressed state" constructions could be enlightening, and will be the subject of future work.

1.1.2 Functional Methods

The Bloch and Nordsieck approach came before the previous perturbative arguments, and involved understanding the behavior of an electron in a background electric field. The background field was constrained only to involve low energy modes. This setup can be used in a more general functional formalism originally due to Fradkin [2]. Here we will briefly discuss the application of this method to QED and show some of the peculiarities of its results. In particular, we derive the form of the electron propagator in the presence of a slowly varying background field. This form of the propagator can be used to derive an expression for the generating functional of the theory, as is done in the next chapter for linearized quantum gravity. Appendices C and D also discuss the relationship of the generating functional to the S-matrix, which will be useful in deriving the soft theorems we have just seen for gravitons. Our presentation closely follows that of Bogoliubov and Shirkov [36], though arguments can be found in other books and papers as well.[1]

The equation of motion for the fermion field ψ in some *fixed* background vector potential A_μ is

$$[\gamma^\mu \left(i\partial_\mu + eA_\mu(x)\right) - m]\,\psi(x) = 0. \tag{1.24}$$

In the Bloch-Nordsieck model, we approximate the gamma matrices (making this a scalar theory of QED) by making the replacement

$$\gamma^\mu \to u^\mu, \qquad u^2 = 1, \tag{1.25}$$

where u^μ is a vector of constant numbers. Its interpretation in momentum space is the velocity of the particle, $u^\mu = p^\mu/m$. This model leads to the definition of the electron propagator on the background

$$[u^\mu \left(i\partial_\mu + eA_\mu(x)\right) - m]\,\mathcal{G}(x, x'|A) = -\delta(x - x'). \tag{1.26}$$

[1] See e.g. [34, 35] or the books of Popov [37] or Fried [1] and references therein.

This equation can be solved using the Schwinger/Fock "proper time" representation of the propagator, giving the formal solution

$$\mathcal{G}(x,x'|A) = i\int_0^\infty ds \exp\left\{is\left[u^\mu\left(i\partial_\mu + eA_\mu(x)\right) - m + i\epsilon\right]\delta(x-x')\right\}. \tag{1.27}$$

The resulting form of $\mathcal{G}(x,x'|A)$ is discussed in detail when we do the calculation for gravity. We then take the result, and functionally integrate over all possible long wavelength configurations of the background A_μ to get the full electron propagator:

$$\mathcal{G}(x,x') = \int \mathcal{D}A\, e^{iS[A]}\mathcal{G}(x,x'|A), \tag{1.28}$$

where $S[A]$ is just the action of the free vector field. The momentum space result for the propagator in the long wavelength approximation turns out to be

$$\mathcal{G}(p) \approx \frac{1}{m - u\cdot p}\left|1 - \frac{u\cdot p}{m}\right|^\zeta, \tag{1.29}$$

where $\zeta \equiv -\frac{\alpha}{2\pi}(3-\xi) + \mathcal{O}(\alpha^2)$, and ξ is a gauge-fixing parameter. The form of ζ in the gauge where $\xi = 0$ was found by Solov'ev [38] to all orders in α, and this result was generalized to arbitrary gauges in [40]. This form shows a correction to the bare propagator by the multiplicative factor $\left|1 - \frac{u\cdot p}{m}\right|^\zeta$, which can be expanded as

$$\left|1 - \frac{u\cdot p}{m}\right|^\zeta = 1 - \frac{\alpha}{2\pi}(3-\xi)\ln\left|1 - \frac{u\cdot p}{m}\right| + \mathcal{O}(\alpha^2) \tag{1.30}$$

showing logarithmic corrections that are reminiscent of those we saw after cancelling off IR divergences. The difference is that now we have none of the intuition of summing up indistinguishable processes to aid us in interpreting this result. Looking closely, we see that the propagator no longer has a simple pole at $u\cdot p = m$. This may appear troublesome, but in the gauge of Yennie and Fried [33], where $\xi = 3$, we see that this odd behavior disappears entirely. Johnson and Zumino [32] remark that the multiplicative factor is due only to the (gauge-dependent) description of scalar and longitudinal modes of A_μ, which should not be interpreted as physical degrees of freedom. Thus we should not be surprised that we can remove this factor without changing the physics, as it is pure gauge. A proof of this at all orders of perturbation theory was given by Braun [39], though that computation missed a change in the wavefunction renormalization that was then corrected in [40].

Before moving on, we mention that functional methods have also already been applied to the computation of eikonal amplitudes in QED [41, 42] and in gravity [45]. We will elaborate upon this approach later in this work, so we do not discuss it in depth here. Such amplitudes result from summing an infinite number of Feynman diagrams [43]. In the two-fermion scattering case, these diagrams are the ladder type graphs obtained by ignoring vertex corrections and the vacuum polarization of the exchanged photons. The result that we reproduce with the functional formalism for gravity is found in [44]:

$$i\mathcal{M} = 8Ep \int d^2x_\perp e^{-iq_\perp \cdot x_\perp} \left(e^{i\chi} - 1\right) \tag{1.31}$$

where E is the energy and p is the center of mass momentum of both particles, q is the momentum transfer, and the "eikonal" χ is

$$\chi \approx -\frac{G\gamma(s)}{Ep} \ln(\mu x_\perp) \tag{1.32}$$

for $\gamma(s) = \frac{1}{2}\left[(s-2m^2)^2 - 2m^4\right]$, s is the usual Mandelstam variable, and μ is a graviton mass serving as an IR regulator. We refer to this paper after recovering this result for further comments on bound state poles in the amplitude and relation to previous results.

We will show in this work that the functional formalism is well-equipped to discuss all of the issues presented in this section. It contains all of the mentioned IR divergences as well as their cancellation, and can at the same time recover IR correlators and eikonal amplitudes. This thesis should be seen as a unification of these ideas and a first step in using the functional approach to better understand linearized quantum gravity.

But first - a quick review of linearized quantum gravity.

1.2 Theory of Quantized Gravity and Matter Fields

Shortly after the first major developments of quantum mechanics, an obvious question presented itself: how might quantum effects appear in Einstein's (also still relatively new) generally covariant theory of gravity [8]? This question is at the heart of the field appropriately named *quantum gravity*.

Carlo Rovelli's "Notes for a brief history of quantum gravity" [12] helpfully break the myriad of approaches to the field into three categories, the "covariant line of research," the "canonical line of research," and the "sum over histories line of research." The approach that we will focus on in this work falls into the "covariant line of research," in that it splits the spacetime metric into a flat background, and a dynamical spin-two field that propagates on that background. In this line of research, the dynamical field is then quantized, and interacts with quantum matter in a way described by conventional, flat spacetime quantum field theory. One simplifying aspect of this approach is that it straightforwardly avoids the question of how to generalize quantum coordinate systems to allow a relativistic description. We will give a brief discussion of the subset of this line of work that is relevant to us, ignoring the other approaches except to refer the interested reader to those related references in the Rovelli notes.

To introduce the necessary elements of this sort of quantum theory of gravity, we first discuss the full and linearized Lagrangian derivation of the Einstein field equations (EFEs) in the presence of matter. We then discuss previous attempts to understand how gravity can be quantized, and how even though the best we can do is a formally nonrenormalizable theory, we can gain understanding at low energies through the lens of effective field theory (EFT). This discussion includes a derivation of the Feynman rules for our theory. Throughout the discussion we will assume a basic understanding of QFT, renormalization, and GR for the sake of brevity, and we assume strictly unmodified Einstein GR taking the connection to be Levi-Cevita and imposing only minimal coupling to matter.

1.2.1 The Lagrangian Formulation of GR

The EFEs can be derived via the stationary action principle from the following (Einstein-Hilbert) action:

$$S[\phi, g_{\mu\nu}] = \int d^4x\sqrt{-g}\left[\frac{2}{\kappa^2}R + \mathcal{L}_\phi\right] \equiv S_{EH}[g_{\mu\nu}] + \int d^4x\sqrt{-g}\mathcal{L}_\phi. \quad (1.33)$$

R is the Ricci scalar, obtained by fully contracting the Riemann curvature tensor. The Riemann tensor is built from the Levi-Cevita connection coefficients ($R^{\mu}_{\nu\alpha\beta} \sim \partial\Gamma - \partial\Gamma + \Gamma\Gamma - \Gamma\Gamma$) which are in turn built from the spacetime metric $g_{\mu\nu}$ and its derivatives ($\Gamma \sim \frac{1}{2}g[\partial g + \partial g - \partial g]$). $\kappa^2 = 32\pi G$, $g \equiv \det(g_{\mu\nu})$, and $\mathcal{L}_\phi$ is the Lagrangian density for any matter in the theory. Generically, ϕ can be shorthand for any types of matter fields, but for concreteness and simplicity we take it to be a single real scalar field with Lagrangian density

$$\mathcal{L}_\phi[\phi] = \frac{1}{2}\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2\right]. \quad (1.34)$$

The stationary action principle demands that $\delta S_{EH}/\delta g^{\mu\nu} = 0$, yielding immediately the EFEs:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa^2}{4}T_{\mu\nu} \quad (1.35)$$

$T_{\mu\nu}$ is the stress-energy tensor of the matter, defined by

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}}. \quad (1.36)$$

This formalism leads to a consistent classical field theory, but how do we go about quantization? For our purposes, it will be sufficient to focus on taking the limit of weak gravitational field.

1.2.2 Linearization

If we focus our discussions on situations with only small spacetime curvature (e.g., a few elementary particles or mesoscopic systems) we can simplify things a great deal. We will break the metric into two pieces: a constant Minkowski background and the dynamical field that encodes the departure of the full metric from that background. In other words,

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x). \tag{1.37}$$

The factor of κ is for convenience in defining a canonically normalized quantum field theory.

So far, this procedure is exact. At this point we introduce an approximation by assuming that $|\kappa h_{\mu\nu}| \ll 1$. What we want is to reproduce the linearized version of the EFEs. To do this, we keep to leading order in κh in the pure gravity action, and only the linear matter coupling term. After some tedious algebra, the full action can be written as $S[\phi, h_{\mu\nu}] = \int d^4x\mathcal{L}$, with

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_\phi + \mathcal{L}_{int}, \tag{1.38}$$

$$\mathcal{L}_g[h] \equiv \frac{1}{2}\partial_\mu h_{\alpha\beta}\partial^\mu h^{\alpha\beta} - \frac{1}{2}\partial^\mu h \partial_\mu h - \partial_\alpha h^{\alpha\gamma}\partial^\beta h_{\beta\gamma} + \partial_\alpha h^{\alpha\beta}\partial_\beta h \tag{1.39}$$

$$\mathcal{L}_\phi[\phi] \equiv \frac{1}{2}\eta^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \frac{1}{2}m^2\phi^2 \tag{1.40}$$

$$\mathcal{L}_{int}[\phi, h] \equiv -\frac{\kappa}{2}h^{\mu\nu}T_{\mu\nu} \tag{1.41}$$

Here, $h \equiv \eta_{\mu\nu}h^{\mu\nu}$ is the trace of the metric perturbation. For a scalar field, we have

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}_\phi \tag{1.42}$$

Varying this new action w.r.t. $h^{\mu\nu}$ gives the linearized version of the EFEs.

This sort of formalism is not new. The above manipulations are found in introductory GR textbooks, and we refer to [10] for introduction and [9] for further information on the classical theory.

1.2.3 Gravitons

Theories that quantize gravity around a flat background date back to at least 1930 [12] with Rosenfeld [13, 14] offering a first attempt at applying quantization rules to the linearized EFEs. The sixties saw further development, notably due to the contributions of Feynman [15] and DeWitt [16–18]. The article of Feynman provides a particularly readable account of the connection of flat space quantization techniques to classical gravitational physics via computation of tree level scattering amplitudes. It also provides a sense of the first confusion resulting from the nonrenormalizability of quantizing gravity in the way we are about to. In order to understand these kinds of calculations, we will proceed with quantizing our linearized theory via the path integral. We can use path integral language to simply state the vacuum-to-vacuum amplitude for the graviton field in the presence of a classical source $I_{\mu\nu}$. This amplitude is known as the *generating functional* (elaborated upon in the appendix). Writing this down for pure gravity, we have

$$\mathcal{Z}_g[I] \equiv \int \mathcal{D}h \, e^{i\int \mathcal{L}_g + i\int h^{\mu\nu} I_{\mu\nu}}, \tag{1.43}$$

where h is now referred to as the *graviton* field. This path integral is Gaussian in h, but there is a problem. After integrating by parts, the path integral can be written in the form

$$\mathcal{Z}_g[I] \equiv \int \mathcal{D}h \, e^{-i\int h^{\mu\nu}\Delta^{-1}_{\mu\nu\alpha\beta}h^{\alpha\beta} + i\int h^{\mu\nu} I_{\mu\nu}}, \tag{1.44}$$

for a differential operator Δ^{-1}, and to evaluate it we must find Δ. However, as is, the inverse of Δ^{-1} cannot be uniquely inverted. This is because h contains redundant gauge degrees of freedom. Only once we pick a gauge can we invert this operator.

Under an infinitesimal change of coordinates, $\delta x^\mu = -\xi^\mu$, the metric perturbation is transformed according to $\delta h^{\mu\nu} = -\partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. The action is invariant under this shift of the coordinate grid (gauge transformation), so we are free to choose any ξ we would like. A convenient choice is the de

Donder or harmonic gauge condition

$$(h^{\alpha\beta})_{,\alpha} = \frac{1}{2}\eta^{\alpha\beta}h_{,\alpha} \tag{1.45}$$

which can be achieved with a choice of coordinates obeying

$$\partial^2\xi^\beta = (h^{\alpha\beta})_{,\alpha} - \frac{1}{2}\eta^{\alpha\beta}h_{,\alpha}. \tag{1.46}$$

In this gauge, the generating functional can be evaluated exactly, but there is one further detail we must mention. In order to properly enforce this gauge condition and stop ourselves from overcounting redundant gauge degrees of freedom in the path integral, we must first implement the Faddeev-Popov gauge fixing procedure. The result of this is [11]

$$\mathcal{Z}_g[I] \equiv \int \mathcal{D}h\,\mathrm{D}_G[h]\delta(G^\beta(h))e^{-i\int h^{\mu\nu}\Delta^{-1}_{\mu\nu\alpha\beta}h^{\alpha\beta}+i\int h^{\mu\nu}I_{\mu\nu}}, \tag{1.47}$$

where $\mathrm{D}_G[h]$ is the Faddeev-Popov determinant corresponding to the gauge-fixing function $G^\beta(h)$, and the delta function simply enforces the constraint (1.45), meaning that for the de Donder gauge we have

$$G^\beta(h) = (h^{\alpha\beta})_{,\alpha} - \frac{1}{2}\eta^{\alpha\beta}h_{,\alpha}. \tag{1.48}$$

For linearized gravity, the determinant only effects a change of overall normalization, so we henceforth ignore it. This leaves

$$\mathcal{Z}_g[I] \equiv \int \mathcal{D}h\,\delta(G^\beta(h))e^{-i\int h^{\mu\nu}\Delta^{-1}_{\mu\nu\alpha\beta}h^{\alpha\beta}+i\int h^{\mu\nu}I_{\mu\nu}}, \tag{1.49}$$

which adequately defines the inverse of Δ^{-1} and allows the integral to be performed, giving

$$\mathcal{Z}_g[I] = \exp\left\{\frac{i}{2}\int d^4x\int d^4y\,I_{\alpha\beta}(x)\Delta^{\alpha\beta\sigma\rho}(x-y)I_{\sigma\rho}(y)\right\}, \tag{1.50}$$

where the *free graviton propagator* Δ is the Green function of the linearized EFEs and is given by

$$\Delta^{\alpha\beta\sigma\rho}(x-x') \equiv x \;\bullet\!\sim\!\sim\!\sim\!\bullet\; x' = \sum_k \frac{e^{ik\cdot(x-x')}}{k^2} P^{\alpha\beta\sigma\rho}, \tag{1.51}$$

where $P^{\alpha\beta\sigma\rho} \equiv \frac{1}{2}[\eta^{\alpha\sigma}\eta^{\beta\rho} + \eta^{\alpha\rho}\eta^{\beta\sigma} - \eta^{\alpha\beta}\eta^{\sigma\rho}]$, and we now have an effective quantum field theory for weak-field gravity on its own. The theory of free (i.e., no coupling to gravity) scalar matter is even simpler, with

$$\begin{aligned} \mathcal{Z}_\phi[J] &\equiv \int \mathcal{D}\phi \, e^{i\int \mathcal{L}_\phi + i\int \phi J} \\ &= \int \mathcal{D}\phi \, e^{-i\int \phi G_0^{-1}\phi + i\int \phi J}, \end{aligned} \tag{1.52}$$

Where $\mathcal{L}_\phi$ is given by equation (1.40), and we are now in Minkowski space. The differential operator in the exponent is easy to write down,

$$G_0^{-1} = \partial^2 + m^2, \tag{1.53}$$

as is its inverse,

$$G_0(x-x') \equiv x \;\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\; x' = \sum_k \frac{e^{ik\cdot(x-x')}}{k^2 - m^2}. \tag{1.54}$$

G_0 is the *free scalar propagator*, and we are almost done constructing the Feynman rules for this theory.

From the interaction part of the Lagrangian density, $\mathcal{L}_{int}$, we can immediately write down the vertex (figure 1.3), here in momentum space, with $q = p_1 - p_2$:

$$\tau_{\mu\nu}(p_1, p_2; q) = \kappa(p_{1(\mu}p_{2\nu)} - \frac{1}{2}\eta_{\mu\nu}[p_1 \cdot p_2 - m^2]) \tag{1.55}$$

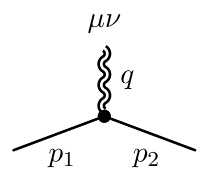


Figure 1.3: Graviton-matter vertex.

So we have established the diagrammar for this theory. This allows us to proceed and calculate gravitational scattering amplitudes using Feynman diagrams [24]. When we do so, we run into the usual UV divergences from computing loop integrals. These can be absorbed into renormalized parameters which are compared to experiment in order to measure their physical values. This makes it so physical quantities can still be predicted in spite of the infinities spit out by the diagrammatics. In order for a theory to have predictive power, it is naively required that the number of necessary renormalized constants needed to remove UV divergences to all orders is finite. Unfortunately, this requirement is not satisfied by gravity [19]. To eliminate UV divergences at every order would require an ever-increasing number of counterterms. The naive argument then says that the theory should have no predictive power. Can we ever hope to make predictions with such a theory, or is this line of attack dead?[2]

Thankfully, it seems like we can recover predictive power if we think of linearized gravity as an *effective field theory* [11, 20, 21]. An EFT is one that is understood to be the low energy limit of some (perhaps unknown) microscopic theory that is valid at all length scales. From this perspective, though renormalization forces upon our theory infinitely many higher order interaction terms in the Lagrangian (including terms beyond the Einstein-Hilbert term), they are all suppressed by ever-increasing powers of $\kappa E \sim E/M_P$, where E characterizes the energy scale at which we are interested in applying our theory and $M_P \sim 10^{18}$GeV is the Planck mass. This means that, although we are formally in trouble, we can truncate our theory as we have done if we promise to only apply it at energy scales well below the Planck

[2] Rovelli says it is dead. Specifically, he claims it died in 1975 [12].

scale. The EFT perspective also has the pleasing consequence that any UV complete theory of quantum gravity should make predictions that agree with ones computed with our truncated theory in the limit of low energy. This method of reinterpretation gets concrete results. These calculations have been shown to reproduce the classical Newtonian potential in two-body scattering processes, and quantum corrections to the Newtonian potential have been derived.[3] This should support our intuition that the theory is valid despite its infinitely many problem terms at higher orders.

So we have discovered a low energy effective theory that reproduces the Newtonian potential, and contains quantum corrections. We are finally ready to investigate the same problems that we saw arise in QED in the last section. To that end we will proceed to apply the same functional techniques we saw there to our quantum theory of gravity.

[3] For more, see [22], and also [23] and references therein.

Chapter 2

Eikonal Methods for Linearized Gravity

In this section we expand upon the introduction to functional methods in QED, as we apply the technique to linearized quantum gravity. Our presentation is similar to those of Fradkin and Fried [1, 2]. We will derive the IR form of the scalar propagator with soft gravitons functionally integrated out, and explain the details of the functional method along the way. Another attractive aspect of the functional approach we have yet to fully discuss is that it also allows a derivation of an incredibly convenient form of the generating functional of the theory in the eikonal limit. This facilitates computation of not just the infrared effective propagator, but in principle *all* of the n-point functions of the theory. This too is done in this chapter. We will not really *need* an expression for the generating functional in this limit to derive many results in this work, but we write it down anyway, and show how to extract any desired correlators. Why? As Schwartz [5] puts it, "The generating functional is the holy grail of any particular field theory: if you have an exact closed-form expression for $\mathcal{Z}$ for a particular theory, you have solved it completely." Our form is close to this ideal - while not exact, it is justified in the soft limit, and while not closed-form in its final incarnation, still very simple to use. We hope that this form for $\mathcal{Z}$ can be of use in other applications beyond the scope of this work.

Our implementation of the functional method involves two explicit assumptions. The first is that at any point, as matter moves through spacetime, the background graviton field is slowly varying compared to the dynamics of the particle. This assumption is certainly well motivated for non-relativistic processes, and also for the isolation of soft graviton effects, which are necessarily long wavelength. The second assumption is that at any point, the likelihood that a virtual graviton will pair produce scalar particles is negligible. In other words, we ignore all loop corrections to the graviton propagator, and simply use the bare one everywhere. This too is justified by the focusing of our attention to soft effects. Virtual quanta would need to attain energies of order $2mc^2$ as in figure 2.1 in order for us to include diagrams involving scalar loops, but this energy scale is far above that which we defined as soft.

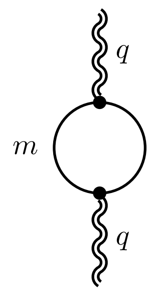


Figure 2.1: In order for virtual gravitons to be modified by loop corrections like this one, where a pair of massive scalar particles are produced, the momentum q must be of order $2mc^2$.

With our assumptions stated, we begin by asking the simplest question of all: how does the scalar field behave in the presence of some particular graviton background?

2.1 Equation of Motion for the Scalar Field

With the effective Lagrangian density $\mathcal{L}[\phi, h] = \mathcal{L}_g[h] + \mathcal{L}_\phi[\phi] + \mathcal{L}_{int}[\phi, h]$, we can isolate the behavior of the matter, conditioned on a particular configuration of the metric. What we will do is "freeze" the perturbation field h in some arbitrary configuration, and get an equation of motion for ϕ. Varying the action w.r.t. a small variation in the field $\phi \to \phi + \delta\phi$ and demanding that $\delta S = 0$ gives an equation of motion for ϕ:

$$\left\{\partial^2 + m^2 + \kappa h^{\alpha\beta}(x)\hat{K}_{\alpha\beta}\right\}\phi(x) = 0, \tag{2.1}$$

$$\hat{K}_{\alpha\beta} \equiv -\partial_\alpha\partial_\beta + \frac{1}{2}\eta_{\alpha\beta}(\partial^2 + m^2). \tag{2.2}$$

where again we have chosen to enforce the de Donder gauge condition. Then the propagator (Green function) for a scalar on a fixed graviton background is defined by:

$$\left\{G_0^{-1} + \kappa h^{\alpha\beta}(x)\hat{K}_{\alpha\beta}\right\}\mathcal{G}(x, x'|h) = -\delta^4(x - x') \tag{2.3}$$

We will need this in what follows and its eikonal form will be discussed later. For now we proceed to show how we can derive the generating functional of the theory in terms of this quantity.

2.2 The Generating Functional

If $J(x)$ and $I_{\alpha\beta}(x)$ are again arbitrary classical sources of the scalar field $\phi(x)$ and the graviton field $h^{\alpha\beta}(x)$ respectively, the full generating functional for the theory with the Lagrangian given above is

$$\mathcal{Z}[J, I] = \int \mathcal{D}\phi \int \mathcal{D}h\, e^{iS_\phi + iS_g + iS_{int} + i\int \phi J + i\int hI}. \tag{2.4}$$

Recall, $S_{int}[\phi, h] = -\frac{\kappa}{2}\int d^4x\, h^{\alpha\beta}T_{\alpha\beta}$. After integrating by parts, this can be rewritten as

$$S_{int}[\phi, h] = -\frac{\kappa}{2}\int d^4x\, h^{\alpha\beta}\phi\hat{K}_{\alpha\beta}\phi. \tag{2.5}$$

Now in this form, we write $\mathcal{Z}$ as

$$\mathcal{Z} = e^{iS_{int}[\frac{\delta}{\delta J}, \frac{\delta}{\delta I}]}\int \mathcal{D}\phi \int \mathcal{D}h e^{iS_\phi + iS_g + i\int \phi J + i\int hI}, \tag{2.6}$$

where we have made the substitutions $\phi(x) \to \frac{-i\delta}{\delta J(x)}$ and $h^{\alpha\beta}(x) \to \frac{-i\delta}{\delta I_{\alpha\beta}(x)}$. Written this way, the path integrals decouple and each can be evaluated explicitly, as done in the previous chapter. This gives

$$\mathcal{Z} = e^{iS_{int}[\frac{\delta}{\delta J},\frac{\delta}{\delta I}]}\mathcal{Z}_\phi[J]\mathcal{Z}_g[I]. \tag{2.7}$$

Now, notice that $e^{iS_{int}}$ is simply a linear shift operator in I, and a quadratic shift operator in J. Because the free generating functionals factorize and are Gaussian, either of these operators can be applied directly. The linear shift in I is easier, but first evaluating the quadratic shift in J is more helpful. Applying this gives

$$\begin{aligned}\mathcal{Z} = &\exp\left\{-\frac{1}{2}\int d^4x \ln\left(1 + \kappa G_0(x,x)\hat{K}_{\alpha\beta}\frac{-i\delta}{\delta I_{\alpha\beta}(x)}\right)\right\} \\ &\times\exp\left\{\frac{i}{2}\int d^4x \int d^4y J(x)\left(\frac{G_0(x,y)}{1+\kappa G_0\hat{K}_{\alpha\beta}\frac{-i\delta}{\delta I_{\alpha\beta}}}\right)J(y)\right\}\mathcal{Z}^0_\phi[J]\mathcal{Z}^0_g[I].\end{aligned} \tag{2.8}$$

One then recognizes that the quantity $\frac{G_0(x,y)}{1+\kappa G_0\hat{K}_{\alpha\beta}\frac{-i\delta}{\delta I_{\alpha\beta}}}$ is the (symbolic) solution to the equation that defines the propagator $\mathcal{G}(x,y|h)$ on a *fixed* background perturbation field, with the field $h^{\mu\nu}(x)$ replaced by $\frac{-i\delta}{\delta I_{\mu\nu}(x)}$. This means that $\mathcal{Z}$ can be written as

$$\begin{aligned}\mathcal{Z} = &\exp\left\{\frac{i}{2}\int d^4x \int_0^\kappa dg\, \mathcal{G}(x,x|g\frac{\delta}{\delta I})\hat{K}_{\alpha\beta}\frac{\delta}{\delta I_{\alpha\beta}(x)}\right\} \\ &\times \exp\left\{\frac{i}{2}\int d^4x \int d^4y\, J(x)\mathcal{G}(x,y|\frac{\delta}{\delta I})J(y)\right\}\mathcal{Z}^0_\phi[J]\mathcal{Z}^0_g[I].\end{aligned} \tag{2.9}$$

The first exponential "interaction operator" above describes the polarization of gravitons ([1], ch.3) which here means diagrams with scalar loops (remember, we have removed all the nonlinearity at the level of the action, so there are no graviton-only polarization contributions). Because of this, we ignore that term in the calculations that follow, setting it to 1, as physical arguments presented at the beginning of this chapter say there should be no such diagrams in the infrared.

Now all that we need is an appropriate expression for the scalar propagator in a fixed, slowly varying background field.

2.3 Propagator on a Fixed Background

Remember, the propagator we want to find is defined by

$$\left\{G_0^{-1} + \kappa h^{\alpha\beta}(x)\hat{K}_{\alpha\beta}\right\}\mathcal{G}(x,x'|h) = -\delta^4(x-x'). \tag{2.10}$$

In momentum space, with $\mathcal{G}(x,x'|h) = \sum_k e^{ik(x-x')}\mathcal{G}_k(x|h)$, the propagator obeys

$$\left\{G_0^{-1}(k) - \hat{U}\right\}\mathcal{G}_k(x|h) = 1. \tag{2.11}$$

Here we are expanding about the bare propagator, $G_0(k) = \frac{1}{k^2-m^2}$, and explicitly:

$$\begin{aligned}\hat{U} \equiv\; & \partial^2 + 2ik^\mu\partial_\mu \\ & +\kappa h^{\alpha\beta}\left[k_\alpha k_\beta - 2ik_\alpha\partial_\beta - \partial_\alpha\partial_\beta - \frac{1}{2}\eta_{\alpha\beta}(k^2 - m^2 - \partial^2 - 2ik^\mu\partial_\mu)\right]\end{aligned} \tag{2.12}$$

At this point, it is also useful to scale the above relation by the scalar mass:

$$\left\{mG_0^{-1}(k) - m\hat{U}\right\}\mathcal{G}_k(x|h) = m \tag{2.13}$$

The reason for this is made clear below. Using the Schwinger/Fock "proper time" representation, we can write the bare propagator as

$$G_0(k) = im\int_0^\infty ds\, e^{-ism(k^2-m^2)} \equiv im\int_0^\infty ds\, G_0(k,s), \tag{2.14}$$

where the exponent is to be given a small negative imaginary part, and it is useful to note that

$$i\partial_s G_0(k,s) = m(k^2 - m^2)G_0(k,s) = mG_0^{-1}(k)G_0(k,s). \tag{2.15}$$

It is also convenient to express the full propagator as

$$\mathcal{G}_k(x|h) = im\int_0^\infty ds\, G_0(k,s)\,\mathcal{Y}(k,s,x|h), \tag{2.16}$$

such that $\mathcal{Y}$ acts to weight the free propagator term under the proper time integral. In order to do this, we must be able to satisfy

$$\left\{mG_0^{-1}(k) - m\hat{U}\right\} i\int_0^\infty ds\, G_0(k,s)\,\mathcal{Y}(k,s,x|h) = 1, \tag{2.17}$$

and it turns out we can, if the weighting factor obeys the Schrödinger equation

$$-\,i\partial_s\mathcal{Y} = m\hat{U}\mathcal{Y}, \qquad \mathcal{Y}(s=0) = 1. \tag{2.18}$$

In deducing the form of $\mathcal{Y}$, we use the ansatz

$$\mathcal{Y} \equiv e^{\chi}, \tag{2.19}$$

with χ inheriting all of the dependencies of $\mathcal{Y}$.

Now, for this scalar theory, the equation of motion for χ is nonlinear and generally intractable. Progress can be made, however, by expressing χ as a power series. In order to do this, note that the coupling $\kappa = \sqrt{32\pi G}$ can be expressed in natural units as $\kappa = \sqrt{32\pi}/M_P$, with M_P the Planck mass. The scaling of the perturbation $\hat{U}$ by the scalar mass m gives naturally a small dimensionless parameter $m\kappa = \sqrt{32\pi}(m/M_P) \ll 1$. So to isolate the eikonal behavior of the propagator, expand χ as a power series in the dimensionless "coupling" in the spirit of the WKB technique:

$$\chi \equiv \sum_{n=1}^{\infty} (m\kappa)^n \chi_n \tag{2.20}$$

To first order in $m\kappa$, this gives an equation of motion for χ_1:

$$-i\partial_s \chi_1 = m\left[\partial^2 + 2ik^\mu \partial_\mu\right]\chi_1 + h^{\alpha\beta}(x)[k_\alpha k_\beta - \frac{1}{2}\eta_{\alpha\beta}(k^2 - m^2)] \tag{2.21}$$

Higher order terms can be found in a similar manner. One interesting thing to note is that, unlike in a linear theory (e.g. QED), having to perform this expansion at this stage due to the second-order equation of motion could lead to a different source of subleading soft effects. However this has not yet been investigated. The ∂^2 term can also be dropped at this stage due to the assumption of a slowly-varying background field.[4] We then write χ in terms of its Fourier transform, $\chi_1 = \sum_q e^{iq\cdot x}\widetilde{\chi}_1(q)$, in which case the solution for $\widetilde{\chi}_1(q)$ reads

$$m\kappa\widetilde{\chi}_1(q) = im\int_0^s ds' e^{-2is'm(k\cdot q)}\widetilde{h}^{\mu\nu}(q)\tau_{\mu\nu}(k), \tag{2.22}$$

where the graviton-scalar vertex in de Donder gauge is again $\tau_{\mu\nu}(p_1, p_2; q) = \kappa(p_{1(\mu}p_{2\nu)} - \frac{1}{2}\eta_{\mu\nu}[p_1 \cdot p_2 - m^2])$. The appearance of $\tau_{\mu\nu}(k) \equiv \tau_{\mu\nu}(k, k; 0)$ is a clear manifestation of the eikonal physics here. Imposing momentum conservation at the vertex and setting $p_1 = p_2 = k$ implies that, if one is to truly take this object as a vertex, the momentum of the emitted graviton is identically zero. This can also be thought of as a manifestation of the inconsistency of the linearized theory, in which the gravitons do not source themselves (i.e. one incorrectly assumes that the stress-energy of gravitons is zero).

[4] If we do not drop this term here, we get a slightly more complicated form factor later. This would consist of replacing the factor $\int_0^s ds' e^{-2is'm(k\cdot q)}$ with $\int_0^s ds' e^{-is'm(q^2+2k\cdot q)}$. This does not change the final results of our work, so for our purposes these forms are equivalent. However if one wishes to consider slightly larger graviton momenta, this difference becomes more important.

Finally, keeping just the first order eikonal correction, we have that

$$\mathcal{G}_k(x|h) \approx im \int_0^\infty ds\, e^{-ism(k^2-m^2)+m\kappa \sum_q e^{iq\cdot x}\widetilde{\chi}_1(q)}, \tag{2.23}$$

and the explicit dependence on the mass scale can be seen to be trivial by choosing the integration variable to be sm rather than s (and similarly for s' in the form factor of $\widetilde{\chi}_1$), to get

$$\mathcal{G}_k(x|h) \approx i \int_0^\infty ds\, e^{-is(k^2-m^2)+\kappa \sum_q e^{iq\cdot x}\widetilde{\chi}_1(q)}. \tag{2.24}$$

How is this different from perturbation theory? In our ansatz (2.19), we chose to leave the s-dependence of χ arbitrary. Choosing a form like

$$\mathcal{Y} \equiv e^{-is\bar{\chi}} \tag{2.25}$$

where $\bar{\chi}$ is no longer a function of proper time, would give

$$\mathcal{G} = \frac{1}{k^2 - m^2 + \bar{\chi}}. \tag{2.26}$$

We would find that, to first order, $\bar{\chi}$ is just the one-loop self energy of the scalar. So in this way, we could immediately recover the usual perturbation theory by keeping ever-increasing orders of corrections to $\bar{\chi}$. However, simply relaxing the dependence on proper time gives strictly nonperturbative results and we now proceed to investigate what effect this has on the generating functional and correlators of the theory.

2.4 Applying the Interaction Operators

So, then, the propagator given on a fixed background, to first order in $m\kappa$, has the form

$$\mathcal{G}_k(x|h) \approx i \int_0^\infty ds\, e^{-is(k^2-m^2)-\sum_q e^{iq\cdot x} f(q)\tau_{\mu\nu}(k,k)\widetilde{h}^{\mu\nu}(q)}, \tag{2.27}$$

where the form factor is

$$f(q) \equiv -i \int_0^s ds' e^{-2is'(k\cdot q)} = \frac{e^{-2is(k\cdot q)} - 1}{2k \cdot q}. \tag{2.28}$$

Plugging (2.27) into (2.9) and setting the first exponential to one, we see that the generating functional in the infrared regime has the approximate form

$$\begin{aligned} \mathcal{Z} \approx \exp &\left\{ \frac{i}{2} \int d^4x \sum_k e^{ik\cdot x} J(x) \widetilde{J}(k) \mathcal{G}_k(x|\frac{\delta}{\delta I}) \right\} \\ \times \exp &\left\{ \frac{i}{2} \sum_q \widetilde{I}_{\mu\nu}(q) \Delta^{\mu\nu\alpha\beta}(q^2) \widetilde{I}_{\alpha\beta}(-q) \right\}, \end{aligned} \tag{2.29}$$

or in convenient shorthand,

$$\mathcal{Z} \approx e^{\frac{i}{2}\int J\mathcal{G}(|\frac{-i\delta}{\delta I})J} e^{\frac{i}{2}\int I\Delta I}. \tag{2.30}$$

This gets a little messy, but since the propagator here is restricted to terms in the exponent that are linear in κh, the result is a straightforward application of the properties of linear shift operators. To see this, write the above as

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} &\left[\frac{1}{2} \int d^4x \sum_k e^{ik\cdot x} J(x) \widetilde{J}(k) \right. \\ &\left. \times \int_0^\infty ds\, e^{-is(k^2-m^2)+i\sum_q e^{iq\cdot x} f(q) \tau_{\mu\nu}(k) \frac{\delta}{\delta \widetilde{I}_{\mu\nu}(q)}} \right]^n \mathcal{Z}_g^0[\widetilde{I}_{\mu\nu}(q)] \end{aligned} \tag{2.31}$$

$$\begin{aligned} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \prod_{a=1}^{n} &\left[\frac{1}{2} \int d^4x_a \sum_{k_a} e^{ik_a\cdot x_a} J(x_a) \widetilde{J}(k_a) \int_0^\infty ds_a\, e^{-is_a(k_a^2-m^2)} \right] \\ &\times e^{i\sum_q \sum_{a=1}^n e^{iq_a\cdot x_a} f_a(q) \tau_{\mu\nu}(k_a) \frac{\delta}{\delta \widetilde{I}_{\mu\nu}(q)}} \mathcal{Z}_g^0[\widetilde{I}_{\mu\nu}(q)]. \end{aligned} \tag{2.32}$$

Recognizing this as a shift operator -

$$\begin{aligned}&\exp\left\{i\sum_q e^{iq\cdot x}f(q)\tau_{\mu\nu}(k)\frac{\delta}{\delta\widetilde{I}_{\mu\nu}(q)}\right\}\mathcal{Z}_g^0[\widetilde{I}_{\mu\nu}(q)]\\&=\mathcal{Z}_g^0\left[\widetilde{I}_{\mu\nu}(q)+ie^{iq\cdot x}f(q)\tau_{\mu\nu}(k)\right]\end{aligned}\tag{2.33}$$

- and writing out the shifted $\mathcal{Z}_g^0$ gives the explicit form for the generating functional:

$$\begin{aligned}\mathcal{Z}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\prod_{a=1}^{n}&\left[\frac{1}{2}\int d^4x_a\sum_{k_a}e^{ik_a\cdot x_a}J(x_a)\widetilde{J}(k_a)\int_0^\infty ds_a\, e^{-is_a(k_a^2-m^2)}\right]\mathcal{Z}_g^0[I]\\ \times\ \exp\Bigg\{&-\sum_{a=1}^{n}\sum_q e^{-iq\cdot x_a}f_a(-q)\tau_{\mu\nu}(k_a)\Delta^{\mu\nu\alpha\beta}(q^2)\widetilde{I}_{\alpha\beta}(q)\\&-\frac{i}{2}\sum_{a,b=1}^{n}\sum_q e^{iq\cdot(x_a-x_b)}f_a(q)f_b(-q)\tau_{\mu\nu}(k_a)\Delta^{\mu\nu\alpha\beta}(q^2)\tau_{\alpha\beta}(k_b)\Bigg\}\end{aligned}\tag{2.34}$$

This is not quite the "exact, closed-form expression" we may dream of, but in fact the form (2.30) is practically just as useful. To demonstrate this, we now run through a couple examples of finding n-point correlators.

2.5 Correlators

Now that we have an explicit form for $\mathcal{Z}$ in the soft graviton regime, we can use it to find the IR form of whichever correlators we are interested in. In particular, we focus here on finding the first couple of correlators, $\mathcal{G}_2$ and $\mathcal{G}_4$. Defining

$$\mathcal{Z}[J,I]\equiv e^{i\mathcal{W}[J,I]},\qquad \mathcal{W}=-i\ln\mathcal{Z},\tag{2.35}$$

the correlators for the scalar field are

$$\mathcal{G}_n(x_1,\ldots,x_n)=\frac{(-i)^n\delta^n\mathcal{Z}[J,I]}{\delta J(x_1)\ldots\delta J(x_n)}|_{I=J=0}.\tag{2.36}$$

and the *connected* correlators are

$$\mathcal{G}_n^{(c)}(x_1,\ldots,x_n) = \frac{(-i)^n \delta^n i\mathcal{W}[J,I]}{\delta J(x_1)\ldots\delta J(x_n)}|_{I=J=0}. \tag{2.37}$$

In practice, it is most convenient to use the real space form of (2.30) to calculate these. Doing things this way results in expressions that are simply repeated applications of linear shift operators, so all of the n-point functions for ϕ are simple to recover.

2.5.1 The Two-Point Function

Following (2.30),

$$\frac{-i\delta\mathcal{Z}}{\delta J(x)} = \left[-\int d^4a\, J(a)\mathcal{G}(x,a|\frac{\delta}{\delta I})\right]\mathcal{Z}. \tag{2.38}$$

Rinsing and repeating:

$$\frac{(-i)^2\delta^2\mathcal{Z}}{\delta J(x)\delta J(x')} = \left[-i\mathcal{G}(x,x'|\frac{\delta}{\delta I}) + \mathbf{0}\right]\mathcal{Z}, \tag{2.39}$$

where the bold $\mathbf{0}$ indicates more terms that vanish when the auxiliary sources are set to zero after applying all functional derivatives. Now all that is left is to evaluate $\mathcal{G}$ acting on $\mathcal{Z}$, as done before in (2.33), and turn off the sources. The result is

$$\begin{aligned}\mathcal{G}_2(r) = &\sum_k e^{ik\cdot r}\int_0^\infty ds\, e^{-is(k^2-m^2)} \\ &\times \exp\left\{-\frac{i}{2}\sum_q f(q)f(-q)\tau_{\alpha\beta}(k)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k)\right\},\end{aligned} \tag{2.40}$$

where r stands for the difference $x-x'$. Though the propagator found with a fixed background was not translationally invariant, this correlator is after integrating out background configurations.

2.5.2 The Four-Point Function

Computing the two particle correlator is more messy, but just as straightforward. By definition,

$$\mathcal{G}_4(w,x,y,z) = \frac{(-i)^4\delta^4\mathcal{Z}}{\delta J(w)\delta J(x)\delta J(y)\delta J(z)}|_{I=J=0}, \tag{2.41}$$

and after computing this we get

$$\mathcal{G}_4(w,x,y,z) = \boldsymbol{G}(w,x|y,z) + \boldsymbol{G}(w,y|x,z) + \boldsymbol{G}(w,z|x,y), \tag{2.42}$$

where

$$\begin{aligned}
\boldsymbol{G}(w,x|y,z) \equiv & \sum_{k_1}\sum_{k_2} e^{ik_1\cdot(w-x)}e^{ik_2\cdot(y-z)}\int_0^\infty ds_1\int_0^\infty ds_2\, e^{-is_1(k_1^2-m^2)}e^{-is_2(k_2^2-m^2)} \\
& \times\exp\left\{-\frac{i}{2}\sum_q f_1(q)f_1(-q)\tau_{\alpha\beta}(k_1)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k_1)\right\}\times \\
& \times\exp\left\{-\frac{i}{2}\sum_q f_2(q)f_2(-q)\tau_{\alpha\beta}(k_2)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k_2)\right\}\times \\
& \times\exp\left\{-i\sum_q e^{iq\cdot(w-y)}f_1(q)f_2(-q)\tau_{\alpha\beta}(k_1)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k_2)\right\}.
\end{aligned} \tag{2.43}$$

Any correlator can be computed in this manner, and using this first approximation to $\mathcal{Z}$ makes each of them straightforward.

2.6 Infrared Behavior of the Two-Point Function

In addition to pulling $\mathcal{G}_2$ from $\mathcal{Z}$ in the manner above, one can also recover it by integrating out the fluctuations of the metric perturbation directly

$$\mathcal{G}(x-y) = \int \mathcal{D}h^{\mu\nu}\,\mathbf{P}[h]\mathcal{G}(x,y|h), \tag{2.44}$$

in analogy with ordinary probability theory. $\mathbf{P}$ is a probability amplitude for a given configuration of h. By the definition of the full correlator for ϕ, $\mathbf{P}$ is found to be $\mathbf{P}[h] = e^{iS_g[h]}$, with $S_g[h]$ the pure gravity action, which in this treatment is only quadratic in h. The integral is then Gaussian in h and is straightforward. From either derivation, we get the form

$$\mathcal{G}(k^2) = \int_0^\infty ds\, e^{-is(k^2-m^2)}\exp\left\{-\frac{i}{2}\sum_q f(q)f(-q)\tau_{\alpha\beta}(k)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k)\right\}. \tag{2.45}$$

Evaluating the integral in the phase - seen in Appendix B - and using $\tau_{\alpha\beta}P^{\alpha\beta\mu\nu}\tau_{\mu\nu} \approx \frac{\kappa^2}{4}k^4 \approx \frac{\kappa^2}{4}m^4$ near the mass shell,

$$\mathcal{G}(k^2) \approx \int_0^\infty ds\, e^{-is(k^2-m^2)-\frac{\kappa^2}{32\pi^2}m^2[is\Lambda-\ln(s\Lambda)]}. \tag{2.46}$$

In the new phase, the term linear in s should be thought of as a mass renormalization. However at this point it must be remarked that these results are gauge-dependent, and as discussed in [1], it should simply be assumed that all physical mass renormalization has been done at this stage, and that those full results should rightfully be gauge-independent. Additionally, writing $\ln(s\Lambda) = \ln(sm) + \ln(\Lambda/m)$ gives a divergent wavefunction renormalization. Finally, the renormalized propagator looks like

$$\mathcal{G}(k^2) = \int_0^\infty ds\, e^{-is(k^2-m^2)+\frac{\kappa^2}{32\pi^2}m^2\ln(sm)} = \int_0^\infty ds\, e^{-is(k^2-m^2)}(sm)^\zeta, \tag{2.47}$$

where $\zeta \equiv \frac{Gm^2}{\pi}$, and thus *in harmonic gauge*:

$$\mathcal{G}(k^2) = -i\Gamma(1+\zeta)(-im)^{\zeta}[k^2 - m^2]^{-(1+\zeta)} \tag{2.48}$$

We emphasize that this result as presented is valid only in harmonic gauge. We have suppressed the dependence on a gauge-fixing parameter to simplify the index gymnastics. However as in the scalar QED result in the introduction, it is still possible to reduce this propagator to a form with a simple pole by a suitable gauge choice, discussed in [66, 67]. In this way one gets rid of spurious apparent IR divergences in the propagator. It would be interesting to repeat the calculations performed in this work using this divergence-free gauge.

The key results of this chapter are equations (2.30), (2.45), and (2.48). The first gives an eikonal representation of the generating functional, and answers the general question of how the matter-gravity theory behaves in the low energy limit. The last gives a nonperturbative result for the new effective propagator of the scalar field which has the same basic form as the result for QED. It does this through the help of the convenient form (2.45). Though a similar form for the QED propagator is known, we have presented a new approach to calculating (2.48), which matches the known result, e.g., eq. 4.7 of [66].

The form (2.45) is crucial because it allows for efficient computation of more physical quantities, like S-matrix elements. This connection is what will allow us to make contact with experiment (via probabilities from squared S-matrix elements) and to current theoretical discussion involving soft theorems. To that end, we proceed by discussing how to use this functional formalism to compute scattering amplitudes. This will also be a convincing proof that this formalism encodes the same soft physics as diagrammatic approaches.

Chapter 3

Amplitudes and Soft Theorems

So far we have obtained expressions for the generating functional and correlators for our theory. How can we use this technology to derive S-matrix elements? In this section we will show how to compute these amplitudes, and how the soft graviton theorem comes about naturally from these calculations. In particular, we show how it emerges as a consequence of the eikonal formulation of the theories in the specific cases of bremsstrahlung facilitated by a classical potential and two-body scattering. We make some comments about the cancellation of IR divergences in these cases, but those results are more or less the same as those discussed for QED in the introduction.

The important thing about the soft photon results at the beginning of this work one should note for this chapter is summarized in equation (1.21). The effect of adding a soft photon emission to an amplitude is a divergent multiplicative factor. The same is true for gravity, with only slight modification to the resulting factor [29]. The soft graviton correction takes some known amplitude $\mathcal{M}$ to

$$\mathcal{M} \times \frac{\sqrt{8\pi G}}{2(2\pi)^{3/2}\sqrt{2|q|}} \left[\sum_i^m \frac{\epsilon^{*\mu\nu}(p_i')_\mu (p_i')_\nu}{p_i' \cdot q - i\delta} - \sum_j^n \frac{\epsilon^{*\mu\nu}(p_j)_\mu (p_j)_\nu}{p_j \cdot q + i\delta} \right] \equiv \mathcal{M} \times \kappa\Omega. \tag{3.1}$$

3.1 Gravitational Bremsstrahlung

Here we will give the simplest possible example of gravitational bremsstrahlung - a massive scalar particle in a small external metric perturbation field h_{ext} emitting n (soft) gravitons, with momenta $\{q_n\}$ and polarizations labeled by $\{\lambda_n\}$.

3.1.1 Constructing the Amplitude

The amplitude for this process (using $\mathcal{B}$ for $\mathcal{B}$remsstrahlung) is

$$\mathcal{B}_n = {}_{\mathrm{IN}}\langle p'; q_1, \lambda_1; \cdots; q_n, \lambda_n | p\rangle_{\mathrm{IN}} = {}_{\mathrm{OUT}}\langle p'; q_1, \lambda_1; \cdots; q_n, \lambda_n | \mathcal{S} | p\rangle_{\mathrm{IN}}\,, \tag{3.2}$$

and we will drop the "IN/OUT" subscripts from here on out. First, consider the case of the emission of a single graviton. The generalization to multiple graviton emission will be easy. We construct the states by

$$\langle p'; q, \lambda | = \langle 0 | \, \hat{a}(p')\hat{b}(q) \tag{3.3}$$

and

$$|p\rangle = \hat{a}^\dagger(p)\,|0\rangle\,. \tag{3.4}$$

The a creation/annihilation operators create/annihilate scalar particles, while the b operators are for the gravitons. We also have (see the appendix) that

$$\mathcal{S} = : e^{\int \phi_{\mathrm{IN}}\hat{\mathcal{D}}\frac{\delta}{\delta J} + \int h^{\mu\nu}_{\mathrm{IN}}\hat{\mathcal{K}}\frac{\delta}{\delta I^{\mu\nu}}} : \mathcal{Z}[J, I]|_{J=I=0} \equiv \bar{\mathcal{S}}\mathcal{Z}|_0, \tag{3.5}$$

where the $:(\cdots):$ denotes normal ordering of operators, and $\mathcal{Z}$ is the full generating functional. $\hat{\mathcal{D}}$ and $\hat{\mathcal{K}}$ are the (free) inverse propagators for the scalar field and the graviton field respectively (to be identified with the differential operators previously called G_0^{-1} and Δ^{-1}), and the IN fields are

$$\phi_{\mathrm{IN}}(x) = \sum_k^{(3)} \frac{1}{\sqrt{2\omega_p}} \left[\hat{a}(k)e^{-ik\cdot x} + \hat{a}^\dagger(k)e^{ik\cdot x}\right]; \quad \omega_p^2 = k^2 - m^2, \tag{3.6}$$

and

$$h_{\mathrm{IN}}^{\mu\nu}(x) = \sum_{k}^{(3)} \frac{1}{\sqrt{2|\vec{k}|}} \sum_{\lambda} \left[\epsilon^{\mu\nu}(k,\lambda)\hat{b}(k)e^{-ik\cdot x} + \epsilon^{\mu\nu *}(k,\lambda)\hat{b}^{\dagger}(k)e^{ik\cdot x} \right]. \quad (3.7)$$

Here the $\epsilon^{\mu\nu} = \epsilon^{\mu}\epsilon^{\nu}$ are the graviton polarization vectors. In this construction, the amplitude is

$$\mathcal{B}_1 = \langle 0| \, \hat{a}(p')\hat{b}(q,\lambda) \, (\bar{\mathcal{S}}\mathcal{Z}|_0) \, \hat{a}^{\dagger}(p) \, |0\rangle \, . \quad (3.8)$$

Commuting the creation and annihilation operators through the S-matrix gives for $\mathcal{B}_1$

$$\begin{aligned} \mathcal{B}_1 = \langle 0| & \left(\int dx \left[\hat{b}(q,\lambda), h_{\mathrm{IN}}^{\mu\nu}(x) \right] \hat{\mathcal{K}} \frac{\delta}{\delta I^{\mu\nu}(x)} \right) \left(\int dy \left[\hat{a}(p'), \phi_{\mathrm{IN}}(y) \right] \hat{\mathcal{D}} \frac{\delta}{\delta J(y)} \right) \\ & \times \left(\int dz \left[\phi_{\mathrm{IN}}(z), \hat{a}^{\dagger}(p) \right] \hat{\mathcal{D}} \frac{\delta}{\delta J(z)} \right) (\bar{\mathcal{S}}\mathcal{Z}|_0) \, |0\rangle \end{aligned} \quad (3.9)$$

$$\begin{aligned} = (2\pi)^{-9/2} \frac{\epsilon^{*}(q,\lambda)}{\sqrt{2|\vec{q}|}} \frac{1}{2\sqrt{\omega_p \omega_{p'}}} \int dx \, e^{iq\cdot x} \hat{\mathcal{K}}_x \int dy \, e^{ip'\cdot y} \hat{\mathcal{D}}_y \int dz \, e^{-ip\cdot z} \hat{\mathcal{D}}_z \\ \times \frac{\delta}{\delta I(x)} \frac{\delta}{\delta J(y)} \frac{\delta}{\delta J(z)} \mathcal{Z}[J,I]|_0. \end{aligned} \quad (3.10)$$

3.1.2 Functional Eikonal Limit

To proceed, we must employ some particular expression for the generating functional. Let us use the eikonal form that we have already derived, eq. (2.30):

$$\mathcal{Z}[J,I] \approx e^{\frac{i}{2}\int J\mathcal{G}(|\frac{-i\delta}{\delta I})J} e^{\frac{i}{2}\int I\Delta I}, \quad (3.11)$$

where we had used

$$\mathcal{G}(x,y|h) \approx \sum_{k} e^{ik\cdot(x-y)} i \int_0^{\infty} ds \, e^{-is(k^2-m^2)+\frac{i}{2}\tau_{\mu\nu}(k)\int_0^s ds' \, h^{\mu\nu}(y+s'k)}, \quad (3.12)$$

with $\tau_{\mu\nu}(k) \approx \kappa k_\mu k_\nu$ (as the scalar is almost on shell: $k^2 - m^2 \approx 0$). Looking at (3.10), we see that the two functional J-derivatives bring down one $\mathcal{G}$, and we replace the h-dependence of $\mathcal{G}$ by $\frac{-i\delta}{\delta I}$, so that $\mathcal{G} \sim e^{-i\int f \frac{\delta}{\delta I}}$. $\mathcal{G}$ then is applied to $e^{\frac{i}{2}\int I\Delta I}$. After using the functional identity (A.5), we get

$$\begin{aligned}\mathcal{B}_1 = i(2\pi)^{-9/2}\frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}}\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\bigg[\int dx\, e^{iq\cdot x}\frac{\delta}{\delta h(x)}e^{\frac{-i}{2}\int\frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}} \\ \times\int dy\, e^{ip'\cdot y}\hat{\mathcal{D}}_y\int dz\, e^{-ip\cdot z}\hat{\mathcal{D}}_z\bigg]\mathcal{G}(y,z|h+h_{\text{ext}})|_{h=0}.\end{aligned} \tag{3.13}$$

Note that we have finally made the dependence on h_{ext} explicit in the above.

3.1.3 On-Shell Results

Assuming that $|\kappa h_{\text{ext}}|$ is small, we keep the first perturbative contribution of the classical source by making the replacement

$$\mathcal{G}(y,z|h+h_{\text{ext}}) \to i\int du\, \mathcal{G}(y,u|h)\tau\cdot h_{\text{ext}}(u)\mathcal{G}(u,z|h). \tag{3.14}$$

This leaves two mass-shell amputations to be performed, with the results

$$\begin{aligned}\int dy\, e^{ip'\cdot y}\hat{\mathcal{D}}_y\mathcal{G}(y,u|h)|_{p'^2\to m^2} &= e^{ip'\cdot u}e^{\frac{i}{2}\int_0^\infty ds\,\tau'\cdot h(u-sp')}, \\ \int dz\, e^{-ip\cdot z}\hat{\mathcal{D}}_z\mathcal{G}(u,z|h)|_{p^2\to m^2} &= e^{-ip\cdot u}e^{\frac{i}{2}\int_0^\infty ds\,\tau\cdot h(u+sp)},\end{aligned} \tag{3.15}$$

where we have assumed that the scalar follows straight line paths from y to x, defining the eikonal limit. The seemingly complicated eikonal correlators have thus given simple Wilson-line-type factors that will be acted upon by the operators corresponding to real graviton radiation. The amplitude then becomes

$$\begin{aligned}\mathcal{B}_1 = i(2\pi)^{-9/2}\,\frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}}\,\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\Bigg[\int du\, e^{iu\cdot(p'-p)}\,\tau\cdot h_{\text{ext}}(u) \\ \times \int dx\, e^{iq\cdot x}\frac{\delta}{\delta h(x)}e^{\frac{-i}{2}\int\frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}}\Bigg]e^{i\int fh}|_{h=0},\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}f_{\mu\nu}(w,u) &\equiv \frac{1}{2}\sum_q e^{-iq\cdot(w-u)}\int_0^\infty ds\,\left[\tau_{\mu\nu}e^{isp\cdot q}+\tau'_{\mu\nu}e^{-isp'\cdot q}\right] \\ &\equiv f^{in}_{\mu\nu}(w,u)+f^{out}_{\mu\nu}(w,u).\end{aligned} \tag{3.17}$$

Now application of the quadratic shift $e^{\frac{-i}{2}\int\frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}}$ is trivial, giving

$$\begin{aligned}\mathcal{B}_1 = i(2\pi)^{-3}\,\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\int du\, e^{iu\cdot(p'-p)}\,\tau\cdot h_{\text{ext}}(u)e^{\frac{i}{2}\int f\Delta f} \\ \times\left((2\pi)^{-3/2}\,\frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}}\int dx\, e^{iq\cdot x}\frac{\delta}{\delta h(x)}e^{i\int fh}|_{h=0}\right).\end{aligned} \tag{3.18}$$

Before expanding upon this result, let us investigate what happens when n gravitons are emitted.

3.1.4 Multiple Graviton Emission

Generalizing to n-graviton radiation, the amplitude becomes simply

$$\begin{aligned}\mathcal{B}_n =& i(2\pi)^{-3}\,\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\int du\, e^{iu\cdot(p'-p)}\,\tau\cdot h_{\text{ext}}(u)e^{\frac{i}{2}\int f\Delta f} \\ &\times\prod_m^n\left((2\pi)^{-3/2}\,\frac{\epsilon^*(q_m,\lambda_m)}{\sqrt{2|\vec{q_m}|}}\int dx_m\, e^{iq_m\cdot x_m}\frac{\delta}{\delta h(x_m)}e^{i\int fh}|_{h=0}\right).\end{aligned} \tag{3.19}$$

For each graviton radiated, as $q \to 0$, we get

$$\int dx\, e^{iq\cdot x} \frac{\delta}{\delta h^{\mu\nu}(x)} e^{i\int fh}|_{h=0} = \sqrt{8\pi G}\left[\frac{p'_\mu p'_\nu}{p'\cdot q - i\delta} - \frac{p_\mu p_\nu}{p\cdot q + i\delta}\right] + \mathcal{O}(q^0) \tag{3.20}$$

so that the full amplitude becomes

$$\begin{aligned}\mathcal{B}_n = &\left(i(2\pi)^{-3}\frac{1}{2\sqrt{\omega_p\omega_{p'}}}\left[\tau\cdot\widetilde{h}_{\text{ext}}(p-p')\right] e^{\frac{i}{2}\int f\Delta f}\right)\\ &\times\prod_m^n\left(\frac{\sqrt{8\pi G}}{(2\pi)^{3/2}\sqrt{2|\vec{q_m}|}}\left[\frac{\epsilon_m^{\mu\nu*}p'_\mu p'_\nu}{p'\cdot q - i\delta} - \frac{\epsilon_m^{\mu\nu*}p_\mu p_\nu}{p\cdot q_m + i\delta}\right]\right).\end{aligned} \tag{3.21}$$

In other words,

$$\mathcal{B}_n = \mathcal{B}_0^{\text{eik.}} \times \prod_m^n\left(\frac{\sqrt{8\pi G}}{(2\pi)^{3/2}\sqrt{2|\vec{q_m}|}}\left[\frac{\epsilon_m^{\mu\nu*}p'_\mu p'_\nu}{p'\cdot q - i\delta} - \frac{\epsilon_m^{\mu\nu*}p_\mu p_\nu}{p\cdot q_m + i\delta}\right]\right). \tag{3.22}$$

We have just derived an example of the soft graviton theorem.

3.1.5 Virtual Graviton Exchange

The rest of the amplitude $\mathcal{B}_0^{\text{eik.}}$ also merits some comments. We have

$$e^{\frac{i}{2}\int f\Delta f} = e^{\frac{i}{2}\int[f^{in}+f^{out}]\Delta[f^{in}+f^{out}]}. \tag{3.23}$$

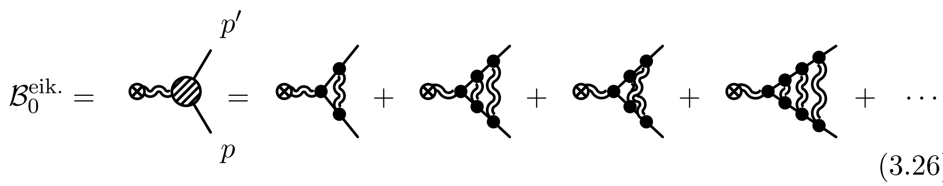
The terms generated by

$$e^{\frac{i}{2}\int f^{in}\Delta f^{in}}, \quad e^{\frac{i}{2}\int f^{out}\Delta f^{out}} \tag{3.24}$$

correspond to a divergent wavefunction renormalization, as well as a renormalization of the mass. To see this, write out the integrals in each of these exponentials and note that they are the same for each particle individually as the ones encountered in the previous chapter when deriving the IR form of the propagator (see the discussion just before equation (2.47)). In this case however, we have already done the overall proper time integrals when performing the mass shell amputation, leaving no dependence on s, and these renormalization constants factor out. As such, we simply assume renormalization has been carried out and drop these parts. The rest,

$$e^{i\int f^{in}\Delta f^{out}}, \tag{3.25}$$

generates vertex corrections at the perturbative point of contact with the external field. Namely, the eikonal limit implicitly sums all graphs like the following:



(3.26)

Though this form contains divergences, these divergences cancel with those coming from real emission. To see this, simply evaluate the integral in $e^{i\int f^{in}\Delta f^{out}}$ and notice that it corresponds to the virtual infrared divergences computed in section 3 of [29], and is precisely the gravitational analog of the Sudakov calculation [27]. With our result for the soft factors in equation (3.22), the calculation of real divergences in section 4 of [29] and the subsequent cancellation of these divergences proceeds identically to those in [29], section 5.

3.2 Two-Body Scattering

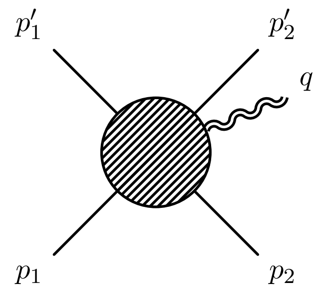
3.2.1 Constructing the Eikonal Amplitude

Here we will be concerned with the radiation of soft gravitons alongside a $2 \to 2$ scattering process. This amplitude (denoted $\mathcal{A}$ to distinguish from the previous discussion) is given by

$$\mathcal{A}_n = \langle p_1'; p_2'; q_1, \lambda_1; \cdots ; q_n, \lambda_n | p_1; p_2\rangle \, , \tag{3.27}$$

and for convenience we start with the single-emission case:

$$\mathcal{A}_1 \equiv \tag{3.28}$$



This amplitude has the expression

$$\begin{aligned}\mathcal{A}_1 = &(2\pi)^{-15/2} \frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}} \frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}} \int dx\, e^{iq\cdot x} \hat{\mathcal{K}}_x \\ &\times \int dy_1\, e^{ip_1'\cdot y_1} \hat{\mathcal{D}}_{y_1} \int dy_2\, e^{ip_2'\cdot y_2} \hat{\mathcal{D}}_{y_2} \int dz_1\, e^{-ip_1\cdot z_1} \hat{\mathcal{D}}_{z_1} \int dz_2\, e^{-ip_2\cdot z_2} \hat{\mathcal{D}}_{z_2} \\ &\times \frac{\delta}{\delta I(x)} \frac{\delta}{\delta J(y_1)} \frac{\delta}{\delta J(y_2)} \frac{\delta}{\delta J(z_1)} \frac{\delta}{\delta J(z_2)} \mathcal{Z}[J,I]|_0.\end{aligned} \tag{3.29}$$

Using the same manipulations as we did for $\mathcal{B}$, we can rewrite this as

$$\begin{aligned}\mathcal{A}_1 = &(2\pi)^{-15/2} \frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}} \frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}} \int dx\, e^{iq\cdot x} \frac{\delta}{\delta h(x)}\, e^{\frac{-i}{2}\int \frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}} \\ &\times \int dy_1\, e^{ip_1'\cdot y_1} \hat{\mathcal{D}}_{y_1} \int dy_2\, e^{ip_2'\cdot y_2} \hat{\mathcal{D}}_{y_2} \int dz_1\, e^{-ip_1\cdot z_1} \hat{\mathcal{D}}_{z_1} \int dz_2\, e^{-ip_2\cdot z_2} \hat{\mathcal{D}}_{z_2} \\ &\times [\mathcal{G}(y_1,z_1|h)\mathcal{G}(y_2,z_2|h) + (y_1 \leftrightarrow y_2)]|_{h\to 0}.\end{aligned} \tag{3.30}$$

We will from now suppress the coordinate dependence by $\mathcal{G}(y_1,z_1|h) \equiv \mathcal{G}_1$, etc. Now, again following [1], ch. 10, we compute not $\mathcal{A}$, but $\partial\mathcal{A}/\partial\kappa^2$. This is so that we can again produce similar Wilson line factors that generate the soft radiation dependence of the amplitude. We also remind ourselves that in this notation, $\tau \sim \kappa$. After some work,

$$\frac{\partial \mathcal{A}_1}{\partial \kappa^2} =$$
$$(2\pi)^{-15/2}\,\frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}}\,\frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}}\int du_1\int du_2\,\Delta^{\alpha\beta\sigma\rho}(u_1-u_2)\int dx\,e^{iq\cdot x}\frac{\delta}{\delta h(x)}\,e^{\frac{-i}{2}\int\frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}}$$
$$\times\int dy_1\,e^{ip_1'\cdot y_1}\hat{\mathcal{D}}_{y_1}\int dy_2\,e^{ip_2'\cdot y_2}\hat{\mathcal{D}}_{y_2}\int dz_1\,e^{-ip_1\cdot z_1}\hat{\mathcal{D}}_{z_1}\int dz_2\,e^{-ip_2\cdot z_2}\hat{\mathcal{D}}_{z_2}$$
$$\times\,(\mathcal{G}_1\frac{\tau^{(1)}_{\alpha\beta}}{\kappa}\mathcal{G}_1)(\mathcal{G}_2\frac{\tau^{(2)}_{\sigma\rho}}{\kappa}\mathcal{G}_2)|_{h\to 0}+\mathbf{0}. \tag{3.31}$$

The $\mathbf{0}$ indicates terms that do not vanish identically, but will not contribute to the final results at leading order.

3.2.2 On-Shell Results

Using (3.15), we put the scalar particles on shell, giving

$$\frac{\partial \mathcal{A}_1}{\partial \kappa^2} =$$
$$(2\pi)^{-6}\,\frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}}\int du_1\int du_2\,\Delta^{\alpha\beta\sigma\rho}(u_1-u_2)\frac{\tau^{(1)}_{\alpha\beta}\tau^{(2)}_{\sigma\rho}}{\kappa^2}\,e^{iu_1\cdot(p_1'-p_1)}e^{iu_2\cdot(p_2'-p_2)}$$
$$\times\,e^{\frac{i}{2}\int[f_1+f_2]\Delta[f_1+f_2]}\left((2\pi)^{-3/2}\frac{\epsilon^*(q,\lambda)}{\sqrt{2|\vec{q}|}}\int dx\,e^{iq\cdot x}\frac{\delta}{\delta h(x)}\,e^{i\int[f_1+f_2]h}|_{h\to 0}\right), \tag{3.32}$$

with

$$(f_{1|2})_{\mu\nu}(w,u_{1|2})\equiv\frac{1}{2}\sum_q e^{-iq\cdot(w-u_{1|2})}\int_0^\infty ds\,\left[\tau^{1|2}_{\mu\nu}e^{isp_{1|2}\cdot q}+\tau'^{1|2}_{\mu\nu}e^{-isp'_{1|2}\cdot q}\right]. \tag{3.33}$$

Note that this result can also be written as

$$(f_{1|2})_{\mu\nu}(w,u_{1|2})\equiv\frac{1}{2}\int_0^\infty ds\,\left[\tau^{1|2}_{\mu\nu}\delta(w-u_{1|2}-sp_{1|2})+\tau'^{1|2}_{\mu\nu}\delta(w-u_{1|2}+sp'_{1|2})\right], \tag{3.34}$$

revealing that these functions are just the classical gravitational sources of particles traveling at constant velocity.

3.2.3 Multiple Graviton Emission

The generalization to n gravitons is the same as before, meaning

$$
\begin{aligned}
&\frac{\partial \mathcal{A}_n}{\partial \kappa^2} = \\
&(2\pi)^{-6}\,\frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}}\int du_1 \int du_2\, \Delta^{\alpha\beta\sigma\rho}(u_1-u_2)\frac{\tau^{(1)}_{\alpha\beta}\tau^{(2)}_{\sigma\rho}}{\kappa^2} \\
&\times e^{iu_1\cdot(p_1'-p_1)}e^{iu_2\cdot(p_2'-p_2)}e^{\frac{i}{2}\int[f_1+f_2]\Delta[f_1+f_2]} \\
&\times \prod_m^n \left((2\pi)^{-3/2}\frac{\epsilon^*(q_m,\lambda_m)}{\sqrt{2|\vec{q_m}|}}\int dx_m\, e^{iq_m\cdot x_m}\,\frac{\delta}{\delta h(x_m)}\; e^{i\int[f_1+f_2]h}|_{h\to 0}\right),
\end{aligned}
\tag{3.35}
$$

and as before, the graviton part simplifies to

$$
\prod_m^n \left(\sum_{j=1}^{2}\frac{\sqrt{8\pi G}}{(2\pi)^{3/2}\sqrt{2|\vec{q_m}|}}\left[\frac{\epsilon_m^{\mu\nu *}(p_j')_\mu(p_j')_\nu}{p_j'\cdot q - i\delta} - \frac{\epsilon_m^{\mu\nu *}(p_j)_\mu(p_j)_\nu}{p_j\cdot q_m + i\delta}\right]\right) \equiv \prod_m^n \kappa\Omega_m,
\tag{3.36}
$$

but the factorization of the soft graviton dependence of $\mathcal{A}$ is not yet established - only for $\partial\mathcal{A}/\partial\kappa^2$ so far. In order to argue that the $\kappa\Omega_m$ factor at the level of the amplitude, we will specialize to the case of small virtual graviton momenta as well.

3.2.4 Virtual Graviton Exchange

Inside of $\exp\left(\frac{i}{2}\int[f_1+f_2]\,\Delta\,[f_1+f_2]\right)$ are terms like $(i,j\in\{1,2\})$ $p_i\Delta p_i$, $p_i\Delta p_i'$, $p_i\Delta p_j$, etc. The $p_i\Delta p_i$ terms again just lead to renormalization of a single particle in the forward limit. The $p_i\Delta p_i'$ terms contribute to renormalizing the vertex, which was seen in the bremsstrahlung calculation, equation (3.26), but here we choose to drop them. The remaining cross terms will generate the ladder type graphs that are known to give the eikonal amplitude when summed [44]. Thus, we approximate

$$e^{\frac{i}{2}\int[f_1+f_2]\Delta[f_1+f_2]} \to e^{i\int f_1\Delta f_2}. \tag{3.37}$$

If the momenta of the virtual radiation are small, we have that $p_i \approx p_i'$, and we also replace

$$\begin{aligned}(f_{1|2})_{\mu\nu}(w,u_{1|2}) &\to \frac{1}{2}\sum_q e^{-iq\cdot(w-u_{1|2})}\int_{-\infty}^{\infty} ds\,\tau_{\mu\nu}^{1|2}e^{-isp_{1|2}\cdot q} \\ &= \frac{1}{2}\int_{-\infty}^{\infty} ds\,\tau_{\mu\nu}^{1|2}\delta(w-u_{1|2}-sp_{1|2}),\end{aligned} \tag{3.38}$$

so that

$$e^{i\int f_1\Delta f_2} \approx e^{i\tau^1_{\alpha\beta}\tau^2_{\sigma\rho}\int_{-\infty}^{\infty} ds_1\int_{-\infty}^{\infty} ds_2\,\Delta^{\alpha\beta\sigma\rho}(u_1-u_2-s_1p_1+s_2p_2)}. \tag{3.39}$$

To evaluate the exponent, use the integral representation of Δ, repeated here for convenience,

$$\Delta^{\alpha\beta\sigma\rho}(x) = \lim_{M\to 0}\sum_k \frac{e^{ik\cdot x}}{k^2-M^2}P^{\alpha\beta\sigma\rho}, \quad P^{\alpha\beta\sigma\rho} \equiv \frac{1}{2}[\eta^{\alpha\sigma}\eta^{\beta\rho}+\eta^{\alpha\rho}\eta^{\beta\sigma}-\eta^{\alpha\beta}\eta^{\sigma\rho}], \tag{3.40}$$

where we have introduced a fictional graviton mass M in order to keep track of divergences later. The limit will remain implicitly. We work in the center-of-mass frame, where $p_1 = (E,0,0,p)$, and $p_2 = (E,0,0,-p)$. The proper time integrals give

$$e^{i\int f_1\Delta f_2} \approx \exp\left(\frac{i\kappa^2\gamma(s)}{16Ep}\sum_k^{(2)}\frac{e^{ik_\perp\cdot u_\perp}}{k_\perp^2-M^2}\right), \tag{3.41}$$

where $\gamma(s) \equiv 2(p_1\cdot p_2)^2 - m^4 = \frac{1}{2}(s-2m^2)^2 - m^4$, to be consistent with the notation in [44], and s is the usual Mandelstam variable, $s \equiv (p_1+p_2)^2$. For some four-vector v, we have defined $v_\perp \equiv (v^1,v^2) \equiv (v^x,v^y)$, and $u \equiv u_1-u_2$. We see then, that taking the eikonal limit has effectively reduced the problem to one with a two-dimensional dependence (i.e., dependence on $u_\perp$, rather than all of u). The two-dimensional integral can also be performed, giving

$$\frac{i\kappa^2\gamma(s)}{16Ep}\sum_k^{(2)}\frac{e^{ik_\perp\cdot u_\perp}}{k_\perp^2-M^2}=\frac{i\kappa^2\gamma(s)}{32\pi Ep}K_0(Mu_\perp)\approx-\frac{i\kappa^2\gamma(s)}{32\pi Ep}\ln(Mu_\perp). \quad (3.42)$$

Then in the limit of small virtual momenta, inserting the same integral representation of the other graviton propagator in (3.35) - the one not in the exponent - the amplitude becomes [5]

$$\begin{aligned}\frac{\partial\mathcal{A}_n}{\partial\kappa^2}&=(2\pi)^{-2}\delta(p_1+p_2-p_1'-p_2')\left[\prod_m^n\kappa\Omega_m\right]\frac{1}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}}\\&\times\int du_\perp\, e^{iu_\perp\cdot Q}\left(8Ep\frac{\partial}{\partial\kappa^2}\right)\exp\left(\frac{i\kappa^2\gamma(s)}{16Ep}\sum_k^{(2)}\frac{e^{ik_\perp\cdot u_\perp}}{k_\perp^2-M^2}\right)\end{aligned} \quad (3.43)$$

We will drop the δ-function henceforth. $Q\equiv(p_1'-p_1)_\perp$ parameterizes the two-dimensional momentum transfer.

3.2.5 Soft Factorization

Displaying the factorization property is now reduced to solving a differential equation. Defining $\mathfrak{a}$ via

$$\mathcal{A}_n\equiv(2\pi)^{-2}\left[\prod_m^n\Omega_m\right]\frac{8Ep}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}}\int du_\perp\, e^{iu_\perp\cdot Q}\,\mathfrak{a}_n(u_\perp,\kappa), \quad (3.44)$$

[5]Really, the δ-function showing up here should be $\delta(p_1+p_2-p_1'-p_2'-\sum_m q_m)$, but to leading order in $1/q$, this reduces to what appears in eq (3.43).

$\mathfrak{a}$ obeys

$$\frac{\partial \mathfrak{a}_n(u_\perp,\kappa)}{\partial \kappa^2} = \kappa^n \frac{\partial}{\partial \kappa^2} \exp\left(\frac{i\kappa^2\gamma(s)}{16Ep} \sum_k^{(2)} \frac{e^{ik_\perp \cdot u_\perp}}{k_\perp^2 - M^2} \right) \equiv \kappa^n \frac{\partial}{\partial \kappa^2} \exp\left(i\kappa^2 \mathfrak{f}\right). \tag{3.45}$$

This equation has the solution [4]

$$\mathfrak{a}_n(u_\perp,\kappa) = \kappa^n \frac{\Gamma\left[\frac{n}{2}+1, -i\kappa^2\mathfrak{f}\right]}{(-i\kappa^2\mathfrak{f})^{n/2}} + \text{const.}, \tag{3.46}$$

where $\Gamma[a,x]$ is the incomplete Gamma function. However, recalling (3.42) we can use the fact that the graviton is massless, and that $\lim_{M\to 0}\mathfrak{f} = +\infty$. We must also enforce the boundary conditions, which state that $\lim_{\kappa\to 0}\mathcal{A}_n = 0$, and that as $n \to 0$, we recover the amplitude with no soft radiation. Then to leading order, [3]

$$\mathfrak{a}_n(u_\perp,\kappa) = \kappa^n \left(e^{i\kappa^2\mathfrak{f}} - 1\right) + \mathcal{O}(\frac{1}{\mathfrak{f}}), \tag{3.47}$$

$$\mathcal{A}_n = (2\pi)^{-2} \left[\prod_m^n \kappa\Omega_m\right] \frac{8Ep}{4\sqrt{\omega_1\omega_1'\omega_2\omega_2'}} \int du_\perp\, e^{iu_\perp\cdot Q} \left[e^{i\chi(s,u_\perp)} - 1\right], \tag{3.48}$$

in which,

$$\chi \equiv \kappa^2\mathfrak{f} \equiv -\frac{\kappa^2\gamma(s)}{32\pi Ep}\ln(Mu_\perp) = -\frac{G\gamma(s)}{Ep}\ln(Mu_\perp), \tag{3.49}$$

in agreement with [44]. Finally, we have shown that the radiative two-body eikonal amplitude factorizes precisely as predicted by the soft theorems:

$$\mathcal{A}_n = \mathcal{A}_0^{\text{eik.}} \times \prod_m^n \left(\sum_{j=1}^2 \frac{\sqrt{8\pi G}}{(2\pi)^{3/2}\sqrt{2|\vec{q_m}|}} \left[\frac{\epsilon_m^{\mu\nu *}(p_j')_\mu (p_j')_\nu}{p_j'\cdot q - i\delta} - \frac{\epsilon_m^{\mu\nu *}(p_j)_\mu (p_j)_\nu}{p_j\cdot q_m + i\delta} \right] \right) \tag{3.50}$$

The soft theorems are universal, in that they do not discriminate based on the spin of the matter or the particular hard process. We have claimed here that they are also encoded in the functional formalism in a natural way. The key result of this section is the derivation of the soft factors for two concrete processes, gravitational bremsstrahlung and two-body eikonal scattering, with the latter being a new result of this formalism. The results are not surprising, however a few questions still remain. The cancellation of infrared divergences is straightforward in the bremsstrahlung case, following closely the pattern of [29], but the same cancellation has not yet been explicitly demonstrated in the two-body scattering case. Additionally, we have throughout this thesis simply stated things at the lowest contributing order when expanding in small momenta. A great deal though has been said about subleading soft effects [59–61] and they have been associated with new symmetries and memory effects in both QED [62] and gravity [63, 64]. It seems that now that the functional formalism has been shown to neatly reproduce known results at leading order, it would be a mistake not to attempt to take it to subleading order as well. The scalar theory also has an extra source of subdominant contributions, namely the higher order terms in the expansion (2.20).

Chapter 4

Conclusion

The aim of this work has been to demonstrate the utility of the functional formalism in understanding the infrared limit of quantum gravity in a unified language. To that end, we show here that it neatly reproduces many important known results from the literature. This should be seen as a first step in using this formalism to fully understand all universal properties of soft gravity, due to the ease of applying this formalism to many aspects of the IR theory.

In this thesis we have introduced the major problems in IR gravitational physics, and solved them using functional techniques. Chapter 1 covered the example of quantum electrodynamics, and discussed how to construct a simple theory of low energy quantized Einstein GR minimally coupled to scalar matter. The theory was then investigated using functional methods and nonperturbative results were obtained in the main body of the work. In Chapter 2, we applied the powerful Fradkin technique to the linearized theory, yielding a useful representation for the generating functional. We then demonstrated the ease at which correlation functions could be recovered from this expression. As an example, the infrared effective form of the propagator was derived and its form discussed. Chapter 3 contained a recipe for computing S-matrix elements with the help of the eikonal form of the generating functional and the two-point function derived from it. We then finally demonstrated that the soft graviton theorem is encoded in this formalism by deriving in parallel the eikonal form of the two-body scattering amplitude and the soft graviton factorization property.

We have already begun to speculate on potential applications of this work beyond the scope of this thesis. Future investigations using this framework may include applications to decoherence due to quantum gravity [58], an attempt at understanding "dressed state" type approaches [51–57], and the relation of the functional formalism to the "infrared triangle" and the black hole information problem as presented in e.g. [46–50]. Additionally, some more technical aspects of this formalism play a central role, but are somewhat opaque and deserve to be better understood. For example, the form of the proper time dependence of χ in (2.22) and the Wilson line factors appearing in the expression of every amplitude, e.g., the $e^{i\int fh}$ factor in eq. (3.16). These forms are crucial for getting correct results, though not transparent in the way that they lead to eikonal physics, and should be investigated further. The other obvious question is that of how this formalism responds to the inclusion of higher order interaction terms in the gravity-matter lagrangian. Because gravity behaves qualitatively differently than QED when these diagrams are included [65], this may be a worthwhile effort.

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Appendix A

Selected Functional Identities

Here we state without proof some necessary functional identities.

The action of an operator linear in functional derivatives on any functional F of some function j can be shown to be

$$e^{\int f \frac{\delta}{\delta j}} F[j] = F[j+f], \tag{A.1}$$

simply shifting the functional dependence of F.

We also need to know the action of a quadratic shift on certain types of functionals. The quadratic shift operator is of the form, e.g.,

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta j} A \frac{\delta}{\delta j}} \tag{A.2}$$

and we have the special cases

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta j} A \frac{\delta}{\delta j}} e^{i\int fj} = e^{\frac{i}{2}\int fAf} e^{i\int jf}, \tag{A.3}$$

when acting on a linear functional, and

$$e^{-\frac{i}{2}\int \frac{\delta}{\delta j} A \frac{\delta}{\delta j}} e^{\frac{i}{2}jDj} = \exp\left\{\frac{i}{2}\int jD(1-AD)^{-1}j - \frac{1}{2}Tr\ln(1-AD)\right\}, \tag{A.4}$$

acting on a Gaussian functional.

Another identity that is extremely useful for evaluating S-matrix elements in this formalism is

$$e^{-i\int f\frac{\delta}{\delta j}}e^{\frac{i}{2}\int j\Delta j} = e^{\frac{i}{2}\int j\Delta j}e^{-\frac{i}{2}\int\frac{\delta}{\delta h}\Delta\frac{\delta}{\delta h}}e^{\int fh}, \tag{A.5}$$

where $h \equiv \int \Delta j$.

Appendix B

Evaluation of Phase Integrals

The exponent in eq. (2.45) has a phase

$$\sum_q f(q)f(-q)\tau_{\alpha\beta}(k,k)\Delta^{\alpha\beta\mu\nu}(q^2)\tau_{\mu\nu}(k,k) = \tau_{\alpha\beta}P^{\alpha\beta\mu\nu}\tau_{\mu\nu}\sum_q \frac{f(q)f(-q)}{q^2}$$
$$\equiv \tau_{\alpha\beta}P^{\alpha\beta\mu\nu}\tau_{\mu\nu}\mathcal{I}, \tag{B.1}$$

where $P^{\alpha\beta\mu\nu} \equiv \frac{1}{2}[\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu}]$. Now using the expression for the form factor in (2.28), we can approach $\mathcal{I}$ in a simple way. First, write $\mathcal{I}$ using the unintegrated expressions for the form factors:

$$\begin{aligned}\mathcal{I} &= -\int_0^s ds' \int_0^s ds'' \sum_q \frac{1}{q^2} e^{-i(s'-s'')(2k\cdot q)} \\ &\equiv -\int_0^s ds' \int_0^s ds'' \sum_q \frac{1}{q^2} e^{-is_-(2k\cdot q)}\end{aligned} \tag{B.2}$$

This expression is $\sim \int_0^\infty QdQ$ and thus still has a divergence in the UV. However, we can move this divergence out of the momentum integrals and into the proper time ones by introducing yet another proper time (and going to Euclidean momentum space)

$$\mathcal{I} = -\int_0^s ds' \int_0^s ds'' \int_0^\infty du \sum_q e^{iuq^2 - is_-(2k\cdot q)}. \tag{B.3}$$

Now the momentum integral is Gaussian and can be done, giving

$$\mathcal{I} = -\frac{i}{4\pi^2} \int_0^s ds' \int_0^{s'} ds'' \int_0^\infty \frac{du}{u^2} \exp\left(-i\frac{k^2 s_-^2}{u}\right), \tag{B.4}$$

The integral over u is elementary when s_-^2 is given a small imaginary part: $s_-^2 \to s_-^2 - i\epsilon$. Actually it is more convenient to take $s_- \to s_- - i\epsilon$, which is equivalent in the limit of small ϵ since $s'' < s'$. The result is

$$\mathcal{I} = -\frac{i}{4\pi^2 k^2} \int_0^s ds' \int_0^{s'} ds'' (s_- - i\epsilon)^{-2}, \tag{B.5}$$

and with a little bit of algebra this can be shown, to leading order, to behave as

$$\mathcal{I} = \frac{i}{4\pi^2 k^2} \left[is\Lambda - \ln(s\Lambda) + \mathcal{O}(\Lambda^0) \right], \tag{B.6}$$

where $\Lambda \equiv \epsilon^{-1}$ may be identified as the UV cutoff avoided in the momentum integrals earlier.

Appendix C

The Generating Functional

This appendix and the next deal with deriving eq. (3.5). We closely follow the arguments given in chapters 3 and 4 of [1].

In classical probability theory, the expectation value of some observable O that depends on a random variable x is given by

$$\langle O \rangle = \int dx\, P(x) O(x), \tag{C.1}$$

where $P(x)$ is the probability of some value of x, with $\int dx\, P(x) = 1$. Quantum field theory is similar, in that we typically consider expectation values of, e.g., field operators at different spacetime points,

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi(x)\, \mathbf{P}[\phi]\, \phi(x_1) \cdots \phi(x_n). \tag{C.2}$$

Here, $\mathbf{P}[\phi] = e^{iS[\phi]}$ is the *probability amplitude* for the field configuration $\phi(x)$. We can consider the following expectation value:

$$\mathcal{Z}[j] \equiv \langle z[j] \rangle \equiv \langle \mathcal{T} e^{i \int \phi j} \rangle \tag{C.3}$$

with $\mathcal{T}$ the time ordering operator. Writing out the functional average,

$$\mathcal{Z}[j] = \mathcal{N} \int \mathcal{D}\phi\, e^{iS[\phi] + i \int \phi j}, \tag{C.4}$$

we see that this is just the usual generating functional, which generates correlation functions in a particular theory (this formalism extends trivially to multiple interacting fields). $\mathcal{N}^{-1} = \int \mathcal{D}\phi\, e^{iS[\phi]}$, to ensure that $\mathcal{Z}[j = 0] =$

$\langle 0|0\rangle = 1$ (using the IN state basis). However, we can learn a bit more by investigating z instead of just $\mathcal{Z}$. We generalize z by defining

$$\overset{b}{\underset{a}{z}}[j] \equiv \mathcal{T} e^{i \int_a^b \phi j}, \tag{C.5}$$

where now $\int_a^b \phi j \equiv \int d^3x \int_{t_a}^{t_b} dt\, \phi(x) j(x)$. Then,

$$\frac{\delta \overset{b}{\underset{a}{z}}}{\delta j(x)} = i\, \overset{b}{\underset{x}{z}}\, \phi(x)\, \overset{x}{\underset{a}{z}}, \tag{C.6}$$

by time ordering, or for $z \equiv \overset{\infty}{\underset{-\infty}{z}}$,

$$\frac{\delta z}{\delta j(x)} = i\, \overset{\infty}{\underset{x}{z}}\, \phi(x)\, \overset{x}{\underset{-\infty}{z}}. \tag{C.7}$$

This property of z will allow a connection with the S-matrix.

Appendix D

The Generating Functional Generates the S-Matrix

First define the usual asymptotic IN/OUT states,

$$|a\rangle_{\mathrm{OUT}} = \lim_{t\to\infty} |a,t\rangle\,, \qquad |a\rangle_{\mathrm{IN}} = \lim_{t\to-\infty} |a,t\rangle \tag{D.1}$$

and the S-matrix via

$$|a\rangle_{\mathrm{OUT}} = \mathcal{S}^\dagger\, |a\rangle_{\mathrm{IN}}\,, \qquad \phi_{\mathrm{OUT}}(x) = \mathcal{S}^\dagger \phi_{\mathrm{IN}}(x)\mathcal{S}. \tag{D.2}$$

Generally, the stationary action principle enforces an on-shell relation like

$$\hat{\mathcal{D}}_x \phi(x) = j(x), \tag{D.3}$$

where $\hat{\mathcal{D}}$ is some differential operator acting w.r.t. the coordinate x, and $j(x)$ is a spacetime-dependent source. The general solution to this equation can be expressed in terms of the Green function $G(x,x')$ satisfying $\hat{\mathcal{D}}_x G(x,x') = \delta(x-x')$ as

$$\phi(x) = \phi_0(x) + \int dy\, G(x,y) j(y) = \phi_0(x) + \int dy\, G(x,y)\hat{\mathcal{D}}_y \phi(y), \tag{D.4}$$

or in terms of the IN/OUT fields,

$$\phi(x) = \phi_{\substack{\mathrm{IN}\\ \mathrm{OUT}}}(x) + \int dy\, G_{\substack{R\\ A}}(x,y)\hat{\mathcal{D}}_y \phi(y), \tag{D.5}$$

Where R (A) denotes the retarded (advanced) propagator. By putting (D.5) into (C.7), we get that

$$\begin{aligned}\frac{\delta z}{\delta j(x)} &= i\, z\, \phi_{\mathrm{IN}}(x) + \int d^4y\, G_R(x,y)\hat{\mathcal{D}}_y \frac{\delta z}{\delta j(y)} \\ &= i\, z\, \phi_{\mathrm{OUT}}(x) + \int d^4y\, G_A(x,y)\hat{\mathcal{D}}_y \frac{\delta z}{\delta j(y)}.\end{aligned} \tag{D.6}$$

It does not matter whether we choose to express this using the IN or OUT fields. Take the difference of the two equivalent expressions to see that

$$0 = i\left(\phi_{\mathrm{OUT}}(x) z - z\phi_{\mathrm{IN}}(x)\right) + \int d^4y\, G(x,y)\hat{\mathcal{D}}_y \frac{\delta z}{\delta j(y)}, \tag{D.7}$$

G now meaning the causal propagator. Multiply this expression by S and use (D.2) to show that

$$[\phi_{\mathrm{IN}}(x), \mathcal{S}z] = i\mathcal{S} \int d^4y\, G(x,y)\hat{\mathcal{D}}_y \frac{\delta z}{\delta j(y)}, \tag{D.8}$$

which can be rewritten as

$$[\phi_{\mathrm{IN}}(x), \mathcal{S}z] = i \int d^4y\, G(x,y)\hat{\mathcal{D}}_y \frac{\delta}{\delta j(y)}(\mathcal{S}z), \tag{D.9}$$

because $\mathcal{S}$ will not depend explicitly on $j(x)$. This relation is not immediately useful, but we will see that is enough to determine the form of $\mathcal{S}z$. To do this, consider another operator similar to z, but with *normal ordering* rather than time ordering. The normal ordering operation is denoted by ::, and demands that, for everything between the colons, creation operators are placed to the left of annihilation operators. E.g.,

$$:\hat{a}^\dagger \hat{a}: = \hat{a}^\dagger \hat{a}, \qquad :\hat{a}\hat{a}^\dagger: = \hat{a}^\dagger \hat{a}, \tag{D.10}$$

etc. We consider the operator

$$:e^{\int \phi_{\mathrm{IN}} f}:, \tag{D.11}$$

and make use of the result

$$\left[\phi_{\mathrm{IN}}(x), : e^{\int \phi_{\mathrm{IN}} f} :\right] = : e^{\int \phi_{\mathrm{IN}} f} : i \int d^4y\, G(x,y) f(y). \tag{D.12}$$

Looking at this as well as (D.9), it appears that

$$\mathcal{S}z = : e^{\int \phi_{\mathrm{IN}} \hat{\mathcal{D}} \frac{\delta}{\delta j}} : g[j], \tag{D.13}$$

for some $g[j]$. In fact, this form is fixed as well from the fact that $\langle : e^{\int \phi_{\mathrm{IN}} f} : \rangle = 1$, and we get

$$\mathcal{S}z[j] = : e^{\int \phi_{\mathrm{IN}} \hat{\mathcal{D}} \frac{\delta}{\delta j}} : \langle \mathcal{S}z[j] \rangle\, . \tag{D.14}$$

Finally, if the source $j(x)$ is set to zero after the functional derivatives are taken, we isolate the form of $\mathcal{S}$ up to normalization:

$$\mathcal{S} = : e^{\int \phi_{\mathrm{IN}} \hat{\mathcal{D}} \frac{\delta}{\delta j}} : \mathcal{Z}[j]|_{j=0}, \tag{D.15}$$

where again, $\mathcal{Z} \equiv \langle z \rangle$. The argument proceeds in the same way for theories with multiple fields.