Geometric properties of the space of Lagrangian self-shrinking tori in $\mathbb{R}^4$

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF

Doctor of Philosophy

in

THE FACULTY OF GRADUATE AND POSTDOCTORAL STUDIES

(Mathematics)

The University of British Columbia
(Vancouver)

May 2017

Abstract

We prove that any sequence $\{F_n : \Sigma \to \mathbb{R}^4\}$ of conformally branched compact Lagrangian self-shrinkers to the mean curvature flow with uniform area upper bound has a convergent subsequence, if the conformal structures do not degenerate. When Σ has genus one, we can drop the assumption on non-degeneracy the conformal structures. When Σ has genus zero, we show that there is no branched immersion of Σ as a Lagrangian self-shrinker, generalizing the rigidity result of [52] in dimension two by allowing branch points.

When the area bound is small, we show that any such Lagrangian self-shrinking torus in $\mathbb{R}^4$ is embedded with uniform curvature estimates.

For a general area bound, we prove that the entropy for the Lagrangian self-shrinking tori can only take finitely many values; this is done by deriving a Łojasiewicz-Simon type gradient inequality for the branched conformal self-shrinking tori.

Using the finiteness of entropy values, we construct a piecewise Lagrangian mean curvature flow for Lagrangian immersed tori, along which the Lagrangian condition is preserved, area is decreasing, and the compact type I singularities with a fixed area upper bound can be perturbed away in finitely many steps. This is a Lagrangian version of the construction for embedded surfaces in $\mathbb{R}^3$ in [17].

In the noncompact situation, we derive a parabolic Omori-Yau maximum principle for a proper mean curvature flow when the ambient space has lower bound on ℓ-sectional curvature. We apply this to show that the image of Gauss map is preserved under a proper mean curvature flow in euclidean spaces with uniform bounded second fundamental form. This generalizes a result of Wang [53] for compact immersions. We also prove a Omori-Yau maximum principle for properly immersed self-shrinkers, which improves a result in [8].

Lay Summary

In this thesis we study the mean curvature flow of Lagrangian submanifolds.

We show several compactness theorems for the space of compact Lagrangian self-shrinkers in $\mathbb{R}^4$. When we restrict to torus, we show that Lagrangian self-shrinking torus in $\mathbb{R}^4$ with small area is embedded with uniform curvature estimates.

For a general area bound, we prove that the entropy for the Lagrangian self-shrinking tori can only take finitely many values; this is done by deriving a Łojasiewicz-Simon type gradient inequality for the self-shrinking tori. Using this, we construct a piecewise Lagrangian mean curvature flow for Lagrangian immersed tori.

In the noncompact situation, we derive a parabolic Omori-Yau maximum principle for a proper mean curvature flow. We apply this to show that the image of Gauss map is preserved under a proper mean curvature flow in euclidean spaces with uniform bounded second fundamental form.

Preface

All of the work presented in this thesis was conducted as I was a PhD student in the Mathematics Department of University of British Columbia.

Materials in Chapter 3,4 are from [11], [12]. These are close and extensive collaboration with Professor Jingyi Chen, with both authors contributing about equally. The first drafts of the manuscript [11], [12] were written by me, and then revised by both authors.

Materials in Chapter 5 are from [38], where I am the sole author.

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Acknowledgments

I would like to express my sincere gratitude to my advisor Professor Jingyi Chen for the continuous support of my Ph.D study, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis.

I would also like to thank my family: my parents and my wife Iris for supporting me throughout writing this thesis and my life in general.

Chapter 1

Introduction

The thesis is divided into three parts. In the first part, we study the space of compact Lagrangian self-shrinking surfaces in $\mathbb{R}^4$. We prove two compactness theorems and show that Lagrangian self-shrinking tori with small area are embedded and have uniform curvature estimates.

In the second part, we study a dynamical property of Lagrangian self-shrinking tori. We show that Lagrangian self-shrinking tori can attain only finitely many entropy values under an area upper bound. Then we define a piecewise Lagrangian mean curvature flow for Lagrangian immersed tori in $\mathbb{R}^4$ which preserves the Lagrangian condition and the Maslov class, decreases area and avoids compact type I singularities with any given area upper bound in finite steps.

In the third part, we study mean curvature flow of noncompact immersions and derive a parabolic Omori-Yau maximum principle for proper mean curvature flow.

1.1 Compactness of the space of Lagrangian self-shrinking surfaces

One of the major challenging problems in the study of Lagrangian mean curvature flow is to formulate a weak version of the mean curvature flow that preserves the Lagrangian condition and goes beyond singular time, as the well-known weak forms of mean curvature flow such as the Brakke flow or the level set approach do not work well in the Lagrangian setting.

In general dimension and codimension, it is known that the rescaled flow at a finite time singularity converges weakly (up to subsequences) to a self-similar solution of mean curvature flow $\{\nu_t : t \in (-\infty, 0)\}$ so that $\nu_t = \sqrt{-t}\nu_{-1}$. If ν_{-1} is a smooth immersion F, it is called a self-shrinker and it satisfies the self-shrinking equation

$$\vec{H} = -\frac{1}{2}F^{\perp}, \tag{1.1}$$

where $\vec{H}$ is the mean curvature vector and $F^{\perp}$ is the normal component of the position vector F.

Given a Lagrangian immersions $F : \Sigma^n \to \mathbb{R}^{2n}$. It is shown in [51] that the Lagrangian condition is preserved along the mean curvature flow when Σ is compact. Therefore, the self-shrinkers arise from Lagrangian mean curvature flow are Lagrangian immersions. When $n = 1$, the Lagrangian condition is automatically satisfied by smooth curves and self shrinking solutions are studied in [1].

In the first part of the thesis, we restrict our attention to compact Lagrangian self-shrinkers in $\mathbb{R}^4$. For arbitrary n, Smoczyk [52] showed that there is no Lagrangian self-shrinking immersion in $\mathbb{R}^{2n}$ with zero first Betti number. In particular, there is no immersed Lagrangian self-shrinking sphere in $\mathbb{R}^4$. To establish compactness properties of the moduli space of compact Lagrangian shrinkers, it is crucial to generalize Smoczyk's result to *branched* Lagrangian immersions as the limit of a sequence of immersions may not be an immersion anymore.

Indeed, the rigidity holds for branched Lagrangian self-shrinking spheres in $\mathbb{R}^4$:

Theorem 1.1.1. *There does not exist any branched conformal Lagrangian self-shrinking sphere in* $\mathbb{R}^4$.

At any immersed point, the mean curvature form $\alpha_H = \iota_{\vec{H}}\omega$ of a Lagrangian self-shrinker satisfies a pair of differential equations that form a first order elliptic system. The key ingredient in the proof of Theorem 1.1.1 is to show that α_H extends smoothly across the branch points. The self-shrinker equation yields an L^∞ bound on α_H and this is useful to show that α_H satisfies the first order elliptic system distributionally on $\mathbb{S}^2$. Smoothness of the extended mean curvature form then follows from elliptic theory.

The main application of Theorem 1.1.1 is to derive compactness results of compact Lagrangian self-shrinkers.

Let Σ be a fixed compact oriented smooth surface and $F_n : \Sigma \to \mathbb{R}^4$ be a sequence of branched conformally immersed Lagrangian self-shrinkers in $\mathbb{R}^4$. Let $\langle \cdot, \cdot \rangle$ be the standard Euclidean metric on $\mathbb{R}^4$ and h_n be the Riemannian metric on Σ which is conformal to the pull back metric $F_n^* \langle \cdot, \cdot \rangle$ on Σ such that either

1. it has constant Gauss curvature -1 if the genus of Σ is greater than one, or
2. (Σ, h_n) is $\mathbb{C}/\{1, a+bi\}$ with the flat metric, where $-\frac{1}{2} < a \leq \frac{1}{2}$, $b \geq 0$, $a^2 + b^2 \geq 1$ and $a \geq 0$ whenever $a^2 + b^2 = 1$.

It is well known that the moduli space of the conformal structures on Σ is parametrized by metrics of the above form.

We now state our compactness result.

Theorem 1.1.2. *Let $F_n : (\Sigma, h_n) \to \mathbb{R}^4$ be a sequence of branched conformally immersed Lagrangian self-shrinkers with a uniform area upper bound Λ. Suppose that the sequence of metrics $\{h_n\}$ converges smoothly to a Riemannian metric h on Σ. Then a subsequence of $\{F_n\}$ converges smoothly to a branched conformally immersed Lagrangian self-shrinker $F_\infty : (\Sigma, h) \to \mathbb{R}^4$.*

Note that there is a universal positive lower bound on the extrinsic diameters for two dimensional branched conformal compact shrinkers (cf. section 3.2). The limit F_∞ cannot be a constant map.

The proof of Theorem 1.1.2 uses the observation that self-shrinkers are in fact minimal immersions into $(\mathbb{R}^4, G)$, where G is a metric on $\mathbb{R}^4$ conformal to the Euclidean metric. The advantage of this viewpoint is that we are then able to use the bubble tree convergence of harmonic maps developed in [42]. In particular, Theorem 1.1.1 shows that no bubble is formed during the process and thus the convergence is smooth.

It is interesting to compare Theorem 1.1.2 with the compactness results of Colding-Minicozzi [18] on embedded self-shrinkers in $\mathbb{R}^3$ and of Choi-Schoen [15], Fraser-Li [25] on embedded minimal surfaces in three dimensional manifold N (with or without boundary) with nonnegative Ricci curvature. In their cases, they use the local singular compactness theorem (see for example Proposition 2.1 of [18]), which says that for any sequence of embedded minimal surfaces Σ_n in N with a uniform (global or local) area or genus upper bound, there is an embedded minimal surface Σ such that a subsequence of $\{\Sigma_n\}$ converges smoothly and locally (with finite multiplicities) to Σ away from finitely many points in Σ. A removable singularity theorem, which is based on the maximum principle and is true only in the codimension one case, is needed to prove the local singular compactness theorem. Hence a similar statement is not available in higher codimension. For embedded minimal surfaces in a 3-manifold, Colding and Minicozzi have proven deep compactness results ([16] and the reference therein).

In the case of arbitrary codimension, the regularity of the limit of a sequence of minimal surfaces (as a stationary varifold given by Geometric Measure Theory) is a subtle issue. Not much is known except for some special cases, such as Gromov's compactness theorem on pseudo holomorphic curves [27].

When $\Sigma = \mathbb{T}$ has genus one, we can drop the assumption on the convergence of conformal structures in Theorem 1.1.2.

Theorem 1.1.3. *Let $F_n : \mathbb{T} \to \mathbb{R}^4$ be a sequence of branched conformally immersed Lagrangian self-shrinking tori with a uniform area upper bound Λ. Then a subsequence of $\{F_n\}$ converges smoothly to a branched conformally immersed Lagrangian self-shrinking torus $F_\infty : \mathbb{T} \to \mathbb{R}^4$.*

Our strategy of proving Theorem 1.1.3 is to rule out the possibility of degeneration of the conformal structures induced by the immersions F_n on $\mathbb{T}$. To do this, we use the general bubble tree convergence results in [10], [13] that allow the conformal structures to degenerate. The key observation is that when the conformal structures degenerate, some homotopically nontrivial closed curves in the torus must be pinched to points. Thus the limiting surface is a finite union of spheres (arising from collapse of the closed curves and from the bubbles at the energy concentration points) which are branched Lagrangian self-shrinkers in $\mathbb{R}^4$, but Theorem 1.1.1 forbids their existence. Therefore, for the genus one case, the conformal structures cannot degenerate, and the desired result then follows from Theorem 1.1.2.

If $\Lambda < 32\pi$, the Willmore functional of a self-shrinker with area upper bound

Λ is less than 8π; a classical theorem of Li and Yau [36] then asserts that all such Lagrangian self-shrinking tori must be embedded and without branch points. Using results of Lamm-Schätzle in [32] and Theorem 1.1.3, we show that the upper bound can be increased above Li-Yau's estimate. We introduce the following definition:

Definition 1.1.1. Let Λ be a positive number. Let $\mathfrak{X}_\Lambda$ be the space of branched conformally immersed Lagrangian self-shrinking tori with area less than or equal to Λ.

Theorem 1.1.4. *There are positive numbers $\varepsilon_0, \varepsilon_1$ and C_0, where $\varepsilon_1 \leq \varepsilon_0$, so that*

1. *(No Branch Points) All elements in $\mathfrak{X}_{32\pi+\varepsilon_0}$ are immersed, and all elements in $\mathfrak{X}_{32\pi+\varepsilon_1}$ are embedded.*
2. *(Curvature Estimates) If $F \in \mathfrak{X}_{32\pi+\varepsilon_0}$, then the second fundamental form of F is bounded by C_0.*

1.2 Finiteness of entropy and piecewise Lagrangian mean curvature flow

In [17], Colding and Minicozzi introduce an entropy functional (see (2.13)) of a hypersurface (cf. [39]) and show that the sphere and the generalized cylinders are the only entropy stable self-shrinking hypersurfaces. Using this and a compactness theorem [18] on the space of embedded self-shrinking surfaces in $\mathbb{R}^3$, they constructed in [17] a piecewise mean curvature flow for embedded surfaces in $\mathbb{R}^3$ (under some assumptions), such that if a uniform diameter estimate holds then the flow shrinks to a round point.

In [33], [34] the authors studied the Lagrangian entropy stability of Lagrangian self-shrinking immersions and obtained entropy instability results. In particular, Li and Zhang showed in [34] that if $F : M^n \to \mathbb{R}^{2n}$ is a closed orientable Lagrangian self-shrinker and the first Betti number of M is greater than 1 then F is Lagrangian entropy unstable [1]. Since there is no simply connected closed Lagrangian self-

[1]More precisely, in [33], [34], the authors study the Lagrangian $\mathscr{F}$-stability of a Lagrangian immersion. That $\mathscr{F}$-instability implies entropy instability is proved in [17] for the hypersurface case and can be generalized to immersions of higher codimension. See [3] and also Chapter 2.

shrinker, all closed orientable Lagrangian self-shrinkers in $\mathbb{R}^4$ are Lagrangian entropy unstable.

When the area upper bound Λ is not small (as in Theorem 1.1.4), it is not known whether any branched conformal Lagrangian self-shrinking torus with nonempty branch locus exists or not. The possible existence of branch points of elements in $\mathfrak{X}_\Lambda$ is a serious obstacle for applications to Lagrangian mean curvature flow as one would hope to perturb the branched Lagrangian surface to a nearby Lagrangian immersion, but such resolution of singularity in the Lagrangian setting, even in dimension two, is not available. Note that it is in general difficult to study nearby branched immersions by deforming them along the normal vector fields. In particular, it is hard to study stability problem of branched Lagrangian self-shrinking immersions as in [17], [34], and Weinstein's Lagrangian neighbourhood theorem [54] does not apply to the branched case. In view of all these and the special feature of the embedded graphic representation of a surface near a self-shrinker in the codimension one case, the idea of the piecewise mean curvature flow introduced in [17] is not directly applicable to the Lagrangian case in $\mathbb{R}^4$, even with the compactness theorems 1.1.2, 1.1.3.

In order to construct a piecewise Lagrangian mean curvature flow for a torus, we observe that one can bypass the issue of branchedness of a limiting surface in $\mathfrak{X}_\Lambda$ by controlling the entropy values $\lambda(F)$ attained by the self-shrinkers, where for a branched immersion $F:\mathbb{T}\to\mathbb{R}^4$ its entropy is defined by

$$\lambda(F)=\sup_{x_0\in\mathbb{R}^4,t_0>0}\frac{1}{4\pi t_0}\int_{\mathbb{T}}e^{-\frac{|F(x)-x_0|^2}{4t_0}}\,d\mu_F.$$

The theorem below is a crucial ingredient in our construction of piecewise Lagrangian mean curvarture flow for tori, but it is also interesting in its own right: it is equivalent to that in the induced metric from $G=e^{-\frac{|x|^2}{4}}\delta_{ij}$ on $\mathbb{R}^4$ the areas of branched Lagrangian self-shrinking tori in $\mathfrak{X}_\Lambda$ can only take a finite number (depending on Λ) of values for any given Λ.

Theorem 1.2.1. *Let $\lambda:\mathfrak{X}_\Lambda\to[0,\infty)$ be the entropy function which sends F to its entropy $\lambda(F)$. Then the image of λ is finite for any given Λ.*

To prove Theorem 1.2.1, we derive a Łojasiewicz-Simon gradient inequality for branched conformal self-shrinking 2-dimensional tori. The celebrated Łojasiewicz-Simon gradient inequality is proved in [49] with important applications to the harmonic map flow and the minimal cones. Since the pioneering work [49], the inequality and its variations have wide applications in geometric problems. For mean curvature flow, Schulze [48] used the inequality to prove a uniqueness result for compact embedded singularity of tangent flow. Colding and Minicozzi [19] derived a Łojasiewicz gradient inequality in a noncompact setting and settled the uniqueness problem for all generic singularities of mean convex mean curvature flow at all singularities.

The classical Łojasiewicz-Simon gradient inequality is established for real analytic functionals over a compact manifold whose Euler-Lagrange operator is elliptic and of order 2. In our case, we are concerned with the entropy functional λ, which is, at a self-shrinker, just the area of the shrinker in $(\mathbb{R}^4, G)$ up to a universal constant. However, in our situation, the self-shrinkers might be branched and the Euler-Lagrange operator of the area functional fails to be elliptic at the branch locus, so Simon's infinite dimensional version of the Łojasiewicz inequality in [49] is not directly applicable. To overcome the difficulty, we consider the real analytic energy functional $\mathscr{E}$ defined on the mapping space $C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$ together with the Teichmüller space of $\mathbb{T}$, and continue to view self-shrinkers as branched minimal immersions in $(\mathbb{R}^4, G)$. The functional $\mathscr{E}$ has been extensively used in minimal surface theory, especially, in showing existence of minimal surfaces. A critical point of $\mathscr{E}$ corresponds to a branched conformal self-shrinking torus. Since the space of conformal structures on a torus is two dimensional, the ellipticity of the L^2-gradient of $\mathscr{E}$ at a critical point of $\mathscr{E}$ for each *fixed* conformal structure enables us to show that the second order derivative $\mathscr{L}$ of $\mathscr{E}$ at the critical point is a Fredholm operator of index zero, which is sufficient to derive the desired gradient inequality. Theorem 1.2.1 is then a direct consequence of the gradient inequality and the compactness Theorem 1.1.3.

We then apply Theorem 1.2.1 to construct a piecewise Lagrangian mean curvature flow for Lagrangian immersed tori $F : \mathbb{T} \to \mathbb{R}^4$ (see Definition 4.2.1). We show that all type I singularities with an arbitrarily given area upper bound can be perturbed in finitely many steps, where a smooth Lagrangian mean curvature flow of a torus restarts at each step, such that the same kind of singularities will not appear in the last step. We remark that the perturbation can be made arbitrarily small while fixing the number of perturbations performed. Note that, in the special case of small area, Theorem 1.1.4 is sufficient since the existence of a nearby Lagrangian immersion of the torus around a limiting surface in $\mathfrak{X}_\Lambda$ (now immersed) follows from the Lagrangian neighbourhood theorem.

Our main result on Lagrangian mean curvature flow in a weak form is

Theorem 1.2.2. *Let $F : \mathbb{T} \to \mathbb{R}^4$ be an immersed Lagrangian torus and let $\Lambda, \delta > 0$ be given constants. Then there exists a piecewise Lagrangian mean curvature flow $\{F_t^i : i = 0, 1, \cdots, k-1\}$ with initial condition F, where $k \leq |\lambda(\mathfrak{X}_\Lambda)| < \infty$, such that the singularity at time t_k is not a type I singularity modelled by a compact self-shrinker with area less than or equal to Λ. Moreover, the Maslov class of each immersion is invariant along the flow.*

Under an additional assumption, we prove a similar result in Theorem 4.2.1 for the case of genus larger than one.

1.3 Parabolic Omori-Yau maximum principle and some applications

In the last chapter, we consider mean curvature flow of noncompact manifold. The main result is a parabolic Omori-Yau maximum principle for mean curvature flow of noncompact manifold. First we recall the Omori-Yau maximum principle for the Laplace operator.

Let (M, g) be a Riemannian manifold and let $u : M \to \mathbb{R}$ be a twice differentiable function. If M is compact, u is maximized at some point $x \in M$. At this point, basic advanced calculus implies

$$u(x) = \sup u, \quad \nabla^M u(x) = 0, \quad \Delta^M u(x) \leq 0.$$

Here ∇^M and Δ^M are respectively the gradient and Laplace operator with respect to the metric g. When M is noncompact, a bounded function might not attain a maximum. In this situation, Omori [41] and later Yau [56] provide some noncompact versions of maximum principle. We recall the statement in [56]:

Theorem 1.3.1. *Let (M,g) be a complete noncompact Riemannian manifold with bounded below Ricci curvature. Let $u : M \to \mathbb{R}$ be a bounded above twice differentiable function. Then there is a sequence $\{x_i\}$ in M such that*

$$u(x_i) \to \sup u, \quad |\nabla u|(x_i) \to 0, \quad \limsup_{i\to\infty} \Delta^M u(x_i) \le 0.$$

Maximum principles of this form are called Omori-Yau maximum principles. The assumption on the lower bound on Ricci curvature in Theorem 1.3.1 has been weaken in, e.g., [9], [43]. On the other hand, various Omori-Yau type maximum principles have been proved for other elliptic operators and on solitons in geometric flows, such as Ricci solition [8] and self-shrinkers in mean curvature flows [14]. The Omori-Yau maximum principles are powerful tools in studying noncompact manifolds and have a lot of geometric applications. We refer the reader to the book [2] and the reference therein for more information.

In this paper, we derive the following parabolic version of Omori-Yau maximum principle for mean curvature flow.

Theorem 1.3.2 (Parabolic Omori-Yau Maximum Principle)**.** *Let $n \ge 2$ and $m \ge 1$. Let $(\overline{M}^{n+m}, \bar{g})$ be an $n+m$-dimensional noncompact complete Riemannian manifold such that the $(n-1)$-sectional curvature of $\overline{M}$ is bounded below by $-C$ for some positive constant C. Let M^n be a n-dimensional noncompact manifold and let $F : M^n \times [0,T] \to \overline{M}$ be a proper mean curvature flow. Let $u : M \times [0,T] \to \mathbb{R}$ be a continuous function which satisfies*

1. $\sup_{(x,t)\in M\times[0,T]} u > \sup_{x\in M} u(\cdot,0)$,
2. *u is twice differentiable in $M \times (0,T]$, and*
3. *(sublinear growth condition) There are $B > 0$, $\alpha \in [0,1)$ and some $y_0 \in \overline{M}$ so that*

$$u(x,t) \le B(1 + d_{\overline{M}}(y_0, F(x,t))^{\alpha}), \quad \forall (x,t) \in M \times [0,T]. \tag{1.2}$$

Then there is a sequence of points $(x_i,t_i)\in M\times(0,T]$ *so that*

$$u(x_i,t_i)\to\sup u,\ \ |\nabla^{M_{t_i}}u(x_i,t_i)|\to 0,\ \ \liminf_{i\to\infty}\left(\frac{\partial}{\partial t}-\Delta^{M_{t_i}}\right)u(x_i,t_i)\geq 0. \quad (1.3)$$

We remark that the above theorem makes no assumption on the curvature of the immersion F_t. See section 5.1 for the definition of ℓ-sectional curvature.

With this parabolic Omori-Yau maximum principle, we derive the following results.

In [53], the author studies the gauss map along the mean curvature flow in the euclidean space. He shows that if the image of the gauss map stays inside a totally geodesic submanifold in the Grassmanians, the same is also true along the flow when the initial immersion is compact. As a first application, we extend Wang's theorem to the noncompact situation.

Theorem 1.3.3. *Let* $F_0 : M^n\to\mathbb{R}^{n+m}$ *be a proper immersion and let* $F : M^n\times[0,T]\to\mathbb{R}^{n+m}$ *be a mean curvature flow of* F_0 *with uniformly bounded second fundamental form. Let* Σ *be a compact totally geodesic submanifold of the Grassmanians of n-planes in* $\mathbb{R}^{n+m}$. *If the image of the Gauss map* γ *satisfies* $\gamma(\cdot,0)\subset\Sigma$, *then* $\gamma(\cdot,t)\subset\Sigma$ *for all* $t\in[0,T]$.

As a corollary, we have the following:

Corollary 1.3.1. *Let* $F_0 : M^n\to\mathbb{R}^{2n}$ *be a proper Lagrangian immersion and let* $F : M\times[0,T]\to\mathbb{R}^{2n}$ *be a mean curvature flow with uniformly bounded second fundamental form. Then* F_t *is Lagrangian for all* $t\in[0,T]$.

The above result is well-known when M is compact [51], [53]. Various forms of Corollary 1.3.1 are known to the experts (see remark 5 in Chapter 5).

The second application is to derive a Omori-Yau maximum principle for the $\mathscr{L}$-operator of a proper self-shrinker. The $\mathscr{L}$ operator is introduced in [17] when the authors study the entropy stability of a self-shrinker. Since then it proves to be an important operator in mean curvature flow. Using Theorem 1.3.2, we prove

Theorem 1.3.4. *Let* $\widetilde{F} : M^n\to\mathbb{R}^{n+m}$ *be a properly immersed self-shrinker and let* $f : M^n\to\mathbb{R}$ *be a twice differentiable function so that*

$$f(x)\leq C(1+|\widetilde{F}(x)|^{\alpha}) \quad (1.4)$$

for some $C > 0$ *and* $\alpha \in [0,1)$*. Then there exists a sequence* $\{x_i\}$ *in* M *so that*

$$f(x_i) \to \sup_M f, \quad |\nabla f|(x_i) \to 0, \quad \limsup_{i\to\infty} \mathscr{L} f(x_i) \le 0. \tag{1.5}$$

The above theorem is a generalization of Theorem 5 in [8] since we assume weaker conditions on f.

Chapter 2

Background in mean curvature flow and Lagrangian immersions

2.1 Mean curvature flow and self-shrinkers

A family of immersions $F_t : \Sigma \to \mathbb{R}^N$ from an n-dimensional manifold Σ to the Euclidean space is said to satisfy the mean curvature flow if

$$\frac{\partial F_t}{\partial t} = \vec{H}. \tag{2.1}$$

Here $\vec{H}$ is the mean curvature vector given by $\vec{H} = \mathrm{tr}A$, the trace of the second fundamental form

$$A(X,Y) = (D_X Y)^{\perp}$$

(here D is the standard connection on $\mathbb{R}^n$ and $\perp$ denotes the normal component of a vector with respect to the immersion F_t). In local coordinates $(x^1 . \cdots x^n)$ of Σ, the second fundamental form A and the mean curvature vector $\vec{H}$ are given by

$$A_{ij} = \left(\partial_{ij} F\right)^{\perp}, \;\; \vec{H} = \sum_{i,j=1}^{n} g^{ij} A_{ij} = \Delta_g F$$

where Δ_g is the Laplace-Beltrami operator in the induced metric g.

When M is compact, standard parabolic PDE theory implies that the mean curvature flow exists for a short time $[0,T)$ and is unique.

An immersion is called self-shrinking (or a self-shrinker) if it satisfies

$$\vec{H} = -\frac{1}{2}F^{\perp}. \tag{2.2}$$

If F is self-shrinking, then up to a family of diffeomorphisms, the family of immersions

$$\{\sqrt{-t}F : t \in [-1,0)\}$$

solves the mean curvature flow. The self-shrinkers model the singularity of mean curvature flow (cf. [29, 30, 55]).

Let G be a metric on $\mathbb{R}^{n+k}$ defined by

$$G(\cdot,\cdot) = e^{-\frac{|x|^2}{2n}}\langle\cdot,\cdot\rangle. \tag{2.3}$$

The following lemma is well-known and is proved by Angenent [4] for hypersurfaces. The proof can be generalized to arbitrary codimension.

Lemma 2.1.1. *The immersion F satisfies equation (2.2) if and only if F is a minimal immersion with respect to the metric G defined in (2.3) on $\mathbb{R}^{n+k}$.*

2.2 Lagrangian immersions

Next we consider immersions $F : \Sigma \to \mathbb{R}^{2n}$. F is called Lagrangian if $F^*\omega = 0$, where

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

is the standard symplectic form on $\mathbb{R}^{2n}$. Let $\langle\cdot,\cdot\rangle, J$ be the standard euclidean metric and complex structure on $\mathbb{R}^{2n}$ respectively. It is known that ω, J and $\langle\cdot,\cdot\rangle$ are related by

$$\langle JX,JY\rangle = \langle X,Y\rangle, \ \ \omega(X,Y) = \langle JX,Y\rangle \tag{2.4}$$

for any $X,Y \in T_x\mathbb{R}^{2n}, x \in \mathbb{R}^{2n}$. Thus F is Lagrangian if and only if J sends the tangent vectors of Σ to the normal vectors. In particular, by (2.4), $J\vec{H}(x)$ is tangent to $F(\Sigma)$ at x for any Lagrangian immersion F. The mean curvature form α_H is the

1-form on Σ defined by: for all $x \in \Sigma$ and $Y \in T_x\Sigma$,

$$\alpha_H(Y) = \omega(\vec{H}(x), (F_*)_x Y) = g(J\vec{H}(x), (F_*)_x Y). \tag{2.5}$$

Let d denotes the exterior differentiation of Σ. For Lagrangian immersions, it is shown in [20] that α_H is closed:

$$d\alpha_H = 0\ . \tag{2.6}$$

If F is also a self-shrinker, we have the following equation for α_H:

Lemma 2.2.1. *Let F be a Lagrangian self-shrinker. Then α_H satisfies*

$$d^*\alpha_H = -\frac{1}{4}\alpha_H(\nabla_g |F|^2), \tag{2.7}$$

where d^* is the formal adjoint of d on Σ with respect to g.

Proof. (see also [7], p.1521) In local coordinates $(x^1, \cdots, x^n)$ of Σ,

$$\alpha_H = \sum_{i=1}^{n} (\alpha_H)_i dx^i$$

where the coefficients are given by

$$\begin{aligned}(\alpha_H)_i &= \langle J\vec{H}, \partial_i F\rangle \\ &= -\frac{1}{2}\langle JF^{\perp}, \partial_i F\rangle \\ &= -\frac{1}{2}\langle JF, \partial_i F\rangle.\end{aligned}$$

Note that we have used (2.2) in the second equality. Now fix a point $p \in \Sigma$ and take the normal coordinates at p. At p, $g_{ij} = g^{ij} = \delta_{ij}$ and $\Gamma_{ij}^k = 0$. At p, we have

$$\begin{aligned} d^* \alpha_H &= -\sum_{i,j=1}^m g^{ij} (\alpha_H)_{i;j} \\ &= -\sum_{i=1}^m (\alpha_H)_{i,i} \\ &= \sum_{i=1}^m \frac{1}{2} \partial_i \langle JF, \partial_i F \rangle \\ &= \sum_{i=1}^m \frac{1}{2} \left(\langle J\partial_i F, \partial_i F \rangle + \langle JF, \partial_{ii}^2 F \rangle \right). \end{aligned}$$

Using $J\partial_i F \perp \partial_i F$ and $\partial_{ii}^2 F = A_{ii}$ for each i at p,

$$\begin{aligned} d^* \alpha_H &= \frac{1}{2} \sum_{i=1}^m \langle JF, A_{ii} \rangle \\ &= \frac{1}{2} \langle JF, \vec{H} \rangle \\ &= -\frac{1}{2} \langle F, J\vec{H} \rangle \\ &= -\frac{1}{2} \alpha_H (F^\top) \\ &= -\frac{1}{4} \alpha_H (\nabla_g |F|^2), \end{aligned}$$

where $F^\top$ is the tangential component of F and we have used

$$F^\top = \sum_{i=1}^m \langle F, \partial_i F \rangle \, \partial_i F = \frac{1}{2} \nabla_g |F|^2$$

at p. □

2.3 $\mathscr{F}$-stability and entropy stability

The entropy λ and $\mathscr{F}$-stability are introduced in [17] for an embedded self-shrinking hypersurfaces and are later carried over in [3], [33], [34] for all codimensions. The Lagrangian case is discussed in [33], [34] and the definition of Lagrangian $\mathscr{F}$-stability is introduced therein. We start with recalling the definition of the $\mathscr{F}$-

functional, the entropy λ and the related stability. When we consider Lagrangian immersions, we will assume $k = n$.

Definition 2.3.1. Let $(x_0, t_0) \in \mathbb{R}^{n+k} \times \mathbb{R}_{>0}$. The $\mathscr{F}$-functional of an immersion $F : \Sigma^n \to \mathbb{R}^{n+k}$ is given by

$$\mathscr{F}_{x_0,t_0}(F) = (4\pi t_0)^{-\frac{n}{2}} \int_{\Sigma} e^{-\frac{|F(x)-x_0|^2}{4t_0}} d\mu_F. \tag{2.8}$$

The $\mathscr{F}$-functional characterizes the self-shrinkers as follows: $F : \Sigma \to \mathbb{R}^{n+k}$ is a self-shrinker if and only if

$$\frac{d}{ds}\mathscr{F}_{x_s,t_s}(F_s)\bigg|_{s=0} = 0$$

for all variations (x_s, t_s, F_s) such that $(x_0, t_0, F_0) = (0, 1, F)$.

We recall that a normal vector field X along a Lagrangian immersion is called a Lagrangian variation if

$$d(\iota_X \omega) = 0. \tag{2.9}$$

Definition 2.3.2. A self-shrinker F is called (Lagrangian) $\mathscr{F}$-stable if for all (Lagrangian) variations F_s, there is a variation (x_s, t_s) of $(0, 1)$ so that

$$\frac{d^2}{ds^2}\mathscr{F}_{x_s,t_s}(F_s)\bigg|_{s=0} \geq 0.$$

In [34], Li and Zhang calculate the second variation of the $\mathscr{F}$-functional of a Lagrangian immersion with respect to the Lagrangian variations. They prove:

Theorem 2.3.1. *Let Σ be a compact orientable n-dimensional manifold whose first Betti number is greater than* 1. *If $F : \Sigma \to \mathbb{R}^{2n}$ is a Lagrangian self-shrinker, then F is Lagrangian $\mathscr{F}$-unstable.*

When $F : \Sigma \to \mathbb{R}^{2n}$ is a Lagrangian immersion, let $F_s : \Sigma \to \mathbb{R}^{2n}$ be a normal variation of F such that each F_s is a Lagrangian immersion. In this case, the normal variational vector field $X = \frac{d}{ds}|_{s=0}F_s$ can be identified with a closed 1-form on Σ by $X \mapsto -\iota_X \omega$. The converse is also true as seen in the following elementary lemma.

Recall that if α is a 1-form on a Riemannian manifold (Σ, g) then $\alpha^\sharp$ is the vector field on Σ uniquely determined by

$$g(\alpha^\sharp, Y) = \alpha(Y), \quad \forall Y \in T\Sigma. \tag{2.10}$$

Lemma 2.3.1. *Let $F : \Sigma^n \to \mathbb{R}^{2n}$ be a compact Lagrangian immersion and let α be a closed 1-form on Σ. Then there is a family of Lagrangian immersions $F_s : \Sigma \to \mathbb{R}^{2n}$ so that $F_0 = F$ and*

$$\left.\frac{d}{ds}\right|_{s=0} F_s = J\alpha^\sharp. \tag{2.11}$$

Proof. Let $\pi : N\Sigma \to \Sigma$ be the normal bundle of the immersion F. Then the mapping

$$\tilde{F}(x, v) = F(x) + v$$

is a local diffeomorphism from a tubular neighbourhood U of the zero section of $N\Sigma$ onto its image in $\mathbb{R}^{2n}$.

Since α is a closed 1-form on Σ, $\beta = (\pi|_U)^*\alpha$ is a closed 1-form on U, and β sends the normal vectors v to zero. The pullback 2-form $\omega_0 = \tilde{F}^*\omega$ on U is closed as ω is closed and it is non-degenerate as $\tilde{F}$ is a locally diffeomorphic and ω is non-degenerate. Let X be the vector field on U dual to β with respect to ω_0, that is,

$$\beta(Y) = -\,\omega_0(X, Y) \tag{2.12}$$

for all vector fields Y on U. Let ϕ_s with $s \in (-\varepsilon, \varepsilon)$ be the one parameter group of diffeomorphisms on U generated by X. Then $F_s := \tilde{F} \circ \phi_s|_\Sigma : \Sigma \to \mathbb{R}^{2n}$ is a family of Lagrangian immersions in $\mathbb{R}^{2n}$ and $F_0 = \tilde{F} \circ \phi_0|_\Sigma = \tilde{F}|_\Sigma = F$.

It remains to verify (2.11). By the definition of $\tilde{F}$, its differential $\tilde{F}_*$ maps the tangent vectors to the zero section $(\Sigma, 0)$ at the point $(x, 0) \in U$ to the tangent vectors to the image surface $F(\Sigma)$ at the point $F(x) \in \mathbb{R}^{2n}$ and it maps the normal vectors to the normal vectors by the identity map at the corresponding points. We need to check $X = J\alpha^\sharp$. Let Y_1, Y_2 be arbitrary tangent vectors to the zero section $(\Sigma, 0)$ at a point $(x, 0)$. Since JY_2 is normal to Σ as Σ is Lagrangian and

$$\omega(X, Y) = \langle JX, Y \rangle,$$

we have

$$\begin{aligned}\alpha(\pi_* Y_1) &= \beta(Y_1 + JY_2)\\ &= -\omega_0(X, Y_1 + JY_2)\\ &= -\omega(\tilde{F}_* X, \tilde{F}_* Y_1 + \tilde{F}_* JY_2)\\ &= -\omega(\tilde{F}_* X, \tilde{F}_* Y_1 + JY_2)\\ &= -\langle J\tilde{F}_* X, \tilde{F}_* Y_1 + JY_2\rangle\\ &= -\langle J\tilde{F}_* X, \tilde{F}_* Y_1\rangle - \langle \tilde{F}_* X, Y_2\rangle\end{aligned}$$

As $\tilde{F}$ is locally diffeomorphic, X is normal to the zero section because Y_2 is arbitrary. Then it follows from the arbitrariness of Y_1 that $-JX = \alpha^\sharp$, by dropping the notion $\tilde{F}_*$. This is the same as $X = J\alpha^\sharp$. □

The entropy of a hypersurface is defined in [17, 39]. The definition for an immersion in any codimension is the same.

Definition 2.3.3. The entropy of an immersion $F : \Sigma \to \mathbb{R}^{n+k}$ is defined as

$$\lambda(F) = \sup_{x_0, t_0} \mathscr{F}_{x_0, t_0}(F). \tag{2.13}$$

It is clear that $\lambda(F)$ is invariant under translations and scalings. Huisken's monotonicity formula [29] implies that $\lambda(F_t)$ is non-increasing if $\{F_t\}$ satisfies the mean curvature flow, and is constant if and only if $\{F_t\}$ is self-shrinking. Analogous to the entropy stability introduced in [17], we define Lagrangian entropy stability of a Lagrangian self-shrinker.

Definition 2.3.4. Let $F : \Sigma \to \mathbb{R}^{2n}$ be a self-shrinker. Then F is called Lagrangian entropy stable if $\lambda(\tilde{F}) \geq \lambda(F)$ for all Lagrangian immersions C^0 close to F.

In [17], it is proved that every $\mathscr{F}$-unstable embedded self-shrinking hypersurface which does not split off a line is entropy unstable. As observed in [3], the exact same proof works for any codimension. According to [34], the second variation formula for the $\mathscr{F}$-functional at a closed Lagrangian self-shrinker can be rewritten in terms of the closed 1-form dual to the Lagrangian variation field. Therefore, when $F : \Sigma \to \mathbb{R}^{2n}$ is a Lagrangian $\mathscr{F}$-unstable self-shrinker, there is a closed 1-form α on Σ so that $\mathscr{F}''(\alpha) < 0$ for all variations (x_s, t_s) of $(0,1)$. To proceed from the Lagrangian $\mathscr{F}$-instability to the Lagrangian entropy instability, one needs to use the actual family F_s of Lagrangian immersions coming from the Lagrangian variation. By Lemma 2.3.1, there is a Lagrangian variation $\{F_s\}$ that corresponds to α. By taking a family of diffeomorphism $\phi_s : \Sigma \to \Sigma$, we can further assume that $\{F_s\}$ is a family of normal variations. Thus the same proof of Theorem 0.15 in [17] can be carried over to show that $F : \Sigma \to \mathbb{R}^{2n}$ is also Lagrangian entropy unstable. We omit the proof here.

Theorem 2.3.2. *Let Σ be compact and $F : \Sigma \to \mathbb{R}^{2n}$ be an immersed Lagrangian self-shrinker. If F is Lagrangian $\mathscr{F}$-unstable, then it is also Lagrangian entropy unstable. In particular, there is a Lagrangian immersion $\widehat{F} : \Sigma \to \mathbb{R}^{2n}$ so that $\lambda(F) > \lambda(\widehat{F})$. Moreover, $\widehat{F}$ can be chosen to be arbitrarily close to F, in the sense of smallness of $\|F - \widehat{F}\|_{C^k}$ for any k.*

Chapter 3

Compactness of the space of compact Lagrangian self-shrinking surfaces

3.1 Lagrangian branched conformal immersions

From now on we consider $n = 2$. Let (Σ, g_0) be a smooth Riemann surface. A smooth map $F : \Sigma \to \mathbb{R}^4$ is called a branched conformal immersion if

1. there is a discrete set $B \subset \Sigma$ such that $F : \Sigma \setminus B \to \mathbb{R}^4$ is an immersion,

2. there is a function $\lambda : \Sigma \to [0, \infty)$ such that $g := F^*\langle\ ,\ \rangle = \lambda g_0$ on Σ, where λ is zero precisely at B, and

3. the second fundamental form A on $\Sigma \setminus B$ satisfies $|A|_g \in L^2(K \setminus B, d\mu)$ for any compact domain K in Σ, where $|\cdot|_g$ and $d\mu$ are respectively the norm and the area element with respect to g (Note that g defines a Riemannian metric on $\Sigma \setminus B$ but not on Σ).

Elements in B are called the branch points of F. A branched conformal immersion $F : \Sigma \to \mathbb{R}^4$ is called Lagrangian and self-shrinking if it is Lagrangian and self-shrinking, respectively, when restricted to $\Sigma \setminus B$. Note that when F is Lagrangian, the mean curvature form α_H is defined only on $\Sigma \setminus B$.

The following proposition is the key result on removable singularity of α_H. Note that in this proposition we do not assume Σ to be compact.

Proposition 3.1.1. *Let $F : \Sigma \to \mathbb{R}^4$ be a branched conformal Lagrangian self-shrinker with the set of branch points B. Then there is a smooth one form $\tilde{\alpha}$ on Σ which extends α_H and $d\tilde{\alpha} = 0$ on Σ.*

Proof. The result is local, so it suffices to consider Σ to be the unit disc $\mathbb{D}$ with a branch point at the origin only. Let (x,y) be the local coordinate of $\mathbb{D}$. We write $\alpha_H = adx + bdy$ for some smooth functions a and b on the punctured disk $\mathbb{D}^* = \{z \in \mathbb{D} : z \neq 0\}$. Let

$$\operatorname{div}(\alpha_H) = \frac{\partial a}{\partial x} + \frac{\partial b}{\partial y}, \quad \nabla_0 = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \tag{3.1}$$

be the divergence and the gradient with respect to the Euclidean metric δ_{ij} on $\mathbb{D}$. As F is conformal, $g_{ij} = \lambda \delta_{ij}$, where $\lambda = \frac{1}{2}|DF|^2$. By restricting to a smaller disk if necessary, we assume that the image $|F|$ and λ are bounded. By (2.2) we have

$$|\vec{H}| \leq \frac{1}{2}|F|,$$

where $|\cdot|$ is taken with respect to $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^4$. As $g = F^*\langle \cdot, \cdot \rangle$, using (2.5) we see that

$$|\alpha_H|_g \leq C \text{ on } \mathbb{D}^*.$$

Using $g_{ij} = \lambda \delta_{ij}$ and $n = 2$, we have

$$\nabla_g = \lambda^{-1} \nabla_0 \text{ and } d_g^* = -\lambda^{-1} \operatorname{div}.$$

Hence equation (2.7) is equivalent to

$$\operatorname{div}(\alpha_H) = \frac{1}{4} \alpha_H (\nabla_0 |F|^2). \tag{3.2}$$

Moreover, as $|\alpha_H|_g = \sqrt{\lambda^{-1}(a^2 + b^2)}$, we also have

$$|a|, |b| \leq |\alpha_H|_g \sqrt{\lambda} \leq C\sqrt{\lambda}.$$

Thus $|a|, |b|$ are bounded on $\mathbb{D}^*$. To simplify notations, let

$$P = \frac{1}{4}\alpha_H(\nabla_0|F|^2).$$

Note that P is also bounded on $\mathbb{D}^*$.

Both equations (2.6) and (3.2) are satisfied pointwisely in $\mathbb{D}^*$. We now show that they are satisfied in the sense of distribution on $\mathbb{D}$. That is, for all test functions $\phi \in C_0^\infty(\mathbb{D})$,

$$\int_{\mathbb{D}} \alpha_H \wedge d\phi = 0 \tag{3.3}$$

and

$$\int_{\mathbb{D}} \alpha_H(\nabla_0\phi)dxdy = -\int_{\mathbb{D}} P\phi dxdy. \tag{3.4}$$

Note that all the integrands in equation (3.3) and (3.4) are integrable, since a, b and P are in $L^\infty(\mathbb{D})$.

First we show (3.3). Let $\psi_r \in C_0^\infty(\mathbb{D})$, $r < 1/2$, be a cutoff function such that $0 \le \psi \le 1$, $|\nabla_0\psi| \le 2/r$ and

$$\psi(x) = \begin{cases} 1 & \text{when } |x| \ge 2r, \\ 0 & \text{when } |x| \le r. \end{cases}$$

Then $\phi\psi_r \in C_0^\infty(\mathbb{D}^*) \cap C^\infty(\mathbb{D})$. Using Stokes' theorem and equation (2.6), we have

$$0 = \int_{\mathbb{D}^*} d(\phi\psi_r\alpha_H) = \int_{\mathbb{D}^*} d(\phi\psi_r) \wedge \alpha_H.$$

This implies

$$\int_{\mathbb{D}^*} \psi_r\alpha_H \wedge d\phi = -\int_{\mathbb{D}^*} \phi\alpha_H \wedge d\psi_r. \tag{3.5}$$

Since $\psi_r \to 1$ on $\mathbb{D}^*$ as $r \to 0$ and α_H, $d\phi$ are bounded,

$$\lim_{r\to 0}\int_{\mathbb{D}^*} \psi_r\alpha_H \wedge d\phi = \int_{\mathbb{D}^*} \alpha_H \wedge d\phi = \int_{\mathbb{D}} \alpha_H \wedge d\phi$$

by Lebesgue's dominated convergence theorem. To estimate the right hand side of (3.5), note that as $d\psi_r$ has support on $\mathbb{D}_{2r} \setminus \mathbb{D}_r$, where $\mathbb{D}_s$ denotes the disk of radius

s. Hence

$$\left| \int_{\mathbb{D}} \phi \alpha_H \wedge d\psi_r \right| \leq \frac{4C \sup|\phi|}{r} \int_{\mathbb{D}_{2r} \setminus \mathbb{D}_r} dx = 12\pi C \sup|\phi| r \to 0 \tag{3.6}$$

as $r \to 0$. Thus (3.3) holds.

To show (3.4), we use the same cutoff function ψ_r. Then $\phi \psi_r \in C_0^\infty(\mathbb{D}^*) \cap C^\infty(\mathbb{D})$. By the divergence theorem,

$$0 = \int_{\mathbb{D}^*} \mathrm{div}(\phi \psi_r \alpha_H)\, dxdy = \int_{\mathbb{D}^*} \alpha_H \big(\nabla_0(\phi \psi_r)\big)\, dxdy + \int_{\mathbb{D}^*} \phi \psi_r \mathrm{div}(\alpha_H)\, dxdy.$$

Now we use (3.2) to conclude

$$-\int_{\mathbb{D}^*} P\phi \psi_r\, dxdy = \int_{\mathbb{D}^*} \psi_r \alpha_H(\nabla_0 \phi)\, dxdy + \int_{\mathbb{D}^*} \phi \alpha_H(\nabla_0 \psi_r)\, dxdy. \tag{3.7}$$

Similarly, we can estimate the second term on the right hand side of (3.7) as for (3.6):

$$\left| \int_{\mathbb{D}^*} \phi \alpha_H(\nabla_0 \psi_r)\, dxdy \right| \leq 12\pi C \sup|\phi| r. \tag{3.8}$$

Using Lebesgue's dominated convergence theorem again, we can set $r \to 0$ in (3.7) to arrive at (3.4).

Writing $\alpha_H = adx + bdy$, the two equations (3.3) and (3.4) are equivalent to

$$\int_{\mathbb{D}} \left(a\frac{\partial \phi}{\partial y} - b\frac{\partial \phi}{\partial x} \right) dxdy = 0, \tag{3.9}$$

$$\int_{\mathbb{D}} \left(a\frac{\partial \phi}{\partial x} + b\frac{\partial \phi}{\partial y} \right) dxdy = -\int_{\mathbb{D}} P\phi\, dxdy, \tag{3.10}$$

for any test functions $\phi \in C_0^\infty(\mathbb{D})$.

For any $\psi \in C_0^\infty(\mathbb{D})$, set $\phi = \frac{\partial \psi}{\partial y}$ in (3.9), $\phi = \frac{\partial \psi}{\partial x}$ in (3.10) and cancel the cross term $b\frac{\partial^2 \psi}{\partial x \partial y}$ by taking summation of the two, we have

$$\int_{\mathbb{D}} a \Delta \psi\, dxdy = -\int_{\mathbb{D}} P \frac{\partial \psi}{\partial x}\, dxdy$$

where Δ is the Laplace operator in the Euclidean metric on $\mathbb{D}$.

Similarly, set $\phi = \frac{\partial \psi}{\partial x}$ in (3.9), $\phi = \frac{\partial \psi}{\partial y}$ in (3.10) and take the difference of the two equations, we obtain

$$\int_{\mathbb{D}} b\Delta\psi\, dxdy = -\int_{\mathbb{D}} P\frac{\partial\psi}{\partial y}\, dxdy.$$

We conclude now that a and b satisfy

$$\begin{aligned}\Delta a &= \frac{\partial P}{\partial x},\\ \Delta b &= \frac{\partial P}{\partial y}\end{aligned} \tag{3.11}$$

on $\mathbb{D}$ in the sense of distribution. Now we apply the elliptic regularity theory for distributional solutions. As F is smooth and $\alpha \in L^\infty(\mathbb{D})$, we have $P \in L^2(\mathbb{D})$. Hence the right hand side of equation (3.11) is in $H^{loc}_{-1}(\mathbb{D})$. By the local regularity theorem ([24], Theorem 6.30), we have $a, b \in H^{loc}_1(\mathbb{D})$. This implies $P \in H^{loc}_1(\mathbb{D})$. Using this, we see that the right hand side of equation (3.11) is in $H^{loc}_0(\mathbb{D})$. By the same local regularity theorem again, these implies $a, b \in H^{loc}_2(\mathbb{D})$. Thus we can iterate this argument and see that $a, b \in H^{loc}_s(\mathbb{D})$ for all positive integers s. By the Sobolev embedding theorem we have $a, b \in C^\infty(\mathbb{D})$. Hence α_H can be extended to a smooth one form $\tilde{\alpha}$ on $\mathbb{D}$ and $d\tilde{\alpha} = 0$ is satisfied on $\mathbb{D}$. □

Now we proceed to the proof of Theorem 1.1.1.

Proof. Let $F : \mathbb{S}^2 \to \mathbb{R}^4$ be a branched conformal Lagrangian self-shrinker with branch points $b_1, \cdots, b_k$. By Proposition 3.1.1, there is a smooth 1-form $\tilde{\alpha}$ on $\mathbb{S}^2$ such that $\tilde{\alpha} = \alpha_H$ on $\mathbb{S}^2 \setminus B$. As $\tilde{\alpha}$ is closed and the first cohomology group of $\mathbb{S}^2$ is trivial, there is a smooth function f on $\mathbb{S}^2$ such that $df = \tilde{\alpha}$. By equation (2.7), f satisfies

$$\Delta_g f = -\frac{1}{4} df(\nabla_g |F|^2) \tag{3.12}$$

on $\mathbb{S}^2 \setminus B$. Note that this equation is elliptic but not uniformly elliptic on $\mathbb{S}^2 \setminus B$. By the strong maximum principle, the maximum of f cannot be attained in $\mathbb{S}^2 \setminus B$ unless f is constant. Let $b \in B$ be a point where f attains its maximum. Let $\mathbb{D}$ be a local chart around b such that $g_{ij} = \lambda\delta_{ij}$ on $\mathbb{D}^*$. As $\Delta_g = \lambda^{-1}\Delta$ and $\nabla_g = \lambda^{-1}\nabla_0$, equation (3.12) can be written

$$\Delta f = \frac{1}{4} df(\nabla_0 |F|^2) \quad \text{on } \mathbb{D}^*. \tag{3.13}$$

As f and $|F|^2$ are smooth on $\mathbb{D}$, the equation (3.13) is in fact satisfied on $\mathbb{D}$. By the strong maximum principle, f is constant as f has an interior maximum at b. Hence $\alpha_H = 0$ and $\vec{H} = 0$. This implies that F is a branched minimal immersion in $\mathbb{R}^4$, which is not possible as $\mathbb{S}^2$ is compact. □

3.2 Proof of Theorem 1.1.2

Let $\{F_n : (\Sigma, h_n) \to \mathbb{R}^4\}$ be a sequence of branched conformal Lagrangian self-shrinkers which satisfy the hypothesis in Theorem 1.1.2. In the following proof, we will view each self-shrinker F_n as a harmonic map from (Σ, h_n) to $(\mathbb{R}^4, G)$. The existence of harmonic 2-spheres in [44] and the bubble tree convergence theorem in [42] require compact target space. To deal with non-compactness of $(\mathbb{R}^4, G)$, we will show that a uniform area bound of the sequence $\{F_n\}$ implies that all $F_n(\Sigma)$ lie in a bounded region. Hence the harmonic maps can be viewed as mappings into a compact Riemannian manifold. This is done in the next two lemmas.

Lemma 3.2.1. *Let F be a compact branched conformal self-shrinker in $\mathbb{R}^4$. Then the image of F lies in a ball of radius R_0 centered at the origin in $\mathbb{R}^4$, where R_0 depends only on $\mu(F)$, the area of F.*

Proof. Let $F : \Sigma \to \mathbb{R}^4$ be a branched conformally immersed self-shrinker. By the self-shrinking equation (2.2),

$$\Delta_g |F|^2 = -|F^{\perp}|^2 + 4 \tag{3.14}$$

holds on $\Sigma \setminus B$, where $g = F^* \langle \cdot, \cdot \rangle$ and B is the finite branch locus.

First, we show that F must intersect the closed ball centered at the origin of $\mathbb{R}^4$ with radius 2. Since F is a branched conformal immersion, there is a nonnegative smooth function φ and a smooth metric g_0 on Σ compatible with the conformal structure h so that $g = \varphi g_0$. Therefore

$$\varphi \Delta_g = \Delta_{g_0}$$

and by (3.14),

$$\Delta_{g_0} |F|^2 = \varphi(-|F^\perp|^2 + 4). \tag{3.15}$$

Unlike (3.14), (3.15) is satisfied everywhere on Σ, as both sides of the equation are continuous and B is finite. Since Σ is compact, the smooth function $|F|^2$ attains its minimum, say at $x_0 \in \Sigma$. Since F is a minimal immersion in $(\mathbb{R}^4, G)$, the tangential component $F^\top$ is well defined even at a branch point and $F^\top(x_0) = 0$. If F is immersed at x_0, the weak maximum principle shows that $\Delta_g |F|^2(x_0) \le 0$. This implies $|F^\perp(x_0)|^2 \le 4$ by (3.14). Since $F^\top(x_0) = 0$,

$$|F(x_0)|^2 = |F^\perp(x_0)|^2 \le 4$$

and we are done. Hence we only need to rule out the case that F is branched at x_0, $|F(x_0)|^2 > 4$ and there does not exist any immersed point $y \in \Sigma$ so that $|F(y)|^2 = |F(x_0)|^2$. Assume this case happens. Since the branch points are isolated, $|F|^2$ has a strict minimum at x_0. Noting again $F^\top(x_0) = 0$ and $|F|^2 = |F^\top|^2 + |F^\perp|^2$, we have $|F^\perp(x)|^2 > 4$ in a neighbourhood of x_0. By (3.15) we have $\Delta_{g_0}|F|^2 \le 0$ in the neighbourhood. However, this contradicts the strong maximum principle and we are done.

Next, we show that the extrinsic distance between any two points on the image of F is bounded above by a constant that depends only on the area upper bound. Note that

$$\Delta_g |F|^2 d\mu_g = d *_g d|F|^2. \tag{3.16}$$

and the Hodge star operator $*_g$ depends only on the conformal class of g, $\Delta_g |F|^2 d\mu_g$ is well-defined on Σ. Thus we integrate (3.14) and use (2.2) to get

$$\mathscr{W}(F) := \frac{1}{4} \int_\Sigma |\vec{H}|^2 d\mu = \frac{1}{4} \mu(F). \tag{3.17}$$

One also note that Simon's diameter estimate [50] holds for 2-varifolds with square integrable generalized mean curvature ((A.16) in [31]). Thus there is a constant C

such that

$$\left(\frac{\mu(F)}{\mathscr{W}(F)}\right)^{\frac{1}{2}} \leq \mathrm{diam} F(\Sigma) \leq C\big(\mu(F)\mathscr{W}(F)\big)^{\frac{1}{2}}, \tag{3.18}$$

where

$$\mathrm{diam} F(\Sigma) := \sup_{x,y\in\Sigma} |F(x)-F(y)|.$$

Together with (3.17), we see that

$$\mathrm{diam} F(\Sigma) \leq \frac{1}{2} C\mu(F).$$

It follows that the image of F lies in $B(R_0)$ for some R_0 depending only on the area upper bound. □

Let $U = B_{R_0+1}$ endowed with the metric G given by

$$G = e^{-\frac{|x|^2}{4}} \langle \cdot, \cdot \rangle. \tag{3.19}$$

The next lemma enables us to apply the results in [42] for harmonic maps into a compact Riemannian manifold.

Lemma 3.2.2. *There is a compact Riemannian manifold* $(N, \overline{g})$ *such that* (U, G) *defined as above can be isometrically embedded into* $(N, \overline{g})$.

Proof. Let $d = \frac{1}{R_0+1}$ and N is the disjoint union of B_{R_0+2} and B_d, with the identification that $x \sim y$ if and only if $y = \frac{x}{|x|^2}$ by the inversion. The manifold N is compact, as it can be identified as the one point compactification of $\mathbb{R}^4$ via the stereographic projection. Let g_1 be any metric on B_d. Let $\rho_1, \rho_2 \in C^\infty(N)$ be a partition of unity subordinate to the open cover $\{B_{R_0+2}, B_d\}$ in N and define a Riemannian metric on N by

$$\overline{g} = \rho_1 G + \rho_2 g_1.$$

As

$$B_{R_0+2} \cap B_d = \{x \in B_{R_0+2} : R_0 + 1 < |x| < R_0 + 2\},$$

$\bar{g} = \rho_1 G + \rho_2 g_1 = G$ on $B_{R_0+1} \subset B_{R_0+2}$. Thus the inclusion $U \subset B_{R_0+1} \subset N$ is an isometric embedding of U. □

The use of harmonic map theory is essential in our proof of Theorem 1.1.2. For the reader's convenience, we recall the terminologies in the construction of the bubbles and the bubble tree for a sequence of harmonic maps from surfaces. The main references are [44], [42].

A smooth map $f : (\Sigma, h) \to (N, \bar{g})$ from a two dimensional Riemannian surface to a Riemannian manifold is called harmonic if it is a critical point of the energy functional

$$E_{h,\bar{g}}(f) = \frac{1}{2}\int_{\Sigma} \|df_x\|^2 d\mu_h,$$

where $df_x : T_x\Sigma \to T_{f(x)}N$ is the differential of f and $\|df_x\|^2$ is locally given by

$$\|df_x\|^2 = g^{\alpha\beta}(f(x))h_{ij}(x)\frac{\partial f^\alpha}{\partial x^i}(x)\frac{\partial f^\beta}{\partial x^j}(x).$$

It is well-known [47]:

Proposition 3.2.1. *Let $f : (\Sigma, h) \to (N, \bar{g})$ be a mapping from a two dimensional Riemannian surface to a Riemannian manifold. Then*

1. *the energy is conformally invariant: $E_{h,\bar{g}}(f) = E_{e^\nu h,\bar{g}}(f)$.*
2. *If f is a branched conformal immersion, then $E_h(f) = \mu(f)$, where $\mu(f)$ is the area of f.*
3. *If f is conformal, then f is a harmonic map if and only if it is a branched minimal immersion.*
4. *If f is a nontrivial harmonic map from a 2-sphere, then f is conformal.*

As in [42], we state the analytic results in [44] that are needed in our proof of Theorem 1.1.2 and in the bubble tree convergence [42].

Proposition 3.2.2. *There are positive constants C and ε_0 that depend on (Σ, h) and $(N, \bar{g})$ such that*

1. *(Sup Estimate) If $f : \Sigma \to N$ is harmonic with $\int_{D(2r)} \|df\|^2 d\mu_h < \varepsilon_0$, then*

$$\sup_{D(r)} \|df_x\|^2 \leq Cr^{-2}\int_{D(2r)} \|df\|^2 d\mu_h. \tag{3.20}$$

2. *(Uniform Convergence) If $\{f_n\}$ is a sequence of harmonic maps from a disk $D(2r)$ with energies less than ε_0 for all n, then a subsequence of $\{f_n\}$ converges in C^1 in $D(r)$.*

3. *(Energy Gap) Any nontrivial harmonic map $f : \mathbb{S}^2 \to N$ has energy $E(f) \geq \varepsilon_0$.*

4. *(Removable Singularities) Any smooth harmonic map from a punctured disk $\mathbb{D}^*$ with finite energy extends to a smooth harmonic map on $\mathbb{D}$.*

Let $f_n : (\Sigma, h_n) \to (N, \bar{g})$ be a sequence of harmonic maps with a uniform energy upper bound E_0. Assume that the metrics h_n converge smoothly to a smooth metric h on Σ. The set of bubbling points (or energy concentration points) of $\{f_n\}$ is defined as

$$S = S_{\{f_n\}} = \bigcap_{\delta>0} \left\{ x \in \Sigma : \liminf_{n\to\infty} \int_{D(x,\delta)} \|df_n\|^2 d\mu_{h_n} \geq \varepsilon_0 \right\}, \tag{3.21}$$

where ε_0 is the constant in (1) of Proposition 3.2.2. Since $h_n \to h$, Theorem 2.3 in [45] asserts that $\{f_n\}$, taking a subsequence if necessary, converges to a harmonic map f in $C^1(M - S, N)$ and S consists of finitely many points.

Next we focus on each $x \in S$ and describe how the bubble tree is constructed at x. In [42], a sequence $\varepsilon_n \to 0$ is chosen and via the exponential map at x, all the geodesic disks $D(0, 2\varepsilon_n)$ are identified as subsets in $\mathbb{R}^2 \cong T_x\Sigma$. Let c_n be the center of mass of $\|df_n\|^2 dh_n$ and $\lambda_n > 0$ be suitably chosen (see Section 1 in [42] for the choices of c_n, λ_n) so that

$$D(c_n, n\lambda_n) \subset D(c_n, \varepsilon_n) \subset D(0, 2\varepsilon_n).$$

Then we define the rescaled mapping $\tilde{f}_n : S_n \to M$ by

$$\tilde{f}_n(y) = f_n(\exp_x(c_n + \lambda_n \sigma(y))), \tag{3.22}$$

where $\sigma : \mathbb{S}^2 \setminus \{p^+\} \to T_xM$ is the stereographic projection from the north pole p^+ and $S_n = \{y \in \mathbb{S}^2 \setminus \{p^+\} : c_n + \lambda_n \sigma(y) \in D(0, 2\varepsilon_n)\}$. Note that the disk $D(c_n, \lambda_n)$ corresponds to the northern hemisphere. The parameters c_n, λ_n are chosen in such a way that $S_n \to \mathbb{S}^2 \setminus \{p^-\}$ as $n \to \infty$, where p^- is the south pole.

For the metrics on S_n pulled back from h_n, $\{\tilde{f}_n\}$ is a sequence of harmonic maps defined on S_n with uniformly bounded energies. Let $\tilde{S} = S_{\{\tilde{f}_n\}}$ be the bubbling set for the sequence $\{\tilde{f}_n\}$ (defined as in (3.21)). Then, as in the case for $\{f_n\}$, Theorem 2.3 in [45] and the removable singularity theorem imply that $\{\tilde{f}_n\}$, by passing to a subsequence if necessary, converges locally in $C^1(\mathbb{S}^2 \setminus \tilde{S} \cup \{p^-\}, N)$ to a harmonic map $s_{x,1} : \mathbb{S}^2 \to N$. The harmonic map $s_{x,1}$ is called a bubble at x.

The above procedure is then performed at each $y \in \tilde{S}$, and this produces bubbles on the bubble $s_{x,1}$. Iterate this procedure. Note that the process would terminate at a finite number of steps since the energy of each nontrivial bubble is at least ε_0.

So far, for each $x \in S$, we have associated to it finitely many harmonic maps $s_{x,i} : \mathbb{S}^2_{x,i} \to (N,g)$, where $i = 1, 2, \cdots i_x$. Now we describe how the original sequence $\{f_n\}$ converges to f_∞ and the $s_{x,i}$'s. Due to formation of bubbles, we renormalize the mapping $\{f_n\}$ in order to formulate C^0 convergence. First we restrict each f_n to

$$\Sigma \setminus \bigcup_{x \in S} \exp_x D(c_n, \varepsilon_n). \tag{3.23}$$

It is shown (Lemma 1.3 in [42]) that for each $x \in S$, the image of $\partial D(c_n, \varepsilon_n)$ under f_n always lies in the ball $B(f_\infty(x), C/n)$ for some positive constant C. Thus we can redefine f_n on the disk $\exp_x D(c_n, \varepsilon_n)$ by coning off the image at each $x \in S$ (see (1.12) in [42]). The resulting map is denoted $\bar{f}_n \in C^0(\Sigma, N)$. Note that $\bar{f}_n$ converges in C^0 to f_∞.

For the first layer of bubbles, we restrict the definition of the rescaled mapping $\tilde{f}_n$ (defined in (3.22)) to $\sigma^{-1}(D(c_n, n\lambda_n))$. Again $\tilde{f}_n$ maps $\partial D(c_n, n\lambda_n)$ to $B(\tilde{f}(p^-), C/n)$ for some C. Hence, as for $\bar{f}_n$, we can similarly cone off the image of $\tilde{f}_n$. The resulting continuous mapping is denoted by

$$\overline{Rf}_{n,x,1} : \mathbb{S}^2_{x,1} \to N.$$

Then similarly define $\overline{Rf}_{n,x,i} : \mathbb{S}^2_{x,i} \to N$, where $i = 2, \cdots, i_x$, for each level of bubbling.

The restriction of f_n to the annular region $A_n = D(c_n, \varepsilon_n) \setminus D(c_n, n\lambda_n)$ is called the neck map. The limit of the images of $f_n|_{A_n}$ will connect the base map $\bar{f}$ and the bubbling maps $s_{x,1}$ by joining $f_\infty(x)$ and $s_{x,1}(p^-)$. It is shown in Lemma 2.1 of [42] that the neck maps $f_n|_{A_n}$ converge to points, and there is no energy lost in this limiting process. The same argument applies to bubbles at all levels of the bubble tree, hence the bubbles are connected to the bubbles at the previous level.

Now we define the bubble tower T for the domain of the renormalized map. First of all, for each $x \in S$, we attach a 2-sphere $\mathbb{S}^2$, and on this $\mathbb{S}^2$ we attach a 2-sphere $\mathbb{S}^2$ at each $y \in \tilde{S}$. Then we repeat this construction on the third layer and so on. Hence we obtain a bubble tower T. Let I be the finite set that indexes all bubbles and $0 \in I$ for Σ. Then the family of maps

$$\{\bar{f}_n, \overline{Rf}_{n,x,i} : \ x \in S, i = 1, 2, \cdots, i_x\}$$

can be described in a simple notation $f_{n,I} : T \to N$. Let $f_I : T \to N$ be the family of maps given by $\{f_\infty, s_{x,i} : \ x \in S, i = 1, 2, \cdots, i_x\}$.

Now we can state the following bubbling convergence theorem:

Theorem 3.2.1. *(Theorem 2.2 in [42]) Let $f_n : (\Sigma, h_n) \to (N, \bar{g})$ be a sequence of harmonic maps from a Riemannian surface (Σ, h_n) to a compact Riemannian manifold $(N, \bar{g})$ with $E_{h_n,\bar{g}}(f_n) \leq E_0$. Assume in addition that the metrics h_n converge to h, then there is a subsequence (still use the same notation) of $\{f_n\}$ and a bubble tower domain T so that the sequence of renormalized maps*

$$\{f_{n,I} : T \to N\} \tag{3.24}$$

converges in $W^{1,2} \cap C^0$ to a smooth harmonic bubble tree map $f_I : T \to N$. Moreover,

1. *(No energy loss) $E_{h_n,\bar{g}}(f_n)$ converges to $\sum_{j \in I} E(f_j)$, and*
2. *(Zero distance bubbling) At each bubble point y (at any level of the bubble tree), the image of the respective base map f_j and the bubble map f_k, where $j, k \in I$, meet at $f_j(y) = f_k(p^-)$.*

Consequently, the image of the limit $f_I : T \to N$ is connected, and the images of the original maps $f_n : \Sigma \to N$ converge pointwisely to the image of f_I.

Note that the above theorem is stated only for $h_n = h$ for all n in [42], as remarked in Section 5 in [42], it still holds if the conformal structures represented by the metrics h_n stay in a bounded domain of the moduli space.

We are now ready to prove Theorem 1.1.2.

Proof of Theorem 1.1.2. Given a sequence of conformally immersed compact Lagrangian self-shrinkers $F_n : (\Sigma, h_n) \to \mathbb{R}^4$ with a uniform area upper bound, by Lemma 3.2.1, the images $F_n(\Sigma)$ lie in a fixed U for all n. By composing with the isometric embedding $(U,G) \to (N,\bar{g})$ in Lemma 3.2.2, we regard each F_n as a map with image in N. As each F_n is a conformal minimal immersion with respect to h_n on Σ and $G = \bar{g}$ on U, by Proposition 3.2.1, $\{F_n : (\Sigma, h_n) \to (N, \bar{g})\}$ is a sequence of harmonic maps. The area $\tilde{\mu}(F_n)$ of $F_n(\Sigma)$ in $(N,\bar{g})$ is given by

$$\tilde{\mu}(F_n) = \int_\Sigma e^{-\frac{|F_n|^2}{4}} d\mu_{F_n^*\langle\cdot,\cdot\rangle}.$$

Therefore, as $e^{-\frac{|F_n|^2}{4}} \leq 1$ we have

$$\tilde{\mu}(F_n) \leq \mu(F_n) < \Lambda.$$

As $F_n : (\Sigma, h_n) \to (N,\bar{g})$ is conformal, the area $\tilde{\mu}(F_n)$ in N is the same as the energy (Proposition 3.2.1): $E_{h_n,\bar{g}}(F_n) = \tilde{\mu}(F_n)$. Therefore, the energies of the harmonic mappings $F_n : (\Sigma, h_n) \to (N,\bar{g})$ are also uniformly bounded by Λ. By assumption, the sequence of metrics h_n converges to a Riemannian metric h. Hence we can apply the theory of bubble tree convergence of harmonic maps discussed above. In particular, by Theorem 3.2.1, the sequence $\{F_n\}$ converges in the sense of bubble tree to a harmonic mapping $F_\infty : (\Sigma, h) \to (N,\bar{g})$ and finitely many harmonic mappings $s_{x,i} : \mathbb{S}^2 \to (N,\bar{g})$. Since each F_n has image in $\overline{U}$, by the C^0 convergence of the renormalized map in Theorem 3.2.1, both F_∞ and $s_{x,i}$ have image in $\overline{U}$. In light of Lemma 3.2.2, F_∞ and $s_{x,i}$ are harmonic mappings into $(\mathbb{R}^4, G)$.

Note that some of these mappings, including F_∞, might be trivial. Let $s_{x,i} : \mathbb{S}^2 \to \mathbb{R}^4$ be a nontrivial bubble. Then we claim that $s_{x,i}$ is a conformally branched Lagrangian self-shrinker. First of all, as $s_{x,i}$ is nontrivial, by Proposition 3.2.1, $s_{x,i}$ is a branched conformal minimal immersion into $(\mathbb{R}^4, G)$. By Lemma 2.1.1, $s_{x,i}$ is a conformally branched self-shrinker.

It remains to show that $s_{x,i}$ is Lagrangian. When $i = 1$, $s_{x,1}$ is the limit of the rescaling map $\tilde{f}_n$ (see (3.22)). Since the rescalings are performed on the domains, each $\tilde{f}_n$ is Lagrangian since it has the same image in $\mathbb{R}^4$ as the Lagrangian immersion F_n at the corresponding points. As $\{\tilde{f}_n\}$ converges locally uniformly in C^1 to $s_{x,1}$ on $\mathbb{S}^2 \setminus \tilde{S} \cup \{p^-\}$, $s_{x,1}$ is Lagrangian when restricted to $\mathbb{S}^2 \setminus \tilde{S} \cup \{p^-\}$. The smoothness of $s_{x,1}$ then implies that $s_{x,1}$ is indeed Lagrangian on $\mathbb{S}^2$. The same argument applies to bubbles at any level, and so all nontrivial $s_{x,i}$ are Lagrangian in $\mathbb{R}^4$.

However, according to Theorem 1.1.1, there does not exist any nontrivial conformally branched Lagrangian self-shrinking immersion of $\mathbb{S}^2$ in $\mathbb{R}^4$. Therefore, there does not exist nontrivial $s_{x,i}$. From the construction of the bubbling convergence, we conclude that the set S of the bubbling points is empty. Thus the convergence $F_n \to F_\infty$ is in $C^1(\Sigma)$.

By Theorem 3.3 in [44], since each $F_n : \Sigma \to N$ is nontrivial by definition, there is an $\varepsilon > 0$ such that $E_{h_n,\bar{g}}(F_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. As the convergence $F_n \to F_\infty$ is in $C^1(\Sigma)$ (or using (2) in Theorem 3.2.1) and $h_n \to h$,

$$E_{h_n,\bar{g}}(F_n) \to E_{h,\bar{g}}(F_\infty) \quad \text{as } n \to \infty. \tag{3.25}$$

Hence $E_{h,\bar{g}}(F_\infty) \geq \varepsilon$ and F_∞ is nontrivial. (Alternatively, one can also use the estimate of the diameter of each self-shrinker and the Hausdorff convergence as in the proof of Theorem 1.1.3 in the next section to show that F_∞ is nontrivial).

Again, as the convergence $F_n \to F_\infty$ is in $C^1(\Sigma)$ and the metrics h_n converge to h smoothly, the harmonic map $F_\infty : (\Sigma, h) \to (N, \bar{g})$ is also conformal as each F_n is. Thus as a mapping into $\mathbb{R}^4$, F_∞ is a conformally branched minimal immersion with respect to the metric G defined by (3.19). The C^1 convergence also implies that F_∞ is Lagrangian in $\mathbb{R}^4$. By Lemma 2.1.1, F_∞ is a conformally branched Lagrangian self-shrinker in $\mathbb{R}^4$.

Lastly, by picking a subsequence if necessary, we have the C^∞ convergence of $\{F_n\}$ as follows. Since $F_n : (\Sigma, h_n) \to (N, \bar{g})$ is harmonic and $\{h_n\}$ converges smoothly to h, using the standard elliptic estimates (Chapter 6 of [26]) and a bootstrapping argument, there are constants $C(m)$ such that

$$||F_n||_{C^m} \leq C(m)$$

for all $n \in \mathbb{N}$. Using the Arzelà-Ascoli theorem and picking a diagonal subsequence, one shows that a subsequence of $\{F_n\}$ converges smoothly to a smooth mapping $\Sigma \to \mathbb{R}^4$, which has to be F_∞. Thus the theorem is proved.

3.3 Proof of Theorem 1.1.3

Proof. Let $\{F_n : \mathbb{T} \to \mathbb{R}^4\}$ be a sequence of conformally branched Lagrangian self-shrinking tori in $\mathbb{R}^4$ with a uniform area upper bound. Let h_n be the metric on the torus $\mathbb{T}$ which is conformal to $F_n^*\langle\cdot,\cdot\rangle$ and with zero Gauss curvature as discussed in section 1.1. If we can show that h_n stays in a bounded domain of the moduli space, then Theorem 1.1.3 follows from Theorem 1.1.2.

Using Lemma 3.2.1, Lemma 3.2.2 and Lemma 2.1.1, we regard each F_n as a minimal immersion in $(N,\bar{g})$, this means that $F_n : (\mathbb{T},h_n) \to (N,\bar{g})$ is conformal and harmonic. Assume the contrary that the conformal structures degenerate. In this case, there is a mapping $\hat{F}_\infty$ from Σ_∞ to N and the image $F_n(\mathbb{T})$ converges in the Hausdorff distance to $\hat{F}_\infty(\Sigma_\infty)$ in N. Here Σ_∞ is a stratified surfaces $\Sigma_\infty = \Sigma_0 \cup \Sigma_b$ formed by the *principal component* Σ_0 and *bubble component* Σ_b. The principal component Σ_0 is formed by pinching several closed, homotopically nontrivial curves in $\mathbb{T}$ and the bubbling component is a union of spheres. There are no necks between the components since F_n is conformal. The map $\hat{F}_\infty$ is continuous on Σ_∞ and harmonic when restricted to each component of Σ_∞. Since all the components intersect each others possibly at finitely many points, $\hat{F}_\infty$ is harmonic except at a finite set $\hat{S}$.

Since the conformal structures determined by the metrics h_n degenerate, at least one homotopically nontrivial closed curve must be pinched to a point as $n \to \infty$. It follows that Σ_0 is a finite union of $\mathbb{S}^2$'s. Each of these 2-spheres is obtained by adding finitely many points to the cylinder $\mathbb{S}^1 \times \mathbb{R}$ that comes from pinching one or two homotopically nontrivial loops: two at the infinity and at most finitely many at the blowup points of the sequence F_n, again by (4) in Proposition 3.2.2. Therefore, $\hat{F}_\infty$ is a finite union of harmonic mappings $\hat{F}_\infty^i$ from the sphere to N.

Since all F_n are conformal, there are no necks between the components. The bubble tree convergence described above are given by the results in [13] or [10]. In [13], the limiting surface is a stratified surface with geodesics connecting the two dimensional components. But together with conformality of each F_n and Proposition 2.6 in [13], one sees that all the geodesics involved have zero length. Alternatively, we can use the compactness theorem in [10] for $\Sigma = \mathbb{T}$: Suppose that $\{f_k\}$ is a sequence of $W^{2,2}$ branched conformal immersions of $(\mathbb{T}, h_k)$ in a compact Riemannian manifold M. If

$$\sup_k \{\mu(f_k) + W(f_k)\} < +\infty$$

where $W(f_k) = \frac{1}{4}\int |H_k|^2$, then either $\{f_k\}$ converges to a point, or there is a stratified sphere Σ_∞ and a $W^{2,2}$ branched conformal immersion $f_\infty : \Sigma_\infty \to M$, such that a subsequence of $\{f_k(\Sigma)\}$ converges to $f_\infty(\Sigma_\infty)$ in the Hausdorff topology, and the area and the Willmore type energy satisfy

$$\mu(f_\infty) = \lim_{k\to+\infty} \mu(f_k) \text{ and } W(f_\infty) \leq \lim_{k\to+\infty} W(f_k).$$

The conditions of the theorem are satisfied by the sequence $\{F_n\}$ as the area $\tilde{\mu}(F_n)$ and the Willmore energy $W(F_n)$ in N (which is zero as each F_n is minimal immersion in N) are uniformly bounded. In our situation, F_k will not converges to a point since $\text{diam}(F_n(T)) \geq 2$ by equations (3.18) and (3.17). Thus the inequality on the Willmore type energies in N

$$W(\hat{F}_\infty) \leq \lim_{n\to\infty} W(F_n) = 0$$

implies that the limiting $\mathbb{S}^2$'s are all branched minimal surfaces in N.

Consequently, the images $F_n(\mathbb{T})$ converge in the Hausdorff distance to the image of finitely many harmonic maps $\mathbb{S}^2 \to N$. These harmonic maps are branched conformal immersions, which are also Lagrangian by similar reasons as in the proof of Theorem 1.1.2. By Theorem 1.1.1, all these harmonic maps are trivial. Hence, the images $F_n(\mathbb{T})$ converge in the Hausdorff distance to a point in $\mathbb{R}^4$. This is impossible by the diameter estimate $\text{diam}(F_n(\mathbb{T})) \geq 2$. This contradiction shows that the conformal structures cannot degenerate and that finishes the proof of Theorem 1.1.3. □

3.4 Lagrangian self-shrinking tori with small area

In this section, we restrict to Lagrangian tori and prove Theorem 1.1.4. We will use a contradiction argument. By doing so we need the compactness theorem 1.1.3 and a factorization result of Lamm and Schätzle [32] concerning branched conformal immersions of tori into $\mathbb{R}^4$ with Willmore energy 8π.

The proof of Theorem 1.1.4 will be divided into the following results. We recall that $\mathfrak{X}_\Lambda$ stands for the space of branched conformally immersed Lagrangian self-shrinking tori of area no larger than Λ.

Proposition 3.4.1. *There is a positive number ε_0 so that if $F \in \mathfrak{X}_{32\pi+\varepsilon_0}$, then F is immersed.*

Proof. Arguing by contradiction, we assume that there is a sequence $F_n : \mathbb{T} \to \mathbb{R}^4$ of branched conformal Lagrangian self-shrinking tori so that

$$\liminf_{n\to\infty} \mu(F_n) \leq 32\pi \tag{3.26}$$

and each F_n has a nonempty set of branch points. Using Theorem 1.1.3, by passing to a subsequence if necessary, the sequence $\{F_n\}$ converges smoothly to a branched conformal Lagrangian self-shrinking torus $F_\infty : \mathbb{T} \to \mathbb{R}^4$. Let B_n be the set of branch points of F_n. Since $\mathbb{T}$ is compact, again by passing to a subsequence if necessary, there is a sequence $\{p_n\}$, where $p_n \in B_n$ for each $n \in \mathbb{N}$, so that $p_n \to p \in \mathbb{T}$. As $DF_n(p_n) = 0$ for all $n \in \mathbb{N}$ and the convergence $F_n \to F_\infty$ is smooth, $DF(p) = 0$ and so p is a branch point of F_∞, where DF, DF_n are the differentials of F, F_n, respectively. By the theorem of Li and Yau (Theorem 6 in [36], see also the appendix in [31] for the generalization to branched immersions), since F_∞ is not embedded,

$$\mathscr{W}(F_\infty) \geq 8\pi. \tag{3.27}$$

On the other hand, from (3.26) and Theorem 1 in [10],

$$\mu(F_\infty) \le \liminf \mu(F_n) \le 32\pi.$$

Together with (3.27) and (3.17) we have $\mathscr{W}(F_\infty) = 8\pi$. Since $\mathscr{F}_\infty$ has a branch point, Proposition 2.3 in [32] implies that F_∞ factors through a branched conformal immersion $g : \mathbb{T} \to \mathbb{S}^2$. It follows that there is a branched conformal Lagrangian self-shrinking sphere $h : \mathbb{S}^2 \to \mathbb{R}^4$ so that $F_\infty = h \circ g$. However, by Theorem 1.1.1, such a mapping h does not exist. This contradicts the existence of the sequence $\{F_n\}$. The proposition is now proved. □

Proposition 3.4.1 and Theorem 1.1.3 lead to

Theorem 3.4.1. *Let ε_0 be as in Proposition 3.4.1. Then the space of all Lagrangian* **immersed** *self-shrinking tori with area less than or equal to $32\pi + \varepsilon_0$ is compact.*

Next we prove part (2) in Theorem 1.1.4.

Corollary 3.4.1. *(Curvature Estimates) There is $C_0 > 0$ so that if $F : \mathbb{T} \to \mathbb{R}^4$ is a Lagrangian immersed self-shrinking torus with area less than or equals to $32\pi + \varepsilon_0$, then the second fundamental form of F is bounded by C_0.*

Proof. Assume this were not true. Then there is a sequence $F_n : \mathbb{T} \to \mathbb{R}^4$ of Lagrangian immersed self-shrinking tori with area less than $32\pi + \varepsilon_0$ so that

$$\max_{F_n(\mathbb{T})} |A_n| \to \infty, \tag{3.28}$$

where A_n is the second fundamental form of the immersion F_n. Using Theorem 3.4.1, a subsequence of $\{F_n\}$ converges smoothly to an immersed self-shrinker F_∞. In particular, we have

$$(g_n)_{ij} = \frac{\partial F_n}{\partial x_i} \cdot \frac{\partial F_n}{\partial x_j} \longrightarrow \frac{\partial F_\infty}{\partial x_i} \cdot \frac{\partial F_\infty}{\partial x_j} = (g_\infty)_{ij}, \text{ as } n \to \infty.$$

Since g_∞ is positive definite as F_∞ is immersed, there is a positive number C so that $g_n \ge C\delta_{ij}$ for all n. So g_n^{-1} are uniformly bounded. Hence

$$\max_{F_n(\mathbb{T})} |A_n|^2 = \max_{F_n(\mathbb{T})} g_n^{ij} g_n^{kl} \langle (A_n)_{ik}, (A_n)_{jl} \rangle$$

are uniformly bounded and (3.28) is impossible. □

To finish the proof of Theorem 1.1.4, it remains to prove the first part in (1).

Proposition 3.4.2. *There is a positive constant $\varepsilon_1 \leq \varepsilon_0$ so that if $F \in \mathfrak{X}_{32\pi+\varepsilon_1}$, then F is embedded.*

Proof. As in the proof of Corollary 3.4.1, assume the contrary that there is a sequence $\{F_n\}$ of immersed, non-embedded Lagrangian self-shrinking tori with $\mu(F_n) \leq 32\pi + \varepsilon_0$ and $\mu(F_n) \to 32\pi$. By Theorem 3.4.1, after passing to a subsequence if necessary, $\{F_n\}$ converges smoothly to an immersed Lagrangian self-shrinking torus $F_\infty : \mathbb{T} \to \mathbb{R}^4$ with area $\mu(F_\infty) = 32\pi$. By (3.17), the Willmore energy of F_∞ is 8π. Since each F_n is non-embedded, there are distinct points $p_n, q_n \in \mathbb{T}$ so that

$$F_n(p_n) = F_n(q_n). \tag{3.29}$$

As $\mathbb{T}$ is compact, we may assume $p_n \to p$ and $q_n \to q$. Taking $n \to \infty$ in (3.29), we have $F_\infty(p) = F_\infty(q)$. First of all, we must have $p = q$: Indeed, if $p \neq q$, then F_∞ is not embedded and that contradicts Theorem 2.2 in [32], which states that any immersion $F : \mathbb{T} \to \mathbb{R}^4$ with $\mathscr{W}(F) = 8\pi$ has to be embedded.

Let d_n be the distance function on $\mathbb{T}$ induced by the pullback metric $F_n^*\langle\cdot,\cdot\rangle$. As $p = q$ and $\{F_n\}$ converges smoothly to F, we have $\ell_n := d_n(p_n, q_n) \to 0$ as $n \to \infty$. Let $\eta_n : [0, \ell_n] \to \mathbb{T}^2$ be a shortest geodesics in $(\mathbb{T}, F_n^*\langle\cdot,\cdot\rangle)$ joining p_n to q_n. Since $F_n(\eta_n(0)) = F_n(\eta_n(\ell_n))$, $F_n \circ \eta_n : [0, \ell_n] \to \mathbb{R}^4$ is a closed curve in $\mathbb{R}^4$ with length ℓ_n. Let $\gamma_n : [0, \ell_n] \to \mathbb{R}^4$ be the translation $\gamma_n(t) = F_n \circ \eta_n(t) - F_n(p_n)$. Then each γ_n is parameterized by arc length and $\gamma_n(0) = \gamma_n(\ell_n) = 0 \in \mathbb{R}^4$. Using the following simple estimates

$$
\begin{aligned}
\ell_n &= \int_0^{\ell_n} \langle \gamma_n'(t), \gamma_n'(t) \rangle dt \\
&= -\int_0^{\ell_n} \langle \gamma_n(t), \gamma_n''(t) \rangle dt + \langle \gamma_n(\ell_n), \gamma_n'(d_n) \rangle - \langle \gamma_n(0), \gamma_n'(0) \rangle \\
&= -\int_0^{\ell_n} \langle \gamma_n(t), \gamma_n''(t) \rangle dt \\
&\leq \int_0^{\ell_n} |\gamma_n(t)| \cdot |\gamma_n''(t)| dt \\
&\leq \ell_n \int_0^{\ell_n} |\gamma_n''(t)| dt,
\end{aligned}
$$

we obtain

$$
\int_0^{\ell_n} |\gamma_n''(t)| \, dt \geq 1.
$$

Since $\ell_n \to 0$, the above inequality implies that there is $s_n \in [0, \ell_n]$ so that $|\gamma_n''(s_n)| \to \infty$ as $n \to \infty$. Since η_n is a geodesic on $(\mathbb{T}, F_n^*\langle \cdot, \cdot \rangle)$,

$$
\gamma_n'' = (F_n \circ \eta_n)'' = \nabla^n_{\eta_n'} \eta_n' + A_n(\eta_n', \eta_n') = A_n(\eta_n', \eta_n'),
$$

where ∇^n is the Levi-Civita connection on $(\mathbb{T}, F_n^*\langle \cdot, \cdot \rangle)$ and A_n is the second fundamental form of $F_n(\mathbb{T})$ in $\mathbb{R}^4$. Thus

$$
|\gamma_n''(t)| \leq |A_n(\eta_n(t))|
$$

and this implies

$$
\max_{F_n(\mathbb{T})} |A_n|^2 \to \infty
$$

as $n \to \infty$. However, this is impossible by Corollary 3.4.1. $\square$

Chapter 4

Finiteness of entropy and piecewise Lagrangian mean curvature flow

4.1 A Łojasiewicz-Simon type gradient inequality for branched self-shrinking tori

In the last chapter we showed that with a small area bound, all Lagrangian self-shrinking tori are embedded. This makes it much easier to study the space $\mathfrak{X}_{32\pi+\varepsilon_0}$, as all nearby Lagrangian self-shrinking tori can be deformed to each other by using the normal vectors fields. However, it is difficult in general to relate two nearby branched conformal immersions, even if they are C^k-close when treated as mappings to the Euclidean space. In particular, it seems difficult to extend the perturbation procedure as in [17, 34], where the stability condition is described by using the normal vector fields, to *branched* conformal self-shrinkers.

In this section, we show that the entropy λ is locally a constant function in the space of branched conformal compact self-shrinking tori $F : \mathbb{T} \to \mathbb{R}^4$. To do this, we derive a Łojasiewicz-Simon type gradient inequality for branched conformal self-shrinking tori $F : \mathbb{T} \to \mathbb{R}^4$. In the genus one case, the explicit expression of the conformal structures in the Teichmüller space makes the computation and the real analyticity of the functional $\mathscr{E}$ transparent. Once this is done, together with the compactness of $\mathfrak{X}_\Lambda$, we conclude the proof of Theorem 1.2.1.

4.1.1 A Fredholm operator of index zero

Let (Σ, g) be a compact Riemannian surface and (M, h) a Riemannian manifold. Given a C^1 mapping $F : \Sigma \to M$, the energy of F is given by

$$\mathscr{E}_{g,h}(F) = \frac{1}{2}\int_\Sigma e_{g,h}(F)d\mu_g,$$

where $e_{g,h}(F)$ is the norm of the differential $DF_x : T_x\Sigma \to T_{F(x)}M$. Locally it is given by

$$e_{g,h}(F) = g^{ij}h_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}.$$

For a fixed h, define $\mathscr{E} : C^1(\Sigma, M) \times \{g : g \text{ is a Riemannian metric on } \Sigma\} \to \mathbb{R}$ by

$$\mathscr{E}(F, g) = \mathscr{E}_{g,h}(F).$$

Lemma 4.1.1. *If $F : (\Sigma, g) \to (M, h)$ is conformal, then g is a critical point of $\mathscr{E}$ with respect to all its smooth variations g_s, where $g_0 = g$. That is,*

$$\frac{d}{ds}\mathscr{E}_{g_s,h}(F)\bigg|_{s=0} = 0.$$

Proof. Let g_s be a family of smooth metrics on Σ so that $g_0 = g$ and $\dot{g} = \frac{d}{ds}g_s\big|_{s=0}$. Then

$$\begin{aligned}\frac{d}{ds}\left(g^{ij}\sqrt{\det g}\right)\bigg|_{s=0} &= -g^{ik}g^{jl}\dot{g}_{kl}\sqrt{\det g} + \frac{1}{2}g^{ij}\sqrt{\det g}g^{kl}\dot{g}_{kl}\\ &= \left(\frac{1}{2}g^{kl}g^{ij} - g^{ik}g^{jl}\right)\dot{g}_{kl}\sqrt{\det g}\end{aligned}$$

Thus

$$\frac{d}{ds}e_{g_s,h}(F)d\mu_{g_s}\bigg|_{s=0} = \left(\frac{1}{2}g^{kl}g^{ij} - g^{ik}g^{jl}\right)\dot{g}_{kl}h_{\alpha\beta}\frac{\partial F^\alpha}{\partial x^i}\frac{\partial F^\beta}{\partial x^j}d\mu_g \qquad (4.1)$$

Since F is conformal,

$$h_{\alpha\beta}\frac{\partial F^{\alpha}}{\partial x^{i}}\frac{\partial F^{\beta}}{\partial x^{j}}=\varphi g_{ij} \tag{4.2}$$

for some function φ on Σ. Put (4.2) into (4.1) and use $g^{ij}g_{ij}=2$ since Σ is two dimensional, we see that $\frac{d}{ds}\mathscr{E}_{g_s,h}(F)\big|_{s=0}=0$, as claimed. □

On the other hand, recall that a branched minimal immersion is (weakly) conformal and harmonic, and we have the following ([45], Theorem 1.8)

Proposition 4.1.1. *If u is critical map of $\mathscr{E}$ with respect to the variations of u and the conformal structures on Σ, then u is a branched minimal immersion.*

Let U be an open subset in the upper half space $\mathbb{H}=\{\tau\in\mathbb{C}\,|\,\operatorname{Im}\tau>0\}$. It is well-known that the upper half space represents the Teichmüller space of the standard torus $\mathbb{T}=\mathbb{R}^2/\{1,i\}$ and we treat U as a local parameterization of the conformal structures on $\mathbb{T}$ near a given one.

Let $0<\alpha<1$ be fixed. Define

$$\begin{aligned}\mathscr{U}&=C^{2,\alpha}(\mathbb{T},\mathbb{R}^4)\times U,\\ \mathscr{C}^{k,\alpha}&=C^{k,\alpha}(\mathbb{T},\mathbb{R}^4)\oplus\mathbb{R}^2\\ \mathscr{W}^{k,p}&=W^{k,p}(\mathbb{T},\mathbb{R}^4)\oplus\mathbb{R}^2\\ \mathscr{L}^2&=\mathscr{W}^{0,2}.\end{aligned}$$

Note that $\mathscr{C}^{k,\alpha},\mathscr{W}^{k,\alpha}$ are Banach spaces [1] with the norms

$$\begin{aligned}\|(\phi,\nu)\|_{k,\alpha}&=\|\phi\|_{C^{k,\alpha}}+|\nu|,\\ \|(\phi,\nu)\|_{\mathscr{W}^{k,p}}&=\|\phi\|_{W^{k,p}}+|\nu|\end{aligned}$$

respectively. When $(M,h)=(\mathbb{R}^4,G)$, where G is as in (2.3), the functional $\mathscr{E}:\mathscr{U}\to\mathbb{R}$ takes the form

$$\mathscr{E}(u,\tau)=\frac{1}{2}\int_{\mathbb{T}}e^{-\frac{|u|^2}{4}}|Du|_{\tau}^{2}d\mu_{\tau},\quad (u,\tau)\in\mathscr{U}. \tag{4.3}$$

[1] All Banach spaces considered in this thesis are real Banach spaces.

Here g_τ is the metric on $\mathbb{T}$ given by

$$g_\tau = \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix}^T \begin{pmatrix} 1 & \tau_1 \\ 0 & \tau_2 \end{pmatrix} \tag{4.4}$$

and

$$d\mu_\tau = d\mu_{g_\tau} = \sqrt{\det g_\tau}\, dxdy, \quad |Du|_\tau^2 = g_\tau^{ij} D_i u \cdot D_j u.$$

The metric g_τ is in the conformal class represented by τ, as it can be seen easily that g_τ is the pullback metric via the linear mapping from $\mathbb{T} = \mathbb{R}^2/\{1,i\}$ to $\mathbb{R}^2/\{1,\tau\}$. Note that for each fixed τ, $\mathscr{E}(\cdot,\tau)$ is the Dirichlet energy functional of the mappings $u : (\mathbb{T}, g_\tau) \to (\mathbb{R}^4, G)$.

By Lemma 2.1.1, minimal surfaces in $(\mathbb{R}^4, G)$ corresponds to self-shrinking surfaces in $\mathbb{R}^4$. Thus Lemma 4.1.1 and Proposition 4.1.1 imply the following

Proposition 4.1.2. *(u,τ) is a critical point of $\mathscr{E}$ if and only if $u : (\mathbb{T}, g_\tau) \to \mathbb{R}^4$ is a branched conformal self-shrinking torus.*

Next we consider the L^2-gradient $\mathscr{M} : \mathscr{U} \to \mathscr{C}^{0,\alpha}$ of $\mathscr{E}$. That is, we find for each $(u,\tau) \in \mathscr{U}$ an element $\mathscr{M}(u,\tau) \in \mathscr{C}^{0,\alpha}$ so that for all $(\phi, \nu) \in \mathscr{C}^{2,\alpha}$,

$$\left.\frac{d}{ds}\right|_{s=0} \mathscr{E}(u + s\phi, \tau + s\nu) = \langle \mathscr{M}(u,\tau), (\phi,\nu)\rangle_{u,\tau}. \tag{4.5}$$

Here we define

$$\langle \phi_1, \phi_2 \rangle_{u,\tau} = \int_{\mathbb{T}} \phi_1 \cdot \phi_2 \, \mathrm{e}^{-\frac{|u|^2}{4}} d\mu_\tau$$

and

$$\langle (\phi_1,\nu_1), (\phi_2,\nu_2) \rangle_{u,\tau} = \langle \phi_1, \phi_2 \rangle_{u,\tau} + \nu_1 \cdot \nu_2. \tag{4.6}$$

Integrating by parts, we see that

$$\mathscr{M}(u,\tau) = \left(-g_\tau^{ij} e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i u) - \frac{1}{4} |Du|_\tau^2 u, \nabla \mathscr{E}_\tau^u \right) \tag{4.7}$$

where $\mathscr{E}^u : U \to \mathbb{R}$ is given by $\mathscr{E}^u(\tau) = \mathscr{E}(u,\tau)$ and $\nabla \mathscr{E}_\tau^u$ is the gradient of $\mathscr{E}^u$ at τ.

Let (u,τ) be a critical point of $\mathscr{E}$, that is, $\mathscr{M}(u,\tau) = 0$. Let

$$\mathscr{L} = \mathscr{L}_{(u,\tau)} : \mathscr{C}^{2,\alpha} \to \mathscr{C}^{0,\alpha}$$

be the Fréchet derivative of $\mathscr{M}$ at (u,τ). We will show that

$$\mathscr{L}(\phi,\nu)=\left(L\phi+\nabla_\nu B,(\nabla^2\mathscr{E}^u_\tau)\nu+\langle\nabla B_\tau,\phi\rangle_{u,\tau}\right) \tag{4.8}$$

where

$$L\phi=-g^{ij}_\tau e^{\frac{|u|^2}{4}}D_j(e^{-\frac{|u|^2}{4}}D_i\phi)-\frac{1}{4}|Du|^2_\tau\phi+\frac{1}{2}g^{ij}_\tau D_j(u\cdot\phi)D_iu-\frac{1}{2}g^{ij}_\tau(D_ju\cdot D_i\phi)u \tag{4.9}$$

and $\nabla^2\mathscr{E}^u_\tau$ is the Hessian of $\mathscr{E}^u$ at τ; furthermore, $B:U\to C^{0,\alpha}(\mathbb{T},\mathbb{R}^4)$ is given by

$$B(\sigma)=-g^{ij}_\sigma\left(e^{\frac{|u|^2}{4}}D_j(e^{-\frac{|u|^2}{4}}D_iu)+\frac{1}{4}(D_iu\cdot D_ju)u\right) \tag{4.10}$$

and ∇B_τ denotes the Fréchet derivative of B at τ and $\nabla_\nu B_\tau$ stands for the Fréchet derivative of B at τ in the direction ν:

$$\nabla_\nu B_\tau=\frac{d}{ds}\bigg|_{s=0}B(\tau+s\nu).$$

To derive (4.8), note that the two terms in the first component of (4.8) arise from direct differentiation of the first component of (4.7) with respect to ϕ and ν. To derive the second component, note that $(\nabla^2\mathscr{E}^u_\tau)\nu$ is just the directional derivative of $\nabla\mathscr{E}^u_\tau$ with respect to ν. Thus we need to show that $\nabla_\phi\nabla\mathscr{E}^u_\tau=\langle\nabla B_\tau,\phi\rangle_{u,\tau}$, where

$$\nabla_\phi\nabla\mathscr{E}^u_\tau=\frac{d}{ds}\bigg|_{s=0}\nabla\mathscr{E}^{u+s\phi}_\tau.$$

Note

$$\nabla\mathscr{E}^u_\tau=\frac{1}{2}\int_{\mathbb{T}}(\nabla g^{ij}_\tau)e^{-\frac{|u|^2}{4}}(D_iu\cdot D_ju)d\mu_\tau+\frac{1}{4}\mathrm{tr}(g_\tau^{-1}\nabla g_\tau)\mathscr{E}(u,\tau),$$

where the second term on the right comes from differentiating the volume form $d\mu_\tau$. Since (u,τ) is a critical point of $\mathscr{E}$, this term vanishes when we differentiate with respect to ϕ. Using this observation and integration by parts,

$$\begin{aligned}
\nabla_\phi \nabla \mathscr{E}^u_\tau &= \frac{1}{2}\int_{\mathbb{T}} (\nabla g^{ij}_\tau) \nabla_\phi \left(e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j u) \right) d\mu_\tau \\
&= \frac{1}{2}\int_{\mathbb{T}} (\nabla g^{ij}_\tau) \left(-\frac{1}{2}(u\cdot\phi) e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j u) + 2 e^{-\frac{|u|^2}{4}} (D_i u \cdot D_j \phi) \right) d\mu_\tau \\
&= -\int_{\mathbb{T}} (\nabla g^{ij}_\tau) \left(\frac{1}{4}(D_i u \cdot D_j u) u + e^{\frac{|u|^2}{4}} D_j (e^{-\frac{|u|^2}{4}} D_i) \cdot \phi \right) e^{-\frac{|u|^2}{4}} d\mu_\tau \\
&= \int_{\mathbb{T}} \nabla B_\tau \cdot \phi \, e^{-\frac{|u|^2}{4}} d\mu_\tau \\
&= \langle \nabla B_\tau, \phi \rangle_{u,\tau}.
\end{aligned}$$

Thus (4.8) is shown.

Lemma 4.1.2. *Let (u,τ) be a critical point of $\mathscr{E}$. For all $(\phi,\nu),(\psi,\eta) \in \mathscr{C}^{2,\alpha}$, we have*

$$\langle \mathscr{L}(\phi,\nu),(\psi,\eta)\rangle_{u,\tau} = \langle (\phi,\nu), \mathscr{L}(\psi,\eta)\rangle_{u,\tau}. \tag{4.11}$$

Proof. Let $\phi,\psi \in C^{2,\alpha}(\mathbb{T},\mathbb{R}^4)$, then from (4.8) and (4.10),

$$\begin{aligned}
\langle L\phi,\psi\rangle_{u,\tau} &= \langle g^{ij}_\tau D_i\phi, D_j\psi\rangle_{u,\tau} - \frac{1}{4}\langle |Du|^2_\tau \phi,\psi\rangle_{u,\tau} \\
&\quad + \frac{1}{2}\int g^{ij}_\tau D_j(u\cdot\phi) D_i u \cdot \psi \, e^{-\frac{|u|^2}{4}} d\mu_\tau - \frac{1}{2}\int g^{ij}_\tau (D_j u \cdot D_i \phi)(u\cdot\psi) \, e^{-\frac{|u|^2}{4}} d\mu_\tau.
\end{aligned} \tag{4.12}$$

Integrating by parts for the third term on the right hand side in (4.12) gives

$$\begin{aligned}
&\frac{1}{2}\int g^{ij}_\tau D_j(u\cdot\phi) D_i u \cdot \psi \, e^{-\frac{|u|^2}{4}} d\mu_\tau \\
&\qquad = -\frac{1}{2}\int g^{ij}_\tau (u\cdot\phi)(D_i u \cdot D_j \psi) \, e^{-\frac{|u|^2}{4}} d\mu_\tau - \frac{1}{2}\int (u\cdot\phi) g^{ij}_\tau D_j (e^{-\frac{|u|^2}{4}} D_i u)\cdot\psi \, d\mu_\tau.
\end{aligned} \tag{4.13}$$

Since $\mathscr{M}(u,\tau) = 0$, we have by (4.7)

$$g^{ij}_\tau D_j (e^{-\frac{|u|^2}{4}} D_i u) = -\frac{1}{4} e^{-\frac{|u|^2}{4}} |Du|^2_\tau u.$$

Putting this into (4.13), we have

$$\begin{aligned}\langle L\phi,\psi\rangle_{u,\tau} &= \langle g_\tau^{ij} D_i\phi, D_j\psi\rangle_{u,\tau} - \frac{1}{4}\langle |Du|_\tau^2\phi,\psi\rangle_{u,\tau} \\ &\quad - \frac{1}{2}\int g_\tau^{ij}(u\cdot\phi)(D_i u\cdot D_j\psi)e^{-\frac{|u|^2}{4}}d\mu_{g_\tau} - \frac{1}{2}\int g_\tau^{ij}(D_j u\cdot D_i\phi)(u\cdot\psi)e^{-\frac{|u|^2}{4}}d\mu_\tau \\ &\quad + \frac{1}{8}\int (u\cdot\phi)(u\cdot\psi)|Du|_\tau^2 e^{-\frac{|u|^2}{4}}d\mu_\tau.\end{aligned}$$

Note that the right hand side is symmetric in ϕ and ψ. Thus

$$\langle L\phi,\psi\rangle_{u,\tau} = \langle \phi, L\psi\rangle_{u,\tau}, \quad \forall \phi,\psi \in C^{2,\alpha}(\mathbb{T},\mathbb{R}^4). \tag{4.14}$$

Using this, we have

$$\begin{aligned}\langle \mathscr{L}(\phi,\nu),(\psi,\eta)\rangle_{u,\tau} &= \langle L\phi + \nabla_\nu B_\tau,\psi\rangle_{u,\tau} + (\nabla^2\mathscr{E}_\tau^u\nu + \langle \nabla B_\tau,\psi\rangle_{u,\tau})\cdot\eta \\ &= \langle L\phi,\psi\rangle_{u,\tau} + \langle \nabla_\nu B_\tau,\psi\rangle_{u,\tau} + \langle \nabla_\eta B_\tau,\phi\rangle_{u,\tau} + (\nabla^2\mathscr{E}_\tau^u\nu)\cdot\eta\end{aligned}$$

Again, the right hand side is symmetric in (ϕ,ν) and (ψ,η). We can now conclude the proof of the lemma. □

Remark 1. Note that the apparent self-adjointness expression for $\mathscr{L}$ in (4.11) only holds in $\mathscr{C}^{2,\alpha}$, and $\mathscr{L}$ is an operator from $\mathscr{C}^{2,\alpha}$ to $\mathscr{C}^{0,\alpha}$. Nevertheless, (4.11) is useful in proving the following theorem.

Theorem 4.1.1. *$\mathscr{L}$ is a Fredholm operator of index zero at a critical point (u,τ) of $\mathscr{E}$.*

Proof. The proof will be divided into several steps.

Step 1. We show that $\dim\ker\mathscr{L}$ is finite.

Consider the first component of $\mathscr{L}$,

$$L\phi + \nabla_\nu B_\tau = 0. \tag{4.15}$$

This equation is bilinear in ϕ,ν. Let S be the subspace of $\mathbb{R}^2$ so that $\nu \in S$ if and only if (4.15) has a solution. If $S = \{(0,0)\}$, then $\dim\ker\mathscr{L} = \dim\ker L < \infty$ since L is elliptic. If not, let $\{\nu_i\}$ be a basis of S. Pick $\phi_i \in C^{2,\alpha}(\mathbb{T},\mathbb{R}^4)$ so that ϕ_i satisfies

(4.15) with $\nu = \nu_i$. Let $(\phi, \nu) \in \ker \mathscr{L}$. Then $\nu \in S$. Write $\nu = \sum_i s^i \nu_i$ for some $s^i \in \mathbb{R}$. Then $\phi - s^i \phi_i \in \ker L$ and thus

$$(\phi, \nu) = (\phi_0, 0) + \sum_i s^i (\phi_i, \nu_i)$$

for some $\phi_0 \in \ker L$. Again, due to the ellipticity of L, $\dim \ker L$ is finite, hence $\ker \mathscr{L}$ is finite dimensional.

Step 2. $\mathscr{L}$ has finite dimensional cokernel. Moreover, $\dim \ker \mathscr{L} = \dim \operatorname{coker} \mathscr{L}$.

We will show that the mapping

$$\ker \mathscr{L} \hookrightarrow \mathscr{C}^{2,\alpha} \hookrightarrow \mathscr{C}^{0,\alpha} \xrightarrow{\pi} \operatorname{coker} \mathscr{L} \tag{4.16}$$

is bijective, where π is the projection to the quotient $\operatorname{coker} \mathscr{L} = \mathscr{C}^{0,\alpha} / \operatorname{Im} \mathscr{L}$.

Firstly, if $(\psi_1, \eta_1), (\psi_2, \eta_2) \in \ker \mathscr{L}$ represent the same element in coker$L$, then there is $(\phi, \nu) \in \mathscr{C}^{2,\alpha}$ so that

$$(\psi, \eta) := (\psi_1 - \psi_2, \eta_1 - \eta_2) = \mathscr{L}(\phi, \nu).$$

Using (4.11),

$$\langle (\psi, \eta), (\psi, \eta) \rangle_{u,\tau} = \langle \mathscr{L}(\phi, \nu), (\psi, \eta) \rangle_{u,\tau} = \langle (\phi, \nu), \mathscr{L}(\psi, \eta) \rangle_{u,\tau} = 0.$$

Thus $(\psi, \eta) = 0$ and so the mapping $\ker \mathscr{L} \to \operatorname{coker} \mathscr{L}$ defined in (4.16) is injective.

Secondly, we show that the mapping $\ker \mathscr{L} \to \operatorname{coker} \mathscr{L}$ is surjective. Let $\overline{\operatorname{Im} \mathscr{L}}$ be the L^2 closure of the image of $\mathscr{L}$ in $\mathscr{L}^2$ with respect to the inner product defined in (4.6). Let $(\psi, \eta) \in \mathscr{C}^{0,\alpha}$ represents an element in $\operatorname{coker} \mathscr{L}$. We decompose (ψ, η) into the component in $\overline{\operatorname{Im} \mathscr{L}}$ and $\overline{\operatorname{Im} \mathscr{L}}^{\perp}$. That is,

$$(\psi, \eta) = (\psi^{\top}, \eta^{\top}) + (\psi^{\perp}, \eta^{\perp}) \tag{4.17}$$

for some $\psi^{\top}, \psi^{\perp} \in L^2(\mathbb{T}, \mathbb{R}^4)$. Note that

$$\langle (\psi^{\perp}, \eta^{\perp}), \mathscr{L}(\phi, \nu) \rangle_{u,\tau} = 0$$

for all $(\phi,\nu)\in\mathscr{C}^{2,\alpha}$. Letting $\nu=0$ and using (4.8), we have

$$\langle L\phi,\psi^{\perp}\rangle_{u,\tau}+\langle\nabla_{\eta^{\perp}}B_{\tau},\phi\rangle_{u,\tau}=0,\quad\forall\phi\in C^{2,\alpha}(\mathbb{T},\mathbb{R}^4).$$

Note that the above equation is of the form

$$\int(-g_{\tau}^{ij}D_{ij}\phi+\mathscr{A}^iD_i\phi+\mathscr{B}\phi)\cdot\psi^{\perp}dxdy=\int\mathscr{F}\cdot\phi\,dxdy,$$

where $\mathscr{A}^i=(\mathscr{A}^i_{\beta\gamma})$ and $\mathscr{B}=(\mathscr{B}_{\beta\gamma})$ are (4×4)-matrix-valued smooth functions and $\mathscr{F}=(\mathscr{F}_{\beta})$ is a $\mathbb{R}^4$-valued smooth function. If we choose $\phi=(\rho,0,0,0)$, where $\rho\in C^{\infty}(\mathbb{T},\mathbb{R})$, we have

$$\int(-g_{\tau}^{ij}D_{ij}\rho+\mathscr{A}^i_{11}D_i\rho+\mathscr{B}_{11}\rho)\psi_1^{\perp}dxdy=D(\rho),\tag{4.18}$$

where

$$D(\rho)=-\int\sum_{k\neq 1}\mathscr{A}^i_{k1}\psi_k^{\perp}D_i\rho\,dxdy-\int\sum_{k\neq 1}\mathscr{B}_{k1}\psi_k^{\perp}\rho\,dxdy+\int\mathscr{F}_1\rho\,dxdy.\tag{4.19}$$

Since $\psi_k^{\perp}$ are in L^2 (noting that the L^2 spaces with respect the area elements $e^{-\frac{|u|^2}{4}}d\mu_{\tau}$ and $dxdy$ coincide over $\mathbb{T}$), as a distribution, D is in H^{-1}_{loc}. Thus the Elliptic Regularity Theorem (Theorem 6.33 in [24]) asserts $\psi_1^{\perp}\in H^1_{loc}$. Similarly, we have $\psi_k^{\perp}\in H^1_{loc}$ for $k=2,3,4$. Putting this information into (4.19), we see that $D\in H^0_{loc}$, and in turn, this implies $\phi_1^{\perp}\in H^2_{loc}$ by the Elliptic Regularity Theorem again. By a standard bootstrapping argument and the Sobolev embedding theorem, we see that $\psi^{\perp}\in C^{2,\alpha}$ (in fact, smooth). Using (4.11) and the definition of $(\psi^{\perp},\eta^{\perp})$, we have

$$(\mathscr{L}(\psi^{\perp},\eta^{\perp}),(\phi,\nu))_{u,\tau}=0$$

for all $(\phi,\nu)\in\mathscr{C}^{2,\alpha}$, thus

$$\mathscr{L}(\psi^{\perp},\eta^{\perp})=0.$$

The smoothness of $(\psi^{\perp},\eta^{\perp})$ asserts $(\psi^{\top},\eta^{\top})\in\mathscr{C}^{0,\alpha}$. If we can show that

$$(\psi^{\top},\eta^{\top})\in\operatorname{Im}\mathscr{L},\tag{4.20}$$

then $\pi(\psi,\eta) = \pi(\psi^\perp,\eta^\perp)$ by (4.17) and it follows that the mapping $\ker\mathscr{L} \to \operatorname{coker}\mathscr{L}$ defined in (4.16) is surjective and we are done. To show (4.20), recall that $(\psi^\top,\eta^\top) \in \overline{\operatorname{Im}\mathscr{L}}$. Thus there is a sequence $(\phi_n,\nu_n) \in \mathscr{C}^{2,\alpha}$ so that $\mathscr{L}(\phi_n,\nu_n) \to (\psi^\top,\eta^\top)$ in $\mathscr{L}^2$. Using the $\mathscr{L}^2$ inner product, we decompose (ϕ_n,ν_n) into

$$(\phi_n,\nu_n) = (\phi_n^{\mathrm{K}},\nu_n^{\mathrm{K}}) + (\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}}), \tag{4.21}$$

where $(\phi_n^{\mathrm{K}},\nu_n^{\mathrm{K}}) \in \ker\mathscr{L}$ and $(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}}) \in \ker\mathscr{L}^\perp$. Then by setting

$$(\psi_n,\eta_n) = \mathscr{L}(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}}) \tag{4.22}$$

and using $\mathscr{L}(\phi_n^{\mathrm{K}},\nu_n^{\mathrm{K}}) = 0$, we have

$$\begin{aligned}(\psi_n,\eta_n) &= \mathscr{L}(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}}) \\ &= \mathscr{L}(\phi_n,\nu_n) \\ &\xrightarrow{\mathscr{L}^2} (\psi^\top,\eta^\top).\end{aligned}$$

The convergence above in particular implies that $\|\psi_n\|_{L^2} \le C$ for some constant C. From the first component of (4.8), which is

$$L\phi_n^{\mathrm{P}} = \psi_n - \nabla_{\nu_n^{\mathrm{P}}} B_\tau,$$

the standard elliptic estimates (Theorem 9.11 in [26]) implies that there are constants $C', C'', C''' > 0$ so that

$$\begin{aligned}\|\phi_n^{\mathrm{P}}\|_{W^{2,2}} &\le C'\left(\|\phi_n^{\mathrm{P}}\|_{L^2} + \|\psi_n - \nabla_{\nu_n^{\mathrm{P}}} B_\tau\|_{L^2}\right) \\ &\le C'\left(\|\phi_n^{\mathrm{P}}\|_{L^2} + \|\psi_n\|_{L^2} + C''|\nu_n^{\mathrm{P}}|\right) \\ &\le C'''\left(\|(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}})\|_{\mathscr{L}^2} + 1\right).\end{aligned} \tag{4.23}$$

Next, we show that the sequence $\{\|(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}})\|_{\mathscr{L}^2}\}$ is bounded. Assume not, then by taking a subsequence if necessary, we have $\|(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}})\|_{\mathscr{L}^2} \to \infty$. Let

$$(\tilde{\phi}_n,\tilde{\nu}_n) = \frac{(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}})}{\|(\phi_n^{\mathrm{P}},\nu_n^{\mathrm{P}})\|_{\mathscr{L}^2}}. \tag{4.24}$$

Then, as (ψ_n, η_n) converges to $(\psi^\top, \eta^\top)$ in $\mathscr{L}^2$,

$$\mathscr{L}(\tilde{\phi}_n, \tilde{\nu}_n) = \frac{(\psi_n, \eta_n)}{\|(\phi_n^{\mathrm{P}}, \nu_n^{\mathrm{P}})\|_{\mathscr{L}^2}} \xrightarrow{\mathscr{L}^2} 0. \tag{4.25}$$

Since $\|(\tilde{\phi}_n, \tilde{\nu}_n)\|_{\mathscr{L}^2} = 1$, we may assume $\tilde{\nu}_n \to \tilde{\nu}$ for some $\tilde{\nu} \in \mathbb{R}^2$. From (4.23) and (4.24), the sequence $\{\|\tilde{\phi}_n\|_{W^{2,2}}\}$ is bounded. Hence, again by taking subsequence if necessary, there is $\tilde{\phi} \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$ so that $\tilde{\phi}_n \to \tilde{\phi}$ in $W^{1,2}(\mathbb{T}, \mathbb{R}^4)$. Using (4.25), we have

$$\begin{cases} L\tilde{\phi} + \nabla_{\tilde{\nu}} B_\tau = 0 & \text{weakly in } W^{1,2}(\mathbb{T}, \mathbb{R}^4), \\ \nabla^2 \mathscr{E}_\tau^u \tilde{\nu} + \langle \nabla B_\tau, \tilde{\phi} \rangle_{u,\tau} = 0 \end{cases}$$

Since $\tilde{\phi} \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$, the first equation is actually satisfied strongly in $W^{2,2}(\mathbb{T}, \mathbb{R}^4)$. Since $\nabla_{\tilde{\nu}} B_\tau$ is smooth, by the elliptic regularity, $\tilde{\phi}$ is smooth. Thus $(\tilde{\phi}, \tilde{\nu}) \in \mathscr{C}^{2,\alpha}$ and $\mathscr{L}(\tilde{\phi}, \tilde{\nu}) = 0$, in other words, $(\tilde{\phi}, \tilde{\nu}) \in \ker \mathscr{L}$. On the other hand, since $(\tilde{\phi}_n, \tilde{\nu}_n) \to (\tilde{\phi}, \tilde{\nu})$ in $\mathscr{L}^2$ and $(\tilde{\phi}_n, \tilde{\nu}_n) \in \ker \mathscr{L}^\perp$, we also have $(\tilde{\phi}, \tilde{\nu}) \in \ker \mathscr{L}^\perp$. Thus $(\tilde{\phi}, \tilde{\nu}) = (0, 0)$. But this is impossible as $\|(\tilde{\phi}, \tilde{\nu})\|_{\mathscr{L}^2} = 1$ since $(\tilde{\phi}_n, \tilde{\nu}_n) \to (\tilde{\phi}, \tilde{\nu})$ in $\mathscr{L}^2$ and $\|(\tilde{\phi}_n, \tilde{\nu}_n)\|_{\mathscr{L}^2} = 1$. The contradiction leads to the conclusion that the sequence $\{\|(\phi_n^{\mathrm{P}}, \nu_n^{\mathrm{P}})\|_{\mathscr{L}^2}\}$ is bounded.

From (4.23), the sequence $\{\|(\phi_n^{\mathrm{P}}, \nu_n^{\mathrm{P}})\|_{\mathscr{W}^{2,2}}\}$ is also bounded. By taking a subsequence if necessary, there is $(\phi, \nu) \in \mathscr{W}^{2,2}$ so that $(\phi_n^{\mathrm{P}}, \nu_n^{\mathrm{P}}) \to (\phi, \nu)$ in $\mathscr{W}^{1,2}$ and

$$\mathscr{L}(\phi, \nu) = (\psi^\top, \nu^\top).$$

The first component of this is given by

$$L\phi + \nabla_\nu B_\tau = \psi^\top.$$

Since $\phi \in W^{2,2}(\mathbb{T}, \mathbb{R}^4)$ and $\psi^\top \in C^{0,\alpha}(\mathbb{T}, \mathbb{R}^4)$, the standard elliptic regularity (Theorem 9.19 in [26]) implies that $\phi \in C^{2,\alpha}(\mathbb{T}, \mathbb{R}^4)$. Thus $(\phi, \nu) \in \mathscr{C}^{2,\alpha}$. This shows $(\psi^\top, \eta^\top) \in \mathrm{Im}\mathscr{L}$. Therefore, the mapping $\ker \mathscr{L} \to \mathrm{coker}\mathscr{L}$ is surjective.

Step 3. From the previous two steps, the bounded operator $\mathscr{L}$ has finite dimensional kernel and cokernel so it is a Fredholm operator of

$$\mathrm{index}\, \mathscr{L} = \dim \ker \mathscr{L} - \dim \mathrm{coker} \mathscr{L} = 0.$$

This completes the proof of the theorem. □

4.1.2 A Łojasiewicz-Simon type inequality

Next we prove a Łojasiewicz-Simon gradient inequality for compact branched self-shrinkers $F:\mathbb{T}\to\mathbb{R}^4$. As in [49], we use the Liapunov-Schmidt reduction argument and the classical Łojasiewicz inequality in [37]. See [23] for a Łojasiewicz-Simon inequality in the abstract setting and the related work in the reference therein.

Let

$$\Pi:\mathscr{L}^2\to\ker\mathscr{L}$$

be the L^2-projection with respect to the L^2 inner product:

$$\langle(\psi_1,\nu_1),(\psi_2,\nu_2)\rangle_{\mathscr{L}^2}=\int_{\mathbb{T}}\psi_1\cdot\psi_2\,dxdy+\nu_1\cdot\nu_2 \tag{4.26}$$

for all $(\psi_1,\nu_1),(\psi_2,\nu_2)\in\mathscr{L}^2$. Recall that $\ker\mathscr{L}$ is a finite dimensional subspace and $\ker\mathscr{L}\subset\mathscr{C}^\infty$. For all $k=0,1,2,\cdots$, we let

$$\Pi_k:\mathscr{C}^{k,\alpha}\to\mathscr{C}^{0,\alpha}$$

be the restriction of Π to $\mathscr{C}^{k,\alpha}$ composed with the inclusion $\ker\mathscr{L}\hookrightarrow\mathscr{C}^{0,\alpha}$.

Lemma 4.1.3. $\Pi_k:\mathscr{C}^{k,\alpha}\to\mathscr{C}^{0,\alpha}$ *is a bounded linear operator for all nonnegative integers k. In particular, there is a positive constant* C_α *so that*

$$\|\Pi_k(\psi,\nu)\|_{0,\alpha}\le C_\alpha\|(\psi,\nu)\|_{k,\alpha} \tag{4.27}$$

for all $(\psi,\nu)\in\mathscr{C}^{k,\alpha}$.

Proof. Let $(\chi_1,\nu_1),\cdots,(\chi_n,\nu_n)\in\ker\mathscr{L}$ be an orthonormal basis of the finite dimensional space $\ker\mathscr{L}$ with respect to the inner product in (4.26). Then for any $(\psi,\nu)\in\mathscr{L}^2$, we have

$$\Pi(\psi,\nu)=\sum_{i=1}^n\langle(\chi_i,\nu_i),(\psi,\nu)\rangle_{\mathscr{L}^2}(\chi_i,\nu_i).$$

Then we have

$$\begin{aligned}\|\Pi_k(\psi,\nu)\|_{0,\alpha} &\le \sum_{i=1}^{n} |\langle(\chi_i,\nu_i),(\psi,\nu)\rangle_{\mathscr{L}^2}|\, \|(\chi_i,\nu_i)\|_{0,\alpha} \\ &\le \left(\sum_{i=1}^{n} \|(\chi_i,\nu_i)\|_{0,\alpha}\right) \|(\psi,\nu)\|_{\mathscr{L}^2}.\end{aligned}$$

Note that we used the Cauchy-Schwarz inequality and that $\|(\chi_i,\nu_i)\|_{\mathscr{L}^2} = 1$. Since

$$\int_{\mathbb{T}} dxdy = 1,$$

we have

$$\|(\psi,\nu)\|_{\mathscr{L}^2} \le \max_{\mathbb{T}^2} |\psi| + |\nu| \le \|(\psi,\nu)\|_{k,\alpha} \tag{4.28}$$

for all nonnegative k. Now (4.27) follows with $C_\alpha = \sum_{i=1}^{n} \|(\chi_i,\nu_i)\|_{0,\alpha}$. □

To simplify notations, in the sequel we use x,y and a,b to denote elements in $\mathscr{C}^{2,\alpha}$ and $\mathscr{C}^{0,\alpha}$ respectively. Let $x_c = (u,\tau)$ be a critical point of $\mathscr{E}$ as before, that is $\mathscr{M}(x_c) = 0$.

Consider the mapping $\mathscr{N} : \mathscr{U} \to \mathscr{C}^{0,\alpha}$ given by

$$\mathscr{N}(x) = \mathscr{M}(x) + \Pi_2(x - x_c). \tag{4.29}$$

Since Π_2 is linear, the differential $D\mathscr{N}$ at x_c is given by

$$D\mathscr{N}_{x_c} = \mathscr{L} + \Pi_2. \tag{4.30}$$

Lemma 4.1.4. *$D\mathscr{N}_{x_c}$ is bijective and its inverse is bounded.*

Proof. First we show that $D\mathscr{N}_{x_c}$ is injective. Let $D\mathscr{N}_{x_c}(x) = 0$. Then by (4.30) we have

$$\mathscr{L}(x) = -\Pi_2 x.$$

Using (4.11), for all $y \in \ker\mathscr{L}$ we have

$$\langle \Pi_2 x, y\rangle_{u,\tau} = -\langle \mathscr{L}x, y\rangle_{u,\tau} = -\langle x, \mathscr{L}y\rangle_{u,\tau} = 0.$$

This means that $\Pi_2 x \in \ker \mathscr{L}$ is orthogonal to $\ker \mathscr{L}$. Therefore, $\Pi_2 x = 0$. Thus $\mathscr{L} x = 0$ and so $x \in \ker \mathscr{L}$. Hence $x = \Pi_2 x = 0$ and $D\mathscr{N}_{x_c}$ is injective.

By Theorem 4.1.1, $\mathscr{L}$ is a Fredholm operator of index zero. Since Π_2 is bounded with a finite dimensional range, Π_2 is a compact operator and $D\mathscr{N}_{x_c} : \mathscr{C}^{2,\alpha} \to \mathscr{C}^{0,\alpha}$ is Fredholm with index zero (Theorem 5.10 in [46]). Together with the fact that $D\mathscr{N}_{x_c}$ is injective, $D\mathscr{N}_{x_c}$ is also surjective. Finally, the bounded inverse theorem (Theorem 3.8 in [46]) asserts that $D\mathscr{N}_{x_c}$ has a bounded inverse. □

By the inverse function theorem for Banach spaces (Theorem 15.2 in [21]), since $\mathscr{N}$ is C^1 ($\mathscr{N}$ is even analytic: see the appendix), there are open neighbourhoods $\mathscr{U}_1$ of x_c in $\mathscr{U}$ and $\mathscr{V}_1$ of 0 in $\mathscr{C}^{0,\alpha}$ so that $\mathscr{N} : \mathscr{U}_1 \to \mathscr{V}_1$ is invertible with a C^1 inverse Ψ. By shrinking $\mathscr{U}_1, \mathscr{V}_1$ if necessary, we assume that $\mathscr{V}_1$ is convex, $\mathscr{U}_1$ is contained in a convex set $\mathscr{U}_2 \subset \mathscr{U}$ and (since $\mathscr{M}$ and Ψ are C^1) there exist two positive constants M_1, M_2 so that

$$\begin{aligned} \|D\Psi(a)\|_{op} &\le M_1, \quad \forall a \in \mathscr{V}_1, \\ \|D\mathscr{M}(x)\|_{op} &\le M_2, \quad \forall x \in \mathscr{U}_2, \end{aligned} \tag{4.31}$$

where $\|\cdot\|_{op}$ denotes the operator norm for the corresponding operator. Using the Fundamental Theorem of Calculus, the above imply

$$\|\Psi(a) - \Psi(b)\|_{2,\alpha} \le M_1 \|a - b\|_{0,\alpha} \tag{4.32}$$

for all $a, b \in \mathscr{V}_1$ and

$$\|\mathscr{M}(x) - \mathscr{M}(y)\|_{0,\alpha} \le M_2 \|x - y\|_{2,\alpha} \tag{4.33}$$

for all $x, y \in \mathscr{U}_1$.

A main technical result in this section is the following Łojasiewicz-Simon type gradient inequality:

Theorem 4.1.2. *There is an open neighbourhood $\mathscr{W}_0 \subset \mathscr{U}$ of x_c, a positive constant C_2 and a constant $\theta \in (0, 1/2)$ depending on $\mathscr{E}$ and x_c so that*

$$|\mathscr{E}(x) - \mathscr{E}(x_c)|^{1-\theta} \le C_2 \|\mathscr{M}(x)\|_{0,\alpha}, \quad \forall x \in \mathscr{W}_0. \tag{4.34}$$

Proof. Since Π_0 is bounded, there is an open neighbourhood $\mathscr{V}_0$ of 0 so that $\mathscr{V}_0, \Pi_0\mathscr{V}_0 \subseteq \mathscr{V}_1$. For all $a \in \mathscr{V}_0$, $\Pi_0 a \in \mathscr{V}_1$. Since $\mathscr{U}_2$ is convex, the line segment joining $\Psi(a)$ and $\Psi(\Pi_0 a)$ is in $\mathscr{U}_2$. The Fundamental Theorem of Calculus and (4.5) yield

$$\begin{aligned}\mathscr{E}(\Psi(a)) - \mathscr{E}(\Psi(\Pi_0 a)) &= -\int_0^1 \frac{d}{dt}(\mathscr{E}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))dt \\ &= -\int_0^1 \langle \mathscr{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a))), \Psi(\Pi_0 a) - \Psi(a)\rangle_{u_t,\tau_t} dt,\end{aligned}$$

where we write

$$(u_t, \tau_t) = \Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)).$$

Using the Cauchy-Schwarz inequality, (4.28), (4.33) and $|t| \leq 1$,

$$\begin{aligned}&|\mathscr{E}(\Psi(a)) - \mathscr{E}(\Psi(\Pi_0 a))| \\ &\quad \leq \|\mathscr{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))\|_{\mathscr{L}^2} \|\Psi(\Pi_0 a) - \Psi(a)\|_{\mathscr{L}^2} \\ &\quad \leq \|\mathscr{M}(\Psi(a) + t(\Psi(\Pi_0 a) - \Psi(a)))\|_{0,\alpha} \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \\ &\quad \leq \big(\|\mathscr{M}(\Psi(a)\|_{0,\alpha} + M_2 t\, \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha}\big) \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \\ &\quad \leq \big(\|\mathscr{M}(\Psi(a)\|_{0,\alpha} + M_2 \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha}\big) \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha}\end{aligned} \tag{4.35}$$

On the order hand, since $a, \Pi_0 a \in \mathscr{V}_1$, by (4.32) we have

$$\|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \leq M_1 \|\Pi_0 a - a\|_{0,\alpha}. \tag{4.36}$$

Using the definition of $\mathscr{N}, \Psi$ and $\Pi_0\Pi_2 = \Pi_2$,

$$a = \mathscr{N}(\Psi(a)) = \mathscr{M}(\Psi(a)) + \Pi_2(\Psi(a) - x_c) \tag{4.37}$$

$$\begin{aligned}\Pi_0 a - a &= \Pi_0 a - \mathscr{M}(\Psi(a)) - \Pi_2(\Psi(a) - x_c) \\ &= \Pi_0\big(a - \Pi_2(\Psi(a) - x_c)\big) - \mathscr{M}\big(\Psi(a)\big).\end{aligned} \tag{4.38}$$

Since Π_0 is bounded by Lemma 4.1.3,

$$\begin{aligned}\|\Pi_0\big(a - \Pi_2(\Psi(a) - x_c)\big)\|_{0,\alpha} &\leq C_\alpha \|a - \Pi_2(\Psi(a) - x_c)\|_{0,\alpha} \\ &= C_\alpha \|\mathscr{M}(\Psi(a))\|_{0,\alpha},\end{aligned}$$

where in the last line we use (4.37) again. Combining this with (4.36) and (4.38), we are led to

$$\|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} \le C_1 \|\mathscr{M}(\Psi(a))\|_{0,\alpha} \tag{4.39}$$

for all $a \in \mathscr{V}_0$ with $C_1 = M_1(C_\alpha + 1)$. Putting this into (4.35), we have

$$|\mathscr{E}(\Psi(a)) - \mathscr{E}(\Psi(\Pi_0 a))| \le C_3 \|\mathscr{M}(\Psi(a))\|_{0,\alpha}^2 \tag{4.40}$$

for all $a \in \mathscr{V}_0$ and for some $C_3 > 0$.

Let $f : \mathscr{V}_1 \cap \ker \mathscr{L} \to \mathbb{R}$ be defined by

$$f(a) = \mathscr{E}(\Psi(a)). \tag{4.41}$$

It is easy to show that $\mathscr{E}$, $\mathscr{M}$ are analytic (a proof is given in the appendix for completeness). Since Π_2 is linear,

$$\mathscr{N} = \mathscr{M} + \Pi_2 - \Pi_2(x_c)$$

is analytic as well. Hence Ψ is analytic by the analytic version of inverse function theorem (Theorem 15.3 in [21]). Consequently, as a composition of analytic functions, f is also analytic, and it is defined on an open set in $\ker \mathscr{L}$, which is finite dimensional. The classical Łojasiewicz inequality [37] then implies that there is an open neighbourhood $\mathscr{V}_2 \subset \mathscr{V}_0$, constants $c > 0$ and $\theta \in (0, 1/2)$ so that

$$|f(\xi) - f(0)|^{1-\theta} \le c|f'(\xi)|, \quad \forall \xi \in \mathscr{V}_2 \cap \ker \mathscr{L}. \tag{4.42}$$

Using (4.41) and (4.5), for all $b \in \mathscr{V}_1 \cap K$ we have

$$f'(b)(\cdot) = \langle \mathscr{M}(\Psi(b)), D\Psi_b(\cdot) \rangle_{u,\tau}.$$

Using (4.28), (4.33) and (4.39),

$$
\begin{aligned}
|f'(\Pi_0 a)| &\le M_1 \|\mathscr{M}(\Psi(\Pi_0 a))\|_{\mathscr{L}^2} \\
&\le M_1 \|\mathscr{M}(\Psi(\Pi_0 a))\|_{0,\alpha} \\
&\le M_1 \big(\|\mathscr{M}(\Psi(\Pi_0 a)) - \mathscr{M}(\Psi(a))\|_{0,\alpha} + \|\mathscr{M}(\Psi(a))\|_{0,\alpha}\big) \\
&\le M_1 \big(M_2 \|\Psi(\Pi_0 a) - \Psi(a)\|_{2,\alpha} + \|\mathscr{M}(\Psi(a))\|_{0,\alpha}\big) \\
&\le C_4 \|\mathscr{M}(\Psi(a))\|_{0,\alpha}
\end{aligned} \tag{4.43}
$$

for some $C_4 > 0$. Now let $\mathscr{W}_0 = \Psi(\mathscr{V}_2)$. Thus for every $x \in \mathscr{W}_0$, there exists an $a \in \mathscr{V}_2$ such that $x = \Psi(a)$. By (4.43), the classical Łojasiewicz inequality (4.42) and (4.40),

$$
\begin{aligned}
C_4 c \|\mathscr{M}(x)\|_{0,\alpha} &\ge c|f'(\Pi_0 a)| \\
&\ge |f(\Pi_0 a) - f(0)|^{1-\theta} \\
&= |\mathscr{E}(\Psi(\Pi_0 a)) - \mathscr{E}(\Psi(a)) + \mathscr{E}(\Psi(a)) - \mathscr{E}(x_c)|^{1-\theta} \\
&\ge |\mathscr{E}(x) - \mathscr{E}(x_c)|^{1-\theta} - C_3 \|\mathscr{M}(x)\|_{0,\alpha}^{2(1-\theta)}.
\end{aligned} \tag{4.44}
$$

Since $2(1-\theta) \ge 1$, (4.34) is established for some $C_2 > 0$ and for all $x \in \mathscr{W}_0$. □

4.1.3 Proof of Theorem 1.2.1

The following lemma is first proved in [17] (Lemma 7.10 therein) when Σ is an n-dimensional self-shrinking embedded hypersurface in $\mathbb{R}^{n+1}$ with polynomial growth. Since a branched conformal immersion is immersed away from finitely many points, the exact same proof holds for compact branched conformally immersed self-shrinkers in $\mathbb{R}^m, m \ge 3$. For the reader's convenience, we sketch the proof of Lemma 4.1.5 in the appendix. Note that the $\mathscr{F}$-functional (2.8) and the entropy (2.13) are also defined for branched immersions of compact surfaces.

Lemma 4.1.5. *Let $F : \Sigma \to \mathbb{R}^m, m \ge 3$, be a compact branched conformally immersed self-shrinking surface. Then the entropy λ defined in (2.13) is maximized at $(x_0, t_0) = (0, 1)$. That is,*

$$
\lambda(F) = \frac{1}{4\pi} \int_\Sigma e^{-\frac{|F|^2}{4}} d\mu. \tag{4.45}
$$

Note that if (F,τ) is a critical point of $\mathscr{E}$, then F is a branched conformally immersed self-shrinking surface. Conformality of F then implies $|DF|_\tau^2 d\mu_\tau = 2d\mu$, where $d\mu$ is the area element of the metric induced by F away from the branch points. Together with (4.3) and Lemma 4.1.5,

$$\mathscr{E}(F,\tau) = \int_{\mathbb{T}} e^{-\frac{|F|^2}{4}} d\mu = 4\pi\lambda(F). \tag{4.46}$$

Now we proceed to prove Theorem 1.2.1.

Proof. Assume the theorem is false. Then there is a sequence $\{F_n\} \in \mathfrak{X}_\Lambda$ with $\lambda(F_i) \neq \lambda(F_j)$ for all $i \neq j$. Let $g_n = F_n^*\langle\cdot,\cdot\rangle$ and let g_{τ_n} be the Riemannian metric on $\mathbb{T}$ which is of the form (4.4) and is conformal to g_n. By Theorem 1.1.3, there is $F \in \mathfrak{X}_C$ and $\tau \in \mathbb{H}$ so that F_n converges smoothly to F and $\tau_n \to \tau$. Thus

$$\|(F_n,\tau_n) - (F,\tau)\|_{2,\alpha} \to 0 \text{ as } n \to \infty.$$

From Proposition 4.1.2 and (4.46) and by setting $x_c = (F,\tau)$ in (4.34), we have $\lambda(F_i) = \lambda(F)$ for all i large enough, since $\mathscr{M}(F_n,\tau_n) = 0$ for all n. That leads to a contradiction. Thus the theorem is proved. $\square$

4.2 Piecewise Lagrangian mean curvature flow

In this section, we extend the definition of the piecewise mean curvature flow in [17] to Lagrangian mean curvature flow for tori in $\mathbb{R}^4$ and construct a piecewise Lagrangian mean curvature flow for a Lagrangian immersed torus $F : \mathbb{T} \to \mathbb{R}^4$.

Definition 4.2.1. Let $F : L \to \mathbb{R}^4$ be a Lagrangian immersion, where L is a compact surface. A piecewise Lagrangian mean curvature flow with initial condition F is a finite collection of smooth Lagrangian mean curvature flows

$$F_t^i : L \to \mathbb{R}^4$$

defined on $[t_i, t_{i+1}]$, $i = 0, 1, \cdots, k-1$, where $0 = t_0 < t_1 < \cdots < t_{k-1} < t_k < \infty$ so that:

1. $F_0^0 = F$,

2. $\mu(F^{i+1}_{t_{i+1}}) = \mu(F^{i}_{t_{i+1}})$,

3. $\lambda(F^{i+1}_{t_{i+1}}) < \lambda(F^{i}_{t_{i+1}})$,

4. there is $\delta > 0$ such that

$$\|F^{i}_{t_{i+1}} - F^{i+1}_{t_{i+1}}\|_{C^0} \leq \delta\sqrt{\mu(F^{i}_{t_{i+1}})} \tag{4.47}$$

for $i = 0, 1, 2, \cdots, k-2$.

Remark 2. Note that if $k = 1$, the piecewise mean curvature flow is just the usual smooth mean curvature flow. The above definition is interesting only if we can characterize the behavior of the flow when $t \to t_k$.

Let $\{F_t : L \to \mathbb{R}^4\}$ be a smooth mean curvature flow defined on $[t_0, T_0)$, where $T_0 < \infty$ and L is a closed surface. Assume that a so-called type I singularity develops at T_0, which means $\sup_{F_t(L)} \|A_t\| \to \infty$ as $t \to T_0$ and there is a positive constant C so that

$$\max_{F_t(L)} |A_t|^2 \leq \frac{C}{\sqrt{T_0 - t}} \tag{4.48}$$

for all $t < T_0$. Let $t_n \to T_0$ and $q_n \in F_{t_n}(L)$ where $\max_{F_n(L)} |A_{t_n}|$ is attained, and suppose $q_n \to q \in \mathbb{R}^4$. Consider the type I rescaling, which is the family of immersions $\widetilde{F}(\cdot, s)$, where $-\log T_0 \leq s < \infty$ and

$$\widetilde{F}(\cdot, s) = \frac{1}{\sqrt{(T_0 - t)}}(F_t(x) - q), \;\; s(t) = -\log(T_0 - t). \tag{4.49}$$

For any sequence $s_j \to \infty$, a subsequence of $\{\widetilde{F}(\cdot, s_j)\}$ converges locally smoothly to a self-shrinking immersion $\overline{F} : \Sigma \to \mathbb{R}^4$ ([29]). In this case, we say that *the type I singularity can be modelled by* $\overline{F}$. It is not known whether $\overline{F}$ is unique: If we choose another sequence $\tilde{s}_k$, $\{\widetilde{F}(\cdot, \tilde{s}_k)\}$ might converge to a different self-shrinker.

Now we prove Theorem 1.2.2.

Proof. Let $F : \mathbb{T} \to \mathbb{R}^4$ be a Lagrangian immersion. By [51], there is a unique smooth Lagrangian mean curvature flow $\{F_t\}$ which is defined on a maximal time interval $[0, T_0)$, where $T_0 < \infty$ as $\mathbb{T}$ is compact.

If the singularity at T_0 is not a type I singularity that can be modelled by a compact self-shrinker with area no larger than Λ, then we set $k=0$ and no perturbation is performed.

Otherwise, the singularity at T_0 is of type I and it can be modelled by a compact self-shrinker with area no larger than Λ. In this case, the inequality (4.48) is satisfied at a point $q \in \mathbb{R}^4$ at time T_0 for some positive constant C and for all $t \in [0, T_0)$, and there is a sequence $s_j \to \infty$ such that $\widetilde{F}(\cdot, s_j)$ as in (4.49) converges locally smoothly to a compact self-shrinker $\overline{F}$ with area no bigger than Λ. To be precise about the convergence, we recall that Lemma 3.3, Corollary 3.2 and Proposition 2.3 in [29] hold for any codimension, and they guarantee that all $\widetilde{F}(\cdot, s_j)$ touch a fixed bounded region, the areas inside a ball $B(R)$ are bounded by $C(R)$ and the second fundamental forms and their derivatives of any order are bounded. Therefore, all the conditions in Theorem 1.3 in [6] are satisfied for the sequence $\{\widetilde{F}(\cdot, s_j)\}$, and the theorem asserts: by passing to a subsequence if necessary, there is a surface Σ and an immersion $\overline{F} : \Sigma \to \mathbb{R}^4$ and a sequence of diffeomorphisms

$$\varphi_j : U^j \to \widetilde{F}(\cdot, s_j)^{-1}(B_j) \subset \mathbb{T},$$

where B_j is the ball of radius j in $\mathbb{R}^4$ centered at the origin, $U_j \subset \Sigma$ are open sets with $U_j \subset\subset U_{j+1}$ and $\Sigma = \bigcup_j U_j$, such that

$$\|\widetilde{F}(\cdot, s_j) \circ \varphi_j - \overline{F}\|_{C^0(U_j)} \to 0$$

and $\widetilde{F}(\cdot, s_j) \circ \varphi$ converges to $\overline{F}$ locally smoothly. In our situation, we have assumed that Σ is compact (as we are dealing with singularity that can be modelled by compact shrinkers). Hence $\Sigma = U_k$ for all k large and thus φ_k are diffeomorphisms from Σ to $\mathbb{T}$, since the torus is connected. To simplify notations, we write $\Sigma = \mathbb{T}$. The diffeomorphisms $\varphi_j : \mathbb{T} \to \mathbb{T}$ have the property that

$$\|\widetilde{F}(\cdot, s_j) \circ \varphi_j - \overline{F}\|_{C^k(\mathbb{T})} \to 0 \tag{4.50}$$

for all $k = 0, 1, 2, \cdots$. Since each $\{F_t\}$ is Lagrangian, the sequence of blowups $\widetilde{F}(\cdot, s_j)$ are also Lagrangian for all j. The above convergence implies that $\overline{F}$ is Lagrangian, hence, $\overline{F} \in \mathfrak{X}_\Lambda$.

Since the entropy λ (2.13) is translation and scaling invariant,

$$\lambda(\widetilde{F}(\cdot,s(t))) = \lambda(F_t). \tag{4.51}$$

Furthermore, by the definition of $\mathscr{F}_{x_0,t_0}$ in (2.8), we see

$$\lambda(\widetilde{F}(\cdot,s_j)\circ\varphi_j) = \lambda(\widetilde{F}(\cdot,s_j)). \tag{4.52}$$

Since $\mathscr{F}_{0,1}$ (see (2.8)) is continuous with respect to the C^1-topology, there is a sequence d_j of positive numbers so that $d_j \to 0$ as $j \to \infty$ and

$$\mathscr{F}_{0,1}(\widetilde{F}(\cdot,s_j)\circ\varphi_j) \geq \mathscr{F}_{0,1}(\overline{F}) - d_j.$$

By definition of λ and Lemma 4.1.5, since $\overline{F}$ is a self-shrinker, from the above we have

$$\lambda(\widetilde{F}(\cdot,s_j)\circ\varphi_j) \geq \lambda(\overline{F}) - d_j. \tag{4.53}$$

As λ is non-increasing along the mean curvature flow, $\lambda(\widetilde{F}(\cdot,s_j))$ is non-increasing in j by (4.51). Together with (4.52) and (4.53), we conclude

$$\lambda(\widetilde{F}(\cdot,s_j)) \geq \lim_{j\to\infty}\lambda(\widetilde{F}(\cdot,s_j)) \geq \lambda(\overline{F}).$$

Fix $\delta > 0$. Let

$$\delta_1 = \frac{\delta\sqrt{\mu(\overline{F})}}{6}, \quad \delta_2 = \min\left\{\frac{1}{2}, \frac{\delta_1}{\|\overline{F}\|_{C^0}+\delta_1}\right\}.$$

Using (4.50), for all $k \geq 1$, there is j_0 so that

$$\|\widetilde{F}(\cdot,s_{j_0})\circ\varphi_{j_0} - \overline{F}\|_{C^k} < \delta_1, \tag{4.54}$$

and

$$\left|\frac{\mu(\widetilde{F}(\cdot,s_{j_0}))}{\mu(\overline{F})} - 1\right| \leq \delta_2. \tag{4.55}$$

By Theorem 2.3.1, $\overline{F}$ is Lagrangian $\mathscr{F}$-unstable. Then by Theorem 2.3.2, there is a Lagrangian immersion $\widehat{F}:\mathbb{T}\to\mathbb{R}^4$ which satisfies

$$\|\widehat{F}-\overline{F}\|_{C^2}<\delta_1, \tag{4.56}$$

$$\left|\frac{\mu(\overline{F})}{\mu(\widehat{F})}-1\right|\leq\delta_2 \tag{4.57}$$

and

$$\lambda(\widehat{F})<\lambda(\overline{F}). \tag{4.58}$$

Now we define the first part of the piecewise Lagrangian mean curvature flow:

(i) The first piece of Lagrangian mean curvature flow is just $F_t^0:=F_t$, where $t\in[0,t_1]$ and $t_1<T_0$ is such that $s(t_1)=s_{j_0}$.

(ii) Define the first perturbation $F_{t_1}^1$ at time t_1 as

$$F_{t_1}^1=\sqrt{T_0-t_1}(\kappa\widehat{F})\circ\varphi_{j_0}^{-1}+q. \tag{4.59}$$

where the dilation factor

$$\kappa=\sqrt{\frac{\mu(\widetilde{F}(\cdot,s_{j_0})}{\mu(\widehat{F})}}.$$

The constant κ is chosen so that

$$\mu(\kappa\widehat{F})=\mu(\widetilde{F}(\cdot,s_{j_0})). \tag{4.60}$$

We check now that (2)-(4) in definition 4.2.1 are satisfied with $i=0$. First note that (2) follows from (4.60) and the definition of $F_{t_1}^0$ and $F_{t_1}^1$. To prove (3), since the entropy (2.13) is scaling and translation invariant, using $\lambda(\overline{F})>\lambda(\widehat{F})$ we obtain

$$\lambda(F_{t_1}^0)=\lambda(\widetilde{F}(\cdot,s_{j_0})\geq\lambda(\overline{F})>\lambda(\widehat{F})=\lambda(F_{t_1}^1).$$

Thus (3) is also shown. Lastly, we show that (4.47) is satisfied with $i=0$. From (4.49) and (4.59), we have

$$\|F_{t_1}^0-F_{t_1}^1\|_{C^0}=\sqrt{T_0-t_1}\|\widetilde{F}(\cdot,s_{j_0})\circ\varphi_{j_0}-\kappa\widehat{F}\|_{C^0}.$$

Note that (4.55) and (4.57) imply

$$|\kappa - 1| \leq B. \tag{4.61}$$

Together with (4.56), (4.54), the definition of δ_2, we have

$$\begin{aligned}
\|\widetilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \kappa \widehat{F}\|_{C^0} &\leq \|\widetilde{F}(\cdot, s_{j_0}) \circ \varphi_{j_0} - \overline{F}\|_{C^0} + \|\overline{F} - \widehat{F}\|_{C^0} + \|(1-\kappa)\widehat{F}\|_{C^0} \\
&\leq 2\delta_1 + \delta_2(\delta_1 + \|\tilde{F}\|_{C^0}) \\
&\leq 3\delta_1,
\end{aligned}$$

where we used the simple estimate

$$\|\widehat{F}\|_{C^0} \leq \|\widehat{F} - \overline{F}\|_{C^0} + \|\overline{F}\|_{C^0}.$$

Thus we have

$$\begin{aligned}
\|F^0_{t_1} - F^1_{t_1}\|_{C^0} &< 3\delta_1 \sqrt{T_0 - t_1} \\
&= 3\delta_1 \sqrt{\frac{\mu(F^0_{t_1})}{\mu(\widetilde{F}(\cdot, s_{j_0}))}} \\
&\leq \frac{1}{2}\delta \sqrt{\mu(F^0_{t_1})} \sqrt{\frac{\mu(\overline{F})}{\mu(\widetilde{F}(\cdot, s_{j_0}))}} \\
&\leq \delta \sqrt{\mu(F^0_{t_1})},
\end{aligned}$$

where in the last step we used $\delta_2 \leq \frac{1}{2}$. Thus (4.47) is shown and this finishes the construction of the first piece of the piecewise Lagrangian mean curvature flow.

Using $F^1_{t_1}$ as initial condition, there is another family $\{F_t : t \in [t_1, T_1)\}$ of smooth Lagrangian mean curvature flow with $F_{t_1} = F^1_{t_1}$. Again, if the condition in Theorem 1.2.2 is satisfied at the singular time T_1 (that is, the singularity at T_1 is not of type I which can be modelled by a compact self-shrinker of area $\leq \Lambda$), then we set $k = 1$, $t_2 = T_1$ and we are done. If not, we carry out exactly the same procedure as above. Thus we have a Lagrangian self-shrinking torus $\overline{F}_1 \in \mathfrak{X}_\Lambda$, some time $t_2 < T_1$ and another Lagrangian immersion $F^2_{t_2}$ so that

$$\lambda(F^1_{t_2}) \geq \lambda(\overline{F}^1) > \lambda(F^2_{t_2}),$$

$$\mu(F_{t_2}^1) = \mu(F_{t_2}^2)$$

and

$$\|F_{t_2}^1 - F_{t_2}^2\|_{C^0} < \delta\sqrt{\mu(F_{t_2}^1)}.$$

Then, again, we apply the smooth Lagrangian mean curvature flow to $F_{t_2}^2$. Note that the above procedure must stop: Indeed, by Theorem 1.2.1, the image of $\lambda : \mathfrak{X}_\Lambda \to \mathbb{R}$ is finite. Moreover, from the above construction, each perturbation is chosen so that the entropy value is strictly less then one of the element in $\lambda(\mathfrak{X}_\Lambda)$. Since λ is non-increasing along the usual mean curvature flow, the above procedure must terminate after k steps for some $k \le |\lambda(\mathfrak{X}_\Lambda)|$. This implies that at t_k, the piecewise Lagrangian mean curvature flow do not encounter a type I singularity which can be modelled by a compact self-shrinker with area less than or equals to Λ.

To prove the last statement of Theorem 1.2.2, recall that the Maslov class of a Lagrangian immersion is given by $2[H] \in H^1(\mathbb{T},\mathbb{Z})$, where H is the mean curvature form and $[H]$ is an integral class as

$$H = d\theta, \tag{4.62}$$

where $\theta : \mathbb{T} \to \mathbb{S}^1$ is the Lagrangian angle of the immersion $F : \mathbb{T} \to \mathbb{R}^4$ [28]. When $\{F_t\}$ is a smooth Lagrangian mean curvature flow, $[H_t]$ is invariant as $[H_t]$ is an integral class and H_t is smooth in t. This fact can also be checked using the evolution of H under the Lagrangian mean curvature flow, see Theorem 2.9 in [51]. From (4.62) it is also clear that the Maslov class is invariant under translation and scaling of the immersion. Thus when there is a type I singularity and $\overline{F} : \mathbb{T} \to \mathbb{R}^4$ is a compact Lagrangian self-shrinker which models the singularity, then $[H_{\overline{F}}] = [H_t]$. Lastly, we recall that in Theorem 2.3.2 the perturbation $\widehat{F}$ is defined using a closed 1-form on $\mathbb{T}$. Hence we also have $[H_{\overline{F}}] = [H_{F_{t_1}^1}]$. Thus the Maslov class is preserved when we perturb the Lagrangian immersion in constructing the piecewise Lagrangian mean curvature flow. This completes the proof of Theorem 1.2.2. $\square$

4.2.1 Generalization to Lagrangian immersion of higher genus surfaces

Theorem 1.2.2 can be extended to genus $g > 1$ if we impose further assumptions on the singularity. Let $c_1, c_2 > 0$ and consider the set $\mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$ of all Lagrangian self-shrinking immersions $\overline{F} : \Sigma_g \to \mathbb{R}^4$ with area $\leq c_1$ and the second fundamental form satisfying $\max_{\overline{F}(\Sigma_g)} |A| \leq c_2$, where Σ_g is a closed orientable surface of genus g with $g > 1$. Using (2.2), there are constants $C(k, c_1, c_2) > 0$ that depend on c_1, c_2, k, such that

$$\max_{\overline{F}(\Sigma_g)} |\nabla^k A| \leq C(k, c_1, c_2)$$

for all $\overline{F} \in \mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$. Thus we can apply Theorem 1.3 in [6] to conclude that $\mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$ is compact in the C^2-topology, in particular, all sequential limits are unbranched. Unbranchedness of any limiting surface guarantees existence of nearby Lagrangian immersions by the Lagrangian neighbourhood theorem. By Theorem 2.3.1 and Theorem 2.3.2 again, the Lagrangian self-shrinkers in $\mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$ are Lagrangian entropy unstable. It follows that all $\overline{F} \in \mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$ are Lagrangian entropy unstable. With these facts, the proof of the following proposition is identical to that of Corollary 8.4 in [17] and is omitted here.

Proposition 4.2.1. *Let $\delta > 0$. Then there is a positive constant c depending only on δ such that for any Lagrangian self-shrinker $\overline{F} \in \mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$, there is a Lagrangian immersion $\widehat{F} : \Sigma_g \to \mathbb{R}^4$ so that $\|\widehat{F} - \overline{F}\|_{C^0} < \delta\sqrt{\mu(\overline{F})}$ and $\lambda(\widehat{F}) < \lambda(\overline{F}) - c$.*

Remark 3. For genus > 1, without assuming uniform boundedness of the second fundamental forms, the compactness Theorem 1.1.2 is not enough to conclude Proposition 4.2.1 due to the assumption on the conformal structures there.

Using Proposition 4.2.1, we can define a piecewise Lagrangian mean curvature flow for a Lagrangian immersion $F : \Sigma_g \to \mathbb{R}^4$, as we did in the genus 1 case. After each perturbation, the entropy decreases by a fixed amount $c > 0$ (Note that this c might depend on δ). Since the entropy is always is positive number, we conclude that the process must terminate in finite time and we have the following

Theorem 4.2.1. *Let $F : \Sigma_g \to \mathbb{R}^4$ be a Lagrangian immersion and $\delta > 0$ be given. Then there exists a piecewise Lagrangian mean curvature flow $\{F^i_t : i = 0, 1, \cdots, k -$*

$1\}$ *with initial condition F, such that the singularity at time t_k is not a type I singularity which can be modelled by a self-shrinker in $\mathfrak{X}^{\mathrm{imm}}_{g,c_1,c_2}$. Moreover, we have the estimates $\|F^i_{t_i} - F^{i+1}_{t_i}\|_{C^0} < \delta\sqrt{\mu(F^i_{t_i})}$ and the Maslov class of each immersion is invariant along the flow.*

4.3 Appendix

4.3.1 Proof of Lemma 4.1.5

Let Σ be a compact surface without boundary and let $F : \Sigma \to \mathbb{R}^4$ be a branched conformal self-shrinker. Define the operator $\mathscr{L}_s$ by

$$\mathscr{L}_s u = \Delta u - \frac{1}{2t_s}\langle (x - x_s)^\top, \nabla u\rangle = e^{\frac{|x-x_s|^2}{4t_s}} \mathrm{div}(e^{-\frac{|x-x_s|^2}{4t_s}} \nabla u). \tag{4.63}$$

Here $(x_s, t_s) \in \mathbb{R}^4 \times \mathbb{R}_{>0}$, ∇, div and Δ are taken with respect to the pullback metric $F^*\langle\cdot,\cdot\rangle$ and u, v are functions on $\mathbb{R}^4$. Note that $\mathscr{L}_s$ is defined away from the set of branch points B. As in [17], we use the square bracket $[\cdot]_s$ to denote

$$[f]_s = \frac{1}{4\pi t_s}\int_\Sigma f e^{-\frac{|x-x_s|^2}{4t_s}} d\mu \tag{4.64}$$

Lemma 4.3.1. *We have*

$$[u\mathscr{L}_s v]_s = -[\langle \nabla u, \nabla v\rangle]_s. \tag{4.65}$$

Proof. Let $B = \{x_1, \cdots, x_n\}$. Let $\varepsilon > 0$ be small and $B_i(\varepsilon)$ be an ε-ball in Σ with center x_i, so that $B_i(\varepsilon) \cap B_j(\varepsilon) = \emptyset$ if $i \neq j$. Then

$$
\begin{aligned}
[u\mathscr{L}v]_s &= \frac{1}{4\pi t_s}\int_\Sigma u\,\mathrm{div}\left(e^{-\frac{|x-x_s|^2}{4t_s}}\nabla v\right)d\mu \\
&= \lim_{\varepsilon\to 0}\frac{1}{4\pi t_s}\int_{\Sigma\setminus\bigcup B_i(\varepsilon)} u\,\mathrm{div}\left(e^{-\frac{|x-x_s|^2}{4t_s}}\nabla v\right)d\mu \\
&= \lim_{\varepsilon\to 0}\frac{1}{4\pi t_s}\left(\sum_i\int_{\partial B_i(\varepsilon)} u\langle\nabla v, n_i\rangle e^{-\frac{|x-x_s|^2}{4t_s}}dl - \int_{\Sigma\setminus\bigcup B_i(\varepsilon)}\langle\nabla u,\nabla v\rangle e^{-\frac{|x-x_s|^2}{4t_s}}d\mu\right) \\
&= -[\langle\nabla u,\nabla v\rangle]_s
\end{aligned}
\tag{4.66}
$$

where n_i is the unit outward normal along $\partial B_i(\varepsilon)$. □

In particular, we have

$$
[u\mathscr{L}_s v]_s = -[\langle\nabla u,\nabla v\rangle]_s = [v\mathscr{L}_s u]_s. \tag{4.67}
$$

Using (4.67), exactly the same argument in [17], pp. 786-788, shows that for all $y\in\mathbb{R}^4$ and $a\in\mathbb{R}$ if we set $(x_s,t_s)=(sy,1+as^2)$ and $g(s)=\mathscr{F}_{x_s,t_s}(F)$ then $g'(s)\le 0$ for all $s>0$ with $1+as^2>0$. Thus $\mathscr{F}_{y,t}(F)\le\mathscr{F}_{0,1}(F)$ for all $(y,t)\in\mathbb{R}^4\times\mathbb{R}_{>0}$ and thus Lemma 4.1.5 is proved.

4.3.2 Analyticity of $\mathscr{E}$ and $\mathscr{M}$

Next we show that both $\mathscr{E}$ and $\mathscr{M}$ defined in (4.3) and (4.5) are analytic. For the definition of continuous symmetric n-linear form and analytic function between Banach spaces, please refer to Chapter 4 in [21]. First we have

Lemma 4.3.2. *Let X, Y Z be Banach spaces, $\mathscr{U}$, $\mathscr{V}$ are open in X, Y respectively, and $f:\mathscr{U}\to\mathbb{R}$, $g:\mathscr{V}\to Z$ are analytic at $x_0\in\mathscr{U}$, $y_0\in\mathscr{V}$ respectively. Then the function*

$$
h:\mathscr{U}\times\mathscr{V}\to Z,\quad h(x,y)=f(x)g(y)
$$

is analytic at (x_0,y_0).

Proof. Since f, g are analytic at x_0, y_0 respectively, then

$$f(x_0+h) = f(x_0) + \sum_{i=1}^{\infty} A_i(h^i), \quad g(y_0+k) = g(y_0) + \sum_{j=1}^{\infty} B_j(k^j) \tag{4.68}$$

for all $\|h\|_X < \varepsilon_1$ and $\|k\|_Y < \varepsilon_2$, and A_i, B_j are continuous multi-linear forms such that

$$\sum_{i=1}^{\infty} \|A_i\| \varepsilon_1^i < +\infty \text{ and } \sum_{j=1}^{\infty} \|B_j\| \varepsilon_2^j < +\infty. \tag{4.69}$$

The absolute convergence of (4.68) implies that

$$h(x_0+h, y_0+k) = f(x_0)g(y_0) + \sum_{n=1}^{\infty} C_n(h,k), \tag{4.70}$$

for all (h,k) such that $\|h\|_X < \varepsilon_1, \|k\|_Y < \varepsilon_2$, where

$$C_n(h,k) = \sum_{i=0}^{n} A_i(h^i) B_{n-i}(k^{n-i}). \tag{4.71}$$

Let $\varepsilon = \frac{1}{2}\min\{\varepsilon_1, \varepsilon_2\}$. Then by definition of $\|C_n\|$ and ε, one has

$$\begin{aligned} \|C_n\| \varepsilon^n &= \sup_{\|h\|+\|k\|=\varepsilon} \|C_n(h,k)\|_Z \\ &\leq \sum_{i=0}^{n} \left(\|A_i\| \varepsilon_1^i\right) \left(\|B_{n-i}\| \varepsilon_2^{n-i}\right). \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} \|C_n\| \varepsilon^n \leq \left(\sum_{i=1}^{\infty} \|A_i\| \varepsilon_1^i\right) \left(\sum_{j=1}^{\infty} \|B_j\| \varepsilon_2^j\right) < +\infty$$

by (4.69). Hence h is also analytic at (x_0, y_0). □

Proposition 4.3.1. *The mapping $\mathscr{E} : \mathscr{U} \to \mathbb{R}$ in (4.3) is analytic.*

Proof. Using (4.4), we have

$$2\mathscr{E}(u,\tau) = \left(\tau_1^2/\tau_2 + \tau_2\right) L_{11}(u) - (2\tau_1/\tau_2) L_{12}(u) + L_{22}(u), \tag{4.72}$$

where

$$L_{ij}(u) = \int_{\mathbb{T}} D_i u \cdot D_j u e^{-\frac{|u|^2}{4}} dxdy. \tag{4.73}$$

Since $\tau \mapsto (\tau_1^2/\tau_2) + \tau_2$ and $\tau \mapsto \tau_1/\tau_2$ are analytic, by Lemma 4.3.2, it suffices to check $L_{ij} : C^{2,\alpha} \to \mathbb{R}$ is analytic. But this is obvious, using the power series expansion of $e^{-\frac{|u|^2}{4}}$. □

Proposition 4.3.2. *The mapping* $\mathscr{M} : \mathscr{U} \mapsto \mathscr{C}^{0,\alpha}$ *in (4.5) is analytic.*

Proof. It suffices to show that both components in (4.7) are analytic. The second component $(u,\tau) \mapsto \nabla \mathscr{E}_\tau^u$ is analytic since $\mathscr{E}$ is analytic by Proposition 4.3.1, here we recall that $\nabla \mathscr{E}_\tau^u$ is the gradient of $\mathscr{E}(u,\tau)$ at τ. Note that the first component can be written as

$$(u,\tau) \mapsto -g_\tau^{ij} \left(D_{ij}u - (u \cdot D_j u) D_i u + \frac{1}{4}(D_i u \cdot D_j u) u \right) \tag{4.74}$$

Since $\tau \mapsto g_\tau^{ij}$ is analytic, the mapping in (4.74) is also analytic by Lemma 4.3.2. □

Chapter 5

Parabolic Omori-Yau maximum principle for mean curvature flow

In this chapter, we prove a parabolic Omori-Yau maximum principle for mean curvature flow and provide some applications. The main reference is [38].

5.1 Proof of the parabolic Omori-Yau maximum principle

We recall the definition of ℓ-sectional curvature in [35]. Let $\overline{M}^N$ be an N-dimensional Riemannian manifold. Let $p \in \overline{M}$, $1 \leq \ell \leq N-1$. Consider a pair $\{w, V\}$, where $w \in T_p\overline{M}$ and $V \subset T_p\overline{M}$ is a ℓ-dimensional subspace so that w is perpendicular to V.

Definition 5.1.1. The ℓ-sectional curvature of $\{w, V\}$ is given by

$$K^{\ell}_{\overline{M}}(w, V) = \sum_{i=1}^{\ell} \langle R(w, e_i)w, e_i \rangle, \tag{5.1}$$

where R is the Riemann curvature tensor on $\overline{M}$ and $\{e_1, \cdots, e_\ell\}$ is any orthonormal basis of V.

We say that $\overline{M}$ has ℓ-sectional curvature bounded from below by a constant C

if

$$K^{\ell}_{\overline{M}}(w,V) \geq \ell C$$

for all pairs $\{w,V\}$ at any point $p \in M$. In [35], the authors prove the following comparison theorem for the distance function r on manifolds with lower bound on ℓ-sectional curvatures.

Theorem 5.1.1. *[Theorem 1.2 in [35]] Assume that $\overline{M}$ has ℓ-sectional curvature bounded from below by $-C$ for some $C > 0$. Let $p \in M$ and $r(x) = d_{\bar{g}}(x,p)$. If x is not in the cut locus of p and $V \subset T_x\overline{M}$ is perpendicular to $\nabla r(x)$, then*

$$\sum_{i=1}^{\ell} \nabla^2 r(e_i,e_i) \leq \ell\sqrt{C}\coth(\sqrt{C}r), \tag{5.2}$$

where $\{e_1,\cdots,e_\ell\}$ is any orthonormal basis of V.

Now we prove Theorem 1.3.2. We recall that F is assumed to be proper, and u satisfies condition (1)-(3) in the statement of Theorem 1.3.2.

Proof of Theorem 1.3.2. Adding a constant to u if necessary, we assume

$$\sup_{x\in M} u(x,0) = 0.$$

By condition (1) in Theorem 1.3.2, we have $u(y,s) > 0$ for some (y,s). Note that $s > 0$. Let $y_0 \in \overline{M}$ and $r(y) = d_{\bar{g}}(y,y_0)$ be the distance to y_0 in $\overline{M}$. Let $\rho(x,t) = r(F(x,t))$. Note that $u(y,s) - \varepsilon\rho(y,s)^2 > 0$ whenever ε is small. Let $(\bar{x}_i,s_i)$ be a sequence so that $u(\bar{x}_i,s_i) \to \sup u \in (0,\infty]$. Let $\{\varepsilon_i\}$ be a sequence in $(0,\varepsilon)$ converging to 0 which satisfies

$$\varepsilon_i\rho(\bar{x}_i,s_i)^2 \leq \frac{1}{i}, \quad i = 1,2,\cdots. \tag{5.3}$$

Define

$$u_i(x,t) = u(x,t) - \varepsilon_i\rho(x,t)^2.$$

Note that $u_i(y,s) > 0$ and $u_i(\cdot,0) \leq 0$. Using condition (3) in Theorem 1.3.2, there is $R > 0$ so that $u_i(x,t) \leq 0$ when $F(x,t) \notin B_R(y_0)$, the closed ball in $\overline{M}$ centered at y_0 with radius R. Since $\overline{M}$ is complete, $B_R(y_0)$ is a compact subset. Furthermore,

F is proper and thus u_i attains a maximum at some $(x_i,t_i) \in M \times (0,T]$. From the choice of $(\bar{x}_i,s_i)$ and ε_i in (5.3),

$$u(x_i,t_i) \geq u_i(x_i,t_i) \geq u_i(\bar{x}_i,s_i) \geq u(\bar{x}_i,s_i) - \frac{1}{i}.$$

Thus we have

$$u(x_i,t_i) \to \sup u.$$

Now we consider the derivatives of u at (x_i,t_i). If $F(x_i,t_i)$ is not in the cut locus of y_0, then ρ is differentiable at (x_i,t_i). Then so is u_i and we have

$$\nabla^{M_{t_i}} u_i = 0 \quad \text{and} \quad \left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right) u_i \geq 0 \quad \text{at } (x_i,t_i). \tag{5.4}$$

(The inequality holds since $t_i > 0$). The first equality implies

$$\nabla^{M_{t_i}} u = \varepsilon_i \nabla^{M_{t_i}} \rho^2 = 2\varepsilon_i \rho (\nabla r)^\top \tag{5.5}$$

at (x_i,t_i), where $(\cdot)^\top$ denotes the projection onto $T_{x_i}M_{t_i}$. Let $\{e_1,\cdots,e_n\}$ be any orthonormal basis at $T_{x_i}M_{t_i}$ with respect to g_{t_i}. Then

$$\Delta^{M_{t_i}} \rho^2 = 2\sum_{i=1}^n |\nabla_{e_i}^{M_{t_i}} r|^2 + 2\rho \sum_{i=1}^n \nabla^2 r(e_i,e_i) + 2\rho \bar{g}(\nabla r, \vec{H}). \tag{5.6}$$

Next we use the lower bound on $(n-1)$-sectional curvature of $\overline{M}$ to obtain the following lemma.

Lemma 5.1.1. *There is $C_1 = C_1(n,C) > 0$ so that*

$$\sum_{i=1}^n \nabla^2 r(e_i,e_i) \leq C_1. \tag{5.7}$$

Proof of lemma. : We consider two cases. First, if γ' is perpendicular to $T_{x_i}M_{t_i}$, write

$$\sum_{i=1}^n \nabla^2 r(e_i,e_i) = \frac{1}{n-1} \sum_{j=1}^n \sum_{i\neq j} \nabla^2 r(e_i,e_i).$$

Since $\overline{M}$ has $(n-1)$-sectional curvature bounded from below by $-C$, we apply

Theorem 5.1.1 to the plane V spanned by $\{e_1, \cdots, e_n\} \setminus \{e_i\}$ for each i. Thus

$$\begin{aligned}\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) &\le \frac{n}{n-1} \sum_{j=1}^{n-1} \sqrt{C} \coth(\sqrt{C}\rho) \\ &= n\sqrt{C} \coth(\sqrt{C}\rho).\end{aligned} \tag{5.8}$$

Second, if γ' is not perpendicular to $T_{x_i}M_{t_i}$, since the right hand side of (5.7) is independent of the orthonormal basis chosen, we can assume that e_1 is parallel to the projection of γ' onto $T_{x_i}M_{t_i}$. Write

$$e_1 = e_1^{\perp} + a\gamma',$$

where $e_1^{\perp}$ lies in the orthogonal complement of γ' and $a = \langle e_1, \gamma' \rangle$. By a direct calculation,

$$\begin{aligned}\nabla^2 r(e_1, e_1) &= (\nabla_{e_1} \nabla r)(e_1) \\ &= e_1 \langle \gamma', e_1 \rangle - \langle \gamma', \nabla_{e_1} e_1 \rangle \\ &= \langle \nabla_{e_1} \gamma', e_1 \rangle \\ &= \langle \nabla_{e_1^{\perp} + a\gamma'} \gamma', e_1^{\perp} + a\gamma' \rangle \\ &= \langle \nabla_{e_1^{\perp}} \gamma', e_1^{\perp} \rangle + a \langle \nabla_{e_1^{\perp}} \gamma', \gamma' \rangle \\ &= \nabla^2 r(e_1^{\perp}, e_1^{\perp}).\end{aligned} \tag{5.9}$$

We further split into two situations. If $e_1^{\perp} = 0$, then the above shows $\nabla^2 r(e_1, e_1) = 0$. Using Theorem 5.1.1 we conclude

$$\begin{aligned}\sum_{i=1}^{n} \nabla^2 r(e_i, e_i) &= \sum_{i=2}^{n} \nabla^2 r(e_i, e_i) \\ &\le (n-1)\sqrt{C} \coth(\sqrt{C}\rho)\end{aligned} \tag{5.10}$$

If $e_1^{\perp} \neq 0$, write $b = \|e_1^{\perp}\|$ and $f_1 = b^{-1} e_1^{\perp}$. Then $\{f_1, e_2, \cdots, e_n\}$ is an orthonormal basis of an n-dimensional plane in $T_{F(x_i,t_i)}\overline{M}$ orthogonal to γ'. Using (5.9),

$$\begin{aligned}\sum_{i=1}^{n}\nabla^2 r(e_i,e_i) &= \nabla^2 r(e_1^{\perp},e_1^{\perp}) + \sum_{i=2}^{n}\nabla^2 r(e_i,e_i)\\ &= b^2\nabla^2 r(f_1,f_1) + \sum_{i=2}^{n}\nabla^2 r(e_i,e_i)\\ &= b^2\left(\nabla^2 r(f_1,f_1) + \sum_{i=2}^{n}\nabla^2 r(e_i,e_i)\right) + (1-b^2)\sum_{i=2}^{n}\nabla^2 r(e_i,e_i).\end{aligned}$$

Now we apply Theorem 5.1.1 again (note that the first term can be dealt with as in (5.8))

$$\begin{aligned}\sum_{i=1}^{n}\nabla^2 r(e_i,e_i) &\le b^2 n\sqrt{C}\coth(\sqrt{C}\rho) + (1-b^2)(n-1)\sqrt{C}\coth(\sqrt{C}\rho)\\ &\le n\sqrt{C}\coth(\sqrt{C}\rho).\end{aligned} \tag{5.11}$$

Summarizing (5.8), (5.10) and (5.11), we have

$$\sum_{i=1}^{n}\nabla^2 r(e_i,e_i) \le n\sqrt{C}\coth(\sqrt{C}\rho) \le C_1$$

for some $C_1 = C_1(n,C) > 0$. Thus the lemma is proved. $\square$

Using Lemma 5.1.1, (5.6) and $\frac{\partial \rho}{\partial t}^2 = 2\rho\bar{g}(\nabla r,\vec{H})$,

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right)\rho^2 &= -2\sum_{i=1}^{n}|\nabla_{e_i}^{M_{t_i}} r|^2 - 2\rho\sum_{i=1}^{n}\nabla^2 r(e_i,e_i)\\ &\ge -2n - 2C_1\rho\end{aligned} \tag{5.12}$$

(5.5) and (5.12) imply that at (x_i,t_i) we have respectively

$$|\nabla u| \le 2\varepsilon_i\rho \tag{5.13}$$

and

$$\left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right)u \ge -2\varepsilon_i(n + C_1\rho). \tag{5.14}$$

Note

$$u(x_i,t_i) - \varepsilon_i \rho(x_i,t_i)^2 = u_i(x_i,t_i) \geq u_i(y,s) > 0.$$

This implies

$$\rho(x_i,t_i)^2 \leq u(x_i,t_i)\varepsilon_i^{-1}.$$

Using the sub-linear growth condition (3) of u and Young's inequality, we have

$$\begin{aligned}\rho(x_i,t_i)^2 &\leq B\varepsilon_i^{-1} + B\varepsilon_i^{-1}\rho(x_i,t_i)^\alpha \\ &\leq B\varepsilon_i^{-1} + \frac{1}{2}\rho(x_i,t_i)^2 + \frac{1}{2}(B\varepsilon_i^{-1})^{\frac{2}{2-\alpha}}.\end{aligned}$$

Thus we get

$$\rho(x_i,t_i)\varepsilon_i \leq \sqrt{2B}\sqrt{\varepsilon_i} + B^{\frac{1}{2-\alpha}}\varepsilon_i^{\frac{1-\alpha}{2-\alpha}}.$$

Together with (5.13), (5.14) and that $\varepsilon_i \to 0$,

$$|\nabla u|(x_i,t_i) \to 0, \quad \liminf_{i\to\infty}\left(\frac{\partial}{\partial t} - \Delta^{M_{t_i}}\right)u(x_i,t_i) \geq 0.$$

This proves the theorem if ρ is smooth at (x_i,t_i) for all i. When ρ is not differentiable at some (x_i,t_i), one applies the Calabi's trick by considering $r_\varepsilon(y) = d_{\bar{g}}(y,y_\varepsilon)$ instead of r, where y_ε is a point closed to y_0. The method is standard and thus is skipped. □

Remark 4. Condition (1) in the above theorem is used solely to exclude the case that u_i is maximized at $(x_i,0)$ for some $x_i \in M$. The condition can be dropped if that does not happen (see the proof of Theorem 1.3.4).

5.2 Preservation of Gauss image

In this section we assume that $F_0 : M^n \to \mathbb{R}^{n+m}$ is a proper immersion. Let $F : M \times [0,T] \to \mathbb{R}^{n+m}$ be a mean curvature flow starting at F_0. We further assume that the second fundamental form are uniformly bounded: there is $C_0 > 0$ so that

$$\|A(x,t)\| \leq C_0, \quad \text{for all } (x,t) \in M \times [0,T]. \tag{5.15}$$

Lemma 5.2.1. *The mapping F is proper.*

Proof. Let $B_0(r)$ be the closed ball in $\mathbb{R}^{n+m}$ centered at the origin with radius r. Then by (2.1) and (5.15) we have

$$\begin{aligned}|F(x,t)-F(x,0)| &= \left|\int_0^t \frac{\partial F}{\partial s}(x,s)ds\right| \\ &= \left|\int_0^t \vec{H}(x,s)ds\right| \\ &\le \sqrt{n}\int_0^t \|A(x,s)\|ds \\ &\le C_0\sqrt{n}T.\end{aligned}$$

Thus if $(x,t) \in F^{-1}(B_0(r))$, then x is in $F_0^{-1}(B_0(r+C_0\sqrt{n}T))$. Let $(x_n,t_n) \in F^{-1}(B_0(r))$. Since F_0 is proper, a subsequence of $\{x_n\}$ converges to $x \in M$. Since $[0,T]$ is compact, a subsequence of (x_n,t_n) converges to (x,t), which must be in $F^{-1}(B_0(r))$ since F is continuous. As $r>0$ is arbitrary, F is proper. □

In particular, the parabolic Omori-Yau maximum principle (Theorem 1.3.2) can be applied in this case.

Let $G(n,m)$ be the real Grassmanians of n-planes in $\mathbb{R}^{n+m}$ and let

$$\gamma : M\times[0,T]\to G(n,m), \quad x\mapsto F_*T_xM \tag{5.16}$$

be the Gauss map of F.

Now we prove Theorem 1.3.3, which is a generalization of a Theorem of Wang [53] to the noncompact situation with bounded second fundamental form.

Proof of Theorem 1.3.3. Let $d : G(n,m)\to\mathbb{R}$ be the distance to Σ. That is $d(\ell) = \inf_{L\in\Sigma} d(L,\ell)$. Since $\gamma(\cdot,0)\subset\Sigma$, we have $d\circ\gamma = 0$ when $t=0$. Using chain rule and (5.15), as $d\gamma = A$,

$$d(\gamma(x,t)) = d(\gamma(x,t)) - d(\gamma(x,0)) = \int_0^t \nabla d\circ d\gamma(x,s)ds \le tC_0.$$

Since $\Sigma\subset G(n,m)$ is compact, there is $\varepsilon_0>0$ so that the open set

$$V = \{\ell\in G(n,m) : d(\ell,\Sigma) < \sqrt{\varepsilon_0}\}$$

lies in a small tubular neighborhood of Σ and the function d^2 is smooth on this neighborhood. Let $T' = \varepsilon_0/2C_0$. Then the image of $f := d^2 \circ \gamma$ lies in this tubular neighborhood if $t \in [0, T']$ and f is a smooth function on $M \times [0, T']$.

The calculation in [53] shows that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq C|A_t|^2 f, \tag{5.17}$$

where $C > 0$ depends on ε_0 and Σ. Together with (5.15) this shows that

$$\left(\frac{\partial}{\partial t} - \Delta\right) f \leq C_1 f$$

for some positive constant C_1.

Let $g = e^{-(C_1+1)t} f$. Then g is bounded, nonnegative and $g(\cdot, 0) \equiv 0$. On the other hand,

$$\left(\frac{\partial}{\partial t} - \Delta\right) g = -(C_1+1)g + e^{-(C_1+1)t}\left(\frac{\partial}{\partial t} - \Delta\right) f \leq -g. \tag{5.18}$$

If g is positive at some point, Theorem 1.3.2 implies the existence of a sequence (x_i, t_i) so that

$$g(x_i, t_i) \to \sup g, \quad \limsup_{i\to\infty}\left(\frac{\partial}{\partial t} - \Delta\right) g(x_i, t_i) \geq 0.$$

Take $i \to \infty$ in (5.18) gives $0 \leq -\sup g$, which contradicts that g is positive somewhere. Thus g and so f is identically zero. This is the same as saying that $\gamma(x,t) \in \Sigma$ for all $(x,t) \in [0, T']$. Note that T' depends only on C_0, so we can repeat the same argument finitely many time to conclude that $\gamma(x,t) \in \Sigma$ for all $(x,t) \in M \times [0, T]$. □

Proof of Corollary 1.3.1. An immersion is Lagrangian if and only if its Gauss map has image in the Lagrangian Grassmanians $LG(n)$, which is a totally geodesic submanifold of $G(n,n)$. The Corollary follows immediately from Theorem 1.3.3. □

Remark 5. Various forms of Corollary 1.3.1 are known to the experts. In [40], the author comments that the argument used in [51] can be generalized to the complete noncompact case, if one assumes the following volume growth condition:

$$\mathrm{Vol}(L_0 \cap B_R(0)) \leq C_0 R^n, \quad \text{for some } C_0 > 0.$$

The above condition is needed to apply the non-compact maximum principle in [22].

5.3 Omori-Yau maximum principle for self-shrinkers

In this section, we improve Theorem 5 in [8] by using Theorem 1.3.2. The proof is more intuitive in the sense that we use essentially the fact that a self-shrinker is a self-similar solution to the mean curvature flow (possibly after reparametrization on each time slice).

First we recall some facts about self-shrinker. A self-shrinker to the mean curvature flow is an immersion $\widetilde{F} : M^n \to \mathbb{R}^{n+m}$ which satisfies

$$\widetilde{F}^{\perp} = -\frac{1}{2}\vec{H}. \tag{5.19}$$

Fix $T_0 \in (-1, 0)$. Let $\phi_t : M \to M$ be a family of diffeomorphisms on M so that

$$\phi_{T_0} = \mathrm{Id}_M, \ \ \frac{\partial}{\partial t}\big(\widetilde{F}(\phi_t(x))\big) = \frac{1}{2(-t)}\widetilde{F}^{\top}(\phi_t(x)), \quad \forall t \in [-1, T_0]. \tag{5.20}$$

Let

$$F(x,t) = \sqrt{-t}\widetilde{F}(\phi_t(x)), \quad (x,t) \in M \times [-1, T_0]. \tag{5.21}$$

Then F satisfies the MCF equation since by (5.19),

$$\begin{aligned}
\frac{\partial F}{\partial t}(x,t) &= \frac{\partial}{\partial t}\big(\sqrt{-t}\widetilde{F}(\phi_t(x))\big) \\
&= -\frac{1}{2\sqrt{-t}}\widetilde{F}(\phi_t(x)) + \sqrt{-t}\frac{\partial}{\partial t}\big(\widetilde{F}(\phi_t(x))\big) \\
&= -\frac{1}{2\sqrt{-t}}\widetilde{F}(\phi_t(x)) + \frac{1}{2\sqrt{-t}}\widetilde{F}^{\top}(\phi_t(x)) \\
&= \frac{1}{\sqrt{-t}}\vec{H}_{\widetilde{F}}(\phi_t(x)) \\
&= \vec{H}_F(x,t).
\end{aligned}$$

Lastly, recall the $\mathscr{L}$ operator defined in [17]:

$$\mathscr{L} f = \Delta f - \frac{1}{2}\langle \nabla f, \widetilde{F}^{\top} \rangle. \tag{5.22}$$

We are now ready to prove Theorem 1.3.4:

Proof of Theorem 1.3.4. Recall $T_0 \in (-1,0)$. Let $u : M \times [-1,T_0] \to \mathbb{R}$ be given by

$$u(x,t) = f(\phi_t(x)), \quad \forall (x,t) \in M \times [-1,T_0]. \tag{5.23}$$

Then

$$u(x,t) \le C(1 + |\widetilde{F}(\phi_t(x)|^{\alpha}) \le C(-T_0)^{-\alpha/2}|F(x,t)|^{\alpha}.$$

Thus we can apply Theorem 1.3.2 (The condition that $u(\cdot,0) \equiv 0$ in Theorem 1.3.2 is used only to exclude the case $t_i = -1$. But since

$$u_i(x,t) = f(\phi_t(x)) - \varepsilon_i|\sqrt{-t}\widetilde{F}(\phi_t(x))|^2,$$

in order that u_i is maximized at (x_i,t_i) we must have $t_i = T_0$. In particular $t_i \neq -1$). Thus there is a sequence (x_i, T_0) so that

$$u(x_i,T_0) \to \sup u, \quad |\nabla^{M_{T_0}} u(x_i,T_0)| \to 0, \quad \liminf_{i\to\infty} \left(\frac{\partial}{\partial t} - \Delta^{M_{T_0}}\right) u(x_i,T_0) \ge 0.$$

Using $\phi_{T_0} = \mathrm{Id}$ and the definition of u, the first condition gives

$$f(x_i) \to \sup f. \tag{5.24}$$

Since $\nabla^{M_{T_0}} = \frac{1}{\sqrt{-T_0}}\nabla^M$, the second condition gives

$$|\nabla^M f(x_i)| \to 0. \tag{5.25}$$

Lastly,

$$\frac{\partial u}{\partial t}(x_i,T_0) = \frac{\partial f}{\partial t}(\phi_t(x))\Big|_{t=T_0} = \frac{1}{2(-T_0)}\langle \nabla f(x_i), \widetilde{F}^{\top}(x_i) \rangle \tag{5.26}$$

and

$$\Delta^{M_{T_0}} u(x_i, T_0) = \Delta^{M_{T_0}} f(x_i) = \frac{1}{-T_0} \Delta^M f(x_i).$$

Thus

$$\left(\frac{\partial}{\partial t} - \Delta^{M_{T_0}}\right) u(x_i, T_0) = \frac{1}{T_0} \mathscr{L} f(x_i)$$

and the result follows. □

Remark 6. Note that the above theorem is stronger than Theorem 5 in [8], where they assume that f is bounded above (which corresponds to our case when $\alpha = 0$).

Remark 7. Our growth condition on f is optimal: the function $f(x) = \sqrt{|x|^2 + 1}$ defined on $\mathbb{R}^n$ (as a self-shrinker) has linear growth, but the gradient of f

$$\nabla f = \frac{x}{\sqrt{|x|^2 + 1}}$$

does not tend to 0 as $f(x) \to \sup f = \infty$.

Remark 8. In Theorem 4 of [8], the authors also derive an Omori-Yau maximum principle on a properly immersed self-shrinker for the Laplace operator. There they assume $u : M \to \mathbb{R}$ satisfies the growth condition

$$\lim_{x \to \infty} \frac{u(x)}{\log\left(\sqrt{|\widetilde{F}(x)|^2 + 4} - 1\right)} = 0.$$

We remark that the condition can be weaken to

$$\lim_{x \to \infty} \frac{u(x)}{|\widetilde{F}(x)| + 1} = 0,$$

since the Laplacian of the function $|\widetilde{F}|^2$ satisfies better estimates: $\Delta |\widetilde{F}|^2 \leq 2n$. Thus one can argue as in p.79 in [2] to conclude Theorem 4 in [8].

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