Renormalization group analysis of self-interacting walks and spin systems

by

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Abstract

The central concern of this thesis is the study of critical behaviour in models of statistical physics in the upper-critical dimension. We study a generalized n-component lattice $|\varphi|^4$ model and a model of weakly self-avoiding walk with nearest-neighbour contact self-attraction on the Euclidean lattice $\mathbb{Z}^d$. By utilizing a supersymmetric integral representation involving boson and fermion fields, the two models are studied in a unified manner.

Our main result, which is contingent on a small coupling hypothesis, identifies the precise leading-order asymptotics of the two-point function, susceptibility, and finite-order correlation length of both models in $d = 4$. In particular, we show that the critical two-point function satisfies mean-field scaling whereas the near-critical susceptibility and finite-order correlation length exhibit logarithmic corrections to mean-field behaviour. The proof employs a renormalization group method of Bauerschmidt, Brydges, and Slade based on a finite-range covariance decomposition and requires two extensions to this method.

The first extension, which is required for the computation of the finite-order correlation length (even for the ordinary weakly self-avoiding walk and $|\varphi|^4$ model), is an improvement of the norms used to control the evolution of the renormalization group. This allows us to obtain improved error estimates in the massive regime of the renormalization group flow.

The second extension involves the identification of critical parameters for models initialized with a non-zero error coordinate coupled to a marginal/relevant coordinate. This allows us, for example, to realize the two-point function and susceptibility for the walk with self-attraction as a small perturbation of the corresponding quantities without self-attraction, whose asymptotic behaviour was determined by Bauerschmidt, Brydges, and Slade. This establishes a form of universality.

Lay Summary

In this thesis, we study two models from statistical physics: the weakly self-avoiding walk with self-attraction, which is a model of a linear polymer in a poor solution; and the lattice $|\varphi|^4$ model, which can be understood as a model of a ferromagnet. Both models are expected to undergo a phase transition at which certain precise quantitative properties, known as critical exponents, should become independent of the fine model specifications. Our main result provides a mathematically rigorous confirmation of several predicted values of critical exponents for both models.

Preface

Sections 1–1.6 are a general introduction to the subject matter of this thesis and we do not claim any originality here. Section 1.7 states our main result, which combines and slightly extends results from the following:

- the article [12], written jointly with Roland Bauerschmidt, Gordon Slade, and Alexandre Tomberg and published in *Annales Henri Poincaré*; and
- the article [13], written jointly with Roland Bauerschmidt and Gordon Slade and published in *Journal of Statistical Physics*.

Section 1.8 includes part of [13].

Chapters 2–5 are based on [12, 13]:

- Chapter 2 includes part of [12,13], and discusses the general theory developed in [10,28–31] and applied and extended in [8, 9, 108];
- Chapter 3 includes part of [12, 13] and Section 3.1 includes an additional discussion regarding an argument of [9];
- Chapter 4 includes part of [12]; in addition, Sections 4.1 and 4.3.4 include discussions regarding some of the ideas in [31] and [29], respectively; and
- Chapter 5 includes part of [13].

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Chapter 1

Introduction

Thermodynamics originated as the study of heat and its transformations into other forms of energy, but can more generally be described as the study of bulk matter. That is, it seeks to answer questions regarding those properties of matter that are defined in terms of large collections of particles. Two familiar examples are density and temperature. The density of a substance is its mass per unit volume. Such a quantity is sometimes said to be *intensive*: it depends only on the nature of the substance under consideration, not on the amount that is present.

There is a small caveat that should be added to this definition: the mass per unit volume of a sample of some substance will only become independent of the volume once this volume is sufficiently large. This makes density a *macroscopic* quantity (as opposed to, e.g. the atomic number). To say that thermodynamics is the study of bulk matter amounts to saying that it deals in macroscopic quantities.

This raises a number of simple questions. For instance, if these quantities are truly intensive, then it should be possible to derive them from the microscopic behaviour of the substance's constituent particles; yet above, we have resorted to discussing "large" quantities of matter in order to make sense of density. In fact, this leads us to another natural question: what constitutes a sample sufficiently large to be considered bulk matter and why should certain measurements of a sample stabilize when the sample is large?

A natural response is that some sort of law of large numbers must be at work. Although the behaviour of a system of particles is not random, it may be sufficiently complex that it is reasonable to view it as such. This is the basic idea of equilibrium statistical mechanics: to leverage the complexity of large systems of particles in order to explain the apparent simplicity of bulk matter as arising from the kind of self-averaging that pervades probability theory. From this point of view, macroscopic quantities are types of averages that arise from some probability distribution. These averages typically depend on temperature or other parameters, so in this sense the main objects of study in statistical mechanics are certain parameterized families of probability measures.

It is useful to organize these measures according to their qualitative properties. Qualitatively similar measures correspond to a *phase* of a substance. For instance, water at 20°C is not significantly different from water at 50°C. On the other hand, steam and water have very different behaviours despite both being H_2O. In fact, the boiling of water is signalled by an abrupt drop in density by a factor of around 1/1000 at normal atmospheric pressure.

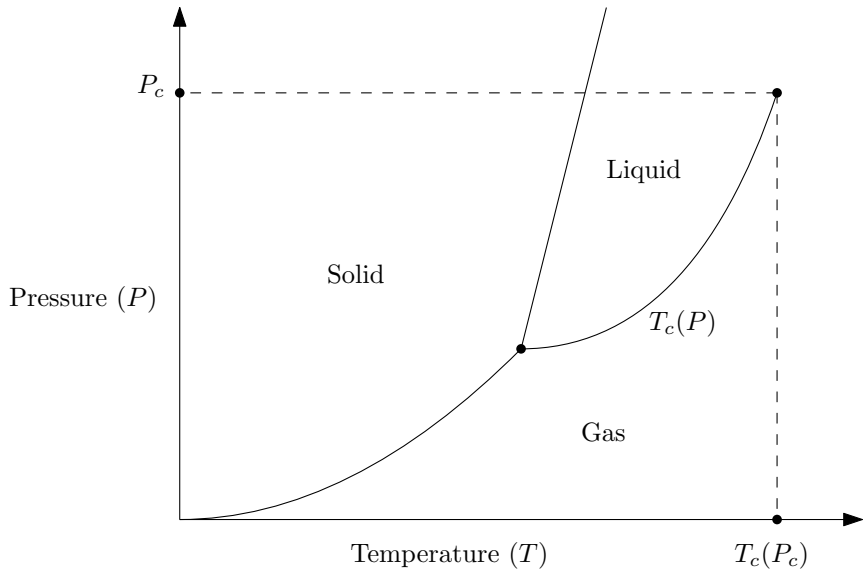

Figure 1.1: The phase diagram of H_2O

A similar effect occurs under different atmospheric conditions as well. The temperature at which water boils varies as a function of pressure $T_c = T_c(P)$ and a *phase transition* is said to occur when the pressure and/or temperature are varied in such a way as to cross the graph of this function; see Figure 1.1. However, as P is increased, the density difference along this curve decreases and there is a *critical point* $(T_c(P_c), P_c)$ at which this difference vanishes.

Since statistical mechanics deals with extremely large, complicated systems of interacting particles, only simplified models of real materials can usually be studied in detail. Such models are useful for building a *qualitative* understanding of the phases of matter and phase transitions. However, the simplifications inherent in their definitions mean that they are usually not suitable for making quantitative predictions.

A remarkable phenomenon, known as *universality*, is that this is no longer entirely true at the critical point. At criticality, many quantities behave in a way that is independent of the fine details of the model being used. Thus, some of the *quantitative* properties of real materials can in principle be predicted exactly by studying models only roughly resembling these materials

In the 1970's Ken Wilson, inspired by ideas in quantum field theory, gave an explanation of universality in terms of the *renormalization group*: an abstract dynamical system that acts on models by averaging out their fine details. This idea was enormously successful and led to his 1982 Nobel Prize.

There have been several rigorous implementations of Wilson's ideas, some of which will be mentioned in Section 1.5.4. The main purpose of this thesis is to discuss extensions of a rigorous renormalization group method of Bauerschmidt, Brydges, and Slade that have been used to study the critical behaviour of a generalized $|\varphi|^4$ model and a model of weakly self-avoiding walk with contact self-attraction (WSAW-SA). We will introduce these models and our main results in the present chapter and discuss the proofs in the remainder of the thesis.

We begin in Section 1.1 with some general background on asymptotic notation and the method of generating functions and Laplace transforms. In Section 1.2, we briefly introduce some of the basic ideas of equilibrium statistical physics. The models we study are defined on graphs, which we discuss in Section 1.3. We define spin systems, in particular the $|\varphi|^4$ model, in Section 1.4. Critical behaviour and the renormalization group is probably most easily explained in the context of such systems, and we give an informal description of these ideas in Section 1.5. In Section 1.6, we introduce the WSAW-SA and discuss its critical behaviour. This gives us all the necessary definitions to state our main results in Section 1.7. Before proceeding to the method of proof, we discuss the close relationship between models of walks and spin systems in Section 1.8; in particular, we recall a representation of the WSAW-SA in terms of a spin system related to the $|\varphi|^4$ model that allows us to unify our treatment of both models. The remainder of the thesis is outlined in Section 1.9.

1.1 Asymptotics

We begin with a short discussion of some useful notation and mathematical background.

1.1.1 Asymptotic notation

Let F and G be a functions on a subset of the real line. For $a \in [-\infty, \infty]$, we write

$$F(x) \sim G(x), \qquad x \to a \tag{1.1.1}$$

$$F(x) = o(G(x)), \qquad x \to a \tag{1.1.2}$$

if, respectively,

$$\lim_{x \to a} \frac{F(x)}{G(x)} = 1 \tag{1.1.3}$$

$$\lim_{x \to a} \frac{F(x)}{G(x)} = 0. \tag{1.1.4}$$

We also write

$$F(x) = O(G(x)) \text{ or } F(x) \leq O(G(x)), \qquad x \to a \tag{1.1.5}$$

if there is a constant $C \geq 0$ such that $|F(x)| \leq CG(x)$ for all x in some neighbourhood of a (in the extended real line). Lastly, we write

$$F(x) \asymp G(x), \qquad x \to a \tag{1.1.6}$$

if $F(x) = O(G(x))$ and $G(x) = O(F(x))$ as $x \to a$, possibly with different constants for the two inequalities. We will sometimes write $F(x) \approx G(x)$ in heuristic arguments where the hope is that a rigorous argument might replace $\approx$ by $\asymp$ or $\sim$.

1.1.2 Generating function and Laplace transform

The *generating function* of a sequence a_n is the function $g(z)$ defined by the power series with coefficients a_n:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{1.1.7}$$

If the function g is sufficiently well understood, then the coefficients can be recovered by differentiation:

$$a_n = \frac{1}{n!} g^{(n)}(0). \tag{1.1.8}$$

This is known as the method of generating functions [114].

In many cases, g cannot be computed exactly. Nevertheless, there is a close relationship between the asymptotics of the sequence a_n as $n \to \infty$ and the function $g(z)$ near its dominant singularities, i.e. its singularities closest to the origin [52]. For instance, if $a_n \sim r^{-n} n^{\alpha}$, then the root test implies that the generating function f has radius of convergence r. If, moreover, $a_n \geq 0$, then $g(z) \sim C(r - z)^{-(\alpha+1)}$ as $z \uparrow r$. This is an example of an *Abelian theorem*. The converse does not always hold; a theorem providing conditions under which the converse is true is known as a *Tauberian theorem*, and is generally harder to prove than its Abelian counterpart.

Likewise, the asymptotics of a function $f(T)$ as $T \to \infty$ can sometimes be recovered from the behaviour of its *Laplace transform*

$$G(\nu) = \int f(T) e^{-\nu T} \, dT \tag{1.1.9}$$

near $\nu_0 := \inf\{\nu : G(\nu) < \infty\}$. There are Abelian and Tauberian theorems for the Laplace transform analogous to those for generating functions. For instance, $f(T) \sim AT^{\alpha}$ implies that $G(\nu) \sim A'(\nu - \nu_0)^{-(\alpha+1)}$ as $\nu \downarrow \nu_0$ and the converse holds when f is monotone (see [113]).

1.2 Equilibrium statistical mechanics

Let (Ω, λ) be a measure space. We view λ as some "natural" measure on Ω. The dynamics of a physical system with state space Ω are often determined by a function H on Ω, known as the *Hamiltonian*. Typically, $H(\omega)$ represents the total energy of the system in state ω; it is thus reasonable to assume that H is bounded below[1].

Example 1.2.1. The canonical example is a system of n point particles in a domain $U \subset \mathbb{R}^3$, for which $\Omega = (U \times \mathbb{R}^3)^n$ and λ is Lebesgue measure on Ω. A state $\omega \in \Omega$ consists of the positions $q_i \in U$ and momenta $p_i \in \mathbb{R}^3$ of the n particles. The Hamiltonian is usually smooth

[1] It is convenient to allow H to take on negative values.

and the dynamics are determined by Hamilton's equations

$$\frac{dq}{dt} = \nabla_p H \tag{1.2.1}$$

$$-\frac{dp}{dt} = \nabla_q H. \tag{1.2.2}$$

Typically, the Hamiltonian has the form

$$H(q,p) = \frac{1}{2m}|p|^2 + U(q). \tag{1.2.3}$$

The first term is the usual definition of kinetic energy and the second term, which depends only on q, is a potential energy function.

In statistical mechanics, the systems of concern consist of a very large number of particles (n large in the previous example). Typically, the state space Ω is very high-dimensional and studying the exact dynamics is infeasible.

A common simplifying assumption is that after a long time has passed, the system will settle into a state of *thermal equilibrium*, meaning that there is no net flow of heat between the system and its surroundings. Thus, the temperature is constant; we denote by β the *inverse temperature*. The *Gibbs measure* for a system with Hamiltonian H at inverse temperature β is the probability measure on Ω given by

$$\mu_\beta(d\omega) = \frac{1}{Z} e^{-\beta H(\omega)} d\lambda(\omega) \tag{1.2.4}$$

when this is well-defined. The normalizing constant

$$Z = \int e^{-\beta H}\, d\lambda \tag{1.2.5}$$

is known as the *partition function*.

1.2.1 Relation to quantum theory

Let us briefly describe the relationship between equilibrium statistical mechanics and quantum field theory. Our presentation is heuristic and involves manipulations of a priori ill-defined objects. A rigorous development of these ideas goes back to [95, 97, 98, 110]; useful references are [105] (for quantum mechanics) and [60] (for quantum field theory).

Let $H(x,p)$ be the Hamiltonian (1.2.3) (we have written x instead of q). In quantum mechanics, the state space is replaced by a Hilbert space, usually $L^2 = L^2(\mathbb{R}^{3n})$; the position x_j is replaced by the *position operator* $\hat{x}_j$ given by multiplication by x_j; and the momentum p_j is replaced by the *momentum operator* $\hat{p}_j = -i\hbar\partial/\partial x_j$ (we ignore details regarding the domains of these operators). The resulting operator $\hat{H} = H(\hat{x}, \hat{p})$ on L^2 determines the evolution of the

wave function $\psi \in L^2$ via the Schrödinger equation

$$i\hbar \frac{d\psi}{dt} = \hat{H}\psi \tag{1.2.6}$$

whose solution with initial condition $\psi(0)$ is given by $\psi(t) = e^{-it\hat{H}/\hbar}\psi(0)$.

Suppose that the solution operator is an integral operator with kernel K_t:

$$e^{-it\hat{H}/\hbar} f = \int K_t(\cdot, y) f(y)\, dy. \tag{1.2.7}$$

The Feynman path integral formulation of quantum mechanics [50] involves writing the kernel as

$$K_t(a,b) = \int_{\mathcal{W}_t(a,b)} e^{(i/\hbar)\int_0^t L(x(s),\dot{x}(s))\, ds}\, \mathcal{D}x, \tag{1.2.8}$$

where $\mathcal{W}_t(a,b)$ is a space of paths $[0,t] \to \mathbb{R}^{3n}$ from a to b equipped with a "Lebesgue measure" $\mathcal{D}x$ and

$$L(x,\dot{x}) = \frac{1}{2}m|\dot{x}|^2 - U(x) \tag{1.2.9}$$

is the *Lagrangian* of the classical system we started with. As such, the integral representation of K_t is ill-defined; for instance, the measure $\mathcal{D}x$ on paths does not exist. However, let us not concern ourselves with this.

Suppose that ψ can be analytically continued to a region of the complex plane containing the positive imaginary axis. Then we might hope the function $t \mapsto \psi(-it)$ has solution operator with kernel of the form

$$K_{-it}(a,b) = \int_{\mathcal{W}_{-it}(a,b)} e^{(i/\hbar)\int_0^{-it} L(x(s),\dot{x}(s))\, ds}\, \mathcal{D}x. \tag{1.2.10}$$

By the change of variables $s = -iu$,

$$(i/\hbar)\int_0^{-it} L(x(s),\dot{x}(s))\, ds = (1/\hbar)\int_0^t L(\tilde{x}(u), i\dot{\tilde{x}}(u))\, du = -(1/\hbar)\int_0^t H(\tilde{x}(u), m\dot{\tilde{x}}(u))\, du \tag{1.2.11}$$

with $\tilde{x}(t) = x(-it)$. Thus, by the fictive change of variables in which $\mathcal{W}_{it}(a,b)$ is replaced by $\mathcal{W}_t(a,b)$, we get

$$K_{-it}(a,b) = \int_{\mathcal{W}_t(a,b)} e^{-(1/\hbar)\int_0^t H(x(u),m\dot{x}(u))\, du}\, \mathcal{D}x \tag{1.2.12}$$

This procedure is known as a *Wick rotation*.

The analogue of this idea in quantum field theory involves replacing integration over paths with integration over fields φ, which are functions on $\mathbb{R}^d$. The corresponding "measures" of the form $e^{-(1/\hbar)\int H}\mathcal{D}\varphi$ should be compared with the Gibbs measures (1.2.4) with Planck's constant $\hbar$ playing the role of temperature. Although H has been replaced by the *integral* $\int H$, we will discuss how natural choices for the Hamiltonian in a Gibbs measure are given by *sums* over

spaces of fields on discrete approximations to $\mathbb{R}^d$ (graphs).

Remark 1.2.2. In fact, it turns out that (1.2.12) is not too unreasonable. For simplicity, let us take $\hbar = 1$ and $m = 1$. Then we would expect $\tilde{\psi}$ to solve

$$\frac{d\tilde{\psi}}{dt} = -\hat{H}\tilde{\psi}. \tag{1.2.13}$$

When $U = 0$, we have $H(x,p) = |p|^2$, so $\hat{H} = -\frac{1}{2}\Delta_{\mathbb{R}^{3n}}$ is the Laplacian on $\mathbb{R}^{3n}$ and (1.2.13) is the heat equation. On the other hand, $\int_0^t H$ is formally a positive-definite quadratic form, so (1.2.12) is a formal Gaussian integral. This is consistent with the well-known fact that a solution $\tilde{\psi}$ of the heat equation on a domain can (under appropriate conditions) be written as $\tilde{\psi}(t,x) = \mathbb{E}(\tilde{\psi}(0,B_t) \mid B_0 = x)$, where B_t is a Brownian motion. More generally, under appropriate hypotheses, the Feynman-Kac formula expresses solutions to (1.2.13) with $U \neq 0$ as averages with respect to a Brownian motion.

1.3 Graphs

An *undirected graph* or simply a *graph* is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set of *vertices* and $\mathcal{E}$ is a set of *edges* $\{x, y\}$ with $x, y \in \mathcal{V}$; we will write $x \sim y$ if $\{x, y\} \in \mathcal{E}$. For simplicity, we will assume that $\mathcal{V}$ is countable and that there are no *self-loops* $\{x\} \in \mathcal{E}$.

A graph *automorphism* is a bijection $f : \mathcal{V} \to \mathcal{V}$ such that $x \sim y$ if and only if $f(x) \sim f(y)$. We will assume that $\mathcal{G}$ is *transitive* meaning that for all pairs of distinct vertices $a, b \in \mathcal{V}$, there exists an automorphism f with $f(a) = b$. We fix a vertex $0 \in \mathcal{V}$ whose precise choice is immaterial due to transitivity.

1.3.1 Functions on graphs

Let us denote the components of an element $\varphi \in (\mathbb{R}^n)^{\mathcal{V}}$ by $\varphi_x^i \in \mathbb{R}$ for $x \in \mathcal{V}$ and $i = 1, \dots, n$. The Euclidean inner product and norm on $(\mathbb{R}^n)^{\mathcal{V}}$ are defined by

$$\varphi \cdot \tilde{\varphi} = \sum_{x \in \mathcal{V}} \varphi_x \cdot \tilde{\varphi}_y = \sum_{i=1}^{n} \sum_{x \in \mathcal{V}} \varphi_x^i \tilde{\varphi}_x^i \tag{1.3.1}$$

$$|\varphi|^2 = \varphi \cdot \varphi. \tag{1.3.2}$$

A $\mathcal{V} \times \mathcal{V}$ matrix $M = (M_{xy})_{x,y \in \mathcal{V}}$ acts on φ component-wise:

$$(M\varphi)_x = \sum_{y \in \mathcal{V}} M_{xy} \varphi_y. \tag{1.3.3}$$

1.3.2 The graph Laplacian

Let us say that a $\mathcal{V} \times \mathcal{V}$ matrix M is *indexed by* $\mathcal{E}$ if $M_{xy} \neq 0$ if and only if $x \sim y$. Throughout this chapter, we let J be a matrix indexed by $\mathcal{E}$ with nonnegative entries. Thus,

$$J_{xy} \geq 0 \tag{1.3.4}$$

with equality if and only if $x \not\sim y$. The pair $(\mathcal{G}, J)$ is an example of a *weighted* graph. We will usually denote this weighted graph simply as $\mathcal{G}$, with J implicit.

Let D be the diagonal $\mathcal{V} \times \mathcal{V}$ matrix with diagonal entries

$$d_x = D_{xx} = \sum_{y \sim x} J_{xy}, \tag{1.3.5}$$

where the sum is over all vertices y adjacent to x. The graph *Laplacian* on $\mathcal{G}$ is defined by

$$-\Delta = D - J. \tag{1.3.6}$$

An important case is when $J_{xy} = \mathbb{1}_{x \sim y}$, for which

$$-\Delta_{xy} = d_x \mathbb{1}_{x=y} - \mathbb{1}_{x \sim y}. \tag{1.3.7}$$

We also define the *massive Laplacian* with squared *mass* $m^2 > 0$ by

$$-\Delta + m^2. \tag{1.3.8}$$

Note that

$$\varphi \cdot (-\Delta\varphi) = \frac{1}{2} \sum_{x,y \in \mathcal{V}} J_{xy} |\varphi_x - \varphi_y|^2 \geq 0, \tag{1.3.9}$$

so $-\Delta$ is positive-semidefinite.

Example 1.3.1. We will often work on the discrete d-dimensional *torus* of side L^N, defined by

$$\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d \tag{1.3.10}$$

for integers $L > 1$ and $N \geq 0$. We view Λ as a graph with $\mathcal{V} = \Lambda$ and $x \sim y$ if x and y have distance 1 on the torus. This graph is transitive and $d_x = 2d$ for all x.

1.3.3 The Green function

If $m^2 > 0$, then $-\Delta + m^2$ is positive-definite, hence invertible with inverse

$$(-\Delta + m^2)^{-1} = (m^2 + D)^{-1} \sum_{n=0}^{\infty} Z^n P^n, \tag{1.3.11}$$

where

$$Z = (m^2 + D)^{-1} D, \quad P = D^{-1} J. \tag{1.3.12}$$

Let z_x denote the diagonal elements of Z. The *Green function* of $\mathcal{G}$ is the kernel of $(-\Delta + m^2)^{-1}$, given by

$$C_{xy} = (m^2 + d_x)^{-1} \sum_{n=0}^{\infty} z_x^n P_{xy}^n. \tag{1.3.13}$$

1.4 Spin systems

We begin by restricting our attention to spin systems in *finite* volume:

$$|\mathcal{V}| < \infty. \tag{1.4.1}$$

Suppose that $S \subset \mathbb{R}^n$ is equipped with a measure $d\lambda^0$, let $\Omega = S^{\mathcal{V}}$, and define the product measure

$$d\lambda(\varphi) = \prod_{x \in \mathcal{V}} d\lambda^0(\varphi_x), \quad \varphi \in \Omega. \tag{1.4.2}$$

We refer to the elements of Ω as *fields* or *spin configurations* on $\mathcal{V}$ with spins in S.

Let $H : \Omega \to \mathbb{R}$ be a function and suppose that e^{-H} is integrable with respect to $d\lambda$. The *spin system* with Hamiltonian $H : \Omega \to \mathbb{R}$ at inverse temperature β is given by the Gibbs measure

$$d\mu_\beta(\varphi) = \frac{1}{Z_\beta} e^{-\beta H(\varphi)} \, d\lambda(\varphi). \tag{1.4.3}$$

We sometimes add an *external field* $h \in \mathbb{R}$ by considering the measure

$$d\mu_{\beta,h}(\varphi) = \frac{1}{Z_{\beta,h}} e^{-\beta\left(H(\varphi) - h\sum_{x \in \mathcal{V}} \varphi_x\right)} \, d\lambda(\varphi). \tag{1.4.4}$$

We will mainly be concerned with *ferromagnetic* spin systems, for which the Hamiltonian is given by

$$H(\varphi) = -\varphi \cdot M\varphi, \quad M_{xy} \geq 0. \tag{1.4.5}$$

The total energy of such a system is a sum of contributions of the form $-2\varphi_x \cdot M_{xy}\varphi_y$ for $x \sim y$. Such a contribution attains its minimum when $\varphi_x = \varphi_y$ and its maximum when $\varphi_x = -\varphi_y$. In this sense, it is energetically favourable for spins to *align* in ferromagnetic systems.

1.4.1 Examples of spin systems

Below we discuss some common examples of spin systems. In many cases, we discuss Hamiltonians that depend on one or more parameters for which adjusting the inverse temperature β is equivalent to rescaling these parameters. In these cases, we will fix

$$\beta = 1 \tag{1.4.6}$$

without loss of generality.

Gaussian measures

Suppose $S = \mathbb{R}^n$ and λ^0 is Lebesgue measure. Then the Hamiltonian H must be bounded below for the Gibbs measure (1.4.3) to be well-defined. Essentially the simplest class of non-constant[2] Hamiltonians are positive-definite quadratic forms. These are given by a positive-definite symmetric $\mathcal{V} \times \mathcal{V}$ matrix C, called the *covariance* matrix, and take the form

$$H_C(\varphi) = \frac{1}{2}\varphi \cdot C^{-1}\varphi. \tag{1.4.7}$$

The corresponding Gibbs measures are *Gaussian* measures.

The partition function can be computed explicitly and the Gibbs measure takes the form

$$\frac{d\varphi}{\sqrt{\det(2\pi C)}} e^{-\frac{1}{2}\varphi \cdot A\varphi}. \tag{1.4.8}$$

Wick's theorem gives an expression for the moments: if $x_1, \ldots, x_{2p} \in \Lambda$, then

$$\int d\mu_C(\varphi) \prod_{k=1}^{2p} \varphi^i_{x_k} = \sum_{\pi} \prod_{kl \in \pi} C_{x_k x_l} \tag{1.4.9}$$

where the sum is over all pairings π of $\{1, \ldots, 2p\}$ (i.e. partitions of this set into 2-element subsets).

The $|\varphi|^4$ model

As in the previous example, let $S = \mathbb{R}^n$ and λ^0 be Lebesgue measure. The next step up in complexity is a quartic Hamiltonian. In particular, we have the Hamiltonian for the *lattice* $|\varphi|^4$ *model* or *Ginzburg-Landau-Wilson model*:

$$H_{g,\nu}(\varphi) = \sum_{x \in \mathcal{V}} \left(\frac{1}{4} g |\varphi_x|^4 + \frac{1}{2}\nu |\varphi_x|^2 + \frac{1}{2}\varphi_x \cdot (-\Delta\varphi)_x \right), \tag{1.4.10}$$

where $g > 0$ and $\nu \in \mathbb{R}$. The expression (1.4.10) should be compared with (1.2.3). With $p = m\dot{q}$ in the latter expression, the kinetic energy takes the form $\frac{1}{2}m|\dot{q}|^2$. The lattice analogue of this quantity in (1.4.10) is

$$\frac{1}{2}\sum_{x \in \mathcal{V}} \varphi_x \cdot (-\Delta\varphi)_x = \frac{1}{4}\sum_{x \in \mathcal{V}} \sum_{y \sim x} (\varphi_y - \varphi_x)^2, \tag{1.4.11}$$

where we have applied (1.3.9) to get the right-hand side.

[2]The Gibbs measure with constant Hamiltonian is just the product measure.

Example 1.4.1 (The GFF)**.** With $g = 0$ and $\nu > 0$,

$$H_{0,\nu}(\varphi) = H_C(\varphi), \qquad C = (-\Delta + \nu)^{-1}. \tag{1.4.12}$$

In other words, the $|\varphi|^4$ model becomes the Gaussian measure with covariance given by the massive Green function. The corresponding spin system is the discrete massive *Gaussian free field* or *GFF*.

The continuum analog of this model with mass 0 (which can be defined on $\mathbb{R}^d$ with $d > 2$) is a simple example of a non-interacting[3] quantum field theory. The $|\varphi|^4$ model was introduced in attempts to construct an *interacting* theory.

The $O(n)$ spin model

Consider the $|\varphi|^4$ model and suppose that $\sum_{y\sim x} J_{xy} = d_0$ for all $x \in \mathcal{V}$ (e.g. this occurs in the setting of (1.3.7) when every vertex in $\mathcal{G}$ has d_0 neighbours). In this case, we can write

$$H_{g,\nu}(\varphi) = \sum_{x\in\mathcal{V}} U_{g,\nu}(\varphi_x) - \frac{1}{2}\varphi \cdot J\varphi, \tag{1.4.13}$$

where the *single-spin potential* $U_{g,\nu}$ is given by

$$U_{g,\nu}(t) = \frac{1}{4}g|t|^4 + \frac{1}{2}(\nu + d_0)|t|^2, \quad t \in \mathbb{R}^n. \tag{1.4.14}$$

We can see from (1.4.13) that the $|\varphi|^4$ model is ferromagnetic. When $\nu + d_0 < 0$, the potential has the shape of a double well with roots at $t = 0$ and on the sphere $|t| = \pm\sqrt{-2(\nu + d_0)/g}$. Thus, as $g \to \infty$ with $\nu = -(d_0 + g/2)$, the Gibbs measure for the $|\varphi|^4$ model converges (weakly) to the Gibbs measure with Hamiltonian

$$H_J(\varphi) = -\frac{1}{2}\varphi \cdot J\varphi, \quad \varphi \in (S^{n-1})^{\mathcal{V}} \tag{1.4.15}$$

where $S^{n-1} \subset \mathbb{R}^n$ is the $(n-1)$-dimensional unit sphere. This is known as the $O(n)$ *spin model*. The special case $n = 1$ is the famous *Ising model*. The cases $n = 2$ and $n = 3$ are known as the *XY model* and the *classical Heisenberg model*.

Remark 1.4.2. The Ising model on $\mathbb{Z}^d$ (defined by a limiting procedure) was introduced by Willhelm Lenz [87]. Lenz suggested it as a problem for his student Ernst Ising, who determined in 1924 [75] that there is no (non-trivial) phase transition when $d = 1$. In [70] Heisenberg, seeking a model that would possess a phase transition, proposed the quantum version of his model (see [16] for a discussion on this). Ironically, it turns out that the classical version of his model does not possess a phase transition in both $d = 1$ and $d = 2$ [92]; and, moreover, in 1936 Rudolf Peierls [100] put forth an argument for the existence of a phase transition in

[3] The corresponding classical model only involves the kinetic term (1.4.11).

the Ising model if $d > 1$. Peierls' argument was made rigorous in [62], but even before that Onsager [96] provided an exact computation of the free energy in $d = 2$, which incontrovertibly demonstrated the existence of a phase transition in two dimensions. The general $O(n)$ model was first studied in [109].

1.4.2 Phase transition in the Ising model

Let $\mathcal{G} = \Lambda_N$ and consider the Hamiltonian for the Ising model with interaction $J_{xy} = \mathbb{1}_{x\sim y}$ in an external field $h \in \mathbb{R}$:

$$H_h^{(N)}(\varphi) = -\frac{1}{2}\sum_{x\sim y}\varphi_x\varphi_y - h\sum_{x\in\mathcal{V}}\varphi_x, \qquad \varphi \in \{\pm 1\}^{\mathcal{V}}. \tag{1.4.16}$$

Let $\langle\cdot\rangle_{\beta,h}^{(N)}$ denote the expectation with respect to the corresponding Gibbs measure $\mu_{\beta,h}^{(N)}$ and let $Z_{\beta,h}^{(N)}$ be the partition function.

The *free energy* is defined (in finite volume) by

$$F_{\beta,h}^{(N)} = -\frac{1}{\beta|\mathcal{V}|}\log Z_{\beta,h}^{(N)}. \tag{1.4.17}$$

The *magnetization* is given by

$$M_{\beta,h}^{(N)} = \frac{1}{|\mathcal{V}|}\sum_{x\in\mathcal{V}}\langle\varphi_x\rangle_{\beta,h}^{(N)} \tag{1.4.18}$$

and can be written in terms of the free energy as

$$M_{\beta,h}^{(N)} = -\frac{\partial}{\partial h}F_{\beta,h}^{(N)}. \tag{1.4.19}$$

When $h = 0$, the Gibbs measure is invariant under the spin-flip transformation $\varphi \mapsto -\varphi$ and it follows that $M_{\beta,0}^{(N)} = 0$ for all β.

In order to study the dependence of the magnetization on h, we define the *magnetic susceptibility*

$$\chi^{(N)}(\beta,h) = \frac{1}{\beta}\frac{\partial}{\partial h}M_{\beta,h}^{(N)}. \tag{1.4.20}$$

By translation-invariance,

$$\chi^{(N)}(\beta,h) = \frac{1}{\beta^2}\sum_{x\in\mathcal{V}}G_x^{(N)}(\beta,h), \tag{1.4.21}$$

where

$$G_x^{(N)}(\beta,h) = \langle\varphi_0\varphi_x\rangle_{\beta,h} - \langle\varphi_0\rangle_{\beta,h}\langle\varphi_x\rangle_{\beta,h} \tag{1.4.22}$$

is the correlation between φ_0 and φ_x, known as the *two-point function*. These are all analytic functions. In order to detect a phase transition, we must take the infinite-volume limit $N \to \infty$.

The infinite-volume free energy, magnetization, susceptibility, and two-point function are defined as the $N \to \infty$ limits of their finite-volume counterparts; we denote them by $F_{\beta,h}$, $M_{\beta,h}$,

$\chi(\beta, h)$, and $G_x(\beta, h)$. Let

$$M_\beta^\pm = \lim_{h \to 0^\pm} M_{\beta,h}. \tag{1.4.23}$$

Then when β is sufficiently large [62,100], there is a non-zero *spontaneous magnetization*, meaning that

$$M_\beta^- < 0 < M_\beta^+. \tag{1.4.24}$$

That is, the magnetization is *discontinuous* at $h = 0$. Equivalently, the free energy is not differentiable[4] at that point. The *critical point* for the Ising model is the inverse temperature

$$\beta_c = \sup\{\beta > 0 : M_\beta^+ = 0\}. \tag{1.4.25}$$

At the critical point, the magnetization is continuous [4,118], i.e. the free energy is differentiable. However, the susceptibility diverges (see [3]), so the free energy is not twice-differentiable. This divergence corresponds to slow decay of the two-point function, which will be discussed further in Section 1.5.

1.4.3 Infinite-volume spin systems

A broad distinction can be made between *first-order* phase transitions in which the free energy has discontinuous first derivative with respect to an external field h and *continuous* phase transitions, in which the free energy is differentiable but non-analytic. As discussed above, the infinite-volume limit must be taken in order for a phase transition to manifest. For the Ising model, we took the infinite-volume limit of the free energy along a sequence of increasing tori.

It is worth mentioning that a more general and elegant approach to the study of infinite-volume spin systems was developed by Dobrushin [41] and Lanford and Ruelle [82]. An excellent introduction to this subject is given in [54] and a comprehensive reference is [59] (see also [86] for spin systems with unbounded 1-component spins).

Loosely speaking, this approach take as fundamental not the Hamiltonian but rather a "potential", which is a collection of functions encoding the microscopic interactions from which the Hamiltonian is to be defined; for instance, for the Ising model, the Hamiltonian is a sum of contributions of the form $J_{xy}\sigma_x\sigma_y$ for $x \sim y$. Given such a potential, a Hamiltonian can be defined on any finite subgraph of $\mathcal{G}$ and a probability measure on Ω is said to be a Gibbs measure or Gibbs state whenever its finite-volume conditional measures are of the form (1.4.3).

This is somewhat in the spirit of Kolmogorov's consistency conditions with the important difference that the resulting collection $\mathscr{G}_\beta$ of Gibbs states at inverse temperature β need not consist of only a single element. This is significant due to the interpretation of distinct elements of $\mathscr{G}_\beta$ as corresponding to different phases. For many systems of interest there is a critical inverse temperature β_c such that $|\mathscr{G}_\beta| > 1$ if and only if $\beta > \beta_c$. The region $\beta > \beta_c$ is typically associated with first-order phase transitions whereas continuous phase transitions usually occur

[4] There is an interchange of limit and derivative implicit in this statement, but it can be justified.

at the critical point β_c.

In this thesis, we are mainly interested in the single-phase regime $\beta \leq \beta_c$. Thus, we will avoid the issue of existence and uniqueness of infinite-volume Gibbs measures by defining observable quantities of interest in infinite volume as limits of their finite-volume counterparts as was done in the previous section. Below we give the definitions we will use throughout the rest of this thesis for (a generalization of) the $|\varphi|^4$ model.

Generalized $|\varphi|^4$ model

We view Λ_N as a subset of $\mathbb{Z}^d$ approximately centred at the origin (say as $[-\frac{1}{2}L^N+1, \frac{1}{2}L^N]^d \cap \mathbb{Z}^d$ if L^N is even and as $[-\frac{1}{2}(L^N-1), \frac{1}{2}(L^N-1)]^d \cap \mathbb{Z}^d$ if L^N is odd). This allows us to preserve translation-invariance of the models that concern us when defining them in finite volume.

We fix $J_{xy} = \mathbb{1}_{x\sim y}$ so that

$$-\Delta_{xy} = 2d\mathbb{1}_{x=y} - \mathbb{1}_{x\sim y}. \tag{1.4.26}$$

We will study a generalization of the $|\varphi|^4$ model whose Hamiltonian on Λ_N is given by

$$V_{g,\gamma,\nu,N}(\varphi) = \sum_{x\in\Lambda_N} \left(\frac{1}{4}(g-\gamma)|\varphi_x|^4 + \frac{1}{2}\nu|\varphi_x|^2 + \frac{1}{2}\varphi_x \cdot (-\Delta\varphi)_x + \frac{1}{4d}\gamma(\nabla|\varphi_x|^2)^2 \right), \tag{1.4.27}$$

where

$$(\nabla|\varphi_x|^2)^2 = \sum_{|e|=1} (\nabla^e|\varphi_x|^2)^2 \tag{1.4.28}$$

and the discrete gradient in the direction of a unit vector $e \in \mathbb{Z}^d$ is defined by

$$\nabla^e f_x = f_{x+e} - f_x. \tag{1.4.29}$$

The expectation with respect to the corresponding Gibbs measure will be denoted $\langle\cdot\rangle_{g,\gamma,\nu,N}$.

Following (1.4.21) and (1.4.22), we define the two-point function and susceptibility by

$$G_{x,N}(g,\gamma,\nu;n) = \frac{1}{n}\langle\varphi_0 \cdot \varphi_x\rangle_{g,\gamma,\nu,N}, \quad G_x(g,\gamma,\nu;n) = \lim_{N\to\infty} G_{x,N}(g,\gamma,\nu;n) \tag{1.4.30}$$

and

$$\chi(g,\gamma,\nu;n) = \lim_{N\to\infty} \sum_{x\in\Lambda_N} G_{x,N}(g,\gamma,\nu;n). \tag{1.4.31}$$

Existence of these limits (which is not known in general) will follow from the proof of our main result.

Remark 1.4.3. We have omitted the term $\langle\varphi_0\rangle_{g,\gamma,\nu,N} \cdot \langle\varphi_x\rangle_{g,\gamma,\nu,N}$ which vanishes in the regime of interest $\beta \leq \beta_c$.

1.5 Critical behaviour and universality

Many systems exhibit singular behaviour at or near the critical temperature in the form of power law scaling of various observable quantities. This is known as *critical behaviour*. For concreteness, we will discuss the generalized $|\varphi|^4$ model on $\mathbb{Z}^d$ given by (1.4.27) with $J_{xy} = \mathbb{1}_{x\sim y}$.

1.5.1 The critical point

We follow the convention (1.4.6) of setting $\beta = 1$ and seek a phase transition for $|\varphi|^4$ model when the parameter ν is varied. We define the *critical point* by

$$\nu_c = \nu_c(g, \gamma; n) = \inf\{\nu : \chi(g, \gamma, \nu; n) < \infty\}. \tag{1.5.1}$$

By (1.4.31), it is reasonable to expect rapid (i.e. summable) decay of $G_x(g, \gamma, \nu; n)$ in $|x|$ for $\nu > \nu_c$ and much slower decay at $\nu = \nu_c$.

In fact, the two-point function is expected to decay exponentially above ν_c. The *correlation length* ξ is defined to be the reciprocal of the exponential rate of decay of the two-point function; concretely, we let

$$\xi(g, \gamma, \nu; n) = \limsup_{k\to\infty} \frac{-k}{\log G_{ke}(g, \gamma, \nu; n)}, \tag{1.5.2}$$

where $e \in \mathbb{Z}^d$ is a unit vector whose choice is irrelevant by invariance of this model under lattice rotations. Roughly speaking, the correlation length acts as a "macroscopic length scale" of the model; it is a measure of the largest scale at which spins are very strongly correlated. Based on the above discussion, we expect ξ to diverge as $\nu \downarrow \nu_c$. This divergence is one of the key features of critical behaviour.

A related quantity is the *correlation length of order p*, defined by

$$\xi_p(g, \gamma, \nu; n) = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^p G_x(g, \gamma, \nu; n)}{\chi(g, \gamma, \nu; n)}\right)^{1/p}. \tag{1.5.3}$$

An analysis of ξ_p requires less control over the behaviour of the two-point function than would be needed to study the correlation length ξ. Our main result includes a statement regarding the critical behaviour of ξ_p in $d = 4$.

Remark 1.5.1. There is a simple heuristic relationship between ξ and ξ_p. Suppose that the two-point function decays exponentially at rate $1/\xi$, possibly with some sub-exponential multiplicative correction; for instance, suppose that

$$G_x(g, \gamma, \nu; n) \approx C|x|^{-\alpha} e^{-|x|/\xi(g,\gamma,\nu;n)}, \tag{1.5.4}$$

in some sense, where α and C are positive constants independent of ν. Then the main contributions to the numerator of (1.5.3) should come from $|x| \le \xi = \xi(g, \gamma, \nu; n)$. For such $|x|$,

$G_x(g,\gamma,\nu;n) \approx C|x|^{-\alpha}$ and so

$$\sum_{x\in\mathbb{Z}^d} |x|^p G_x(g,\gamma,\nu;n) \approx C \sum_{|x|\le\xi} |x|^{-(\alpha-p)} \approx C\xi(g,\gamma,\nu;n)^{d-\alpha+p}. \tag{1.5.5}$$

It follows then from the definition that

$$\xi_p^p(g,\gamma,\nu;n) \approx \xi^p(g,\gamma,\nu;n). \tag{1.5.6}$$

1.5.2 Critical behaviour

For simplicity, let us drop g, γ, and n from the notation. It is predicted that there exist constants η, $\bar\gamma$, and $\bar\nu$ (unrelated to γ and ν), known as *critical exponents*, such that

$$G_x(\nu_c) \sim C_1|x|^{-(d-2+\eta)}, \qquad |x|\to\infty \tag{1.5.7}$$

$$\chi(\nu) \sim C_2(\nu-\nu_c)^{-\bar\gamma}, \qquad \nu\downarrow\nu_c \tag{1.5.8}$$

$$\xi(\nu) \sim C_3(\nu-\nu_c)^{-\bar\nu}, \qquad \nu\downarrow\nu_c \tag{1.5.9}$$

$$\xi_p(\nu) \sim C_4(\nu-\nu_c)^{-\bar\nu}, \qquad \nu\downarrow\nu_c \tag{1.5.10}$$

where the C_i are constants that may depend on g, γ, and n (and p when $i=4$). The critical exponents, on the other hand, are expected to be *universal* in the sense that they only depend on "large-scale properties" of the model such as its symmetries and the global geometry of the underlying graph. In particular, for the n-component $|\varphi|^4$ model on $\mathbb{Z}^d$, these exponents should only depend on n and d and be independent of g and γ when $g>0$ and γ is sufficiently small (depending on g). In fact, analogous relations are expected to hold for the $O(n)$ spin model, with the *same* critical exponents.

These and other relations are all believed to be manifestations of the existence of a universal *scaling limit* for the $|\varphi|^4$ model and other models in its *universality class*. That is, any spin system in this class, when appropriately rescaled, is expected to converge in distribution to a unique continuum random field. In this sense, the study of critical behaviour involves far-reaching generalizations of the central limit theorem.

Example 1.5.2 (The Gaussian free field)**.** On $\mathbb{Z}^d$, (1.3.13) and (1.4.9) imply that

$$G_x(0,0,m^2;n) = (m^2+2d)^{-1}\sum_{k=0}^{\infty} z^k P_{0x}^k, \tag{1.5.11}$$

where $z = 2d/(m^2+2d)$ and $P=(2d)^{-1}J$ is the transition matrix for the simple random walk X on $\mathbb{Z}^d$. Thus, for $m^2>0$,

$$\chi(0,0,m^2;n) = (m^2+2d)^{-1}\sum_{k=0}^{\infty} z^k = (m^2+2d)^{-1}(1-z)^{-1} = \frac{1}{m^2}. \tag{1.5.12}$$

It follows that $\nu_c(0,0;n) = 0$ and $\bar{\gamma} = 1$ for this model. It can also be shown that $\bar{\nu} = 1/2$ and $\eta = 0$ in this case (e.g. see [83] for η). In Appendix B, we give a new proof that $\bar{\nu} = 1/2$ for $\xi_p(0,0,m^2;n)$.

1.5.3 Critical exponents

As mentioned above, the critical exponents are generally expected to depend on d and n. Below we discuss some of the conjectured values and known results.

Dimension $d = 1$

In $d = 1$, nearest-neighbour models typically do not possess a phase transition. However, phase transitions may occur for sufficiently long-range models [44].

Dimensions $d = 2$

In $d = 2$, the Mermin-Wagner theorem [92] implies that the $O(n)$ model does not possess a first-order phase transition for $n \geq 2$. However, the case $n = 2$ is expected to possess a *Kosterlitz-Thouless* phase transition [81] (Kosterlitz and Thouless, together with Duncan Haldane, were awarded the 2016 Nobel Prize in Physics for this and related ideas). Rigorous results relating to this kind of phase transition include [40, 46, 47, 56]. For $n = 1$, there is a phase transition; in fact, Onsager [96] gave an exact formula for the free energy of the Ising model.

The 1-component planar models are expected to possess *conformally invariant* scaling limits. As a consequence of these conformal symmetries, the predicted critical exponents are rational numbers: $\bar{\gamma} = 56/32$, $\bar{\nu} = 1$, and $\eta = 1/4$. Recent years have shown rapid progress in this direction, stimulated by the identification by Schramm [104] of a 1-parameter family of conformally invariant random planar curves now known as the *Schramm-Loewner evolution* with parameter κ, or SLE_κ. For instance, it was shown in [34] that the interface curve (between $+1$ and -1 spins) for the Ising model on a bounded simply connected domain with Dobrushin boundary conditions[5] converges to SLE_3 in an appropriate scaling limit.

Dimension $d = 3$

Very little is known rigorously about three-dimensional models. In fact, it was only recently proved that the Ising model's spontaneous magnetization vanishes continuously at the critical point in three dimensions [3]. The critical exponents are not expected to take on rational values. Approximate values for the exponents have been computed by non-rigorous methods in [45, 64, 85].

[5] Positive spins along one side of the boundary and negative spins on the other.

Dimensions $d > 4$

If $d > 4$, the critical exponents for the $O(n)$ and $|\varphi|^4$ models are predicted to become independent of d and n and to take on the values of the corresponding exponents for the Gaussian free field, i.e. $\bar{\gamma} = 1$, $\bar{\nu} = 1/2$, and $\eta = 0$. This phenomenon is known as *mean-field behaviour* and dimension 4 is called the *upper-critical dimension* for this class of models. For $n = 1, 2$ it is known that $\eta = 0$ for the *continuum limit* of these models [2, 55] if it exists. On the lattice, it has been shown that $\eta = 0$ for a spread-out version of the Ising model [102] and for the 1-component $|\varphi|^4$ model with small coupling strength [103]. Extensions to $n = 1, 2$ are upcoming in [23].

Dimension $d = 4$

Dimension 4 is the case of primary interest in quantum field theory, where one dimension plays the role of time. It is predicted in dimension 4 that a number of observables possess multiplicative logarithmic corrections to mean-field scaling. An exception is the two-point function, which is expected to undergo mean-field scaling ($\eta = 0$); this was shown for the 1-component $|\varphi|^4$ model in [57] and [48]. Logarithmic corrections to scaling of the susceptibility and correlation length of the 1-component model were identified in [66, 69].

Recently, Bauerschmidt, Brydges, and Slade [10, 28–31] have developed a renormalization group method for studying the n-component $|\varphi|^4$ model in 4 dimensions; this method works for any n and, in a certain sense, extends to models of self-interacting walks, interpreted as $n = 0$ (more on this in Section 1.8). In particular, they computed logarithmic corrections to scaling of the susceptibility and specific heat and also identified the continuum massive GFF as the scaling limit in the near-critical regime [7]. Using an extension of this method, Slade and Tomberg computed asymptotics for critical correlation functions in [108]; in particular, they showed that $\eta = 0$. In this thesis, we discuss extensions of this method that have been used to study the finite-order correlation length [12] as well as more general models of walks [13].

1.5.4 The renormalization group

In [79], Leo Kadanoff considered a coarse-graining procedure for studying the Ising model in which disjoint blocks in $\mathbb{Z}^d$ of side $L \ll \xi$ are replaced by single spins. He argued that spins inside such blocks are so strongly correlated that the model obtained by making this replacement should behave approximately like an Ising model with a new ("renormalized") interaction.

At the critical point, $\xi = \infty$ and the transformation T can be iterated indefinitely resulting in a dynamical system on a space of models: the one-parameter semigroup $(T^j)_{j \in \mathbb{Z}_+}$ is known as the *renormalization group*[6]. This was the basis for Ken Wilson's generalizations of Kadanoff's idea in [115, 116].

[6]The name renormalization *group* is attributed by Wilson in [115] to the work of Gell-Mann and Low [58]. The relationship between these two approaches is discussed in [76].

The coarse-graining procedure of Kadanoff can be viewed as an approximate method for computing integrals with respect to a Boltzmann weight $e^{-\beta H}$ by successively integrating out fluctuations that are "small" in the sense that they are localized in space. In Wilson's approach fluctuations are instead localized in Fourier space.

Wilson's method results in a dynamical system on a space of models that acts by appropriately integrating out small fluctuations, followed by a rescaling step used to make this system autonomous (i.e. independent of the "scale" j). He argued that the action of this dynamical system would leave the long-range behaviour of the models invariant. Consequently, critical models lying in the same orbit would belong to the same universality class. Thus, such models should possess the same critical exponents and scaling limit. Moreover, this scaling limit should be invariant under the action of the renormalization group, i.e. it should arise as a fixed point of this dynamical system. Therefore, the set of points that flow towards it form its stable manifold.

In addition to these rather broad statements regarding the nature of universality and scaling limits, Wilson demonstrated that critical exponents could be computed by a careful analysis of the asymptotics of the renormalization group near its fixed points. He claimed that such an analysis could be performed by approximating this infinite-dimensional dynamical system by a *finite-dimensional* system spanned by certain *marginal and relevant* directions. The remaining *irrelevant* directions would contract under the action of the renormalization group.

By analyzing this finite-dimensional approximation, Wilson determined that there is a unique hyperbolic fixed point (in the terminology of dynamical systems theory) in dimensions $d > 4$, corresponding to the Gaussian free field and mean-field behaviour. As the dimension is lowered below 4, a bifurcation occurs in which the Gaussian fixed point splits into two fixed points. One of these corresponds to Gaussian behaviour but is unstable. The other is hyperbolic and corresponds to anomalous scaling behaviour; it is sometimes known as the *Wilson-Fisher* fixed point [117]. At the bifurcation point $d = 4$, there is only one (Gaussian) fixed point and logarithmic corrections to mean-field scaling arise from the fact that this fixed point is not hyperbolic.

There are many difficulties in making Wilson's ideas rigorous and several different approaches exist. For instance, a rigorous implementation of Kadanoff's *block-spin* renormalization group was developed in [57] and used to show that $\eta = 0$ if $d = 4$ and $n = 1$. This was also shown independently in [48], using related ideas. By extensions to the block-spin approach, logarithmic corrections to mean-field scaling in this case were identified in [69]. This thesis concerns the renormalization group method of Bauerschmidt, Brydges, and Slade [10, 28–31], which we will discuss further in Chapter 2.

Example 1.5.3. Let us mention how renormalization group ideas can be used to prove a version of the classical central limit theorem. This idea appears in [77, 80] and has been extended to prove stable limit laws in [88].

Let X_n denote a sequence of independent identically distributed continuous random variables

with mean 0 and variance 1. For such X_n, let $Y_n = 2^{-n/2}(X_1 + \cdots + X_{2^n})$. Then the map $Y_n \mapsto Y_{n+1}$ induces a renormalization group map R on density functions given by a convolution and rescaling of the form $(Rf)(x) = \sqrt{2}(f * f)(\sqrt{2}x)$. It is easily verified that the standard Gaussian density f_* is a fixed point of R (in fact, it is the unique fixed point).

In this context, a subsequential version of the central limit theorem follows if we can show that, for all f in an appropriate space of densities, $R^n f$ converges to the fixed point f_*. A local version of this statement can be proved by analyzing the linearized map $DR(f_*)$. A computation shows that this map has eigenfunctions H_i given by Hermite polynomials with eigenvalues $\lambda_i = 2^{1-i/2}$ for integers $i \geq 0$. The relevant directions (in the sense of Wilson's renormalization group) are those in which the linearized map is expanding; thus, the relevant subspace is spanned in this case by H_0 and H_1. Similarly, the irrelevant directions are spanned by H_i with $i > 2$ while H_2 spans the marginal subspace.

1.6 Self-interacting walks

Before introducing the walks studied in this thesis, we mention the following important example of a self-interacting walk.

Example 1.6.1 (Self-avoiding walk)**.** Let $\omega : \{0, \ldots, n\} \to \mathcal{V}$ be a *discrete-time walk* of length n on $\mathcal{G}$, meaning that $\omega_i \sim \omega_{i+1}$ for all i. Let us denote by $\widehat{\mathcal{W}}_n$ the collection of such walks and set $\widehat{\mathcal{W}} = \bigcup_n \widehat{\mathcal{W}}_n$. We say that ω is *self-avoiding* if $\omega_i \neq \omega_j$ for all $i \neq j$. Let $\mathcal{S}_n$ denote the collection of n-step self-avoiding walks with $\omega_0 = 0$. We equip $\mathcal{S}_n$ with the uniform measure μ_n for each n. This gives us a simple model of a linear polymer in a good solution. The self-avoidance constraint models the excluded volume effect of matter.

The uniform measures do not form a consistent family due to the possibility of "traps". That is, the equality

$$\mu_{|\omega|}(\omega) = \sum_{\bar{\omega} \supset \omega} \mu_{|\bar{\omega}|}(\bar{\omega}) \tag{1.6.1}$$

does not hold for all $\omega \in \widehat{\mathcal{W}}$ (the sum here is over all self-avoiding walks extending ω). For instance, consider the self-avoiding walk $\omega \in \widehat{\mathcal{W}}_7$ on $\mathbb{Z}^d$ in Figure 1.2. This walk has positive probability under μ_7 but, since there are no self-avoiding walks extending ω, the sum on the right-hand side of (1.6.1) is 0.

As a result, there is no straightforward way to apply the usual methods of stochastic processes to study the self-avoiding walk. The existence of traps also contributes to the combinatorial difficulty of this model; for instance, if $c_n = |\mathcal{S}_n|$, then it is not clear how to express c_{n+1} as a simple function of c_n.

We do know, however, that the sequence c_n is sub-multiplicative: $c_{m+n} \leq c_m c_n$. This follows from the fact that a self-avoiding walk can be split at any point into two self-avoiding walks, but the concatenation of two self-avoiding walks is not necessarily self-avoiding. Thus, $\log c_n$ is

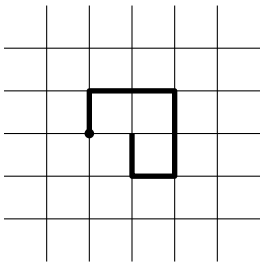

Figure 1.2: A trapped self-avoiding walk

subadditive and Fekete's lemma for subadditive sequences implies the existence of the limit

$$c(\mathcal{G}) = \lim_{n\to\infty} n^{-1} \log c_n. \tag{1.6.2}$$

We call $c(\mathcal{G})$ the *connective constant* of $\mathcal{G}$. By definition, $c(\mathcal{G})$ is the reciprocal of the radius of convergence of the susceptibility, which is defined as the generating function

$$\chi(z) = \sum_{n=0}^{\infty} c_n z^n \tag{1.6.3}$$

of the sequence c_n. Recalling Section 1.1, it is natural to study the asymptotics of the susceptibility as $z \to c(\mathcal{G})^{-1}$.

Due to some of the difficulties involved in studying the self-avoiding walk, alternatives to this model have been proposed. In fact, models of discrete-time walks can be defined in quite a bit of generality in terms of measures on $\widehat{\mathcal{W}}_n$. It is sometimes useful to work instead with models of continuous-time walks parameterized by intervals $[0, T]$. In both cases, we can conveniently define Gibbs measures directly in infinite-volume with respect to the base measure induced by simple random walk. However, instead of discussing models of walks in great generality, we will proceed directly to the case of interest in this thesis.

1.6.1 Weakly self-avoiding walk with self-attraction

Let X denote the continuous-time simple random walk on $\mathcal{G}$ conditioned to start at 0. This is the $\mathcal{V}$-valued Markov process X with generator Δ. In other words,

$$\mathbb{P}(X_t = y \mid X_0 = x) = (e^{t\Delta})_{xy}. \tag{1.6.4}$$

Define the *local time* up to time T at $x \in \mathcal{V}$ by

$$L_T^x = \int_0^T \mathbb{1}_{X(S)=x}\, dS. \tag{1.6.5}$$

We define the *intersection local time*

$$I_T = \int_0^T \int_0^T \mathbb{1}_{X(S_1)=X(S_2)}\, dS_1 dS_2 = \sum_{x\in\mathcal{V}} (L_T^x)^2 \tag{1.6.6}$$

and the *contact self-attraction*

$$C_T = \int_0^T \int_0^T \mathbb{1}_{X(S_1)\sim X(S_2)}\, dS_1 dS_2 = \sum_{x\in\mathcal{V}} \sum_{y\sim x} L_T^x L_T^y \tag{1.6.7}$$

up to time T.

Given a parameter $g > 0$, and $\gamma \in \mathbb{R}$, let

$$U_{g,\gamma}(f) = g \sum_{x\in\mathcal{V}} f_x^2 - \frac{\gamma}{2d} \sum_{x\in\mathcal{V}} \sum_{y\sim x} f_x f_y \tag{1.6.8}$$

for $f : \mathcal{V} \to \mathbb{R}$. The *weakly self-avoiding walk with self-attraction* (WSAW-SA) is defined by the Hamiltonian

$$U_{g,\gamma,T} = U_{g,\gamma} \circ L_T = gI_T - \frac{\gamma}{2d} C_T \tag{1.6.9}$$

which induces a Gibbs measure with respect to the measure of X. We will refer to the case $\gamma = 0$ as the *weakly self-avoiding walk* (WSAW).

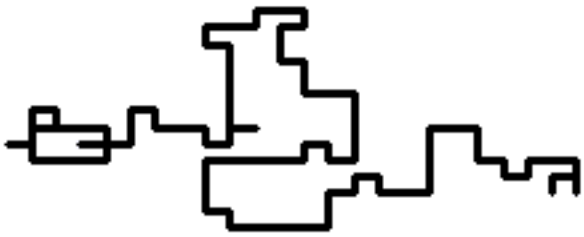

Figure 1.3: Monte Carlo simulation of discrete-time WSAW with 100 steps

We let

$$c_T = E_0\left(e^{-U_{g,\gamma,T}}\right), \quad c_T(x) = E_0\left(e^{-U_{g,\gamma,T}} \mathbb{1}_{X_T=x}\right). \tag{1.6.10}$$

The *two-point function* and *susceptibility* are defined as the Laplace transforms of these weights:

$$G_x(g,\gamma,\nu) = \int_0^\infty c_T(x) e^{-\nu T}\, dT \tag{1.6.11}$$

and

$$\chi(g,\gamma,\nu) = \int_0^\infty c_T e^{-\nu T}\, dT = \sum_{x\in\mathbb{Z}^d} G_x(g,\gamma,\nu). \tag{1.6.12}$$

Note that (1.6.12) is more-or-less consistent with (1.4.31) We will establish an exact analogue of (1.4.31) in Proposition 1.8.4. The relationship between the two-point function for walks and spin systems will be discussed in Section 1.8.2.

We also have a version of the *correlation length of order p*:

$$\xi_p(g,\gamma,\nu) = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^p G_x(g,\gamma,\nu)}{\chi(g,\gamma,\nu)}\right)^{1/p}. \tag{1.6.13}$$

Note that

$$\xi_p^p(g,\gamma,\nu) = \frac{\int_0^\infty \langle |X(T)|^p\rangle c_T e^{-\nu T}\, dT}{\int_0^\infty c_T e^{-\nu T}\, dT}, \tag{1.6.14}$$

where

$$\langle F(X)\rangle = \frac{1}{c_T} E_0(F(X) e^{-U_{g,\gamma,T}}) \tag{1.6.15}$$

is the expectation induced by the weights (1.6.10). Thus, ξ_p is related to the Laplace transform of the mean p-th displacement $\langle |X(T)|^p\rangle$. On $\mathbb{Z}^d$, a version of the correlation length can also be defined exactly as in (1.5.2).

Remark 1.6.2. The discrete-time version of the WSAW-SA is straightforward to define in terms of discrete-time simple random walk; when $\gamma = 0$, it is known as the *Domb-Joyce model* or *discrete-time weakly self-avoiding walk*. A sample of the Domb-Joyce model with 100 steps is shown in Figure 1.3. The SAW can be recovered as an appropriate limit of the Domb-Joyce model or the continuous-time WSAW [18].

Alternative representation

For $f : \mathbb{Z}^d \to \mathbb{R}$, let

$$|\nabla f_x|^2 = \sum_{|e|=1} |\nabla^e f_x|^2, \qquad |\nabla f|^2 = \sum_{x\in\mathbb{Z}^d} |\nabla f_x|^2. \tag{1.6.16}$$

Then by (1.3.9)

$$\sum_{x\in\mathbb{Z}^d} f_x \Delta f_x = -\frac{1}{2}|\nabla f|^2. \tag{1.6.17}$$

It follows that

$$\sum_{x\in\mathbb{Z}^d}\sum_{e\in\mathcal{U}} f_x f_{x+e} = 2d\sum_{x\in\mathbb{Z}^d} f_x^2 + \sum_{x\in\mathbb{Z}^d} f_x\Delta f_x = 2d\sum_{x\in\mathbb{Z}^d} f_x^2 - \frac{1}{2}\sum_{x\in\mathbb{Z}^d} |\nabla f_x|^2 \tag{1.6.18}$$

and so we get the useful representation:

$$U_{g,\gamma}(f) = (g-\gamma)\sum_{x\in\mathbb{Z}^d} f_x^2 + \frac{\gamma}{4d}\sum_{x\in\mathbb{Z}^d}\sum_{e\in\mathcal{U}} |\nabla^e f_x|^2. \tag{1.6.19}$$

In particular,

$$U_{g,\gamma,T} = (g-\gamma)I_T + \frac{\gamma}{4d}|\nabla L_T|^2. \tag{1.6.20}$$

1.6.2 Predicted behaviour

We can view the susceptibility (1.6.12) as the partition function for a measure on walks of *any* length (sometimes called a *grand ensemble*). When ν reaches the critical point

$$\nu_c = \nu_c(g,\gamma) = \inf\{\nu : \chi(g,\gamma,\nu) < \infty\}, \tag{1.6.21}$$

we expect the susceptibility to diverge. The susceptibility is a partition function for walks with ν playing the role of an external field, so this divergence would be indicative of a phase transition as discussed in Section 1.4.3. In a certain sense [42, 61], paths in the $\nu > \nu_c$ phase should scale as geodesics while paths for $\nu < \nu_c$ should be space-filling.

In fact, it is not clear how to show that $\chi(g,\gamma,\nu_c) = \infty$ in general, although this can be established for $\gamma = 0$ (see [9, Lemma A.1]). For γ sufficiently small, this divergence will be part of our main result.

The two-point function, susceptibility, and correlations lengths of the self-avoiding walk and (discrete- or continuous-time) WSAW-SA (with γ small depending on g) on $\mathbb{Z}^d$ are all expected to scale according to analogues of (1.5.7)–(1.5.10). Moreover, the discussion in Section 1.1 suggests that

$$c_T \sim C_5 e^{-\nu_c T} T^{\bar{\gamma}-1}, \tag{1.6.22}$$

$$\langle |X_T|^2 \rangle \sim C_6 T^{2\bar{\nu}}. \tag{1.6.23}$$

The critical exponents $\bar{\gamma}, \bar{\nu}, \eta$ are expected to be universal; in particular, they should only depend on d in this context.

Below, we discuss the predicted values of the exponents for γ small before turning our attention to the case of large γ. A more detailed reference is [90]. The values of $\bar{\nu}$ were first predicted[7] by the chemist Paul Flory [53], who later won the 1974 Nobel Prize in Chemistry for his work on polymers.

Dimension $d = 1$

For the SAW, dimension 1 is trivial: the only self-avoiding walks are straight lines. This is not the case for the WSAW, see e.g. the survey [73].

[7] Flory's prediction for $d = 3$ is no longer generally accepted.

Dimension $d = 2$

In $d = 2$, the predicted values of the critical exponents are

$$\bar{\nu} = 3/4, \quad \bar{\gamma} = 43/32, \quad \eta = 5/24. \tag{1.6.24}$$

It was shown in [84] that the scaling limit of SAW, if it exists and is conformally invariant, is given by $\mathrm{SLE}_{8/3}$, which is consistent with the predicted exponents given above.

It is not immediately clear how to make sense of the supercritical regime ($\nu < \nu_c$ for WSAW-SA). However, the authors of [42] considered SAW on a discretized bounded planar domain. They showed that the scaling limit of supercritical SAW conditioned to start and end on the boundary of the domain is space-filling (their results extend to all dimensions $d \geq 2$).

Dimension $d = 3$

In $d = 3$, again very little is known rigorously. Approximate values of the self-avoiding walk critical exponents (assuming their existence) have been obtained by running simulations, see e.g. [35, 36].

Dimension $d > 4$

The upper-critical dimensions for these models is $d = 4$ and the mean-field exponents are the same as for models in the Ising universality class, namely $\bar{\gamma} = 1$, $\bar{\nu} = 1/2$, $\eta = 0$. In other words, self-avoiding walk is expected to scale like simple random walk in dimensions above 4.

Brydges and Spencer introduced the lace expansion in [32] and used it to show that $\bar{\nu} = 1/2$ if $d > 4$ for the discrete-time WSAW. By vastly extending this method, Hara and Slade [67, 68] showed for the SAW that $\bar{\gamma} = 1$, $\bar{\nu} = 1/2$ (for the mean-squared displacement, correlation length, and correlation length of order 2), $\eta = 0$, and the scaling limit is Brownian motion.

Even above the upper-critical dimension, very little is known about WSAW-SA with $\gamma \neq 0$. Exceptions include [65, 111].

Remark 1.6.3. Define the free *bubble diagram*[8] to be the $\ell^2(\mathbb{Z}^d)$ norm of the massive Green function $x \mapsto C_{0x}$. Thus, by (1.3.13), if $z = 2d/(m^2 + 2d)$, then

$$B_{m^2} = \|C\|_{\ell^2(\mathbb{Z}^d)} = (m^2 + 2d)^{-2} \sum_{m,n=0}^{\infty} z^{m+n} \, \mathbb{P}(X_m = Y_n), \tag{1.6.25}$$

where X and Y are independent simple random walks started at 0. In other words, B_{m^2} is proportional to the expected number of intersections between the traces of two such random walks *killed* at rate $1 - z$. The upper-critical dimension can be "guessed" as follows. First, we

[8] This is sometimes represented by a graph consisting of two edges (forming a "bubble") joining a vertex labeled 0 (denoting the origin) to an unlabeled vertex (denoting an arbitrary point x that is summed over $\mathbb{Z}^d$).

make the convenient definitions

$$\mathsf{B}_{m^2} = (n+8)B_{m^2}, \quad \mathsf{b} = \frac{n+8}{16\pi^2}. \tag{1.6.26}$$

Then it is an exercise in Fourier analysis to show that in the limit $m^2 \downarrow 0$ (equivalently, $z \uparrow 1$)

$$\mathsf{B}_{m^2} \sim \begin{cases} \mathsf{b} \log m^{-2}, & d = 4 \\ \mathsf{B}_0, & d > 4, \end{cases} \tag{1.6.27}$$

which suggests the value $d_c = 4$ of the upper-critical dimension.

Dimension $d = 4$

Weakly self-avoiding walk on a hierarchical lattice was studied by a renormalization group method in [20,24,25,61]. As discussed in Section 1.5.4, Bauerschmidt, Brydges, and Slade have recently made great strides in the case $d = 4$ on the Euclidean lattice using a new renormalization group method. This method was first applied to walks in [8,9], where the susceptibility and two-point function were studied.

Phase diagram

Let $d \geq 2$. The predicted behaviour of self-avoiding walk with attraction is discussed in [74, 112]. The predicted phase diagram and critical exponents are shown in Figure 1.4. Generally speaking, one might expect the self-attraction to dominate when $\gamma > g$ so that the walk typically remains in a bounded region, i.e. $\bar{\nu} = 0$. A proof of this fact appears in [71] for a related model in $d = 1$. The authors of [71] also conjecture that $\bar{\nu} = 1/(d+1)$ for $g = \gamma$ and this is established for $d = 1$ in [72].

While it is natural to expect the self-avoidance to dominate when $\gamma < g$, a *collapse transition* is believed to occur as γ crosses the θ-curve $g \mapsto \gamma_\theta(g)$. The value of the critical exponent $\bar{\nu}$ for $\gamma = \gamma_\theta$ is predicted to be given by $\bar{\nu}_\theta = 4/7$ if $d = 2$ and $\bar{\nu}_\theta = 1/2$ if $d \geq 3$ (with a logarithmic correction in $d = 3$). However, very little is known rigorously about the θ-curve. Only recently has a collapse transition been shown to exist for a model of *prudent* self-avoiding walk with self-attraction [101].

Remark 1.6.4. De Gennes [38] related the behaviour of polymers in poor solvents to the tricritical behaviour of certain spin systems. Such systems typically possess two phase transitions: one corresponding to each critical point on a line of critical points and one corresponding to a *tricritical* point given by an endpoint of this line. For the WSAW-SA, the tricritical point should be given by $(g, \gamma_\theta, \nu_c)$. Moreover, the upper-critical dimension for such tricritical behaviour is $d = 3$, consistent with the predicted values of $\bar{\nu}_\theta$.

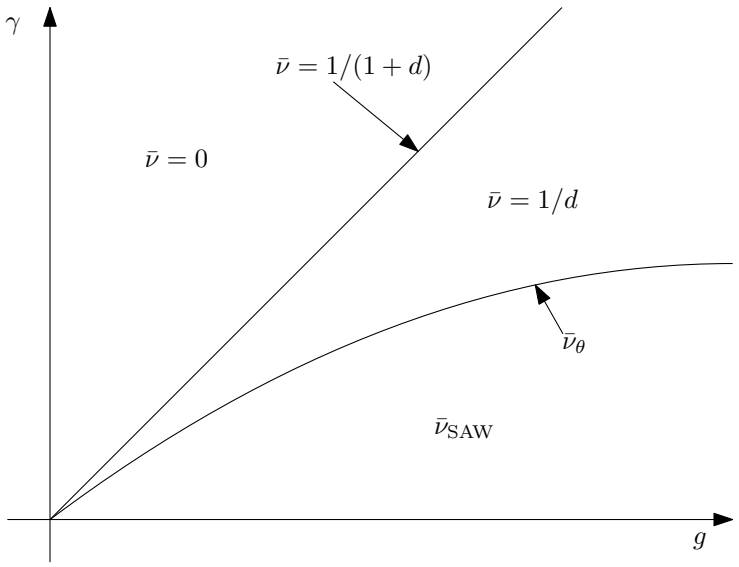

Figure 1.4: Phase diagram for the WSAW-SA

1.7 Main result

For any integer $n \geq 1$, let $G_x(g, \gamma, \nu; n)$ denote the two-point point function for the version of the $|\varphi|^4$ model defined by (1.4.30). We let $G_x(g, \gamma, \nu; 0)$ denote the two-point function of the WSAW-SA, defined in (1.6.11); this notation will be explained in Section 1.8. We employ similar conventions for the susceptibility, correlation length of order p, and critical point, which we denote by $\chi(g, \gamma, \nu; n)$, $\xi_p(g, \gamma, \nu; n)$, $\nu_c(g, \gamma; n)$, respectively, with $n \geq 0$ an integer. When $n = 0$, these correspond to the WSAW-SA, whereas for $n \geq 1$ they correspond to the $|\varphi|^4$ model. The following theorem is the main result of this thesis.

Theorem 1.7.1. *Let $d = 4$ and $n \geq 0$. For L sufficiently large (depending on n), there exists $g_* > 0$ and a positive function $\gamma_* : (0, g_*) \to \mathbb{R}$ such that whenever $0 < g < g_*$ and $|\gamma| < \gamma_*(g)$, there are constants $A_{g,\gamma,n}$ and $B_{g,\gamma,n}$ such that the following hold:*

(i) The critical two-point function decays as

$$G_x(g, \gamma, \nu_c; n) = A_{g,\gamma,n}|x|^{-2}\big(1 + O((\log|x|)^{-1})\big) \quad \text{as } |x| \to \infty, \tag{1.7.1}$$

with $A_{g,\gamma,n} = (4\pi)^{-2}(1 + O(g))$ as $g \downarrow 0$.

(ii) The susceptibility diverges as

$$\chi(g, \gamma, \nu_c + \varepsilon; n) \sim B_{g,\gamma,n}\varepsilon^{-1}(\log \varepsilon^{-1})^{(n+2)/(n+8)}, \quad \varepsilon \downarrow 0 \tag{1.7.2}$$

with $B_{g,\gamma,n} = ((n+8)g/16\pi^2)^{(n+2)/(n+8)}(1 + O(g))$ as $g \downarrow 0$.

(iii) For any $p > 0$, if L is chosen large and g_* small (both depending on p), then the correlation

length of order p diverges as

$$\xi_p(g,\gamma,\nu_c+\varepsilon;n) \sim B_{g,\gamma,n}^{1/2}\mathsf{c}_p\varepsilon^{-1/2}(\log\varepsilon^{-1})^{(n+2)/2(n+8)}, \quad \varepsilon\downarrow 0 \tag{1.7.3}$$

with

$$\mathsf{c}_p^p = \int_{\mathbb{R}^4} |x|^p(-\Delta_{\mathbb{R}^4}+1)_{0x}^{-1}\,dx. \tag{1.7.4}$$

The $\gamma = 0$ cases of (i) and (ii) were proved by Bauerschmidt, Brydges, and Slade in [8,9]; in fact, the $n = 1$ case of their results was first obtained in [48,57,66,69]. The $n > 0$ case with $\gamma \neq 0$ is a new result in this thesis. We will only discuss the proof of the $\gamma \geq 0$ case, which is of primary interest. The proof of the $\gamma < 0$ case with $n = 0$ can be found in [13] and the extension to $n \geq 1$ is straightforward.

Remark 1.7.2.

1. The behaviour (1.7.3) is consistent with predictions for the correlation length ξ and should be understood as a step towards a rigorous understanding of ξ in four dimensions (even in the case $\gamma = 0$).

2. The statement of Theorem 1.7.1 does not provide a quantitative upper bound on γ. However, it should be possible to extend this result to all $|\gamma| \leq Cg^3$ for some constant C. We did not pursue this extension here as we expect that the results of Theorem 1.7.1 may well hold for γ larger than $O(g^3)$. Indeed, these results should hold for all γ below the θ-curve and we know of no particular reason to expect the θ-curve to scale like g^3.

3. We expect that the results of [108] on higher-order correlation functions can be extended to the γ-dependent case considered here by the methods used to prove Theorem 1.7.1. For instance, it should be possible to show that

$$\frac{1}{n}\langle|\varphi_0|^2;|\varphi_x|^2\rangle_{g,\gamma,\nu_c} \sim C_{g,\gamma,n}|x|^{-4}(\log|x|)^{-2\left(\frac{n+2}{n+8}\right)} \tag{1.7.5}$$

1.8 Relations between models

One way to understand universality is via representation theorems that relate different models. For instance, the *Kac-Siegert transformation* can be used to write the partition function of the $O(n)$ model as a partition function for a perturbation of the $|\varphi|^4$ model (we will discuss this further in Section 6.1.2). In the other direction, the Simon-Griffiths construction [106] can be used to approximate the 1-component $|\varphi|^4$ model as a suitable limit of Ising models. Such theorems do not necessarily imply universality (in the sense that models related in this way have the same critical exponents or scaling limit), but tend to be suggestive of it and may in some cases be used as the basis for the proof of a universality-type result.

We have already noted in Example 1.5.2 the close relationship between the simple random walk and the Gaussian free field, which ultimately stems from the representation of matrix

powers in terms of walks and which is familiar to anyone who has studied Markov chains. Namely, if M is a $\mathcal{V} \times \mathcal{V}$ matrix, then

$$M_{ab}^n = \sum_{x_1,\dots,x_n \in \mathcal{V}} M_{ax_1} M_{x_1 x_2} \dots M_{x_n b}. \tag{1.8.1}$$

When M is indexed by $\mathcal{E}$, the sum above can be replaced by a sum over n-step walks from a to b on $\mathcal{G}$. When the entries of M are nonnegative, such a sum acquires a probabilistic interpretation as an expectation with respect to the random walk whose steps are weighted by the entries of M.

It was discovered by Symanzik [110] that certain spin systems could be represented as models of interacting walks in a background of interacting loops. Symanzik used this insight to study quantum field theories in terms of walks. Such representations were also studied, e.g. in [21,43]. A comprehensive reference is [49].

In the opposite direction, one can consider studying walks by looking for corresponding spin systems. In [37], de Gennes argued that the self-avoiding walk corresponds to an $n \to 0$ "limit" of the $O(n)$ spin model and used this to predict the values of its critical exponents. Since n is the number of components of the spins, the $O(n)$ model is only well-defined for n a positive integer and it is not clear how to make sense of such a limit.

Parisi and Sourlas [99] and McKane [91] discovered an alternative approach to the predictions of de Gennes. They argued that the weakly self-avoiding walk two-point function could be represented as the two-point function for a version of the $|\varphi|^4$ model, involving boson and fermion fields (we discuss these below). The formal appearance of $n = 0$ quantities was then explained as a consequence of a symmetry between the bosons and fermions known as *supersymmetry*.

In Section 1.8.1, we provide a brief description of the heuristic relation between spin systems and self-avoiding walk. Then in Section 1.8.2, we describe the rigorous representation of WSAW-SA as a supersymmetric field theory.

1.8.1 The $n \to 0$ limit

The heuristic relation between self-avoiding walk and spin systems is most easily treated on finite graphs $\mathcal{G}$ of degree 3 so we restrict our attention to this case. In addition, we consider a version of the $O(n)$ model with spins normalized to lie on the sphere of radius $\sqrt{n}$, which we equip with the uniform measure. We denote the product measure on the resulting configuration space by

$$d\sigma = \prod_{x \in \mathcal{V}} d\sigma_x. \tag{1.8.2}$$

Remark 1.8.1. This normalization of the spins was in fact used when the $O(n)$ model was originally introduced in [109]. Moreover, in [78], it was shown that this normalization is necessary for the study of the $n \to \infty$ limit.

The *high-temperature expansion* of a spin system is based on the expansion of the Boltzmann weight $e^{-\beta H}$ about $\beta = 0$. For the $O(n)$ spin model with interaction $J_{xy} = \mathbb{1}_{x\sim y}$, neglecting higher-order terms in the high-temperature expansion yields

$$\begin{aligned} Z &= \int d\sigma \prod_{xy\in\mathcal{E}} e^{\beta\sigma_x\cdot\sigma_y} \\ &\approx \int d\sigma \prod_{xy\in\mathcal{E}} (1+\beta\sigma_x\cdot\sigma_y) \\ &= \sum_{E\subset\mathcal{E}} \beta^{|E|} \int d\sigma \prod_{xy\in E} \sigma_x\cdot\sigma_y \end{aligned} \tag{1.8.3}$$

By reflection-invariance, the last integral above is non-zero if and only if every vertex in the product over E appears an even number of times. On a graph of degree 3, this is only possible if E is a (possibly empty) collection of mutually avoiding (i.e. disjoint) self-avoiding loops (walks from a vertex to itself that are self-avoiding everywhere except this vertex).

Moreover, for any loop L, invariance under orthogonal transformations and the fact that spins have radius $\sqrt{n}$ implies that

$$\int d\sigma \prod_{xy\in L} \sigma_x\cdot\sigma_y = \sum_{i=1}^{n} \prod_{x\in\mathcal{V}(L)} \int d\sigma_x\, (\sigma_x^i)^2 = n, \tag{1.8.4}$$

where $\mathcal{V}(L)$ is the set of vertices in L. Thus,

$$Z \approx 1 + \sum_{N\geq 1} \frac{n^N}{N!} \sum_{L_1,\dots,L_N} \beta^{|L_1|+\cdots+|L_N|}, \tag{1.8.5}$$

where the inner sum is over all collections of disjoint loops $L_1,\dots,L_N$ and permutations of these loops are accounted for by the $1/N!$ factor. Notice that the final expression on the right-hand side of (1.8.5) makes sense for any N and equals 1 when $n = 0$.

The two-point function for the $O(n)$ model can be defined analogously to (1.4.30) and (1.4.22). By a similar expansion as was used to study the partition function above, the numerator in the two-point function becomes

$$n^{-1} \int d\sigma(\sigma_a\cdot\sigma_b) \prod_{xy\in\mathcal{E}} e^{\beta\sigma_x\cdot\sigma_y} \approx n^{-1} \sum_{E\subset\mathcal{E}} \beta^{|E|} \int d\sigma(\sigma_a\cdot\sigma_b) \prod_{xy\in E} \sigma_x\cdot\sigma_y. \tag{1.8.6}$$

Once again, every vertex must appear twice on the right-hand side in order to make a non-zero contribution to the sum. Due to the presence of the factor $\sigma_a\cdot\sigma_b$, this means (unless $a = b$) that the sum can be replaced by a sum over subsets E containing a self-avoiding walk from a to b together with with a (possibly empty) family of mutually avoiding self-avoiding loops that also avoid this walk. (As a very simple example, if $a \sim b$, then there is a non-zero contribution

from $E = \{a, b\}$.) For any such configuration E containing N loops,

$$\int d\sigma \, (\sigma_a \cdot \sigma_b) \prod_{xy \in E} \sigma_x \cdot \sigma_y = n^{1+N}. \tag{1.8.7}$$

The extra factor of n arises from the walk in E but is cancelled by the normalization in (1.8.6). Thus, after formally setting $n = 0$ (so that $Z = 1$), the two-point function is approximately given by

$$1 + \sum_{\omega \in \mathcal{S}_n(a,b)} \beta^{|\omega|}, \tag{1.8.8}$$

which is the two-point function, i.e. the generating function for all self-avoiding walks from a to b.

1.8.2 Self-avoiding walk representation

In this section we describe an integral representation of the of WSAW-SA on the discrete torus Λ. We begin with the necessary background on Grassmann integration, which was introduced in [15]. However, we follow the treatment of [26] in terms of differential forms.

Boson and fermion fields

Let ϕ_x, $\bar{\phi}_x$ denote complex variables indexed by $x \in \Lambda$. We refer to $(\phi, \bar{\phi})$ as a *boson* field. Let u_x, v_x denote the real and imaginary parts of ϕ_x and define the differentials $d\phi_x = du_x + i dv_x$ and likewise for $d\bar{\phi}_x$. We multiply differential forms in the usual way via the anticommutative wedge product $\wedge$ but drop this in our notation; in particular,

$$d\bar{\phi}_x d\phi_x = 2i du_x dv_x. \tag{1.8.9}$$

Example 1.8.2. Let C be a positive-definite symmetric $\Lambda \times \Lambda$ matrix. The *complex Gaussian measure* with covariance C is the probability measure on $\mathbb{R}^{2\Lambda}$ given by

$$d\mu_C(\phi, \bar{\phi}) = \frac{d\bar{\phi} d\phi}{\det(2\pi i C)} e^{-\phi \cdot A \bar{\phi}} \tag{1.8.10}$$

where $A = C^{-1}$ and

$$d\bar{\phi} d\phi \coloneqq \prod_{x \in \Lambda} d\bar{\phi}_x d\phi_x \tag{1.8.11}$$

The order in which the product over $x \in \Lambda$ is taken does not matter since the $d\bar{\phi}_x d\phi_x$ commute. The complex Gaussian satisfies a version of Wick's theorem. In particular,

$$\int \bar{\phi}_x \phi_y \, d\mu_C(\phi, \bar{\phi}) = C_{xy}. \tag{1.8.12}$$

Let

$$\psi_x = \frac{1}{\sqrt{2\pi i}} d\phi_x, \quad \bar{\psi}_x = \frac{1}{\sqrt{2\pi i}} d\bar{\phi}_x, \tag{1.8.13}$$

where we fix a choice of complex square root. We refer to $(\psi_x, \bar{\psi}_x)_{x\in\Lambda}$ as a *fermion* field. A differential form that is the product of a function of $(\phi, \bar{\phi})$ with p differentials is said to have *degree* p. A sum of forms of even degree is said to be *even*.

We introduce a copy $\bar{\Lambda}$ of Λ and we denote the copy of $X \subset \Lambda$ by $\bar{X} \subset \bar{\Lambda}$. We also denote the copy of $x \in \Lambda$ by $\bar{x} \in \bar{\Lambda}$ and define $\phi_{\bar{x}} = \bar{\phi}_x$ and $\psi_{\bar{x}} = \bar{\psi}_x$. Then any differential form F can be written

$$F = \sum_{\vec{y}} F_{\vec{y}}(\phi, \bar{\phi}) \psi^{\vec{y}} \tag{1.8.14}$$

where the sum is over finite sequences $\vec{y}$ over $\Lambda \sqcup \bar{\Lambda}$, and $\psi^{\vec{y}} = \psi_{y_1} \dots \psi_{y_p}$ when $\vec{y} = (y_1, \dots, y_p)$. Here, we take the sequences to be ordered in some fixed but arbitrary fashion. We let F^0 denote the 0-degree (bosonic) part of F, given by the coefficient $F_{\vec{y}}$ with $\vec{y} = \varnothing$ the empty sequence.

In order to apply the results of [8, 9, 12], we require smoothness of the coefficients $F_{\vec{y}}$ of F. For Theorem 1.7.1(i,ii), we need these coefficients to be C^{10}, and for Theorem 1.7.1(iii) we require a p-dependent number of derivatives for the analysis of ξ_p. In either case, we let $p_{\mathcal{N}}$ denote the desired degree of smoothness. We will discuss this further in Section 4.2.2.

We let $\mathcal{N}^{\varnothing}$ be the algebra of even forms (i.e. differential forms of even degree) with sufficiently smooth coefficients and we let $\mathcal{N}^{\varnothing}(X) \subset \mathcal{N}^{\varnothing}$ be the sub-algebra of even forms only depending on fields in X. Thus, for $F \in \mathcal{N}^{\varnothing}(X)$, the sum in (1.8.14) runs over sequences $\vec{y}$ over $X \sqcup \bar{X}$.

Now let $F = (F_j)_{j\in J}$ be a finite collection of even forms indexed by a set J and write $F^0 = (F_j^0)_{j\in J}$. Given a C^∞ function $f : \mathbb{R}^J \to \mathbb{C}$, we define $f(F)$ by its Taylor expansion about F^0:

$$f(F) = \sum_{\alpha} \frac{1}{\alpha!} f^{(\alpha)}(F^0)(F - F^0)^\alpha. \tag{1.8.15}$$

The summation terminates as a finite sum, since $\psi_x^2 = \bar{\psi}_x^2 = 0$ by anticommutativity.

We define the integral $\int F$ of a differential form F in the usual way as the Riemann integral of its top-degree part (which may be regarded as a function of the boson field). In particular, given a positive-definite symmetric $\Lambda \times \Lambda$ matrix C with inverse $A = C^{-1}$, we define the *Gaussian expectation* (or *super-expectation*) of F by

$$\mathbb{E}_C F = \int e^{-S_A} F, \tag{1.8.16}$$

where

$$S_A = \sum_{x\in\Lambda} \Big(\phi_x (A\bar{\phi})_x + \psi_x (A\bar{\psi})_x \Big). \tag{1.8.17}$$

The super-expectation has the following self-normalizing property:

$$\mathbb{E}_C 1 = \int e^{-S_A} = 1. \tag{1.8.18}$$

Moreover, if F is a degree-0 form, then

$$\mathbb{E}_C F = \int F \, d\mu_C. \tag{1.8.19}$$

There is also a version of Wick's theorem for fermions. In particular,

$$\int e^{-S_A} \bar{\psi}_x \psi_x = C_{xx}. \tag{1.8.20}$$

Proofs of the statements (1.8.18)–(1.8.20) can be found in [26].

For $F = f(\phi, \bar{\phi})\psi^{\vec{y}}$, we let

$$\theta F = f(\phi + \xi, \bar{\phi} + \bar{\xi})(\psi + \eta)^{\vec{y}}, \tag{1.8.21}$$

where ξ is a new boson field, $\eta = (2\pi i)^{-1/2} d\xi$ a new fermion field, and $\bar{\xi}, \bar{\eta}$ are the corresponding conjugate fields. We extend θ to all $F \in \mathcal{N}^{\varnothing}$ by linearity and define the convolution operator $\mathbb{E}_C\theta$ by letting $\mathbb{E}_C\theta F \in \mathcal{N}^{\varnothing}$ denote the Gaussian expectation of θF with respect to $(\xi, \bar{\xi}, \eta, \bar{\eta})$, with $\phi, \bar{\phi}, \psi, \bar{\psi}$ held fixed.

Integral representation of the two-point function

An integral representation formula applying to general local time functionals is given in [20,26]. We state the result we need in the proposition below. A direct proof can be obtained by a small modification to the proof in [108, Appendix A].

We define the differential forms:

$$\tau_x = \phi_x \bar{\phi}_x + \psi_x \bar{\psi}_x \tag{1.8.22}$$

$$\tau_{\Delta,x} = \frac{1}{2}\Big(\phi_x(-\Delta\bar{\phi})_x + (-\Delta\phi)_x\bar{\phi}_x + \psi_x(-\Delta\bar{\psi})_x + (-\Delta\psi)_x\bar{\psi}_x\Big) \tag{1.8.23}$$

$$|\nabla\tau_x|^2 = \sum_{|e|=1} (\nabla^e \tau)_x^2. \tag{1.8.24}$$

The forms τ_x are special due to the following remarkable property of the super-expectation (see [26]):

$$\int e^{-S_A} F(\tau) = F(0). \tag{1.8.25}$$

Note that (1.8.18) is an immediate consequence of this fact. Recall (1.6.19) and define

$$V_{g,\gamma,\nu,N} = U_{g,\gamma}(\tau) + \sum_{x\in\Lambda_N} \Big(\nu\tau_x + \tau_{\Delta,x}\Big) \tag{1.8.26}$$

Proposition 1.8.3. *Let $d > 0$ and $g > 0$. For $\gamma < g$ and $\nu \in \mathbb{R}$,*

$$G_{x,N}(g, \gamma, \nu; 0) = \int e^{-V_{g,\gamma,\nu,N}} \bar{\phi}_0 \phi_x. \tag{1.8.27}$$

Finite-volume approximation

In order to make use of the integral representation above, we must approximate the WSAW-SA on $\mathbb{Z}^d$ by a model on Λ_N.

Let X^{L^N} denote the simple random walk on Λ_N. For $F_T = F_T(X)$ any one of the functions L^x_T, I_T, C_T of X defined in (1.6.5)–(1.6.7), we write $F_{N,T} = F_T(X^{L^N})$. For instance, with $n = L^N$,

$$L^x_{N,T} = \int_0^T \mathbb{1}_{X^n_t = x}\, dt, \quad I_{N,T} = \sum_{x\in\Lambda_N} (L^x_{N,T})^2. \tag{1.8.28}$$

As before, we identify the vertices of Λ_N with nested subsets of $\mathbb{Z}^d$, centred at the origin (approximately if L is even), with Λ_{N+1} paved by L^d translates of Λ_N. We denote the expectation of X^{L^N} started from $0 \in \Lambda_N$ by $E_0^{\Lambda_N}$ and define

$$c_{N,T}(x) = E_0^{\Lambda_N}\left(e^{-U_{g,\gamma,T}}\mathbb{1}_{X(T)=x}\right), \quad x \in \Lambda_N \tag{1.8.29}$$

$$c_{N,T} = E_0^{\Lambda_N}\left(e^{-U_{g,\gamma,T}}\right). \tag{1.8.30}$$

The finite-volume two-point function and susceptibility are defined by

$$G_{x,N}(g,\gamma,\nu;0) = \int_0^\infty c_{N,T}(x)e^{-\nu T}\, dT, \tag{1.8.31}$$

$$\chi_N(g,\gamma,\nu;0) = \int_0^\infty c_{N,T}e^{-\nu T}\, dT. \tag{1.8.32}$$

The proof of the following proposition is given in Appendix A.

Proposition 1.8.4. *Let $d > 0$, $g > 0$ and $\gamma < g$. For all $\nu \in \mathbb{R}$,*

$$\lim_{N\to\infty} G_{x,N}(g,\gamma,\nu;0) = G_x(g,\gamma,\nu;0) \tag{1.8.33}$$

and

$$\lim_{N\to\infty} \chi_N(g,\gamma,\nu;0) = \chi(g,\gamma,\nu;0). \tag{1.8.34}$$

In fact, χ_N and χ are analytic in $\mathrm{Re}\nu > \nu_c$ and $\chi_N \to \chi$ uniformly on compact subsets of this domain.

1.9 Outline

Chapter 2 introduces the elements and formalism of the renormalization group method developed in [10, 28–31]. However, we proceed differently from these papers in two regards.

Firstly, in Section 2.4.3 we employ a different choice of norm weights from that used in [31], where the renormalization group map was constructed. However, these new weights cannot be used with the same norms as in [31]. In Chapter 4, we explain how to overcome this obstacle by a new choice of norm and we provide a detailed verification that the estimates

on the renormalization group map are improved by this choice. The result is summarized as Theorem 2.7.1, which is the first main technical achievement of this thesis. The improved estimates that we obtain are required for the proof of Theorem 1.7.1(iii), even when we take $\gamma = 0$. This result first appeared in [12].

Secondly, the initial coordinates for the renormalization group that we define in Section 2.6.1 involve a non-trivial error coordinate that captures the self-attraction term in the WSAW-SA and the $\gamma(\nabla|\phi_x|^2)^2$ term in the generalized $|\varphi|^4$ model. This error coordinate is coupled to a coordinate capturing the relevant and marginal directions and a version of the implicit function theorem is consequently required for the identification of critical parameters such that the renormalization group can be initialized on its stable manifold when $\gamma \neq 0$. The construction of these critical parameters is carried out in Chapter 5 and the result is summarized as Theorem 2.8.1. This is the second main technical achievement of this thesis and is required for the proof of Theorem 1.7.1 with $\gamma \neq 0$. This result first appeared in [13] for $n = 0$; here, we have extended it to all $n \geq 0$. However, we restrict our attention to the more interesting case of $\gamma \geq 0$ for simplicity.

Prior to proving Theorems 2.7.1 and 2.8.1, we show in Chapter 3 how to obtain Theorem 1.7.1 as a consequence of these results. The proof of Theorem 1.7.1(iii) is a novel contribution even in the case $\gamma = 0$, which first appeared in [12].

The proof of Theorem 1.7.1(i)–(ii) was previously obtained for $\gamma = 0$ by Bauerschmidt, Brydges, and Slade in [7–9] and Slade and Tomberg in [108]. The extension to $\gamma \neq 0$ is an adaptation of the proofs found in those papers but involves (in addition to the proof of Theorem 2.8.1) a change of variables result stated and proved in Section 3.1.1.

We conclude in Chapter 6 with a discussion of some open problems.

Chapter 2

Renormalization group method

This chapter introduces the elements of the renormalization group method developed in the series of papers [10, 28–31] and applied in [7–9, 108]. We will often state results from these papers without proof.

The main contribution of this thesis, which is based on the work in [12,13], is the improvement of the estimates in Theorem 2.7.1 and the extension to $\gamma_0 \neq 0$ in Theorem 2.8.1.

2.1 Notation

To unify our treatment of the two models, we define the forms $\tau_x, \tau_{\Delta,x}, |\nabla\tau_x|^2$ according to (1.8.22)–(1.8.24) if $n = 0$ and

$$\tau_x = \tfrac{1}{2}|\varphi_x|^2, \quad \tau_{\Delta,x} = \tfrac{1}{2}\varphi_x \cdot (-\Delta\varphi)_x, \quad |\nabla\tau_x|^2 = \sum_{|e|=1} (\nabla^e|\varphi_x|^2)^2 \tag{2.1.1}$$

if $n \geq 1$. Then by (1.4.27), (1.6.19), and (1.8.26),

$$V_{g,\gamma,\nu,N} = \sum_{x\in\Lambda_N} \Big((g-\gamma)\tau_x^2 + \nu\tau_x + \tau_{\Delta,x} + \tfrac{1}{4d}\gamma|\nabla\tau_x|^2 \Big) \tag{2.1.2}$$

for any choice of n. We write

$$\langle F\rangle_{g,\gamma,\nu,N} = \begin{cases} \displaystyle\int F e^{-U_{g,\gamma,\nu,N}}, & n = 0 \\ \displaystyle\frac{1}{Z_{g,\gamma,\nu,N}} \int F(\varphi) e^{-U_{g,\gamma,\nu,N}}\, d\varphi, & n \geq 1. \end{cases} \tag{2.1.3}$$

The action S_A is defined by (1.8.17) if $n = 0$ and

$$S_A = \frac{1}{2}\sum_{x\in\Lambda} \varphi_x \cdot (A\varphi)_x \tag{2.1.4}$$

if $n \geq 1$. In either case, if $A = -\Delta + m^2$, then

$$S_A = \sum_{x\in\Lambda} (\tau_{\Delta,x} + m^2\tau_x). \tag{2.1.5}$$

Thus, if $\mathbb{E}_C\theta$ is the super-expectation (1.8.16) for $n = 0$ and Gaussian integration over $(\mathbb{R}^n)^\Lambda$ if $n \geq 1$, then for $\nu > 0$,

$$\langle F \rangle_{0,0,m^2,N} = \mathbb{E}_C F, \qquad C = (-\Delta + m^2)^{-1}. \tag{2.1.6}$$

By (1.4.30), (1.8.33), and (1.8.27),

$$G_x(g, \gamma, \nu; n) = \lim_{N\to\infty} G_{x,N}(g, \gamma, \nu; n), \tag{2.1.7}$$

where

$$G_{x,N}(g, \gamma, \nu; n) = \begin{cases} \langle \bar\phi_0 \phi_x \rangle_{g,\gamma,\nu,N}, & n = 0 \\ \langle \varphi_0 \cdot \varphi_x \rangle_{g,\gamma,\nu,N} & n \geq 1. \end{cases} \tag{2.1.8}$$

By Proposition 1.8.4 and (1.4.31), for any integer $n \geq 0$,

$$\chi(g, \gamma, \nu; n) = \lim_{N\to\infty} \chi_N(g, \gamma, \nu; n) \tag{2.1.9}$$

$$\chi_N(g, \gamma, \nu; n) = \sum_{x\in\Lambda_N} G_{x,N}(g, \gamma, \nu; n). \tag{2.1.10}$$

$$\xi_p(g, \gamma, \nu; n) = \left(\frac{\sum_{x\in\mathbb{Z}^d} |x|^p G_x(g, \gamma, \nu; n)}{\chi(g, \gamma, \nu; n)} \right)^{1/p} \tag{2.1.11}$$

$$\nu_c = \nu_c(g, \gamma; n) = \inf\{\nu : \chi(g, \gamma, \nu; n) < \infty\}. \tag{2.1.12}$$

2.2 Reformulation of the problem

In preparation for our application of the renormalization group, we write the two-point function and susceptibility in terms of appropriate perturbations of Gaussian measures.

Given $m^2 > 0$ and $z_0 > -1$, let

$$g_0 = (g - \gamma)(1 + z_0)^2, \quad \nu_0 = \nu(1 + z_0) - m^2, \quad \gamma_0 = \frac{1}{4d}\gamma(1 + z_0)^2. \tag{2.2.1}$$

We discuss the role of (m^2, z_0) some more in Remark 2.3.2.

We fix two points $0, x \in \Lambda$ and introduce *observable fields* $\sigma_0, \sigma_x \in \mathbb{R}$. We distinguish these from the *bulk* fields φ, ϕ, $\bar\phi$, ψ, $\bar\psi$. We also make a distinction between bosonic fields φ, ϕ, $\bar\phi$, σ, and fermionic fields ψ, $\bar\psi$.

For any $y \in \Lambda$, we define the polynomials

$$V^+_{0,y} = g_0\tau_y^2 + \nu_0\tau_y + z_0\tau_{\Delta,y} - f_0\sigma_0\mathbb{1}_{y=0} - f_x\sigma_x\mathbb{1}_{y=x}, \quad U^+_y = |\nabla\tau_y|^2 \tag{2.2.2}$$

where

$$f_u = \begin{cases} \bar{\phi}_0, & n = 0, u = 0 \\ \phi_x, & n = 0, u = x \\ \varphi^1_u, & n \geq 1. \end{cases} \tag{2.2.3}$$

These are examples of local polynomials, which are polynomials in the fields and their derivatives at a point $y \in \Lambda$. For any such local polynomial V_y, we will usually write

$$V(X) = \sum_{y \in X} V_y. \tag{2.2.4}$$

Let

$$Z_0 = \prod_{y \in \Lambda} e^{-(V^+_{0,y} + \gamma_0 U^+_y)} \tag{2.2.5}$$

and

$$Z_N = \mathbb{E}_C \theta Z_0 \tag{2.2.6}$$

where the covariance is given by $C = (-\Delta + m^2)^{-1}$ as in (2.1.6). In particular,

$$\mathbb{E}_C Z_0 = Z^0_N(0). \tag{2.2.7}$$

Recall here that Z^0_N denotes the 0-degree part of Z_N (when $n \geq 1$, $Z^0_N = Z_N$). This is a function of the bulk bosonic fields, which we have set to 0 on the right-hand side of (2.2.7).

Recall that the Gaussian convolution operator $\mathbb{E}_C \theta$ was defined in Section 1.8.2. We define a test function $\mathbb{1} : \Lambda_N \to \mathbb{R}$ by $\mathbb{1}_y = 1$ for all y. If F is a sufficiently smooth function of the bosonic fields (i.e. $F = F(\phi, \bar{\phi})$ if $n = 0$ and $F = F(\varphi)$ if $n \geq 1$), let

$$D^2 F(0; \mathbb{1}, \mathbb{1}) = \frac{\partial^2}{\partial s \partial t}\Big|_0 \begin{cases} F(s\mathbb{1}, t\mathbb{1}), & n = 0 \\ F(s\mathbb{1} + t\mathbb{1}), & n \geq 1 \end{cases} \tag{2.2.8}$$

where the derivative is evaluated with all fields (bulk and observable) and s, t set to 0. Let $F(0)$ denote F evaluated at 0 bulk field. We denote by $D^2_{\sigma_0 \sigma_x} F(0)$ the second partial derivative of $F(0)$ with respect to the observable fields σ_0, σ_x evaluated at $\sigma_0 = \sigma_x = 0$.

Proposition 2.2.1. *Let $d > 0$, $\gamma, \nu \in \mathbb{R}$, $g > 0$ and $\gamma < g$. If the relations (2.2.1) hold, then*

$$G_{x,N}(g, \gamma, \nu; n) = (1 + z_0) D^2_{\sigma_0 \sigma_x} \log \mathbb{E}_C Z_0 \tag{2.2.9}$$

and

$$\chi_N(g, \gamma, \nu; n) = (1 + z_0) \hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0; n), \tag{2.2.10}$$

with

$$\hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0; n) = \frac{1}{m^2} + \frac{1}{m^4} \frac{1}{|\Lambda|} \frac{D^2 Z^0_N(0; \mathbb{1}, \mathbb{1})}{Z^0_N(0)}. \tag{2.2.11}$$

Proof. We prove the case $n = 0$ and drop the parameter n from the notation. Note that by (1.8.25), $Z_N(0)\big|_{\sigma_0=\sigma_x=0} = 1$ in this case. The proof for $n \geq 1$ is similar and involves only ordinary integration with respect to real boson fields.

We make the change of variables $\phi_x \mapsto (1+z_0)^{1/2}\phi_x$ and likewise for $\bar{\phi}_x, \psi_x, \bar{\psi}_x$ in (1.8.27), and obtain

$$G_{x,N}(g,\gamma,\nu) = (1+z_0)\int e^{-\sum_{x\in\Lambda}\left(g_0\tau_x^2+\gamma_0|\nabla\tau_x|^2+\nu(1+z_0)\tau_x+(1+z_0)\tau_{\Delta,x}\right)}\bar{\phi}_a\phi_b. \tag{2.2.12}$$

Note here that the Jacobian factor is automatically accounted for by the change of variables in the fermionic fields. For any $m^2 \in \mathbb{R}$, it follows that

$$G_{x,N}(g,\gamma,\nu) = (1+z_0)\int e^{-\sum_{x\in\Lambda}(\tau_{\Delta,x}+m^2\tau_x)}Z_0\bar{\phi}_0\phi_x \tag{2.2.13}$$

(m^2 simply cancels with ν_0 on the right-hand side). We use this with $m^2 > 0$, so that the inverse matrix $C = (-\Delta+m^2)^{-1}$ exists and

$$G_{x,N}(g,\gamma,\nu) = (1+z_0)\mathbb{E}_C(Z_0\bar{\phi}_0\phi_x) \tag{2.2.14}$$

by (2.1.6). The identity (2.2.9) follows by the standard procedure of writing the moments of an integral as a derivative of a moment-generating function.

Summation of (2.2.14) over $x \in \Lambda_N$ gives the formula $\chi_N(g,\gamma,\nu) = (1+z_0)\sum_{x\in\Lambda}\mathbb{E}_C(Z_0\bar{\phi}_0\phi_x)$. Call the right-hand side $\hat{\chi}_N(g,\gamma,\nu)$. To show that this is consistent with (2.2.11), begin by noting that

$$\hat{\chi}_N(g,\gamma,\nu) = |\Lambda|^{-1}\frac{D^2\Sigma(0;\mathbb{1},\mathbb{1})}{Z_N(0)}, \tag{2.2.15}$$

where

$$\Sigma(J,\bar{J}) = \mathbb{E}_C(Z_0 e^{J\cdot\bar{\phi}+\phi\cdot\bar{J}}). \tag{2.2.16}$$

Completing the square yields

$$\Sigma(J,\bar{J}) = e^{J\cdot C\bar{J}}Z_N^0(CJ,C\bar{J}) \tag{2.2.17}$$

and differentiating this expression gives

$$D^2\Sigma(0;\mathbb{1},\mathbb{1}) = (\mathbb{1},C\mathbb{1}) + D^2Z_N^0(0;C\mathbb{1},C\mathbb{1}) \tag{2.2.18}$$

The result then follows from the fact that

$$C\mathbb{1} = A^{-1}\mathbb{1} = m^{-2}\mathbb{1}. \tag{2.2.19}$$

■

2.3 Progressive integration

By Proposition 2.2.1, our task is to understand the Gaussian expectation $Z_N = \mathbb{E}_C Z_0$ and its derivatives to leading order, uniformly in the volume Λ_N and the mass m^2 near 0.

We proceed using the covariance decomposition

$$C = C_1 + \cdots + C_{N-1} + C_{N,N} \tag{2.3.1}$$

constructed in [5]; a similar decomposition was also constructed in [22]. The covariances $C_1, \dots, C_{N-1}$ are independent of the volume Λ_N. The final covariance $C_{N,N}$ *does* depend on the volume; so, for instance, $C_{N,N} \neq C_{N,N+1}$. Nevertheless, we will often write $C_N \coloneqq C_{N,N}$ when the volume is implicit.

The covariances C_j have the following important *finite-range property*:

$$C_{j;xy} = 0 \text{ if } |x - y| \geq \tfrac{1}{2} L^j. \tag{2.3.2}$$

Thus, if ζ is a Gaussian field with covariance C_j, then ζ_x is independent of ζ_y whenever $|x-y| \geq \frac{1}{2}L^j$. In particular, if F_x, F_y are functions of the fields at x, y, respectively, then

$$\mathbb{E}_{C_{j+1}}(F_x F_y) = (\mathbb{E}_{C_{j+1}} F_x)(\mathbb{E}_{C_{j+1}} F_y). \tag{2.3.3}$$

In addition, we have the following covariance bounds (this is a restatement of [10, Proposition 6.1(a)]).

Proposition 2.3.1. *Let $d > 2$, $L \geq 2$, $j \geq 1$, $\bar{m}^2 > 0$. For multi-indices α, β with ℓ^1 norms $|\alpha|_1, |\beta|_1$ at most some fixed value p, for any k, and for $m^2 \in [0, \bar{m}^2]$,*

$$|\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}| \leq c(1 + m^2 L^{2(j-1)})^{-k} L^{-(j-1)(d-2+|\alpha|_1+|\beta|_1)}, \tag{2.3.4}$$

where $c = c(p, k, \bar{m}^2)$ is independent of m^2, j, L. The same bound holds for $C_{N,N}$ if $m^2 L^{2(N-1)} \geq \varepsilon$ for some $\varepsilon > 0$, with c depending on ε but independent of N.

It is a basic property of the Gaussian distribution that a sum of independent Gaussian random variables with covariances C' and C'' is itself Gaussian with covariance $C' + C''$. It follows that for any boson field F,

$$\mathbb{E}_{C'+C''} \theta F = \mathbb{E}_{C'} \theta \circ \mathbb{E}_{C''} \theta F. \tag{2.3.5}$$

This extends to any sufficiently smooth form F (see [28]). It follows that

$$Z_N = \mathbb{E}_{C_N} \theta \circ \mathbb{E}_{C_{N-1}} \theta \circ \ldots \circ \mathbb{E}_{C_1} \theta Z_0. \tag{2.3.6}$$

We define the *renormalization group map* $Z_j \mapsto Z_{j+1}$ by

$$Z_{j+1} = \mathbb{E}_{C_{j+1}} \theta Z_j, \quad j < N. \tag{2.3.7}$$

Remark 2.3.2. The key to understanding Z_N for large N is the careful choice of *critical* initial conditions (m^2, z_0) in (2.2.1). Viewed as functions of (g, γ, ν), these define a stable manifold for the dynamical system induced by the renormalization group map and the fixed point for this stable manifold is the Gaussian measure with covariance $(1+z_0)(-\Delta+m^2)^{-1}$. However, we have scaled out the factor $1+z_0$ in the change of variables performed in the proof of Proposition 2.2.1. Indeed, for $n \geq 1$ the exponent in (2.2.12) contains the term $-\frac{1}{2}(1+z_0)\sum_{x\in\Lambda} \varphi_x \cdot [(-\Delta+m^2)\varphi]_x$.

The construction of the critical parameters for $\gamma \neq 0$ will be carried out in Section 3.1.1 and is a key step in the proof of Theorem 1.7.1.

2.4 The space of field functionals

For the analysis of the dynamical system (2.3.7), we require a suitable space on which this system evolves.

Let $\mathcal{N}^{\varnothing}$ be defined as in Section 1.8.2 if $n = 0$ and

$$\mathcal{N}^{\varnothing} = \mathcal{N}^{\varnothing}(\Lambda) = C^{p_{\mathcal{N}}}((\mathbb{R}^n)^{\Lambda}, \mathbb{R}) \tag{2.4.1}$$

if $n \geq 1$. Recall that $p_{\mathcal{N}}$ is the smoothness parameter discussed in Section 1.8.2.

We extend $\mathcal{N}^{\varnothing}$ to a space $\mathcal{N}$ that includes functions of the observable fields σ_0 and σ_x, which we identify to order $1, \sigma_0, \sigma_x, \sigma_0\sigma_x$ (this is sufficient for computing the derivative in (2.2.9)). Formally, we let $\mathcal{N}'$ denote the extension of $\mathcal{N}^{\varnothing}$ whose elements may depend smoothly on σ_0, σ_x. In other words, if $n \geq 1$, then $\mathcal{N}'$ consists of functions of $(\varphi, \sigma_0, \sigma_x)$ that are $C^{p_{\mathcal{N}}}$ in φ and C^{∞} in σ_0, σ_x. Likewise, for $n = 0$, a similar statement is true of the coefficients $F_{\vec{y}}$ in (1.8.14). Letting $\mathcal{I} \subset \mathcal{N}'$ be the ideal consisting of elements whose formal expansion to order $1, \sigma_0, \sigma_x, \sigma_0\sigma_x$ is 0, we define $\mathcal{N} = \mathcal{N}'/\mathcal{I}$. Then $\mathcal{N}$ has the direct sum decomposition

$$\mathcal{N} = \mathcal{N}^{\varnothing} \oplus \mathcal{N}^{a} \oplus \mathcal{N}^{b} \oplus \mathcal{N}^{ab}, \tag{2.4.2}$$

where $\mathcal{N}^{a}$ consists of elements of the form $\sigma_a F$ with $F \in \mathcal{N}^{\varnothing}$ and a similar statement is true of $\mathcal{N}^{b}, \mathcal{N}^{ab}$. Thus, every $F \in \mathcal{N}$ has the form

$$F = F_{\varnothing} + \sigma_0 F_0 + \sigma_x F_x + \sigma_0\sigma_x F_{0x}, \quad F_{\varnothing}, F_0, F_x, F_{0x} \in \mathcal{N}^{\varnothing}. \tag{2.4.3}$$

There are natural projections $\pi_\alpha : \mathcal{N} \to \mathcal{N}_\alpha$ with $\alpha = \varnothing, 0, x, 0x$ such that $\pi_\alpha F = F_\alpha$. For $X \subset \Lambda$, we let $\mathcal{N}(X)$ denote the subspace of $\mathcal{N}$ consisting of field functionals that only depend on fields in X.

In order to control the evolution of Z_j on $\mathcal{N}$, we make use of a family $\|\cdot\|_{T_{\phi,j}(\mathfrak{h}_j)}$ of scale-

dependent seminorms defined in terms of a sequence of weights $\mathfrak{h}_j > 0$; the field ϕ lies in $\mathbb{C}^\Lambda$ if $n = 0$ and $(\mathbb{R}^n)^\Lambda$ if $n \geq 1$. For convenience, we will simply write $\|\cdot\|_{T_\phi(\mathfrak{h}_j)}$ with the scale j implied by the choice of parameter $\mathfrak{h}_j$.

We given the precise definitions below for $n = 0$. The case $n \geq 1$ involves only minor changes, which we describe in Remark 2.4.1.

2.4.1 Test functions

Recall the notation introduced in Section 1.8.2. A *test function* g is defined to be a function $(\vec{x}, \vec{y}) \mapsto g_{\vec{x},\vec{y}}$, where $\vec{x}$ and $\vec{y}$ are finite sequences of elements in $\Lambda \sqcup \bar{\Lambda}$. When $\vec{x}$ or $\vec{y}$ is the empty sequence $\varnothing$, we drop it from the notation as long as this causes no confusion; e.g., we may write $g_{\vec{x}} = g_{\vec{x},\varnothing}$. The length of a sequence $\vec{x}$ is denoted $|\vec{x}|$. Gradients of test functions are defined component-wise. Thus, if $\vec{x} = (x_1, \dots, x_m)$ and $\alpha = (\alpha_1, \dots, \alpha_m)$ with each $\alpha_i \in \mathbb{N}_0^{\mathcal{U}}$, and similarly for $\vec{y} = (y_1, \dots, y_n)$ and $\beta = (\beta_1, \dots, \beta_n)$, then

$$\nabla^{\alpha,\beta}_{\vec{x},\vec{y}} g_{\vec{x},\vec{y}} = \nabla^{\alpha_1}_{x_1} \dots \nabla^{\alpha_m}_{x_m} \nabla^{\beta_1}_{y_1} \dots \nabla^{\beta_n}_{y_n} g_{x_1,\dots,x_m,y_1,\dots,y_n}. \tag{2.4.4}$$

We fix a positive constant $p_\Phi \geq 4$ and restrict our attention to test functions that vanish when $|\vec{x}| + |\vec{y}| > p_{\mathcal{N}}$. The $\Phi_j = \Phi(\mathfrak{h}_j)$ norm on such test functions is defined by

$$\|g\|_{\Phi_j} = \sup_{\vec{x},\vec{y}} \mathfrak{h}_j^{-(|\vec{x}|+|\vec{y}|)} \sup_{\alpha,\beta:|\alpha|_1+|\beta|_1 \leq p_\Phi} L^{j(|\alpha|_1+|\beta|_1)} |\nabla^{\alpha,\beta} g_{\vec{x},\vec{y}}|, \tag{2.4.5}$$

where $|\alpha|_1$ denotes the total order of the differential operator ∇^α. Thus, for any test function g and for sequences $\vec{x}, \vec{y}$ with $|\vec{x}| + |\vec{y}| \leq p_{\mathcal{N}}$ and corresponding α, β with $|\alpha|_1 + |\beta|_1 \leq p_\Phi$,

$$|\nabla^{\alpha,\beta} g_{\vec{x},\vec{y}}| \leq \mathfrak{h}_j^{|\vec{x}|+|\vec{y}|} L^{-j(|\alpha|_1+|\beta|_1)} \|g\|_{\Phi_j}. \tag{2.4.6}$$

2.4.2 The T_ϕ seminorm

If $n = 0$, then for any $F \in \mathcal{N}^\varnothing$, there are *unique* functions $F_{\vec{y}}$ of $(\phi, \bar{\phi})$ that are anti-symmetric under permutations of $\vec{y}$, such that

$$F = \sum_{\vec{y}} \frac{1}{|\vec{y}|!} F_{\vec{y}}(\phi, \bar{\phi}) \psi^{\vec{y}}. \tag{2.4.7}$$

Given a sequence $\vec{x}$ with $|\vec{x}| = m$, we define

$$F_{\vec{x},\vec{y}} = \frac{\partial^m F_{\vec{y}}}{\partial \phi_{x_1} \dots \partial \phi_{x_m}}. \tag{2.4.8}$$

We define a ϕ-dependent pairing of elements of $\mathcal{N}$ with test functions by

$$\langle F, g \rangle_\phi = \sum_{\vec{x},\vec{y}} \frac{1}{|\vec{x}|!|\vec{y}|!} F_{\vec{x},\vec{y}}(\phi, \bar{\phi}) g_{\vec{x},\vec{y}}. \tag{2.4.9}$$

Let $B(\Phi)$ denote the unit Φ-ball in the space of test functions. Then the $T_\phi = T_\phi(\mathfrak{h}_j)$ seminorm on $\mathcal{N}^\varnothing$ is defined by

$$\|F\|_{T_\phi} = \sup_{g \in B(\Phi_j)} |\langle F, g \rangle_\phi|. \tag{2.4.10}$$

Remark 2.4.1. If $n \geq 1$, a test function is a function g on sequences over $\Lambda \times \{1, \dots, n\}$. For any such sequence $\vec{x} = ((x_1, i_1), \dots, (x_m, i_m))$, we write $|\vec{x}| = m$ and set

$$F_{\vec{x}} = \frac{\partial^m F}{\partial \varphi^{i_1}_{x_1} \dots \partial \varphi^{i_m}_{x_m}} \tag{2.4.11}$$

and

$$\langle F, g \rangle_\varphi = \sum_{|\vec{x}| \leq p_{\mathcal{N}}} \frac{1}{|\vec{x}|!} F_{\vec{x}}(\varphi) g_{\vec{x}}. \tag{2.4.12}$$

Then the T_φ seminorm can be defined as in (2.4.10).

To extend the T_ϕ seminorm to $\mathcal{N}$, we make use of an additional sequence of parameters $\mathfrak{h}_{\sigma,j}$. For any $F \in \mathcal{N}$ of the form (2.4.3), we let

$$\|F\|_{T_\phi} = \|F_\varnothing\|_{T_\phi} + (\|F_0\|_{T_\phi} + \|F_x\|_{T_\phi})\mathfrak{h}_\sigma + \|F_{0x}\|_{T_\phi}\mathfrak{h}^2_\sigma. \tag{2.4.13}$$

By its definition, the T_ϕ seminorm controls the values of F and its derivatives (up to order $p_{\mathcal{N}}$) at ϕ. For instance, we will make use of the following facts.

Lemma 2.4.2. *If $F \in \mathcal{N}^\varnothing$, then $|F^0(0)| \leq \|F\|_{T_0}$. For $F \in \mathcal{N}$,*

$$|D^2 F^0(0; \mathbb{1}, \mathbb{1})| \leq 2\|F\|_{T_0(\mathfrak{h}_j)} \|\mathbb{1}\|^2_{\Phi_N(\mathfrak{h}_j)} = 2\|F\|_{T_0(\mathfrak{h}_j)} \mathfrak{h}_j^{-1} \tag{2.4.14}$$

and

$$|D^2_{\sigma_0 \sigma_x} F^0(0)| \leq \mathfrak{h}^{-2}_{\sigma,j} \|F\|_{T_0}. \tag{2.4.15}$$

An essential property of the T_ϕ seminorm is the following *product property*, which is essential to fully take advantage the factorization property (2.3.3) that follows from the finite-range property of the covariance decomposition.

Proposition 2.4.3. *If $F, G \in \mathcal{N}$, then $\|FG\|_{T_\phi} \leq \|F\|_{T_\phi} \|G\|_{T_\phi}$.*

Remark 2.4.4. This follows essentially from the fact that the series expansion of the product of two functions is the product of their respective series expansions (see [28]). This is part of the reason the T_ϕ seminorm was defined in terms of the pairing (2.4.9).

2.4.3 Norm weights

Control of the T_ϕ seminorm is needed for all values of ϕ in order to obtain control of the convolution (2.3.7) sufficient for iteration of the renormalization group map. This will be

discussed further in Section 4.2.2.

For now, we turn our attention to the special case of the T_0 seminorm. Recalling (2.2.7), it is natural to choose the weights $\mathfrak{h}_j$ so that $\mathbb{E}_{C_{j+1}}F$ is of order $\|F\|_{T_0(\mathfrak{h}_j)}$. By Wick's theorem (1.4.9), for a 1-component field φ,

$$\mathbb{E}_{C_{j+1}}\varphi_x^{2p} = (2p-1)!!C_{j+1;00}^p \tag{2.4.16}$$

and similar statements hold for complex and fermionic fields by the analogues of Wick's theorem for such fields. On the other hand, by definition of the T_0 seminorm,

$$\|\varphi_x^{2p}\|_{T_0(\mathfrak{h}_j)} \asymp \mathfrak{h}_j^{2p}. \tag{2.4.17}$$

This suggests defining $\mathfrak{h}_j$ so that $|C_{j+1;00}| \le O(\mathfrak{h}_j^2)$.

The key to our analysis of the correlation length is that we make a choice of norm weights that takes full advantage of the k-dependence in the covariance bounds (2.3.4). With $k = s+1$, this estimate together with the elementary bound

$$(1+m^2L^{2j})^{-k} \le c_L L^{-2(s+1)(j-j_m)_+} \tag{2.4.18}$$

imply that

$$|C_{j;xy}| \le O(L^{-j(d-2)-s(j-j_m)_+}), \tag{2.4.19}$$

where j_m is the *mass scale*, defined by

$$j_m = \lfloor \log_L m^{-1} \rfloor. \tag{2.4.20}$$

Based on this, when $d = 4$, we define the following weights:

$$\ell_j = \ell_0 L^{-j-s(j-j_m)_+}, \quad \ell_{\sigma,j} = \ell_{j\wedge j_x}^{-1} 2^{(j-j_x)_+}\tilde{g}_j, \tag{2.4.21}$$

where

$$j_x = \max\{0, \lfloor \log_L(2|x|) \rfloor\} \tag{2.4.22}$$

is the *coalescence scale* and the sequence $\tilde{g}_j = \tilde{g}_j(m^2, g_0)$ will be discussed in Section 2.5.3. The origin of the definition of $\ell_{\sigma,j}$ is discussed in [30, Remark 3.3].

We will set $\mathfrak{h}_j = \ell_j$ to estimate "small" fields. These are fields which are assumed not to deviate too much from their expected value. A different norm parameter $\mathfrak{h}_j = h_j$ will be used to control "large" fields. This will be discussed in Section 4.2.2.

Remark 2.4.5. The parameter $\tilde{g}_j$ is used to overcome what [1] refers to as the "fibred norm problem". Briefly, the norms used to control the renormalization group trajectory must be decoupled from the initial parameter g_0. Ultimately, we will set $\tilde{g}_0 = g_0$ (see Remark 5.2.3).

2.4.4 Symmetries

It is useful to restrict our attention to field functionals $F \in \mathcal{N}$ that obey certain symmetry conditions preserved by Gaussian expectation (and which are obeyed by V_0^+).

We let any automorphism E of Λ act on $\mathcal{N}$ by $EF(\varphi) = F(E\varphi)$ with $(E\varphi)_x = \varphi_{Ex}$. We say that $F \in \mathcal{N}$ is *Euclidean-invariant* if $EF = F$ for all such automorphisms.

If $n = 0$, we define the *gauge flow* $(q, \bar{q}) \mapsto (e^{-2\pi it}q, 2^{2\pi it}\bar{q})$, where $q = \phi_x, \psi_x, \sigma$ with $\sigma_0 = \sigma$ and $\sigma_x = \bar{\sigma}$ for all $x \in \Lambda$. A form $F \in \mathcal{N}$ is said to be *gauge-invariant* if it is invariant under the gauge flow. We also define the *supersymmetry generator*

$$Q = (2\pi i)^{1/2} \sum_{x \in \Lambda} \left(\psi_x \frac{\partial}{\partial \phi_x} + \bar{\psi}_x \frac{\partial}{\partial \bar{\phi}_x} - \phi_x \frac{\partial}{\partial \psi_x} + \bar{\phi}_x \frac{\partial}{\partial \bar{\psi}_x}. \right) \tag{2.4.23}$$

A form $F \in \mathcal{N}$ is said to be *supersymmetric* if $QF = 0$.

If $n \geq 1$, we let an $n \times n$ matrix T act on $\mathcal{N}$ by $TF(\varphi) = F(T\varphi)$, where $(T\varphi)_x = T(\varphi_x)$. We say that $F \in \mathcal{N}$ is $O(n)$*-invariant* if $TF = F$ for all orthogonal matrices T.

2.5 Perturbative coordinate

As mentioned in Section 1.5.4, one of Wilson's key insights was that the renormalization group could be well-approximated by a finite-dimensional dynamical system. In this section, we reformulate Wilson's insights in terms of the covariance decomposition and define a subspace on which this finite-dimensional system will evolve.

2.5.1 Dimensional analysis

We call $M_x \in \mathcal{N}$ a local monomial if it is a monomial in φ_x and its (discrete) gradients. For instance, for a 1-component field, such M_x has the form

$$M_x = (\nabla^{\alpha_1}\varphi_x) \dots (\nabla^{\alpha_p}\varphi_x). \tag{2.5.1}$$

The T_0 seminorm of a local monomial M_x essentially just counts the number of fields and derivatives in M_x. For instance, for M_x as above,

$$\|M_x\|_{T_0(\ell_j)} = O\big(L^{-j(|\alpha|+p[\varphi])}\big) \tag{2.5.2}$$

where $|\alpha| = |\alpha_1| + \dots + |\alpha_p|$ and

$$[\varphi] = \frac{d-2}{2} \tag{2.5.3}$$

is the *scaling dimension* of the field. Based on this observation, we define the *dimension* of M_x by

$$[M_x] = |\alpha| + p[\varphi]. \tag{2.5.4}$$

Note here that we have neglected the rapid decay of fields above the mass scale.

By (2.3.4), φ is approximately constant on blocks of side L^j. In a sense, the fields on a block B act as a unit and this contributes to a volume factor $|B| = L^{jd}$. This leads us to compare the dimension of a monomial with the dimension d of the lattice. We say that M_x is *relevant* if $[M_x] < d$, *marginal* if $[M_x] = d$, and *irrelevant* if $[M_x] > d$.

Remark 2.5.1. Note that the self-attraction term $|\nabla\tau_x|^2$ is irrelevant in the above sense. However, this does not mean that the inclusion of this term should not have an effect on the critical behaviour of the model under consideration (indeed, this term is responsible for the phase diagram given by Figure 1.4). Rather, the notion of irrelevance is an asymptotic one: irrelevant terms are only "unimportant" at very large scales j. At scale $j = 0$ there is little difference between a relevant and an irrelevant term, which is why we must choose the coefficient γ of $|\nabla\tau_x|^2$ to be small in Theorem 1.7.1.

2.5.2 Local field polynomials

For $y \in \Lambda$, we supplement (1.8.22)–(1.8.24) and (2.1.1) by defining

$$\tau_{\nabla\nabla,y} = \begin{cases} \frac{1}{2}\sum_{e\in\mathcal{U}} \left((\nabla^e\phi)_y(\nabla^e\bar{\phi})_y + (\nabla^e\psi)_y(\nabla^e\bar{\psi})_y\right), & n = 0 \\ \frac{1}{4}\sum_{|e|=1} \nabla^e\varphi_y \cdot \nabla^e\varphi_y, & n \geq 1. \end{cases} \tag{2.5.5}$$

When $n = 0$, it can be shown that the only marginal and relevant local monomials that are Euclidean-invariant and supersymmetric are constant multiples of

$$1, \quad \tau_x, \quad \tau_x^2, \quad \tau_{\Delta,x}, \quad \tau_{\nabla\nabla,x}. \tag{2.5.6}$$

When $n \geq 1$, these are the only marginal and relevant monomials that are Euclidean-invariant and $O(n)$-invariant (see [10]).

The marginal and relevant contributions to the evolution of the renormalization group will be tracked by a *local polynomial* (a sum of local monomials) of the form $\sum_{y\in\Lambda} U_y$, where (recall (2.2.3))

$$\begin{aligned} U_y = {} & g\tau_y^2 + \nu\tau_y + z\tau_{\Delta,y} + u \\ & - \mathbb{1}_{y=0}\lambda_0 f_0\sigma_0 - \mathbb{1}_{y=x}\lambda_x f_x\sigma_x \\ & - \tfrac{1}{2}(\mathbb{1}_{y=0}q_0 + \mathbb{1}_{y=x}q_x)\sigma_0\sigma_x. \end{aligned} \tag{2.5.7}$$

We have omitted $\tau_{\nabla\nabla}$ as (1.3.9) gives

$$\sum_{x\in\Lambda} \tau_{\nabla\nabla,x} = \sum_{x\in\Lambda} \tau_{\Delta,x}. \tag{2.5.8}$$

Remark 2.5.2. When $n = 0$, we can also omit u since constant terms are not produced by the Gaussian super-expectation. For example, $\mathbb{E}_C\theta\tau_x$ has constant part 0 by (1.8.12) and (1.8.20). More generally, this is a consequence of the supersymmetry identity (1.8.25).

We define $\mathcal{U}$ to be the space of all polynomials of the form U_y. Given $X \subset \Lambda$, we let

$$\mathcal{U}(X) = \{U(X) : U \in \mathcal{U}\}, \tag{2.5.9}$$

where $U(X)$ is defined as in (2.2.4). We also make use of the subspace $\mathcal{V}$ of polynomials with $u = y = q_0 = q_x = 0$. We will usually denote an element of $\mathcal{V}$ as V. For $U \in \mathcal{U}$, we define the map $U \mapsto U^{(0)} \in \mathcal{V}$, which sets $u = q_0 = q_x = 0$.

We define the $\mathcal{U} = \mathcal{U}_j$ norm by

$$\|U\|_{\mathcal{U}} = \max\Big\{|g|, L^{2j}|\nu|, |z|, L^{4j}|u|, \ell_j\ell_{\sigma,j}(|\lambda_0| \vee |\lambda_x|),\ \ell_{\sigma,j}^2(|q_0| \vee |q_x|)\Big\} \tag{2.5.10}$$

on $U \in \mathcal{U}$, which depends on the parameters ℓ_j and $\ell_{\sigma,j}$. The $\mathcal{U} = \mathcal{U}_j$ norm is equivalent to the $T_0(\ell_j)$ seminorm on $\mathcal{U}(B)$ when $|B| = L^{jd}$:

$$\|U\|_{\mathcal{U}} \asymp \|U(B)\|_{T_0(\ell_j)} = L^{jd}\|U_y\|_{T_0(\ell_j)}. \tag{2.5.11}$$

2.5.3 Perturbative flow

Here we discuss how to maintain the form $Z_j \approx e^{-V_j(\Lambda)}$ to second order with $V_j \in \mathcal{V}$. The basic idea begins with the *cumulant expansion*

$$\mathbb{E}_C\theta e^{-V(\Lambda)} \approx e^{-\mathbb{E}_C\theta V(\Lambda) + \frac{1}{2}\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda))}, \tag{2.5.12}$$

where

$$\mathbb{E}_C(F;G) = \mathbb{E}_C(FG) - (\mathbb{E}_C F)(\mathbb{E}_C G) \tag{2.5.13}$$

is the *truncated expectation*. In [29] an operator Loc_x is defined so that $\mathrm{Loc}_x F$ is an approximation of F by a local polynomial at x. We make the split

$$\frac{1}{2}\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda)) = \frac{1}{2}\,\mathrm{Loc}_x\,\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda)) + \frac{1}{2}(1 - \mathrm{Loc}_x)\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda)) \tag{2.5.14}$$

With $e^F \approx 1 + F$, we get

$$\mathbb{E}_C\theta e^{-V(\Lambda)} \approx e^{-\mathbb{E}_C\theta V(\Lambda) + \frac{1}{2}\,\mathrm{Loc}_x\,\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda))}\Big(1 + \tfrac{1}{2}(1 - \mathrm{Loc}_x)\mathbb{E}_C(\theta V(\Lambda);\theta V(\Lambda))\Big). \tag{2.5.15}$$

Based on this idea, in [10] a map[1] $U_{\mathrm{pt}} : \mathcal{V} \to \mathcal{U}$ of the form

$$U_{\mathrm{pt}}(V) = \mathbb{E}_C\theta V - P \tag{2.5.16}$$

[1] In [10], U_{pt} maps into a larger space including $\tau_{\nabla\nabla}$. Here, following (2.5.8), we define U_{pt} by composing that map with the map that replaces $z\tau_\Delta + y\tau_{\nabla\nabla}$ by $(z + y)\tau_\Delta$.

with P a local polynomial quadratic in V chosen so that the approximation

$$Z_j \approx e^{-U_j(\Lambda)}(1 + W_j) \tag{2.5.17}$$

can be maintained with $W_j = W_j(V)$ a non-local remainder. Precisely,

$$P_x = \mathrm{Loc}_x \, \mathbb{E}_C \theta W_j(V, x) + \frac{1}{2} \mathrm{Loc}_x [\mathbb{E}_C \theta(V_x V(\Lambda)) - (\mathbb{E}_C \theta V_x)(\mathbb{E}_C \theta V(\Lambda))] \tag{2.5.18}$$

and

$$W_j(V, x) = \frac{1}{2}(1 - \mathrm{Loc}_x)[\mathbb{E}_{w_j} \theta(V_x V(\Lambda)) - (\mathbb{E}_{w_j} \theta V_x)(\mathbb{E}_{w_j} \theta V(\Lambda))]. \tag{2.5.19}$$

By [30, (4.57)],

$$\|W_j\|_{T_0(\ell_j)} \leq O(\vartheta_j)\|V\|_{\mathcal{V}}^2, \tag{2.5.20}$$

where ϑ_j is a parameter that decays exponentially above the mass scale and will be discussed in Section 2.7. We will elaborate on the meaning of (2.5.17) in Section 2.6.

The map U_{pt} depends on the covariance C and in practice we set $C = C_{j+1}$ and obtain a sequence $U_{\mathrm{pt}} = U_{\mathrm{pt},j+1}$. By successively iterating these maps, we generate a sequence of coupling constants that we refer to as the *perturbative flow*. The equations defining this flow can be computed exactly by way of Feynman diagrams or with a computer program [6]. In [10], these flow equations are summarized and it is shown that a change of variables can be used to triangularize the resulting system of equations up to third-order errors. Below, we summarize these transformed flow equations for g, λ, and q.

The flow of g

The (transformed) perturbative flow of g takes the form

$$\bar{g}_{j+1} = \bar{g}_j - \beta_j \bar{g}_j^2, \qquad \bar{g}_0 = g_0 \tag{2.5.21}$$

where

$$\beta_j = (n + 8) \sum_{x \in \mathbb{Z}^d} (w_{j+1;0x}^2 - w_{j;0x}^2), \quad w_j = \sum_{i=1}^{j} C_i. \tag{2.5.22}$$

The sequence β_j is closely related to the free bubble diagram (1.6.26). Indeed, using the telescope nature of $\sum_j \beta_j$, we can show that

$$\sum_{j=1}^{\infty} \beta_j = \mathsf{B}_{m^2}. \tag{2.5.23}$$

The logarithmic divergence of the bubble diagram in (1.6.27) is reflected in the behaviour of g and, ultimately, in the appearance of logarithmic corrections in Theorem 1.7.1. Precisely, the

results of [11], were used to show in [9, Proposition 6.1] that

$$\bar{g}_j = O((\log m^{-1})^{-1}) \text{ for } j \geq j_m, \quad \bar{g}_{j_x} = O((\log |x|)^{-1}) \text{ for } j_x \leq j_m. \tag{2.5.24}$$

Remark 2.5.3. A heuristic argument is as follows: Using Proposition 2.3.1, it is straightforward to show that

$$\beta_j = O(L^{-j(d-4)-s(j-j_m)_+}). \tag{2.5.25}$$

Thus, a crude approximation to the flow of $\bar{g}$ is the recursion

$$y_{j+1} = y_j - c\mathbb{1}_{j\leq j_m} y_j^2, \quad c > 0. \tag{2.5.26}$$

Comparing this to the differential equation $\dot{y} = -cy^2$, which has solutions of the form $y(t) = (C + ct)^{-1}$, it is reasonable to expect that $y_j \approx (cj)^{-1}$ for $j \leq j_m$. By definition, $y_j = y_{j_m}$ for $j > j_m$. Thus, $y_j \approx (c \log m^{-1})^{-1}$ for $j \geq j_m$. A similar argument can be used to study y_{j_x}.

Following [9, (6.15)], we define the parameter $\tilde{g}_j$ in (2.4.21) as a function of two variables $(\tilde{m}^2, \tilde{g}_0)$ by

$$\tilde{g}_j(\tilde{m}^2, \tilde{g}_0) = \bar{g}_j(0, \tilde{g}_0)\mathbb{1}_{j\leq j_{\tilde{m}}} + \bar{g}_{j_{\tilde{m}}}(0, \tilde{g}_0)\mathbb{1}_{j>j_{\tilde{m}}}. \tag{2.5.27}$$

These parameters play an important role in Section 2.7 and in the proof of Theorem 2.8.1.

The flow of λ and q

It was shown in [10, (3.34)–(3.35)] (for $n = 0$) and [108, Proposition 3.2] (for $n \geq 1$) that, with $C = C_{j+1}$ and $u = 0, x$,

$$\lambda_{u,\mathrm{pt}} = \begin{cases} (1 - \delta[\nu w^{(1)}])\lambda_u, & j + 1 < j_x \\ \lambda_u, & j + 1 \geq j_x \end{cases} \tag{2.5.28}$$

$$q_{\mathrm{pt}} = q + \lambda_0 \lambda_x C_{0x}, \tag{2.5.29}$$

where

$$\delta[\nu w^{(1)}] = (\nu + 2gC_{00})w_{j+1}^{(1)} - \nu w_j^{(1)}, \qquad w_j^{(1)} = \sum_{x\in\Lambda} \sum_{i=1}^{j} C_{i;0x}. \tag{2.5.30}$$

Note that $q_{\mathrm{pt}} = q$ for $j + 1 < j_x$.

2.6 Non-perturbative coordinate

Let $\mathbb{V}$ denote either Λ_N or $\mathbb{Z}^d$. We allow $\mathcal{N}$ to depend on $\mathbb{V}$. If $\mathbb{V} = \Lambda$, then $\mathcal{N} = \mathcal{N}(\Lambda)$ was defined in Section 2.4. Otherwise, we set

$$\mathcal{N}(\mathbb{Z}^d) = \bigcup_{\text{finite } X\subset\mathbb{V}} \mathcal{N}(X). \tag{2.6.1}$$

We set $N(\mathbb{V}) = N$ if $\mathbb{V} = \Lambda_N$ and $N(\mathbb{V}) = \infty$ if $\mathbb{V} = \mathbb{Z}^d$. For $j \le N(\mathbb{V})$ (meaning $j < \infty$ if $N(\mathbb{V}) = \infty$), we partition $\mathbb{V}$ into disjoint *scale-j blocks* of side length L^j, each of which is a translate of the block $\{x \in \Lambda : 0 \le x_i < L^j, i = 1, \dots, d\}$. A scale-$j$ *polymer* is a union of scale-j blocks. Given a scale-j polymer X and $k \le j$, we let $\mathcal{B}_k(X)$ (respectively, $\mathcal{P}_k(X)$) denote the set of all scale-k blocks (respectively, scale-k polymers) in X. We sometimes write $\mathcal{B}_j = \mathcal{B}_j(\mathbb{V})$ and $\mathcal{P}_j = \mathcal{P}_j(\mathbb{V})$ when the volume $\mathbb{V}$ is implicit.

Any map $F : \mathcal{B}_j \to \mathcal{N}$ can be extended to a map on $\mathcal{P}_j$ by *block-factorization*:

$$F(X) = F^X := \prod_{B \in \mathcal{B}_j(X)} F(B). \tag{2.6.2}$$

Given maps $F, G : \mathcal{P}_j(\Lambda) \to \mathcal{N}$ (sometimes called *polymer activities*), we define the *circle product* $F \circ G : \mathcal{P}_j(\Lambda) \to \mathcal{N}$ by

$$(F \circ G)(X) = \sum_{Y \in \mathcal{P}_j(X)} F(X \setminus Y) G(Y). \tag{2.6.3}$$

The circle product is commutative, associative, and has the identity element

$$\mathbb{1}_\varnothing(X) = \begin{cases} 1, & X = \varnothing \\ 0, & X \neq \varnothing \end{cases}. \tag{2.6.4}$$

We track Z_j using *renormalization group coordinates* $u_j, q_{0,j}, q_{x,j} \in \mathbb{R}$, $I_j, K_j : \mathcal{P}_j \to \mathcal{N}$ such that

$$Z_j = e^{\zeta_j}(I_j \circ K_j)(\Lambda), \qquad \zeta_j = -u_j|\Lambda| + \tfrac{1}{2}(q_{0,j} + q_{x,j})\sigma_0\sigma_x. \tag{2.6.5}$$

The coordinate $I_j = I_j(V, \cdot)$ is defined by setting

$$I_j(V, B) = e^{-V(B)}(1 + W_j(B, V)), \quad X \in \mathcal{P}_j, \quad V \in \mathcal{V} \tag{2.6.6}$$

and extending this by block-factorization. Thus, (2.6.5) gives a rigorous implementation of (2.5.17).

Before defining the space in which K_j lies, we need the following notions:

- We call a nonempty polymer $X \in \mathcal{P}_j$ *connected* if for any $x, x' \in X$, there is a sequence $x = x_0, \dots, x_n = x' \in X$ such that $|x_{i+1} - x_i|_\infty = 1$ for $i = 0, \dots, n-1$. Let $\mathcal{C}_0 = \mathcal{C}_0(\mathbb{V})$ denote the set of connected polymers.

- For $X \in \mathcal{P}_j$, let $|X|_j$ denote the number of scale-j blocks in X. We call a connected polymer $X \in \mathcal{C}_j$ a *small set* if $|X|_j \le 2^d$. Let $\mathcal{S}_j = \mathcal{S}_j(\mathbb{V})$ denote the collection of small sets. The *small set neighbourhood* $X^\square$ of a polymer X is defined by

$$X^\square = \bigcup_{Y \in \mathcal{S}_j : Y \cap X \neq \varnothing} Y. \tag{2.6.7}$$

- Two polymers X, Y *do not touch* if $\min(|x-y|_\infty : x \in X, y \in Y) > 1$. We let $\mathrm{Comp}(X)$ denote the set of maximal connected components that do not touch in X.

We say that a map $F : \mathcal{P}_j \to \mathcal{N}$ is *Euclidean-covariant* if $E(F(X)) = F(EX)$ for all $X \in \mathcal{P}_j$ and all automorphisms E of $\mathbb{V}$. We also say that F is *gauge-invariant*, *supersymmetric*, or *$O(n)$-invariant* if $F(X)$ is gauge-invariant, supersymmetric, or $O(n)$-invariant, respectively.

Definition 2.6.1. For $j \le N(\mathbb{V})$, let $\mathcal{CK}_j = \mathcal{CK}_j(\mathbb{V})$ denote the real vector space of maps $K : \mathcal{C}_j(\mathbb{V}) \to \mathcal{N}(\mathbb{V})$ satisfying the following properties:

- Field Locality: If $X \in \mathcal{C}_j$, then $K(X) \in \mathcal{N}(X^\square)$. Also: (i) $\pi_\alpha K(X) = 0$ unless $\alpha \in X$ for $\alpha = 0, x$; (ii) $\pi_{0x} K(X) = 0$ unless $a \in X$ and $x \in X^\square$ or vice versa; and (iii) $\pi_{0x} K(X) = 0$ if $X \in \mathcal{S}_j$ and $j < j_x$.
- Symmetry: (i) $\pi_\varnothing K$ is Euclidean-covariant; (ii) if $n = 0$, then K is gauge-invariant and $\pi_\varnothing K$ is supersymmetric and has no constant part; if $n \ge 1$, then $\pi_\varnothing K$ is $O(n)$-invariant.

We let $\mathcal{K}_j = \mathcal{K}_j(\mathbb{V})$ denote the real vector space of functions $K \in \mathcal{CK}_j$ with the following additional property:

- Component factorization: If $X \in \mathcal{P}_j$, then $K(X) = \prod_{Y \in \mathrm{Comp}(X)} K(Y)$.

Addition in $\mathcal{CK}_j$ is defined by $(F_1 + F_2)(X) = F_1(X) + F_2(X)$. We extend any $F \in \mathcal{CK}_j$ to $\mathcal{K}_j$ by defining $F(X) = \prod_{Y \in \mathrm{Comp}(X)} F(Y)$.

2.6.1 Initial coordinates

At scale $j = 0$, we are given V_0^+ as defined in (2.2.2) and we set $\zeta_0 = 0$. In particular, the initial values of u, q_0, q_x are zero, and the initial values of λ_0, λ_x are 1. By definition, $W_0 = 0$. For $X \subset \Lambda$, we define

$$I_0^+(X) = I_0(V_0^+, X) = \prod_{y \in X} e^{-V_{0,y}^+}, \qquad K_0^+(X) = \prod_{y \in X} I_{0,y}^+(e^{-\gamma_0 U_y^+} - 1), \tag{2.6.8}$$

where $I_{0,y}^+ = I_0^+(\{y\})$. It is straightforward to verify that $K_0 \in \mathcal{K}_0$. Moreover, by (2.2.5),

$$Z_0 = \prod_{y \in \Lambda} \Big(I_{0,y}^+ + I_{0,y}^+(e^{-\gamma_0 U_y^+} - 1) \Big) = (I_0^+ \circ K_0^+)(\Lambda). \tag{2.6.9}$$

The second equality here follows from the binomial expansion formula

$$\prod_{y \in \Lambda} (F_y + G_y) = \sum_{X \subset \Lambda} \Big(\prod_{y \in \Lambda \setminus X} F_y \Big) \Big(\prod_{z \in X} G_z \Big). \tag{2.6.10}$$

Thus, Z_0 takes the form (2.6.5) and we seek $(u_j, q_{0,j}, q_{x,j}, V_j, K_j)$ such that this continues to hold as the scale advances.

Equivalently, given (V_j, K_j), we must define $(\delta u_{j+1}, \delta q_{0,j+1}, \delta q_{x,j+1}, V_{j+1}, K_{j+1})$ so that

$$\mathbb{E}_{j+1}\theta(I_j \circ K_j)(\Lambda) = e^{-\delta\zeta_{j+1}}(I_{j+1} \circ K_{j+1})(\Lambda). \tag{2.6.11}$$

Moreover, we need K_j to contract as the scale advances, under an appropriate norm. The construction of (scale-dependent) maps U_+ and K_+ such that (2.6.11) holds with

$$(\delta u_{j+1}, \delta q_{0,j+1}, \delta q_{x,j+1}, V_{j+1}) = U_+(V_j, K_j), \quad K_{j+1} = K_+(V_j, K_j) \tag{2.6.12}$$

is the main accomplishment of [31].

2.7 Renormalization group step

In [31, Section 1.7.3], a sequence of norms $\|\cdot\|_{\mathcal{W}_j} = \|\cdot\|_{\mathcal{W}_j(\tilde{m}^2,\tilde{g}_j,\Lambda)}$ parameterized by $(\tilde{m}^2, \tilde{g}_j)$ is defined on $\mathcal{K}_j$. These are defined in terms of the $T_\phi(\mathfrak{h}_j)$ norms with parameters $\mathfrak{h}_j = \ell_j, h_j$. In order to make use of the improved norm parameters with $s > 1$, we must modify the definition of $\mathcal{W}_j$ when j is above the mass scale. The precise definition of the $\mathcal{W}_j$ norm adapted to our current setting will be discussed in Section 5.1.3. We note here only the fact that the $\mathcal{W}_j(\Lambda)$ norm dominates the $T_0(\ell_j)$ norm in the following sense:

$$\|F(\Lambda)\|_{T_0(\ell_j)} \le \|F\|_{\mathcal{W}_j}. \tag{2.7.1}$$

We let $\mathcal{W}_j = \mathcal{W}_j(\mathbb{V})$ denote the space of $K \in \mathcal{K}_j(\mathbb{V})$ with finite $\mathcal{W}_j$ norm and denote the ball of radius r in the normed space $\mathcal{W}_j$ by $B_{\mathcal{W}_j}(r)$.

Let

$$j_\Omega = j_\Omega(m^2) = \inf\{k \ge 0 : |\beta_j| \le 2^{-(j-k)}\|\beta\|_\infty \text{ for all } j\} \tag{2.7.2}$$

and note that, by (2.5.25), $j_\Omega < \infty$ for $m^2 > 0$. We define

$$\vartheta_j = \vartheta_j(m^2) = 2^{-(j-j_\Omega)_+} \tag{2.7.3}$$

and write $\tilde{\vartheta}_j = \vartheta_j(\tilde{m}^2)$. Given constants $\alpha > 0$ and $C_{\mathcal{D}} > 0$, we define the (finite-volume) renormalization group domains

$$\mathcal{D}_j = \{V \in \mathcal{V} : g > C_{\mathcal{D}}^{-1}\tilde{g}_j,\ \|V\|_{\mathcal{U}} < C_{\mathcal{D}}\tilde{g}_j\}, \tag{2.7.4}$$

$$\mathbb{D}_j = \mathbb{D}_j(\mathbb{V}) = \mathcal{D}_j \times B_{\mathcal{W}_j}(\alpha\tilde{\vartheta}_j\tilde{g}_j^3). \tag{2.7.5}$$

The domain $\mathcal{D}_j$ is independent of the volume $\mathbb{V}$ while $\mathbb{D}_j$ depends on $\mathbb{V}$ through $\mathcal{W}_j$.

In the statement of the following theorem, we fix the scale j and consider maps $U_+ = U_{j+1}$ and $K_+ = K_{j+1}$ that act on the domain $\mathbb{D}_j$ and map into $\mathcal{U}$, $\mathcal{K}_{j+1}$, respectively. We will drop the scale j from the notation for objects at scale j and replace $j+1$ with $+$. When $\mathbb{V} = \Lambda$, we

take $j < N$. The deviation of the map U_+ from the perturbative map U_{pt} is denoted by R_+:

$$R_+(V,K) = U_+(V,K) - U_{\mathrm{pt}}(V). \tag{2.7.6}$$

The renormalization group map depends also on the mass m^2 through its dependence on the covariance C_{j+1}. We let $\tilde{\mathbb{I}}_j(\tilde{m}^2)$ be the neighbourhood of $\tilde{m}^2$ defined by

$$\tilde{\mathbb{I}}_j = \tilde{\mathbb{I}}_j(\tilde{m}^2) = \begin{cases} [\frac{1}{2}\tilde{m}^2, 2\tilde{m}^2] \cap \mathbb{I}_j & (\tilde{m}^2 \neq 0) \\ [0, L^{-2(j-1)}] \cap \mathbb{I}_j & (\tilde{m}^2 = 0) \end{cases}, \tag{2.7.7}$$

where $\mathbb{I}_j = [0,\delta]$ if $j < N$ and $\mathbb{I}_N = [\delta L^{-2(N-1)}, \delta]$.

Theorem 2.7.1. *Let $d = 4$, $n \geq 0$, and $\mathbb{V} = \Lambda$ or $\mathbb{Z}^d$. Fix $s > 1$ (or $s = 0$). Let $C_{\mathcal{D}}$ and L be sufficiently large. There exist $M > 0$, $\delta > 0$, and $\kappa = O(L^{-1})$ such that for $\tilde{g} \in (0,\delta)$ and $\tilde{m}^2 \in \mathbb{I}_+$, and with the domain $\mathbb{D}$ defined using any $\alpha > M$, the maps*

$$R_+ : \mathbb{D} \times \tilde{\mathbb{I}}_+ \to \mathcal{U}, \quad K_+ : \mathbb{D} \times \tilde{\mathbb{I}}_+ \to \mathcal{W}_+ \tag{2.7.8}$$

are analytic in (V,K) and satisfy the estimates

$$\|R_+\|_{\mathcal{U}} \leq M\tilde{\vartheta}_+\tilde{g}_+^3, \qquad \|K_+\|_{\mathcal{W}_+} \leq M\tilde{\vartheta}_+\tilde{g}_+^3 \tag{2.7.9}$$

and

$$\|D_K K_+\|_{L(\mathcal{W},\mathcal{W}_+)} \leq \kappa. \tag{2.7.10}$$

When $\mathbb{V} = \Lambda$, these maps define $(V,K) \mapsto (U_+,K_+)$ obeying (2.6.11).

Remark 2.7.2. When the improved weights are used, a new norm must be employed above the mass scale. This will be discussed in Section 4.2.2. A technical requirement of this new norm is that we set $s > 1$ rather than $s > 0$ as Lemma 4.2.2 fails with $s \in (0,1)$. This issue is absent when $s = 0$ as we do not change the norms in this case. Since we are ultimately interested only in the cases $s = 0$ and s large, we have not attempted to handle the case $s \in (0,1)$.

With $s = 0$ in the choice of weights ℓ_j and $\ell_{\sigma,j}$, this theorem was the main achievement of [31]. The statement in [31] with $s = 0$ additionally contains bounds on the derivatives of the maps R_+ and K_+. Our improvements apply to these bounds as well, but we do not state them here as we will not make direct use of these bounds. One of the main novelties in this thesis is the case $s > 1$, for which the bounds on the observables derived from (2.7.9) are greatly improved beyond the mass scale.

Note that the maps R_+ and K_+ themselves are *independent* of s. The proof of Theorem 2.7.1 involves showing that the inductive estimates (2.7.9) hold for any s. In some cases, we will make use of these estimates both with $s > 1$ and $s = 0$. The proof for $s > 1$ is an adaptation of

the proof of the $s = 0$ case contained in[2] [30, 31]. Some steps in this proof continue to hold unchanged whereas others require some modification. As mentioned above, a major change that is required is a new definition of $\mathcal{W}_j$ above the mass scale. A detailed verification that the proof holds for $s > 1$ is carried out in Chapter 4.

2.8 Renormalization group flow

Theorem 2.7.1 allows us to perform a single renormalization group step. The fact that K_+ is a contraction, as expressed by the estimate (2.7.10), was used in [9, Proposition 7.1] to construct *critical* initial conditions ν_0^c, z_0^c depending on (m^2, g_0, n) such that the renormalization group map can be iterated indefinitely (this was shown for $n = 0$ in [9] but extends without difficulty to $n \geq 1$ as discussed in [7]). This results in a sequence (U_j, K_j) generated by the renormalization group map, hence whose elements lie in the domains $\mathbb{D}_j$. This was proved with $s = 0$, but the sequence itself is independent of s and continues to exist in our setting. In particular, Theorem 2.7.1 shows that this sequence satisfies improved estimates. Thus, there is no difficulty in extending [9, Proposition 7.1] to the s-dependent domains used here.

However, in order to study the WSAW-SA and the generalized $|\varphi|^4$ model, we must extend [9, Proposition 7.1] to $\gamma_0 \neq 0$. We state this extension as Theorem 2.8.1 below, which is one of the main contributions of this thesis. Its proof, which depends on the results of [11] together with a specially tailored version of the implicit function theorem, is the subject of Chapter 5. We note that, for $n = 0$, this proof first appeared in [13]; the proof for $n \geq 1$ is new to this thesis.

Let $\delta > 0$ and suppose $r : [0, \delta] \to [0, \infty)$ is a continuous *positive-definite* function; by this we mean[3] that $r(x) > 0$ if $x > 0$ and $r(0) = 0$. We define

$$D(\delta, r) = \{(w, x, y) \in [0, \delta]^3 : y \leq r(x)\} \tag{2.8.1}$$

and we let $C^{0,1,+}(D(\delta, r))$ denote the space of continuous functions $f = f(w, x, y)$ on $D(\delta, r)$ that are C^1 in (x, y) away from $y = 0$, C^1 in x everywhere, and whose right-derivative in y at $y = 0$ exists. In our applications, we take $w = m^2$, $x = g_0$ or g, and $y = \gamma_0$ or γ.

Theorem 2.8.1. *There exists a domain $D(\delta, \hat{r})$ (with $\delta > 0$ and $\hat{r}$ positive-definite) and functions $\hat{\nu}_0^c, \hat{z}_0^c \in C^{0,1,+}(D(\delta, \hat{r}))$ such that for any $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$ with $g_0 > 0$ and $m^2 \in [\delta L^{-2(N-1)}, \delta)$, the following holds: if $(U_0, K_0) = (V_0^+, K_0^+)$ with $(\nu_0, z_0) = (\hat{\nu}_0^c, \hat{z}_0^c)$, then for any $N \in \mathbb{N}$, there exists a sequence $(U_j, K_j) \in \mathbb{D}_j(m^2, g_0)$ such that*

$$(U_{j+1}, K_{j+1}) = (U_{j+1}(V_j, K_j), K_{j+1}(V_j, K_j)) \text{ for all } j < N \tag{2.8.2}$$

[2] For $n \geq 1$, there is an additional step to deal with observables. This is dealt with in the proof of [108, Theorem 5.1] and is unchanged in the present context.

[3] Note that our usage of this term is different from that in the theory of quadratic forms.

and (2.6.11) *is satisfied. Moreover, the sequence* $U_j, j = 1, \dots, N$ *is independent of the volume* Λ *and*

$$\hat{\nu}_0^c = O(g_0), \quad \hat{z}_0^c = O(g_0) \tag{2.8.3}$$

uniformly in (m^2, γ_0).

Note that in the statement of Theorem 2.8.1 flow, we have evaluated the domains $\mathbb{D}_j$ at $(\tilde{m}^2, \tilde{g}_0) = (m^2, g_0)$, where m^2 is the mass in the covariance C and $g_0 = g(1 + z_0^c)^2$.

2.9 Bibliographic remarks

The notion of a polymer used in Section 2.6 was introduced in [63]. The utility of multi-scale decompositions of a singular covariance as a sum of regular covariances in the context of the renormalization group was probably first clearly articulated in [14]. The use of expansions of the form (2.6.3) together with carefully weighted norms to achieve rigorous control of the renormalization group map goes back to Brydges and Yau [33]. This method was extended by Dimock and Hurd, see e.g. [39,40]. Finite-range decompositions were first used with this method to study a continuum model in [94], following a suggestion of Brydges. *Lattice* covariance decompositions were constructed in [22] and used in [93] to study the renormalization group flow for the supersymmetric field theory corresponding to WSAW; however, critical exponents were not computed. Critical exponents for a version of weakly self-avoiding walk on a hierarchical lattice were computed by a renormalization group method in [24,25] (such hierarchical models go back to [44]).

Chapter 3

Analysis of critical behaviour

In this chapter, we prove Theorem 1.7.1 using Theorem 2.8.1. For simplicity, we drop the parameter n from the notation. In order to employ Theorem 2.8.1, we fix

$$\nu_0 = \hat{\nu}_0^c(m^2, g_0, \gamma_0), \quad z_0 = \hat{z}_0^c(m^2, g_0, \gamma_0). \tag{3.0.1}$$

Then Theorem 2.8.1 defines a sequence

$$(U_j, K_j) \in \mathbb{D}_j, \quad 0 \le j \le N \tag{3.0.2}$$

for any N. Moreover, U_j is independent of the volume, so we actually have an *infinite* sequence

$$U_j \in \mathcal{D}_j \quad j \ge 0. \tag{3.0.3}$$

Throughout this section we write U_j as

$$U_{j;x} = g_j\tau_y^2 + \nu_j\tau_y + z_j\tau_{\Delta,y} + u_j - \lambda_{0,j}f_0\mathbb{1}_{y=0} - \lambda_{x,j}f_x\mathbb{1}_{y=x} - \tfrac{1}{2}(\mathbb{1}_{y=0}q_{0,j} + \mathbb{1}_{y=x}q_{x,j})\sigma_0\sigma_x. \tag{3.0.4}$$

3.1 Susceptibility

The proof of Theorem 1.7.1(ii) involves some small changes to the proof of the $\gamma_0 = 0$ case in [9]. Rather than specifying the individual changes that need to be made, here we sketch the complete argument.

Since the only scale-N blocks are the empty set and Λ, at scale $j = N$ the representation (2.6.5) becomes

$$Z_N = e^{\zeta_N}(I_N(\Lambda) + K_N(\Lambda)). \tag{3.1.1}$$

In particular, (2.7.4)–(2.7.5), (2.5.10), (2.5.20), and (2.7.1) imply that

$$u_N|\Lambda_N| = O(1), \qquad \nu_N = O(L^{-2N}g_N), \tag{3.1.2}$$

$$\|W_N\|_{T_0(\ell_N)} \le O(\vartheta_N g_N^2), \qquad \|K_N\|_{T_0(\ell_N)} \le O(\vartheta_N g_N^3). \tag{3.1.3}$$

Now by (3.1.1) and (2.2.11) together with the definitions of I_N and V_N,

$$\hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0^c, z_0^c) = \frac{1}{m^2} + \frac{1}{m^4|\Lambda|}\frac{-\nu_N|\Lambda| + D^2W_N^0(0;\mathbb{1},\mathbb{1}) + D^2K_N^0(0;\mathbb{1},\mathbb{1})}{(1 + W_N^0(0) + K_N^0(0))}. \tag{3.1.4}$$

Remark 3.1.1. In fact, with a bit more work it can be shown that $D^2 W_N^0(0; \mathbb{1}, \mathbb{1}) = 0$. However, we will not need this here.

Using Lemma 2.4.2 with $s = 0$ (recall (2.4.21)) together with (3.1.3), we see that the last term vanishes as $N \to \infty$ leaving

$$\hat{\chi}(m^2, g_0, \gamma_0, \nu_0, z_0) = \lim_{N \to \infty} \hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0) = \frac{1}{m^2}. \tag{3.1.5}$$

In order to identify the asymptotics of m^2 as ν approaches the critical point, we will need information about the derivative of $\hat{\chi}$ with respect to ν_0. Let us denote by F' the derivative of a function F with respect to ν_0. By (3.1.4), the derivative $\hat{\chi}_N'$ will contain a term $-\nu_N'/m^4$. An argument using Lemma 2.4.2 shows that the remaining terms are of strictly higher order. Together with a careful analysis of the derivatives of the renormalization group flow with respect to the initial condition ν_0 (as in [9, Section 8] for $\gamma = 0$), we get

$$\hat{\chi}'(m^2, g_0, \gamma_0, \nu_0^c, z_0^c) \sim -\frac{1}{m^4} \frac{c(\hat{g}_0, \gamma_0)}{(\hat{g}_0 \mathsf{B}_{m^2})^{(n+2)/(n+8)}} \quad \text{as } (m^2, g_0, \gamma_0) \to (0, \hat{g}_0, \hat{\gamma}_0), \tag{3.1.6}$$

where c is a continuous function. The bubble diagram B_{m^2} was defined in (1.6.26) and its logarithmic divergence as $m^2 \downarrow 0$ is ultimately the source of the logarithmic corrections in Theorem 1.7.1.

Remark 3.1.2. There is one aspect of the proof of (3.1.6) that must be modified when $\gamma_0 = 0$: This is the verification of the third bound in the base case ($j = 0$) of the inductive hypothesis [9, (8.34)]. This will be done in Section 5.1.4 (see Remark 5.1.7).

3.1.1 Change of parameters

We wish to recover the asymptotics of χ from (3.1.5) and (3.1.6). By (2.2.10),

$$\chi_N(g, \gamma, \nu) = (1 + z_0) \hat{\chi}_N(m^2, g_0, \gamma_0, \nu_0, z_0), \tag{3.1.7}$$

whenever the variables on the left- and right-hand sides satisfy

$$g_0 = (g_0 - \gamma)(1 + z_0)^2, \quad \nu_0 = \nu(1 + z_0) - m^2, \quad \gamma_0 = \frac{1}{4d} \gamma (1 + z_0)^2. \tag{3.1.8}$$

On the other hand, (3.1.5) is contingent on the initialization of the renormalization group with the critical parameters

$$\nu_0 = \hat{\nu}_0^c(m^2, g_0, \gamma_0), \quad z_0 = \hat{z}_0^c(m^2, g_0, \gamma_0). \tag{3.1.9}$$

Given g, γ, ν, the relations (3.1.8) leave free two of the variables $(m^2, g_0, \gamma_0, \nu_0, z_0)$. More generally, if any three of the variables $(g, \gamma, \nu, m^2, g_0, \gamma_0, \nu_0, z_0)$ are fixed, then two of the remaining variables are free. In the following two propositions, which together form an extension

of [9, Proposition 4.2], we fix three variables and show that the addition of the constraints (3.1.9) allows us to uniquely specify the two remaining variables. For this, we make use of the following version of the implicit function theorem, which we prove in Appendix C.

Proposition 3.1.3. *Let $\delta > 0$, and let r_1, r_2 be continuous positive-definite functions on $[0, \delta]$. Recalling (2.8.1), set*

$$D(\delta, r_1, r_2) = \{(w, x, y, z) \in D(\delta, r_1) \times \mathbb{R}^n : |z| \le r_2(x)\}, \tag{3.1.10}$$

and let F be a continuous function on $D(\delta, r_1, r_2)$ that is C^1 in (x, z). Suppose that for all $(\bar{w}, \bar{x}) \in [0, \delta]^2$ there exists $\bar{z}$ such that both $F(\bar{w}, \bar{x}, 0, \bar{z}) = 0$ and $D_Y F(\bar{w}, \bar{x}, 0, \bar{z})$ is invertible. Then there is a continuous positive-definite function r on $[0, \delta]$ and a continuous map $f : D(\delta, r) \to \mathbb{R}^n$ that is C^1 in x and such that $F(w, x, y, f(w, x, y)) = 0$ for all $(w, x, y) \in D(\delta, r)$. Moreover, if F is left-differentiable (respectively, right-differentiable) in y at some point (w, x, y, z), then f is left-differentiable (respectively, right-differentiable) at (w, x, y).

Our first application of this result is Proposition 3.1.4, in which the three fixed variables are (m^2, g_0, γ).

Proposition 3.1.4. *There exist $\delta_* > 0$, a continuous positive-definite function $r_* : [0, \delta_*] \to [0, \infty)$, and continuous functions $(\nu^*, g_0^*, \gamma_0^*, \nu_0^*, z_0^*)$ defined for $(m^2, g, \gamma) \in D(\delta_*, r_*)$, such that (3.1.8) and (3.1.9) hold with $\nu = \nu^*$ and $(g_0, \gamma_0, \nu_0, z_0) = (g_0^*, \gamma_0^*, \nu_0^*, z_0^*)$. Moreover,*

$$g_0^* = g_0 + O(g_0^2), \quad \nu_0^* = O(g_0), \quad z_0^* = O(g_0). \tag{3.1.11}$$

Proof. Suppose we have found the desired continuous functions (g_0^*, γ_0^*) and that g_0^* satisfies the first bound in (3.1.11). Then the functions defined by

$$\nu_0^* = \hat{\nu}_0^c(m^2, g_0^*, \gamma_0^*), \quad z_0^* = \hat{z}_0^c(m^2, g_0^*, \gamma_0^*), \quad \nu^* = \frac{\nu_0^* + m^2}{1 + z_0^*} \tag{3.1.12}$$

are continuous, satisfy (3.1.8), and satisfy the remaining bounds in (3.1.11) by (2.8.3).

In order to construct (g_0^*, γ_0^*), we first solve the third equation of (3.1.8), and then solve the first equation of (3.1.8). To this end, we begin by defining

$$f_1(m^2, g_0, \gamma, \gamma_0) = \gamma_0 - (4d)^{-1}\gamma(1 + \hat{z}_0^c(m^2, g_0, \gamma_0))^2 \tag{3.1.13}$$

for $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$ and $|\gamma| \le \hat{r}(g_0)$. Although f_1 is well-defined for any $\gamma \in \mathbb{R}$, we restrict the domain in preparation for our application of Proposition 3.1.3. Note that f_1 is C^1 in γ and $f_1(\cdot, \cdot, \gamma, \cdot) \in C^{0,1,+}(D(\delta, \hat{r}))$ for any γ. The equation $f_1(m^2, g_0, \gamma, \gamma_0) = 0$ has the solution $\gamma_0 = 0$ when $\gamma = 0$ and, for any $\gamma_0 \neq 0$,

$$\frac{\partial f_1}{\partial \gamma_0} = 1 - (2d)^{-1}\gamma(1 + \hat{z}_0^c(m^2, g_0, \gamma_0))\frac{\partial \hat{z}_0^c}{\partial \gamma_0}. \tag{3.1.14}$$

By Theorem 2.8.1, the one-sided γ_0 derivatives of $\hat{z}_0^c$ exist at $\gamma_0 = 0$. Thus, the γ_0 derivative of f_1 is well-defined and equal to 1 when $\gamma = 0$ for any small γ_0 (including $\gamma_0 = 0$). It follows by Proposition 3.1.3 (with $w = m^2$, $x = g_0$, $y = \gamma$, $z = \gamma_0$ and $r_1 = r_2 = \hat{r}$) that there exists a continuous function $\gamma_0^{(1)}(m^2, g_0, \gamma)$ on $D(\delta, r^{(1)})$ (for some continuous positive-definite function $r^{(1)}$ on $[0, \delta]$) such that $f_1(m^2, g_0, \gamma, \gamma_0^{(1)}) = 0$. Moreover, $\gamma_0^{(1)}$ is C^1 in (g_0, γ).

Next, we define

$$f_2(m^2, g, \gamma, g_0) = g_0 - (g - \gamma)(1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)}(m^2, g, \gamma)))^2 \tag{3.1.15}$$

for $(m^2, g_0, \gamma) \in D(\delta, r^{(1)})$ and $g \in [0, \delta_*]$, where $\delta_* > 0$ will be made sufficiently small below. Then $f_2(m^2, g, \gamma, g_0) = 0$ is solved by $(\gamma, g_0) = (0, g_0^*(m^2, g, 0))$, where $g_0^*(m^2, g, 0)$ was constructed in [9, (4.35)]. By [9, (4.37)], $g_0^* = g + O(g^2)$, so we may restrict the domain of f_2 so that $|g_0| \le 2g$. Moreover,

$$\frac{\partial f_2}{\partial g_0} = 1 - 2(g - \gamma)(1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)})) \left(\frac{\partial \hat{z}_0^c}{\partial g_0} + \frac{\partial \hat{z}_0^c}{\partial \gamma_0} \frac{\partial \gamma_0^{(1)}}{\partial g_0} \right). \tag{3.1.16}$$

Differentiating both sides of

$$\gamma_0^{(1)} = \frac{1}{4d} \gamma (1 + \hat{z}_0^c(m^2, g_0, \gamma_0^{(1)}))^2, \tag{3.1.17}$$

and solving for $\frac{\partial \gamma_0^{(1)}}{\partial g_0}$, gives

$$\frac{\partial \gamma_0^{(1)}}{\partial g_0} = \frac{\gamma (1 + \hat{z}_0^c) \frac{\partial \hat{z}_0^c}{\partial g_0}}{2d - \gamma (1 + \hat{z}_0^c) \frac{\partial \hat{z}_0^c}{\partial \gamma_0}}, \tag{3.1.18}$$

where $\hat{z}_0^c$ and its derivatives are evaluated at $(m^2, g_0, \gamma_0^{(1)})$. Thus, $\frac{\partial \gamma_0^{(1)}}{\partial g_0} = 0$ when $\gamma = 0$. It follows that $\partial f_2 / \partial g_0$ is well-defined when $(\gamma, g_0) = (0, g_0^*(m^2, g, 0))$ and equals

$$1 - 2g(1 + \hat{z}_0^c(m^2, g_0^*, 0)) \frac{\partial \hat{z}_0^c}{\partial g_0}(m^2, g_0^*, 0), \tag{3.1.19}$$

which is positive when δ_* is small, by (2.8.3). Thus, by Proposition 3.1.3 (with $w = m^2$, $x = g$, $y = \gamma$, $z = g_0$ and $r_1 = r^{(1)}$, $r_2(g) = 2g$), there exists a function $g_0^*(m^2, g, \gamma) \in C^{0,1,+}(D(\delta_*, r^{(2)}))$ (for some continuous positive-definite function $r^{(2)}$ on $[0, \delta_*]$) such that $f_2(m^2, g, \gamma, g_0^*) = 0$.

By the fact that g_0^* solves $f_2 = 0$,

$$g_0^* = (g - \gamma) + O((g - \gamma)^2). \tag{3.1.20}$$

Since $|\gamma| \le r^{(2)}(g_0)$ and $r^{(2)}(g_0)$ can be taken as small as desired, this implies the first estimate in (3.1.11). Thus, by taking r_* sufficiently small, if $|\gamma| \le r_*(g_0)$, then $|\gamma| \le r^{(2)}(g_0^*(m^2, g, \gamma))$.

Thus, for $g < \delta_*$ and $|\gamma| \le r_*(g)$, we can define

$$\gamma_0^*(m^2, g, \gamma) = \gamma_0^{(1)}(m^2, g_0^*(m^2, g, \gamma), \gamma), \tag{3.1.21}$$

which completes the proof. ∎

Using Proposition 3.1.4, it is possible to identify the critical point ν_c, as follows. By (3.1.5), (3.1.7), Proposition 1.8.4, and Proposition 3.1.4,

$$\chi(g, \gamma, \nu^*) = \frac{1 + z_0^*}{m^2} = \frac{1 + O(g)}{m^2}. \tag{3.1.22}$$

Thus, with $\nu = \nu^*$, we see that $\chi < \infty$ when $m^2 > 0$, and $\chi = \infty$ when $m^2 = 0$. By (2.1.12), this implies that

$$\nu_c(g, \gamma) = \nu^*(0, g, \gamma) = O(g), \quad \nu_c(g, \gamma) < \nu^*(m^2, g, \gamma) \quad (m^2 > 0). \tag{3.1.23}$$

It follows that

$$\chi(g, \gamma, \nu_c) = \infty, \tag{3.1.24}$$

which is a fact that cannot be concluded immediately from the definition (2.1.12).

In (3.1.22), χ is evaluated at $\nu^* = \nu^*(m^2, g, \gamma)$. However, in the setting of Theorem 1.7.1, we need to evaluate χ at a *given* value of ν and then take $\nu \downarrow \nu_c$. To do so, we must determine a choice of m^2 in terms of ν such that (3.1.8) is satisfied and this choice must approach 0 (as it should by (3.1.23)) right-continuously as $\nu \downarrow \nu_c$. The following proposition carries out this construction. In the following, the functions $\tilde{m}^2, \tilde{g}_0$ should not be confused with the parameter $\tilde{m}^2, \tilde{g}_0$ that appeared previously in the $\mathcal{W}_j$ norms (these norms are not used in this chapter).

Proposition 3.1.5. *Write $\nu = \nu_c + \varepsilon$. There exist functions $\tilde{m}^2, \tilde{g}_0, \tilde{\gamma}_0, \tilde{\nu}_0, \tilde{z}_0$ of $(\varepsilon, g, \gamma) \in D(\delta_*, r_*)$ (all right-continuous as $\varepsilon \downarrow 0$) such that (3.1.8) and (3.1.9) hold with*

$$(m^2, g_0, \gamma_0, \nu_0, z_0) = (\tilde{m}^2, \tilde{g}_0, \tilde{\gamma}_0, \tilde{\nu}_0, \tilde{z}_0). \tag{3.1.25}$$

Moreover,

$$\tilde{m}^2(0, g, \gamma) = 0, \qquad \tilde{m}^2(\varepsilon, g, \gamma) > 0 \quad (\varepsilon > 0) \tag{3.1.26}$$

$$\tilde{g}_0 = g + O(g^2), \quad \tilde{\nu}_0 = O(g), \quad \tilde{z}_0 = O(g). \tag{3.1.27}$$

Proof. The proof is a minor modification of the proof in [9], using Proposition 3.1.4. Define

$$\tilde{m}^2 = \tilde{m}^2(\varepsilon, g, \gamma) = \inf\{m^2 > 0 : \nu^*(m^2, g, \gamma) = \nu_c(g, \gamma) + \varepsilon\}, \tag{3.1.28}$$

on $D(\delta_*, r_*)$. By continuity of ν^*, the infimum is attained and

$$\nu_c(g,\gamma) + \varepsilon = \nu^*(\tilde{m}^2(\varepsilon, g, \gamma), g, \gamma). \tag{3.1.29}$$

From the above expression, continuity of ν^*, and (3.1.23), it follows that $\tilde{m}^2$ is right-continuous as $\varepsilon \downarrow 0$. It is immediate that (3.1.26) holds. Also, the functions of (ε, g, γ) defined by

$$\tilde{\nu}_0 = \nu_0^*(\tilde{m}^2, g, \gamma), \quad \tilde{z}_0 = z_0^*(\tilde{m}^2, g, \gamma), \tag{3.1.30}$$

$$\tilde{g}_0 = (g-\gamma)(1+\tilde{z}_0)^2, \quad \tilde{\gamma}_0 = \frac{1}{4d}\gamma(1+\tilde{z}_0)^2 \tag{3.1.31}$$

are right-continuous as $\varepsilon \downarrow 0$ and satisfy (3.1.8). The bounds (3.1.27) follow from the definitions and (3.1.11), and the proof is complete. ∎

3.1.2 Conclusion of the argument

We sketch the remainder of the argument, which follows as in [9, Section 4]. By Proposition 1.8.4, (3.1.7), (3.1.5), and Proposition 3.1.5,

$$\chi(g,\gamma,\nu) = \frac{1+\tilde{z}_0}{\tilde{m}^2}. \tag{3.1.32}$$

Similarly, from (3.1.6) (using (3.1.32)), we get

$$\chi'(g,\gamma,\nu) \sim -\chi^2(g,\gamma,\nu)\frac{c_0(g,\gamma)}{(\tilde{g}_0 \mathsf{B}_{\tilde{m}^2})^{\frac{n+2}{n+8}}} \tag{3.1.33}$$

with $c_0(g,\gamma) = \lim_{\varepsilon\downarrow 0} c(\tilde{g}_0, \tilde{\gamma}_0)$. By exactly the same argument as in [9, Section 4.3], the differential relation (3.1.33) can be solved, which gives the result of Theorem 1.7.1(ii).

Remark 3.1.6. It is a consequence of (1.7.2) and (3.1.32) that

$$\tilde{m}^2 \sim \tilde{A}_{g,n}^{-1}\varepsilon(\log \varepsilon^{-1})^{-\frac{n+2}{n+8}} \quad \text{as } \varepsilon \downarrow 0. \tag{3.1.34}$$

3.2 Two-point function

Our analysis of the two-point function and finite-order correlation length is based on the following proposition.

Proposition 3.2.1. *Let $d = 4$, $n \geq 0$, $\varepsilon \in (0, \delta)$ with δ sufficiently small, and $\nu = \nu_c + \varepsilon$. Let $x \in \mathbb{Z}^4$ with $x \neq 0$. Fix $s = 0$ or $s > 1$. For L sufficiently large and for $g > 0$ sufficiently small (depending on s),*

$$\frac{1}{1+\tilde{z}_0} G_x(g,\gamma,\nu) = (1 + O(\bar{g}_{j_x}))G_x(0,0,\tilde{m}^2) + R_x(\tilde{m}^2) \tag{3.2.1}$$

and the remainder R_x satisfies the bound

$$|R_x(m^2)| \le \frac{O(\bar{g}_{j_x})}{|x|^2} \times \begin{cases} 1, & (m|x| \le 1) \\ (m|x|)^{-2s}, & (m|x| \ge 1) \end{cases} \tag{3.2.2}$$

with the constant depending on L and s.

Proof. Let D_{σ_0} and D_{σ_x} denote differentiation with respect to σ_0 and σ_x, respectively, evaluated with all fields set to 0. By (2.2.9), (3.1.1), and (2.6.5),

$$\frac{1}{1+\tilde{z}_0} G_{x,N}(g,\gamma,\nu) = \frac{1}{2}(q_{0,N} + q_{x,N}) + \frac{D^2_{\sigma_0\sigma_x} K^0_N}{1+K^0_N} - \frac{\left(D_{\sigma_0} K^0_N\right)\left(D_{\sigma_x} K^0_N\right)}{(1+K^0_N)^2}, \tag{3.2.3}$$

where the quantities on the right-hand side are evaluated at $(\tilde{m}^2, \tilde{g}_0, \tilde{\gamma}_0, \tilde{\nu}_0, \tilde{z}_0)$. No W_N term appears on the right-hand side since W_N is quadratic in V and V has no constant part. By (3.1.3) and Lemma 2.4.2, the last two terms vanish as $N \to \infty$ leaving

$$\frac{1}{1+\tilde{z}_0} G_x(g,\gamma,\nu) = \frac{1}{2}(q_{0,\infty} + q_{x,\infty}). \tag{3.2.4}$$

Now it is a straightforward computation using (2.5.28)–(2.5.30) and (2.7.6) to show that

$$q_{u,\infty} = \lambda_{0,j_x}\lambda_{x,j_x} G_x(0,0,\tilde{m}^2) + \sum_{i=j_x}^{\infty} R_i^{q_u}, \quad u = 0, x \tag{3.2.5}$$

where $R_i^{q_u}$ is the coefficient of $\mathbb{1}_{y=u}\sigma_0\sigma_x$ (recall (2.5.7)) in $R_{+,i}$. Moreover, as in [108, (5.30)] and [108, Corollary 6.4],

$$\lambda_{u,j_x} = 1 + O(\vartheta_{j_x}\bar{g}_{j_x}). \tag{3.2.6}$$

It follows that

$$\frac{1}{1+\tilde{z}_0} G_x(g,\gamma,\nu) = (1 + O(\bar{g}_{j_x})) G_x(0,\tilde{m}^2) + R_x \tag{3.2.7}$$

with

$$R_x = \frac{1}{2}\sum_{i=j_x}^{\infty} (R_i^{q_0} + R_i^{q_x}). \tag{3.2.8}$$

By the first bound of (2.7.9) and the definition (2.5.10) of the $\mathcal{V}$ norm,

$$|R_{+,i}^{q_u}| \le O(\ell_{\sigma,i}^{-2}\vartheta_i\bar{g}_i^3). \tag{3.2.9}$$

We insert the definition of $\ell_{\sigma,j}$ from (2.4.21) into (3.2.9). We also use $\tilde{g}_j^{-2} = O(\bar{g}_j^{-2})$, $\vartheta_i \le 1$, $\ell_0^2 \le O(1)$, as well as $\bar{g}_j \le O(\bar{g}_{j_x})$ for $j \ge j_x$. The definitions of the coalescence scale j_x and the

mass scale j_m imply that $L^{-2j_x} \le O(|x|^{-2})$ and $L^{-(j_x-j_m)_+} \le O((m|x|)^{-1})$. All this leads to

$$\sum_{j=j_x}^{\infty} |R_j^{q_u}| \le L^{-2j_x-2s(j_x-j_m)_+} \sum_{j=j_x}^{\infty} O(\bar{g}_j) 4^{-(j-j_x)} \le |x|^{-2}(m|x|)^{-2s} O(\bar{g}_{j_x}). \tag{3.2.10}$$

This gives the desired estimate (3.2.2). ∎

A version of this result with $s = 0$ and $\gamma = 0$ was obtained in [8, 108]. This version is sufficient for studying the *critical* two-point function with $\gamma = 0$. With the extension to $\gamma \neq 0$, we can complete the proof of the first part of Theorem 1.7.1.

Proof of Theorem 1.7.1(i). We apply Proposition 3.2.1 with $s = 0$ to get

$$\frac{1}{1+\tilde{z}_0} G_x(g,\gamma,\nu) = (1 + O(\bar{g}_{j_x})) G_x(0,0,\tilde{m}^2) + R_x(\tilde{m}^2). \tag{3.2.11}$$

By Proposition 3.1.5, $\tilde{m}^2 = 0$ when $\nu = \nu_c$. Since $R_x(0) = O(\bar{g}_{j_x}) G_x(0,0,0)$,

$$\frac{1}{1+\tilde{z}_0} G_x(g,\gamma,\nu_c) = (1 + O(\bar{g}_{j_x})) G_x(0,0,0) \tag{3.2.12}$$

and the result follows from (2.5.24). ∎

3.3 Finite-order correlation length

An elementary ingredient in the proof of Theorem 1.7.1(iii) is the following result for the $g = 0$ case, which is independent of $n \ge 0$. For simplicity, we restrict attention to dimensions $d > 2$, as only $d = 4$ is used here. A proof is provided in Appendix B.

Proposition 3.3.1. *Let c_p be the constant defined by (1.7.4). For all dimensions $d > 2$ and all $p > 0$, as $m^2 \downarrow 0$,*

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(0,0,m^2) = \mathsf{c}_p^p m^{-(p+2)} (1 + O(m)). \tag{3.3.1}$$

In particular, $\xi_p(0,0,\varepsilon) = \mathsf{c}_p \varepsilon^{-1/2}(1 + O(\varepsilon^{1/2}))$ as $\varepsilon \downarrow 0$.

Proof of Theorem 1.7.1. We multiply (3.2.1) by $|x|^p$, sum over $x \in \mathbb{Z}^4$, and use (3.1.22) to obtain

$$\xi_p^p(g,\gamma,\nu) = \sum_{x \in \mathbb{Z}^4} |x|^p \frac{G_x(g,\gamma,\nu)}{\chi(g,\gamma,\nu)} = \tilde{m}^2 \sum_{x \in \mathbb{Z}^4} |x|^p \Big(G_x(0,0,m^2) + r_x(\tilde{m}^2) \Big), \tag{3.3.2}$$

with

$$r_x(m^2) = O(\bar{g}_{j_x})G_x(0,0,m^2) + R_x(m^2). \tag{3.3.3}$$

By Proposition 3.3.1, this gives (as $\tilde{m}^2 \downarrow 0$)

$$\xi_p^p(g,\gamma,\nu) \sim \mathsf{c}_p^p \tilde{m}^{-p} + \tilde{m}^2 \sum_{x\in\mathbb{Z}^4} |x|^p r_x(\tilde{m}^2). \tag{3.3.4}$$

By (3.1.34), it suffices to prove that the first term on the right-hand side of (3.3.4) is dominant.

For the term $O(\bar{g}_{j_x})G_x(0,0,m^2)$ in (3.3.3), we apply (2.5.24) to obtain

$$\begin{aligned}&\sum_{x\in\mathbb{Z}^4} \bar{g}_{j_x}|x|^p G_x(0,0,\tilde{m}^2) \\ &\quad\le \sum_{x:0<j_x\le j_{\tilde{m}}} \frac{c|x|^p}{\log|x|} G_x(0,0,\tilde{m}^2) + \frac{c}{\log\tilde{m}^{-1}} \sum_{x:j_x>j_{\tilde{m}}} |x|^p G_x(0,0,\tilde{m}^2).\end{aligned} \tag{3.3.5}$$

In the first term, we use $G_x(0,0,m^2) \le G_x(0,0,0) \le O(|x|^{-2})$. The restriction $j_x \le j_{\tilde{m}}$ ensures that $|x| \le O(\tilde{m}^{-1})$. Therefore the first term is bounded above by a multiple of $(\tilde{m}^{-1})^{d+p-2}(\log\tilde{m}^{-1})^{-1}$, which suffices. For the term with $j_x > j_{\tilde{m}}$, we extend the sum to $x \in \mathbb{Z}^4$ and apply Proposition 3.3.1 to obtain a bound of the same form as for the first term.

For the term R_x of (3.3.3), we use Proposition 3.2.1 to see that

$$|R_x(\tilde{m}^2)| = O(\bar{g}_{j_x})L^{-2j_x - 2s(j_x - j_{\tilde{m}})_+}. \tag{3.3.6}$$

We divide $\mathbb{Z}^4$ into shells $S_1 = \{x : |x| < \frac{1}{2}L\}$ and, for $j \ge 2$, $S_j = \{x : \frac{1}{2}L^{j-1} \le |x| < \frac{1}{2}L^j\}$. The number of points in S_j is bounded by $O(L^{4j})$. Note that j_x is the unique scale so that

$$x \in S_{j_x+1}. \tag{3.3.7}$$

By (3.3.6) with $s > \frac{1}{2}(p+2)$ and (3.3.7),

$$\sum_{x\in\mathbb{Z}^4} |x|^p |R_x(\tilde{m}^2)| = \sum_{j=1}^{\infty} \sum_{x\in S_j} |x|^p |R_x(\tilde{m}^2)| = \sum_{j=1}^{\infty} L^{4j+pj-2j-2s(j-j_{\tilde{m}})_+} O(\bar{g}_j), \tag{3.3.8}$$

with an L-dependent constant. By Lemma 3.3.2 below (with $a = p+2$ and $b = 1$), we obtain

$$\tilde{m}^2 \sum_{x\in\mathbb{Z}^4} |x|^p |R_x(\tilde{m}^2)| = O\big(\tilde{m}^{-p}(\log\tilde{m}^{-1})^{-1}\big). \tag{3.3.9}$$

The first term on the right-hand side of (3.3.4) therefore dominates, and the proof is complete. ∎

The estimate used to obtain (3.3.9) is given by the following lemma, which is stated more

generally for use in the proof of Proposition 3.3.1.

Lemma 3.3.2. *Let $L > 1$, $2s > a > 0$, $b \geq 0$, and let $\bar{g}_0 > 0$ be sufficiently small. Then*

$$\sum_{j=1}^{\infty} L^{aj-2s(j-j_m)_+} \bar{g}_j^b = O(m^{-a} \bar{g}_{j_m}^b) = O(m^{-a} (\log m^{-1})^{-b}). \tag{3.3.10}$$

Proof. We divide the sum at the mass scale as

$$\sum_{j=1}^{\infty} L^{aj-2s(j-j_m)_+} \bar{g}_j^b = \sum_{j=1}^{j_m} L^{aj} \bar{g}_j^b + \sum_{j=j_m+1}^{\infty} L^{aj-2s(j-j_m)} \bar{g}_j^b. \tag{3.3.11}$$

For the second sum on the right-hand side, we use $\bar{g}_j = O(\bar{g}_{j_m})$ for $j > j_m$ by (2.5.24), and obtain a bound consistent with the first equality of (3.3.10). For the first term, we use the crude bound $\bar{g}_i / \bar{g}_{i+1} = 1 + O(g_0)$ (by [11, Lemma 2.1]), and find

$$\sum_{j=1}^{j_m} L^{aj} \bar{g}_j^b \leq L^{aj_m} \bar{g}_{j_m}^b \sum_{j=1}^{j_m} ((1 + O(\bar{g}_0)) L^{-a})^{j_m - j} = O(L^{aj_m} \bar{g}_{j_m}^b), \tag{3.3.12}$$

for sufficiently small $\bar{g}_0 > 0$. This proves the first equality in (3.3.10). The second equality then follows since $\bar{g}_{j_m} = O(\log m^{-1})$ by (2.5.24). ∎

Chapter 4

The renormalization group step

In this chapter we discuss the proof of Theorem 2.7.1 with $s > 1$ in the definitions (2.4.21) of the weights ℓ, ℓ_σ. The proof involves numerous changes to results in [29–31]. Consequently, the arguments presented here will not be completely self-contained; for instance, we will not detail the construction of the renormalization group map, which makes up the bulk of [31]. However, we will begin in Section 4.1 with an *informal* overview of some of the key ideas used in these papers.

The details of the proof begin in Section 4.2, where we define new norms above the mass scale and show that they satisfy two key hypotheses required by results in [30]. The use of these new norms is essential: the old norms fail to work with the new weights for technical reasons discussed briefly in Remark 4.2.3. In Section 4.3, we discuss the changes that must be made in [29, 30] with the new weights. Of particular importance is the adaptation of the proof of the *crucial contraction* to these new weights, which is discussed in Section 4.3.4.

Throughout this chapter, we use the notation appropriate for the spin field $\varphi \in (\mathbb{R}^n)^\Lambda$ for $n \geq 1$; only notational modifications are needed for $n = 0$. Since we are dealing with a single renormalization group step, we will often drop the index j of the current scale and write a subscript $+$ to indicate objects at the next scale $j + 1$.

4.1 Simplified renormalization group step

For this discussion, let us drop λ, q, u from the notation and write $U = V$. In this setting our goal, given (V, K), is to construct (V_+, K_+) such that, with $I = I(V)$ and $I_+ = I(V_+)$,

$$\mathbb{E}_+ \theta (I \circ K)(\Lambda) = (I_+ \circ K_+)(\Lambda) \tag{4.1.1}$$

with $K \mapsto K_+$ contractive in some sense; this is needed not only to control the error produced by K in the computation of critical exponents (e.g. recall the use of Lemma 2.4.2 in Section 3) but also so that the map $(V, K) \mapsto (V_+, K_+)$ can be iterated an arbitrary number of times as in Theorem 2.8.1. The algebraic problem (4.1.1) admits a multitude of solutions and the main difficulty is the construction of a solution with good analytic properties.

A possible definition of the map $(V, K) \mapsto V_+$ is suggested by perturbation theory, as discussed in Section 2.5.3. For now, let us suppose that $V_+ = V_{\mathrm{pt}}$ is the correct definition and set $I_+ = I_{\mathrm{pt}} = I(V_{\mathrm{pt}})$. Then we have the following procedure for the construction of K_+ in terms of I_{pt} such that (4.1.1) holds.

Recall (2.6.2). For $B \in \mathcal{B}$, let $\delta I(B) = \theta I(B) - I_{\mathrm{pt}}(B)$ and extend this to $X \in \mathcal{P}$ by imposing block-factorization. Then by (2.6.10) and associativity of the circle product,

$$\begin{aligned}(I \circ K)(\Lambda) &= \sum_{X \in \mathcal{P}} (I_{\mathrm{pt}} + \delta I)^{\Lambda \setminus X} K(X) \\ &= [(I_{\mathrm{pt}} \circ \delta I) \circ K](\Lambda) \\ &= [I_{\mathrm{pt}} \circ (\delta I \circ K)](\Lambda). \end{aligned} \tag{4.1.2}$$

Note that the fluctuation fields at scale j have been integrated out in the definition of V_{pt}. Thus,

$$\mathbb{E}_+ \theta (I \circ K)(\Lambda) = [I_{\mathrm{pt}} \circ \tilde{K}](\Lambda) \tag{4.1.3}$$

with

$$\tilde{K} = \mathbb{E}_+(\delta I \circ \theta K). \tag{4.1.4}$$

This has the form (4.1.1) but with the circle product on the right-hand side at the wrong scale. This is remedied by a simple resummation:

$$\mathbb{E}_+ \theta (I \circ K)(\Lambda) = \sum_{Y \in \mathcal{P}} I_{\mathrm{pt}}^{\Lambda \setminus Y} \tilde{K}(Y) = \sum_{X \in \mathcal{P}_+} I_{\mathrm{pt}}^{\Lambda \setminus X} K_+(X) = (I_{\mathrm{pt}} \circ K_+)(\Lambda) \tag{4.1.5}$$

where

$$K_+(X) = \sum_{Y \in \bar{\mathcal{P}}(X)} I_{\mathrm{pt}}^{X \setminus Y} \tilde{K}(Y). \tag{4.1.6}$$

Here, $\bar{\mathcal{P}}(X)$ is the collection of polymers $Y \in \mathcal{P}(X)$ whose *polymer closure* is X, i.e. for which X is the smallest polymer in $\mathcal{P}_+$ containing Y.

4.1.1 Main contributions to K_+

By expanding the circle product in the definition of $\tilde{K}$, we can write

$$K_+(X) = I_{\mathrm{pt}}^X [h(X) + k(X) + R(X)] \tag{4.1.7}$$

where (letting $\bar{\mathcal{C}}(X) = \mathcal{C} \cap \bar{\mathcal{P}}(X)$)

$$h(X) = \sum_{Y \in \bar{\mathcal{P}}(X)} I_{\mathrm{pt}}^{-Y} \mathbb{E}_+ \delta I(Y) \tag{4.1.8}$$

$$k(X) = \sum_{Y \in \bar{\mathcal{C}}(X)} I_{\mathrm{pt}}^{-Y} \mathbb{E}_+ \theta K(Y) \tag{4.1.9}$$

and R is the remainder. If δI and K are sufficiently small (in an appropriate sense), then it is reasonable to view the terms h and k as first-order contributions to K_+. Note that the sum defining k is restricted to connected polymers; by component-factorization, terms involving the

values of K on disconnected polymers should be higher-order in this sense. Our main task then is to bound h and k.

Perturbative contribution and covariance bound

By the definition (2.6.6) of I, we expect that $\delta I = O(\delta V)$ where $\delta V = \theta V - V_{\mathrm{pt}}$. Indeed, a version of this statement is true by [30, Proposition 2.7]. Thus, in order to bound the contribution h defined in (4.2.13), we must consider the size of δV. By definition,

$$\|\delta V(B)\|_{T_0(\ell)} \leq \|\theta V(B) - V(B)\|_{T_0(\ell)} + \|V(B) - V_{\mathrm{pt}}(B)\|_{T_0(\ell)}. \tag{4.1.10}$$

By (2.5.16), $V_{\mathrm{pt}} - V = (\mathbb{E}\theta V - V) - P(V)$ and the first term on the right-hand side can be bounded term-by-term. For instance, the difference between a single quartic term and its expectation is given by

$$\varphi_x^4 - \mathbb{E}\theta\varphi_x^4 = 6C_{00}\varphi_x^2 + 3C_{00}^2. \tag{4.1.11}$$

Covariance terms such as those above can be estimated using Proposition 2.3.1. The method of [30] is more flexible, however, and does not require the precise bounds in (2.3.4). Rather, necessary bounds on the covariance and its derivatives are encoded in the hypothesis [30, (1.73)] on the $\Phi(\ell)$ norm estimate of the covariance. This constraint naturally ensures that the $T_0(\ell)$ seminorm estimates properly reflect the size of the expectation of a field as discussed in Section 2.4.3. Our main bound on the covariance, which extends [30, (1.73)], will be stated in Section 4.2.1.

Remark 4.1.1. The generality provided by predicating the results of [30] on a norm estimate on the covariance is very useful, e.g. as in [107].

Extraction and contraction

The term k in (4.1.9) is the contribution to K_+ that is linear in K. Thus, its control is essential to obtaining the contractivity estimate (2.7.10).

In the simple case that $K = 0$, we will have $K_+ = I_{\mathrm{pt}}h$, which is a Taylor remainder that contains terms at all orders in the fields. Thus, it includes relevant and marginal terms as well as non-local irrelevant terms. The size and number of such terms will prevent K from shrinking under the action of the renormalization group unless they are somehow dealt with. This is done by using the operator Loc mentioned in Section 2.5.3 to *extract* a marginal/relevant part from K prior to integration.

Thus, the true definition of the renormalization group map constructed in [31] involves several more steps than (4.1.6). In fact, the definition of the map $(V, K) \mapsto (V_+, K_+)$ is a composition of 6 maps, called Maps 1–6. In Maps 1–2, the operator Loc is used to perform extraction. Map 3 then implements the expectation and change of scale in (4.1.6).

Once a sufficiently large portion of the marginal and relevant terms have been extracted from K in Maps 1–2, we expect by the discussion in Section 2.5.1 that the expectation in Map 3 should cause K to contract. The fact that irrelevant terms shrink under expectation and change of scale is formally captured by [30, Proposition 2.8], which we refer to as the *crucial contraction*. In Section 4.3.4, we prove that the crucial contraction continues to hold when $s > 1$.

Remark 4.1.2. In order to maintain the form (4.1.1), the extraction step in Maps 1–2 must make a corresponding adjustment to V. This adjustment results in a map $V \mapsto V_+$, given by

$$V_+ = V_{\mathrm{pt}}(V - Q), \quad Q(B) = \sum_{B \subset Y \in \mathcal{S}} \mathrm{Loc}_{Y,B}\, I^{-Y} K(Y), \tag{4.1.12}$$

where $\mathrm{Loc}_{Y,B}\, F$ is the restriction to $B \subset Y$ of the polynomial on Y determined by $\mathrm{Loc}_Y F$.

The map V_+ is a perturbation of V_{pt} whose size is determined by the norm of K. In particular, for $(V, K) \in \mathbb{D}$ we get the first bound of (2.7.9) (recall (2.7.6)), which ensures that the flow of coupling constants exhibits the same qualitative behaviour as the perturbative flow.

On the other hand, if K is large, then the resulting perturbation may have a non-trivial effect on the flow of coupling constants, resulting in different critical behaviour than that predicted by the perturbative flow.

4.2 Improved norm

In this section, we prove an improved covariance estimate, which indicates why it is possible to use the improved weights (2.4.21). This leads to a discussion of simplified norm pairs beyond the mass scale. A lemma concerning the fluctuation-field regulator indicates why the simplification is possible.

4.2.1 Covariance bounds

Recall from (2.4.21) that

$$\ell_j = \ell_0 L^{-j-s(j-j_m)_+}, \quad \ell_{\sigma,j} = \ell_{j\wedge j_x}^{-1} 2^{(j-j_x)_+} \tilde{g}_j. \tag{4.2.1}$$

The analysis of [30, 31] uses the norm parameters ℓ_j and $\ell_{\sigma,j}$ with $s = 0$. To distinguish these from our new choice (4.2.1) of ℓ_j and $\ell_{\sigma,j}$, we write

$$\ell_j^{\mathrm{old}} = \ell_0 L^{-j}, \quad \ell_{\sigma,j}^{\mathrm{old}} = (\ell_{j\wedge j_x}^{\mathrm{old}})^{-1} 2^{(j-j_x)_+} \tilde{g}_j. \tag{4.2.2}$$

We may regard a covariance C_j in the decomposition (2.3.1) as a test function depending

on two arguments x, y, and with this identification its $\Phi_j(\ell_j)$ norm is

$$\|C_j\|_{\Phi_j(\ell_j)} = \ell_j^{-2} \sup_{x,y\in\Lambda} \sup_{|\alpha|_1+|\beta|_1\le p_\Phi} L^{(|\alpha|_1+|\beta|_1)j} |\nabla_x^\alpha \nabla_y^\beta C_{j;x,y}|. \tag{4.2.3}$$

The following lemma justifies our choice of ℓ_j in (4.2.1), by showing that the bound [30, (1.73)], proved there only for the $s = 0$ version ℓ_j^{old} of (4.2.2), remains true with the stronger choice of norm parameter ℓ_j that permits arbitrary $s \ge 0$. Recall that the sequence ϑ_j was defined in (2.7.3).

Lemma 4.2.1 (Extension of [30, (1.73)])**.** *Given $\mathfrak{c} \in (0, 1]$, ℓ_0 can be chosen large (depending on $L, \mathfrak{c}, s$) so that*

$$\|C_j\|_{\Phi_j(\ell_j)} \le \min(\mathfrak{c}, \vartheta_j). \tag{4.2.4}$$

Proof of Lemma 4.2.1. For $d = 4$, insertion of (2.3.4) into (4.2.3) gives

$$\|C_j\|_{\Phi_j(\ell_j)} \le cL^{p_\Phi} \ell_j^{-2} (1 + m^2 L^{2(j-1)})^{-k} L^{-2(j-1)}. \tag{4.2.5}$$

With $s = 0$ in (4.2.1), (4.2.5) gives $\|C_j\|_{\Phi_j(\ell_j)} \le c_L \ell_0^{-2} (1 + m^2 L^{2(j-1)})^{-k}$ for an L-dependent constant c_L (whose value may now change from line to line). We insert (2.4.18) and the definition $\ell_j = \ell_0 L^{-j-s(j-j_m)_+}$ from (4.2.1) into (4.2.5), to conclude that there exists $c_0 = c_0(s, L)$ such that

$$\|C_j\|_{\Phi_j(\ell_j)} \le c_0 \ell_0^{-2} L^{-2(j-j_m)_+}. \tag{4.2.6}$$

By definition of ϑ_j, $L^{-2(j-j_m)_+}$ is bounded by a multiple of ϑ_j. It thus suffices to choose ℓ_0 large enough that $\ell_0^2 \ge c_0 \mathfrak{c}^{-1}$. ∎

4.2.2 New choice of norm beyond the mass scale

A field φ can be viewed as a test function supported on sequences with $|\vec{x}| = 1$ and $|\vec{y}| = 0$. In particular,

$$\|\varphi\|_{\Phi_j(\ell_j)} = \ell_j^{-1} \sup_{x\in\Lambda} \sup_{1\le i\le n} \sup_{|\alpha|_1\le p_\Phi} L^{j|\alpha|_1} |\nabla^\alpha \varphi_x^i|, \tag{4.2.7}$$

As in [30, (1.36)], we use the localized version of (4.2.7), defined for subsets $X \subset \Lambda$ by

$$\|\varphi\|_{\Phi_j(X)} = \inf\{\|\varphi - f\|_{\Phi_j} : f \in \mathbb{C}^\Lambda \text{ such that } f_x = 0 \ \forall x \in X\}. \tag{4.2.8}$$

A similar definition is given for general test functions. Given $X \subset \Lambda$ and $\varphi \in (\mathbb{R}^n)^\Lambda$, we recall from [30, (1.38)] that the *fluctuation-field regulator* G_j is defined by

$$G_j(X, \varphi) = \prod_{x\in X} \exp\left(|B_x|^{-1} \|\varphi\|^2_{\Phi_j(B_x^\square, \ell_j)} \right), \tag{4.2.9}$$

where $B_x \in \mathcal{B}_j$ is the unique block that contains x, and hence $|B_x| = L^{dj}$. The *large-field regulator* is defined in [30, (1.41)] by

$$\tilde{G}_j(X,\varphi) = \prod_{x\in X} \exp\left(\frac{1}{2}|B_x|^{-1}\|\varphi\|^2_{\tilde{\Phi}_j(B_x^\square,\ell_j)}\right). \tag{4.2.10}$$

The $\tilde{\Phi}_j$ seminorm appearing on the right-hand side of (4.2.10) will be defined in Section 4.3.4. The two regulators serve as weights in the *regulator norms* of [30, Definition 1.1]. The regulator norms are defined, with $\mathtt{t} \in (0,1]$ and for $F \in \mathcal{N}(X^\square)$ by

$$\|F\|_{G_j(\ell_j)} = \sup_{\varphi\in(\mathbb{R}^n)^\Lambda} \frac{\|F\|_{T_{\varphi,j}(\ell_j)}}{G_j(X,\varphi)}, \tag{4.2.11}$$

$$\|F\|_{\tilde{G}^{\mathtt{t}}_j(h_j)} = \sup_{\varphi\in(\mathbb{R}^n)^\Lambda} \frac{\|F\|_{T_{\varphi,j}(h_j)}}{\tilde{G}^{\mathtt{t}}_j(X,\varphi)}. \tag{4.2.12}$$

The parameter ℓ_j that appears in the regulators (4.2.9)–(4.2.10) and in the numerator of (4.2.11) was taken to be ℓ_j^{old} in [30], but now we use ℓ_j instead. As in [30], the parameter h_j and its observable counterpart $h_{\sigma,j}$ are given by

$$h_j = k_0 \tilde{g}_j^{-1/4} L^{-j}, \quad h_{\sigma,j} = (\ell^{\mathrm{old}}_{j\wedge j_x})^{-1} 2^{(j-j_x)_+} \tilde{g}_j^{1/4}. \tag{4.2.13}$$

In [30], estimates on $\|\cdot\|_{j+1}$ are given in terms of $\|\cdot\|_j$, where the pair $(\|\cdot\|_j, \|\cdot\|_{j+1})$ refers to either of the norm pairs

$$\|F\|_j = \|F\|_{G_j(\ell_j^{\mathrm{old}})} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{T_{0,j+1}(\ell^{\mathrm{old}}_{j+1})}, \tag{4.2.14}$$

or

$$\|F\|_j = \|F\|_{\tilde{G}_j(h_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{\tilde{G}^{\mathtt{t}}_{j+1}(h_{j+1})}. \tag{4.2.15}$$

We will show that, above the mass scale j_m, the results of [30] hold with both norm pairs in (4.2.14) and (4.2.15) replaced by the single new norm pair

$$\|F\|_j = \|F\|_{G_j(\ell_j)} \quad \text{and} \quad \|F\|_{j+1} = \|F\|_{G_{j+1}(\ell_{j+1})}, \tag{4.2.16}$$

with the improved ℓ_j of (4.2.1) with $s > 1$ fixed as large as desired.

The use of two norm pairs adds intricacy to [30, 31]. The pair (4.2.14) is insufficient, on its own, because the scale-$(j+1)$ norm is the T_0 seminorm which controls only small fields, and an estimate in this norm does not imply an estimate for the G_{j+1} norm. The norm pair (4.2.15) is used to supplement the norm pair (4.2.14), and estimates in both of the scale-$(j+1)$ norms can be combined to provide an estimate for the G_{j+1} norm. This then sets the stage for the next renormalization group step. Above the mass scale, the use of (4.2.16) now bypasses many issues. For example, for $j > j_m$ the $\mathcal{W}_j$ norm of [31, (1.45)] is replaced simply by the $\mathcal{F}_j(G)$

norm, and there is no need for the $\mathcal{Y}_j$ norm of [31, (2.12)] nor for [31, Lemma 2.4].

The need for both norm pairs (4.2.14)–(4.2.15) is discussed in [30, Section 1.2.1] and is related to the so-called *large-field problem*. Roughly speaking, the norm pair (4.2.15) is used to take advantage of the quartic term in the interaction to suppress the effects of large values of the fields. This approach relies on the fact that the interaction polynomial is dominated by the quartic term in the h-norm, as expressed by [30, (1.91)], together with the lower bound [30, (1.90)] on the quartic term. However, above the mass scale, large fields are naturally suppressed by the rapid decay of the covariance. This idea is captured in Lemma 4.2.2 below, which replaces [30, Lemma 1.2] above the mass scale. The regulators in its statement are defined by (4.2.9) with the s-dependent ℓ_j of (4.2.1).

Lemma 4.2.2 (Replacement for [30, Lemma 1.2])**.** *Let $X \subset \Lambda$ and assume that $s > 1$. For any $q > 0$, if L is sufficiently large depending on q, then for $j_m \le j < N$,*

$$G_j(X,\varphi)^q \le G_{j+1}(X,\varphi). \tag{4.2.17}$$

Proof. By (4.2.9), it suffices to show that, for any scale-j block B_j and any scale-$(j+1)$ block B_{j+1} containing B_j,

$$q\|\varphi\|^2_{\Phi_j(B_j^\square,\ell_j)} \le L^{-4}\|\varphi\|^2_{\Phi_{j+1}(B_{j+1}^\square,\ell_{j+1})}. \tag{4.2.18}$$

In fact, since $\|\varphi\|_{\Phi_j(B_j^\square,\ell_j)} \le \|\varphi\|_{\Phi_j(B_{j+1}^\square,\ell_j)}$ by definition, it suffices to prove the above bound with B_j replaced by B_{j+1} on the left-hand side. According to the definition of the norm in (4.2.8), to show this it suffices to prove that

$$q\|\varphi\|^2_{\Phi_j(\ell_j)} \le L^{-4}\|\varphi\|^2_{\Phi_{j+1}(\ell_{j+1})} \tag{4.2.19}$$

(then we replace φ by $\varphi - f$ in the above and take the infimum).

By definition,

$$\|\varphi\|_{\Phi_j(\ell_j)} \le \ell_j^{-1}\ell_{j+1} \sup_{x\in\Lambda}\sup_{|\alpha|\le p_\Phi} \ell_{j+1}^{-1}L^{(j+1)|\alpha|}|\nabla^\alpha\varphi_x|, \tag{4.2.20}$$

with the inequality due to replacement of $L^{j|\alpha|}$ on the left-hand side by $L^{(j+1)|\alpha|}$ on the right-hand side. Since $\ell_j^{-1}\ell_{j+1} = L^{-1-s\mathbb{1}_{j\ge j_m}}$,

$$\|\varphi\|_{\Phi_j(\ell_j)} \le L^{-1-s\mathbb{1}_{j\ge j_m}}\|\varphi\|_{\Phi_{j+1}(\ell_{j+1})}. \tag{4.2.21}$$

Thus,

$$q\|\varphi\|^2_{\Phi_j(\ell_j)} \le qL^{-4}L^{2-2s\mathbb{1}_{j\ge j_m}}\|\varphi\|^2_{\Phi_{j+1}(\ell_{j+1})}, \tag{4.2.22}$$

and then (4.2.19) follows once L is large enough that $qL^{2-2s} \le 1$. ∎

Remark 4.2.3. The elimination of the h-norm after the mass scale is more than a convenience. It becomes a necessity when we improve the ℓ-norm. Briefly, the reason is as follows. In the

proof of [31, Lemma 2.4], the ratio ℓ_σ/h_σ must be bounded. For this, we would need to increase h_σ beyond the mass scale (since ℓ_σ has been increased). This forces a compensating decrease in h beyond j_m, to keep the product hh_σ bounded for stability (as in Section 4.3.2 below). But if we do this, we lose the lower bound required on $\epsilon_{g\tau^2}$ required for stability in the h-norm (see [30, (3.8)]).

4.3 Proof of Theorem 2.7.1

In this section, we show that Theorem 2.7.1 holds, thereby completing the proof of Proposition 3.2.1. The key steps in the proof of the $s = 0$ case of Theorem 2.7.1 are contained in [30,31]. Our main objective in this section is to show that the results in [30, 31] continue to hold with the new norm parameters $\ell_j, \ell_{\sigma,j}$. To this end, we may and do use the fact that the estimates of [30] have already been established with the old norm parameters.

In the following, we indicate the changes in the analysis of [30,31] that arise due to the new choice of norm parameters (4.2.1) beyond the mass scale, and due to the reduction from two norm pairs to one. This requires repeated reference to previous papers. In such references, we will sometimes use the notation from those papers without defining it here.

4.3.1 Norm parameter ratios

The analysis of [30] assumes that the norm parameters $\mathfrak{h}_j, \mathfrak{h}_{\sigma,j}$, for $\mathfrak{h} = \ell$ or $\mathfrak{h} = h$, satisfy the estimates [30, (1.79)]; these assert that

$$\mathfrak{h}_j \geq \ell_j, \qquad \frac{\mathfrak{h}_{j+1}}{\mathfrak{h}_j} \leq 2L^{-1}, \qquad \frac{\mathfrak{h}_{\sigma,j+1}}{\mathfrak{h}_{\sigma,j}} \leq \text{const} \begin{cases} L & (j < j_x) \\ 1 & (j \geq j_x). \end{cases} \tag{4.3.1}$$

We do not change $\mathfrak{h}_j$ or $\mathfrak{h}_{\sigma,j}$ for j below the mass scale, so there can be no difficulty until above the mass scale. Above the mass scale, the parameters $h_j, h_{\sigma,j}$ are eliminated, and requirements involving them become vacuous. Thus, for (4.3.1), we need only verify the second and third inequalities for the case $\mathfrak{h} = \ell$. By definition,

$$\frac{\ell_{j+1}}{\ell_j} = L^{-(1+s\mathbb{1}_{j\geq j_m})}, \qquad \frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}} = \frac{\tilde{g}_{j+1}}{\tilde{g}_j} \times \begin{cases} L^{1+s\mathbb{1}_{j\geq j_m}} & (j < j_x) \\ 2 & (j \geq j_x). \end{cases} \tag{4.3.2}$$

According to [30, (1.77)], $\frac{1}{2}\tilde{g}_{j+1} \leq \tilde{g}_j \leq 2\tilde{g}_{j+1}$. Thus, the second estimate of (4.3.1) is satisfied (the ratio being improved when $j \geq j_m$), while the third is *not* when $s > 1$ and $j_m < j_x$. This potentially dangerous third estimate in (4.3.1) is used to prove the scale monotonicity lemma [30, Lemma 3.2], as well as the crucial contraction. We discuss [30, Lemma 3.2] next, and return to the crucial contraction in Section 4.3.4 below.

[30, Lemma 3.2] There is actually no problem with the scale monotonicity lemma. Indeed, for the case $\alpha = ab$ of the proof of [30, Lemma 3.2], the hypothesis that $\pi_{0x}F = 0$ for $j < j_x$ ensures that this case only relies on the dangerous estimate for $j \geq j_x$ where the danger is absent in (4.3.2). For the cases $\alpha = a$ and $\alpha = b$ of the proof of [30, Lemma 3.2], what is important is the inequality $\ell_{\sigma,j+1}\ell_{j+1} \leq \text{const}\, \ell_{\sigma,j}\ell_j$, which continues to hold with (4.2.1) for all scales j, both above and below the mass scale, since the products in this inequality are the same for the new and the old choices of ℓ. So [30, Lemma 3.2] continues to hold with the choice (4.2.1). In addition,

$$\|F\|_{T_\varphi(\ell_j)} \leq \|F\|_{T_\varphi(\ell_j^{\text{old}})}. \tag{4.3.3}$$

This strengthened special case of the first inequality of [30, (3.6)] (strengthened due to the constant 1 on the right-hand side of (4.3.3) compared to the generic constant in [30, (3.6)]) can be seen from an examination of the proof of the $\alpha = a, b$ case of [30, Lemma 3.2], together with the observation that $\ell_{\sigma,j}\ell_j = \ell_{\sigma,j}^{\text{old}}\ell_j^{\text{old}}$ by definition.

4.3.2 Stability domains

In [30, (1.83)], an extension of the domain (2.7.4) is defined. By some abuse of notation, we will also denote this extended domain by $\mathcal{D}_j$. We modify $\mathcal{D}_j$ only for the coupling constant q, by replacing r_q in [30, (1.84)] by

$$L^{2j_x+2s(j_x-j_m)_+}2^{2(j-j_x)}r_{q,j} = \begin{cases} 0 & j < j_x \\ C_{\mathcal{D}} & j \geq j_x. \end{cases} \tag{4.3.4}$$

[30, Proposition 1.5] With (4.3.4), [30, Proposition 1.5] as it pertains to $\mathfrak{h} = \ell$ (omitting all reference to $\mathfrak{h} = h$) continues to hold beyond the mass scale by the same proof. In particular, with the smaller choice for the domain of q, [30, (3.14)] holds with the larger s-dependent $\ell_{\sigma,j}$.

Note that we do not need to change the domain of λ. This is because the bound [30, (3.13)] continues to hold with the new norm parameters. Indeed, while ℓ_j and $\ell_{\sigma,j}$ have been modified, their product $\ell_j\ell_{\sigma,j}$ has not. This guarantees that the T_0 seminorm $\|\sigma\bar{\varphi}_a\|_{T_0} = \ell_\sigma\ell$ remains identical to what it was with the old norm parameters, and therefore there is no new stability requirement arising from this.

The choice (4.3.4) places a more stringent requirement on the domain than does the $s = 0$ version. To see that this requirement is actually met by the renormalization group flow, we note a minor improvement to the proof of [31, Lemma 6.2(ii)], where the bound $|\delta q| \leq cL^{-2j}$ is used to show that $v(X)$ (defined there) satisfies

$$\|v(X)\| \leq cL^{-2j}(\ell_{\sigma,j}^{\text{old}})^2 \leq c'. \tag{4.3.5}$$

Here the factor L^{-2j} arises as a bound on the covariance $C_{j+1;00}$ in the perturbative flow [30, (3.35)] of q and it can therefore be improved to $L^{-2j-2s(j-j_m)_+}$ by Lemma 4.2.1. Thus also with

$\ell^{\mathrm{old}}, \ell_\sigma^{\mathrm{old}}$ replaced by ℓ, ℓ_σ, the required bound $\|v(X)\| \le c'$ remains valid.

4.3.3 Extension of stability analysis

In this and the next section, we verify that the results of [30, Section 2] remain valid with ℓ^{old} replaced by ℓ. In this section, we deal with the results whose proofs need only minor modification.

First, we note that the supporting results of [30, Section 4] hold with the new norms. Indeed, it is immediate from (4.3.3) that analogues of [30, Proposition 4.1] and [30, Lemmas 3.4, 4.11–4.12] hold with the new ℓ_j. Moreover, [30, Lemma 4.7] and [30, Proposition 4.10] hold for general values of the parameters $\mathfrak{h}_j$ (which are implicit in the $T_{0,j}$-norm). We discuss [30, Proposition 4.9] in Section 4.3.4 below, and the remaining results of [30, Section 4] do not make use of norms.

[30, Proposition 2.1] With $\mathfrak{h} = \ell$, [30, (2.1)] continues to hold with the same proof; in fact the proof does not depend on the explicit choice of $\mathfrak{h}$. We do not need [30, (2.2)] as it is only applied with $\mathfrak{h} = h$.

[30, Proposition 2.2] The only change to the proof is for the case $j_* = j + 1$. To get [30, (2.9)], we proceed as previously in the case $\mathfrak{h} = h$ but applying Lemma 4.2.2 rather than [30, Lemma 1.2] following [30, (5.22)]. In the same way, we get [30, (2.10)] and the remaining parts of the proposition follow without changes to the proof.

[30, Proposition 2.3] Again the only required change in the proof is the use of Lemma 4.2.2 in the case $j_* = j + 1$, for which as previously we use Lemma 4.2.2 instead of [30, Lemma 1.2].

[30, Proposition 2.4] No changes need to be made to the proof. In fact, it is necessary *not* to use the $\mathfrak{h} = \ell$ case of the estimate [30, (5.32)]. Instead, the $\mathfrak{h} = \ell^{\mathrm{old}}$ case of this estimate should be used for g_Q. This is possible since the renormalization group map, and in particular the coupling constants, are independent of the choice of norm.

[30, Proposition 2.5] Using (4.3.3), we see that the proof continues to hold above the mass scale. The only change to the proof is that in the application of [30, Proposition 2.2], j should be replaced by $j + 1$ in [30, (2.9)] with $j_* = j + 1$ (corresponding to the G_{j+1} norm). This yields [30, (6.6)] with a G_{j+1} norm on the left-hand side.

[30, Proposition 2.6] A version of [30, Lemma 6.1] with the new ℓ continues to hold. This lemma makes use of $\hat{\ell}$, which superficially depends on the choice of ℓ in its definition [30, (3.17)]. However, brief scrutiny of [30, (3.17)] reveals that the apparent dependence on ℓ actually cancels and there is in fact no dependence. Similarly, [30, Lemma 3.4] continues to hold without any changes to its proof. The proof of [30, Proposition 2.6] then applies without change.

[30, Proposition 2.7] With the new choice of ℓ (and $\mathcal{G} = G$), [30, Lemma 7.1] continues to hold with no changes to its proof. Thus, by [30, (3.6)] and [30, Lemma 7.1],

$$\begin{aligned}
&\|\mathbb{E}_{j+1}\delta I^X \theta F(Y)\|_{T_{\varphi,j+1}(\ell_{j+1})} \\
&\quad \le \|\mathbb{E}_{j+1}\delta I^X \theta F(Y)\|_{T_{\varphi,j}(\ell_j)} \\
&\quad \le \alpha_{\mathbb{E}}^{|X|_j+|Y|_j}(C_{\delta V}\bar{\epsilon})^{|X|_j}\|F(Y)\|_{G_j(\ell_j)}G_j(X\cup Y,\varphi)^5. \qquad (4.3.6)
\end{aligned}$$

By Lemma 4.2.2, $G_j(X\cup Y,\varphi)^5 \le G_{j+1}(X\cup Y,\varphi)$. Now we divide both sides by $G_{j+1}(X\cup Y,\varphi)$ and take the supremum over φ to complete the proof.

4.3.4 Extension of the crucial contraction

The proof of the "crucial contraction" [30, Proposition 2.8] makes use of the third estimate in (4.3.1), which is now violated above the mass scale due to our new choice of ℓ_j. On the other hand, the second estimate of (4.3.1) is improved by the new choice and compensates for the degraded third estimate, as we explain in this section.

The operator Loc

There is a certain kind of duality between the space Φ of test functions and the space $\mathcal{N}$ of field functionals induced by the pairing (2.4.9). By exploiting this, in [30] the operator Loc is defined as a kind of adjoint to an operator Tay_a, which replaces test functions by a lattice Taylor expansion at $a \in \Lambda$. Non-constant polynomials are not well-defined on the whole torus Λ, but such a Taylor expansion can nevertheless be defined for test functions supported on sequences whose components lie in a sufficiently "small" subset of Λ. These are referred to in [30] as *coordinate patches*. By definition, they are nonempty and any small set is contained in a coordinate patch. (In this chapter, we are ultimately only concerned with the case of small sets).

Suppose we fix a coordinate patch $\Lambda' \subset \Lambda$. By definition, it can be identified with a rectangle in $\mathbb{Z}^d$. Then given a local monomial M of the form (2.5.1), we define $p_M \in \Phi$ by

$$p_M(x_1,\dots,x_p) = x_1^{\alpha_1}\dots x_p^{\alpha_p}, \qquad x_1,\dots,x_p \in \Lambda' \qquad (4.3.7)$$

and set $p_M(\vec{x}) = 0$ if $|\vec{x}| \ne p$ or if the lattice points in $\vec{x}$ do not all lie in Λ'. Following (2.5.4), we define the *dimension* of such a monomial to be the dimension of M, i.e. $|\alpha| + p[\varphi]$. We let $d_+ \ge 0$ and let $\Pi = \Pi[\Lambda']$ denote the span of the monomials of this form with dimension at most d_+. For $X \subset \Lambda'$, we can also define $\Pi(X) = \Pi[\Lambda'](X)$ as the space of test functions that agree with an element of Π on X. Thus, $\Pi(X) \supset \Pi$.

For $a \in \Lambda'$, in [29] the operator Tay_a is defined as a map $\mathrm{Tay}_a : \Phi \to \Pi$ by a lattice analogue of Taylor expansion. Although the monomials p_M form an obvious basis with respect to which

this expansion can be performed, a different basis[1] is used in [29]. The operator $\mathrm{Tay}_a g$ satisfies lattice analogues of the usual properties of Taylor polynomials. We will not discuss this further as we will only use the fact that $\mathrm{Tay}_a g \in \Pi$ here.

The following, which is a restatement of [30, Proposition 1.5] specialized to small sets, defines $\mathrm{Loc}_X F$ as the unique element of $\mathcal{V}(X)$ that agrees with F to order d_+ in the sense of the pairing.

Proposition 4.3.1. *Let $X \in \mathcal{S}$ be a small set and let $F \in \mathcal{N}(X)$. Then there is a unique polynomial $V \in \mathcal{V}$ such that*

$$\langle F, g\rangle_0 = \langle V(X), g\rangle_0 \tag{4.3.8}$$

for all $g \in \Pi$. Moreover, V is independent of the choice of coordinate patch used to define Π. We write $\mathrm{Loc}_X F = V(X)$.

Define the seminorm

$$\|g\|_{\tilde{\Phi}(X)} = \inf\{\|g - f\|_\Phi : f \in \Pi(X)\} \tag{4.3.9}$$

on Φ (this is used in the definition of the large-field regulator (4.2.10)). We will need the following lemma, which is a restatement of [29, Lemma 2.6].

Lemma 4.3.2. *Let $X \in \mathcal{S}$ be a small set and let $g \in \Phi$. There exists $f \in \Pi(X)$ such that, with $h = g - f$, we have $\|g\|_{\tilde{\Phi}(X)} \le \|h\|_\Phi \le (1+\epsilon)\|g\|_{\tilde{\Phi}(X)}$ and $\|f\|_\Phi \le (2+\epsilon)\|g\|_\Phi$.*

Proof of the crucial contraction

Below the mass scale, we continue to use the crucial contraction as stated in [30, Proposition 2.8] in terms of two norm pairs. Next, we state a version of the crucial contraction for use above the mass scale using the new norm pair (4.2.16). Throughout this section, we sometimes write the dimension as d for emphasis, although we only consider $d = 4$. We define

$$\tilde{I}_{\mathrm{pt}}(B) = e^{-V_{\mathrm{pt}}(B)}(1 + W_{j+1}(V_{\mathrm{pt}}, B)), \quad B \in \mathcal{B}_j \tag{4.3.10}$$

and extend this to $\tilde{I}_{\mathrm{pt}}(X) = \tilde{I}^X_{\mathrm{pt}}$ for $X \in \mathcal{P}_{j+1}$ by block-factorization.

Proposition 4.3.3 (Improvement of [30, Proposition 2.8])**.** *Let $j_m \le j < N$ and $V \in \mathcal{D}_j$. Let $X \in \mathcal{S}_j$ and let U be the polymer closure of X. Let $F(X) \in \mathcal{N}(X^\square)$ be such that $\pi_\alpha F(X) = 0$ when $\alpha \notin X$ $(\alpha = 0, x)$ and such that $\pi_{0x} F(X) = 0$ unless $j \ge j_x$. There is a constant C (independent of L) such that*

$$\|\tilde{I}^{U\setminus X}_{\mathrm{pt}} \mathbb{E}_{C_{j+1}} \theta F(X)\|_{G_{j+1}(\ell_{j+1})} \le C\Big((L^{-d-1} + L^{-1}\mathbb{1}_{X\cap\{0,x\}\neq\varnothing})\kappa_F + \kappa_{\mathrm{Loc}F}\Big), \tag{4.3.11}$$

with $\kappa_F = \|F(X)\|_{G_j(\ell_j)}$ and $\kappa_{\mathrm{Loc}F} = \|\tilde{I}^X_{\mathrm{pt}} \mathrm{Loc}_X \tilde{I}^{-X}_{\mathrm{pt}} F(X)\|_{G_j(\ell_j)}$.

[1] This basis is similar to the *Newton polynomial basis* used in the calculus of finite differences.

An ingredient in the proof of Proposition 4.3.3 is [29, Lemma 3.6], which is the $s = 0$ version of the following lemma. The proof of Lemma 3.6 with $s = 0$ is based on the assumption $\ell_{j+1}/\ell_j \le cL^{-1}$ (we take $[\varphi_i] = 1$; the parameters $\ell_{\sigma,j}$ are not used). For our new values of ℓ, the stronger assumption $\ell_{j+1}/\ell_j \le L^{-1-s\mathbb{1}_{j\ge j_m}}$ holds. The unique change to the proof occurs in the transition from [29, (3.42)] to [29, (3.43)], where the ratio ℓ_{j+1}/ℓ_j is used.

In the following, we let $\Phi = \Phi_j(\mathfrak{h}_j)$ and $\Phi' = \Phi_{j+1}(\mathfrak{h}_{j+1})$. We employ similar conventions for $\Phi(X)$ and $\tilde{\Phi}(X)$. The constant d'_+ is defined in [29, (1.38)] and in this context becomes $d'_+ = d_+ + 1$. The enlargement X_+ of a polymer $X \in \mathcal{P}_j$ is defined by replacing each block $B \in \mathcal{B}_j(X)$ by a cube of twice the side length of B (minus 1 if L^j is odd) that is centred at B.

Lemma 4.3.4 (Improvement of [29, Lemma 3.6])**.** *With the same hypotheses and notation as in [29, Lemma 3.6],*

$$\|g\|_{\tilde{\Phi}(X)} \le \bar{C}_3 L^{-(1+s\mathbb{1}_{j\ge j_m})d'_+} \|g\|_{\tilde{\Phi}'(X_+)}. \tag{4.3.12}$$

Proof. Assume without loss of generality that X is connected. Let $f \in \Pi(X)$ and $h \in \Phi$ be as in Lemma 4.3.2. Thus, $g = f + h$ and so we have $g - (h - \mathrm{Tay}_a h) = f + \mathrm{Tay}_a h \in \Pi(X)$, where a is the largest point which is lexicographically no larger than any point in X. By definition of the $\tilde{\Phi}(X)$ seminorm,

$$\|g\|_{\tilde{\Phi}(X)} = \|h - \mathrm{Tay}_a h\|_{\tilde{\Phi}(X)} \le \|h - \mathrm{Tay}_a h\|_{\Phi(X)}. \tag{4.3.13}$$

By the bound on h (from Lemma 4.3.2), it suffices to show that

$$\|h - \mathrm{Tay}_a h\|_{\Phi(X)} \le \frac{1}{2} \bar{C}_3 L^{-(1+s\mathbb{1}_{j\ge j_m})d'_+} \|g\|_{\tilde{\Phi}'(X_+)}. \tag{4.3.14}$$

To this end, let $r = h - \mathrm{Tay}_a h$. By [29, Lemma 3.3] with $t = 1/2$, there exists $K > 1$ such that

$$\|r\|_{\Phi(X)} \le \sup_{\vec{x}\in\mathbf{X}_+} (K\ell_j^{-1})^{\vec{x}} \sup_{|\beta|_\infty \le p_\Phi} L^{j|\beta|_1} |\nabla^\beta r_{\vec{x}}| \tag{4.3.15}$$

where $A^{\vec{x}} = A^{|\vec{x}|}$ and $\mathbf{X}_+$ is the set of sequences whose components lie in X_+. In other words, we can estimate the $\Phi(X)$ norm of r in terms of the values of r and its derivatives in the enlargement X_+ of X. With the new ratio (4.3.2), we can rewrite this as

$$\|r\|_{\Phi(X)} \le \sup_{\vec{x}\in\mathbf{X}_+} (K\ell_{j+1}^{-1})^{\vec{x}} \sup_{|\beta|_\infty \le p_\Phi} L^{-(|\vec{x}|+|\vec{x}|s\mathbb{1}_{j\ge j_m}+|\beta|_1)} L^{(j+1)|\beta|_1} |\nabla^\beta r_{\vec{x}}|, \tag{4.3.16}$$

replacing [29, (3.43)].

By definition, for the empty sequence $\varnothing$, $(\mathrm{Tay}_a h)_\varnothing = h_\varnothing$, and thus $r_\varnothing = 0$. It follows that we can take $|\vec{x}| \ge 1$ in the supremum over $\vec{x} \in \mathbf{X}_+$ in (4.3.16). Thus,

$$\|r\|_{\Phi(X)} \le L^{-s\mathbb{1}_{j\ge j_m}} \sup_{\vec{x}\in\mathbf{X}_+} (K\ell_{j+1}^{-1})^{\vec{x}} \sup_{|\beta|_\infty \le p_\Phi} L^{-(|\vec{x}|+|\beta|_1)} L^{(j+1)|\beta|_1} |\nabla^\beta r_{\vec{x}}|. \tag{4.3.17}$$

The quantity

$$\sup_{\vec{x}\in\mathbf{X}_+} (K\ell_{j+1}^{-1})^{\vec{x}} \sup_{|\beta|_\infty\le p_\Phi} L^{-(|\vec{x}|+|\beta|_1)} L^{(j+1)|\beta|_1} |\nabla^\beta r_{\vec{x}}| \tag{4.3.18}$$

is identical to the right-hand side of [29, (3.43)] when $[\varphi_i] = 1$ and is bounded in the same way. Namely, it is shown in [29] that this quantity can be bounded by a constant times

$$L^{-d'_+} \|h\|_{\Phi'(X_+)}, \tag{4.3.19}$$

which completes the proof. ∎

Roughly speaking, the L-dependent factor in (4.3.12) implements the dimensional gain for irrelevant directions in a renormalization group step when passing from one scale to the next. In other words, we may regard the dimension of the field as improving from 1 below the mass scale to $1 + s$ above the mass scale. The $s = 0$ version of Lemma 4.3.4 is adapted to the scaling at the critical point, where $m^2 = 0$. In the noncritical case $m^2 > 0$, the dimensional gain improves greatly for $j > j_m$ as apparent from (2.3.4), and is captured more accurately by the general-s version of (4.3.12). As a consequence of the former improvement we have the following two further improvements.

[29, Proposition 1.19] The improvement in Lemma 4.3.4 propagates to [29, Proposition 1.19], which now holds as stated except with $\gamma_{\alpha,\beta}$ improved to

$$\gamma_{\alpha,\beta} = \left(L^{-(d'_\alpha + s\mathbb{1}_{j\ge j_m})} + L^{-(A+1)}\right) \left(\frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}}\right)^{|\alpha\cup\beta|}. \tag{4.3.20}$$

The right-hand side can be estimated as follows. By (4.3.2),

$$\frac{\ell_{\sigma,j+1}}{\ell_{\sigma,j}} \le 4 \begin{cases} L^{1+s\mathbb{1}_{j\ge j_m}} & j < j_x \\ 1 & j \ge j_x, \end{cases} \tag{4.3.21}$$

and hence

$$\gamma_{\alpha,\beta} \le C'' \left(L^{-(d'_\alpha + s\mathbb{1}_{j\ge j_m})} + L^{-(A+1)}\right) \times \begin{cases} L^{(1+s\mathbb{1}_{j\ge j_m})(|\alpha\cup\beta|)} & j < j_x \\ 1 & j \ge j_x. \end{cases} \tag{4.3.22}$$

[30, Proposition 4.9] As we explain next, using (4.3.20) and identical notation to that defined in and around [30, Proposition 4.9], the proposition holds as stated also for the improved norms, provided we take $A \ge 5 + s$. For this, what is required is to show that under the

hypotheses of [30, Proposition 4.9], the $\gamma_{\alpha,\beta}$ that arise in its proof obey

$$\gamma_{\alpha,\beta} \leq C \begin{cases} L^{-5} & |\alpha \cup \beta| = 0 \\ L^{-1} & |\alpha \cup \beta| = 1, 2. \end{cases} \tag{4.3.23}$$

For $|\alpha \cup \beta| = 0$, the first term of (4.3.22) obeys the bound of (4.3.23), since $d'_{\varnothing} = d+1$. For the remaining cases, $d'_{\alpha} = 2$ for $j < j_x$ and $d'_{\alpha} = 1$ for $j \geq j_x$. For $|\alpha \cup \beta| = 2$, the assumption that F_1, F_2, F_1F_2 have no component in $\mathcal{N}_{0x}$ unless $j \geq j_x$ means that we are in the case with no growth due the ratio $\ell_{\sigma,j+1}/\ell_{\sigma,j}$ in (4.3.22), and its first term again obeys the bound (4.3.23) with room to spare. Finally, when $|\alpha \cup \beta| = 1$, the first term of (4.3.22) also obeys the estimate (4.3.23), and again with room to spare. Concerning the second term of (4.3.22), given our choice of A and the fact that we need only consider the growing factor in (4.3.22) for $|\alpha \cup \beta| = 1$, it suffices to observe that

$$L^{-(A+1)} L^{1+s\mathbb{1}_{j \geq j_m}} \leq L^{-5}. \tag{4.3.24}$$

This completes the proof of the improved version of [30, Proposition 4.9].

Proof of Proposition 4.3.3. We complete the proof of Proposition 4.3.3 by modifying the proof of [30, Proposition 2.8] above the mass scale. The estimate [30, (7.22)] follows from [30, Proposition 2.7] as an estimate in terms of the modified norm pair (4.2.16), for which [30, Proposition 2.7] was verified in Section 4.3.3. The bound [30, (7.25)] with improved γ is obtained by applying the improved version of [30, Proposition 4.9]. In the remainder of the proof of [30, Proposition 2.8], we specialize each occurrence of $\mathcal{G}$ to the case $\mathcal{G} = G$ and we conclude by obtaining an analogue of [30, (7.31)] with $\tilde{G}$ replaced by G by applying Lemma 4.2.2 rather than [30, Lemma 1.2].

An additional detail is that it is required that we choose the parameter defining the space $\mathcal{N}$ to obey $p_{\mathcal{N}} > A$. Since we have changed A (depending on s), we must make a corresponding change to $p_{\mathcal{N}}$. This does not pose problems (beyond the previously discussed requirement that g needs to be chosen small depending on p), as this parameter may be fixed to be an arbitrary and sufficiently large integer (see [108, Section 7.1.3] where this point is addressed in a different context). Similarly, the value of A is immaterial and can be any fixed number in the proof of [30, Proposition 2.8]. ∎

Chapter 5

Critical initial conditions

In this chapter, we prove Theorem 2.8.1. We begin in Section 5.1.1 by showing that K_0^+ satisfies the inductive assumption required by Theorem 2.7.1. In Section 5.2, we discuss a general version of this theorem for a parameter K_0 that is independent of the coupling constants. This theorem is then applied with $K_0 = K_0^+$ by solving a set of implicit equations in Section 5.3.

Throughout this chapter, we take $n \geq 1$, drop n from the notation, and denote fields by φ. The $n = 0$ case is dealt with in [13]. We also deal only with the bulk flow (so we set $\sigma_0 = \sigma_x = 0$). The construction of the observable flow follows as in [108] (with the same critical initial conditions) once the bulk flow has been constructed.

5.1 Initial coordinates for the renormalization group

We establish norm estimates on K_0^+ in Sections 5.1.1–5.1.3. The initial coordinate K_0^+ depends on the coupling constants $(g_0, \gamma_0, \nu_0, z_0)$ of (2.2.1) and regularity of K_0 as a function of these variables is shown in Section 5.1.4.

5.1.1 Properties of the T_φ seminorm

We will need several properties of the T_φ seminorm, whose proofs can be found in [28]. We have already mentioned the product property in Proposition 2.4.3. An immediate consequence is that $\|e^{-F}\|_{T_\varphi} \leq e^{\|F\|_{T_\varphi}}$. This is improved in [28, Proposition 3.8], which states that

$$\|e^{-F}\|_{T_\varphi} \leq e^{-2F(\varphi)+\|F\|_{T_\varphi}}. \tag{5.1.1}$$

We will also use [28, Proposition 3.10], which states that if $F \in \mathcal{N}$ is a polynomial in φ of total degree $A \leq p_{\mathcal{N}}$, then

$$\|F\|_{T_\varphi} \leq \|F\|_{T_0}(1 + \|\varphi\|_\Phi)^A. \tag{5.1.2}$$

Let $x^\square$ denote the small set neighbourhood of a singleton $\{x\}$ and recall that the $\Phi_x \equiv \Phi(x^\square)$ norm of $\varphi \in (\mathbb{R}^n)^\Lambda$ was defined in (4.2.8). By taking the infimum in (5.1.2) over all possible re-definitions of φ_y for $y \notin x^\square$, we get

$$\|F\|_{T_\varphi} \leq \|F\|_{T_0}(1 + \|\varphi\|_{\Phi_x})^A \tag{5.1.3}$$

when $F \in \mathcal{N}(x^\square)$.

5.1.2 Bounds on K_0

The main estimate on $K^+_{0,x}$ is given by the following proposition. Recall that $\mathcal{D}_j$ was defined in (2.7.4).

Proposition 5.1.1. *Suppose that $V_0^+ \in \mathcal{D}_0$, with $\tilde{g}_0$ sufficiently small. If $0 \le \gamma_0 \le \tilde{g}_0$, then (with constants that may depend on L)*

$$\|K^+_{0,x}\|_{G_0} = O(\gamma_0), \quad \|K^+_{0,x}\|_{\tilde{G}_0} = O(\gamma_0/g_0). \tag{5.1.4}$$

The form of the estimates (5.1.4) can be anticipated from the definition of K_0^+. The upper bound arises from the small size of $e^{-\gamma_0 U_x^+} - 1$. For small fields, hence small U_x^+, this is of order γ_0, as reflected by the G_0 norm estimate of (5.1.4). For large fields, namely fields of size $|\varphi| \approx \tilde{g}_0^{-1/4}$, the difference $e^{-\gamma_0 U_x^+} - 1$ is roughly of size $\gamma_0 |\varphi|^4 \approx \gamma_0/\tilde{g}_0$. This effect is measured by the $\tilde{G}_0$ norm.

Before proving the proposition, we write

$$K^+_{0,x} = I^+_{0,x} J^+_x \tag{5.1.5}$$

where, by the fundamental theorem of calculus,

$$I^+_{0,x} = e^{-V^+_{0,x}} \tag{5.1.6}$$

$$J^+_x = e^{-\gamma_0 U^+_x} - 1 = -\int_0^1 \gamma_0 U^+_x e^{-t\gamma_0 U^+_x}\, dt. \tag{5.1.7}$$

Let $F \in \mathcal{N}(x^\square)$ be a polynomial of degree at most $p_{\mathcal{N}}$. Then the stability estimates [30, (2.1)–(2.2)] imply that there exists $c_3 > 0$ and, for any $c_1 \ge 0$, there exist positive constants C, c_2 such that if $V_0^+ \in \mathcal{D}_0$ then

$$\|I^+_{0,x} F\|_{T_\varphi(\mathfrak{h}_0)} \le C \|F\|_{T_0(\mathfrak{h}_0)} \begin{cases} e^{c_3 g_0 \left(1 + \|\varphi\|^2_{\Phi_x(\ell_0)}\right)} & \mathfrak{h}_0 = \ell_0 \\ e^{-c_1 k_0^4 \|\varphi\|^2_{\Phi_x(h_0)}} e^{c_2 k_0^4 \|\varphi\|^2_{\tilde{\Phi}_x(\ell_0)}} & \mathfrak{h}_0 = h_0. \end{cases} \tag{5.1.8}$$

This essentially reduces our task to estimating J_x^+. The next lemma is an ingredient for this.

Lemma 5.1.2. *There is a universal constant $\tilde{C}$ such that*

$$\|U^+_x\|_{T_\varphi(\mathfrak{h}_0)} \le 2U^+_x + \tilde{C}\mathfrak{h}_0^4 (1 + \|\varphi\|^2_{\Phi_x(\mathfrak{h}_0)}). \tag{5.1.9}$$

Proof. Let

$$M^+ = M^+_e = (\nabla^e \tau_x)^2 \tag{5.1.10}$$

so that $U_x^+ = \sum_{e \in \mathcal{U}} M_e^+$. It suffices to prove (5.1.9) with U_x^+ replaced by M^+ on both sides. In addition, we can replace the Φ_x norm by the Φ norm; the bound with the Φ_x norm then follows in the same way that (5.1.3) is a consequence of (5.1.2), since $M^+ \in \mathcal{N}(x^\square)$.

By definition of the T_φ seminorm,

$$\|\nabla^e|\varphi_x|^2\|_{T_\varphi} \le \nabla^e|\varphi_x|^2 + 2\mathfrak{h}_0(|\varphi_x| + |\varphi_{x+e}|) + 2\mathfrak{h}_0^2. \tag{5.1.11}$$

With the product property and (2.4.6), this implies that

$$\|M^+\|_{T_\varphi} \le M^+ + 2|\nabla^e|\varphi_x|^2|(2\mathfrak{h}_0(|\varphi_x| + |\varphi_{x+e}|)) + O(\mathfrak{h}_0^4)(1 + \|\varphi\|_\Phi^2). \tag{5.1.12}$$

By the inequality

$$2|ab| \le |a|^2 + |b|^2 \tag{5.1.13}$$

and another application of (2.4.6),

$$2|\nabla^e|\varphi_x|^2|(2\mathfrak{h}_0(|\varphi_x| + |\varphi_{x+e}|)) \le M^+ + O(\mathfrak{h}_0^2\|\varphi\|_\Phi^2), \tag{5.1.14}$$

and the bound on M^+ follows.

■

An immediate consequence of Lemma 5.1.2, using (5.1.1), is that for any $s \ge 0$,

$$\|e^{-sU_x^+}\|_{T_\varphi(\mathfrak{h}_0)} \le e^{\tilde{C}s\mathfrak{h}_0^4\left(1+\|\varphi\|^2_{\Phi_x(\mathfrak{h}_0)}\right)}. \tag{5.1.15}$$

Proof of Proposition 5.1.1. According to the definition of the regulator norms in (4.2.11)–(4.2.12), it suffices to prove that, under the hypothesis on γ_0,

$$\|K_{0,x}^+\|_{T_\varphi(\mathfrak{h}_0)} = O(\gamma_0\mathfrak{h}_0^4)\begin{cases} e^{\|\varphi\|^2_{\Phi_x}} & (\mathfrak{h}_0 = \ell_0) \\ e^{\frac{\mathfrak{t}}{2}\|\varphi\|_{\tilde{\Phi}}} & (\mathfrak{h}_0 = h_0). \end{cases} \tag{5.1.16}$$

For $t \in [0,1]$, let $\tilde{I}_x^+(t) = e^{-t\gamma_0 U_x^+}$. By (5.1.5), (5.1.7), and the product property,

$$\|K_{0,x}^+\|_{T_\varphi(\mathfrak{h}_0)} \le \gamma_0\|I_{0,x}^+U_x^+\|_{T_\varphi(\mathfrak{h}_0)} \sup_{t\in[0,1]} \|\tilde{I}_x^+(t)\|_{T_\varphi(\mathfrak{h}_0)}. \tag{5.1.17}$$

By (5.1.8) and Lemma 5.1.2, there exists $c_3 > 0$, and, for any $c_1 \ge 0$ there exists $c_2 > 0$, such that

$$\|I_{0,x}^+U_x^+\|_{T_\varphi(\mathfrak{h}_0)} \le O(\mathfrak{h}_0^4)\begin{cases} e^{c_3 g_0\|\varphi\|^2_{\Phi_x(\ell_0)}} & \mathfrak{h}_0 = \ell_0 \\ e^{-c_1k_0^4\|\varphi\|^2_{\Phi_x(h_0)}}e^{c_2k_0^4\|\varphi\|^2_{\tilde{\Phi}_x(\ell_0)}} & \mathfrak{h}_0 = h_0. \end{cases} \tag{5.1.18}$$

The constant in $O(\gamma_0\mathfrak{h}_0^4)$ may depend on c_1, but this is unimportant. Also, by (5.1.15),

$$\sup_{t\in[0,1]} \|\tilde{I}_x^+(t)\|_{T_\varphi(\mathfrak{h}_0)} \le e^{\tilde{C}\gamma_0\mathfrak{h}_0^4\left(1+\|\varphi\|^2_{\Phi_x(\mathfrak{h}_0)}\right)}. \tag{5.1.19}$$

Thus, for $\mathfrak{h}_0 = \ell_0$, the total exponent in our estimate for the right-hand side of (5.1.17) is

$$\tilde{C}\gamma_0\ell_0^4 + (c_3 g_0 + \tilde{C}\gamma_0\ell_0^4)\|\varphi\|^2_{\Phi_x(\ell_0)}. \tag{5.1.20}$$

This gives the $\mathfrak{h}_0 = \ell_0$ version of (5.1.16) provided that g_0 is small and γ_0 is small depending on L.

For $\mathfrak{h}_0 = h_0$, the total exponent in our estimate for the right-hand side of (5.1.17) is

$$\tilde{C}\gamma_0 k_0^4 \tilde{g}_0^{-1} + (\tilde{C}\gamma_0 k_0^4 \tilde{g}_0^{-1} - c_1 k_0^4)\|\varphi\|^2_{\Phi_x(h_0)} + c_2 k_0^4 \|\varphi\|^2_{\tilde{\Phi}_x(\ell_0)}. \tag{5.1.21}$$

This gives the $\mathfrak{h}_0 = h_0$ version of (5.1.16) provided that $\gamma_0 \le \tilde{g}_0$, $c_1 \ge \tilde{C}$, and $c_2 k_0^4 \le \mathtt{t}/2$.

All the provisos are satisfied if we choose $c_1 \ge \tilde{C}$, k_0 small depending on c_1 and $\tilde{g}_0$ small. ■

Remark 5.1.3. By a small modification to the proof of Proposition 5.1.1, it can be shown that if $M_x \in \mathcal{N}(x^\square)$ is a monomial of degree $r \le p_\mathcal{N} - 4$ (so that $M_x U_x^+$ has degree at most $p_\mathcal{N}$), then

$$\|M_x K^+_{0,x}\|_{\mathcal{G}_0} = O(\gamma_0 \mathfrak{h}_0^{4+r}). \tag{5.1.22}$$

5.1.3 Unified bound on K_0

We begin by recalling the definition of the $\mathcal{W}_j$ norm from [31]. It follows from the product property of the T_φ seminorm that the regulator norms obey the following version of the product property:

$$\|F_1 F_2\|_{\mathcal{G}_j} \le \|F_1\|_{\mathcal{G}_j}\|F_2\|_{\mathcal{G}_j} \text{ for } F_i \in \mathcal{N}(X_i^\square) \text{ with } X_1 \cap X_2 = \varnothing. \tag{5.1.23}$$

Given a map $K \in \mathcal{K}_j$, we define the $\mathcal{F}_j(\mathcal{G})$ norms (for $\mathcal{G} = G, \tilde{G}$) by

$$\|K\|_{\mathcal{F}_j(G)} = \sup_{X \in \mathcal{C}_j} \tilde{g}_j^{-f_j(a,X)} \|K(X)\|_{G_j} \tag{5.1.24}$$

$$\|K\|_{\mathcal{F}_j(\tilde{G})} = \sup_{X \in \mathcal{C}_j} \tilde{g}_j^{-f_j(a,X)} \|K(X)\|_{\tilde{G}_j^{\mathrm{t}}}, \tag{5.1.25}$$

with

$$f_j(a,X) = a(|X|_j - 2^d)_+ = \begin{cases} a(|X|_j - 2^d) & \text{if } |X|_j > 2^d \\ 0 & \text{otherwise} \end{cases} \tag{5.1.26}$$

(recall that $|X|_j$ is the number of j-blocks in X). Here a is a small constant; its value is discussed below [31, (1.46)]. The $\mathcal{W}_j = \mathcal{W}_j(\tilde{g}_j, \mathbb{V})$ norm is then defined by

$$\|K\|_{\mathcal{W}_j} = \max\Big\{\|K\|_{\mathcal{F}_j(G)}, \tilde{g}_j^{9/4}\|K\|_{\mathcal{F}_j(\tilde{G})}\Big\}. \tag{5.1.27}$$

Proposition 5.1.4. *If $V_0^+ \in \mathcal{D}_0$ with $\tilde{g}_0$ sufficiently small (depending on L), and if $\gamma_0 \le O(\tilde{g}_0^{1+a'})$ for some $a' > a$, then $\|K_0^+\|_{\mathcal{W}_0} \le O(\gamma_0)$, where all constants may depend on L.*

Proof. Let $X \in \mathcal{C}_0$. By the product property and Proposition 5.1.1,

$$\|K_0^+(X)\|_{\mathcal{G}_0} \le (c\gamma_0 \mathfrak{h}_0^4)^{|X|} = (c\gamma_0 \mathfrak{h}_0^4)^{|X| \wedge 2^d} (c\gamma_0 \mathfrak{h}_0^4)^{(|X|-2^d)_+}. \tag{5.1.28}$$

For $\mathcal{G}_0 = G_0$, we use $\mathfrak{h}_0 = \ell_0$, $(c\gamma_0 \mathfrak{h}_0^4)^{|X| \wedge 2^d} \le O(\gamma_0)$, and

$$(c\gamma_0 \mathfrak{h}_0^4)^{(|X|-2^d)_+} \le (c' \tilde{g}_0)^{(1+a')(|X|-2^d)_+} \le \tilde{g}_0^{f_0(a,X)}. \tag{5.1.29}$$

For $\mathcal{G}_0 = \tilde{G}_0$, we use $\mathfrak{h}_0 = h_0 = O(\tilde{g}_0^{-1/4})$ and, since $a' > a$,

$$(c\gamma_0 \mathfrak{h}_0^4)^{(|X|-2^d)_+} \le (c' \tilde{g}_0)^{a'(|X|-2^d)_+} \le \tilde{g}_0^{f_0(a,X)}. \tag{5.1.30}$$

Since $\gamma_0 \le \tilde{g}_0$, it follows from (5.1.28) that

$$\tilde{g}_0^{9/4} \|K_0^+\|_{\mathcal{F}_0(\tilde{G})} \le \tilde{g}_0^{9/4} O(\gamma_0 \tilde{g}_0^{-1}) \le \gamma_0, \tag{5.1.31}$$

and the proof is complete. ∎

5.1.4 Smoothness of K_0

Given any map $F : D \to \mathcal{W}_0(\tilde{g}_0, \mathbb{Z}^d)$ for $D \subset \mathbb{R}$ an open interval, let us write $F_X : D \to \mathcal{N}(X^\square)$ and $F_X^\varphi : D \to \mathbb{R}$ for the maps defined by partial evaluation of F at X and at (X, φ), respectively.

Lemma 5.1.5. *Let $D \subset \mathbb{R}$ be open and $F : D \to \mathcal{W}_0(\tilde{g}_0, \mathbb{Z}^d)$ be a map. Suppose that F_X^φ is C^2 for all $X \in \mathcal{C}_0$ and $\varphi \in (\mathbb{R}^n)^\Lambda$, and define $F^{(i)} : D \to \mathcal{W}_0(\tilde{g}_0, \mathbb{Z}^d)$ by $(F^{(i)}(t))_X^\varphi = (F_X^\varphi)^{(i)}(t)$ for $i = 1, 2$, where the right-hand side denotes the i^{th} derivative of F_X^φ. If $\|F^{(i)}(t)\|_{\mathcal{W}_0} < \infty$ for $i = 1, 2$ and $t \in D$, then $F^{(1)}$ is the (Fréchet) derivative of F.*

Proof. For $t, t + s \in D$, define $R(t, s) \in \mathcal{W}_0$ by

$$R_X^\varphi(t, s) = F_X^\varphi(t + s) - F_X^\varphi(t) - s(F_X^\varphi)'(t). \tag{5.1.32}$$

By Taylor's theorem, for any φ and X,

$$R_X^\varphi(t, s) = s^2 \int_0^1 (F_X^\varphi)''(t + us)(1 - u)\, du. \tag{5.1.33}$$

It follows that

$$\|R(t, s)\|_{\mathcal{W}_0} \le |s|^2 \sup_{u \in [0,1]} \|F''(t + us)\|_{\mathcal{W}_0} \le O(|s|^2), \tag{5.1.34}$$

so F is differentiable and its derivative satisfies $(F')^{\varphi}_X = (F^{\varphi}_X)'$. Continuity of F' follows similarly, since, by the fundamental theorem of calculus,

$$\|F'(t+s) - F'(t)\|_{\mathcal{W}_0} \leq |s| \sup_{u \in [t,t+s]} \|F''(u)\|_{\mathcal{W}_0} \leq O(|s|), \tag{5.1.35}$$

which suffices. ∎

Let us view K_0^+ as a map

$$(g_0, \gamma_0, \nu_0, z_0) \mapsto K_0^+ \in \mathcal{W}_0(\tilde{g}_0, \mathbb{Z}^d). \tag{5.1.36}$$

for $(g_0, \gamma_0, \nu_0, z_0)$ satisfying the hypotheses of Proposition 5.1.4. The map K_0 is in fact analytic away from $\gamma_0 = 0$. However, we only prove the following, which is what we need later.

Proposition 5.1.6. *Suppose that $V_0^+ \in \mathcal{D}_0$, with $\tilde{g}_0$ sufficiently small (depending on L) and $\gamma_0 \leq O(\tilde{g}_0^{1+a'})$ for some $a' > a$. Then the map $K_0^+(g_0, \gamma_0, \nu_0, z_0)$ is jointly continuous in its four variables, is C^1 in (g_0, ν_0, z_0), and (when $\gamma_0 \neq 0$) is C^1 in $(g_0, \gamma_0, \nu_0, z_0)$, with partial derivatives with respect to $t = g_0$, ν_0, and z_0 satisfying*

$$\|\partial K_0^+ / \partial t\|_{\mathcal{W}_0} = O(\gamma_0). \tag{5.1.37}$$

Moreover, K_0^+ is right-differentiable in γ_0 at $\gamma_0 = 0$.

Proof. Let t denote any one of the coupling constants g_0, γ_0, ν_0 or z_0. We drop the subscript 0 and superscript $+$, and let $K(t)$ denote K_0^+ viewed as a function of t, with the remaining coupling constants fixed. Then K^{φ}_X is smooth for any φ, X. If t is g_0, ν_0 or z_0, then

$$(K^{\varphi}_x)' = -M_x(\varphi) K^{\varphi}_x, \quad (K^{\varphi}_x)'' = M_x^2(\varphi) K^{\varphi}_x, \tag{5.1.38}$$

where M_x is τ_x^2, τ_x or $\tau_{\Delta,x}$, respectively. The maximal degree of M_x is 4, so (5.1.22) implies that

$$\|K'_x\|_{\mathcal{G}_0} \leq O(\gamma_0 \mathfrak{h}_0^8), \quad \|K''_x\|_{\mathcal{G}_0} \leq O(\gamma_0 \mathfrak{h}_0^{12}). \tag{5.1.39}$$

For t denoting γ_0, we write $U = U^+$ and $V_0 = V_0^+$. Then

$$(K^{\varphi}_x)' = -U_x(\varphi) e^{-V_x(\varphi) - \gamma_0 U_x(\varphi)}, \quad (K^{\varphi}_x)'' = U_x^2(\varphi) e^{-V_x(\varphi) - \gamma_0 U_x(\varphi)}, \tag{5.1.40}$$

and (5.1.8) and (5.1.15) imply that

$$\|K'_x\|_{\mathcal{G}_0} \leq O(\mathfrak{h}_0^4), \quad \|K''_x\|_{\mathcal{G}_0} \leq O(\mathfrak{h}_0^8). \tag{5.1.41}$$

By definition, $K_X = \prod_{x \in X} K_x$ so, for derivatives with respect to any one of the four variables

(with $\gamma_0 \neq 0$ when differentiating with respect to γ_0),

$$(K_X^\varphi)' = \sum_{x\in X} (K_x^\varphi)' K_{X\setminus x}^\varphi, \quad (K_X^\varphi)'' = \sum_{x\in X} ((K_x^\varphi)'' K_{X\setminus x}^\varphi + (K_x^\varphi)'(K_{X\setminus x}^\varphi)'). \tag{5.1.42}$$

Thus, by the product property, (5.1.39), and Proposition 5.1.1,

$$\|K_X'\|_{\mathcal{G}_0} \leq O(|X|)\gamma_0 \mathfrak{h}_0^8 (\gamma_0 \mathfrak{h}_0^4)^{|X|-1}. \tag{5.1.43}$$

when differentiating with respect to g_0, ν_0, or z_0. The bound (5.1.37) then follows from the hypothesis on γ_0. Similarly, using (5.1.41),

$$\|K_X'\|_{\mathcal{G}_0} \leq O(|X|)\mathfrak{h}_0^4 (\gamma_0 \mathfrak{h}_0^4)^{|X|-1} \tag{5.1.44}$$

when differentiating with respect to γ_0 away from $\gamma_0 = 0$. In both cases, we have

$$\|K_X''\|_{\mathcal{G}_0} \leq O(|X|^2)\mathfrak{h}_0^8 (\gamma_0 \mathfrak{h}_0^4)^{(|X|-2)\wedge 0}. \tag{5.1.45}$$

Thus, by Lemma 5.1.5, K is C^1 in any of its variables. Therefore, K is C^1 in (g_0, ν_0, z_0) on the whole domain and in all the variables when $\gamma_0 \neq 0$.

To show right-continuity in γ_0 at $\gamma_0 = 0$, fix (g_0, ν_0, z_0) and define $F \in \mathcal{W}_0$ by

$$F(X) = \begin{cases} -U_x e^{-V_{0,x}} & X = \{x\} \\ 0 & |X| > 1, \end{cases} \tag{5.1.46}$$

where $U_x, V_{0,x}$ are defined above. Let $K'(\gamma_0)$ denote the γ_0 derivative of K evaluated at $\gamma_0 > 0$. Then (5.1.40) and (5.1.42) imply that

$$F(X) - K_X'(\gamma_0) = \begin{cases} U_x K_x(\gamma_0) & X = \{x\} \\ \sum_{x\in X} K_x'(\gamma_0) K_{X\setminus x}(\gamma_0) & |X| > 1. \end{cases} \tag{5.1.47}$$

Thus, by (5.1.22), (5.1.41), and Proposition 5.1.1,

$$\|F(X) - K_X'(\gamma_0)\|_{\mathcal{G}_0} \leq \begin{cases} O(\gamma_0 \mathfrak{h}_0^8) & X = \{x\} \\ O(|X|)\mathfrak{h}_0^4 (\gamma_0 \mathfrak{h}_0^4)^{|X|-1} & |X| > 1. \end{cases} \tag{5.1.48}$$

It follows that

$$\lim_{\gamma_0 \downarrow 0} \|F - K'(\gamma_0)\|_{\mathcal{W}_0} = 0, \tag{5.1.49}$$

i.e., F is the right-derivative of K in γ_0 at $\gamma_0 = 0$. ∎

Remark 5.1.7. With γ_0 sufficiently small, the bound (5.1.37) verifies the condition

$$\|\partial K_0^+/\partial \nu_0\|_{\mathcal{W}_0} \le O(g_0^3) \tag{5.1.50}$$

required in the proof of [9, Lemma 8.6] (see [9, (8.34)]), which is in turn needed in Section 3.1.2.

5.2 Renormalization group flow

The following theorem is an extension of [9, Proposition 7.1] to non-trivial K_0. This extension is possible with only minor modifications to the proof of the $K_0 = \mathbb{1}_\varnothing$ case, due to the generality allowed by the main result of [11].

The theorem provides, for any $N \ge 1$ and for initial error coordinate K_0 in a specified domain, a choice of initial condition (ν_0^c, z_0^c) for which there exists a finite-volume renormalization group flow $(V_j, K_j) \in \mathbb{D}_j$ for $0 \le j \le N$. In order to ensure a degree of consistency amongst the sequences (V_j, K_j), which depend on the volume Λ_N, a notion of consistency must be imposed upon the collection of initial error coordinates $K_{0,\Lambda} \in \mathcal{K}_0(\Lambda)$ for varying Λ. Specifically, the family $K_{0,\Lambda}$ is required to satisfy the property $(\mathbb{Z}^d)$ of [31, Definition 1.15]. We refer to any such family as a Λ-family. As discussed in [31, Definition 1.15], any Λ-family induces an infinite-volume error coordinate $K_{0,\mathbb{Z}^d} \in \mathcal{K}_0(\mathbb{Z}^d)$ in a natural way.

Remark 5.2.1. Roughly, the requirement that the (K_Λ) form a Λ-family is that if $\Lambda \subset \Lambda'$, then K_Λ and $K_{\Lambda'}$ agree on polymers $X \in \mathcal{P}_j(\Lambda)$. However, some care must be taken with this when the polymer X "wraps around" the torus. This issue is handled using coordinate patches, as was done for discussing "torus polynomials" in (4.3.7).

Theorem 5.2.2. *Let $d = 4$. There exists a constant $a_* > 0$ and continuous functions ν_0^c, z_0^c of (m^2, g_0, K_0), defined for $(m^2, g_0) \in [0,\delta]^2$ (for some $\delta > 0$ sufficiently small) and for any $K_0 \in \mathcal{W}_0(m^2, g_0, \mathbb{Z}^d)$ with $\|K_0\|_{\mathcal{W}_0(m^2,g_0,\mathbb{Z}^d)} \le a_* g_0^3$, such that the following holds for $g_0 > 0$: if $K_{0,\Lambda} \in \mathcal{K}_0(\Lambda)$ is a Λ-family that induces the infinite-volume coordinate K_0, and if*

$$V_0 = V_0^c(m^2, g_0, K_0) = (g_0, \nu_0^c(m^2, g_0, K_0), z_0^c(m^2, g_0, K_0)), \tag{5.2.1}$$

then for any $N \in \mathbb{N}$ and $m^2 \in [\delta L^{-2(N-1)}, \delta]$, there exists a sequence $(V_j, K_j) \in \mathbb{D}_j(m^2, g_0, \Lambda)$ such that

$$(V_{j+1}, K_{j+1}) = (V_{j+1}(V_j, K_j), K_{j+1}(V_j, K_j)) \text{ for all } j < N \tag{5.2.2}$$

and (2.6.5) is satisfied. Moreover, ν_0^c, z_0^c are continuously differentiable in $g_0 \in (0,\delta)$ and $K_0 \in B_{\mathcal{W}_0(m^2,g_0,\Lambda)}(a_* g_0^3)$, and

$$\nu_0^c(m^2,0,0) = z_0^c(m^2,0,0) = 0, \quad \frac{\partial \nu_0^c}{\partial g_0} = O(1), \quad \frac{\partial z_0^c}{\partial g_0} = O(1), \tag{5.2.3}$$

where the estimates above hold uniformly in $m^2 \in [0,\delta]$.

Proof. The proof results from small modifications to the proofs of [9, Proposition 7.1] and then to [9, Proposition 8.1], where (in both cases) we relax the requirement that $K_0 = \mathbb{1}_\varnothing$, which was chosen in [9] due to the fact that $K_0 = \mathbb{1}_\varnothing$ when $\gamma = 0$. The more general condition that $K_0 \in B_{\mathcal{W}_0(m^2,g_0,\Lambda)}(a_* g_0^3)$ comes from the hypothesis of [11, Theorem 1.4] when $(m^2, g_0) = (\tilde{m}^2, \tilde{g}_0)$. By [11, Remark 1.5], no major changes to the proof result from this choice of K_0. The following paragraph outlines in more detail the modifications to the proof of [9, Proposition 7.1].

By [11, Theorem 1.4] and [11, Corollary 1.8], for any $(\tilde{m}^2, \tilde{g}_0) \in (0, \delta)^2$ and for any $\tilde{K}_0 \in B_{\mathcal{W}_0(\tilde{m}^2,\tilde{g}_0,\mathbb{Z}^d)}(a_* \tilde{g}_0^3)$, there is a neighbourhood $\mathsf{N}(\tilde{g}_0, \tilde{K}_0)$ of $(\tilde{g}_0, \tilde{K}_0)$ such that for all $(m^2, g_0, K_0) \in \tilde{\mathbb{I}}(\tilde{m}^2) \times \mathsf{N}(\tilde{g}_0, \tilde{K}_0)$, there is an infinite-volume renormalization group flow

$$(\check{V}_j, K_j) = \check{x}_j^d(\tilde{m}^2, \tilde{g}_0, \tilde{K}_0; m^2, g_0, K_0) \tag{5.2.4}$$

in *transformed variables* $(\check{V}_j, K_j)$. The transformed variables are defined in [9, Section 6.6] and a flow in the original variables can be recovered from the transformed flow. The global solution is defined by $\check{x}_j^c(m^2, g_0, K_0) = \check{x}_j^d(m^2, g_0, K_0; m^2, g_0, K_0)$ (or $\check{x}^c \equiv 0$ if $g_0 = 0$). By [11, Remark 1.5], the proof of regularity of $\check{x}^c$ can proceed as in [9]. The functions (ν_0^c, z_0^c) are given by the (ν_0, z_0) components of $\check{x}_0^c = (\check{V}_0, K_0) = (V_0, K_0)$. ∎

Remark 5.2.3. The proof of [9, Proposition 7.1], hence of Theorem 5.2.2, makes important use of the parameter $\tilde{g}_0$ in order to prove regularity of the renormalization group flow in g_0. However, once the flow has been constructed, we can and do set $\tilde{g}_0 = g_0$.

We wish to apply this theorem with $(g_0, K_0) = (g_0, K_0^+)$. We have already remarked that $K_0^+ \in \mathcal{K}_0$. It also follows from the definition that the finite-volume coordinates $K_{0,\Lambda}^+$ form a Λ-family.

Moreover, by Proposition 5.1.4, if γ_0 is sufficiently small (depending on g_0; we now take $\tilde{g}_0 = g_0$) then $K_0 = K_0^+$ satisfies the bound required by Theorem 5.2.2. However, we cannot apply the theorem immediately with this choice of K_0, due to the fact that K_0^+ depends on (g_0, ν_0, z_0). We resolve this issue in the next section.

5.3 Critical parameters

We wish to initialize the renormalization group with (ν_0, z_0) a solution to the system of equations

$$\nu_0 = \nu_0^c(m^2, g_0, K_0^+(g_0, \gamma_0, \nu_0, z_0)), \tag{5.3.1}$$

$$z_0 = z_0^c(m^2, g_0, K_0^+(g_0, \gamma_0, \nu_0, z_0)). \tag{5.3.2}$$

Such a choice of (ν_0, z_0) will be critical for K_0^+, where K_0^+ is itself evaluated at this same choice of (ν_0, z_0).

When $\gamma_0 = 0$, we get $K_0^+ = \mathbb{1}_\varnothing$, so K_0^+ no longer depends on (ν_0, z_0) and this system is solved by $(\nu_0^c(m^2, g_0, 0), z_0^c(m^2, g_0, 0))$ for any (small) $m^2, g_0 \geq 0$. Local solutions for $\gamma_0 \neq$

0 can then be constructed using a version of the implicit function theorem from [89] that allows for the continuous but non-smooth behaviour of K_0^+ in γ_0. In order to obtain global solutions with certain desired regularity properties (needed in the next section), we make use of Proposition 3.1.3, which is based on the implicit function theorem from [89].

Recall that $D(\delta, r)$ was defined in (2.8.1) by

$$D(\delta, r) = \{(w, x, y) \in [0, \delta]^3 : y \leq r(x)\}. \tag{5.3.3}$$

Proposition 5.3.1. *There exists a continuous positive-definite function $\hat{r} : [0, \delta] \to [0, \infty)$ and continuous functions $\hat{\nu}_0^c, \hat{z}_0^c \in C^{0,1,+}(D(\delta, \hat{r}))$ such that the system (5.3.1)–(5.3.2) is solved by $(\nu_0, z_0) = (\hat{\nu}_0^c, \hat{z}_0^c)$ whenever $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$. Moreover, these functions satisfy the bounds*

$$\hat{\nu}_0^c = O(g_0), \quad \hat{z}_0^c = O(g_0) \tag{5.3.4}$$

uniformly in (m^2, γ_0).

Proof. Let

$$F(m^2, g_0, \gamma_0, \nu_0, z_0) = (\nu_0, z_0) - (\nu_0^c(m^2, g_0, K_0), z_0^c(m^2, g_0, K_0)), \tag{5.3.5}$$

where $K_0 = K_0^+(g_0, \gamma_0, \nu_0, z_0)$. Then for $\delta > 0$ small and an appropriate constant $c > 0$ (depending on a_*), F is well-defined on

$$\{(m^2, g_0, \gamma_0, \nu_0, z_0) : (m^2, g_0, \gamma_0) \in D(\delta, cg_0^3), |\nu_0|, |z_0| \leq C_{\mathcal{D}} g_0\}. \tag{5.3.6}$$

Indeed, for $(m^2, g_0, \gamma_0, \nu_0, z_0)$ in this domain, Proposition 5.1.4 (with $\tilde{g}_0 = g_0$) implies that (m^2, g_0, K_0) is in the domain of (ν_0^c, z_0^c). By Theorem 5.2.2 and Proposition 5.1.6, F is C^1 in (g_0, ν_0, z_0) and also in γ_0 away from $\gamma_0 = 0$, continuous in m^2, and is right-differentiable in γ_0 at $\gamma_0 = 0$.

For fixed $(\bar{m}^2, \bar{g}_0) \in [0, \delta]^2$, set $(\bar{\nu}_0, \bar{z}_0) = (\nu_0^c(\bar{m}^2, \bar{g}_0, 0), z_0^c(\bar{m}^2, \bar{g}_0, 0))$ so that

$$F(\bar{m}^2, \bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0) = (0, 0). \tag{5.3.7}$$

By (5.1.37), at $(\bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0)$,

$$\frac{\partial K_{0,x}}{\partial \nu_0} = \frac{\partial K_{0,x}}{\partial z_0} = 0. \tag{5.3.8}$$

It follows that $D_{\nu_0, z_0} F(\bar{m}^2, \bar{g}_0, 0, \bar{\nu}_0, \bar{z}_0)$ is the identity map on $\mathbb{R}^2$. The existence of $\delta, \hat{r}$ and $\hat{\nu}_0^c, \hat{z}_0^c$ follows from Proposition 3.1.3 with $w = m^2, x = g_0, y = \gamma_0, z = (\nu_0, z_0)$, and with $r_1(g_0) = cg_0^3$, $r_2(g_0) = C_{\mathcal{D}} g_0$.

By the fundamental theorem of calculus, for any $0 < a < \gamma_0$,

$$\hat{\nu}_0^c(m^2, g_0, \gamma_0) = \hat{\nu}_0^c(m^2, g_0, a) + \int_a^{\gamma_0} \frac{\partial \hat{\nu}_0^c}{\partial \gamma_0}(m^2, g_0, t)\, dt. \tag{5.3.9}$$

Taking the limit $a \downarrow 0$ and using (5.2.3), we obtain

$$|\hat{\nu}_0^c(m^2, g_0, \gamma_0)| \leq O(g_0) + \gamma_0 \sup_{t \in (0,\gamma_0]} \left| \frac{\partial \hat{\nu}_0^c}{\partial \gamma_0}(m^2, g_0, t) \right|. \tag{5.3.10}$$

The supremum above is bounded by a constant and so the first estimate of (5.3.4) follows from the fact that $|\gamma_0| \leq \hat{r}(g_0)$ (since $\hat{r}(g_0)$ can be taken as small as desired). ■

Proof of Theorem 2.8.1. By Proposition 5.1.4, and by taking $\hat{r}$ smaller if necessary, $K_0 = K_0^+$ satisfies the estimate required by Theorem 5.2.2 whenever $(m^2, g_0, \gamma_0) \in D(\delta, \hat{r})$. The existence of the sequence (2.8.2) then follows from Theorem 5.2.2 and Proposition 5.3.1. ■

Chapter 6

Conclusion

We end with a discussion of some open problems that may be accessible by extensions to the renormalization group method discussed in this thesis. We will try to point out some of the main obstacles that must be overcome.

6.1 Other models

In order to apply the renormalization group to the models we have considered, we had to express them as perturbations of a Gaussian measure whose covariance admits an appropriate finite-range decomposition. Here we discuss other models that can be written in this way.

6.1.1 Long-range models

In [117], Wilson and Fisher suggested studying models in $d_c - \epsilon$ dimensions with $\epsilon > 0$ small and $d_c = 4$ the upper-critical dimension. Building on this, approximate values for 3-dimensional critical exponents were computed in [45,64,85]. One approach to the rigorous implementation of this idea involves studying models in dimension d (an integer) whose upper-critical dimension is $d_c + \epsilon$. This is not as problematic as considering models in fractional dimensions, as the upper-critical dimension d_c need not be the actual dimension of some ambient space.

Given a massless covariance C', the upper-critical dimension is simply a number d_c such that some class of models scales like a Gaussian model with covariance C' if and only if $d > d_c$. For instance, suppose we choose C' to decay like

$$C'_{0x} \asymp |x|^{-(d-\alpha)} \tag{6.1.1}$$

with $\alpha = \frac{1}{2}(d + \epsilon)$. Such a choice is given by

$$C' = (-\Delta)^{-\alpha/2} \tag{6.1.2}$$

for $\alpha \in (0, 2)$. Then recalling Remark 1.6.3, we might expect that $d_c = 2\alpha$. In particular, if $\alpha = \frac{1}{2}(d + \epsilon)$ with $d \leq 3$, then $d = d_c - \epsilon$.

This approach has been used to implement the renormalization group below the upper-critical dimensions in [1, 19, 27, 94]. Recently, Slade [107] has extended the approach discussed in this thesis to compute *anomalous* (non-Gaussian) critical exponents for long-range versions of the weakly self-avoiding walk and the $|\varphi|^4$ model. In particular, he showed that, as $\nu \downarrow \nu_c$

for these models, the susceptibility χ satisfies

$$\chi \asymp (\nu - \nu_c)^{-\left(1+\frac{n+2}{n+8}\frac{\epsilon}{\alpha}+O(\epsilon^2)\right)}. \tag{6.1.3}$$

By extensions of [107] to use observable fields, we think it should be possible to identify the scaling behaviour of the two-point function and possibly other correlation functions for these long-range models. In particular, this would make it possible to confirm the intriguing prediction of [51], which states that

$$\eta = 2 - \alpha \tag{6.1.4}$$

if $d = d_c - \epsilon$ for small ϵ. In other words, unlike the susceptibility, deviations from mean-field behaviour of the two-point function cannot be detected to any order in ϵ.

Remark 6.1.1. Models at and above the upper-critical dimension exhibit *asymptotic freedom*. In our context, this means that $\|K_j\|_{\mathcal{W}_j} \to 0$, $\nu_j, z_j \to 0$, and $g_j \to 0$ in the massless regime $m^2 = 0$. Below d_c, we do not have asymptotic freedom, as reflected by the lack of exact asymptotics in (6.1.3). In some ways, this is advantageous (see [107]), but in others it creates additional difficulties that must be overcome.

6.1.2 The $O(n)$ model and self-avoiding walk

Recall that the Hamiltonian of the $O(n)$ model was defined in (1.4.15). On Λ, it takes the form

$$H_J(\sigma) = -\frac{1}{2}\sigma \cdot J\sigma, \quad \sigma \in S^{n-1}. \tag{6.1.5}$$

This was derived from the $|\varphi|^4$ model by taking a suitable $g \to \infty$ limit. The restriction to small coupling g is deeply embedded into the method we use, but the Kac-Siegert transformation (see [17]) offers an alternative approach to the study of this model.

Namely, let $\Omega = (S^{n-1})^\Lambda$ and let $d\sigma$ denote the product measure on Ω, where S^{n-1} is equipped with the uniform measure. The partition function of the $O(n)$ model is given by

$$Z_J = \int_\Omega e^{-H_J(\sigma)}\, d\sigma. \tag{6.1.6}$$

When J is a positive-definite symmetric matrix, the Gaussian measure $d\mu_J(\varphi)$ with covariance J is well-defined and satisfies the elementary identity

$$e^{-H_J(\sigma)} = e^{\frac{1}{2}\sigma\cdot J\sigma} = \int_{(\mathbb{R}^n)^\Lambda} e^{\sigma\cdot\varphi}\, d\mu_J(\varphi). \tag{6.1.7}$$

Interchanging the order of integration, we can write

$$Z_J = \int_{(\mathbb{R}^n)^\Lambda} e^{-\sum_{x\in\Lambda} L(\varphi_x)}\, d\mu_J(\varphi), \tag{6.1.8}$$

where

$$L(t) = -\log \int_{S^{n-1}} e^{\sigma_0 \cdot t} \, d\sigma_0, \quad t \in \mathbb{R}^n \tag{6.1.9}$$

is the negative logarithm of the Laplace transform of the sphere. Since L is a rotation- and reflection-invariant analytic function and $L(0) = 0$, it has the form

$$L(t) = \nu|t|^2 + g|t|^4 + \sum_{k=3}^{\infty} c_{2k}|t|^{2k}. \tag{6.1.10}$$

Letting $J = (-\Delta + \gamma^2)^{-1}$, we have

$$d\mu_J(\varphi) \propto e^{-\frac{1}{2}(\gamma^2|\varphi|^2 + \varphi \cdot (-\Delta\varphi))}. \tag{6.1.11}$$

Thus, we can express the partition function as a perturbed $|\varphi|^4$ model.

By a procedure as in Section 2.2, the analysis of this model can be reformulated in terms of the evolution of an effective interaction Z_j with initial condition $Z_0 = (I_0 \circ K_0)(\Lambda)$. Once again, the initial error coordinate K_0 will be coupled to I_0, but we expect that the critical parameters ν_0^c, z_0^c can be identified by an implicit function theorem argument as in Section 5.3.

However, estimates on K_0 (which are straightforward to obtain by a more careful computation of (6.1.10)) indicate that K_0 is not of order g_0^3, which is required to invoke Theorem 5.2.2. In fact, $K_0 = O(g_0^{3/2})$.

One approach to possibly overcoming this issue is the following: First, note that the irrelevant error coordinate should shrink by a factor of $O(L^{-1})$ after each renormalization group step. Thus, after the first $j_g = \lfloor \log_L g^{-3/2} \rfloor$ steps, we should be left with an error coordinate K_j of size $O(g^3)$. A careful analysis of the renormalization group flow is required during these first j_g steps. However, the flow of coupling constants in this regime need only be computed to first order; indeed, any second-order terms would in any case be of higher order than the error term, which is of order $g^{3/2}$.

Remark 6.1.2. Similarly, it is possible to re-cast the strictly self-avoiding walk as a perturbation of weakly self-avoiding walk using a supersymmetric integral representation obtained in [26]. The covariance of the form $(-\Delta + \gamma^2)^{-1}$ in this case corresponds to a model of *spread-out* self-avoiding walk with exponentially decaying jump probabilities. Once again, the initial error coordinate is not of order $O(g_0^3)$.

6.2 Other observable quantities

Here we discuss some open problems concerning the models studied in this thesis.

6.2.1 The correlation length

Our results concerning the finite-order correlation lengths ξ_p are insufficient for recovering the predicted behaviour of the *true* correlation length ξ. The estimate (3.2.2) shows that the errors in the approximation (3.2.1) of the two-point function decay at any desired polynomial rate, but this is not sufficient for studying ξ, which would need exponentially decaying errors. The current estimates follow from the covariance bounds (2.3.4) on the decomposition of [5]. Although it may not be possible to improve the bounds for this particular decomposition, this should be possible for the decomposition of [22] (see [22, p. 445]).

However, even if this were possible, exponentially decaying errors would require that the weights ℓ_j decay like e^{-cL^j} above the mass scale. This, in turns, would cause the weights $\ell_{\sigma,j}$ defined by (2.4.21) to *grow* so quickly that the third bound in (4.3.1) would fail in a major way. Thus, it seems new ideas would be needed to study the correlation length (note, however, that the correlation length for the 1-component $|\varphi|^4$ model was successfully studied by a block-spin renormalization group method in [69]).

6.2.2 Inversion of the Laplace transform

One of the main motivations for studying the susceptibility and finite-order correlation length for walks is the possibility of recovering information about the growth of the partition function c_T and the mean-squared distance $\langle |X(T)|^2 \rangle$ as $T \to \infty$. In particular, recalling the discussion in Section 1.1, one may try to derive logarithmic corrections to the predicted scaling relations (1.6.22)–(1.6.23) as a consequence of Theorem 1.7.1(ii)–(iii).

This approach was successfully used in [24], where the mean-squared displacement of a hierarchical model of weakly self-avoiding walk is recovered by inversion of the Laplace transform. This requires control over the two-point function in a sector of the complex plane larger than what has been achieved here on the Euclidean lattice.

6.2.3 The broken symmetry phase

The authors of [61] studied weakly self-avoiding walk on a four-dimensional hierarchical lattice in the phase $\nu < \nu_c$. They employed a renormalization group method similar to the one used here in order to show that the walks exhibit a broken supersymmetry in this phase. In particular, they showed that the asymptotic density

$$\rho(g,\nu) = \lim_{x\to\infty} \lim_{N\to\infty} \frac{1}{G_{0,N}(g,0,\nu)} \int_0^\infty c_{N,T}(0) L_T^x \, dT \tag{6.2.1}$$

of such a walk is non-zero for $\nu = \nu_c - \varepsilon$ with $\varepsilon > 0$ small. More precisely, they obtained logarithmic corrections to mean-field behaviour given by $\rho(g, \nu_c - \varepsilon) \sim C\varepsilon(\log \varepsilon^{-1})^{1/2}$. It would be of great interest to see whether an analogue of this fact could be proved on the Euclidean lattice by an extension of the method described in this thesis.

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Appendix A

Finite-volume approximation

In this appendix, we prove Proposition 1.8.4. Since we are only dealing with walks, we can drop the parameter $n = 0$ from our notation. We will use n to denote a different quantity below.

A.1 A monotonicity lemma

Let P^n be the projection of $\mathbb{Z}^d$ onto the discrete torus of side n, which we denote $\mathbb{Z}^d_n$. Then P^n has a natural action on the path space $(\mathbb{Z}^d)^{[0,\infty)}$. We let $X^n = P^n(X)$ be the projection of X and note that X^n is a simple random walk on $\mathbb{Z}^d_n$.

We call $h = (h_x)_{x\in\mathbb{Z}^d}$ a *field of path functionals* if $h_x : (\mathbb{Z}^d)^{[0,\infty)} \to \mathbb{R}$ is a function on continuous-time paths for each $x \in \mathbb{Z}^d$; a simple example is given by the local time functional. We assume that the *random* field $h(X) = (h_x(X))_{x\in\mathbb{Z}^d}$ has finite support almost surely, i.e., with probability 1, $h_x(X) = 0$ for all but finitely many x. Denote by $h(X^n)$ the corresponding random field for X^n, i.e., for $x \in \mathbb{Z}^d_n$,

$$h_x(X^n) = \sum_{y\in\mathbb{Z}^d} h_{x+ny}(X). \tag{A.1.1}$$

Given a positive integer k, we define $Q_k \subset \mathbb{Z}^d$ by $Q_k = \{y \in \mathbb{Z}^d : 0 \le y_i < k,\ i = 1, \dots, d\}$. Then, for integers $n, k \ge 1$,

$$\sum_{y\in Q_k} h_{x+ny}(X^{kn}) = \sum_{y\in Q_k}\sum_{z\in\mathbb{Z}^d} h_{x+ny+knz}(X) = \sum_{y\in\mathbb{Z}^d} h_{x+ny}(X) = h_x(X^n), \tag{A.1.2}$$

and it follows by summation over $x \in \mathbb{Z}^d_n$ that

$$\sum_{x\in\mathbb{Z}^d_{kn}} h_x(X^{kn}) = \sum_{x\in\mathbb{Z}^d_n} h_x(X^n). \tag{A.1.3}$$

Lemma A.1.1. *Let $n, k \ge 1$ and let f and g be nonnegative fields of path functionals with finite support almost surely. Then*

$$\sum_{x\in\mathbb{Z}^d_{kn}} f_x(X^{kn})g_x(X^{kn}) \le \sum_{x\in\mathbb{Z}^d_n} f_x(X^n)g_x(X^n). \tag{A.1.4}$$

Proof. By (A.1.3) and (A.1.2),

$$\sum_{x\in\mathbb{Z}^d_{kn}} f_x(X^{kn})g_x(X^{kn}) = \sum_{x\in\mathbb{Z}^d_n}\sum_{y\in Q_k} f_{x+ny}(X^{kn})g_{x+ny}(X^{kn}). \tag{A.1.5}$$

By nonnegativity and two more applications of (A.1.2),

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d_n}\sum_{y\in Q_k} f_{x+ny}(X^{kn})g_{x+ny}(X^{kn}) &\le \sum_{x\in\mathbb{Z}^d_n}\left(\sum_{y\in Q_k} f_{x+ny}(X^{kn})\right)\left(\sum_{y\in Q_k} g_{x+ny}(X^{kn})\right)\\ &= \sum_{x\in\mathbb{Z}^d_n} f_x(X^n)g_x(X^n).\end{aligned} \tag{A.1.6}$$

■

A.2 Convergence of the finite-volume approximation

For $L \ge 2$ and $N \ge 1$ note that Λ_N is the torus $\mathbb{Z}^d_n$ with $n = L^N$. Thus, X^{L^N} is the simple random walk on Λ_N. For $F_T = F_T(X)$ any one of the functions L^x_T, I_T, C_T of X defined in (1.6.5)–(1.6.7), we write $F_{N,T} = F_T(X^{L^N})$. For instance, with $n = L^N$,

$$L^x_{N,T} = \int_0^T \mathbb{1}_{X^n_t = x}\, dt, \quad I_{N,T} = \sum_{x\in\Lambda_N} (L^x_{N,T})^2. \tag{A.2.1}$$

We apply Lemma A.1.1 with $k = L$ and $n = L^N$ for three choices of f, g:

$$I_{N+1,T} \le I_{N,T} \qquad (f_x = g_x = L^x_T), \tag{A.2.2}$$

$$C_{N+1,T} \le C_{N,T} \qquad (f_x = \textstyle\sum_{e\in\mathcal{U}} L^{x+e}_T,\ g_x = L^x_T), \tag{A.2.3}$$

$$\sum_{x\in\Lambda_{N+1}} |\nabla^e L^x_{N+1,T}|^2 \le \sum_{x\in\Lambda_N} |\nabla^e L^x_{N,T}|^2 \quad (f_x = g_x = |\nabla^e L^x_T|). \tag{A.2.4}$$

Summation of (A.2.4) over unit vectors $e \in \mathbb{Z}^d$ also gives

$$\sum_{x\in\Lambda_{N+1}} |\nabla L^x_{N+1,T}|^2 \le \sum_{x\in\Lambda_N} |\nabla L^x_{N,T}|^2. \tag{A.2.5}$$

The following is a re-statement of Proposition 1.8.4.

Proposition A.2.1. *Let $d > 0$, $g > 0$ and $\gamma < g$. For all $\nu \in \mathbb{R}$,*

$$\lim_{N\to\infty} G_x(g,\gamma,\nu) = G_x(g,\gamma,\nu) \tag{A.2.6}$$

and

$$\lim_{N\to\infty} \chi_N(g,\gamma,\nu) = \chi(g,\gamma,\nu). \tag{A.2.7}$$

In fact, χ_N and χ are analytic in $\mathrm{Re}\nu > \nu_c$ and $\chi_N \to \chi$ uniformly on compact subsets of this domain.

Proof. It suffices to prove (A.2.6) and (A.2.7). Analyticity is a property of the Laplace transform and uniform convergence on compact sets follows from Montel's theorem. For pointwise convergence, we will only prove the case $\gamma \geq 0$. The proof for $\gamma < 0$ can be found in [13].

Fix $x \in \mathbb{Z}^d$, and consider N sufficiently large that x can be identified with points in Λ_N. By (1.6.20), (A.2.2) and (A.2.5)

$$c_{N,T}(x) \leq c_{N+1,T}(x). \tag{A.2.8}$$

Thus, (A.2.6) follows by monotone convergence, once we show that

$$\lim_{N\to\infty} c_{N,T}(x) = c_T(x). \tag{A.2.9}$$

To show this, recall that we are identifying the vertices of Λ_N with nested subsets of $\mathbb{Z}^d$. We can thus define $\partial\Lambda_N$ to be the inner vertex boundary of Λ_N. We set

$$c^*_{N,T}(x) = E_0^{\Lambda_N}\left(e^{-U_{g,\gamma,T}}\mathbb{1}_{X(T)=b}\mathbb{1}_{\{X([0,T])\cap\partial\Lambda_N\neq\varnothing\}}\right) \tag{A.2.10}$$

$$c^*_{T}(x) = E_0\left(e^{-U_{g,\gamma,T}}\mathbb{1}_{X(T)=b}\mathbb{1}_{\{X([0,T])\cap\partial\Lambda_N\neq\varnothing\}}\right). \tag{A.2.11}$$

Since walks which do not reach $\partial\Lambda_N$ make equal contributions to both $c_T(x)$ and $c_{N,T}(x)$, we have

$$c_T(x) - c^*_T(x) = c_{N,T}(x) - c^*_{N,T}(x). \tag{A.2.12}$$

Thus,

$$|c_T(x) - c_{N,T}(x)| = |c^*_T(x) - c^*_{N,T}(x)| \leq c^*_T(x) + c^*_{N,T}(x). \tag{A.2.13}$$

Let $P_0^{\Lambda_N}$ and P_0 be the measures associated with $E_0^{\Lambda_N}$ and E_0, respectively. With Y_t a rate-$2d$ Poisson process with measure P,

$$\begin{aligned} c^*_T(x) + c^*_{N,T}(x) &\leq P_0(X([0,T])\cap\partial\Lambda_N \neq \varnothing) + P_0^{\Lambda_N}(X([0,T])\cap\partial\Lambda_N\neq\varnothing) \\ &\leq 2\mathsf{P}(Y_T \geq \mathrm{diam}(\Lambda_N)) \to 0 \end{aligned} \tag{A.2.14}$$

as $N\to\infty$. This completes the proof of (A.2.6).

Finally, by monotone convergence of G_N to G, for $\nu \in \mathbb{R}$,

$$\lim_{N\to\infty}\chi_N(g,\gamma,\nu) = \sum_{x\in\mathbb{Z}^d}\lim_{N\to\infty} G_{x,N}(g,\gamma,\nu)\mathbb{1}_{x\in\Lambda_N} = \chi(g,\gamma,\nu), \tag{A.2.15}$$

which proves (A.2.7). ∎

Appendix B

Moments of the free Green function

In this appendix we prove Proposition 3.3.1.

B.1 Main result

The following is a re-statement of Proposition 3.3.1. Since we are only dealing with the free Green function, we set

$$G_x(m^2) = G_x(0, 0, m^2). \tag{B.1.1}$$

Proposition B.1.1. *Let c_p be the constant defined by (1.7.4). For all dimensions $d > 2$ and all $p > 0$, as $m^2 \downarrow 0$,*

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(0, m^2) = \mathsf{c}_p^p m^{-(p+2)}(1 + O(m)). \tag{B.1.2}$$

In particular, $\xi_p(0, \varepsilon) = \mathsf{c}_p \varepsilon^{-1/2}(1 + O(\varepsilon^{1/2}))$ as $\varepsilon \downarrow 0$.

The last sentence in the the proposition follows immediately from (B.1.2) and the fact that $\chi(0, m^2) = m^{-2}$ (recall (1.5.12)), so it suffices to prove (B.1.2).

The case $p = 2$ of (B.1.2) can be obtained easily from the identity

$$\sum_{x \in \mathbb{Z}^d} |x|^2 G_x(m^2) = -\Delta_{\mathbb{R}^d} \hat{G}(0), \tag{B.1.3}$$

where $\hat{G}$ is the Fourier transform of G. Higher even moments could in principle be computed by further differentiating $\hat{G}$. We adopt a different approach for general $p > 0$, based on the finite range decomposition of $(-\Delta_{\mathbb{Z}^d} + m^2)^{-1}$ given in [5, 22]. This finite range decomposition also provides the basis for the renormalization group method.

B.2 Riemann sum approximation

We will make use of the following elementary result.

Lemma B.2.1. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function with support in a box of side t centred at the origin. Then there is a constant C such that for any $\epsilon > 0$,*

$$\left| \int_{\mathbb{R}^d} f(x)\, dx - \epsilon^d \sum_{x \in \mathbb{Z}^d} f(\epsilon x) \right| \leq C(t\epsilon)^d. \tag{B.2.1}$$

Proof. For any $x \in \mathbb{Z}^d$, let $S_x(\epsilon)$ denote the square of side ϵ centred at $\epsilon x \in \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} f(x)\, dx = \sum_{x \in \mathbb{Z}^d} \int_{S_x(\epsilon)} f(y)\, dy. \tag{B.2.2}$$

By the mean value theorem, there exists $y_x = y_x(\epsilon) \in S_x(\epsilon)$ such that

$$\int_{S_x(\epsilon)} f(y)\, dy = \epsilon^d f(y_x). \tag{B.2.3}$$

Thus,

$$\left| \int_{\mathbb{R}^d} f(x)\, dx - \epsilon^d \sum_{x \in \mathbb{Z}^d} f(\epsilon x) \right| \le \epsilon^d \sum_{x \in \mathbb{Z}^d} |f(y_x) - f(\epsilon x)|. \tag{B.2.4}$$

By the Lipschitz condition on f, each summand on the right-hand side is $O(\epsilon)$. By the support assumption on f, there are at most $O(t^d/\epsilon)$ such summands and the result follows. ∎

B.3 Covariance decomposition

The finite-range decomposition of the finite-volume covariance discussed in Section 2.3 is derived from a decomposition of the infinite-volume covariance (whose construction is the main result of [5]) of the form

$$G_x(m^2) = \sum_{j=1}^{\infty} C_{j;x}(m^2). \tag{B.3.1}$$

Recall that the finite-range property refers to the fact that $C_{j;x}(m^2) = 0$ if $|x| \ge \frac{1}{2}L^j$, where $L > 1$ is fixed arbitrarily. We review some properties of this decomposition, from [5, 10], before proving Proposition B.1.1. The positive-definiteness of the finite range decomposition is not needed here, and L need not be large.

The terms $C_{j;x}(m^2)$ are defined in [10, Section 6.1] by

$$C_{j;x}(m^2) = \begin{cases} \displaystyle\int_0^{\frac{1}{2}L} \phi_t^*(x; m^2)\, \frac{dt}{t} & (j = 1) \\ \displaystyle\int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t^*(x; m^2)\, \frac{dt}{t} & (j \ge 2) \end{cases} \tag{B.3.2}$$

(in [10], the notation $C_{j;0,x}$ and $\phi_t^*(0, x; m^2)$ was used instead). Here, ϕ_t^* is a function of $x \in \mathbb{R}^d$ and $m^2 > 0$ given in [5, Example 1.1]. It satisfies the finite range property that $\phi_t^*(x; m^2) = 0$ for $|x| > t$. It was also shown in [5] that there exists a function ϕ_t satisfying the same finite range property but giving a decomposition of the *continuum* Green function:

$$(-\Delta_{\mathbb{R}^d} + m^2)^{-1}_{0x} = \int_0^{\infty} \phi_t(x; m^2) \frac{dt}{t}. \tag{B.3.3}$$

Moreover, by [5, (1.37)], for $|x| \le t$,

$$\phi_t^*(x; m^2) = \phi_t(x; m^2) + O(t^{-(d-1)}(1 + m^2 t^2)^{-k}). \tag{B.3.4}$$

This allows us to approximate the discrete Green function by the continuum one, for which the moments are easily computed. We have set the constant c present in [5] equal to 1, which we can do by rescaling ϕ_t^*.

As t approaches 0, the error bound in (B.3.4) degenerates. However, to estimate (B.1.1), it suffices to restrict to $x \neq 0$. Then, since $x \in \mathbb{Z}^d$, the finite range property permits replacement of the lower bound in the range of integration for $j = 1$ in (B.3.2) by $\frac{1}{2}$, and the contribution due to $j = 1$ can be estimated in the same way as the terms $j \ge 2$.

Also, by [5, (1.34)], for any k there is a constant C_k such that

$$|D_x \phi_t(x; m^2)| \le C_k t^{-(d-1)}(1 + m^2 t^2)^{-k}. \tag{B.3.5}$$

We fix a choice of k which obeys $k > \frac{1}{2}(p + 1)$ and use only this choice. By [5, (1.38)], there exists a function $\bar{\phi}$ such that

$$\phi_t(x; m^2) = t^{-(d-2)} \bar{\phi}\left(\frac{x}{t}; m^2 t^2\right). \tag{B.3.6}$$

B.4 Proof of main result

Proof of Proposition 3.3.1. We begin by writing

$$\sum_{x \in \mathbb{Z}^d} |x|^p G_x(m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} C_{j;x}(m^2) = M(m^2) + E(m^2), \tag{B.4.1}$$

where the main and error terms are respectively

$$M(m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \, \frac{dt}{t}, \tag{B.4.2}$$

$$E(m^2) = \sum_{x \in \mathbb{Z}^d} |x|^p \sum_{j=1}^{\infty} \left(C_{j;x} - \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x; m^2) \frac{dt}{t} \right). \tag{B.4.3}$$

We first compute the main term M. By (B.3.6),

$$\phi_t(x; m^2) = m^{d-2} \phi_{mt}(mx; 1). \tag{B.4.4}$$

Therefore, by Riemann sum approximation,

$$\sum_{x\in\mathbb{Z}^d} |x|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x;m^2)\,\frac{dt}{t} \tag{B.4.5}$$

$$= m^{-(p+2)} m^d \sum_{x\in\mathbb{Z}^d} |mx|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_{mt}(mx;1)\,\frac{dt}{t} \tag{B.4.6}$$

$$= m^{-(p+2)} \int_{\mathbb{R}^d} |x|^p \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_{mt}(x;1)\,\frac{dt}{t} + O(L^{(p+1)j} L^{-2k(j-j_m)_+}),$$

where the error estimate follows from (B.3.5) and (2.4.18). Summation over j gives

$$M(m^2) = \mathsf{c}_p^p m^{-(p+2)} + O(m^{-(p+1)}), \tag{B.4.7}$$

where we used (B.3.3) for the first term, and we used $2k > p+1$ and Lemma 3.3.2 for the second term.

For the error term, it follows from (B.3.2), (B.3.4), and the observation that the lower bound in the range of integration for the $j=1$ term in (B.3.2) can be changed to $\frac{1}{2}$ that

$$C_{j;x} = \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \phi_t(x;m^2)\frac{dt}{t} + O(L^{-j(d-1)}(1+m^2L^{2j})^{-k})\mathbb{1}_{|x|\le L^j}. \tag{B.4.8}$$

Therefore, again using (2.4.18), we have

$$E(m^2) = \sum_{j=1}^{\infty} \sum_{|x|\le L^j} |x|^p O(L^{-j(d-1)} L^{-2k(j-j_m)_+}) \tag{B.4.9}$$

$$= \sum_{j=1}^{\infty} O(L^{(p+1)j} L^{-2k(j-j_m)_+}). \tag{B.4.10}$$

With $2k > p+1$ and Lemma 3.3.2, this gives $E(m^2) = O(m^{-(p+1)})$, and the proof is complete. ∎

Appendix C

An implicit function theorem

In this appendix, we prove Proposition 3.1.3.

C.1 Implicit function theorem with a parameter

We make use of [89, Chapter 4, Theorem 9.3], which is a version of the implicit function theorem that allows for a continuous, rather than differentiable, parameter. While the precise statement of [89, Chapter 4, Theorem 9.3] takes this parameter from an open subset of a Banach space, by [89, Chapter 4, Theorem 9.2], the parameter can in fact be taken from an arbitrary metric space. With this minor change, we restate [89, Chapter 4, Theorem 9.3] as the following proposition.

Proposition C.1.1. *Let A be a metric space, let W, X be Banach spaces, and let $B \subset W$ be an open subset. Let $F : A \times B \to X$ be continuous, and suppose that F is C^1 in its second argument. Let $(\alpha, \beta) \in A \times B$ be a point such that $F(\alpha, \beta) = 0$ and $D_2F(\alpha, \beta)^{-1}$ exists. Then there are open balls $M \ni \alpha$ and $N \ni \beta$ and a unique continuous mapping $f : M \to N$ such that $F(\xi, f(\xi)) = 0$ for all $\xi \in M$.*

We also use the following lemma, which is a small modification of [89, Chapter 3, Theorem 11.1]. In particular, it considers functions that may only be left- or right-differentiable.

Lemma C.1.2. *Let F be a mapping as in the previous proposition with $A \subset \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$. In addition, suppose that F is left-differentiable (respectively, right-differentiable) in α_2 at (α, β), with $\alpha = (\alpha_1, \alpha_2)$. If f is a continuous mapping defined in a neighbourhood of α, such that $F(\xi, f(\xi)) = 0$, then f is left-differentiable (respectively, right-differentiable) in α_2 at α.*

C.2 Main result

The above results lead to the following proposition, which is a re-statement of Proposition 3.1.3. Recall that $D(\delta, r)$ is defined in (2.8.1) by

$$D(\delta, r) = \{(w, x, y) \in [0, \delta]^3 : y \leq r(x)\}. \tag{C.2.1}$$

Proposition C.2.1. *Let $\delta > 0$, and let r_1, r_2 be continuous positive-definite functions on $[0, \delta]$. Set*

$$D(\delta, r_1, r_2) = \{(w, x, y, z) \in D(\delta, r_1) \times \mathbb{R}^n : |z| \leq r_2(x)\}, \tag{C.2.2}$$

and let F be a continuous function on $D(\delta, r_1, r_2)$ that is C^1 in (x, z). Suppose that for all $(\bar{w}, \bar{x}) \in [0, \delta]^2$ there exists $\bar{z}$ such that both $F(\bar{w}, \bar{x}, 0, \bar{z}) = 0$ and $D_Y F(\bar{w}, \bar{x}, 0, \bar{z})$ is invertible. Then there is a continuous positive-definite function r on $[0, \delta]$ and a continuous map $f : D(\delta, r) \to \mathbb{R}^n$ that is C^1 in x and such that $F(w, x, y, f(w, x, y)) = 0$ for all $(w, x, y) \in D(\delta, r)$. Moreover, if F is left-differentiable (respectively, right-differentiable) in y at some point (w, x, y, z), then f is left-differentiable (respectively, right-differentiable) at (w, x, y).

Proof. Take any $(\bar{w}, \bar{x}) \in [0, \delta] \times (0, \delta]$ and let $R(\bar{w}, \bar{x})$ be the maximal radius s such that for all $(w, x, y) \in B(\bar{w}, \bar{x}, 0; s)$ there exists z such that both $F(w, x, y, z) = 0$ and $D_Z F(w, x, y, z)$ is invertible. By continuity of $(D_Z F(w, x, y, z))^{-1}$ near $(\bar{w}, \bar{x}, 0, \bar{z})$, and by Proposition C.1.1 (applied to the restriction of F to $A \times B$, for some $A \ni (\bar{w}, \bar{x}, 0)$ and an open set $B \ni \bar{z}$), we have $R(\bar{w}, \bar{x}) > 0$ and there is a continuous function

$$f_{\bar{w}, \bar{x}} : B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})) \to \mathbb{R}^n \tag{C.2.3}$$

such that $F(w, x, y, f_{\bar{w}, \bar{x}}(w, x, y)) = 0$ for all $(w, x, y) \in B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x}))$. Moreover, the unique solution to $F(w, x, y, z) = 0$ is given by $z = f_{\bar{w}, \bar{x}}(w, x, y)$ for all such (w, x, y). By an application of Lemma C.1.2 (with $\alpha_1 = (w, x), \alpha_2 = y$), we see that $f_{\bar{w}, \bar{x}}$ is left- or right-differentiable in y wherever F is. By another application of Lemma C.1.2 (with $\alpha_1 = (w, y), \alpha_2 = x$), we see that $f_{\bar{w}, \bar{x}}$ is C^1 in x.

Set $R(\bar{w}, 0) = 0$ for all $\bar{w} \in [0, \delta]$, and let

$$D_f = \bigcup_{(\bar{w}, \bar{x}) \in [0, \delta]^2} B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})). \tag{C.2.4}$$

We define $f(w, 0, 0) = 0$ and, for $x > 0$,

$$f(w, x, y) = f_{\bar{w}, \bar{x}}(w, x, y) \quad \text{for} \quad (w, x, y) \in B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x})). \tag{C.2.5}$$

By uniqueness, this function is well-defined. Continuity of f at $(w, 0, 0)$ follows from the fact that $|f(w, x, y)| \leq r_2(x)$. The remaining desired regularity properties of f follow from those of the $f_{\bar{w}, \bar{x}}$. It remains to show that $D(\delta, r) \subset D_f$ for some continuous positive-definite function r on $[0, \delta]$.

First, let us show that R is continuous on $[0, \delta]^2$. Let $\bar{x} > 0$ and fix $0 < \epsilon < R(\bar{w}, \bar{x})$. Then for any $(\bar{w}', \bar{x}') \in [0, \delta] \times (0, \delta]$ such that $|(\bar{w}, \bar{x}) - (\bar{w}', \bar{x}')| < \epsilon$, we have $B(\bar{w}', \bar{x}', 0; R(\bar{w}, \bar{x}) - \epsilon) \subset B(\bar{w}, \bar{x}, 0; R(\bar{w}, \bar{x}))$ by maximality of R. It follows that $R(\bar{w}', \bar{x}') \geq R(\bar{w}, \bar{x}) - \epsilon$. By a similar argument, $R(\bar{w}', \bar{x}') \leq R(\bar{w}, \bar{x}) + \epsilon$, so $|R(\bar{w}, \bar{x}) - R(\bar{w}', \bar{x}')| \leq \epsilon$. Thus, R is continuous on $[0, \delta] \times (0, \delta]$. Continuity at $\bar{x} = 0$ follows from the fact that $R(\bar{w}, \bar{x}) \leq r_1(\bar{x})$ uniformly in $\bar{w}$.

For $\bar{x} \in [0, \delta]$, let

$$r(\bar{x}) = \inf(R(\bar{w}, \bar{x}) : \bar{w} \in [0, \delta]). \tag{C.2.6}$$

Since $R(\cdot, \bar{x})$ is continuous, $r(\bar{x}) > 0$ for $\bar{x} > 0$. Moreover, $0 \leq r(0) \leq r_1(0) = 0$, so r is positive-definite. Continuity of r follows from joint continuity of R. For any $(w, x, y) \in D(\delta, r)$ (with this choice of r),

$$|(w, x, y) - (w, x, 0)| = |y| < r(x) \leq R(w, x), \tag{C.2.7}$$

so $(w, x, y) \in B(w, x, 0; R(w, x))$. We conclude that $D(\delta, r) \subset D_f$. ■