Long-time dynamics for the energy-critical Harmonic Map Heat Flow and Nonlinear Heat Equation

by

Dimitrios Roxanas

A thesis submitted in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

in

The Faculty of Graduate and Postdoctoral Studies
(Mathematics)

The University of British Columbia
(Vancouver)

May 2017

Abstract

The main focus of this thesis is on *critical* parabolic problems, in particular, the harmonic map heat flow into the 2-sphere

$$\vec{u}_t = \Delta\vec{u} + |\nabla\vec{u}|^2\vec{u},\ \vec{u} : \mathbb{R}^2 \to \mathbb{S}^2,$$

and the focusing nonlinear heat equation

$$u_t = \Delta u + |u|^{\frac{4}{d-2}}u,\ u : \mathbb{R}^d \to \mathbb{R},\ d \geq 3.$$

The focus of this work has been on long-time dynamics, stability and singularity formation, and the investigation of the role of special, soliton-like solutions, to the asymptotic behaviour of solutions.

Harmonic Map Heat Flow: We consider m-corotational solutions to the harmonic map heat flow from $\mathbb{R}^2$ to $\mathbb{S}^2$. We first work in a class of maps with trivial topology and energy of the initial data below two times the energy of the stationary harmonic map solutions. We give a new proof of global existence and decay. The proof is based on the "concentration-compactness plus rigidity" approach of Kenig and Merle and relies on the dissipation of energy and a new profile decomposition.

We also treat m-corotational maps ($m \geq 4$) with non-trivial topology and energy of the initial data less than three times the energy of the stationary harmonic map solutions. Through a new stability argument we rule out finite-time blow-up and show that the global solution asymptotically converges to a harmonic map.

Nonlinear Heat Equation: We also study solutions of the focusing energy-critical nonlinear heat equation $u_t - \Delta u - |u|^2 u = 0$ in $\mathbb{R}^4$. We show that solutions emanating from initial data with energy and $\dot{H}^1-$norm below those of the stationary solution W are global and decay to zero. We first show that global solutions dissipate to zero. The proof is based on a refined small data theory, L^2-dissipation and an approximation argument. We then follow the "concentration-compactness plus rigidity" roadmap of Kenig and Merle (and in particular the approach taken by Kenig and Koch for Navier-Stokes) to exclude finite-time blow-up. Our proof extends to all dimensions $d \geq 3$.

Lay Summary

This thesis focuses on the study of some evolution Partial Differential Equations. These are equations that govern the evolution in-time of quantities appearing in the modeling of phenomena in Science and Engineering, for example density or size of population. In particular, we study two nonlinear Heat Equations, the Harmonic Map Heat flow and a Heat Equation with a polynomial nonlinear term. In general, when working with an equation that models a real-life phenomenon, that a solution ceases to exist in finite-time signifies a possible flaw in the modeling step, and consequently the potential unsuitability of this equation, in its current form, as a good approximation of the natural phenomenon observed. In this work, we give criteria under which the solutions to the above equations, not only exist for all times, but also enjoy specific desirable properties after, possibly, long time.

Preface

Much of the original material in the following document is adapted from two of the author's research preprints: [74] and [75] (the research is joint work with his thesis supervisor, Dr Stephen Gustafson). In particular, all of Chapter 2, which evolves around the proof of Theorem 2.2, along with section 3.2, where the proof of Theorem 3.4 is presented, form the main content of [74], "*Global regularity and asymptotic convergence for the higher-degree* 2d *corotational harmonic map heat flow to* $\mathbb{S}^2$". Chapter 4 is adapted from [75], "*Global, decaying solutions below the ground state for a critical heat equation*". The manuscripts have been written and will be submitted for publication soon.

Table of Contents

Acknowledgements

Primarily, I would like to thank my supervisor Stephen Gustafson. No part of this thesis would have been possible without his insight, feedback, and at times, critical intervention. I am very grateful for his patience, time and advice, his shaping of my mathematical thinking, and also his support towards my career goals.

I want to thank the members of my supervisory committee, Tai-Peng Tsai, Young-Heon Kim for their valuable feedback, support, and of course, their time. I also want to take this opportunity to thank Ailana Fraser, Malabika Pramanik and Yaniv Plan for their support, and the great courses I took with them; they really made a difference in my mathematical understanding. Moreover, I would like to thank my instructors-in-charge Dan Coombs, Andrew Rechnitzer, Greg Martin and Fok-Shuen Leung, for all the experience and teaching advice they shared with me, as well as for the time and effort they spent on organizing and troubleshooting, which resulted in my having more time for my research.

I would also like to acknowledge the abundant support I received from St John's Graduate College, my moving there was definitely one of the best decisions I have ever made. In particular, I want to thank my many friends, especially Abhi, Derek, Edmundo, Felipe, Lena and Sheng for their friendship and encouragement, and also the Principals for their work and their support in difficult times. I am also pleased to acknowledge the financial support from "St John's College Reginald and Annie Van Fellowship 2016".

I wish to thank my parents, brothers and sister, and my extended family for their love and understanding, and all of my friends for their ample support all these years. A special mention goes to Dimitris and Pam, Maxime, Myrto, Samuli, Spyros and Shaya.

Last but definitely not least, I would like to thank Laura. I am incredibly grateful for the good times, the abundant support and all the encouragement, especially when facing very stressful situations and hard decisions. Thank you for everything!

Chapter 1

Introduction

1.1 General introduction

Nonlinear evolution equations (partial differential equations with a time variable) arise throughout the sciences, describing the dynamics of various systems. Classical examples include nonlinear heat, wave and Schrödinger equations. Evolution equations have also been used as an important tool in geometry; for example, the application of the Ricci flow to the Poincaré conjecture, and generally the use of heat flows applied to prove results in global differential geometry.

The main focus of this thesis is on *critical* parabolic problems, in particular, the harmonic map heat flow into the 2-sphere

$$\begin{cases} \vec{u}_t = \Delta \vec{u} + |\nabla \vec{u}|^2 \vec{u}, \;\; \vec{u} : [0,T) \times \mathbb{R}^2 \to \mathbb{S}^2, \\ \vec{u}(0,x) = \vec{u}_0(x) \end{cases}$$

and the focusing nonlinear heat equation

$$\begin{cases} u_t = \Delta u + |u|^{\frac{4}{d-2}} u, \;\; d \geq 3, \;\; u : [0,T) \times \mathbb{R}^d \to \mathbb{R}, \\ u(0,x) = u_0(x) \end{cases}$$

In order to place this work into a broader context, we first take a more general look at the study of nonlinear evolution equations, many of which originate in physics. From a mathematical perspective their study centers around the following fundamental questions in connection with the Cauchy (initial data) problem: for given initial data, does there exist a local-in-time solution? Does this solution exist for all time? If it ceases to exist after finite time (blowup), why? What is the asymptotic behavior of a global-in-time solution? Does it behave like a free linear solution in the long run? Two basic features of a nonlinear heat equation play an important role when one tries to answer these questions: the scaling invariance of the equation and the presence of a monotone quantity, which we will call "the energy". The behaviour of the energy with respect to the scaling of the equation suggests a classification of the equation as being either energy sub-critical, energy critical, or energy super-critical.

To both equations there is an associated energy functional of the general form $E(u) = \displaystyle\int |\nabla u|^2 + \int F(u)$, which is dissipated by solutions, and a scaling transformation of the form $u_\lambda(t, x) = \lambda^\beta u(\lambda^2 t, \lambda x)$, some β, which leaves each equation and its associated energy invariant. This invariance suggests that the dissipation of the Laplacian, Δu, is balanced by the corresponding nonlinearity. In cases where the dissipation dominates, the solution should be global and decaying, behaving like the linear heat equation $u_t = \Delta u$. If however the nonlinearity is stronger, the solution may exhibit nonlinear behaviour, such as soliton formation or blow-up by concentration. As a result, the questions of global existence and decay of solutions become very delicate and require special technology to answer. We aim to identify the threshold below which the dissipation is dominating, resulting in global solutions that eventually relax to equilibria, with the goal of writing criteria on the initial data which will guarantee either existence for all times and decay, or singularity formation. In both the cases of the Harmonic Map Heat Flow and the one of the Nonlinear Heat Equation, the threshold is defined in terms of the corresponding stationary solutions. Stationary solutions are global-in-time, but have no time decay, and they have been the objects on which blowing-up solutions have been built, e.g., ([114, 115, 118]).

For both problems we work with functions whose (a priori) regularity matches that of the dissipative energy, i.e., Sobolev spaces of one derivative, which we will call the "energy space". For several of the results obtained, the approach is largely motivated by the recent developments in critical dispersive equations and in particular, the "concentration-compactness plus rigidity" roadmap of Kenig and Merle [79]. Unlike the nonlinear Schrödinger equation for example, heat equations don't enjoy conservation of L^2 ("mass"). This implies the need for a modified approach to the dispersive technology (as in [78]). For example, the irreversibility of the heat equation together with the lack of mass conservation, require a more involved approach in the "rigidity" argument in Chapter 4. On the other hand, we gain a smoothing effect and the monotonicity of the energy. To use it, especially for decay questions, we need to rule out the only obstruction to strict decrease of energy, namely the stationary solutions.

All the results described below are on *critical*, in a scaling sense, problems. When working on such questions, new difficulties (compared to subcritical problems) are encountered for both the local and the global theory. This is mostly because we are now more or less forced to work exclusively with scale-invariant norms, which limits the tools available. For example, the time of existence given by the local theory will depend on the profile of the data - not just on its norm. Because of this, even a priori bounds are not sufficient by themselves to upgrade local to global wellposedness: bounded energy norm does not exclude the possibility that the solution could concentrate at a point in finite time causing the lifespan of the local theory to shrink to zero as we try to iterate the local result to extend the solution. In our framework, the energy norm does stay bounded, and blow-up can only happen because of concentration. For global wellposedness and decay our task becomes to find a good way of measuring concentration and show it cannot occur, at least below some natural threshold.

Before we go more into the details of this work, let us briefly mention some other challenging aspects of these two problems, beyond their critical nature. For the *harmonic map heat flow*, the one challenge is the geometry and topology of the target. We are concerned with the most physically relevant case of maps with values in $\mathbb{S}^2$. It is known [47] that for manifolds with negative scalar curvature, there is no obstruction to all solutions being globally smooth; while for $\mathbb{S}^2$ and other targets with positive or sign-changing curvature the existence of non-trivial static solutions -harmonic maps- presents a possible obstruction [122]. From now on we will restrict this discussion to the case of $\mathbb{S}^2$. We will be working within the class of *corotational* solutions (see Chapter 2 for details). Within this class, one is led to make a choice of boundary conditions; these correspond to assignments for $\vec{u}(0), \vec{u}(\infty)$. Different choices of these conditions result in dramatically different results. Roughly speaking, there are combinations of topologies and energies that prohibit the existence of obstructions to global regularity and decay, while others make it possible for singularities to occur. A second factor, whose importance is discussed in Chapters 2 and 3, is the degree $\mathbb{Z}^* \ni m := \frac{1}{4\pi}\int_{\mathbb{R}^2} (\partial_1\vec{u} \wedge \partial_2\vec{u}) \cdot \vec{u}$ which measures the number of times the map "wraps" around the unit-sphere. We will see that the higher $|m|$ is, the more favourable decay we can expect, in a sense to be quantified in later chapters.

For the case of the *nonlinear heat equation*, the additional challenges stem from the focusing nature of the nonlinearity. The energy functional in this case, say, for $d = 4$, is given by $E(u) = \displaystyle\int_{\mathbb{R}^4} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{4}|u|^4\right) dx$. We refer to the term $\displaystyle\frac{1}{2}\int_{\mathbb{R}^4} |\nabla u|^2 dx$ as the *kinetic energy* and to the term $\displaystyle\frac{1}{4}\int_{\mathbb{R}^4} |u|^4 dx$ as the *potential energy*. Notice that the potential energy is negative which reflects the focusing nature of the nonlinearity. The energy does not provide any a priori control on the critical norm, $\|u\|_{\dot{H}^1}$, which complicates matters significantly.

1.1.1 Landau-Lifshitz and Harmonic Map Heat Flow

Just as the harmonic map equation is a geometric analogue of the classsical Laplace equation for harmonic functions, so the classical linear evolution PDEs, the Heat, the Wave and Schrödinger equations, have geometric "map" analogues: the Harmonic Map Heat Flow, Wave Map and Schrödinger Map equations. These equations are nonlinear when the target space geometry is nontrivial. Quite remarkably, these equations are all of physical (as well as mathematical) interest, at least when the target space is the 2-sphere, arising in the study of ferromagnets (and antiferromagnets), liquid crystals, and general relativity. Part of this dissertation is devoted to our results for map evolution equations, focusing on the harmonic map heat flow, a special case of the Landau-Lifshitz family of equations, which also includes the Schrödinger map equation as a special case; our aim is to address the basic global questions: singularity formation versus global regularity, and long-time asymptotics.

We will begin by giving a brief history, focusing mostly on recent developments. Let us begin with the harmonic map equation. From the outset, we fix a specific choice of domain and target manifold for our maps:

$$\mathbf{u} : \mathbb{R}^n \to \mathbb{S}^2,$$

mostly $n = 2$. We realize $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$, $\mathbb{S}^2 := \{\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3 : |\mathbf{u}| = 1\} \subset \mathbb{R}^3$ (*Notation*: vectors will be bold-faced throughout.) Harmonic maps are critical points of the Dirichlet energy functional

$$E(\mathbf{u}) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2 dx = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{j=1}^{n} \sum_{k=1}^{3} \left(\frac{\partial u_k}{\partial x_j}\right)^2 dx$$

and so (if smooth enough) solve the corresponding Euler-Lagrange equation

$$0 = -\mathcal{E}'(\mathbf{u}) = P^u \Delta \mathbf{u} = \Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u}, \tag{1.1}$$

where $P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 : \boldsymbol{\xi} \cdot \mathbf{u} = 0\}$$

to $\mathbb{S}^2$ at $\mathbf{u}$. Equation 1.1 is the equation for harmonic maps between $\mathbb{R}^n$ and $\mathbb{S}^2$. It generalizes Laplace's equation to maps.

Map evolution equations.

Now we let our maps vary with time as well, so that for each time $t \geq 0$,

$$u(\cdot, t) : \mathbb{R}^n \to \mathbb{S}^2,$$

or equivalently $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ with the pointwise constraint $|\mathbf{u}(x,t)| \equiv 1$.

Harmonic map heat-flow: Harmonic maps (between general Riemannian manifolds) have for many years been of interest to differential geometers, and in order to study them [47] introduced the gradient-flow equations for the energy $\mathcal{E}$, the *harmonic map heat flow* equations $\frac{\partial u}{\partial t} = -\mathcal{E}'(u)$, which in our setting reads

$$\frac{\partial \mathbf{u}}{\partial t} = \Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u} \tag{1.2}$$

The harmonic map heat flow generalizes the linear heat equation to maps.

Landau-Lifshitz equations: Physically, equation (1.2) is the special case $b = 0$ of the Landau-Lifshitz (sometimes Landau-Lifshitz-Gilbert) equations modeling the dynamics of the magnetization $\mathbf{u}$ in an isotropic ferromagnet ([89, 85]):

$$\frac{\partial \mathbf{u}}{\partial t} = aP^{\mathbf{u}}\Delta \mathbf{u} + b\mathbf{u} \times P^{\mathbf{u}}\Delta \mathbf{u} = a\Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u} + b\mathbf{u} \times \Delta \mathbf{u}, \tag{1.3}$$

with $a \geq 0, b \in \mathbb{R}$, and where $\times$ denotes the usual cross-product in $\mathbb{R}^3$. In fact, equation (1.3) itself is a special case of a more general equation incorporating additional physical effects such as anisotropy, and demagnetization.

Schrödinger maps: The opposite limiting case ($a = 0$, i.e., no dissipation) of (1.3) can be written as

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \Delta \mathbf{u} = -J^{\mathbf{u}}\mathcal{E}'(\mathbf{u}) \tag{1.4}$$

where the operator

$$J^{\mathbf{u}} := \mathbf{u} \times \cdot : T_{\mathbf{u}}\mathbb{S}^2 \to T_{\mathbf{u}}\mathbb{S}^2$$

gives a rotation through $\pi/2$ on the tangent plane $T_{\mathbf{u}}\mathbb{S}^2$, and so endows $\mathbb{S}^2$ with a complex structure. Thus equation (1.4) can immediately be written for general maps from Riemannian manifolds into Kähler manifolds ([20, 48]). Since it generalizes the linear Schrödinger equation to maps, (1.4) is known as the Schrödinger map (sometimes Schrödinger flow) equation.

Wave maps: Finally, the wave map equation is $P^{\mathbf{u}}(\frac{\partial^2 \mathbf{u}}{\partial t^2} - \Delta \mathbf{u}) = 0$, which in our setting is

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right)\mathbf{u} + \left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|^2 - |\nabla \mathbf{u}|^2\right)\mathbf{u} = 0 \tag{1.5}$$

generalizes the linear wave equation to maps. The Wave Map problem is one of the most interesting and challenging nonlinear hyperbolic problems. It has a natural formulation as the Euler-Lagrange system for a map between manifolds. The particular case of three dimensions is of special interest in high energy physics and sometimes called by physicists "the nonlinear sigma model". The nuclei of atoms are held together by forces mediated by the pi mesons. These are a set of three particles whose masses are small compared to the nuclei themselves, so to a first approximation they can be considered to be massless, i.e., travelling at the speed of light. If interactions among them are ignored, the pi mesons are described by a field satisfying the wave equation. Interactions would add nonlinearities. A remarkable fact of physics is that the interactions among the pi mesons are described, to a good approximation ([99, 68, 69, 94]), by considering the target manifold to be the sphere $\mathbb{S}^3$ and replacing the wave equation by the corresponding wave map equation. But its main interest, aside from the inherent mathematical one, is in general relativity, where it is studied as a (comparatively simple) model for understanding singularity formation (for some background, see, e.g., [119]). There have been several attempts to modify this model to remove the possibility of finite-time blow up and to retain topological solitons, the most famous modification is due to Skyrme. For an account of recent works we refer to [31, 55, 90, 96].

The energy landscape and equivariant symmetry.

The energy $\mathcal{E}(\mathbf{u})$ plays a central role in all of our analysis. We begin by observing that the energy behaves well under the various dynamics introduced above.

Energy identity. Formally taking the dot product of (1.3) with

$$\mathcal{E}'(\mathbf{u}) = -\Delta\mathbf{u} - |\nabla\mathbf{u}|^2\mathbf{u} \in T_{\mathbf{u}}\mathbb{S}^2$$

and integrating in space and time yields the basic energy identity

$$\mathcal{E}(\mathbf{u}(t)) + a\int_0^t \int_{\mathbb{R}^n} |\Delta\mathbf{u} + |\nabla\mathbf{u}|^2\,\mathbf{u}|^2 dxdt = \mathcal{E}(\mathbf{u}(0)). \tag{1.6}$$

For (1.4) ($a = 0$) this means energy conservation, while for $a > 0$ (including the (1.2) case ($b = 0$)), the energy is nonincreasing. A conserved Hamiltonian functional for (1.5) is obtained by adding $\frac{1}{2}\int_{\mathbb{R}^n}\left|\frac{\partial\mathbf{u}}{\partial t}\right|^2 dx$ to $\mathcal{E}(\mathbf{u})$.

Two space dimensions is energy critical. The energy scales the following way

$$\mathcal{E}(\mathbf{u}(\cdot)) = \lambda^{2-n}\mathcal{E}(\mathbf{u}(\frac{\cdot}{\lambda}))$$

for $\lambda > 0$, which makes the space dimension $n = 2$ "energy critical". This has important consequences (see below) and in particular leads to the intuition that $n = 2$ should be a borderline case for the formation of singularities for our map dynamics. So $n = 2$ turns out to be particularly interesting mathematically (and $n = 2$ and $n = 3$ are physically the most interesting space dimensions). For these reasons, we specialize to $n = 2$ from now on.

Equivariant symmetry. Since the analysis of our flow is a challenging problem, a good starting point is to impose some symmetry. For now we will restrict our attention to *m-equivariant* maps $\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3$, of the form

$$\mathbf{u}(r,\theta) = e^{m\theta R}\mathbf{v}(r)$$

where (r, θ) are polar coordinates on $\mathbb{R}^2, \mathbf{v} : [0, \infty) \to \mathbb{S}^2,\ R$ is the matrix generating rotations around the u_3-axis and $m \in \mathbb{Z}$. We will also consider a subset of m-equivariant maps: the set of *m-corotational* maps; these are maps of the form

$$\mathbf{u}(r, \theta) = (\cos(m\theta)\sin(u(r, t)), \sin(m\theta)\sin(u(r, t)), \cos(u(r, t))),$$

for $u : [0, \infty] \to \mathbb{R}$. Then, if $\mathbf{u}(r, t)$ solves (1.2), $u(r, t)$ satisfies the equation

$$u_t = u_{rr} + \frac{1}{r}u_r - m^2\frac{\sin 2u}{2r^2}. \tag{1.7}$$

This subclass is preserved by (1.2), as well as by the Wave Map, and is much used in the corresponding literature, since the map equations reduce to a scalar PDE for $u(r, t)$. This subclass is notably not preserved by (1.3) or (1.4), just as the wave and heat equations preserve real functions, while the Schrödinger equation (or a heat-Schrödinger mix) does not. We will work in the m-equivariant class for most of what follows in this introduction.

Topological lower bound on the energy. There is a well-known energy lower bound

$$\mathcal{E}(\mathbf{u}) \geq 4\pi|deg(\mathbf{u})|$$

where $\mathbb{Z}^* \ni m = deg(\mathbf{u}) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^2}(\partial_1\mathbf{u} \wedge \partial_2\mathbf{u}) \cdot \mathbf{u}$ is the degree of the map $\mathbf{u}$, considered (compactifying the domain $\mathbb{R}^2$ via stereographic projection) as a map from $\mathbb{S}^2$ to itself. This bound is particularly easy to understand when $\mathbf{u}$ is an m-equivariant map, so that

$$\mathcal{E}(\mathbf{u}) = \pi\int_0^\infty \left(\left|\frac{\partial\mathbf{v}}{\partial r}\right|^2 + \frac{m^2}{r^2}(v_1^2 + v_2^2)\right) rdr.$$

If $\mathcal{E}(\mathbf{u}) < \infty$, then $\mathbf{v}$ is continuous, and the limits $\lim\limits_{r\to 0}\mathbf{v}(r)$ and $\lim\limits_{r\to\infty}\mathbf{v}(r)$ exist (see [70]), and so we must have $\mathbf{v}(0), \mathbf{v}(\infty) = \pm\hat{k}$, where $\hat{k} = (0, 0, 1)$. Without loss of generality we fix $\mathbf{v}(0) = -\hat{k}$. The two cases $\mathbf{v}(\infty) = \pm\hat{k}$ then correspond to different topological classes of maps. We denote by Σ_m the class of m-equivariant maps with

$\mathbf{v}(\infty) = \hat{k}$:

$$\Sigma_m = \{\mathbf{u} : \mathbb{R}^2 \to \mathbb{S}^2 | \mathbf{u} = e^{m\theta R}\mathbf{v}(r), \mathcal{E}(\mathbf{u}) < \infty, \mathbf{v}(0) = -\hat{k}, \mathbf{v}(\infty) = \hat{k}\}.$$

For $\mathbf{u} \in \Sigma_m$, the energy $\mathcal{E}(\mathbf{u})$ can be rewritten by "completing the square":

$$\mathcal{E}(\mathbf{u}) = \pi \int_0^\infty \left(\left| \frac{\partial \mathbf{v}}{\partial r} \right|^2 + \frac{m^2}{r^2} |J^{\mathbf{v}} R\mathbf{v}|^2 \right) rdr = \pi \int_0^\infty \left| \frac{\partial \mathbf{v}}{\partial r} - \frac{|m|}{r} J^{\mathbf{v}} R\mathbf{v} \right|^2 rdr + \mathcal{E}_{\min},$$

with

$$\mathcal{E}_{\min} = 2\pi \int_0^\infty \mathbf{v}_r \cdot \frac{|m|}{r} J^{\mathbf{v}} R\mathbf{v} rdr = 2\pi |m| \int_0^\infty \mathbf{v}_{3r} dr = 4\pi |m|.$$

Thus we arrive at the conclusion that

$$\mathbf{u} \in \Sigma_m \Rightarrow \mathcal{E}(\mathbf{u}) \geq 4\pi |m|$$

Harmonic maps: This topological lower bound is clearly saturated if and only if

$$\frac{\partial \mathbf{v}}{\partial r} = \frac{|m|}{r} J^{\mathbf{v}} R\mathbf{v}$$

and the minimal energy is attained precisely an explicit two-parameter family of harmonic maps:

$$\mathcal{O}_m := \{e^{(m\theta+\alpha)R} h(\frac{r}{s}) : s > 0, \alpha \in \mathbb{R}\}$$

where

$$\mathbf{h}(r) = \begin{pmatrix} h_1(r) \\ 0 \\ h_3(r) \end{pmatrix} \qquad h_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, \qquad h_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}} \tag{1.8}$$

The rotation parameter α is determined only up to shifts of 2π (i.e., really $\alpha \in \mathbb{S}^1$). Note that $\mathcal{O}_m$ is just the orbit of the harmonic map $e^{m\theta R} h(r)$ under the symmetries of the energy $\mathcal{E}$ which preserve equivariance: scaling and rotation. Of course, these harmonic maps are static solutions of all of the map evolution equations introduced above.

Recent History:

here we describe some of the important results for the various map dynamics described above, continuing to focus on maps from $\mathbb{R}^2$ to $\mathbb{S}^2$.

Harmonic map heat-flow: Of the map evolution problems we are considering, (1.2) has been studied the longest, and is certainly the best understood. The question of singularity formation and characterization of possible blow-up has attracted a lot of attention in the last 30 years or so. On a compact manifold domain, Struwe [122] showed that if blow-up is to occur, it can only happen in an energy concentration scenario, and the concentration of energy results in the bubbling off of a non-trivial harmonic map at a finite number of points

$$\mathbf{u}(t_n, a_n + \lambda(t_n)x) \xrightarrow{t_n \nearrow T} \mathbf{Q}, \ \ \lambda(t_n) \to 0,$$

locally in space; but otherwise, weak solutions to (1.2) with finite energy data, exist globally and are smooth: if at time T a singularity occurs the flow develops a bubble which then separates, and restarts from the weak limit (as $t \nearrow T$). If another singularity occurs at a later time, the flow goes through the same process. At least the full energy of a harmonic map is lost every time a singularity occurs, and no bubbling can occur when there is energy less than the lowest energy of a non-trivial harmonic map.

Later work by Qing [110], Ding and Tian [38], Qing and Tian [111], Topping [125, 126] (see also the book [97]) showed that the convergence to a "body map" (what remains after the bubbles are removed) is strong away from the singular points, and also that near the bubble points all the energy is accounted for by the body map and the bubbles - statements of the same flavour were also made for $t \to \infty$. We again refer to the above works for more details.

Working in the subclass of the corotational solutions with $m = 1$, and on a disk, [22] showed that, indeed, finite time blow-up does occur in some situations, the methods relying on the maximum principle and sub(super)-solutions.

Until recently, not much information was available on the admissible rates of blow-up, as well as the relation between the possibility of singularity formation and the degree m or a good description of the blow-up profile. A formal analysis by Van de Berg, Hulshof and King [9], and a rigorous construction verifying their results by Raphael and Schweyer [114], [115] shows that (if T is the blow-up time), for 1-corotational maps, initial data u_0, $L \in \mathbb{N}^*$

$$u(t,r) - Q(\frac{r}{\lambda(t)}) \to u^*, \text{as } \ t \to T, \ \text{ in } \ \dot{H}^1,$$

$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, c(u_0) > 0,$$

with the case $L = 1$ providing the generic blow-up rate.

On the other hand, Grotowski and Shatah ([65]), using maximum principle methods as well, and assuming particular bounds on the initial data, showed that on the unit disc in $\mathbb{R}^2$ blow-up will not occur in finite-time for degrees $m \geq 2$. One of our goals is to extend this result to all of $\mathbb{R}^2$ and give a maximum principle-free proof.

Landau-Lifshitz equation. Once the Schrödinger-type term ($b \neq 0$) is included in (1.3), our understanding diminishes considerably. Though the problem is still dissipative, maximum principle-type arguments are not available, and even partial regularity results become more difficult and weaker (see, e.g., [84] and the references therein). Singularity formation is an open question, partly because the corotational class is no longer preserved. Indeed, the (1.2) blow-up may not provide a reliable guide for the (1.3) problem. In the recent papers [70, 71, 66, 72], for equivariant maps from $\mathbb{R}^2$ to $\mathbb{S}^2$, if $m \geq 3$, near-minimal energy solutions are shown not to form singularities and to converge to harmonic maps as t goes to infinity. When $m = 2$ this asymptotic stability may fail: in the case of heat-flow with a further symmetry restriction, it is shown that more exotic asymptotics are possible, including infinite-time concentration (blow-up).

Schrödinger maps. In the absence of dissipation ($a = 0$), the analysis becomes still more difficult. Even the local theory is just beginning to be understood ([5, 48, 76, 103, 120]; see also [77, 108] for the "modified Schrödinger map" case).

For the class of data we consider in the next subsection, m-equivariant solutions with energy near the minimal energy $4\pi|m|$, an energy-space local wellposedness result is given in [71]. It is worth remarking that the existence time provided by this theorem depends not on the energy (reflecting the energy-space critical nature of the equation in dimension $n = 2$), but rather on more refined information about the initial data: the "length scale" of the $\dot{H}^1$-nearest harmonic map (see [71] for details).

Very few *global* results are known, and these only the equivariant case. The global results of [70, 71] we describe in the next section, showing asymptotic stability of harmonic maps for $m \geq 3$, can be thought of as above threshold analogues of [20] for large energy, where the problem is considerably enriched by the presence of the harmonic map family. The case of energy below the energy of a harmonic map has been treated ([7, 8]), for maps with values either in $\mathbb{S}^2$ or $\mathbb{H}^2$, and also the case $m = 0$ in [73]. We also mention the recent important paper [106] which established finite-time blow-up for the case $m = 1$.

Wave maps: Wave maps have received more attention for a longer time than have Schrödinger maps. There is a large literature, especially concerning local questions, which we will not attempt to summarize here (see, e.g., [119] for some background). Because of the close connection with the problem we are focusing on, we mention only that the possibility of finite-time blow-up for the energy-space critical ($n = 2$) wave maps was established only quite recently, first in [116] for higher degree equivariant maps, and then in [87] for degree $m = 1$. In a series of works, Duyckaerts, Kenig and Merle ([41], [42],[43],[44]) have established soliton resolution for all energies in the case of the radial *wave equation* in $3+1$ dimensions, completely describing global, type I and type II blow-up solutions. Their results build on the Kenig-Merle "concentration-compactness plus rigidity" roadmap ([79], [80]) and the newly devised "energy channel method" for the wave equation. In the spirit of this work, similar results have been obtained for the corotational Wave Map equation in 2+1 dimensions (to the 2-sphere) by Côte-Kenig-Merle [32], Côte-Kenig-Lawrie-Schlag [33], [34], [35], and Côte [30]; for other targets and geometries we refer to [86, 91, 92, 93]. Most of these results rely on the presence of an underlying wave structure: they make frequent use of the finite-speed of propagation as well as the energy channel method.

Note however, that the second order (in time) nature of the above equations is not strong enough to prevent finite-time blow-up; blow-up solutions have been exhibited in [87, 88], [113], [116] (even for high m).

Main Results

In this thesis we specialize to the harmonic map heat flow for corotational maps of degree m :

$$\mathbf{u}(\cdot,t):(r,\theta)\to(\cos(m\theta)\sin(u(r,t)),\sin(m\theta)\sin(u(r,t)),\cos(u(r,t))) \tag{1.9}$$

Then, $u(r,t)$ satisfies

$$\begin{cases} u_t = u_{rr}+\frac{1}{r}u_r - m^2\frac{\sin 2u}{2r^2} \\ u(0,r)=u_0(r) \end{cases} \tag{1.10}$$

The energy is given by

$$E(u(t,r)) := \pi\int_0^\infty (u_r^2+m^2\frac{\sin^2(u)}{r^2})rdr.$$

To streamline the presentation of our results we make the following definitions

$$E_0 := \{u_0 : E(u_0)<2E(Q); u(0)=0, \lim_{r\to\infty} u(r)=0\},$$

which contains no non-trivial harmonic maps; maps in E_0 have degree 0.

$$E_1 := \{u_0 : E(Q)\le E(u_0)<3E(Q); u(0)=\pi, \lim_{r\to\infty} u(r)=0\},$$

Maps in E_1 have degree m; the harmonic maps are also contained in this class.
The harmonic maps are explicitly given by

$$Q^s(r)=\pi-2\arctan((r/s)^m), s>0.$$

These are the maps which minimize the energy within E_1.

In Chapter 2 we first address the below threshold case of E_0. The purpose of this work is twofold: to extend the classical analysis to non-compact domains, and more importantly to revisit Struwe's theory [122] in terms of the "concentration-compactness plus rigidity approach" of Kenig and Merle [79] with the aim of eventually obtaining more detailed results. We prove the following theorem (Chapter 2):

Theorem 1.1. *In the class E_0, solutions to (1.10) with $m \geq 2$, exist globally in time and are smooth; furthermore, $u(t) \xrightarrow{t\to\infty} 0$ in energy.*

We first establish local well-posedness for maps in E_0. Decay of global solutions follows from an approximation by an appropriately chosen linear solution. We then adapt the "concentration-compactness plus rigidity" approach to show global existence and decay. The key tool here is a profile decomposition for a bounded sequence in the energy space, applied on an appropriate sequence of initial data. There was no readily available profile decomposition directly applicable to our setting; the difficulty stems mainly from the absence of some Sobolev embeddings in dimension two, so we take an indirect approach by first establishing estimates on the linear evolution in higher dimensions which then connect back to our problem through a reduction trick.

Given the previous result, it is natural to ask what happens in the presence of non-trivial topologies and higher energies. In the equivariant setting, global regularity and asymptotic stability were shown ([72]) for $m-$equivariant solutions with $m \geq 3$, for data $\mathbf{u}_0$ with $E(\mathbf{u}_0) = E(Q) + \delta, \delta \ll 1$.

However, our class E_1 includes harmonic maps and maps with energy much larger than the energy of the harmonic map, hence the approach taken in the above works (using "coordinate systems" around the family of harmonic maps, reflecting the near-minimality of the energy) is hard to implement. We restrict ourselves to the corotational setting and take a different approach to prove (Chapter 3)

Theorem 1.2. *The solution to (1.10) with $m \geq 4$, and $u_0 \in E_1$ is global and smooth with $u(r,t) \to Q^{s_\infty}(r)$, as $t \to \infty$ for some $s_\infty > 0$.*

As in [122], if the solution blows-up in finite time, it does so by bubbling off a non-trivial harmonic map. Following [110], if $u(t) \in E_1$ is a solution blowing up at time T, with $E(u) < 3E(Q)$, there exists a sequence of times $t_n \to T$, a sequence of scales λ_n and a map $w_0 \in E_0$, such that $u(t_n, r) = Q(\frac{r}{\lambda_n}) + w_0(r) + \xi(t_n)$, with $\xi(t_n) \to 0$ in X^2 as $n \to \infty$.

We will start with data $u(t_n)$, where t_n is very close to the blow-up time T and then use a perturbative approach,based on modulation decomposing the solution u as $u(t,r) = Q(\frac{r}{s(t)}) + w(t,r) + \xi(t,r)$, where w is the solution corresponding to initial data $w_0(r)$. We can eventually show that concentration is not possible in this case, hence the solution extends past T, providing a contradiction.

1.1.2 Nonlinear Heat Equation

For the rest of this section we will focus on the nonlinear heat equation

$$u_t - \Delta u - |u|^{p-1} u = 0. \tag{1.11}$$

Our study of the heat equation was not motivated by a particular physical or geometric application. We were however intrigued by the relative lack of results in the literature concerning critical problems, and because of the similarities with the Nonlinear Schrödinger equation on an algebraic level, we decided to investigate the use of the dispersive roadmap provided by the "concentration-compactness plus rigidity" approach of Kenig and Merle [79]. The resulting methodology is very different from the classical works on the nonlinear heat equation: we make no use of maximum and comparison principles or use techniques such as the "intersection number", nor do we rely directly on a rescaling/blowing-up argument like Struwe's [122] for the Harmonic Map Heat Flow to characterize singularities. Our approach combines elements from the analysis of the energy-critical NLS ([79, 82]), as well as ideas from the literature on the Navier-Stokes system ([56, 78]), with new insights and deviations from known arguments required at several places.

The study of nonlinear heat equations has been a subject of intense work and the literature is vast; it would be impossible to provide a complete list, so we content ourselves with a brief review focused on the case of domain $\mathbb{R}^d-$case and refer the reader to the recent book [112] for a comprehensive review of the literature. For treatments of the Cauchy problem in L^p and Sobolev spaces under various assumptions on the nonlinearity and the initial data we refer to the works [128, 129, 13]. Most of the work on semilinear heat equations has been on subcritical problems $p < \frac{2d}{d-2}$. For the semilinear heat equation, no matter how weak is the initial regularity (let's say in L^p spaces), blow-up occurs always in L^∞ due to the regularizing effect, see [128]. The seminal papers [62],[63],[64] of Giga and Kohn introduced the study of heat equations through similarity variables and characterized blow-up solutions. In continuation of these works, Merle [104] gave a first construction of a solution with arbitrarily given blow-up points and together with Zaag, they provided detailed uniform estimates for the blow-up rate, descriptions of the blow-up set, and stability results for the blow-up profile; we refer to [107] and the references therein. We remark that the blow-up in the subcritical case for $L^\infty-$solutions is known to be of Type I (in the sense that $\limsup_{t\to T_{\max}}(T_{\max}-t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty} < +\infty$) and Type I blow-up solutions are known to behave like self-similar solutions near the blow-up point. More precisely, at any point $a \in \mathbb{R}^d$ where $|u(a,t)| \xrightarrow{t\to T_{\max}} \infty$, one can find a bounded solution $\psi(y)$ of

$$\Delta\psi - \frac{1}{2}y\cdot\nabla\psi - \frac{1}{p-1}\psi + |\psi|^{p-1}\psi = 0, y \in \mathbb{R}^d \tag{1.12}$$

such that u behaves like a self-similar solution (i.e., of the form $u(x,t) = (T-t)^{-\frac{1}{p-1}}\psi(\frac{x-a}{\sqrt{T-t}})$) in a certain "local" sense:

$$u(a+\sqrt{T_{\max}-t}y, t) \sim (T_{\max}-t)^{-\frac{1}{p-1}}\psi(y), \text{ as } t \to T_{\max}.$$

For supercritical problems, we refer to the series of works by Matano and Merle [100, 101, 102]. It is shown that there is no Type II blow-up for $3 \le d \le 10$, while for $d \ge 11$ it is possible for algebraic non-linearities with a large enough exponent ($p > p_{JL} := 1 + \frac{4}{d-4-2\sqrt{d-1}}$, the Joseph-Lundgren exponent). It is also shown that a Type I blowing up solution behaves like a self-similar solution, while a Type II converges (in some sense) to a stationary solution. We also refer to the recent results [30] ($d \geq 11$, bounded domain) and [26] and to the recent preprint [10] for results in Morrey spaces.

For the critical case, there are a few blow-up constructions [52, 118] in the case of domain $\mathbb{R}^d$, which inspired our project. For a bounded domain, we point to the recent construction [28] ($d \geq 5$) of a solution blowing up by bubbling in infinite time, and on $\mathbb{R}^3$, the infinite-time blow-up construction [36]. We also refer to the work [53] dealing with the continuation problem for reaction-diffusion equations under various assumptions and range of exponents. We finally mention the recent result of Collot, Merle and Raphael [27], where a complete classification of solutions near the stationary solution W for $d \geq 7$ is provided. In particular, they show that Type II blow-up is ruled out in $d \geq 7$ "near" W.

For a set of criteria (of a different nature to those we provide below) for global existence/blow-up in terms of the initial data we refer the reader to [18] where the authors prove that given $0 < \alpha < \frac{2}{d}$, there exists a function ψ with the following properties: the solution of the equation $u_t = \Delta u + |u|^a u$, $x \in \mathbb{R}^d$ with the initial condition $u_0 = \psi$ is global. On the other hand, the solution with the initial condition $u_0 = \lambda\psi$ blows up in finite time if $\lambda > 0$ is either sufficiently small or sufficiently large.

Finally, for results on the relation between the regularity of the nonlinear term and the regularity of the corresponding solutions we refer to the work [19].

Main Results

In what follows, we consider the focusing energy-critical nonlinear heat equation in four space dimensions: $u_t - \Delta u - |u|^2 u = 0$. We are interested in $\dot{H}^1$ solutions of the Cauchy Problem

$$\begin{aligned} &u_t = \Delta u + |u|^2 u \\ &u(t_0, x) = u_0(x) \in \dot{H}^1(\mathbb{R}^4) \end{aligned} \tag{1.13}$$

The energy for $u \in \dot{H}^1$ is defined by

$$E(u) := \int_{\mathbb{R}^4} \frac{1}{2}|\nabla u|^2 - \frac{1}{4}|u|^4 dx.$$

We refer to the gradient term in E as the kinetic energy, and the second term as the potential energy. Note that the potential energy is negative, which expresses the focusing nature of the nonlinearity. Equation (1.13) is the L^2-gradient flow for E. The energy is dissipated

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{R}^4} u_t^2 dx \leq 0$$

and that the scaling $u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)$ leaves both the equation and the energy invariant reflect the energy-critical nature of the problem.
The stationary equation $\Delta W + |W|^2 W = 0$ is known to admit the solution

$$W = W(x) = \frac{1}{1 + \frac{|x|^2}{8}}$$

which (along with its rescalings and translations) plays an important role in our analysis. By the work of Aubin-Talenti [2, 123], W is known to saturate the Sobolev inequality:

$$\forall u \in \dot{H}^1, \|u\|_{L^4} \leq \left[\frac{\|W\|_{L^4}}{\|\nabla W\|_{L^2}}\right] \|\nabla u\|_{L^2}, \tag{1.14}$$

One motivation for this project was the following result :

Theorem 1.3. *(Schweyer [118]) Let W be the Talenti-Aubin solution. Then for any $a^* > 0$, there is a radially symmetric initial datum $u_0 \in H^1(\mathbb{R}^4)$, with $E(W) < E(u_0) < E(W) + a^*$, so that the corresponding solution to (1.13) blows-up in finite time $T = T(u_0)$.*

We observe that the solution $u(t,x) = W(x) = \frac{1}{(1+\frac{|x|^2}{8})} \in \dot{H}^1(\mathbb{R}^4)$ is a stationary solution to the equation which, while global, doesn't decay. On the other hand, the local wellposedness theory shows that all maximal-lifespan solutions with sufficiently small kinetic energy are global and decay to zero. The above results and observations suggest that the static solution W will play an important role in determining which initial data lead to globally smooth and decaying solutions and which ones lead to singularity formation. In particular, we show in Chapter 4 that below the threshold set by the static solution all solutions are global and that, in addition, infinite-time blow-up is not possible and global solutions have to decay to zero in $\dot{H}^1$.

Theorem 1.4. *Let $u_0 \in \dot{H}^1(\mathbb{R}^4)$ satisfy*

$$\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2} \text{ and } E(u_0) < E(W). \tag{1.15}$$

Then, the corresponding solution u to (1.13) is global and $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0$.

We give the proof for the case $d = 4$, but the result can be easily transferred to solutions in any $d \geq 3$. Our proof makes no use of parabolic comparison principles and can be used verbatim for complex-valued solutions. Assuming $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$ and $E(u_0) < E(W)$ we first show that global solutions decay to zero. This is done through a reduction to a refined small-data theory and arguments from the local-in-time theory. The result is new in the framework of heat equations and of independent interest. To rule out blow-up, our approach is again based on the "concentration-compactness plus rigidity" roadmap of Kenig and Merle [79] as implemented for the Navier-Stokes in [78]. Ruling out finite-time blow-up is different from the NLS case: we use compactness and the asymptotic vanishing of the L^2-norm on balls as in [79]; however there is no mass conservation in our case and the irreversibility of the equation requires a more involved backwards uniqueness step as in [78]. Another technical challenge is in precluding the scenario where the center of concentration is moving off to infinity. Infinite-time blow-up can be ruled out by appealing to our decay result.

We have also proved a blow-up criterion by a mild variant of a classical argument in [95], which, modulo a decay assumption on the initial data, shows that our results are sharp

Proposition 1.5. *Let $u_0 \in H^1(\mathbb{R}^4)$ and $\delta_0 > 0$ such that*

$$E(u_0) \leq (1-\delta_0)E(W) \ \text{ and } \ \|\nabla u_0\|_{L^2} \geq \|\nabla W\|_{L^2} \tag{1.16}$$

Then the corresponding solution u blows-up in finite-time.

Whether removing the L^2 assumption is possible is an interesting question for future consideration.

Chapter 2

Harmonic Map Heat Flow-Below Threshold

2.1 Introduction

This chapter is devoted to the presentation of our investigations on the global regularity problem for the harmonic map heat flow. In particular, we will present global wellposedness and decay results below a certain threshold.

The harmonic map heat flow is given by the equation

$$\mathbf{u}_t = \Delta \mathbf{u} + |\nabla \mathbf{u}|^2 \mathbf{u}, \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x) \tag{2.1}$$

where $\mathbf{u}(\cdot, t) : \mathbb{R}^2 \to \mathbb{S}^2$, and $\mathbb{S}^2$ is the unit 2-sphere

$$\mathbb{S}^2 := \{\mathbf{u} = (u_1, u_2, u_3) : |\mathbf{u}| = 1\} \subset \mathbb{R}^3,$$

Δ denotes the Laplace operator in $\mathbb{R}^2$ and $|\nabla \mathbf{u}|^2 = \sum_{j=1}^{2} \sum_{i=1}^{3} (\frac{\partial u_i}{\partial x_j})^2$
Equation (2.1) is the L^2-gradient flow of the energy functional

$$E(\mathbf{u}) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \mathbf{u}|^2 dx$$

Taking, at least formally, the scalar product with $\mathbf{u}_t = \Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u}$ and integrating over $\mathbb{R}^2 \times [0,t)$, we obtain:

$$E(\mathbf{u}(t)) + \int_0^t \int_{\mathbb{R}^2} |\mathbf{u}_t|^2 = E(\mathbf{u}(0))$$

which implies that the energy is non-increasing.
A more geometric way to write (2.1) is

$$\mathbf{u}_t = \sum_{j=1}^{2} D_j \partial_j \mathbf{u} = P^{\mathbf{u}} \Delta\mathbf{u},$$

where $P^{\mathbf{u}}$ denotes the orthogonal projection from $\mathbb{R}^3$ onto the tangent plane to $\mathbb{S}^2$ at $\mathbf{u}$:

$$T_{\mathbf{u}}\mathbb{S}^2 := \{\boldsymbol{\xi} \in \mathbb{R}^3 : \boldsymbol{\xi} \cdot \mathbf{u} = 0\},$$

$\partial_j = \frac{\partial}{\partial x_j}$ is the usual partial derivative and D_j the covariant derivative (affine connection) acting on vector fields $\xi(x) \in T_{\mathbf{u}(x)}\mathbb{S}^2$:

$$D_j\xi := P^{\mathbf{u}}\partial_j\xi = \partial_j\xi - (\partial_j\xi \cdot \mathbf{u})\mathbf{u} = \partial_j\xi + (\partial_j\mathbf{u} \cdot \xi)\mathbf{u}.$$

Equation (2.1) is a particular case of the harmonic map heat flow between Riemannian manifolds, introduced by Eells and Sampson [47]. Static solutions of (2.1) are harmonic maps from $\mathbb{R}^2$ to $\mathbb{S}^2$.

The reason why we chose to work in two dimensions, beyond the physical relevance, is that this is when the energy $E(\mathbf{u})$ is invariant under scaling,

$$E(\mathbf{u}(\cdot)) = E(\mathbf{u}(\frac{\cdot}{\lambda}))$$

which is interesting with respect to the questions of singularity formation and global existence.

The question of singularity formation and characterization of possible blow-up has attracted a lot of attention in the last 30 years or so. On a compact manifold domain, Struwe [122] showed that if blow-up is to occur, it can only happen in an energy concentration scenario, and the concentration of energy results in the bubbling off of a non-trivial harmonic map at a finite number of points

$$\mathbf{u}(t_n, a_n + \lambda(t_n)x) \to \mathbf{Q}, \ \ \lambda(t_n) \to 0$$

locally in space; but otherwise, weak solutions to (1.2) with finite energy data, exist globally and are smooth. Later work by Qing [110], Ding and Tian [38], Qing and Tian [111], Topping [125, 126] (see also the book [97]) showed that the convergence to a "body map" (what remains after the bubbles are removed) is strong away from the singular points, and also that near the bubble points all the energy is accounted for by the body map and the bubbles - statements of the same flavour were also made for $t \to \infty$. We again refer to the above works for more details.

Working in the subclass of the corotational solutions with $m = 1$, and on a disk, [22] showed that, indeed, finite time blow-up does occur in some situations, the methods relying on the maximum principle and sub(super)-solutions. A formal analysis by Van de Berg, Hulshof and King [9], and a rigorous construction verifying their results by Raphael and Schweyer [114], [115] shows that (if T is the blow-up time), for 1-corotational maps, initial data u_0, $L \in \mathbb{N}^*$

$$u(t,r) - Q(\frac{r}{\lambda(t)}) \to u^*, \text{as } \ t \to T, \ \text{ in } \ \dot{H}^1,$$

$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, c(u_0) > 0,$$

with the case $L = 1$ providing the generic blow-up rate.

On the other hand, Grotowski and Shatah ([65]), using maximum principle methods as well, and assuming particular bounds on the initial data, showed that on the unit disc in $\mathbb{R}^2$ blow-up will not occur in finite-time for degrees $m \geq 2$. One of our goals is to extend this result to the case of domain $\mathbb{R}^2$ and give a maximum principle-free proof.

In this work we will specialize to maps with some symmetry, namely co-rotational maps of degree m, i.e., maps of the form

$$\mathbf{u}(r,\theta) = (\cos(m\theta)\sin(u(r,t)), \sin(m\theta)\sin(u(r,t)), \cos(u(r,t))),$$

Since $|\nabla \mathbf{u}|^2 = u_r^2 + \frac{m^2}{r^2}\sin^2(u)$ (easy to check), and writing the Laplacian in polar coordinates, plugging the above ansatz into the equation gives

$$\begin{aligned}
\cos(m\theta)\cos(u(t,r))u_t = {} & \cos(m\theta)\sin(u(t,r))u_r^2 + \cos(m\theta)\sin(u(t,r))\frac{m^2}{r^2}\sin^2(u(t,r)) \\
& + \cos(m\theta)(-\sin(u(t,r))u_r^2) + \cos(m\theta)\cos(u(t,r))u_{rr} \\
& + \cos(m\theta)\cos(u(t,r))\frac{u_r}{r} - \cos(m\theta)\frac{m^2}{r^2}\sin(u(t,r))
\end{aligned}$$

or

$$\begin{aligned}
\cos(u(t,r))u_t &= \cos(u(t,r))u_{rr} + \cos(u(t,r))\frac{u_r}{r} \\
&\quad + \sin(u(t,r))\frac{m^2}{r^2}\sin^2(u(t,r)) - \frac{m^2}{r^2}\sin(u(t,r)) \\
&= \cos(u(t,r))u_{rr} + \cos(u(t,r))\frac{u_r}{r} - \sin(u(t,r))\frac{m^2}{r^2}\cos^2(u(t,r)) \\
&= \cos(u(t,r))u_{rr} + \cos(u(t,r))\frac{u_r}{r} - \frac{m^2}{2r^2}\sin(2(u(t,r))\cos(u(t,r)),
\end{aligned}$$

which simplifies to the following equation for the angle u :

$$u_t = u_{rr} + \frac{1}{r}u_r - m^2\frac{\sin 2u}{2r^2} \tag{2.2}$$

Moreover, if (2.2) holds, then the other components of $\vec{u}$ are also easily seen to satisfy the heat-flow equation. We make the following definitions:

$$\begin{aligned}
&\Delta_r u = u_{rr} + \frac{1}{r}u_r, \text{the radial Laplacian in } \mathbb{R}^2, \\
&\Delta_m u = (\Delta_r - \frac{m^2}{r^2})u,
\end{aligned} \tag{2.3}$$

thus, we can rewrite the equation as

$$u_t = (\Delta_r - \frac{m^2}{r^2})u + \frac{m^2}{r^2}(u - \frac{\sin 2u}{2}),$$

and symbolically as $u_t = \Delta_m u + F(u)$, denoting the new nonlinear term by $F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2})$.

The energy for these maps is given by

$$E(u(t,r)) := \pi \int_0^\infty (u_r^2 + m^2 \frac{\sin^2(u)}{r^2}) r dr.$$

The stationary solutions are $Q(r) = \pi - 2\arctan(r^m)$, and for any $s > 0, Q(\frac{r}{s})$ is also a solution. These maps minimize the energy within their topology class, and they are very important objects when addressing the questions of global existence and singularity formation, in that they are the objects which provide the natural thresholds for global wellposedness and decay (in a sense to become precise later).

We make the following definitions

$$E_0 := \{u : E(u) < 2E(Q); u(0) = 0, \lim_{r\to\infty} u(r) = 0\},$$
$$E_1 := \{u : E(Q) \leq E(u_0) < 3E(Q); u(0) = \pi, \lim_{r\to\infty} u(r) = 0\},$$

Remark 2.1. *The assumption of finite energy can be easily shown to be sufficient to guarantee the existence of the above limits.*

The main goal of this chapter is the proof of the following theorem:

Theorem 2.2. *Assuming $u_0 \in E_0$, and $m \geq 2$ the corresponding solution $u(r,t)$ to* (2.2) *exists globally in time and is smooth; furthermore, $E(u(t)) \xrightarrow{t\to\infty} 0$.*

We give a proof which is suited to non-compact domains which makes use of the "concentration-compactness plus rigidity approach" of Kenig and Merle [79], originally developed for dispersive equations. We will establish both local and global wellposedness for maps in E_0. Our results rest upon the extended literature on critical dispersive equations concerning global existence and scattering, and also provide alternative ways to look at parabolic flows. We won't try to provide a complete list of references for all the recent work on the issues of global existence and scattering for dispersive equations, which has become a very active area of research following the breakthrough ideas of Bourgain [12] and of Colliander-Keel-Staffilani-Takaoka-Tao [24] (the "induction on the energy" method), and in particular Kenig and Merle [79] (the "concentration-compactness plus rigidity" approach). We refer the reader to the excellent notes [82] for more background information.

We will now present an outline of the approach we take in this chapter, with more details to follow in the subsequent sections.

First, we establish local wellposedness for maps in E_0. Despite the presence of some results in the literature, they are written within the classical parabolic framework so we prefer to redo the theory specializing to the co-rotational class and in a way that can be compared to the dispersive literature for related problems. In particular, this is mostly apparent in our blow-up alternative which, in this formulation, bonds well with the Kenig-Merle approach.

A "concentration-compactness plus rigidity" procedure excludes the possibility of finite-time blow-up. The key tools here are a profile decomposition for a bounded sequence in an $\dot{H}^1$-like space, applied on an appropriate sequence of initial data, and a stability-under-small-perturbations result. As a result, we establish the existence of a putative minimal counterexample; the dissipation of energy contradicts its existence implying global existence and decay to zero.

There was no readily available profile decomposition directly applicable to our setting, so this was one of the main intermediate tasks to achieve. The difficulty stems mainly from the absence of some Sobolev embeddings in dimension two, so we take an indirect approach by first establishing estimates on the linear evolution in higher dimensions which then connect back to our problem through a reduction trick.

2.2 Some analytical ingredients

2.2.1 Energy properties of maps in E_0

We will first show that maps in the class E_0, the energy space is naturally endowed with the X^2-norm :

$$\|u\|_{X^2}^2 = \int_0^\infty \left(|u_r|^2 + m^2|\frac{u}{r}|^2\right) rdr.$$

This equivalence only holds for the class E_0.
By finiteness of the energy, the limits at zero and infinity exist.

Lemma 2.3. *If $u \in E_0$, with $E(u) \leq 2E(Q) - \delta_1$, for some $\delta_1 > 0$, there is a $\delta_2 = \delta_2(\delta_1) > 0$ such that*

$$|u(r)| \leq \pi - \delta_2, \tag{2.4}$$

Proof. As in [119] we define

$$G(u) := \pi \int_0^u m|\sin(s)|ds$$

and

$$E_{r_1}^{r_2}(u) := \pi \int_{r_1}^{r_2} (u_r^2 + \frac{m^2 \sin^2(u)}{r^2})rdr.$$

Then for all $r_1, r_2 \in [0, \infty)$, by the Fundamental Theorem of Calculus and Young's inequality:

$$\begin{aligned} |G(u(r_2)) - G(u(r_1))| &= \left|\int_{r_1}^{r_2} \frac{\partial}{\partial r} G(u(r))dr\right| = \pi \left|\int_{r_1}^{r_2} m|\sin u|u_r dr\right| \\ &\leq \frac{\pi}{2} \int_{r_1}^{r_2} (\frac{m^2 \sin^2(u)}{r^2} + u_r^2)rdr \leq \frac{1}{2} E_{r_1}^{r_2}(u) \end{aligned} \tag{2.5}$$

In this case $G(u(\infty)) = G(0) = 0$, $G(u(0)) = G(\pi) = 0$. From (2.5) for any $r > 0$:

$$|G(u(r))| = |G(u(r)) - G(u(0))| \leq \frac{1}{2} E_0^r(u)$$

and

$$|G(u(r))| = |G(u(\infty)) - G(u(r))| \leq \frac{1}{2} E_r^\infty(u).$$

Thus

$$2|G(u(r))| \leq \frac{1}{2} E(u) \leq E(Q) - \frac{\delta_1}{2},$$

hence for every r:

$$|G(u(r))| \leq \frac{1}{2} E(Q) - \frac{\delta_1}{4}.$$

Note that G is odd, increasing on $[-\pi, \pi]$ and, $G(\pi) = 2m\pi = \frac{E(Q)}{2}$ and since $G^{-1}(-\frac{E(Q)}{2}) < u(r,t) < G^{-1}(\frac{E(Q)}{2})$, there is a $\delta_2 > 0$

$$|u(r)| \leq \pi - \delta_2$$

as claimed. □

This information is enough in order to prove the desired equivalence between the energy and the X^2-norm. In particular, we prove:

Proposition 2.4. *The class E_0 is naturally endowed with the norm $\| \cdot \|_{X^2}$, in the sense that, given $\delta_1 > 0$, there exist $C_1, C_2 > 0$ such that $C_2 \|u\|^2_{X^2} \leq E(u) \leq C_1 \|u\|^2_{X^2}$, for all $u \in E_0$ with $E_0 \leq 2E(Q) - \delta_1$.*

Proof. By elementary calculus; the previous lemma granting the uniform constants. □

We remark that in E_0, there are no nontrivial harmonic maps as a result of the boundary conditions and the monotonicity of the stationary solutions (see [29]).

2.2.2 Local wellposedness for maps in E_0

From now on, unless otherwise specified, all the norms will be considered with the measure rdr. We define $rL^p := \{\text{measurable } z : z = rf, f \in L^p\}$, and the spaces X^p equipped with the norm

$$\|u\|_{X^p}^p := \int_0^\infty (|u_r|^p + m^2|\frac{u}{r}|^p)rdr.$$

We say that a function $u : I \times \mathbb{R} \to \mathbb{R}, I = [0,T)$ is a solution to the problem

$$\begin{cases} u_t = (\Delta_r - \frac{m^2}{r^2})u + F(u), \\ u(0) = u_0 \in X^2, \end{cases} \tag{2.6}$$

where $F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2})$, if it lies in $C_t X_r^2 \cap L_t^4 r L_r^4(K)$, for every compact $K \subset I$, and obeys the following Duhamel formula for every $t \in I$:

$$u(t) = e^{t\Delta_m}u_0 + \int_0^t e^{(t-s)\Delta_m}F(u)ds. \tag{2.7}$$

We can summarize the local theory in the following theorem:

Theorem 2.5. *(Local wellposedness)*

1. *(Local Existence) Let $u_0 \in X^2$. There exists an $\epsilon > 0$ such that, if $I = [0,T)$, and $\|e^{t\Delta_m}u_0\|_{L_t^4(I;rL^4)} < \epsilon$, then there exists a unique solution to (2.6) with $u \in C(I;X^2), \|u\|_{L_t^4(I;rL^4)} \leq 2\epsilon$. To each initial datum u_0 we can associate a maximal interval of existence $I = [0, T_{max}(u_0))$, where $T_{max}(u_0)$ can be $+\infty$.*

2. *(Blow-up Criterion) $T_{max}(u_0) < +\infty \Rightarrow \|u\|_{L_t^4([0,T_{max}(u_0)];rL_r^4)} = +\infty$.*

3. *(Dissipation) If $T_{max}(u_0) = +\infty$ and $\|u\|_{L_t^4([0,\infty];rL_r^4)} < +\infty$, then $\|u(t)\|_{X^2} \to 0$, as $t \to +\infty$.*

4. *(Small data implies global existence and dissipation) If $\|u_0\|_{X^2}$ is sufficiently small, the solution is global and decays to zero, in the above sense.*

5. *(Continuous Dependence on Initial Data) The solution depends continuously on the initial data. Furthermore, T_{max} is a lower semi-continuous function of the initial data.*

The proof relies on the space-time estimates established in [66]. From now on, we will be referring to them as "the space-time estimates":

For $H = -\Delta_r + \frac{m^2}{r^2}, d = 2, m \geq 2:$

(i)

$$\|e^{-tH}\phi\|_{L^p_r} \lesssim t^{-(1/a-1/p)}\|\phi\|_{L^a}, 1 \leq a \leq p \leq \infty \tag{2.8}$$

(ii)

$$\|e^{-tH}\phi\|_{L^q_t L^p_r} \leq C\|\phi\|_{L^a}, \frac{1}{q} = \frac{1}{a} - \frac{1}{p}, 1 < a \leq q \tag{2.9}$$

(iii)

$$\|\int_0^t e^{-(t-s)H} f(s)ds\|_{L^q_t L^p_r} \lesssim \|f\|_{L^{\tilde{q}'}_t L^{\tilde{p}'}_r} \tag{2.10}$$

for admissible pairs (q,p),$(\tilde{q}, \tilde{p})$ and the Hölder-dual pair $(\tilde{q}', \tilde{p}')$.

(iv)

$$\|e^{-tH}\phi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} + \|\int_0^t e^{-(t-s)H} f(s)ds\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \|\phi\|_{X^2} + \|f\|_{L^1_t X^2 + L^2_t X^1} \tag{2.11}$$

In dimension $d = 2$, an admissible pair (q,p) satisfies : $\frac{1}{q} + \frac{1}{p} = \frac{1}{2}$, including the endpoint $(2, 2, \infty)$.

The proof of the local wellposedness is a mild variant of the classical work of Cazenave-Weissler [21] for the critical NLS, and is based on a fixed point theorem. The strategy is standard but we will give the details of the proof for completeness.

Proof. We will show local existence and uniqueness, give a blow-up alternative and then also prove decay for global solutions.

Local Existence: Define the solution operator

$$\Phi_{u_0}(v) := e^{t\Delta_m}u_0 + \int_0^t e^{(t-s)\Delta_m}F(v)ds.$$

We will use Banach's Fixed Point Theorem by showing $\Phi_{u_0}(\cdot)$ is a contraction mapping on $B_{a,b} := \{v \text{ on } I \times [0,\infty) : \|v\|_{L^4(I;rL^4)} \le a, \|v\|_{L^\infty(I;X^2(\mathbb{R}^2))} \le b\}$, a,b to be determined. In what follows, standard pointwise estimates based on the Taylor expansions of trigonometric functions and the Mean Value Theorem are employed. We will first show that $\Phi_{u_0}(\cdot)$ maps $B_{a,b}$ to itself (for a particular choice of a, b). Set $M := \|u_0\|_{X^2}$. Using Duhamel's formula, the space-time estimates, Hölder's inequality and the assumption on the evolution of the initial data:

$$\begin{aligned}
\|\Phi(u)\|_{L^\infty X^2} &\lesssim M + \|\frac{F}{r}\|_{L^2L^1} + \|F_r\|_{L^2L^1} \lesssim \|\frac{u^3}{r^3}\|_{L^2L^1} + \|\frac{u_r u^2}{r^2}\|_{L^2L^1} \\
&\lesssim M + \|u\|_{L^\infty X^2} \cdot \|u\|^2_{L^4rL^4} \lesssim M + b \cdot a^2 \\
\Rightarrow \|\Phi(u)\|_{L^4rL^4} &< \epsilon + \|F(u)\|_{L_t^{4/3}rL_r^{4/3}(I)} \lesssim \epsilon + \|u\|^3_{L_t^4L_r^4} \lesssim \epsilon + a^3
\end{aligned}$$

Denote by C the largest of all the implied constants, and make the following choices: pick $b = 2CM$ and choose a such that $C \cdot a^2 \le \frac{1}{2}$. Then pick $\epsilon = \frac{a}{2}$. Under these choices, Φ maps $B_{a,b}$ to itself.

Note that the intersection space can be equipped with the metric $d(u,v) = \|u - v\|_{L^4rL^4} + \|u - v\|_{L^\infty X^2}$ which makes the space $L^\infty X^2 \cap L^4rL^4$ complete. It remains to be shown that the mapping above is a contraction:

$$\begin{aligned}
\|\Phi(u) - \Phi(v)\|_{L^\infty X^2} &\lesssim \left\|\frac{F(u) - F(v)}{r}\right\|_{L^{4/3}L^{4/3}} + \|(F(u) - F(v))_r\|_{L^{4/3}L^{4/3}} \\
&\lesssim \left\|\frac{u - v}{r}\frac{(|u| + |v|)^2}{r^2}\right\|_{L^{4/3}L^{4/3}} + \left\||v_r|\frac{|u - v|}{r}\frac{(|u| + |v|)}{r}\right\|_{L^{4/3}L^{4/3}} \\
&\lesssim \|u - v\|_{L^4rL^4}\|u + v\|^2_{L^4rL^4} + \|v_r\|_{L^4rL^4}\|u - v\|_{L^4rL^4}\||u| + |v|\|_{L^4rL^4} \\
&\le C(\|u\|_{L^4rL^4}, \|v\|_{L^4rL^4}, \|v\|_{L^\infty X^2})\ \|u - v\|_{L^4rL^4},
\end{aligned}$$

and $\|\Phi(u)-\Phi(v)\|_{L^4rL^4}$ is treated the same way. The constant C above, depends in a multiplicative way on $\|u\|_{L^4rL^4}, \|v\|_{L^4rL^4}$; hence, by further shrinking ϵ we can arrange for Φ to be a contraction. Continuity in time follows again directly from the space-time estimates.

Maximal Interval of Existence

If $u^{(1)}, u^{(2)}$ are solutions of (2.6) on the interval $I=[t_0,T)$ with the same initial data at t_0, then $u^{(1)} \equiv u^{(2)}$ on I. To see that, for any τ we can partition $[0,\tau)$ into finitely many subintervals I_j, such that $\|u^{(i)}\|_{L^4rL^4(I_j)} \leq 2\epsilon$. Then, if $t_0 \in I_{j_0}$ for some j_0, the uniqueness of the fixed point in the proof above, yields an interval $\hat{I} \ni t_0$, where $u^{(1)} \equiv u^{(2)}$. A continuation argument proves uniqueness in all of I. This allows us to associate a maximal interval to given initial data, i.e., $I(u_0)=[0,0+T_{max}(u_0)),\ T_{max}(u_0)>0$, where

$$T_{max}(u_0)=\sup\{T(u_0)>0 : \text{there is a solution of (2.6) on } [t_0,t_0+T(u_0)]\}$$

Blow-Up Criterion: If $T_{\max}(u_0)<+\infty$, then $\|u\|_{L_t^4([0,T_{\max}(u_0));rL_r^4)}=+\infty$.

Assume, for contradiction, that $T_{max}(u_0)<+\infty$ but $\|u\|_{L^4rL^4([0,T_{max}(u_0))}<+\infty$. For some fixed $\tilde{\epsilon}$, subdivide the interval $[0,T_{max}(u_0))$ into a finite number of intervals I_j so that $\|u\|_{L^4rL^4(I_j)} \leq \tilde{\epsilon}$. Picking a time τ close to $T_{max}(u_0)$ (how close is to be determined) and solving the equation on $[\tau,T_{max}(u_0))$ using Duhamel's formula: $u(t)=e^{(t-\tau)\Delta_m}u(\tau)+\int_\tau^t e^{(t-s)\Delta_m}F(u(s))ds$, and solve for $e^{(t-\tau)\Delta_m}u(\tau)$. To estimate $\|e^{(t-\tau)\Delta_m}u(\tau)\|_{L^4rL^4([\tau,T_{max}(u_0)))}$: arguing as before

$$\|e^{(t-\tau)\Delta_m}u(\tau)\|_{L^4rL^4([\tau,T_{max}(u_0)))} \leq \|u\|_{L^4rL^4([\tau,T_{max}(u_0)))} + \ C\|u\|^3_{L^4rL^4([\tau,T_{max}(u_0)))},$$

where the constant C is independent of τ, T_{max}. Arranging for τ to be such that $C\|u\|^3_{L^4rL^4([\tau,T_{max}(u_0)))} \leq \|u\|_{L^4rL^4([\tau,T_{max}(u_0)))} < \tilde{\epsilon}$, with $\tilde{\epsilon} \leq \frac{\epsilon}{4}$ (where ϵ is in the existence proof) we get: $\|e^{(t-\tau)\Delta_m}u(\tau)\|_{L^4rL^4([\tau,T_{max}(u_0)))} \leq \frac{\epsilon}{2}$. Thus, by continuity, we can find ϵ_0 such that

$$\|e^{(t-\tau)\Delta_m}u(\tau)\|_{L^4rL^4([\tau,T_{max}(u_0)+\epsilon_0)))} < \epsilon,$$

and hence, the solution extends past $T_{max}(u_0)$, which contradicts the assumed maximality of this lifespan.

Proof of the dissipation result:

We will compare with the linear evolution: to begin, observe that by assumption, for any $\epsilon > 0$, there is a $T > 0 : \|u\|^3_{L^4 r L^4([T,+\infty))} < \|u\|_{L^4 r L^4([T,+\infty))} < \frac{\epsilon}{2}$.
We first show that for such a T, $\|u(t) - e^{(t-T)\Delta_m} u(T)\|_{X^2} \to 0$, as $t \to +\infty$ which would imply the statement since

$$\|u(t)\|_{X^2} \leq \|u(t) - e^{(t-T)\Delta_m} u(T)\|_{X^2} + \|e^{(t-T)\Delta_m} u(T)\|_{X^2}$$

and the last term goes to zero (for all $T > 0$) by Young's inequality using the explicit kernel for the linear evolution and the density of $L^1 \cap L^2$ in L^2. In particular, for every ϵ, there is $N > 0$ such that, for all $n > N : \|\phi_n - u(T)\|_{X^2} < \frac{\epsilon}{4}$, where $\phi_n \in X^1 \cap X^2$. However for data $\phi_n \in X^1$, putting the kernel in X^2 in the application of Young's inequality, we get $\|e^{(t-T)\Delta_m} \phi_n\|_{X^2} < \frac{\epsilon}{4}, t > \tilde{T}$, for all n, some $\tilde{T}$ corresponding to this ϵ, while for all times t, we get $\|e^{(t-T)\Delta_m}(u(T) - \phi_n)\|_{X^2} < \frac{\epsilon}{4}$, by (2.8) and density.

Define

$$w(t) = u(t) - e^{(t-T)\Delta_m} u(T).$$

Next, rearranging and using $w(T) = 0$ we get

$$u(t) - w(t) = e^{(t-T)\Delta_m}[u(T) - w(T)]$$

and thus, w satisfies

$$w_t = \Delta_m w + F(u)$$

(since u is a solution). Writing Duhamel's formula with initial time $t_0 = T$ and again using $w(T) = 0$, and (2.10) :

$$w(t) = \int_T^t e^{(t-s)\Delta_m} F(u) ds.$$

$$\|u(t) - e^{(t-T)\Delta_m} u(T)\|_{X^2} = \|w(t)\|_{X^2} \leq \|w\|_{L^\infty X^2} \leq \|F(u)\|_{L^{4/3} r L^{4/3}}$$
$$\lesssim \|u\|^3_{L^4 r L^4([T,t])} \lesssim \epsilon$$

completing the proof, since ϵ was arbitrary.

$\square$

Notice that our local theory combined with the previous section on the equivalence of the X^2 and the energy topology implies that if $u_0 \in E_0$ the boundary conditions persist in time, i.e. $u(t,\cdot) \in E_0$ throughout its lifespan.

2.2.3 Stability under perturbations

An important consequence of the local existence proof, is the following Perturbation Theorem on which much of this work is relying. We are interested in developing a *stability* theory, in the sense that we would like to prove the existence of a solution to (2.6), given an approximate one. This will be done in two steps, the first one (Short-time perturbations) assuming some "smallness", and the second one (Long-time perturbations), without the smallness assumption, iterating the argument of the first step. To the best of our knowledge, theorems of this type were first proved by Colliander-Keel-Staffilani-Tao-Takaoka [23] (and later improved and simplified by Tao-Visan, cf. [124].) In what follows, define $z_0 := z(t=0)$. Moreover, we will be using the standard notation $\|z\|_{S(I)}$ for admissible, and $\|z\|_{N(I)}$ for the dual norms.

Theorem 2.6. *(Short-time perturbations)*
Let $I \subset \mathbb{R}$ be a time interval of the form $[0,T)$. Let $\tilde{u}$ be defined on $I \times [0,\infty)$ and satisfy

$$\tilde{u}_t - \Delta_m \tilde{u} + F(\tilde{u}) = e,$$
$$\|\tilde{u}\|_{L^\infty(I;X^2)} < +\infty, \tag{2.12}$$

and let $u_0 \in X^2$. Assume also the smallness conditions

$$\|\tilde{u}\|_{L^4(I;rL^4)} \leq \epsilon_0 \tag{2.13}$$

$$\|e^{t\Delta_m}(u_0 - \tilde{u}_0)\|_{L^4(I;rL^4)} \leq \epsilon \tag{2.14}$$

$$\|e\|_{L^{4/3}(I;rL^{4/3})} \leq \epsilon \tag{2.15}$$

for some $0 < \epsilon \leq \epsilon_0$. Then, there exists a solution of (2.6) on $I \times [0, \infty)$ with $u(0) = u_0$ satisfying

$$\|u - \tilde{u}\|_{L^4(I;rL^4)} \lesssim \epsilon. \tag{2.16}$$

Remark 2.7. *By* (2.9), *the assumption $\|e^{t\Delta_m}(u_0 - \tilde{u}_0)\|_{L^4 rL^4(I)} \leq \epsilon$ is redundant if $\|u_0 - \tilde{u}_0\|_{X^2}$ is small.*

Proof. By the wellposedness theory reviewed above, it suffices to prove (2.16) as an *a priori* estimate. Let $w = u - \tilde{u}$. Then, w satisfies the following initial value problem

$$\begin{cases} w_t - \Delta_m w = F(\tilde{u} + w) - F(\tilde{u}) - e, \\ w_0 = u_0 - \tilde{u}_0 \end{cases}$$

For $t \in I$, define $A(t) := \|F(\tilde{u} + w) - F(\tilde{u})\|_{L^{4/3} rL^{4/3}([0,t])}$.

We have $F(\tilde{u} + w) - F(\tilde{u}) = \frac{m^2}{r^2} w + \frac{m^2}{2r^2}[\sin(2\tilde{u}) - \sin[2(\tilde{u} + w)]]$.
Using once more the pointwise bounds coming from the Taylor expansion:

$$\|F(\tilde{u} + w) - F(\tilde{u})\|_{L^{4/3} rL^{4/3}} \lesssim \|\frac{(|w|\tilde{u}^2 + |\tilde{u}|w^2 + |w|^3)}{r^3}\|_{L^{4/3} L^{4/3}}$$

$$\lesssim \|w\|^3_{L^4 rL^4} + \|\frac{\tilde{u}^2 w}{r^3}\|_{L^{4/3} L^{4/3}} + \|\frac{\tilde{u} w^2}{r^3}\|_{L^{4/3} L^{4/3}}$$

and this by Hölder's:

$$A(t) \lesssim \|w\|^3_{L^4 rL^4} + \|w\|^2_{L^4 rL^4} \|\tilde{u}\|_{L^4 rL^4} + \|w\|_{L^4 rL^4} \|\tilde{u}\|^2_{L^4 rL^4}. \tag{2.17}$$

Also, from the integral formula for w, the space-time estimates and the assumptions

(2.14) and (2.15),

$$\|w\|_{L^4 r L^4(I)} \lesssim \epsilon + A(t). \tag{2.18}$$

Combining (2.17) and (2.18), by a continuity argument, picking ϵ_0 small enough, we can deduce

$$A(t) \lesssim \epsilon$$

which finishes the proof of 2.16. □

Subdividing an arbitrary interval I so that the smallness assumption (2.13) applies, and by verifying the assumptions of the previous theorem, we can keep iterating the process to cover the whole interval, having only to adjust ϵ a finite number of times; in particular, we can prove the following:

Theorem 2.8. *(Long-time perturbations) Let $I = [0, T)$, a time interval. Let $\tilde{u}$ be defined on $I \times [0, \infty)$ and satisfy*

$$\tilde{u}_t - \Delta_m \tilde{u} + F(\tilde{u}) = e, \text{ in } I \times \mathbb{R}^2$$

$$\|\tilde{u}\|_{L^\infty(I;X^2)} \leq M \tag{2.19}$$

$$\|\tilde{u}\|_{L^4(I;rL^4)} \leq L \tag{2.20}$$

Let u_0 be such that

$$\|u_0 - \tilde{u}_0\|_{X^2} \leq M' \tag{2.21}$$

for some constants $M, M', L > 0$. Assume also the smallness conditions

$$\|e^{(t-t_0)\Delta_m}(u_0 - \tilde{u}_0)\|_{L^4(I;rL^4)} \leq \epsilon \tag{2.22}$$

$$\|e\|_{L^{4/3}(I;rL^{4/3})} \leq \epsilon \tag{2.23}$$

for some $0 < \epsilon \leq \epsilon_1 = \epsilon_1(M, M', L)$. Then, there exists a solution of (2.6) with $u(0) = u_0$ satisfying

$$\|u - \tilde{u}\|_{L^4 r L^4(I)} \leq C(M', M, L)M' \tag{2.24}$$

Proof. Subdivide I into $J = J(\epsilon_0, L)$ subintervals $I_j = [t_j, t_{j+1}], 0 \leq j < J$ such that

$\|\tilde{u}\|_{L^4 r L^4(I_j)} \leq \epsilon_0$, where $\epsilon_0 = \epsilon_0$ is as in the previous theorem.
Consider the first subinterval $I_1 = [0, t_1]$ (assume for simplicity $t_0 = 0$. By the previous theorem, for ϵ small enough (depending only on M') we have

$$\|u - \tilde{u}\|_{L^4 r L^4(I_1)} \leq C(M')M' \tag{2.25a}$$

$$\|F(u) - F(\tilde{u})\|_{L^{4/3} r L^{4/3}(I_1)} \leq C(M')\epsilon \tag{2.25b}$$

Now, to proceed with I_2 : in order to apply the Short-time Perturbations Theorem 2.6, we need to verify the stated assumptions.
To estimate $\|u(t_1) - \tilde{u}(t_1)\|_{X^2}$: writing Duhamel's formula for $w = u - \tilde{u}$ on $I_1 = [0, t_1]$, for $t = t_1$ we get:

$$\|w(t_1)\|_{X^2} = \|u(t_1) - \tilde{u}(t_1)\|_{X^2} \leq \|e^{t_1 \Delta_m} w(0)\|_{X^2}$$
$$+ \| \int_0^{t_1} e^{(t_1 - s)\Delta_m} [F(u) - F(\tilde{u})] ds\|_{X^2} + \| \int_0^{t_1} e^{(t_1 - s)\Delta_m} e ds\|_{X^2}.$$

To estimate $\| \int_0^{t_1} e^{(t_1 - s)\Delta_m} [F(u) - F(\tilde{u})] ds\|_{X^2}$, we use (2.10) to get $\|F(u) - F(\tilde{u})\|_{L^{4/3} r L^{4/3}([0, t_1])} \leq C_1(M')\epsilon$ by the result for the first interval.

By (2.10): $\| \int_0^{t_1} e^{(t_1 - s)\Delta_m} e ds\|_{X^2} \leq \|e\|_{L^{4/3} r L^{4/3}([0, t_1])} \leq \|e\|_{L^{4/3} r L^{4/3}(I)} \leq \epsilon$.

For $\|e^{t_1 \Delta_m} w(0)\|_{X^2} \leq \|e^{t \Delta_m} w(0)\|_{L^\infty X^2([0, t_1])} \leq \|w(0)\|_{X^2} = \|u(0) - \tilde{u}(0)\|_{X^2} \leq M'$, by assumption. Combining all the above estimates

$$\|w(t_1)\|_{X^2} \leq C_1(M')\epsilon + M' + \epsilon.$$

Trivially by the assumption, $\|e\|_{L^{4/3} r L^{4/3}(I_2)} \leq \epsilon$, so it remains to estimate

$$\|e^{(t - t_1)\Delta_m} w(t_1)\|_{L^4 r L^4(I_2)}.$$

To this end, using Duhamel's formula on $[0, t_1]$, for $t = t_1$, we get:

$$\|e^{(t-t_1)\Delta_m} w(t_1)\|_{L^4 r L^4(I_2)} = \|e^{(t-t_1)\Delta_m}[e^{t_1\Delta_m} w(0) + \int_0^{t_1} e^{(t_1-s)\Delta_m}[F(u) - F(\tilde{u})]ds$$

$$+ \int_0^{t_1} e^{(t_1-s)\Delta_m} e ds]\|_{L^4 r L^4(I_2)}$$

$$\leq \|e^{(t-t_1)\Delta_m} e^{t_1\Delta_m} w(0)\|_{L^4 r L^4(I_2)} + \left\| e^{(t-t_1)\Delta_m} \int_0^{t_1} e^{(t_1-s)\Delta_m}[F(u) - F(\tilde{u})]ds \right\|_{L^4 r L^4(I_2)}$$

$$+ \left\| e^{(t-t_1)\Delta_m} \int_0^{t_1} e^{(t_1-s)\Delta_m} e ds \right\|_{L^4 r L^4(I_2)}.$$

We can see that

$\int_0^{t_1} e^{(t_1-s)\Delta_m}[F(u) - F(\tilde{u})]ds, \int_0^{t_1} e^{(t_1-s)\Delta_m} e ds$ are in rL^2. Thus, by (2.8)

$$\leq \|e^{t\Delta_m} w(t_0)\|_{L^4 r L^4(I_2)} + \| \int_0^{t_1} e^{(t_1-s)\Delta_m}[F(u) - F(\tilde{u})]ds\|_{rL^2} + \| \int_0^{t_1} e^{(t_1-s)\Delta_m} e ds\|_{rL^2}$$

$$\leq \|e^{t\Delta_m} w(t_0)\|_{L^4 r L^4(I_2)} + \|F(u) - F(\tilde{u})\|_{L^{4/3} r L^{4/3}(I_1)}$$

$$+ \|e\|_{L^{4/3} r L^{4/3}(I_1)} \leq 2\epsilon + C_1(M')\epsilon$$

employing (2.10) (on I_1) and the same arguments as before. Hence, we can apply the Short-time Perturbations Theorem 2.6 to conclude the result for I_2 as well. Obviously, we can keep iterating the process to cover the whole interval having only to adjust ϵ a finite number of times and thus, complete the proof. $\square$

Using the last theorem we can easily deduce continuous dependence on initial data, by the following argument: choose $u_0 = u_{0,n}, u = u_n, u_{0,n} \xrightarrow{X^2} \tilde{u}_0$, for n that are large enough for

$$\|u_{0,n} - \tilde{u}_0\|_{X^2} \leq \epsilon \leq \epsilon_1(M, L),$$

where ϵ_1 is the one required by the previous theorem.

Also, applying (2.9): $\|e^{(t-t_0)\Delta_m}(u_{0,n} - \tilde{u}_0)\|_{L^4 r L^4(I)} \lesssim \epsilon$; since u_n are solutions of

(2.6), $e \equiv 0$. So the above theorem affirms that the solutions u_n exist on I, and

$$\|u_n - \tilde{u}\|_{L^4 r L^4(I)} \xrightarrow{n\to\infty} 0.$$

2.2.4 Profile decomposition

The main goal of this paragraph is to prove the following proposition which is the main tool (along with the Perturbation Theorem) used to establish global existence and decay. Profile Decompositions have become one of the main tools in the treatment of global behaviour for critical equations. The idea is to characterize the loss of compactness in some embedding, and in some way recover some compactness. It can be traced back to ideas in [98], [13], [121], [117] and their modern "evolution" counterparts [3], [79] and [80].

Proposition 2.9. *Let $\{u_n\}_n$ be a bounded sequence in X^2. Then, after possibly passing to a subsequence (in which case, we suppress notation and rename it u_n again), there exist a family of radial functions $\{\phi^j\}_{j=1}^{\infty} \subset X^2$ and scales $\lambda_n^j > 0$ such that:*

$$u_n(x) = \sum_{j=1}^{J} \phi^j(\frac{x}{\lambda_n^j}) + w_n^J(x), \tag{2.26}$$

$w_n^J \in X^2$ is such that:

$$\lim_{J\to\infty} \limsup_{n} \|e^{t\Delta_m} w_n^J\|_{L_t^4 r L_r^4} = 0, \tag{2.27}$$

$$w_n^J(\lambda_n^j x) \rightharpoonup 0, \quad in \ \ X^2, \ \ \forall j \leq J. \tag{2.28}$$

Moreover, the scales are asymptotically orthogonal, in the sense that

$$\frac{\lambda_n^j}{\lambda_n^i} + \frac{\lambda_n^i}{\lambda_n^j} \to +\infty, \ \ \forall i \neq j \tag{2.29}$$

Furthermore, for all $J \geq 1$ *we have the following decoupling properties:*

$$\|u_n\|_{X^2}^2 = \sum_{j=1}^{J} \|\phi^j\|_{X^2}^2 + \|w_n^J\|_{X^2}^2 + o_n(1) \tag{2.30}$$

$$E(u_n) = \sum_{j=1}^{J} E(\phi^j) + E(w_n^J) + o_n(1). \tag{2.31}$$

The procedure through which one establishes such a decomposition has become standard by now, for example see [3],[81], thus we will only present the equation-specific parts of the argument.

There are two general roadmaps to follow in establishing such a decomposition. To get the convergence of the error w_n^J in the appropriate space-time norm, one can either use directly a refinement of space-time estimates on the linear propagator, or a refinement on a Sobolev inequality through which the refinement of the space-time estimates will follow, making use of interpolation arguments. The first approach is a more modern one, yet it would require more work in our case. Modification of arguments used in the Schrödinger case cannot directly be made due to the lack of an analogue of the restriction theorems used. For the second approach, a remark is that the case of dimension two is very special due to the lack of the usual embeddings.

Our strategy evolves around a refinement of a Sobolev inequality in which this dimension issue is directly addressed. We first establish (2.27) for the homogeneous linear heat equation for radial functions in higher dimensions. We make use of a refined Sobolev inequality, first proved in [3] for $d = 3$ and later generalized to $d > 3$ in [14]. Then, through an isomorphism between $\dot{H}^1$ and X^2, we connect the above estimate to our spaces for the 2d problem, and use interpolation again to obtain the desired convergence in the norm in which the blow-up criterion was stated.

Definition 2.10. *We call a pair* (q, p) L^2*-admissible in dimension* d*, if the following relation is satisfied*

$$\frac{2}{q} + \frac{d}{p} = \frac{d}{2}, \tag{2.32}$$

and $\dot{H}^1$-admissible if

$$\frac{2}{q} + \frac{d}{p} = \frac{d-2}{2}. \tag{2.33}$$

We define the following Besov norm on L^2 :

$$I_k(f) := \left(\int_{2^k \le |\xi| \le 2^{k+1}} |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \qquad \|f\|_B := \sup_{k \in \mathbb{Z}} I_k(f).$$

The following refinement of the Sobolev inequality was proved in [14] (Lemma 3.1)

Lemma 2.11. *(Refined Sobolev) For $d \ge 3$ there is a constant $C = C(d) > 0$ such that for every $u \in \dot{H}^1(\mathbb{R}^d)$, we have*

$$\|u\|_{L^p} \le C \|\nabla u\|_{L^2}^{\frac{2}{p}} \|\nabla u\|_B^{1-\frac{2}{p}}, \tag{2.34}$$

where $p = \frac{2d}{d-2}$ (sometimes denoted by 2^*).

The next result, proved originally in [59], provides a decomposition of bounded sequences in $L^2(\mathbb{R}^d)$ (for a different, but equivalent Besov norm). Here, we specialize to radial functions.

Proposition 2.12. *Let $\{f_n\}_n$ be a bounded sequence of radially symmetric functions in $L^2(\mathbb{R}^d), d \ge 3$. Then, there exist a subsequence (still denoted by $\{f_n\}_n$), a sequence of scales $\{\lambda_n^j\}_n \subset (0, \infty)$ satisfying (2.29), and bounded radial $\{g^j\}_j, \{r_n^J\}_n \subset L^2(\mathbb{R}^d)$, such that for every $J \ge 1, x \in \mathbb{R}^d$*

$$f_n(x) = \sum_{j=1}^{J} \frac{1}{(\lambda_n^j)^{d/2}} g^j(\frac{x}{\lambda_n^j}) + r_n^J(x), \tag{2.35a}$$

$$\|f_n\|_{L^2}^2 = \sum_{j=1}^{J} \|\frac{1}{(\lambda_n^j)^{d/2}} g^j(\frac{x}{\lambda_n^j})\|_{L^2}^2 + \|r_n^J\|_{L^2}^2 + o_n(1), \tag{2.35b}$$

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|r_n^J\|_B = 0. \tag{2.35c}$$

Applying the above result to a bounded radially symmetric sequence $\{v_n\} \subset \dot{H}^1(\mathbb{R}^d)$, we conclude that there is a subsequence (again denoted by $\{v_n\}$), a family of scales λ_n^j as before (satisfying the orthogonality property (2.29)), and a family of (bounded in $\dot{H}^1$) radial functions ψ^j such that

$$v_n(x) = \sum_{j=1}^{J} \psi_n^j(x) + \tilde{w}_n^J(x), \forall J \geq 1,, \tag{2.36}$$

where $\psi_n^j(x) := \frac{1}{(\lambda_n^j)^{\frac{d}{2}-1}} \psi^j(\frac{x}{\lambda_n^j})$,

$$\|v_n\|_{\dot{H}^1}^2 = \sum_{j=1}^{J} \|\psi^j\|_{\dot{H}^1}^2 + \|\tilde{w}_n^J\|_{\dot{H}^1}^2 + o_n(1)$$

and

$$\limsup_n \|\nabla \tilde{w}_n^J\|_B \xrightarrow{J \to \infty} 0. \tag{2.37}$$

Let us consider the homogeneous heat equation on $\mathbb{R}^d$

$$v_t = \Delta v \tag{2.38}$$

and denote the linear propagator by $S(t) := e^{t\Delta}$, i.e. the solution with given initial data v_0, is $v(t) = S(t)v_0$. Evolving (2.36) by the linear propagator we get:

$$S(t)v_n = \sum_{i=1}^{J} S(t)\psi_n^j + S(t)\tilde{w}_n^J.$$

Our first goal is to estimate $S(t)\tilde{w}_n^J$ in an appropriate space-time norm. If σ is a function on $\mathbb{R}^d$, we define $\sigma(D)$ by

$$\widehat{\sigma(D)f}(\xi) := \sigma(\xi)\hat{f}(\xi).$$

Also define $\sigma_k(\xi) := \chi_{2^k \leq |\xi| \leq 2^{k+1}}(\xi), k \in \mathbb{Z}$. Then if z a solution to (2.38), by commutation of Fourier multipliers with derivatives, $\sigma_k(\xi)z$ is also a solution to (2.38). By dissipation we have, for any k, that

$$\|\nabla(\sigma_k(\xi)z(t))\|_{L^2} \leq \|\nabla(\sigma_k(\xi)z_0)\|_{L^2}.$$

Using Plancherel's identity and properties of the Fourier transform and the definition of multipliers:

$$\|\nabla(\sigma_k z)\|_{L^2} = \|\widehat{\nabla(\sigma_k z)}\|_{L^2} = \||\xi|\sigma_k \hat{z}\|_{L^2} = I_k(\nabla z)$$

Hence, by taking supremum in k,

$$\|\nabla z(t)\|_B \leq \|\nabla z_0\|_B.$$

Applying this general observation to the evolution starting with initial data $\tilde{w}_n^J$:

$$\|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x} \leq \|\nabla \tilde{w}_n^J\|_B.$$

Then, due to (2.37) we conclude

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x} = 0. \tag{2.39}$$

For every $t > 0$, (2.34) gives:

$$\begin{aligned}
\|S(t)\tilde{w}_n^J\|_{L_x^{\frac{2d}{d-2}}} &\lesssim \|\nabla(S(t)\tilde{w}_n^J)\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_B^{1-\frac{2(d-2)}{2d}} \\
&\lesssim \|\nabla \tilde{w}_n^J\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_B^{1-\frac{2(d-2)}{2d}} \\
\Rightarrow \|S(t)\tilde{w}_n^J\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} &\lesssim \|\nabla \tilde{w}_n^J\|_{L^2}^{\frac{2(d-2)}{2d}} \cdot \|\nabla(S(t)\tilde{w}_n^J)\|_{L_t^\infty B_x}^{1-\frac{2(d-2)}{2d}}.
\end{aligned}$$

Since, by assumption, $\|\nabla \tilde{w}_n^J\|_{L^2}$ is uniformly bounded, using (2.39):

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|S(t)\tilde{w}_n^J\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} = 0. \tag{2.40}$$

The radial Laplacian in dimension $d = 2m+2$ is $\Delta_m = \partial_{rr} + \frac{2m+1}{r}\partial_r = \frac{1}{r^{2m+1}}\partial_r(r^{2m+1}\partial_r)$. Letting $v = \frac{u}{r^m}$, if u is radial and $u_t = \Delta_m u$ in $\mathbb{R}^2$, then v solves $v_t = \Delta v$. It is straightforward to verify, using the Hardy inequality in dimension $d = 2m+2$, that this map is an isomorphism between our space X^2 and $\dot{H}^1_{\text{rad}}(\mathbb{R}^{2m+2})$ (e.g., see Lemma 4 in [32]).

The reason we are using this transform is so that we can make use of (2.2.4) by connecting the spaces used in our LWP theory in two space dimensions with the ones involved in the preceding argument. First, we make the following observation: suppose (r, p) is an L^2-admissible pair in $d = 2$, i.e., $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$. Then

$$\|\frac{u}{r}\|^r_{L^r_t(I;L^p_r(rdr))} = \int_I \left(\int_0^\infty \frac{r^{1-p}}{r^{2m+2}}|u|^p r^{2m+1} dr\right)^{r/p}$$

Using the above transformation $u = v \cdot r^m$ we get:

$$\int_I \left(\int_0^\infty \frac{r^{1-p}}{r^{2m+2}} r^{mp} |v|^p r^{2m+1} dr\right)^{r/p} = \int_I \left(\int_0^\infty r^{p(m-1)-2m}|v|^p r^{2m+1} dr\right)^{r/p}.$$

Letting $p = \frac{2m}{m-1}$, and thus $r = 2m$, we observe that

$$\|\frac{u}{r}\|_{L^r_t(I;L^p_r(rdr))} = \|v\|_{L^r_t(I;L^p_r(\mathbb{R}^{2m+2}, r^{2m+1}dr)} \tag{2.41}$$

for this choice of (r,p) (which is an $\dot{H}^1-$admissible pair in dimension $2m+2$). This observation is the connecting link between the two-dimensional problem and the higher-dimensional estimates. So for u_n bounded in X^2, $v_n = \frac{u_n}{r^m}$ is bounded in $\dot{H}^1(\mathbb{R}^d)$ and we have (2.36). First, one has to show that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|S(t)\tilde{w}^J_n\|_{L^{2m}(I;L^{\frac{2m}{m-1}}(\mathbb{R}^{2m+2}, r^{2m+1}dr))} = 0. \tag{2.42}$$

For this we use interpolation and the previous result (2.40) for $d = 2m+2$. We have

$$\|S(t)\tilde{w}^J_n\|_{L^{2m}_t L^{\frac{2m}{m-1}}} \le \|S(t)\tilde{w}^J_n\|^{\frac{m-1}{m}}_{L^\infty_t L_r^{\frac{2m+2}{m}}} \cdot \|S(t)\tilde{w}^J_n\|^{\frac{1}{m}}_{L^2_t L_r^{\frac{2(m+1)}{m-1}}}$$

Taking $\lim_{J\to\infty} \limsup_{n\to\infty}$, and noting that the second term is uniformly bounded (by the standard space-time estimates for the heat equation, see, e.g., [60]), the claim (2.42) follows. Undoing the transformation $u_n = r^m v_n$ in (2.36) yields

$$u_n(x) = \sum_{j=1}^{J} \phi^j(\frac{r}{\lambda_n^j}) + w_n^J,\ w_n^J = r^m \tilde{w}_n^J.$$

Now, again by interpolation

$$\|\frac{u}{r}\|_{L_t^4 L_r^4} \leq \|\frac{u}{r}\|^{\frac{m}{2(m-1)}}_{L_t^{2m} L_r^{\frac{2m}{m-1}}} \cdot \|\frac{u}{r}\|^{\frac{m-2}{2(m-1)}}_{L_t^2 L_r^\infty}.$$

Invoking (2.42), we get (2.27), i.e.,

$$\lim_{J\to\infty} \limsup_n \|e^{t\Delta_m} w_n^J\|_{L_t^4 r L_r^4} = 0.$$

The rest of the proof of the profile decomposition follows the same arguments as in the references cited at the beginning of this section and is thus omitted. We still have to show the asymptotic energy splitting with this choice of a space, i.e.(2.31):

Proof. From now on, we will be systematically dropping the π factor in the definition of the energy. Expanding using the definition,

$$\begin{aligned} E(u_n) = \int_0^\infty [&\sum_{j=1}^{J} (\frac{1}{\lambda_n^j})^2 (\phi_r^j(\frac{r}{\lambda_n^j}))^2 + \sum_{j=1, i<j}^{J} \frac{1}{\lambda_n^j \lambda_n^i} \phi_r^j(\frac{r}{\lambda_n^j}) \phi_r^i(\frac{r}{\lambda_n^i}) + (w_{n,r}^J(r))^2 \\ &+ 2\sum_{j=1}^{J} w_{n,r}^J(r) \frac{1}{\lambda_n^j} \phi_r^j(\frac{r}{\lambda_n^j}) + \frac{m^2}{r^2} \sin^2(\sum_{j=1}^{J} \phi^j(\frac{r}{\lambda_n^j}) + w_{n,r}^J(r))] r dr, \end{aligned}$$

We want to show that $E(u_n) - \sum_{j=1}^{J} E(\phi^j) - E(w_n^J)$ is $o_n(1)$.

Expanding this out

$$\int_0^\infty \sum_{j=1,i<j}^J \frac{1}{\lambda_n^j\lambda_n^i}\phi_r^j(\frac{r}{\lambda_n^j})\phi_r^i(\frac{r}{\lambda_n^i})rdr + 2\int_0^\infty [\sum_{j=1}^J w_{n,r}^J(r)\frac{1}{\lambda_n^j}\phi_r^j(\frac{r}{\lambda_n^j})]rdr$$
$$+\int_0^\infty \frac{m^2}{r^2}[\sin^2(\sum_{j=1}^J \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)) - \sum_{j=1}^J \sin^2(\phi^j(\frac{r}{\lambda_n^j})) - \sin^2(w_n^J(r))]rdr$$

For the first two sums it suffices to look at single pairs and show they all are $o_n(1)$. Using an approximation argument, we can assume every function involved is in C_c^∞ and that all the supports lie in some ball $B(0,R)$. The argument is standard so we only give a sketch: for the first sum, we just change variables, assuming without loss of generality $s_n^{i,j} := \frac{\lambda_n^j}{\lambda_n^i}$ goes to zero. Then, by Hölder, each term in the sum is bounded by $\|\phi_r^i\|_{X^2}\int_0^{s_n^{i,j}R}(\phi_r^j(r))^2rdr = o_n(1)$. For the second one, change variables again and employ the weak convergence of $w_n^J(\lambda_n^j r)$ to zero, for all $j \leq J$, i.e., (2.28). For the rest we will use of the trigonometric identity

$$\sin^2(a+b) - \sin^2(a) - \sin^2(b) = \frac{1}{2}\sin(2a)\sin(2b) - 2\sin^2(a)\sin^2(b)$$

and the derived from it inequality

$$|\sin^2(a+b) - \sin^2(a) - \sin^2(b)| \leq C|a||b| \tag{2.43}$$

for some $C > 0$.

We want to show that

$$\left|\int_0^\infty \frac{m^2}{r^2}[\sin^2(\sum_{j=1}^J \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)) - \sum_{j=1}^J \sin^2(\phi^j(\frac{r}{\lambda_n^j})) - \sin^2(w_n^J(r))]rdr\right| = o_n(1)$$

Using (2.43) $J-1$ times, this can be reduced to showing the following two estimates:

$$\int_0^\infty \frac{|\phi^j(\frac{r}{\lambda_n^j})||\phi^i(\frac{r}{\lambda_n^i})|}{r^2} rdr = o_n(1), i \neq j \tag{2.44}$$

$$\int_0^\infty \frac{|w_n^J(r)|\phi^j(\frac{r}{\lambda_n^j})|}{r^2} rdr = o_n(1), \text{for any } j \leq J \tag{2.45}$$

The proof of (2.44) follows the same rescaling argument as before. For (2.45), since the only information we have for the scales is their relation to each other and since we cannot assume convergence to zero, infinity or even existence of the limit at all, a change of variables will not be enough. Observe however, that the obvious change of variables makes the term $|w_n^J(\lambda_n^j r')|$ appear, which suggests that somehow we have to use the weak convergence to zero of the term which appears in the absolute value. Because of the absolute values we can't directly obtain the result, yet the situation is not hopeless: a change of variables gives $\displaystyle\int_0^\infty |w_n^J(\lambda_n^j r')||\phi^j(r')|\frac{r'dr'}{(r')^2}$. Observe that by Hölder's inequality and the definition of the X^2-norm, the integrand is in $L^1(rdr)$, independent of n, since the sequence w_n^J is uniformly bounded in X^2.

Then, again by the fact that $\|\frac{\phi^j}{r}\|_{L^2(rdr)} < +\infty$, for every $\epsilon > 0$ we can find an $R = R(\epsilon)$ (which, since we will be working with very small ϵ, can be picked bigger than 1) such that,

$$\left(\int_{r\geq R} |\frac{\phi^j(r)}{r}|^2 rdr\right)^{1/2} < \frac{\epsilon}{3M},$$

$$\left(\int_{r\leq \frac{1}{R}} |\frac{\phi^j(r)}{r}|^2 rdr\right)^{1/2} < \frac{\epsilon}{3M},$$

where

$$M := \sup_n \|w_n^J\|_{X^2} < +\infty.$$

Now, we split the integral at hand in three regions, using (for all $\epsilon > 0$) this $R = R(\epsilon)$. The first and the last, by Hölder's and the choice of R, are $< \frac{\epsilon}{3}$, while for the one in the middle we have by the obvious bounds for r and the fact that $\frac{1}{R} < 1$ by choice, that

$$\int_{\frac{1}{R}\leq r\leq R}|w_n^J(\lambda_n^j r)||\phi^j(r)|\frac{rdr}{(r)^2}<\int_{\frac{1}{R}\leq r\leq R}|w_n^J(\lambda_n^j r)||\phi^j(r)|dr.$$

But, on a bounded domain $\Omega=[a,b], a>0$, it is very easy to see that a $u\in X^2([a,b])$ is a function in $H^1([a,b])$.
Indeed,

$$\begin{aligned}
\|u\|^2_{H^1([a,b])} &= \int_{a\leq r\leq b}|u_r|^2dr+\int_{a\leq r\leq b}|u|^2dr=\int_{a\leq r\leq b}\frac{|u_r|^2}{r}rdr+|\frac{u}{r}|^2r^2dr\\
&\leq \frac{1}{a}\int_{a\leq r\leq b}|u_r|^2rdr+b\int_{a\leq r\leq b}|\frac{u}{r}|^2rdr\\
&\leq C\left(\int_{a\leq r\leq b}|u_r|^2rdr+\int_{a\leq r\leq b}|\frac{u}{r}|^2rdr\right),\;\; C=\max\{\frac{1}{a},b\};
\end{aligned}$$

but this is exactly $C\|u\|_{X^2([a,b])}$.
Now, we can invoke the continuous embedding of $X^2([a,b])$ in $H^1([a,b])$ we just proved, and the compact one of $H^1([a,b])$ in $L^2([a,b])$, to get that $w_n^J(\lambda_n^j r)$ goes to 0 in $L^2([a,b])$.

With all this in hand, going back to the integral in question, we have that it is smaller by

$$\frac{2\epsilon}{3}+\left(\int_{\frac{1}{R}\leq r\leq R}|w_n^J(\lambda_n^j r)|^2dr\right)^{1/2}\cdot\left(\int_{\frac{1}{R}\leq r\leq R}|\phi^j(r)|^2dr\right)^{1/2},$$

which in turn is below $\frac{2\epsilon}{3}+\left(\int_{\frac{1}{R}\leq r\leq R}|w_n^J(\lambda_n^j r)|^2dr\right)^{1/2}\cdot\|\phi^j\|_{L^2}$. Taking n large enough, because of the strong convergence to zero that we proved earlier:

$$\left(\int_{\frac{1}{R}\leq r\leq R}|w_n^J(\lambda_n^j r|^2dr\right)^{1/2}<\frac{\epsilon}{3\|\phi^j\|_{L^2}},$$

which completes the proof. $\square$

2.3 Minimal blow-up solution

For $u_0 \in E_0$ we define

$$E_c = \inf\{E(u_0) : u \text{ solves (2.6) with } u(0) = u_0, \|u\|_{L^4 r L^4([0,T_{\max}))} = +\infty\};$$

i.e., we consider the infimum of the energies of initial data such that the corresponding solutions fail to be global or decay to zero, in the sense discussed in the local theory. Note that $T_{\max}$ can be finite (blow-up), or infinite (global but not decaying solution). Observe that Theorem 2.2 is equivalent to $E_c \geq 2E(Q)$. Also, a priori, $E_c > 0$. To see that, note that we have shown that for maps in this class the energy is comparable to the X^2-norm so small energy implies small (in X^2- norm) initial data, and by the space-time estimates $\|e^{t\Delta_m} u_0\|_{L^4 r L^4(\mathbb{R}^+)}$ is small, which in turn implies that $u(t)$ is global and decays to zero.

We will follow the contradiction approach of Kenig and Merle; we will assume $E_c < 2E(Q)$ and proceed in two main steps: first, we show that a unique critical element exists. A critical element is initial data with energy E_c giving rise to a solution that that either fails to exist globally or decay to zero. The proof is based on the profile decomposition (2.9) and the long-time perturbation theorem 2.8 (which allows us to construct true solutions from approximate ones, thus obtaining profile decompositions for the solution of the full non-linear equation), and the variational information coming from the energy. The second step, is the so called "rigidity" part. In this part, we show that such a critical element cannot exist, thus reaching a contradiction. This is the content of the next section. Unlike the results in the "dispersive" literature, where a lot of work is required, our rigidity part is trivial because of the dissipation of energy.

We now turn out attention to the existence of a critical element. In particular, we prove the following proposition, which is the main goal of this section:

Proposition 2.13. *Assume $E_c < 2E(Q)$. There exists a function $u_{0,c}$ in X^2 with $E(u_{0,c}) = E_c < 2E(Q)$, such that if $u_c(t,r)$ is the solution of (2.6) with initial data $u_{0,c}$ and maximal interval of existence $I = [0, T_{max}(u_{0,c}))$, then $\|u_c\|_{L^4 r L^4(I)} = +\infty$.*

Proof. Let $\{u_{0,n}\}_n$ be a sequence in X^2 such that $E(u_{0,n}) \to E_c, n \to \infty$ from above, and the corresponding solutions u_n of (2.6) with maximal intervals of existence $I_n = [0, T_{\max}(u_{0,n}))$ satisfy $\|u_n\|_{L^4 r L^4(I_n)} = +\infty$. By the comparability of the energy and the X^2-norm the sequence $\{u_{0,n}\}_n$ is bounded in X^2, just by the convergence of $E(u_{0,n})$. Thus, passing to a subsequence, if necessary, we have the following profile decomposition

$$u_{0,n}(r) = \sum_{j=1}^{J} \phi^j(\frac{r}{\lambda_n^j}) + w_n^J(r)$$

with the stated properties in Proposition 2.9.

For future reference, define a nonlinear profile $v^j : I^j \times [0,\infty) \to \mathbb{R}$ associated to ϕ^j, to be the maximal-lifespan solution to (2.2) with initial data ϕ^j.

For each $j, n \geq 1$, define $v_n^j : I_n^j \times [0,\infty) \to \mathbb{R}$, by $v_n^j(r,t) = v^j(\frac{t}{(\lambda_n^j)^2}, \frac{r}{\lambda_n^j})$, where we define $I_n^j := \{t \in \mathbb{R}^+ : \dfrac{t}{(\lambda_n^j)^2} \in I^j\}$. I.e., each v_n^j is a solution to (2.2) with initial data $v_n^j(0) = \phi^j(\frac{r}{\lambda_n^j})$.

We also have the energy decoupling in the limit

$$E(u_{0,n}) = \sum_{j=1}^{J} E(\phi^j) + E(w_n^J) + o_n(1), \ \forall J.$$

Taking $\overline{\lim}_n$, we get

$$E_c = \sum_{j=1}^{J} E(\phi^j) + \overline{\lim_n} E(w_n^J)$$

which by the positivity of every term implies $\displaystyle\sum_{j=1}^{J} E(\phi^j) \leq E_c$, for any J. This, in turn, for the same reason gives

$$\sup_j E(\phi^j) \leq E_c.$$

We eventually want to show that $\phi^j = 0, j \geq 2$ and $E(\phi^1) = E_c$. We consider the following possibilities:

Case 1: there is an $\epsilon > 0$ such that

$$\sup_j E(\phi^j) \leq E_c - \epsilon,$$

from which we will derive a contradiction. Since $\sup_j E(\phi^j) < E_c$, by the definition of the E_c, all the ϕ^j 's give rise to global and decaying to 0 solutions. Now, define the approximate solution (to $u_n(t)$ of the nonlinear equation with initial data $u_{0,n}$)

$$u_n^J(r,t) = \sum_{j=1}^{J} v_n^j(r,t) + e^{t\Delta_m} w_n^J(r).$$

What we want to show is that u_n^J is a good approximate solution to u_n (for n, J sufficiently large) in the sense of the Long-time Perturbations Theorem 2.8. This would imply that $u_n(t)$ is global, which is a contradiction.
First, to see that $\|u_n^J\|_{L^4rL^4(\mathbb{R}^+)} < +\infty$: for any $\epsilon > 0$, (2.27) provides a J such that

$$\begin{aligned}\overline{\lim_n}\|u_n^J\|_{L^4rL^4(\mathbb{R}^+)} &\leq \overline{\lim_n}\|\sum_{j=1}^{J} v_n^j\|_{L^4rL^4(\mathbb{R}^+)} + \overline{\lim_n}\|e^{t\Delta_m} w_n^J\|_{L^4rL^4(\mathbb{R}^+)} \\ &\leq \sum_{j=1}^{J} \|v^j\|_{L^4rL^4(\mathbb{R}^+)} + \epsilon.\end{aligned}$$

To conclude the claim, we will show that the latter norms are bounded uniformly in J. We can split the sum into two parts (for every fixed J); one over $1 \leq j \leq J_0$, and the rest. Let ϵ_0 be such that Theorem 2.5 affirms that if $\|u_0\|_{X^2} \leq \epsilon_0$, then the corresponding solution u is global and decays to zero, and pick J_0 such that

$$\sum_{j \geq J_0} E(\phi^j) \leq \epsilon_0,$$

which we can do again by the summability of the series; this shows that for $j \geq J_0$, the corresponding L^4rL^4-norms are uniformly (in J) bounded by some $M(\epsilon_0)$, and then choose the maximum bound among this M and the J_0-1 bounds for the others. Hence,

$$\|u_n^J\|_{L^4 r L^4(\mathbb{R}^+)} \leq C,$$

(for some C and n large enough).

We need to show that the assumptions in the Long-time Perturbations Theorem 2.8 hold. From the profile decomposition, $\|u_n^J(0) - u_n(0)\|_{X^2} = 0, \forall J, n$ and also, $\|e^{t\Delta_m}(u_n^J(0) - u_n(0))\|_{L^4 r L^4(\mathbb{R}^+)} = 0, \forall J, n$. The perturbed PDE for $u_n^J(t)$ is

$$\partial_t u_n^J - \Delta_m u_n^J = \sum_{j=1}^{J} F(v_n^j),$$

hence the error is given by

$$e_n^J = F(u_n^J) - \sum_{j=1}^{J} F(v_n^j),$$

where F is the nonlinear term $F(u) = \frac{m^2}{r^2}(u - \frac{\sin 2u}{2})$. We will show that the error is small in the dual norm $\|\cdot\|_{L_t^{4/3} r L_r^{4/3}}$, again for sufficiently large n and J.
By the explicit formula for F, the error is

$$e_n^J = \frac{m^2}{2r^2}(2u_n^J - \sin(2u_n^J) - \sum_{j=1}^{J}(2v_n^j - \sin(2v_n^j)).$$

For simplicity, define $W_n^J(r,t) := e^{t\Delta_m} w_n^J(r)$. We will make use of the following trigonometric relations:

$$\begin{aligned} |\sin(2u) + \sin(2v) - \sin(2u+2v)| &= |2\sin(2u)\sin^2(v) + 2\sin(2v)\sin^2(u)| \\ &\lesssim |u||v|^2 + |v||u|^2. \end{aligned}$$

Using the above estimate:

$$
\begin{aligned}
&|\sum_{j=1}^{J}\sin(v_n^j)+\sin(W_n^J)-\sin(\sum_{j=1}^{J}v_n^j+W_n^J)\pm\sin(\sum_{j=1}^{J}v_n^j)| \\
&\qquad\lesssim |\sum_{j=1}^{J}\sin(v_n^j)-\sin(\sum_{j=1}^{J}v_n^j)|+|W_n^J||\sum_{j=1}^{J}v_n^j|^2+|W_n^J|^2|\sum_{j=1}^{J}v_n^j|
\end{aligned}
$$

Define $A:=|W_n^J|\;|\sum_{j=1}^{J}v_n^j|^2+|W_n^J|^2\;|\sum_{j=1}^{J}v_n^j|,\;\; B:=|\sum_{j=1}^{J}\sin(v_n^j)-\sin(\sum_{j=1}^{J}v_n^j)|.$

By Hölder's:

$$
\begin{aligned}
\|A\|_{L^{4/3}rL^{4/3}} &\le \|W_n^J\|_{L^4rL^4}\;\|\sum_{j=1}^{J}v_n^j\|^2_{L^4rL^4}+\|W_n^J\|^2_{L^4rL^4}\;\|\sum_{j=1}^{J}v_n^j\|_{L^4rL^4} \\
&\le \|W_n^J\|_{L^4rL^4}(\sum_{j=1}^{J}\|v_n^j\|_{L^4rL^4})^2+\|W_n^J\|^2_{L^4rL^4}(\sum_{j=1}^{J}\|v_n^j\|_{L^4rL^4}).
\end{aligned}
$$

But, by (2.27)

$$
\lim_{J\to\infty}\limsup_{n\to\infty}\|W_n^J\|_{L^4rL^4}=0,
$$

and hence, by the scaling invariance of the L^4rL^4-norm,

$$
\lim_{J\to\infty}\limsup_{n\to\infty}\|A\|_{L^{4/3}rL^{4/3}}=0.
$$

For term B, again by adding and subtracting $\sin(\sum_{j=1}^{J-1}v_n^j)$ we get, using the trigonometric inequality:

$$
|\sum_{j=1}^{J-1}\sin(v_n^j)-\sin(\sum_{j=1}^{J-1}v_n^j)|+|v_n^J||\sum_{j=1}^{J-1}v_n^j|^2+|v_n^J|^2|\sum_{j=1}^{J-1}v_n^j|
$$

We will show how to treat the second term, and after that the procedure can be easily iterated. It consists of terms of the form $|v_n^j||v_n^j|^2$ and $|v_n^j|^2|v_n^j|$ (employing Young's inequality for the product terms $|v_n^j||v_n^i|$). We treat terms of the first type, namely $\| \frac{|v_n^j|^2}{r^2} \frac{|v_n^j|}{r} \|_{L^{4/3}L^{4/3}}$ (and the others follow in the same way).

As before, we can employ an approximation argument and therefore assume everything is smooth and compactly supported in space-time, say on $[0,T] \times [0,R]$. Without loss of generality, assume $s_n^{J,j} := \frac{\lambda_n^J}{\lambda_n^j} \to 0$. Changing variables (in space and time) and Hölder's inequality we get that the above is controlled by

$$\|v^j\|_{L^4 r L^4}^2 \, \|v^j\|_{L^4 r L^4([0,(s_n^{J,j})^2 T]\times[0,(s_n^{J,j})R])} \xrightarrow{n\to\infty} 0.$$

Everything else can be treated the same way, proving

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|e_n^J\|_{L^{4/3} r L^{4/3}} = 0.$$

Thus, we have shown that $u_n^J(t)$ is a good approximate solution and hence, by the Long-time Perturbations Theorem 2.8, $u_n(t)$ has to be global for sufficiently large n and J. As noted before, this contradicts the way the sequence of initial data has been picked. So, Case 1 led to a contradiction and the only remaining possibility is the alternative

Case 2:

$$\sup_j E(\phi^j) = E_c$$

This immediately implies (possibly after a relabeling) that $\phi^j = 0, j \geq 2$ and the profile decomposition simplifies to

$$u_{0,n}(r) = \phi^1\left(\frac{r}{\lambda_n^1}\right) + w_n^1(r).$$

By the energy splitting and the fact that $E(u_{0,n}) \to E_c$, we get

$$\overline{\lim_n} E(w_n^1) = 0. \tag{2.46}$$

By (2.46) and the comparability of the energy and the X^2-norm, defining $\tilde{u}_{0,n}(r) := u_{0,n}(\lambda_n^1 r)$ we get $\tilde{u}_{0,n} \to \phi^1$, strongly in X^2. As the proof shows, $E(\phi^1) = E_c$, and it turns out that ϕ^1 gives rise to a blowing-up or non-decaying solution. To see that, assume it doesn't and again employ the Long-time perturbation Theorem 2.8 to reach a contradiction. Thus, ϕ^1 is the critical element we were looking for. $\square$

2.4 Rigidity

In this short section, we will show that the critical element found in the previous section cannot possibly exist, hence completing the contradiction argument and the proof of Theorem 2.2.

Proposition 2.14. *The critical element found in Proposition 2.13 cannot exist.*

Proof. For a solution $u(t)$ emanating from initial data u_0, for all times t in the maximal interval of existence

$$E(u(t)) \leq E(u_0),$$

with equality if and only if u is a stationary solution.

So, we consider the following two cases for initial data ϕ^1, for which we know $E(\phi) = E_c \in (0, 2E(Q))$:
Scenario 1: ϕ^1 is a harmonic map, in which case $u(t)$ is just a stationary solution. However, in this topology class and range of energies, there is no non-trivial harmonic map (e.g. by [29], Proposition 1), hence the solution is global.
Scenario 2: ϕ^1 is not a harmonic map. Then, the energy is strictly decreasing and immediately drops below E_c for any $t_1 > 0$. Employing uniqueness of solutions and the definition of E_c, the new solution starting at t_1 is global and decays, contradicting the properties of ϕ. We have thus reached a contradiction which concludes the proof of the theorem. $\square$

Having treated the below threshold case, the next goal becomes the investigation of higher energies and non-trivial topologies. This is the content of the next chapter.

Chapter 3

Harmonic Map Heat Flow-Above Threshold

3.1 Introduction

This chapter is devoted to our treatment of an above threshold case in the corotational harmonic map heat flow. We show that the solutions in this scenario are globally smooth and asymptotically relax to a stationary solution.

3.1.1 Global results for Landau-Lifshitz and Schrödinger maps

To put our results into context (and because we will make use of some of the related arguments), we will first state some recent results concerning the question of global regularity vs singularity formation for the Landau-Lifshitz (LL) family of equations

$$\frac{\partial \mathbf{u}}{\partial t} = aP^{\mathbf{u}}\Delta\mathbf{u} + b\mathbf{u} \times P^{\mathbf{u}}\Delta\mathbf{u} = a\Delta\mathbf{u} + |\nabla\mathbf{u}|^2\mathbf{u} + b \times \Delta\mathbf{u}, \tag{3.1}$$

with $a \geq 0, b \in \mathbb{R}$, which of course includes as special cases the harmonic map heat-flow ($a = 1, b = 0$) and the Schrödinger map ($a = 0, b = 1$). This is mostly work from the papers [70, 71] which address the Schrödinger case, and the paper [66] which addresses the heat-flow case. This introduction concerns m-equivariant maps with energy near the minimal energy $\mathcal{E}_{\min} = 4\pi|m|$ (the harmonic map energy), and so the standing assumption on the initial data, unless otherwise specified, will be

$$\mathbf{u}_0 \in \Sigma_m, \quad \mathcal{E}(\mathbf{u}_0) = 4\pi|m| + \delta_0^2, \delta_0 \ll 1$$

Let

$$\mathbf{u}(t) \in C([0,T); \Sigma_m)$$

(Σ_m is topologized with the energy ($\dot{H}^1$ norm) be the solution of 3.1 corresponding to the initial data $\mathbf{u}_0$ (which is a priori just a local-in-time solution-see [71] for local wellposedness for this class of data).

The next result shows that when the degree is sufficiently high, singularities will not form, and moreover, solutions converge to specific harmonic maps as $t \to \infty$.

Theorem 3.1. *([66, 70, 71] global regularity and asymptotic stability for high degree). For 3.1 with $a > 0$, assume $|m| \geq 3$. For $(a = 0)$, assume $|m| \geq 4$. The number δ_0 is sufficiently small. Then*

1. *there is no finite-time blow-up: the solution can be extended to $\boldsymbol{u} \in C([0,\infty); \Sigma_m)$.*
2. *For any $r \in (2,\infty], p \in [2,\infty)$ with $\frac{1}{r} + \frac{1}{p} = \frac{1}{2}$, we have*

$$\|\nabla[\boldsymbol{u}(x,t) - e^{(m\theta+\alpha(t))R}\boldsymbol{h}(\frac{r}{s(t)})]\|_{L_t^r L_x^p((\mathbb{R}^2\times[0,\infty))} \lesssim C_p\delta_0$$

(if $a > 0$ we may include $(r,p) = (2,\infty)$)

3. *furthermore, there exist $s_+ > 0$ and α_+ with*

$$s(t) \to s_+, \alpha(t) \to \alpha_+, \text{ as } t \to \infty.$$

We remark that if $T < \infty$, then T is the maximal existence time ($\mathbf{u}(t)$ doesn't extend past T as a solution continuous into Σ_m) if and only if

$$\liminf_{t \nearrow T^-} s(t) = 0.$$

This theorem can be viewed, on one hand, as an *orbital stability* result for the family $\mathcal{O}_m$ of harmonic maps (at least up to the possible blow-up time), and on the other hand as a characterization of blow-up for energy near $\mathcal{E}_{\min}$: solutions blow-up if and only if the "length-scale" $s(t)$ goes to zero. The space-time estimates above imply asymptotic convergence of the solutions to the family of harmonic maps in a space-time norm ("dispersive") sense, which is the best we can expect for the Schrödinger case $a = 0$).

The analysis of [70, 71, 66] in the equivariant setting fails to extend to $m \leq 3$ and a new approach was required. Handling the case $m = 3$ was one of the main results of [72]:

Theorem 3.2. *Let $m \geq 3, a = a + ib \in \mathbb{C}/\{0\}$, and $a \geq 0$. Then there exists $\delta > 0$ such that for any $\boldsymbol{u}(0, x) \in \Sigma_m$ with $\mathcal{E}(\boldsymbol{u}(0)) \leq 4\pi m + \delta^2$, there is a unique global solution $\boldsymbol{u} \in C([0, \infty); \Sigma_m)$ of 3.1, satisfying $\nabla \boldsymbol{u} \in L^2_{t,loc}([0, \infty); L^\infty_x)$. Moreover, for some $\mu \in \mathbb{C}$ we have*

$$\|\boldsymbol{u}(t) - e^{m\theta R}\boldsymbol{h}[\mu]\|_{L^\infty_x} + a\mathcal{E}(\boldsymbol{u}(t) - e^{m\theta R}\boldsymbol{h}[\mu]) \to 0, \ \text{as } t \to \infty, \tag{3.2}$$

where $e^{m\theta R}\boldsymbol{h}(r/s)$ describes all the harmonic maps in Σ_m.

Every solution with energy close to the minimum converges to one of the harmonic maps uniformly in x as $t \to \infty$. Even for the higher degrees $m \geq 4$ this result is stronger than the previous ones ([70, 71, 66]), where the convergence was given only in time average. Moreover, note that in the dissipative case ($a > 0$), solutions converge to a harmonic map also in the energy norm, while this is impossible for the conservative Schrödinger flow ($b = 0$).

The analysis for the case $m = 2$ is trickier and the results weaker: in the harmonic map heat flow case the strong asymptotic stability result of the previous theorem for

$m \geq 3$ is no longer valid; instead, more exotic asymptotics are possible, including infinite-time concentration (blow-up).

Concerning blow-up:

Theorem 3.3. *([66] harmonic map heat flow blow-up for $m = 1$). Let $m = 1$. For any $\delta > 0$, there exists $\boldsymbol{u}_0 \in \Sigma_1$ with $0 < \mathcal{E}(\boldsymbol{u}_0) - 4\pi \leq \delta^2$ such that the corresponding solution of the harmonic map heat flow blows up in finite time, in the sense that, for example, $\|\nabla \boldsymbol{u}(\cdot, t)\|_{L^\infty_x} \to \infty$.*

This result is an adaptation of the blow-up proof of [22] for a disk domain, to the case of $\mathbb{R}^2$. Explicit blowing-up solutions (for $m = 1$) were provided recently by Raphael and Schewyer [114, 115], showing that (if T is the blow-up time), for 1-corotational maps, initial data u_0, $L \in \mathbb{N}^*$

$$u(t,r) - Q(\frac{r}{\lambda(t)}) \to u^*, \text{as } \ t \to T, \ \text{ in } \ \dot{H}^1,$$
$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{\frac{2L}{2L-1}}}, c(u_0) > 0.$$

For a recent construction of a blowing-up solution on a bounded domain we refer to [37].

3.2 Corotational maps in E_1

Moving to higher energies and non-trivial topologies one also encounters, other than harmonic maps, objects with energy much larger than the energy of the harmonic map; the approach taken in the above works (using "coordinate systems" about the family of harmonic maps, reflecting the near-minimality of the energy) cannot be implemented. In what follows all maps are in the class E_1 (unless otherwise stated),

$$E_1 := \{u : 2E(Q) \leq E(u_0) < 3E(Q); u(0) = \pi, \lim_{r\to\infty} u(r) = 0\}$$

In this section we restrict ourselves to the corotational setting and take a different approach to prove

Theorem 3.4. *The solution to the m-corotational harmonic map heat flow in the class E_1 with $m \geq 4$, is global and smooth with $E(u(\cdot,t) - Q(\frac{\cdot}{s_\infty})) \to 0$, as $t \to \infty$ for some $s_\infty > 0$.*

As before, we consider solutions $u(r,t)$ of

$$u_t = u_{rr} + \frac{1}{r}u_r + \frac{m^2}{2r^2}\sin(2u) \tag{3.3}$$

with finite energy

$$E(u) := \frac{1}{2}\int_0^\infty \left(u_r^2 + \frac{m^2}{r^2}\sin^2(u)\right) r\, dr < \infty$$

and boundary conditions

$$u(0,t) = \pi, \quad u(\infty,t) = 0. \tag{3.4}$$

Denote the static solution with these boundary conditions as

$$Q(r) = \pi - 2\tan^{-1}(r^m),$$

and define the following quantities

$$h(r) := \sin(Q(r)) = \frac{2r^m}{1+r^{2m}}, \quad \hat{h}(r) := \cos(Q(r)) = \frac{r^{2m}-1}{r^{2m}+1}$$

For later use, we also record the easy computations $h_r = -\frac{m}{r}h\hat{h}$ and $\hat{h}_r = \frac{m}{r}h^2$.

Denote rescalings as

$$Q^s(r) := Q(r/s), \quad h^s(r) = h(r/s), \quad etc., \quad s > 0.$$

Recall that the energy space (for maps with trivial topology) is:

$$X^2 = \{w : [0,\infty) \mapsto \mathbb{R} \mid \int_0^\infty \left(w_r^2 + \frac{w^2}{r^2} \right) r \, dr < \infty \}.$$

The results in this section are valid in the range $m \geq 4$. It is an open question whether this behaviour is also expected of maps with $m = 2, 3$.

3.2.1 Main results

As we have discussed earlier, the mechanism of singularity formation is that the solution blows-up because of energy concentration, by bubbling off a harmonic map. The first important step is to show that concentration cannot happen at infinity (note that because of the corotational symmetry a singularity can only occur at the origin or at infinity). This is the content of the next lemma:

Lemma 3.5. *Let u be a finite energy smooth solution on (3.3) on $(0,T)$. No energy concentration at spatial infinity is possible:*

$$\lim_{R\to\infty} \limsup_{t\to T} E(u(t); B_R^c) = 0$$

Before we present the proof of the lemma, some remarks are in order: among other things, this result grants us direct access to many of the classical results in the literature that were concerned with the case of a compact domain, for example the local theory irrespective of the topology assumptions, which also enables the use of the same classical arguments to characterize blowing-up solutions (which only requires local analysis).

Proof. The energy dissipation relation

$$E(u(t_2)) - E(u(t_1)) = \int_{t_1}^{t_2} \frac{d}{dt} E(u(s)) ds = \int_{t_1}^{t_2} -\|u_t\|_{L^2}^2 ds \qquad (3.5)$$

for $t_2 > t_1$ will be of use.

We will first choose a radial smooth cut-off function ψ such that

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq 1 \\ 1 & \text{if } r \geq 2 \end{cases}$$

and define $\psi_R(r) := \psi(\dfrac{r}{R})$.

If there was energy concentration at spatial infinity at time $t = T$:

$$\limsup_{t \nearrow T} E(u(t); B_R^c) \geq \delta > 0, \forall R,$$

for some $\delta > 0$. So we could find sequences of radii $R_n \nearrow \infty$ and times $t_n \nearrow T$ such that $\lim_n E(u(t_n), B_{R_n}^c) \geq \delta > 0$. We also define the "exterior" energy in terms of the above cut-off:

$$\hat{E}_R(t) := \int_0^\infty \psi_R(r) \left(u_r^2 + \frac{m^2}{r^2} \sin^2(u) \right) r dr$$

By the finiteness of the energy, for any $t_0 < T$ there is an $R_0 > 1$, such that $\hat{E}_{R_0}(t_0) \leq \frac{\delta}{4}$. By assumption, there is $T > t_1 > t_0 : \hat{E}_{R_0}(t_1) \geq \frac{\delta}{2}$.

Now, by direct calculation: $\dfrac{d}{dt}\hat{E}_{R_0}(t) = -\displaystyle\int_0^\infty \psi_{R_0} u_t^2 r dr - \int_0^\infty u_r u_t \frac{d\psi_{R_0}}{dr} r dr.$

Putting everything together and using $\dfrac{d}{dr}\psi_R(r) = \dfrac{1}{R}\psi'(\dfrac{r}{R})$:

$$\begin{aligned}
\frac{\delta}{4} &\leq \int_{t_0}^{t_1} \frac{d}{dt} E(u(t); B_{R_0}^c) dt = -\int_{t_0}^{t_1} \int_0^\infty \psi_{R_0} u_t^2 r dr - \int_{t_0}^{t_1} \int_0^\infty u_r u_t \psi'_{R_0} r dr. \\
&\leq \left(\int_{t_0}^{t_1} \int_0^\infty u_t^2 r dr \right)^{1/2} \cdot \left(\int_{t_0}^{t_1} \int_0^\infty u_r^2 \left(\psi'_{R_0}\right)^2 \right)^{1/2} \\
&\lesssim \frac{1}{R_0} (t_1 - t_0)^{1/2} E^{1/2}(u_0),
\end{aligned}$$

(because of 3.5) which yields a contradiction taking $t_0 \nearrow T$. $\square$

Following [122] (or [6] directly in the corotational setting) and using Lemma 3.5 :

Lemma 3.6. *If the solution blows-up in finite time, say T,*

$$u(t_j, s(t_j)r) \to Q, \;\; s(t_j) \to 0$$

in X^2_{loc} along a sequence $\{t_j\}_n \nearrow T$.

However working locally removes any knowledge of the topology of the map, which is determined by the behavior of the map at spatial infinity. We will improve the above result in the corotational setting by working globally in space in the energy topology. Here we are forced to account for the topological restrictions of non-trivial degree maps, and in fact we shall use these restrictions, along with our degree zero theory (see the previous chapter), to our advantage.

The improved convergence is as given in the following Proposition (a "no neck" proposition), which is a direct adaptation of Theorem 1.1 in [110]. The only possibility not excluded in that work, and the only potential hindrance to a global-in-space convergence, was the loss of energy at infinity; but this is already ruled out by Lemma 3.5.

Proposition 3.7. *Let $u_0 \in E_1$ and $u(t)$ the corresponding solution to (3.3) blowing up at time $t = T$ with*

$$E(u_0) < 3E(Q). \tag{3.6}$$

Then there exists a sequence of times $t_j \nearrow T$, a sequence of scales $s_j = o(\sqrt{T - t_j})$, a map $w_0 \in E_0$, and a decomposition

$$u(r, t_j) = Q(\frac{r}{s_j}) + w_0(r) + \xi(r, t_j) \tag{3.7}$$

such that $\xi(t_j) : \xi(t_j, 0) = \lim_{r\to\infty} \xi(t_j, r) = 0$ and $\xi(t_j) \to 0$, in X^2 as $j \to \infty$.

In other words, if an m-corotational heat-flow with $E(u(\cdot, 0)) < 3E(Q)$ forms a first singularity at time $t = T < \infty$, we can conclude that there are $t_j \to T-$ and $0 < s_j \to 0$ such that $E(u(t_j) - Q^{s_j} - w_0) \to 0$ for some $w_0 \in X^2$ with $E(w_0) < 2E(Q)$. Note that the combination of the energy bound (3.6) and the boundary conditions

(3.4) prohibit the formation of more than one bubble.

The rest of the section is devoted to proving that finite-time blow-up cannot happen; in particular, since the only possibility is blow-up by energy-concentration, it suffices to show the following proposition:

Proposition 3.8. *Assume $m \geq 4$. Suppose $u(r,t)$ is a smooth solution of* (3.3) *on* $[0,T)$ *such that along some sequence $t_j \to T-$, there are $s_j > 0$ such that*

$$u(\cdot, t_j) - Q^{s_j} \to w_0 \text{ in } X^2. \tag{3.8}$$

for some $w_0 \in X^2$ with $E(w_0) < 2E(Q)$. Then $s_j \not\to 0$.

The argument has two main themes. First, it is a variant of the kind of local wellposedness-"stability" argument which is typical for nonlinear dispersive PDE, except that here we are in a topologically non-trivial setting, in which the static solution family is present. Second, because of this, we need to linearize about a time-dependent rescaled static solution (*modulation theory*), and specifically to extend the [72] proof of asymptotic stability of equivariant harmonic maps to solutions at higher energies, which are far from the static ones.

Its proof is based on two main ingredients. For the first of these, we introduce the solution $w(r,t)$ of (3.3) with initial data at $t = t_j$ given by w_0:

$$\begin{array}{c} w_t - w_{rr} - \frac{1}{r} w_r - \frac{m^2}{2r^2}\sin(2w) = 0 \\ w(r,t_j) = w_0(r) \in X^2, \quad E(w_0) < 2E(Q) \end{array}. \tag{3.9}$$

By the results of Chapter 2, we know that w is a global, smooth solution with

$$\|w\|_{L_t^\infty X^2 \cap L_t^2 (X^\infty \cap r X^2)([t_j,\infty)} < \infty. \tag{3.10}$$

For later use, we record one consequence of the higher regularity gained after the initial time:

$$\frac{w}{r^2} \in L_t^2 X^2([t^*, \infty)) \quad \text{for every } t^* > t_j. \tag{3.11}$$

This follows from the observations that by standard parabolic regularity estimates (for example by performing energy-type estimates on the differentiated PDE), the function $v(x,t) = w(r,t)e^{im\theta}$ satisfies $D^2 v \in L_t^2 L_x^2([t^*, \infty))$, and that w/r^2 and w_r/r are controlled pointwise by $|D^2 v|$ (for any $m \geq 2$).

For fixed $s > 0$, Q^s is also a (static) solution of (3.3). Since the PDE is nonlinear, of course the sum $Q^s + w$ is *not* a solution:

$$\begin{aligned}
&\left(\partial_t - \partial_r^2 - \frac{1}{r}\partial_r - \frac{m^2}{2r^2}\sin(2\,\cdot\,)\right)(Q^s + w) \\
&\quad = \frac{m^2}{2r^2}\left(\sin(2Q^s) + \sin(2w) - \sin(2Q^s + 2w)\right) \\
&\quad = \frac{m^2}{2r^2}\left(\sin(2Q^s)(1 - \cos(2w)) + \sin(2w)(1 - \cos(2Q^s)\right) =: Eqn(Q^s + w).
\end{aligned}$$

However, $Q^{s(t)} + w$ is a good *approximate solution* over short time intervals in the sense:

Lemma 3.9. $\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1} \lesssim \|w\|_{L_t^2 X^\infty} + \|w\|_{L_t^4 X^4}^2$ *and therefore by* (3.10),

$$\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1[t_j, T)} \to 0 \quad \text{as} \quad t_j \to T- \,. \tag{3.12}$$

Remark: We do not need it here, but if $0 < s(t) \ll 1$, then $Q^{s(t)} + w$ is a good approximate solution *globally*, in the sense that $\|Eqn(Q^{s(t)} + w)\|_{L_t^2 X^1[t_j, \infty)} \to 0$ as $\sup\limits_{t \in [t_j, \infty)} s(t) \to 0$.

Proof. This is an easy consequence of the elementary pointwise estimates

$$\begin{aligned}
|Eqn(Q^s + w)| &\lesssim \frac{1}{r^2}\left(h^s w^2 + (h^s)^2 w\right) \\
|\partial_r Eqn(Q^s + w)| &\lesssim \frac{1}{r^3}\left(h^s w^2 + (h^s)^2 w\right) + \frac{1}{r^2}\left(h^s |w||w_r| + (h^s)^2 |w_r|\right).
\end{aligned}$$

Then using $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$, $\|\frac{h^s}{r^2}\|_{L^1} \lesssim 1$, and Hölder, the Lemma follows. $\square$

Lemma 3.10. *(Linear Estimates) Assume $m \geq 4$. Let $\xi(\cdot, t) \in X^2$ be a solution of the inhomogeneous linearized equation about Q^s, where $s = s(t) > 0$ is a differentiable function of time,*

$$\left\{ \begin{array}{c} \partial_t \xi + H^s \xi = f(r,t) \\ \xi(r,0) = \xi_0(r) \end{array} \right\} \qquad H^s := -\partial_r^2 - \frac{1}{r}\partial_r - \frac{m^2}{r^2}\cos(2Q^s),$$

which also satisfies the orthogonality condition

$$\big(\xi(\cdot,t),\ h^{s(t)}\big)_{L^2_{rdr}} \equiv 0. \tag{3.13}$$

Then we have the estimates

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \|\xi_0\|_{X^2} + \|f\|_{L^1_t X^2 + L^2_t X^1} + \|\dot{s}\|_{L^2_t}. \tag{3.14}$$

Proof. The idea comes from [72] where it appeared as a linearization of a *generalized Hasimoto transformation*, while here we apply it directly at the linear level: exploit the factorized form of the linearized operator

$$H^s = (L^s)^* L^s, \quad L^s = \partial_r + \frac{m}{r}\cos(Q^s) = h^s \partial_r (h^s)^{-1},$$

the fact that the reverse factorization is positive,

$$(L^s)(L^s)^* = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2}\big(1 + m^2 - 2m\cos(Q^s)\big) \geq -\partial_r^2 - \frac{1}{r}\partial_r + \frac{(m-1)^2}{r^2}. \tag{3.15}$$

Applying L^s to the linearized equation produces

$$\partial_t \eta + L^s (L^s)^* \eta = L^s f + (\partial_t L^s)\eta \qquad \eta := L^s \xi,\ \ \partial_t L^s = \frac{m^2}{r}\frac{1}{s}(h^s)^2 \dot{s}$$

Multiplying this equation by η, integrating over space and time, and using (3.15) gives

$$\|\eta\|_{L_t^\infty L^2}^2 + \|\eta\|_{L_t^2 X^2}^2 \lesssim \|L^s \xi_0\|_{L^2}^2 + \|(L^s f)\eta\|_{L_t^1 L^1} + \|\frac{1}{s}(h^s)^2 \frac{\eta}{r}\|_{L_t^2 L^1} \|\dot{s}\|_{L_t^2}.$$

Using Hölder on the right, then Young's inequality, as well as $\|\frac{1}{s}(h^s)^2\|_{L^2} \lesssim 1$, yields

$$\|\eta\|_{L_t^\infty L^2 \cap L_t^2 X^2} \lesssim \|\xi_0\|_{X^2} + \|f\|_{L_t^1 X^2 + L_t^2 X^1} + \|\dot{s}\|_{L_t^2}.$$

Finally, in [72] it was shown that we can invert L^s under the orthogonality condition (3.13) to bound ξ:

$$m \geq 4, \ \ (\xi,\ h^s)_{L^2} = 0 \quad \Longrightarrow \quad \|\xi\|_{X^p} \lesssim \|L^s \xi\|_{L^p}, \ \ 2 \leq p \leq \infty.$$

Together with the standard embedding $\|\eta\|_{L^\infty} \lesssim \|\eta\|_{X^2}$ this completes the proof. □

Proof. (of the Proposition 3.8) Let $w(r,t)$ be as in (3.9). For $t \in [t_j, T)$, the idea is to write the solution $u(r,t)$ in the form

$$u(r,t) = Q^{s(t)}(r) + w(r,t) + \xi(r,t), \tag{3.16}$$

where $s(t) > 0$ is chosen so that the orthogonality condition (3.13) holds. The fact that we can make such a choice follows from a standard implicit function theorem argument:

Lemma 3.11. *There is $\epsilon_0 > 0$ such that for any $s_0 > 0$ and any $\xi \in X^2$ with $\|\xi\|_{X^2} \leq \epsilon_0$, there is $0 < s = s(\xi, s_0)$ such that*

$$Q^{s_0} + \xi = Q^s + \tilde{\xi} \quad \textit{with} \ \left(\tilde{\xi},\ h^s\right)_{L^2_{rdr}} = 0, \quad \left|\frac{s}{s_0} - 1\right| + \|\tilde{\xi}\|_{X_2} \lesssim \|\xi\|_{X^2} \leq \epsilon_0.$$

Proof. First take $s_0 = 1$. For $s > 0$ and $\xi \in X^2$ define

$$g(s;\xi) := (Q - Q^s + \xi,\ h^s)_{L^2_{rdr}},$$

a smooth function of s and ξ because the spatial decay of $h(r)$ implies $\|rh(r)\|_{L^2_{rdr}} <$

∞ (provided $m > 2$). We observe $g(1;0) = 0$, and

$$\partial_s g(1;0) = \Big((-\frac{m}{s} h^s,\ h^s) + (Q - Q^s + \xi,\ \partial_s h^s) \Big) |_{s=1,\xi=0} = -m \|h\|^2_{L^2_{rdr}} \neq 0,$$

so by the Implicit Function Theorem there is $\epsilon_0 > 0$ such that for all ξ with $\|\xi\|_{X^2} \leq \epsilon_0$, there is $s = s(\xi)$ with $|s-1| \lesssim \|\xi\|_{X^2}$ such that $g(s;\xi) = 0$. Then also $\tilde{\xi} := \xi + Q - Q^s \implies \|\tilde{\xi}\|_{X^2} \lesssim \|\xi\|_{X^2} + |s-1| \lesssim \|\xi\|_{X^2}$. The case of general $s_0 > 0$ follows from simple rescaling, and the scale invariance of the X^2-norm. $\square$

This lemma shows that as long as

$$\inf_{s>0} \|u(\cdot,t) - w(\cdot,t) - Q^s\|_{X^2} < \epsilon_0, \tag{3.17}$$

we may write u in the form (3.16), with orthogonality (3.13) holding.

In particular, (3.8) implies that for any $0 < \delta_0 < \epsilon_0$, by taking j large enough, and therefore $\|u(\cdot,t_j) - Q^{s(t_j)} - w_0\|_{X^2}$ small enough, we may write

$$u(t_j,r) = Q^{s(0)} + w_0(r) + \xi(r,0), \quad \big(\xi(\cdot,0),\ h^{s(0)}\big) = 0, \quad \|\xi(\cdot,0)\|_{X^2} \leq \delta_0. \tag{3.18}$$

So by continuity, (3.17) holds on some non-empty time interval $I = [t_j,\tau)$, $t_j < \tau \leq T$, on which we may write $u(r,t)$ as in (3.16) with orthogonality (3.13).

Moreover by regularity of $u(r,t)$, by shrinking τ even more if needed, we may also assume

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty([t_j,\tau))} \leq \delta_0^{\frac{2}{3}}, \tag{3.19}$$

which in particular implies (3.17) for δ_0 sufficiently small.

We will use a standard "continuity argument". That is, we will carry out all our estimates over the time interval $I = [t_j,\tau)$ under the assumption (3.19), and then conclude that we may take $\tau = T$ provided δ_0 is chosen sufficiently small.

Inserting (3.16) into the PDE and using standard trigonometric identities yields the following equation for ξ:

$$(\partial_t + H^s)\xi = -m\frac{\dot{s}}{s}h^s + Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right), \tag{3.20}$$

where

$$V = \cos(2Q^s)(\cos(2w) - 1)) - \sin(2Q^s)\sin(2w),$$

and N contains only terms super-linear in ξ coming from the terms

$$\begin{aligned}&\cos(2Q^s)[2(w+\xi) - \sin(2(w+\xi))] \text{ and}\\ &\sin(2Q^s)[1 - \cos(2(w+\xi))].\end{aligned}$$

Rather than write out all the terms of N explicitly, we just record the elementary estimates

$$\begin{aligned}&|N| \lesssim (h^s + |w|)\xi^2 + |\xi|^3\\ &|N_r| \lesssim (1+|w|)(|w_r| + \frac{h^s}{r})\xi^2 + \frac{1}{r}|\xi|^3 + (h^s + |w|)|\xi||\xi_r| + \xi^2|\xi_r|.\end{aligned} \tag{3.21}$$

Our goal is to estimate all the terms on the right side of (3.20) in appropriate space-time norms, so that we may apply the linear estimates (3.14).

For the first term, using $\|\frac{1}{s}h^s\|_{X^1} \lesssim 1$, we have

$$\| -m\frac{\dot{s}}{s}h^s\|_{L^2_t X^1} \lesssim \|\dot{s}\|_{L^2_t}. \tag{3.22}$$

The main estimates for V are

$$|V| \lesssim w^2 + h^s|w| \implies \|\frac{1}{r^2}V\|_{L^2} \lesssim \|\frac{w}{r}\|^2_{L^4} + \|\frac{w}{r}\|_{L^\infty},$$

using $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$, and

$$\begin{aligned}&|V_r| \lesssim |w||w_r| + \frac{1}{r}(h^s)^2w^2 + h^s|w_r| + \frac{1}{r}h^s|w|\\ &\implies \|\frac{1}{r}V_r\|_{L^2} \lesssim \|\frac{w}{r}\|_{L^4}\|w_r\|_{L^4} + \|\frac{w}{r}\|^2_{L^4} + \|w_r\|_{L^\infty} + \|\frac{w}{r}\|_{L^\infty},\end{aligned}$$

using $\|h^s\|_{L^\infty} \lesssim 1$ and $\|\frac{h^s}{r}\|_{L^2} \lesssim 1$. Combining these, we obtain a spatial-norm estimate on the linear term on the right side of (3.20),

$$\|\frac{m^2}{2r^2}V\sin(2\xi)\|_{X^1} \lesssim \left(\|w\|_{X^4}^2 + \|w\|_{X^\infty}\right)\|\xi\|_{X^2},$$

and from there a space-time estimate:

$$\|\frac{m^2}{2r^2}V\sin(2\xi)\|_{L_t^2X^1} \lesssim \left(\|w\|_{L_t^4X^4}^2 + \|w\|_{L_t^2X^\infty}\right)\|\xi\|_{L_t^\infty X^2}, \tag{3.23}$$

where, recall, the time interval over which these norms are taken is $I = [t_j, \tau)$.

Finally, from (3.21), we estimate the nonlinear terms:

$$\|\frac{1}{r^3}N\|_{L^1} \lesssim \left(\|\frac{h^s}{r}\|_{L^2} + \|\frac{w}{r}\|_{L^2}\right)\|\frac{\xi}{r}\|_{L^4}^2 + \|\frac{\xi}{r}\|_{L^3}^3 \lesssim \|\frac{\xi}{r}\|_{L^4}^2 + \|\frac{\xi}{r}\|_{L^3}^3,$$

and using $\|w\|_{L^\infty} \lesssim \|w\|_{X^2} \lesssim 1$, and $\|h_s\|_{X^2} \lesssim 1$,

$$\|\frac{1}{r^2}N_r\|_{L^1} \lesssim \|\frac{\xi}{r}\|_{L^4}^2 + \|\frac{\xi}{r}\|_{L^3}^3 + \|\frac{\xi}{r}\|_{L^4}\|\xi_r\|_{L^4} + \|\frac{\xi}{r}\|_{L^4}^2\|\xi_r\|_{L^2}.$$

These last two give

$$\|\frac{1}{r^2}N\|_{X^1} \lesssim \|\xi\|_{X^4}^2 + \|\xi\|_{X^3}^3 + \|\xi\|_{X^4}^2\|\xi\|_{X^2},$$

and then the spacetime estimate:

$$\|\frac{m^2}{2r^2}N\|_{L_t^2X^1} \lesssim \|\xi\|_{L_t^4X^4}^2(1 + \|\xi\|_{L_t^\infty X^2}) + \|\xi\|_{L_t^6X^3}^3. \tag{3.24}$$

Now applying the linear estimates (3.14) to (3.20), using (3.18), (3.22), (3.12) (taking j larger as needed), (3.23), and (3.24), as well as (3.10), we get

$$\begin{aligned} \|\xi\|_{L_t^\infty X^2 \cap L_t^2X^\infty} \leq C\Big(\delta_0 + \|\dot{s}\|_{L_t^2} + \left(\|w\|_{L_t^4X^4}^2 + \|w\|_{L_t^2X^\infty}\right)\|\xi\|_{L_t^\infty X^2} \\ + \|\xi\|_{L_t^4X^4}^2(1 + \|\xi\|_{L_t^\infty X^2}) + \|\xi\|_{L_t^6X^3}^3\Big). \end{aligned} \tag{3.25}$$

By (3.10), by choosing j larger still if needed we can assure that on the interval

$[t_j, T) \supset I$,

$$C\left(\|w\|^2_{L^4_t X^4([t_j,T)} + \|w\|_{L^2_t X^\infty([t_j,T)}\right) < \frac{1}{2}, \tag{3.26}$$

so that the estimate (3.25) becomes

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \delta_0 + \|\dot{s}\|_{L^2_t} + \|\xi\|^2_{L^\infty_t X^2 \cap L^2_t X^\infty} + \|\xi\|^3_{L^\infty_t X^2 \cap L^2_t X^\infty},$$

and then by using (3.19),

$$\|\xi\|_{L^\infty_t X^2 \cap L^2_t X^\infty} \lesssim \delta_0 + \|\dot{s}\|_{L^2_t}. \tag{3.27}$$

It remains to estimate $\dot{s}$. For this, we differentiate the orthogonality relation (3.13), rewritten as

$$\left(\xi, \frac{1}{s} h^s\right)_{L^2_{rdr}}$$

for convenience of calculation, with respect to t, and use the equation (3.20) for ξ:

$$0 = \left(\xi,\ -\dot{s}\frac{1}{r^2}(r(r^2 h)')^s\right) + \left(-m\frac{\dot{s}}{s}h^s + Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right),\ \frac{1}{s}h^s\right)$$

where we used $H^s h^s = 0$. The first term is bounded by

$$|\dot{s}|\|\frac{\xi}{r}\|_{L^2}\|\frac{1}{r}(r(r^2 h)')^s\|_{L^2} \lesssim |\dot{s}|\|\xi\|_{X^2},$$

while

$$\left(-m\frac{\dot{s}}{s}h^s, \frac{1}{s}h^s\right) = -m\dot{s}\|\frac{1}{r}(rh)^s\|^2_{L^2} = -m\dot{s}\|h\|^2_{L^2},$$

so

$$\left(m\|h\|^2_{L^2} + O(\|\xi\|_{X^2})\right)\dot{s} = \left(Eqn(Q^s + w) + \frac{m^2}{2r^2}\left(V\sin(2\xi) + N\right),\ \frac{1}{s}h^s\right). \tag{3.28}$$

Then by (3.19) and $\|r\frac{1}{s}h^s\|_{L^\infty} = \|rh\|_{L^\infty} \lesssim 1$,

$$|\dot{s}| \lesssim \|Eqn(Q^s + w)\|_{X^1} + \|\frac{1}{r^2}V\sin(2\xi)\|_{X^1} + \|\frac{1}{r^2}N\|_{X^1},$$

and so by (3.12), (3.23), (3.10), (3.21) and (3.19):

$$\|\dot{s}\|_{L^2_t} \lesssim \delta_0 + \left(\|w\|^2_{L^4_t X^4} + \|w\|_{L^2_t X^\infty}\right)\|\xi\|_{L^\infty_t X^2}.$$

Using (3.27) then shows

$$\|\dot{s}\|_{L^2_t} \leq C\left(\delta_0 + \left(\|w\|^2_{L^4_t X^4} + \|w\|_{L^2_t X^\infty}\right)\|\dot{s}\|_{L^2_t}\right).$$

As above, by taking j larger if needed we can ensure (3.26) and so (using again (3.19))

$$\|\dot{s}\|_{L^2_t} + \|\xi\|_{L^\infty_t X^2 \cap L^2_t \cap X^\infty} \lesssim \delta_0. \tag{3.29}$$

This now shows that in our bootstrap assumption (3.19), since we take $\delta_0 \ll \delta_0^{2/3}$, we may indeed take $\tau = T$, and all of our previous estimates hold on the full time interval $[t_j, T)$.

It remains to show that $s(t)$ stays bounded away from zero. Recall the pointwise bounds used above

$$|V| \lesssim w^2 + h^s|w|,\ |N| \lesssim (h^s + |w|)\xi^2 + |\xi|^3,\ Eqn(Q^s + w) \lesssim \frac{1}{r^2}\left(h^s w^2 + (h^s)^2 w\right).$$

We isolate the term in the equation (3.28) for s coming from the part of $Eqn(Q^s + w)$ which behaves linearly in w, and write:

$$\frac{\dot{s}}{s} = v_1 + v_2 + v_3$$

where

$$|v_1| \lesssim \|\frac{w}{r^3}\|_{L^2}\|\frac{1}{s}(rh^3)^s\|_{L^2} \lesssim \|\frac{|w|}{r^2}\|_{X^2} \in L^2_t,$$

$$|v_2| \lesssim \|\frac{w^2}{r^2}\|_{L^\infty}\|\frac{1}{s^2}(h^2)^s\|_{L^1} \lesssim \|w\|^2_{X^\infty} \in L^1_t$$

and

$$
\begin{aligned}
|v_3| &\lesssim \|\frac{w^2}{r^2}\|_{L^2}\|\frac{\xi}{r}\|_{L^\infty}\|\frac{1}{s}(rh)^s\|_{L^2} + \|\frac{w}{r}\|_{L^\infty}\|\frac{\xi}{r}\|_{L^\infty}\|\frac{1}{s^2}h^s\|_{L^1} \\
&\quad + \|\frac{\xi^2}{r^2}\|_{L^\infty}\|\frac{1}{s^2}(h^s)^3\|_{L^1} + \|\frac{\xi^2}{r^2}\|_{L^2}(\|\frac{w}{r}\|_{L^\infty} + \|\frac{\xi}{r}\|_{L^\infty})\|\frac{1}{s}(rh)^s\|_{L^2} \\
&\lesssim \|w\|_{X^4}^2\|\xi\|_{X^\infty} + \|w\|_{X^\infty}\|\xi\|_{X^\infty} + \|\xi\|_{X^\infty}^2 + \|\xi\|_{X^4}^2(\|w\|_{X^\infty} + \|\xi\|_{X^\infty}) \\
&\quad \in L^1_t.
\end{aligned}
$$

So $\frac{\dot{s}}{s} \in L^1_t + L^2_t$ over $[t^*, T)$, and by the Fundamental Theorem of Calculus, and Cauchy-Schwarz,

$$
\sup_{t^* \le t < T} \left| \log\left(\frac{s(t)}{s(t^*)}\right) \right| \le \|\frac{\dot{s}}{s}\|_{L^1_t([t^*,T))} + \sqrt{t - t^*}\|\frac{\dot{s}}{s}\|_{L^2_t([t^*,T))} < \infty,
$$

so that $s(t)$ remains bounded away from zero, as required. $\square$

Now notice that Proposition 3.8 directly prohibits concentration, and so shows that an $m \ge 4$-equivariant heat-flow cannot form a finite-time singularity. Hence such a heat-flow is global. Moreover, it cannot form a singularity at infinite time $t = \infty$, since such this would produce a sequence $t_j \to \infty$, with $0 < s_j \to 0$ or ∞, along which $u(\cdot, t_j) - Q^{s_j} \to v_0$ with $X^2 \ni v_0$ a *static solution*, hence $v_0 \equiv 0$. This is however prohibited by the asymptotic stability result of [72]. Hence we must have $E(u(\cdot, t) - Q^{s_\infty}) \to 0$ for some $s_\infty > 0$.

3.2.2 Conclusions and future directions

The map equations discussed in this introduction are of both physical and geometric interest, and yet it is only very recently that the global behavior of solutions is starting to be understood, and that only in very limited settings. Much work remains to be done.

In continuation of this work, we plan to investigate the general case and in particular address the following problem:

Problem: Characterize blowing-up and global solutions to the *corotational* harmonic map heat flow (from $\mathbb{R}^2 \to \mathbb{S}^2$) with arbitrary initial energy and any choice of topology.

Despite some technical difficulties, we believe this characterization to be within the reach of our techniques. Results of this flavour have already been established in the case of corotational Wave Maps (WM) (e.g., [30, 33, 34]).

A more ambitious next step is to address the general problem in the equivariant class:

Problem: Characterize blowing-up and global solutions to the *equivariant* harmonic map heat flow/Landau-Lifshitz (from $\mathbb{R}^2 \to \mathbb{S}^2$) with arbitrary initial energy and any choice of topology, and in particular, investigate the case of large data, not just close to the family of stationary solutions, which has been addressed in [71, 72].

The very interesting case of Schrödinger Maps is quite challenging and seems to require the development of new techniques.

Problem: The construction of either blowing-up or global solutions, which is still largely unexplored. For the full Landau-Lifshitz equation, once the Schrödinger-type term ($b \neq 0$) is included, our understanding diminishes considerably. Though the problem is still dissipative, maximum principle-type arguments are not readily applicable, and even partial regularity results become more difficult and weaker. Singularity formation is an open question, partly because the corotational class is no longer preserved. Indeed, the harmonic map heat flow blow-up may not provide a reliable guide for the Landau-Lifshitz problem. We do however venture the conjecture:

Conjecture: The Landau-Lifshitz equation from $\mathbb{R}^2 \to \mathbb{S}^2$, is globally well-posed, at least when $m \geq 4$.

We have access to potentially useful linear estimates, but some work is required on a technical level: a choice of gauge must be made to write the equation in a form amenable to estimates and the profile decomposition should take into account the compatibility condition imposed by this choice of gauge. Nevertheless, these are issues that have been addressed before for the case of the Schrödinger Map Equation [7] and we expect these results to provide a guide.

The full physical model. The Landau-Lifshitz equation described above is a fairly simple model, in that it incorporates only the "exchange energy", and (for $a > 0$) dissipation. The question of how physically important effects such as anisotropy, and demagnetization (a nonlocal term) affect solutions has barely been addressed.

Chapter 4

Global solutions of a focusing energy-critical Heat Equation in $\mathbb{R}^4$

4.1 Introduction

This chapter is devoted to identifying the natural threshold below which all solutions of a nonlinear heat equation are global and decay.

In what follows, we will mostly consider the focusing energy-critical nonlinear heat equation in four space dimensions:

in particular, we consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^2 u \\ u(0,x) = u_0(x) \in \dot{H}^1(\mathbb{R}^4) \end{cases} \tag{4.1}$$

for $u(x,t) \in \mathbb{C}$ with initial data in the energy space

$$\dot{H}^1(\mathbb{R}^4) = \{u \in L^4(\mathbb{R}^4; \mathbb{C}) \mid \|u\|_{\dot{H}^1}^2 = \int_{\mathbb{R}^4} |\nabla u(x)|^2 \, dx < \infty\}.$$

This is the L^2 gradient-flow equation for an *energy*, defined for $u \in \dot{H}^1$ as

$$E(u) = \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{4} |u|^4 \right) dx,$$

and so in particular the energy is (formally) dissipated along solutions of (4.1):

$$\frac{d}{dt} E(u(t)) = - \int_{\mathbb{R}^4} |u_t|^2 \, dx \leq 0. \tag{4.2}$$

We refer to the gradient term in E as the *kinetic energy*, and the second term as the *potential energy*. The fact that the potential energy is negative expresses the focusing nature of the nonlinearity. Problem (4.1) is *energy-critical* in the sense that the scaling

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x), \quad \lambda > 0 \tag{4.3}$$

leaves invariant the equation, the potential energy, and in particular the kinetic energy, which is the square of the *energy norm* $\| \cdot \|_{\dot{H}^1}$.

Static solutions of (4.1), which play a key role here, solve the elliptic equation

$$\Delta W + |W|^2 W = 0. \tag{4.4}$$

The function

$$W = W(x) = \frac{1}{(1 + \frac{|x|^2}{8})} \; \in \dot{H}^1(\mathbb{R}^4), \;\; \notin L^2(\mathbb{R}^4)$$

is a well-known solution. Its scalings by (4.3), and spatial translations of these are again static solutions, and multiples of these are well-known [2, 123] to be the unique extremizers of the Sobolev inequality

$$\forall u \in \dot{H}^1, \; \|u\|_{L^4} \leq C \|\nabla u\|_{L^2}, \quad C = \frac{\|W\|_{L^4}}{\|\nabla W\|_{L^2}} = \frac{1}{\|\nabla W\|_2^2} \text{ the best constant.} \tag{4.5}$$

As for time-dependent solutions, a suitable local existence theory – see Theorem 4.6 for details – ensures the existence of a unique smooth solution $u \in C(I; \dot{H}^1(\mathbb{R}^4))$ on a maximal time interval $I = [0, T_{max}(u_0))$. The main result of this chapter states that initial data lying 'below' W gives rise to global smooth solutions of (4.1) which decay to zero:

Theorem 4.1. *Let $u_0 \in \dot{H}^1(\mathbb{R}^4)$ satisfy*

$$E(u_0) \leq E(W), \qquad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}. \tag{4.6}$$

Then the solution u of (4.1) *is global ($T_{max}(u_0) = \infty$) and satisfies*

$$\lim_{t \to \infty} \|u(t)\|_{\dot{H}^1} = 0. \tag{4.7}$$

The conditions (4.6) define a non-empty set, since by the Sobolev inequality (4.5) it includes all initial data of sufficiently small kinetic energy. Moreover, conditions (4.6) are sharp for global existence and decay in several senses. Firstly, if the kinetic energy inequality is replaced by equality, W itself provides a non-decaying (though still global) solution. Secondly, if the kinetic energy inequality is reversed, and under the additional assumption $u_0 \in L^2(\mathbb{R}^4)$, by a slight variant of a classical argument [95] we find that the solution blows up in finite time:

Theorem 4.2. *Let $u_0 \in H^1(\mathbb{R}^4)$ with*

$$E(u_0) < E(W), \qquad \|\nabla u_0\|_{L^2} \geq \|\nabla W\|_{L^2}.$$

Then the solution u of (4.1) *has finite maximal lifespan: $T_{max}(u_0) < \infty$.*

Thirdly, for any $a^* > 0$, [118] constructed finite-time blow-up solutions with initial data $u_0 \in H^1(\mathbb{R}^4)$ satisfying $E(W) < E(u_0) < E(W) + a^*$. See also [52] for formal constructions of blow-up solutions close to W.

It follows from classical variational bounds – see Lemma 4.9 – and energy dissipation (4.2), that any solution u on a time interval $I = [0, T)$ whose initial data satisfies (4.6), necessarily satisfies

$$\sup_{t \in I} \|\nabla u\|_{L^2} < \|\nabla W\|_{L^2}. \tag{4.8}$$

So it will suffice to show that the conclusions of Theorem 4.1 hold for any solution satisfying (4.8). Indeed, we will prove:

1. If $I = [0, \infty)$ and (4.8) holds, then $\lim_{t\to\infty} \|\nabla u(t)\|_{L^2} = 0$. This is given as Theorem 4.10.

2. For any solution satisfying (4.8), $T_{max}(u(0)) = \infty$. This is given as Corollary 4.18.

That static solutions provide the natural threshold for global existence and decay, as in (4.8), is a classical phenomenon (eg. [122]) for critical equations, particularly well-studied in the setting of parabolic problems, mostly on compact domains, (e.g., [40, 61, 97, 125]) via 'blow-up'-type arguments: first, failure of a solution to extend smoothly is shown, by a local regularity estimate, to imply (kinetic) energy concentration; then, near a point of concentration, rescaled subsequences are shown to converge locally to a non-trivial static solution; finally, elliptic/variational considerations prohibit non-trivial static solutions below the threshold.

The main purpose of our work is twofold: first, to establish the global-regularity-below-threshold result Theorem 4.1 on the full space $\mathbb{R}^4$; second, to do so not by way of the classical strategy sketched above, but instead via Kenig-Merle's [79, 80] "concentration-compactness plus rigidity" approach to critical *dispersive* equations, similar to Kenig-Koch's [78] implementation for the Navier-Stokes equations.

The argument is structured as follows. First, in Section 4.3, we prove the energy-norm decay of *global* solutions which satisfy (4.8), Theorem 4.10. The strategy is that employed for the Navier-Stokes equations in [56]: reduce the problem to establishing the decay of *small* solutions (which is a refinement of the local theory) by exploiting the L^2-dissipation relation, using a solution-splitting argument to overcome the fact that the solution fails to lie in L^2. Second, in Section 4.4, we prove the existence and compactness (modulo symmetries) of a "critical" element – a counterexample to global existence and decay, which is minimal with respect to $\sup_t \|\nabla u(t)\|_{L^2}$, following closely the work [82]. See Theorem 4.13. The technical tools are a profile decomposition compatible with the heat equation (described in Section 4.2.2) and a perturbation result for the linear heat equation, based on the local theory (Proposition 4.7). Finally, in Section 4.5, we exclude the possibility of a compact solution with finite maximal existence time in Theorem 4.17. This part is based on classical parabolic tools. We first show that the centre of compactness remains bounded, by exploiting energy dissipation. Then a local small-energy regularity criterion, together with backwards uniqueness and unique continuation theorems as in [78], imply the triviality of the critical element.

There is a vast literature on the semilinear heat equation $u_t = \Delta u + |u|^{p-1}u$. We content ourselves here with a brief review focused on the case of domain $\mathbb{R}^d$, and refer the reader to the recent book [112] for a more comprehensive review of the literature (until 2007). For treatments of the Cauchy problem in L^p and Sobolev spaces under various assumptions on the nonlinearity and the initial data, see [128, 129, 13].

Much of the work concerns (energy) subcritical ($p < \frac{d+2}{d-2}$) problems. The seminal papers [62, 63, 64] introduced the study of heat equations through similarity variables and characterized blow-up solutions. In continuation of these works, [104] gave a first construction of a solution with arbitrarily given blow-up points, and see [107] (and references therein) for estimates of the blow-up rate, descriptions of the blow-up set, and stability results for the blow-up profile. We remark that blow-up in the subcritical case for $L^\infty-$solutions is known to be of Type I, in the sense that $\limsup\limits_{t\to T_{\max}}(T_{\max} - t)^{\frac{1}{p-1}}\|u(\cdot,t)\|_{L^\infty} < +\infty$, and Type I blow-up solutions are known to behave like self-similar solutions near the blow-up point. For a different set of criteria for global existence/blow-up in terms of the initial data we refer the reader to [18]. For results on the relation between the regularity of the nonlinear term and the regularity of the corresponding solutions, see [19].

For supercritical problems, [100, 101, 102] show that there is no Type II blow-up for $3 \leq d \leq 10$, while for $d \geq 11$ it is possible if p is large enough. It is also shown that a Type I blow-up solution behaves like a self-similar solution, while a Type II converges (in some sense) to a stationary solution. We also refer to the recent results [30] ($d \geq 11$, bounded domain), and [26] and to the preprint [10] for results in Morrey spaces.

For the critical case, we have already mentioned the finite-time blow-up constructions [52, 118], and we point recent contructions of infinite-time blowup (bubbling) on bounded domains ($d \geq 5$) [28] , and on $\mathbb{R}^3$ [36]. The work [53] deals with the continuation problem for reaction-diffusion equations. We finally mention the recent result [27], where a complete classification of solutions sufficiently close to the stationary solution W is provided for $d \geq 7$: such solutions either exhibit Type-I blow-up; dissipate to zero; or converge to (a slightly rescaled, translated) W. In particular, Type II blow-up is ruled out in $d \geq 7$ near W.

Remark 4.3. *Theorem 4.1 extends to the energy critical problem for the nonlinear heat equation in general dimension $d \geq 3$:*

$$\begin{aligned} u_t &= \Delta u + |u|^{\frac{4}{d-2}} u \\ u(t_0, x) &= u_0(x) \in \dot{H}^1(\mathbb{R}^d) \end{aligned} \tag{4.9}$$

For simplicity of presentation, we will give the proof only for the case $d = 4$. As will be apparent from the proof, the result can be easily transferred to solutions of (4.9) *(and we will note the specific parts of the proof that are not dimension-independent and remark on the modifications that are required).*

Remark 4.4. *Our proof makes no use of any parabolic comparison principles, and so applies to complex-valued solutions. That said, for ease of writing some estimates we will sometimes replace the nonlinearity $|u|^2 u$ with u^3, though the estimates remain true in the $\mathbb{C}$-valued case.*

4.2 Some analytical ingredients

4.2.1 Local theory

We first make precise what we mean by a *solution* in the energy space:

Definition 4.5. *A function $u : I \times \mathbb{R}^4 \to \mathbb{C}$ on a time interval $I = [0, T)$ $(0 < T \leq \infty)$ is a solution of (4.1) if $u \in (C_t \dot{H}^1_x \cap L^6_{t,x})([0, t] \times \mathbb{R}^4)$; $\nabla u \in L^3_{x,t}([0, t] \times \mathbb{R}^4)$;*

$D^2u,\ u_t \in L^2_tL^2_x([0,t]\times\mathbb{R}^4)$ for all $t\in I$; and the Duhamel formula

$$u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-s)\Delta}F(u(s))ds, \tag{4.10}$$

is satisfied for all $t\in I$, where $F(u)=|u|^2u$. We refer to the interval I as the lifespan of u. We say that u is a maximal-lifespan solution if the solution cannot be extended to any strictly larger interval. We say that u is a global solution if $I=\mathbb{R}^+ := [0,+\infty)$.

We will often measure the space-time size of solutions on a time interval I in $L^6_{x,t}$, denoting

$$S_I(u) := \int_I\int_{\mathbb{R}^4} |u(t,x)|^6 dxdt, \quad \|u\|_{S(I)} := S_I(u)^{\frac{1}{6}} = \left(\int_I\int_{\mathbb{R}^4} |u(t,x)|^6 dxdt\right)^{\frac{1}{6}}.$$

A local wellposedness theory in the energy space $\dot{H}^1(\mathbb{R}^4)$, analogous to that for the corresponding critical nonlinear Schrödinger equation (see e.g., [21]), is easily constructed, based on the Sobolev inequality and space-time estimates for the heat equation on $\mathbb{R}^4$ ([60]),

$$\begin{aligned}
&\|e^{t\Delta}\phi\|_{L^p_x(\mathbb{R}^4)} \lesssim t^{-2(1/a-1/p)}\|\phi\|_{L^a}, \qquad 1\le a\le p\le\infty\\
&\|e^{t\Delta}\phi\|_{L^q_tL^p_x(\mathbb{R}^+\times\mathbb{R}^4)} \lesssim \|\phi\|_{L^a}, \qquad \frac{1}{q}+\frac{2}{p}=\frac{2}{a}, \quad 1<a\le q\\
&\Big\|\int_0^t e^{(t-s)\Delta}f(s)ds\Big\|_{L^q_tL^p_x(\mathbb{R}^+\times\mathbb{R}^4)} \lesssim \|f\|_{L^{\tilde{q}'}_tL^{\tilde{p}'}_x(\mathbb{R}^+\times\mathbb{R}^4)},\\
&\qquad \frac{1}{q}+\frac{2}{p}=\frac{1}{\tilde{q}}+\frac{2}{\tilde{p}}=1, \qquad \frac{1}{q}+\frac{1}{q'}=\frac{1}{p}+\frac{1}{p'}=1,
\end{aligned} \tag{4.11}$$

and $(\tilde{q}',\tilde{p}')$ the dual to any admissible pair $(\tilde{q},\tilde{p})$.

We also refer the reader to [11, 128] for a treatment of the Cauchy problem in the critical Lebesque space $L^{\frac{2d}{d-2}}$; the arguments directly adapt to show wellposedness in $\dot{H}^1$. One can use a fixed-point argument to construct local-in-time solutions for arbitrary initial data in $\dot{H}^1(\mathbb{R}^4)$; however, as usual when working in critical scaling spaces, the time of existence depends on the profile of the initial data, not merely on its $\dot{H}^1$-norm. We summarize:

Theorem 4.6. *(Local well-posedness) Assume* $u_0 \in \dot{H}^1(\mathbb{R}^4)$.

1. *(Local existence) There exists a unique, maximal-lifespan solution to the Cauchy Problem* (4.1) *in* $I \times \mathbb{R}^4, I = [0, T_{max}(u_0))$

2. *(Continuous dependence) The solution depends continuously on the initial data (in both the* $\dot{H}^1$ *and the* S_I*-induced topologies). Furthermore,* T_{max} *is a lower-semicontinuous function of the initial data.*

3. *(Blow-up criterion) If* $T_{max}(u_0) < +\infty$*, then* $\|u\|_{S([0,T_{max}(u_0)))} = +\infty$

4. *(Energy dissipation) the energy E(u(t)) is a non-increasing function in time. More precisely, for* $0 < t < T_{max}$*,*

$$E(u(t)) + \int_0^t \int_{\mathbb{R}^4} |u_t|^2 \, dx \, dt = E(u_0). \tag{4.12}$$

5. *(Small data global existence) There is* $\epsilon_0 > 0$ *such that if* $\|e^{t\Delta}u_0\|_{S(\mathbb{R}^+)} \leq \epsilon_0$*, the solution* u *is global,* $T_{max}(u_0) = \infty$*, and moreover*

$$\|u\|_{S(\mathbb{R}^+)} + \|\nabla u\|_{(L_t^\infty L_x^2 \cap L_{x,t}^3)(\mathbb{R}^+ \times \mathbb{R}^4)} + \|D^2 u\|_{L_{x,t}^2(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim \epsilon_0. \tag{4.13}$$

   This occurs in particular when $\|u_0\|_{\dot{H}^1(\mathbb{R}^4)}$ *is sufficiently small.*

An extension of the proof of the local existence theorem implies the following stability result (see, e.g., [83]):

Proposition 4.7. *(Perturbation result)*
For every $E, L > 0$ *and* $\epsilon > 0$ *there exists* $\delta > 0$ *with the following property: assume* $\tilde{u} : I \times \mathbb{R}^4 \to \mathbb{R}$*,* $I = [0, T)$*, is an approximate solution to (4.1) in the sense that*

$$\|\nabla e\|_{L_{t,x}^{\frac{3}{2}}(I \times \mathbb{R}^4)} \leq \delta, \quad e := \tilde{u}_t - \Delta\tilde{u} - |\tilde{u}|^2\tilde{u},$$

and also

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} \leq E \quad \textit{and} \quad \|\tilde{u}\|_{S(I)} \leq L,$$

then if $u_0 \in \dot{H}_x^1(\mathbb{R}^4)$ *is such that*

$$\|u_0 - \tilde{u}(0)\|_{\dot{H}_x^1(\mathbb{R}^4)} \leq \delta,$$

there exists a solution $u : I \times \mathbb{R}^4 \to \mathbb{R}$ *of* (4.1) *with* $u(0) = u_0$*, and such that*

$$\|u - \tilde{u}\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} + \|u - \tilde{u}\|_{S(I)} \leq \epsilon$$

4.2.2 Profile decomposition

The following proposition is the main tool (along with the Perturbation Proposition 4.7) used to establish the existence of a critical element. The idea is to characterize the loss of compactness in some critical embedding; it can be traced back to ideas in [98], [13], [121], [117] and their modern "evolution" counterparts [3], [79] and [80].

Proposition 4.8. *(Profile Decomposition)*
Let $\{u_n\}_n$ *be a bounded sequence of functions in* $\dot{H}^1(\mathbb{R}^4)$*. Then, after possibly passing to a subsequence (in which case, we rename it* u_n*), there exists a family of functions* $\{\phi^j\}_{j=1}^\infty \subset \dot{H}^1$*, scales* $\lambda_n^j > 0$ *and centers* $x_n^j \in \mathbb{R}^4$ *such that:*

$$u_n(x) = \sum_{j=1}^J \frac{1}{\lambda_n^j}\phi^j\left(\frac{x - x_n^j}{\lambda_n^j}\right) + w_n^J(x),$$

$w_n^J \in \dot{H}^1(\mathbb{R}^4)$ *is such that:*

$$\lim_{J\to\infty} \limsup_n \|e^{t\Delta} w_n^J\|_{L^6_{t,x}(\mathbb{R}^+\times\mathbb{R}^4)} = 0, \tag{4.14}$$

$$\lambda_n^j w_n^J(\lambda_n^j x + x_n^j) \rightharpoonup 0, \quad \textit{in} \ \ \dot{H}^1(\mathbb{R}^4), \quad \forall j \leq J. \tag{4.15}$$

Moreover, the scales are asymptotically orthogonal, in the sense that

$$\frac{\lambda_n^j}{\lambda_n^i}+\frac{\lambda_n^i}{\lambda_n^j}+\frac{|x_n^i-x_n^j|^2}{\lambda_n^j\lambda_n^i}\to+\infty,\quad \forall i\neq j \tag{4.16}$$

Furthermore, for all $J\geq 1$ *we have the following decoupling properties:*

$$\|u_n\|_{\dot H^1}^2=\sum_{j=1}^{J}\|\phi^j\|_{\dot H^1}^2+\|w_n^J\|_{\dot H^1}^2+o_n(1) \tag{4.17}$$

and

$$E(u_n)=\sum_{j=1}^{J}E(\phi^j)+E(w_n^J)+o_n(1) \tag{4.18}$$

The proof follows exactly the same steps with the proof presented in Chapter 2 - before the transformation that was connecting the higher-dimensional to the two-dimensional estimates- and it can be easily finished from (2.40) through the standard heat space-times estimates and the same interpolation procedure as before.

4.2.3 Variational estimates

The elementary variational inequalities we use are summarized here:

Lemma 4.9. *(Variational Estimates)*

1. *If*

$$\|\nabla u_0\|_{L^2}^2\leq\|\nabla W\|_{L^2}^2,\quad E(u_0)\leq(1-\delta_0)E(W),\ \ \delta_0>0,$$

then there exists $\bar\delta=\bar\delta(\delta_0)>0$ *such that for all* $t\in[0,T_{max}(u_0))$*, the solution of* (4.1) *satisfies*

$$\int|\nabla u(t)|^2\leq(1-\bar\delta)\int|\nabla W|^2. \tag{4.19}$$

2. *If* (4.19) *holds, then*

$$\int(|\nabla u(t)|^2-|u(t)|^4)dx\geq\bar\delta\int|\nabla u(t)|^2 \tag{4.20}$$

and moreover $E(u(t)) \geq 0$.

Proof. The second statements are an immediate consequence of the sharp Sobolev inequality (4.5):

$$\int (|\nabla u(t)|^2 - |u(t)|^4)dx \geq \left[1 - \left(\frac{\|\nabla u(t)\|_{L^2}}{\|\nabla W\|_{L^2}}\right)^2\right] \|\nabla u\|_{L^2}^2 \gtrsim \|\nabla u\|_{L^2}^2$$

while the first follows easily from Sobolev and energy dissipation (4.12); see, e.g., Lemma 3.4/Theorem 3.9 in [79]. $\square$

4.3 Asymptotic decay of global solutions

In this section we prove the following theorem:

Theorem 4.10. *If $u \in C([0,\infty); \dot{H}^1(\mathbb{R}^4))$ is a solution to equation (4.1) which moreover satisfies*

$$\sup_{t\geq 0} \|\nabla u(t)\|_{L^2} < \|\nabla W\|_{L^2}, \tag{4.21}$$

then

$$S_{\mathbb{R}_+}(u) < \infty \quad \textit{and} \quad \lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0.$$

Proof. The general strategy, drawn from the techniques of [56] for the Navier-Stokes equations, is as follows. We first show that global solutions for which $S_{\mathbb{R}^+}(u) < \infty$ – which includes *small* solutions by the small data theory (4.13) – decay to zero in the $\dot{H}^1-$norm. Second, we impose the extra assumption of H^1- data, so that we may exploit the L^2-dissipation relation to show finiteness of $\|\nabla u\|_{L^2_{x,t}}$, which in turns allows us to reduce matters to the case of small $\dot{H}^1$ data. Finally, to remove this extra assumption, we split the initial data in frequency, and estimate a perturbed equation.

Proposition 4.11. *If u is a global solution of (4.1) with $S_{\mathbb{R}^+}(u) < \infty$, then*

$$\lim_{t\to\infty} \|\nabla u(t,\cdot)\|_{L^2} = 0. \tag{4.22}$$

Proof. Let $u \in (C_t\dot{H}^1_x \cap L^6_{t,x})(\mathbb{R}_+ \times \mathbb{R}^4)$ be a global solution to (4.1). Just as one proves the blow-up criterion for the local theory Theorem 4.6, we first show:

Claim 4.12. $\|\nabla u\|_{L^3_{t,x}(\mathbb{R}^+\times\mathbb{R}^4)} < \infty$

Proof. Since $u \in L^6_{t,x}(\mathbb{R}^+ \times \mathbb{R}^4)$, given $\eta > 0$, we may subdivide $R^+ = [0, \infty)$ into a finite number of subintervals $I_j = [a_j, a_{j+1})$, $j = 0, 1, \ldots, J$, $0 = a_0 < a_1 < \cdots < a_J = \infty$, on which $\|u\|_{L^6_{t,x}}(I_j) \leq \eta$. Taking ∇ in the Duhamel formula (4.10) and using (4.11):

$$\begin{aligned}\|\nabla u\|_{(L^3_{t,x}\cap L^\infty_t L^2_x)}(I_0) &\leq C\|e^{t\Delta}\nabla u_0\|_{L^2} + C\|\int_0^t S(t-s)\nabla(u^3)ds\|_{L^3_{t,x}(I_0)} \\ &\leq C\|u_0\|_{\dot{H}^1} + C\|u^2\nabla u\|_{L^{3/2}_{t,x}(I_0)} \\ &\leq C\|u_0\|_{\dot{H}^1} + C\|u\|^2_{L^6_{t,x}(I_0)}\|\nabla u\|_{L^3_{t,x}(I_0)},\end{aligned}$$

so by choosing $\eta < \frac{1}{\sqrt{2C}}$ we ensure

$$\|\nabla u\|_{(L^3_{t,x}\cap L^\infty_t L^2_x)(I_0)} \leq 2C\|u_0\|_{\dot{H}^1}.$$

In particular $\|u(a_1)\|_{\dot{H}^1} \leq 2C\|u_0\|_{\dot{H}^1}$, and so we may repeat this argument on the next interval I_1 to find $\|\nabla u\|_{(L^3_{t,x}\cap L^\infty_t L^2_x)(I_1)} \leq (2C)^2\|u_0\|_{\dot{H}^1}$, and, continuing, $\|\nabla u\|_{(L^3_{t,x}\cap L^\infty_t L^2_x)(I_j)} \leq (2C)^{j+1}\|u_0\|_{\dot{H}^1}$, for $j = 0, 1, \ldots, J$. The claim follows. □

Now denote the linear evolution by $S(t) = e^{t\Delta}$, so the solution in Duhamel form is written

$$u(t) = S(t)u_0 + \int_0^t S(t-s)u^3(s)ds$$

Let

$$\mathrm{I} := S(t)u_0, \quad \mathrm{II} := \int_0^\tau S(t-s)u^3(s)ds, \quad \mathrm{III} := \int_\tau^t S(t-s)u^3(s)ds,$$

for some τ to be determined later.

For term I we will take advantage of the decay of the heat propagator. By density, we can approximate ∇u_0 by $v \in L^1 \cap L^2$ and use a standard heat estimate:

$$\begin{aligned}\|\mathrm{I}\|_{\dot{H}^1} = \|S(t)\nabla u_0\|_{L^2} &\le \|S(t)(\nabla u_0 - v)\|_{L^2} + \|S(t)v\|_{L^2} \\ &\le \|\nabla u_0 - v\|_{L^2} + \|S(t)v\|_{L^2}.\end{aligned}$$

The first term can be made arbitrary small by the choice of v, while for the second, by (4.11), $\|S(t)v\|_{L^2} \to 0$ as $t \to \infty$, hence

$$\|I\|_{\dot{H}^1} \to 0 \text{ as } t \to \infty.$$

We now treat term III, which will allow us to fix τ. By the claim, for any $\epsilon > 0$, we can find τ such that $\|u\|_{L^6_{t,x}([\tau,\infty)\times\mathbb{R}^4)}, \|\nabla u\|_{L^3_{t,x}([\tau,\infty)\times\mathbb{R}^4)} \le \epsilon$. Since we are considering the limit $t \to \infty$, we may assume $t > \tau \gg 1$, and so by the same estimate of the nonlinear term as in the proof of the claim,

$$\|III\|_{\dot{H}^1} \lesssim \|u\|^2_{L^6_{t,x}([\tau,t)\times\mathbb{R}^4)} \|\nabla u\|_{L^3_{t,x}([\tau,t)\times\mathbb{R}^4)} \lesssim \epsilon^3.$$

Having fixed τ in this manner, we turn to term II. First notice that

$$II = \int_0^\tau S(t-s)u^3(s)ds = S(t-\tau)\int_0^\tau S(\tau - s)u^3(s)ds.$$

Since $\int_0^\tau S(\tau-s)u^3(s)ds \in \dot{H}^1$ (by $u \in L^6_{x,t}$ and (4.11)), the same approximation argument used for term I shows

$$\|II\|_{\dot{H}^1} = \|S(t-\tau)\int_0^\tau S(\tau-s)u^3(s)ds\|_{\dot{H}^1} \xrightarrow{t\to\infty} 0.$$

Since ϵ was arbitrary, (4.22) follows. $\square$

Now if we assume $u_0 \in H^1(\mathbb{R}^4)$, multiplying (4.1) by u and integrating over space-time yields the L^2 dissipation relation

$$\|u(t)\|^2_{L^2} = \|u_0\|^2_{L^2} + 2\int_0^t \int_{\mathbb{R}^4} [u^4 - |\nabla u|^2] dxds. \tag{4.23}$$

Because of (4.21), we have the variational estimate (4.20) and so for some $\bar{\delta} > 0$,

$$\sup_{t\geq 0} \|u(t)\|_{L^2}^2 + 2\bar{\delta}\|\nabla u\|_{L^2_{t,x}(\mathbb{R}_+\times\mathbb{R}^4)}^2 \leq \|u_0\|_{L^2}^2$$

This estimate immediately implies that for any $\epsilon_0 > 0$, there is some time t_0 such that $\|u(t_0)\|_{\dot{H}^1} \leq \epsilon_0$, and we can directly apply the small data result (4.13) (with initial time $t = t_0$) to conclude that $S_{\mathbb{R}^+}(u) < \infty$, and so by Proposition 4.11, $\lim_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0$, as required.

To remove the extra assumption $u_0 \in L^2$, split

$$u_0 = w_0 + v_0, \qquad \|w_0\|_{\dot{H}^1} \ll 1, \qquad v_0 \in H^1.$$

Define $w(t)$ to be the solution to (4.1) with initial data w_0:

$$\begin{aligned} w_t &= \Delta w + w^3 \\ w(0,x) &= w_0(x) \in \dot{H}^1(\mathbb{R}^d) \end{aligned}$$

From the small data theory (4.13), $w \in C_t\dot{H}^1_x(\mathbb{R}_+\times\mathbb{R}^4)$ is global, with

$$\|w\|_{L^6_{t,x}(\mathbb{R}_+\times\mathbb{R}^4)} + \|\nabla w\|_{(L^\infty_t L^2_x \cap L^3_{t,x})(\mathbb{R}_+\times\mathbb{R}^4)} \lesssim \|\nabla w_0\|_{L^2} \ll 1 \tag{4.24}$$

and by Proposition 4.11, $\|w(t)\|_{\dot{H}^1} \xrightarrow{t\to\infty} 0$.

Defining v by $v := u - w$, it will be a solution of the perturbed equation

$$v_t - \Delta v = v^3 + 3w^2 v + 3wv^2.$$

Just as in the derivation of the L^2-dissipation relation (4.23), multiply by v and integrate in space-time:

$$\|v(t)\|_{L^2}^2 - \|v_0\|_{L^2}^2 + 2\int_0^t \|\nabla v\|_{L^2}^2 = 2\int_0^t \|v\|_{L^4}^4 + 6\int_0^t\int_{\mathbb{R}^4} w^2v^2 + 6\int_0^t\int_{\mathbb{R}^4} wv^3.$$

By (4.24), picking $\|\nabla w_0\|_{L^2}$ small enough, ensures that condition (4.21) holds also for v: $\sup_{t\geq 0}\|\nabla v(t)\|_{L^2} < \|\nabla W\|_{L^2}$. Hence by (4.20), for some $\bar{\delta} > 0$,

$$\|v(t)\|_{L^2}^2 + \bar{\delta}\int_0^t \|\nabla v\|_{L^2}^2 \lesssim \|v_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^4} w^2 v^2 + 6\int_0^t \int_{\mathbb{R}^4} wv^3,$$

and so by Hölder and Sobolev,

$$\begin{aligned}\|v(t)\|_{L^2}^2 + \bar{\delta}\|\nabla v\|_{L^2L^2}^2 &\lesssim \|v_0\|_{L^2}^2 + \|w\|_{L^\infty L^4}^2\|v\|_{L^2L^4}^2 + \|w\|_{L^\infty L^4}\|v\|_{L^\infty L^4}\|v\|_{L^2L^4} \\ &\lesssim \|v_0\|_{L^2}^2 + \|\nabla w\|_{L^\infty L^2}^2\|\nabla v\|_{L^2L^2}^2 + \|\nabla w\|_{L^\infty L^2}\|\nabla v\|_{L^\infty L^2}\|\nabla v\|_{L^2L^2}.\end{aligned}$$

So by (4.24), choosing $\|\nabla w_0\|_{L^2}$ small enough yields $\int_0^\infty \|\nabla v\|_{L^2}^2 dt < \infty$, and hence there is $T > 0$ for which $\|\nabla v(T)\|_{L^2} < \|w_0\|_{\dot{H}^1}$ and so $\|\nabla u(T)\|_{L^2} \leq 2\|\nabla w_0\|_{L^2}$. Choosing $\|\nabla w_0\|_{L^2}$ smaller still, if necessary, we are able to apply the small data result (4.13) to conclude $S_{\mathbb{R}_+}(u) < \infty$, and moreover by Proposition 4.11,

$$\lim_{t\to\infty}\|u(t)\|_{\dot{H}^1} = 0,$$

concluding the proof of the theorem. □

4.4 Minimal blow-up solution

For any $0 \leq E_0 \leq \|\nabla W\|_2^2$, we define

$$L(E_0) := \sup\{S_I(u) \mid u \text{ a solution of (4.1) on } I \text{ with } \sup_{t\in I}\|\nabla u(t)\|_2^2 \leq E_0\},$$

where $I = [0,T)$ denotes the existence interval of the solution in question. $L : [0, \|\nabla W\|_2^2] \to [0,\infty]$ is a continuous (this follows from Proposition 4.7), non-decreasing function with $L(\|\nabla W\|_2^2) = \infty$. Moreover, from the small-data theory (4.13),

$$L(E_0) \lesssim E_0^3 \text{ for } E_0 \leq \epsilon_0.$$

Thus, there exists a unique critical kinetic energy $E_c \in (0, \|\nabla W\|_2^2]$ such that

$$L(E_0) < \infty \text{ for } E_0 < E_c, \quad L(E_0) = \infty \text{ for } E_0 \geq E_c.$$

In particular, if $u : I \times \mathbb{R}^4 \to \mathbb{R}$ is a maximal-lifespan solution, then

$$\sup_{t \in I} \|\nabla u(t)\|_2^2 < E_c \implies u \text{ is global, and } \|u\|_{S(\mathbb{R}^+)} \leq L(\sup_{t \in I} \|\nabla u(t)\|_2^2) < \infty.$$

The goal of this section is the proof of the following theorem:

Theorem 4.13. *There is a maximal-lifespan solution $u_c : I \times \mathbb{R}^4 \to \mathbb{R}$ to (4.1) such that $\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 = E_c$, $\|u_c\|_{S(I)} = +\infty$. Moreover, there are $x(t) \in \mathbb{R}^4, \lambda(t) \in \mathbb{R}^+$, such that*

$$K = \left\{ \frac{1}{\lambda(t)} u_c \left(t, \frac{x - x(t)}{\lambda(t)} \right) \mid t \in I \right\} \tag{4.25}$$

is precompact in $\dot{H}^1$.

For the proof of this theorem we closely follow the arguments in [82] . The extraction of this minimal blow-up solution (and its compactness up to scaling and translation) will be a consequence of the following proposition:

Proposition 4.14. *Let $u_n : I_n \times \mathbb{R}^4$ be a sequence of solutions to (4.1) such that*

$$\limsup_n \sup_{t \in I_n} \|\nabla u_n\|_2^2 = E_c \text{ and } \lim_{n \to \infty} \|u_n\|_{S(I_n)} = +\infty. \tag{4.26}$$

where I_n are of the form $[0, T_n)$. Denote the initial data by $u_n(x, 0) = u_{n,0}(x)$. Then the sequence $\{u_{n,0}\}_n$ converges, modulo scaling and translations, in $\dot{H}^1$ (up to an extraction of a subsequence).

Proof. The sequence $\{u_{n,0}\}_n$ is bounded in $\dot{H}^1$ by (4.26) so applying the profile decomposition (up to a further subsequence) we get

$$u_{n,0}(x) = \sum_{j=1}^{J} \frac{1}{\lambda_n^j} \phi^j (\frac{x - x_n^j}{\lambda_n^j}) + w_n^J(x)$$

with the properties listed in Proposition 4.8.

Define the nonlinear profiles $v^j : I^j \times \mathbb{R}^4 \to \mathbb{R}$, $I^j = [0, T^j_{max})$, associated to ϕ^j by setting them to be the maximal-lifespan solutions of (4.1) with initial data $v^j(0) = \phi^j$.

Also, for each $j, n \geq 1$ we introduce $v^j_n : I^j_n \times \mathbb{R}^4 \to \mathbb{R}$ by

$$v^j_n(t) = \frac{1}{\lambda^j_n} v^j \left(\frac{t}{(\lambda^j_n)^2}, \frac{x - x^j_n}{\lambda^j_n} \right), \quad I^j_n := \{t \in \mathbb{R} : \frac{t}{(\lambda^j_n)^2} \in I^j\}.$$

Each v^j_n is a solution with $v^j_n(0) = \frac{1}{\lambda^j_n}\phi(\frac{x-x^j_n}{\lambda^j_n})$ and maximal lifespan $I^j_n = [0, T^{n,j}_{\max})$, $T^{n,j}_{max} = (\lambda^j_n)^2 T^j_{max}$.

For large n, by the asymptotic decoupling of the kinetic energy (property (4.17)), there is a $J_0 \geq 1$ such that $\|\nabla\phi^j\|_2^2 \leq \epsilon_0$ for all $j \geq J_0$, where ϵ_0 is as in Theorem 4.6, 4. Hence, for $j \geq J_0$, the solutions v^j_n are global and decaying to zero, and moreover

$$\sup_{t \in \mathbb{R}^+} \|\nabla v^j_n\|_2^2 + \|v^j_n\|^2_{S(\mathbb{R}^+ \times \mathbb{R}^4)} \lesssim \|\nabla\phi^j\|_2^2 \tag{4.27}$$

by the small data theory (4.13).

Claim 4.15. *(There is at least one bad profile). There exists* $1 \leq j_0 < J_0$ *such that* $\|v^{j_0}\|_{S(I^{j_0})} = \infty$.

For contradiction, assume that for all $1 \leq j < J_0$

$$\|v^j\|_{S(I^j)} < \infty \tag{4.28}$$

which by the local theory implies $I^j = I^j_n = [0, \infty)$ for all such j and for all n. The goal is to deduce a bound on $\|u_n\|_{S(I_n)}$for sufficiently large n. To do so, we will use Proposition 4.7, for which we first need to introduce a good approximate solution. Define

$$u^J_n(t) = \sum_{j=1}^{J} v^j_n(t) + e^{t\Delta} w^J_n. \tag{4.29}$$

We will show that for n and J large enough this is a good approximate solution (in the sense of Proposition 4.7) and that $\|u_n^J\|_{S([0,+\infty))}$ is uniformly bounded. The validity of both points implies that the true solutions u_n should not satisfy (4.26), reaching a contradiction.

First observe

$$\sum_{j\geq 1}\|v_n^j\|_{S([0,\infty))}^2 = \sum_{j=1}^{J_0-1}\|v_n^j\|_{S([0,\infty))}^2 + \sum_{j\geq J_0}\|v_n^j\|_{S([0,\infty))}^2 \tag{4.30}$$

$$\lesssim 1 + \sum_{j\geq J_0}\|\nabla\phi^j\|_2^2 \lesssim 1 + E_c \tag{4.31}$$

where we have used (4.28), property (4.17) and (4.26).

Now, using the above and (4.14) in Proposition 4.8:

$$\lim_{J\to\infty}\overline{\lim_{n}}\,\|u_n^J\|_{S([0,+\infty))} \lesssim 1 + E_c. \tag{4.32}$$

For convenience, denote

$$\|u\|_{\tilde{S}(I)} := \|\nabla u\|_{L^3_{x,t}(I\times\mathbb{R}^4)}.$$

Under the assumption (4.28), we can also obtain

$$\|v^j\|_{\tilde{S}(I^j)} < \infty,$$

and so similarly we have

$$\lim_{J\to\infty}\overline{\lim_{n}}\,\|u_n^J\|_{\tilde{S}([0,+\infty))} < \infty.$$

To apply Proposition 4.7, it suffices to show that u_n^J asymptotically solves (4.1) in the sense that

$$\lim_{J\to\infty}\overline{\lim_{n}}\,\|\nabla[(\partial_t-\Delta)u_n^J - F(u_n^J)]\|_{L^{\frac{3}{2}}_{t,x}([0,+\infty)\times\mathbb{R}^4)} = 0$$

which reduces (adding and subtracting the term $F(\sum_{j=1}^J v_n^j)$ and using the triangle inequality) to proving

$$\lim_{J\to\infty}\overline{\lim_{n}}\,\|\nabla[\sum_{j=1}^{J}F(v_n^j)-F(\sum_{j=1}^{J}v_n^j)]\|_{L^{\frac{3}{2}}_{t,x}([0,+\infty)\times\mathbb{R}^4)}=0 \tag{4.33}$$

and

$$\overline{\lim_{n}}\,\|\nabla[F(u_n^J-e^{t\Delta}w_n^J)-F(u_n^J)]\|_{L^{\frac{3}{2}}_{t,x}([0,+\infty)\times\mathbb{R}^4)}=0. \tag{4.34}$$

The following easy pointwise estimate will be of use:

$$|\nabla[(\sum_{j=1}^{J}F(v_j)-F(\sum_{j=1}^{J}v_j)]|\lesssim_J\sum_{i\neq j}|\nabla v_j||v_i|^2. \tag{4.35}$$

We have shown that for all $j\geq 1$ and n large enough $v_n^j\in\tilde{S}([0,\infty))$, so using property (4.16)

$$\overline{\lim_{n}}\||v_n^j|^2\nabla v_n^i\|_{L^{\frac{3}{2}}_{t,x}([0,\infty)\times\mathbb{R}^4)}=0$$

for all $i\neq j$; thus

$$\overline{\lim_{n}}\|\nabla[(\sum_{j=1}^{J}F(v_j)-F(\sum_{j=1}^{J}v_j)]\|_{L^{\frac{3}{2}}_{t,x}}\lesssim_J\lim_{n\to\infty}\overline{\lim_{n}}\sum_{i\neq j}\|\nabla v_n^j|v_n^i|^2\|_{L^{\frac{3}{2}}_{t,x}}=0$$

settling (4.33).

$$\|\nabla[F(u_n^J-e^{t\Delta}w_n^J)-F(u_n^J)]\|_{L^{\frac{3}{2}}_{t,x}}\lesssim\|\nabla e^{t\Delta}w_n^J\|_{L^3_{t,x}}\|e^{t\Delta}w_n^J\|^2_{L^6_{t,x}}+\||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L^{\frac{3}{2}}_{t,x}}+$$

$$\|\nabla u_n^J\|_{L^3_{t,x}}\|e^{t\Delta}w_n^J\|^2_{L^6_{t,x}}+\|\nabla u_n^J\|_{L^3_{t,x}}\|e^{t\Delta}w_n^J\|_{L^6_{t,x}}\|u_n^J\|_{L^6_{t,x}}$$

The first, third and fourth terms are easy to be seen to converge to zero (using the space-time estimates, the fact that w_n^J is bounded in $\dot{H}^1$ and (4.14)), so (4.34) is reduced to showing

$$\lim_{J\to\infty}\overline{\lim_{n}}\||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L^{\frac{3}{2}}_{t,x}}=0.$$

By Hölder and the space-time estimates,

$$\begin{aligned}
\||u_n^J|^2\nabla e^{t\Delta}w_n^J\|_{L^{\frac{3}{2}}_{t,x}} &\lesssim \|u_n^J\|^{\frac{3}{2}}_{L^6_{t,x}}\|\nabla e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^3_{t,x}}\|u_n^J\nabla e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^2_{t,x}}\\
&\lesssim \|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^2_{t,x}} + \|e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^6_{t,x}}\|\nabla e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^3_{t,x}}\\
&\lesssim \|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^2_{t,x}} + \|e^{t\Delta}w_n^J\|^{\frac{1}{2}}_{L^6_{t,x}}.
\end{aligned}$$

Again due to (4.14) it suffices to prove

$$\lim_{J\to\infty}\overline{\lim_{n}}\|(\sum_{j=1}^J v_n^j)\nabla e^{t\Delta}w_n^J\|_{L^2_{t,x}} = 0.$$

For any $\eta > 0$ by summability, we see that there exists $J' = J'(\eta) \geq 1$ such that $\sum_{j\geq J'}\|v_n^j\|_{S([0,\infty))} \leq \eta$. For this J',

$$\overline{\lim_{n}}\|\left(\sum_{j=J'}^J v_n^j\right)\nabla e^{t\Delta}w_n^J\|^6_{L^2_{t,x}} \lesssim \overline{\lim_{n}}\left(\sum_{j\geq J'}\|v_n^j\|_{S([0,\infty))}\right)\|\nabla e^{t\Delta}w_n^J\|^6_{L^3_{t,x}} \lesssim \eta.$$

As $\eta > 0$ is arbitrary, it suffices to show

$$\lim_{J\to\infty}\overline{\lim_{n}}\|v_n^j\nabla e^{t\Delta}w_n^J\|_{L^2_{t,x}} = 0,\ 1 \leq j \leq J'.$$

Changing variables and assuming (by density) $v^j \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^4)$, by Hölder and the scale-invariance of the norms, proving (4.34) reduces to proving

$$\lim_{J\to\infty}\overline{\lim_{n}}\|\nabla e^{t\Delta}w_n^J\|_{L^2_{t,x}(K)} = 0,$$

for any compact $K \in \mathbb{R}^+ \times \mathbb{R}^4$. The proof is a direct modification of the proof of the Lemma 2.5 in [83].

We have verified all the requirements of the stability proposition (4.7), hence we conclude that

$$\|u_n\|_{S([0,\infty))} \lesssim 1 + E_c$$

contradicting (4.26).

The problem now is that the kinetic energy is not conserved. The difficulty arises from the possibility that the S-norm of several profiles is large over short times, while their kinetic energy does not achieve the critical value until later. To finish the proof of proposition we have to prove that only one profile is responsible for the blow-up.

We can now (after possibly rearranging the indices) assume there exists $1 \leq J_1 < J_0$ such that

$$\|v^j\|_{S(I^j)} = \infty, 1 \leq j \leq J_1 \text{ and } \|v^j\|_{S([0,\infty))} < \infty, j > J_1$$

Again, we follow the combinatorial argument of [83]: for each integer $m, n \geq 1$, define an integer $j = j(m,n) \in \{1, ..., J_1\}$ and an interval K_n^m of the form $[0, \tau]$ by

$$\sup_{1\leq j\leq J_1} \|v_n^j\|_{S(K_n^m)} = \|v_n^{j(m,n)}\|_{S(K_n^m)} = m \tag{4.36}$$

By the pigeonhole principle, there is a $1 \leq j \leq J_1$ such that for infinitely many m one has $j(m,n) = j_1$ for infinitely many n. Reordering the indices, if necessary, we may assume $j_1 = 1$. By the definition of the critical kinetic energy

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \sup_{t\in K_n^m} \|\nabla v_n^1(t)\|_2^2 \geq E_c \tag{4.37}$$

By (4.36), all v_n^j have finite S-norms on K_n^m for each $m \geq 1$. In the same way as before, we check again that the assumptions of Proposition 4.7 are satisfied to conclude that for J and n large enough, u_n^J is a good approximation to u_n on K_n^m. In particular we have for each $m \geq 1$,

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|u_n^J - u_n\|_{L_t^\infty \dot{H}_x^1(K_n^m \times \mathbb{R}^4)} = 0 \tag{4.38}$$

Lemma 4.16. *(Kinetic energy decoupling for later times). For all $J \geq 1$ and $m \geq 1$,*

$$\limsup_{n\to\infty} \sup_{t\in K_n^m} |\|\nabla u_n^J(t)\|_2^2 - \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_2^2 - \|\nabla w_n^J\|_2^2| = 0 \tag{4.39}$$

Proof. Fix $J \geq 1$ and $m \geq 1$. Then, for all $t \in K_n^m$,

$$\|\nabla u_n^J(t)\|_2^2 =< \nabla u_n^J(t), \nabla u_n^J(t) >= \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_2^2 + \|\nabla w_n^J\|_2^2$$

$$+\sum_{j\neq j'} < \nabla v_n^j(t), \nabla v_n^{j'}(t) > +2\sum_{j=1}^{J} < \nabla e^{t\Delta} w_n^J, \nabla v_n^j(t) >$$

It suffices to prove (for all sequences $t_n \in K_n^m$) that

$$< \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n) > \xrightarrow{n\to\infty} 0 \tag{4.40}$$

and

$$< \nabla e^{t_n\Delta} w_n^J, \nabla v_n^j(t_n) > \xrightarrow{n\to\infty} 0. \tag{4.41}$$

Since $t_n \in K_n^m \subset [0, T_{\max}^{n,j})$, for all $1 \leq j \leq J_1$, we have $t_{n,j} := \frac{t_n}{(\lambda_n^j)^2} \in I^j$ for all $j \geq 1$. For $j > J_1$ the lifespan is $\mathbb{R}^+$. By refining the sequence using the standard diagonalization argument, we can assume that $t_{n,j}$ converges ($+\infty$ is also possible) for every j.

We deal with (4.40) first. If both $t_{n,j}, t_{n,j'} \to \infty$, necessarily $j, j' > J_1$ and $v^j, v^{j'}$ are global solutions satisfying the kinetic energy bound (4.21), so by Theorem (4.10) $\|v^j\|_{\dot{H}^1}, \|v^{j'}\|_{\dot{H}^1} \xrightarrow{t\to\infty} 0$. Employing Hölder's inequality and the scaling invariance of the $\dot{H}^1$-norm, we get (4.40) for this case. When $t_{n,j} \to \infty$ but $t_{n,j'} \to \tau_{j'}$: using the continuity of the flow in $\dot{H}^1$ we can be, for the limit, replacing $\nabla\{\frac{1}{\lambda_n^{j'}} v^{j'}(t_{n,j'}, \frac{x - x_n^j}{\lambda_n^{j'}})\}$ with $\nabla\{\frac{1}{\lambda_n^{j'}} v^{j'}(\tau_{j'}, \frac{x - x_n^j}{\lambda_n^{j'}})\}$. By an L^2- approximation, we can also assume we are working with smooth, compactly supported functions. In this case, we can bound $< \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n) >$ by $\|v^j(t_{n,j})\|_{\dot{H}^1} \|v^{j'}(\tau_{j'})\|_{\dot{H}^1} \to 0$, as $n \to \infty$. The remaining case is when both $t_{n,j}$ and $t_{n,j'}$ converge to finite $\tau_j, \tau_{j'}$ in the interior of $I^j, I^{j'}$ respectively. We can replace as above $t_{n,j}, t_{n,j'}$ by $\tau_j, \tau_{j'}$ respectively, and perform a change of variables:

$$< \nabla v_n^j(t_n), \nabla v_n^{j'}(t_n) >= \int (\frac{\lambda_n^j}{\lambda_n^{j'}})^2 \nabla v^j(\tau_j, x), \nabla v^{j'}(\tau_{j'}, \frac{\lambda_n^j}{\lambda_n^{j'}} x + \frac{x_n^j - x_n^{j'}}{\lambda_n^{j'}}) dx$$

which is going to zero assuming, without loss of generality that $\dfrac{\lambda_n^j}{\lambda_n^{j'}} \to 0$ and the functions in the integrand have been assumed to be compactly supported, thus concluding the case (4.40).

For the case (4.41), performing a change of variable:

$$< \nabla e^{t_n \Delta} w_n^J, \nabla v_n^j(t_n) >=< \nabla e^{t_{n,j}\Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)], \nabla v^j(t_{n,j}) > .$$

When $t_{n,j} \to \infty$, using Hölder, the $L^2 - L^2$ heat estimate (and the boundedness of w_n^J in $\dot{H}^1$ coming from the profile decomposition) and Theorem 4.10 as before, we get to the result. For the case $t_{n,j} \to \tau_j < +\infty$, we can, as before, replace $t_{n,j}$ by its limit τ_j in the integral $\displaystyle\int \nabla e^{t_{n,j}\Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)] \cdot \nabla v^j(\tau_j, x) dx.$. One can show (see Lemma 2.10 in [82], by an easy modification of Lemma 3.63 in [105]) using (4.14) that $e^{t_{n,j}\Delta}[\lambda_n^j w_n^J(\lambda_n^j x + x_n^j)] \rightharpoonup 0$ in $\dot{H}^1$, which concludes the proof of the case (4.41) and hence the proof of the Lemma. □

By (4.26), (4.38), (4.39), we get

$$E_c \geq \limsup_{n\to\infty} \sup_{t\in K_n^m} \|\nabla u_n(t)\|_{L^2}^2 = \lim_{J\to\infty} \limsup_{n\to\infty} \{\|\nabla w_n^J(t)\|_{L^2}^2 + \sup_{t \in K_n^m} \sum_{j=1}^{J} \|\nabla v_n^j(t)\|_{L^2}^2\}.$$

Taking a limit in m and employing (4.37), we see that we actually have equality everywhere. This implies that $J_1 = 1, v_n^j \equiv 0, \forall j \geq 2, w_n := w_n^1 \xrightarrow{\dot{H}^1} 0$. So $u_n(0,x) = \frac{1}{\lambda_n}\phi(\frac{x - x_n^1}{\lambda_n^1}) + w_n(x)$, for some functions $\phi, w_n \in \dot{H}^1, w_n \xrightarrow{s} 0$ in $\dot{H}^1$.

We have shown (for that sequence of solutions u_n we found) that for the corresponding sequence of initial data $u_{n,0} : \lambda_n^1 u_{n,0}(\lambda_n^1 x + x_n^1) \xrightarrow{\dot{H}^1} \phi$. This finishes the proof of Proposition 4.14. $\square$

Now, we are in a position to prove Theorem 4.13.

Proof. By the definition of such a threshold we can find a sequence of solutions $u_n : I_n \times \mathbb{R}^4 \to \mathbb{R}$, with I_n compact, so that

$$\sup_n \sup_{t \in I_n} \|\nabla u_n(t)\|_{L^2}^2 = E_c \text{ and } \lim_n \|u_n\|_{S(I_n)} = +\infty.$$

An application of Proposition 4.14 shows that (up to symmetries) the corresponding sequence of initial data converges to some ϕ, strongly in $\dot{H}^1$. By a further rescaling and translation we can take $\lambda_n^1 \equiv 1, x_n^1 \equiv 0$.

Let $u_c : I \times \mathbb{R}^4 \to \mathbb{R}$ be the maximal-lifespan solution with initial data ϕ. Since $u_{n,0} \xrightarrow{\dot{H}^1} \phi$, employing the stability Proposition 4.7, $I \subset \liminf I_n$, and $\|u_n - u_c\|_{L_t^\infty \dot{H}_x^1(K \times \mathbb{R}^4)} \xrightarrow{n\to\infty} 0$, for all compact $K \subset I$. Thus, by (4.26):

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 \leq E_c \tag{4.42}$$

Applying the stability Proposition 4.7 once again we can also see that $\|u_c\|_{S(I)} = \infty$. Hence, by the definition of the critical kinetic energy level, E_c,

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 \geq E_c \tag{4.43}$$

In conclusion,

$$\sup_{t \in I} \|\nabla u_c(t)\|_{L^2}^2 = E_c. \tag{4.44}$$

and

$$\|u_c\|_{S(I)} = +\infty. \tag{4.45}$$

Let $I := [0, T^*)$, where $T^* := T_{\max}(\phi)$, ϕ the data that corresponds to the previously found critical element u_c.

Finally, the compactness modulo symmetries (4.25) follows from another application of Proposition 4.14. We omit the standard proof (see for example [79] or [83]). $\square$

4.5 Rigidity

The main result of this section is the following theorem ruling out finite-time blowup of compact (modulo symmetries) solutions. Note this is a considerably *stronger* statement than we require, since it is not limited to solutions with below-threshold kinetic energy:

Theorem 4.17. *If u is a solution to* (4.1) *on maximal existence interval $I = [0, T^*)$, such that $K := \left\{ \frac{1}{\lambda(t)} u(t, \frac{x - x(t)}{\lambda(t)}) \mid t \in I \right\}$ is precompact in $\dot{H}^1$ for some $x(t) \in \mathbb{R}^4$, $\lambda(t) \in \mathbb{R}^+$, then $T^* = +\infty$.*

As a corollary, we can complete the proof of the main result Theorem 4.1 by showing:

Corollary 4.18. *For any solution satisfying* (4.8)*, $T_{max}(u(0)) = \infty$.*

Proof. By Theorem 4.17, the solution u_c produced by Theorem 4.13 must be global: $T_{max}(u_c(0)) = \infty$. But since $\|u_c\|_{S(\mathbb{R}^+)} = \infty$, Theorem 4.10 shows $E_c = \|\nabla W\|_2^2$, and the Corollary follows. $\square$

The rest of the section is devoted to the proof of the Theorem 4.17. Our proof is inspired by the work of Kenig and Koch [78] for the Navier-Stokes system, and it's based on classical parabolic tools – local smallness regularity, backwards uniqueness, and unique continuation – though implemented in a somewhat different way. In particular, we will make use of the following two results, proved in [49], [50] (also see [51]):

Theorem 4.19. *(Backwards Uniqueness) Fix any R, δ, M, and $c_0 > 0$. Let $Q_{R,\delta} := (\mathbb{R}^4 \setminus B_R(0)) \times (-\delta, 0)$, and suppose a vector-valued function v and its distributional derivatives satisfy $v, \nabla v, \nabla^2 v \in L^2(\Omega)$ for any bounded subset $\Omega \subset Q_{R,\delta}$, $|v(x,t)| \leq$*

$e^{M|x|^2}$ *for all* $(x,t) \in Q_{R,\delta}$*,* $|v_t - \Delta v| \leq c_0(|\nabla v| + |v|)$ *on* $Q_{R,\delta}$*, and* $v(x,0) = 0$ *for all* $x \in \mathbb{R}^4 \setminus B_R(0)$*. Then* $v \equiv 0$ *in* $Q_{R,\delta}$*.*

Theorem 4.20. *(Unique Continuation) Let* $Q_{r,\delta} := B_r(0) \times (-\delta, 0)$*, for some* $r, \delta > 0$*, and suppose a vector-valued function* v *and its distributional derivatives satisfy* $v, \nabla v, \nabla^2 v \in L^2(Q_{r,\delta})$ *and there exist* $c_0, C_k > 0, (k \in \mathbb{N})$ *such that* $|v_t - \Delta v| \leq c_0(|\nabla v| + |v|)$ *a.e. on* $Q_{r,\delta}$ *and* $|v(x,t)| \leq C_k(|x| + \sqrt{-t})^k$ *for all* $(x,t) \in Q_{r,\delta}$*. Then* $v(x,0) \equiv 0$ *for all* $x \in B_r(0)$*.*

As well, we establish the following:

Lemma 4.21. *(Local Smallness Regularity Criterion) There exist positive absolute constants* ϵ_0, c_k *for* $k \in \mathbb{N}$ *with the following property: if* u *is a solution of equation (4.1) on* Q_1*, where* $Q_r := B_r(0) \times (-r^2, 0)$ *for* $r > 0$*, and satisfies*

$$\|u\|_{L^\infty_t(\dot{H}^1_x \cap L^4_x)(Q_1)} < \epsilon_0$$

then u *is smooth on* $\overline{Q_{\frac{1}{2}}}$ *with bounds on all derivatives,*

$$\max_{\overline{Q_{\frac{1}{2}}}} |D^k u| \leq c_k.$$

Proof. Assume $\|u\|_{L^\infty_t(\dot{H}^1_x \cap L^4_x)(Q_1)} < \epsilon$, for ϵ small enough (to be picked). Define

$$\|u\|^2_{X(Q_1)} := \|\nabla u\|^2_{L^\infty_t L^2_x \cap L^2_t L^4_x(Q_1)} + \|u\|^2_{L^\infty_t L^4(Q_1)} + \|D^2 u\|^2_{L^2_t L^2_x(Q_1)}.$$

Assuming for ease of writing that u is real-valued, differentiating (4.1) and defining $\tilde{u} := \nabla u$, we get

$$\tilde{u}_t = \Delta \tilde{u} + 3u^2 \tilde{u}. \tag{4.46}$$

Consider a smooth, compactly supported spatial cut-off function $\phi_0(x)$ such that $\operatorname{supp}(\phi_0) \subset B_1(0)$ and $\phi_0 \equiv 1$ on $B_{\rho_0}(0)$, for some $\frac{1}{2} < \rho_0 < 1$ to be chosen. Multiplying the above equation by $\phi_0^2 \tilde{u}$ and integrating in space-time (from now on, unless otherwise specified, $t \in [-1,0]$):

$$\int_{-1}^{t}\int_{|x|\leq 1}(\tilde{u}_t - \Delta\tilde{u})\phi_0^2\tilde{u}\,dxdt = 3\int_{-1}^{t}\int_{|x|\leq 1}(u^2\tilde{u})\phi_0^2\tilde{u}\,dxdt$$

$$\Rightarrow \frac{1}{2}\|\phi_0\tilde{u}(t)\|_{L^2}^2 + \int_{-1}^{t}\int_{|x|\leq 1}\phi_0^2|\nabla\tilde{u}|^2\,dxdt$$

$$= \frac{1}{2}\|\phi_0\tilde{u}(0)\|_{L^2}^2 + 3\int_{-1}^{t}\int_{|x|\leq 1}\phi_0^2u^2\tilde{u}^2dxdt + 2\int_{-1}^{t}\int_{|x|\leq 1}\phi_0\nabla\phi_0(\tilde{u}\nabla\tilde{u})dxdt.$$

For the sake of brevity, let us define $v_0 := \phi_0\tilde{u} = \phi_0\nabla u$ and thus (always on the same cylinder):

$$\|v_0\|_{L_t^\infty L_x^2}^2 + \|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2}^2 \lesssim \|v_0(0,x)\|_{L^2}^2 + \|u^2\|_{L_t^\infty L_x^2}\|v_0^2\|_{L_t^1L_x^2} + \|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2}\|\tilde{u}\|_{L_t^2L_x^2}$$
$$= \|v_0(0,x)\|_{L^2}^2 + \|u\|_{L_t^\infty L_x^4}^2\|v_0\|_{L_t^2L_x^4}^2 + \|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2}\|\tilde{u}\|_{L_t^2L_x^2}$$

By the smallness assumed on the cylinder Q_1 and an application of Young's inequality, for any $\delta > 0$ (and also using Hölder and the boundedness of the domain):

$$\|v_0\|_{L_t^\infty L_x^2}^2 + \|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2}^2 \lesssim \epsilon^2 + \epsilon^2\|v_0\|_{L_t^2L_x^4(Q_1)}^2 + \delta^2\|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2(Q_1)} + \frac{\|\tilde{u}\|_{L_t^\infty L_x^4(Q_1)}^2}{\delta^2}$$

$$\Rightarrow \|v_0\|_{L_t^\infty L_x^2}^2 + \|\phi_0\nabla\tilde{u}\|_{L_t^2L_x^2}^2 \lesssim \epsilon^2 + \frac{\epsilon^2}{\delta^2} + \epsilon^2\|v_0\|_{L_t^2L_x^4}^2$$

if δ is chosen small enough. Since $\nabla v_0 = \phi_0\nabla\tilde{u} + \nabla\phi_0\,\tilde{u}$:

$$\|\nabla v_0\|_{L^2} \lesssim \|\phi_0\nabla\tilde{u}\|_{L^2} + \|\nabla\phi_0\|_{L^4}\|\tilde{u}\|_{L^4}$$

and so using the Sobolev inequality,

$$\|v_0\|_{L_t^\infty L_x^2}^2 + \|\nabla v_0\|_{L_t^2L_x^2}^2 + \|v_0\|_{L_t^2L_x^4}^2 \lesssim \epsilon^2 + \epsilon^2\|v_0\|_{L_t^2L_x^4}^2.$$

Choosing ϵ small enough yields

$$\|u\|_{X(Q_{\rho_0})} \lesssim \epsilon.$$

Define another smooth compactly supported cut-off function $\phi_1(x) \leq \phi_0(x)$, with support in B_{ρ_0}, and $\phi_1 \equiv 1$ on $B_{\rho_1}(0)$, some $\frac{1}{2} < \rho_1 < \rho_0 < 1$ to be chosen. Let $\hat{v} := D^2 u$, and $v_1 := \phi_1 \hat{v}$.

Remark 4.22. *We will be abusing notation from this point onwards. For the pointwise operations and estimates we are actually considering the mixed partial derivatives $\partial_k \partial_j u, j, k = 1, ..., 4$ but we will be writing $D^2 u$ all the same without taking care to specify the matrix element at hand. In the end, we are using standard matrix norms.*

Differentiating (4.46), multiplying by $\phi_1^2 \hat{v}$, and integrating over space gives

$$\begin{aligned} \frac{1}{2}\partial_t \int \phi_1^2 \hat{v}^2 dx + \int \phi_1^2 |\nabla \hat{v}|^2 dx = 3 \int \phi_1^2 u^2 \hat{v}^2 dx \\ + 6 \int \phi_1^2 u \tilde{u}^2 \hat{v} dx + 2 \int \phi_1 \nabla \phi_1 \cdot \hat{v} \nabla \hat{v} dx \end{aligned} \tag{4.47}$$

Since by the previous step, $\|\nabla v_0\|_{L^2 L^2(Q_{\rho_0})} \lesssim \epsilon$, we can find $-1 < t_1 < -\rho_0^2$ such that $\|\nabla v_0(\cdot, t_1)\|_{L^2(B_{\rho_0})} \lesssim \epsilon$ (where the implied constant may depend on ρ_0), so that

$$\|\phi_1 \hat{v}(\cdot, t_1)\|_{L^2} = \|\phi_1 D^2 u(\cdot, t_1)\|_{L^2} \leq \|\nabla v_0(\cdot, t_1)\|_{L^2(B_{\rho_0})} \lesssim \epsilon.$$

Integrating (4.47) in t from t_1 to 0, and using the estimates from the previous step:

$$\begin{aligned} \|v_1\|_{L_t^\infty L^2}^2 + \|\phi_1 \nabla \hat{v}\|_{L_t^2 L_x^2}^2 &\lesssim \|\phi_0 u\|_{L^\infty L^4}^2 \|v_1\|_{L^2 L^4}^2 + \|\phi_0 u\|_{L^\infty L^4} \|v_0\|_{L^\infty L^4} \|v_0\|_{L^2 L^4} \|v_1\|_{L^2 L^4} \\ &\quad + \|\hat{v} \nabla \hat{v} \phi_1 \nabla \phi_1\|_{L_t^1 L_x^1} + \epsilon^2 \\ &\lesssim \epsilon^2 \|v_1\|_{L^2 L^4}^2 + \epsilon^3 \|v_1\|_{L^2 L^4} + \|\phi_1 \nabla \hat{v}\|_{L^2 L^2} \|\nabla \phi_1 \hat{v}\|_{L^2 L^2} + \epsilon^2 \end{aligned}$$

where everywhere here the time interval is $[t_1, 0]$. We have

$$\nabla v_0 = \phi_0 D^2 u + \nabla \phi_0 \nabla u = \phi_0 \hat{v} + \nabla \phi_0 \tilde{u}$$

and so

$$|\phi_0\hat{v}| \lesssim |\nabla v_0| + |\nabla\phi_0\tilde{u}| \;\Rightarrow\; |\nabla\phi_1\hat{v}| \lesssim |\frac{\nabla\phi_1}{\phi_0}||\phi_0\hat{v}| \lesssim |\nabla v_0| + |\nabla\phi_0\tilde{u}|.$$

Thus

$$\|\nabla\phi_1\hat{v}\|_{L^2L^2} \lesssim \|\nabla v_0\|_{L^2L^2} + \epsilon \lesssim \epsilon.$$

By Young's inequality once more, for some $\delta_1 > 0$ sufficiently small,

$$\|v_1\|^2_{L^\infty L^2} + \|\phi_1\nabla\hat{v}\|^2_{L^2L^2} \lesssim \epsilon^2\|v_1\|^2_{L^2L^4} + \delta_1^2\|\phi_1\nabla\hat{v}\|^2_{L^2L^2} + \delta_1^2\|v_1\|^2_{L^2L^4} + \frac{\epsilon^2}{\delta_1^2}.$$

Using Sobolev again as above, $\|v_1\|^2_{L^\infty L^2} + \|v_1\|^2_{L^2L^4} + \|\nabla v_1\|^2_{L^2L^2} \lesssim \epsilon$. In particular

$$\|D^2u\|_{X(Q_{\rho_1})} \lesssim \epsilon.$$

This process can be iterated a given finite number of times, to show that for given $k > 0$, there are $\epsilon_0 = \epsilon(k)$, $C = C(k)$, such that if $\|u\|_{L^\infty(\dot{H}^1\cap L^4)(Q_1)} = \epsilon < \epsilon_0$, then $\|D^k u\|_{X(Q_{1/2})} \leq C\epsilon$. $\square$

We proceed now with the proof of Theorem 4.17.

Proof. Let us assume that the conclusion is false, i.e., $T^* < +\infty$. Note first that

$$\lambda(t) \to +\infty.$$

In fact, $\liminf_{t\to T^*-} \sqrt{T^* - t}\ \lambda(t) > 0$, since if $\sqrt{T^* - t_n}\lambda(t_n) \to 0$ along a sequence $t_n \nearrow T^*$, by the compactness assumption (and up to subsequence)

$$v_n(x) := \frac{1}{\lambda(t_n)} u_c(t_n, \frac{x - x(t_n)}{\lambda(t_n)}) \xrightarrow{\dot{H}^1} \exists\, v(x) \in \dot{H}^1.$$

Let $\hat{T} > 0$ be the maximal existence time for the solution of the Cauchy problem (4.1) with initial data $v(x)$. Define $w_n(t,x)$ to be the solutions with initial data $w_n(x,t_n) = v_n(x)$ prescribed at time t_n, and denote their maximal lifespans as $[t_n, T_n^{\max})$. By continuous dependence on initial data, $0 < \hat{T} \leq \liminf(T_n^{\max} - t_n)$. But from scaling:

$$\begin{aligned} T_n^{\max} - t_n = T^{\max}\left(\frac{1}{\lambda(t_n)} u_c(t_n, \frac{\cdot - x(t_n)}{\lambda(t_n)})\right) &= \frac{1}{\lambda^2(t_n)} T^{\max}(u_c(t_n, \cdot)) \\ = \lambda^2(t_n)(T^* - t_n) \to 0, \end{aligned}$$

a contradiction.

By compactness in $\dot{H}^1$, and the continuous embedding $\dot{H}^1 \hookrightarrow L^4$, for every $\epsilon > 0$, there is a $R_\epsilon > 0$ such that for all $t \in I := [0, T^*)$:

$$\int_{|x - x(t)| \geq \frac{R_\epsilon}{\lambda(t)}} \left(|\nabla u_c(t,x)|^2 + |u_c(t,x)|^4\right) dx < \epsilon \tag{4.48}$$

Fix any $\{t_n\} \subset [0, T^*), t_n \nearrow T^*$, and let $\lambda_n = \lambda(t_n) \to \infty$ and $\{x_n\} = \{x(t_n)\} \subset \mathbb{R}^4$, so that (up to subsequence)

$$v_n(x) = \frac{1}{\lambda_n} u_c(\frac{x - x_n}{\lambda_n}, t_n) \xrightarrow{\dot{H}^1} \bar{v}, \text{ some } \bar{v} \in \dot{H}^1,$$

and also in L^4 by Sobolev embedding.
We also make and prove the following claim as in [78]

Claim 4.23. *For any $R > 0$,*

$$\lim_{n \to \infty} \int_{|x| \leq R} |u_c(x, t_n)|^2 dx = 0.$$

Proof.

$$\begin{aligned} \int_{|x| \leq R} |u_c(x, t_n)|^2 dx &= \int_{|x| \leq R} |\lambda_n v_n(\lambda_n x + x_n)|^2 dx \\ &= \frac{1}{\lambda_n^2} \int_{|y - x_n| \leq \lambda_n R} |v_n(y)|^2 dy = \frac{1}{\lambda_n^2} \|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))}. \end{aligned}$$

Denoting $B_r := B_r(0)$, for any $\epsilon > 0$,

$$\frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))} = \frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))\cap B_{\epsilon\lambda_n R})} + \frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))\cap B^c_{\epsilon\lambda_n R})}.$$

Using Hölder's inequality, and the compactness, we get

$$\begin{aligned}
\frac{1}{\lambda_n^2}\|v_n\|^2_{L^2(B_{\lambda_n R}(x_n))} &\lesssim \frac{1}{\lambda_n^2}\|v_n\|^2_{L^4(\mathbb{R}^4)}|B_{\epsilon\lambda_n R}|^{\frac{1}{2}} + \frac{1}{\lambda_n^2}\|v_n\|^2_{L^4(B_{\lambda_n R}(x_n)\cap B^c_{\epsilon\lambda_n R})}|B_{\lambda_n R}(x_n)|^{\frac{1}{2}} \\
&\lesssim \epsilon^2 R^2\|\bar{v}\|^2_{L^4(\mathbb{R}^4)} + \frac{|B_{\lambda_n R}(x_n)|^{\frac{1}{2}}}{\lambda_n^2}\left(\|v_n-\bar{v}\|^2_{L^4(\mathbb{R}^4)} + \|\bar{v}\|^2_{L^4(B_{\lambda_n R}(x_n)\cap B^c_{\epsilon\lambda_n R})}\right) \\
&\lesssim \epsilon^2 R^2\|\bar{v}\|^2_{L^4(\mathbb{R}^4)} + R^2\|v_n-\bar{v}\|^2_{L^4(\mathbb{R}^4)} + R^2\|\bar{v}\|^2_{L^4(B^c_{\epsilon\lambda_n R})} \\
&\lesssim \epsilon^2 R^2 + R^2\|v_n-\bar{v}\|^2_{L^4(\mathbb{R}^4)} + R^2\|\bar{v}\|^2_{L^4(B^c_{\epsilon\lambda_n R})}.
\end{aligned}$$

The first term is arbitrarily small with ϵ, the second one goes to zero (as $n \to \infty$) because of the compactness, and for a fixed ϵ, the last one goes to zero since $\lambda_n \to \infty$. □

We also prove that the center of compactness $x(t)$ is bounded:

Proposition 4.24. $\sup\limits_{0\le t<T^*}|x(t)| < \infty.$

Proof. We will first make the assumption that

$$\underline{E} := \inf_{t\in[0,T^*)} E(u_c(t)) > 0, \tag{4.49}$$

and later show that this is indeed the case for compact blowing-up solutions, without any size restriction. Note that under the assumptions of our Theorem 4.1, i.e., in the below threshold case, we certainly have that $\underline{E} > 0$. This can be easily deduced by the variational estimates in Lemma 4.9 and the small data theory.

The energy dissipation relation

$$E(u(t_2)) + \int_{t_1}^{t_2}\|u_t\|^2_{L^2}\,ds = E(u(t_1)) \le E(u(0)) \tag{4.50}$$

for $t_2 > t_1 > 0$ will be of use. We will assume for contradiction that there is a sequence of times $t_n \nearrow T^* : |x(t_n)| \to \infty$.
Choose a radial smooth cut-off function ψ such that

$$\psi(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \\ 1 & \text{if } |x| \geq 2 \end{cases}$$

and define $\psi_R(x) := \psi(\frac{|x|}{R})$. Choosing any $t_0 \in (0, T^*)$, we can find $R_0 \geq 1$ such that

$$\int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t_0)|^2 - \frac{1}{4} |u_c(t_0)|^4 \right) \psi_{R_0}(x)\, dx \leq \frac{1}{4} \underline{E}. \tag{4.51}$$

Since $|x(t_n)| \to \infty$ and $\lambda(t_n) \to \infty$, for any $\epsilon > 0$, $B_{\frac{R\epsilon}{\lambda(t_n)}}(x(t_n)) \subset B^c_{2R_0}$ for n large enough, and so by (4.48):

$$\lim_{t \nearrow T^*} \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t)|^2 - \frac{1}{4} |u_c(t)|^4 \right) \psi_{R_0}(x) dx = \underline{E},$$

hence we can find a $t_1 \in (t_0, T^*)$ such that

$$\int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t_1)|^2 - \frac{1}{4} |u_c(t_1)|^4 \right) \psi_{R_0}(x)\, dx \geq \frac{1}{2} \underline{E}. \tag{4.52}$$

Combining (4.51) and (4.52):

$$\int_{t_0}^{t_1} \frac{d}{dt} \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t)|^2 - \frac{1}{4} |u_c(t)|^4 \right) \psi_{R_0}(x)\, dx dt \geq \frac{1}{4} \underline{E}. \tag{4.53}$$

On the other hand:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t)|^2 - \frac{1}{4} |u_c(t)|^4 \right) \psi_{R_0}(x)\, dx &= \int_{\mathbb{R}^4} \left(\nabla u_c \cdot \nabla (u_c)_t - u_c^3 (u_c)_t \right) \psi_{R_0}(x)\, dx \\ &= \int_{\mathbb{R}^4} \left(\nabla u_c \cdot \nabla (u_c)_t - ((u_c)_t - \Delta u_c)(u_c)_t \right) \psi_{R_0}(x)\, dx \\ &= - \int_{\mathbb{R}^4} (u_c)_t^2 \psi_{R_0}\, dx - \int_{\mathbb{R}^4} (u_c)_t \nabla u_c \cdot \nabla \psi_{R_0}\, dx \lesssim \int_{\mathbb{R}^4} |(u_c)_t| |\nabla u_c|\, dx, \end{aligned}$$

since $|\nabla \psi_{R_0}(x)| \lesssim \dfrac{1}{R_0} \leq 1$. So by Hölder,

$$\begin{aligned}
&\int_{t_0}^{t_1} \frac{d}{dt} \int_{\mathbb{R}^4} \left(\frac{1}{2} |\nabla u_c(t)|^2 - \frac{1}{4} |u_c(t)|^4 \right) \psi_{R_0}(x)\, dx dt \\
&\qquad \lesssim \|\nabla u_c\|_{L_t^\infty L^2} \sqrt{t_1 - t_0}\, \|(u_c)_t\|_{L^2 L^2 [t_0, t_1] \times \mathbb{R}^4} \\
&\qquad \lesssim \|(u_c)_t\|_{L^2 L^2([t_0, T^*) \times \mathbb{R}^4)}
\end{aligned} \tag{4.54}$$

where we have uniformly bounded the kinetic energy of u_c by once more employing the compactness. Combining (4.53) and (4.54) yields:

$$0 < \frac{1}{4} \underline{E} \lesssim \|(u_c)_t\|_{L^2 L^2([t_0, T^*) \times \mathbb{R}^4)} \to 0 \text{ as } t_0 \nearrow T^* \tag{4.55}$$

by the energy dissipation relation (4.50), a contradiction.
Now we show (4.49). Choose a smooth radial cut-off function ϕ such that

$$\phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}$$

and define $\phi_R(x) := \phi(\frac{|x|}{R})$, and

$$I_R(t) := \int |u_c(x,t)|^2 \phi_R(x) dx, \ t \in [0, T^*).$$

We then have

$$I_R'(t) = \int \phi_R (|u_c|^4 - |\nabla u_c|^2) dx - \frac{1}{R} \int u_c \nabla u_c \nabla \phi(\frac{x}{R}) dx$$

and by Sobolev, Hardy and the compactness, we can immediately deduce that

$$|I_R'(t)| \leq C,$$

C a universal constant. Integrating from t_0 to $T^* > t > t_0 \geq 0$:

$$|I_R(t) - I_R(t_0)| \leq C(t - t_0).$$

By Claim 4.23, we get that $I_R(t) \to 0$ as $t \to T^*$, for all $R > 0$. Hence

$$|I_R(t_0)| \leq C(T^* - t_0).$$

Since this bound is uniform in R, by taking $R \to \infty$, we conclude $u_c(t_0) \in L^2$, and so indeed $u_c(t) \in L^2, t \in [0, T^*)$. Defining then $I(t) := \frac{1}{2} \int |u_c(t,x)|^2 dx$, by direct calculation we get

$$I'(t) = -\int \left(|\nabla u_c|^2 - |u_c|^4\right) dx = -K(u_c(t))$$

(K is defined below). Now for any sequence $\{t_n\}_n \nearrow T^*$, let (up to subsequence)

$$\frac{1}{\lambda(t_k)} u_c(\frac{x - x_k}{\lambda(t_k)}, t_k) \xrightarrow{\dot{H}^1} \bar{v} \in \dot{H}^1.$$

Set

$$K(u) := \int \left(|\nabla u|^2 - |u|^4\right) dx = 2E(u) - \frac{1}{2} \int |u|^4 dx.$$

Proceeding by contradiction, we suppose $\underline{E} \leq 0$. If so,

$$K(\bar{v}) = \lim_{k \to \infty} K(u_c(t_k)) = 2\underline{E} - \frac{1}{2} \int |\bar{v}|^4 \leq -\frac{1}{2} \int |\bar{v}|^4 < 0,$$

since $\bar{v} \equiv 0$ would contradict the assumption $T^* < \infty$. So

$$I'(t_k) = -K(u_c(t_k)) \to -K(\bar{v}) > 0$$

Thus $I'(t) > 0$ for all t sufficiently close to T^*; otherwise, we could find a subsequence along which $I' \leq 0$, and the preceding argument would provide a contradiction. So

$I(t)$ is increasing for t near T^*. But we have also shown that

$$\lim_{t\to T^*} I(t) = \lim_{t\to T^*} \lim_{R\to\infty} I_R(t) = \lim_{R\to\infty} \lim_{t\to T^*} I_R(t) = 0,$$

a contradiction. Thus we have shown that $\underline{E} > 0$, completing the proof that $|x(t)|$ remains bounded. $\square$

Since $|x(t)|$ remains bounded while $\lambda(t) \xrightarrow{t\to T^*} \infty$, by the compactness we can find an $R_0 > 0$ large enough such that for all $x, |x| \geq R_0$:

$$\|u_c\|_{L_t^\infty \dot{H}_x^1 \cap L_t^\infty L_x^4(\Omega_{T^*})} < \epsilon_0,$$

where $\Omega_{T^*} := (0, T^*) \times B_{\sqrt{T^*}}(x_0)$.

By an appropriate scaling and shifting argument, the Regularity Lemma 4.21 shows that u_c is smooth on $\Omega := (\mathbb{R}^4 \setminus B_{R_0}(0)) \times [\frac{3}{4}T^*, T^*]$, with uniform bounds on derivatives. Since u is continuous up to T^* outside B_{R_0}, Claim 4.23 implies that $u_c(x, T^*) \equiv 0$, in the exterior of this ball. Since u_c is bounded and smooth in Ω, an application of the Backwards Uniqueness Theorem 4.19 implies that $u_c \equiv 0$ in Ω. Define $\tilde{\Omega} := \mathbb{R}^4 \times (\frac{3}{4}T^*, \frac{7}{8}T^*]$. Applying the Unique Continuation Theorem 4.20 on a cylinder of sufficiently large spatial radius, centered at a point of Ω, implies $u_c \equiv 0$ in $\tilde{\Omega}$. By the uniqueness guaranteed by the local wellposedness theory we get that $u_c \equiv 0$, which contradicts (4.45). $\square$

Note that our results extend to any dimension bigger than $d = 2$ with obvious modifications to the proofs. The profile decomposition statement we used is valid for all $d \geq 3$, and we refer to [124] for a proof of the stability result (4.7) in higher dimensions.

4.6 Blow-up

In this section we give various criteria on the initial data so that the corresponding solutions blow-up in finite-time.

The following result is well-known [95, 4, 17] but we give the proof for the convenience of the reader.

Proposition 4.25. *Solutions of*

$$\begin{aligned} u_t &= \Delta u + |u|^{p-1}u, \qquad 1 < p \le 2* = \frac{2d}{d-2} \\ u(x,0) &= u_0(x) \in H^1(\mathbb{R}^d) \end{aligned} \tag{4.56}$$

with

$$E(u_0) := \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u_0|^2 dx - \frac{1}{p+1}|u_0|^{p+1} \right) dx < 0$$

must blow-up in finite-time, in the sense that there is no global solution $u \in C([0,\infty); H^1(\mathbb{R}^d))$.

Notice that we can always find such initial data, e.g., if $u_0 = \lambda f$, $f \in H^1(\mathbb{R}^d) \cap L^{p+1}(\mathbb{R}^d)$ we can force the negative energy assumption picking λ large enough.

Proof. We first derive some identities satisfied as long as a solution remains regular.

Multiplying the equation (4.56) first by u and then by u_t and integrating by parts we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}^d} |u|^2 dx \right) = \int_{\mathbb{R}^d} |u|^{p+1} dx - \int_{\mathbb{R}^d} |\nabla u|^2 dx \tag{4.57}$$

and

$$\int_{\mathbb{R}^d} |u_t|^2 dx = \frac{d}{dt}\left(\frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx - \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx \right) = -\frac{d}{dt}E(u(t)), \tag{4.58}$$

the energy dissipation relation. For convenience we define $J(t) := -E(t)$ and hence by (4.58) we have that $J'(t) := \int_{\mathbb{R}^d} |u_t|^2 dx \ge 0$ and by the assumption on the energy $J(0) > 0$. It will be also useful to write $J(t)$ as

$$J(t) = J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dx dt \tag{4.59}$$

Define

$$I(t) = \int_0^t \int_{\mathbb{R}^d} |u|^2 dx dt + A \tag{4.60}$$

with $A > 0$, to be chosen later. With this definition

$$I'(t) = \int_{\mathbb{R}^d} |u|^2 dx \tag{4.61}$$

and

$$I''(t) = 2\left(\int_{\mathbb{R}^d} |u|^{p+1} dx - \int_{\mathbb{R}^d} |\nabla u|^2 dx\right) \tag{4.62}$$

Since $p > 1, \delta := \frac{1}{2}(p-1) > 0$; a comparison with the energy functional yields

$$I''(t) \geq 4(1+\delta)J(t) = 4(1+\delta)\left(J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dxdt\right). \tag{4.63}$$

We can also rewrite

$$I'(t) = \int_{\mathbb{R}^d} |u|^2 dx = \int_{\mathbb{R}^d} |u_0|^2 dx + 2Re \int_0^t \int_{\mathbb{R}^d} \bar{u} u_t dxdt.$$

For any $\epsilon > 0$ the Young and Hölder inequalities give

$$(I'(t))^2 \leq 4(1+\epsilon)\left(\int_0^t \int_{\mathbb{R}^d} |u|^2 dxdt\right)\left(\int_0^t \int_{\mathbb{R}^d} |u_t|^2 dxdt\right) + (1+\frac{1}{\epsilon})\left(\int_{\mathbb{R}^d} |u_0|^2 dx\right)^2 \tag{4.64}$$

Combining (4.63),(4.59),(4.64), for any $\alpha > 0$ we obtain:

$$\begin{aligned} I''(t)I(t) - (1+\alpha)(I'(t))^2 \geq\ & 4(1+\delta)\left[J(0) + \int_0^t \int_{\mathbb{R}^d} |u_t|^2 dxdt\right]\left[\int_0^t \int_{\mathbb{R}^d} |u|^2 dxdt + A\right] \\ & - 4(1+\epsilon)(1+\alpha)\left[\int_0^t \int_{\mathbb{R}^d} |u|^2 dxdt\right]\left[\int_0^t \int_{\mathbb{R}^d} |u_t|^2 dxdt\right] \\ & - (1+\frac{1}{\epsilon})(1+\alpha)\left[\int_{\mathbb{R}^d} |u_0|^2 dx\right]^2. \end{aligned} \tag{4.65}$$

Choose α, ϵ small enough for $1+\delta \geq (1+\alpha)(1+\epsilon)$. Since $J(0) > 0$ picking A large enough we can ensure $I''(t)I(t) - (1+\alpha)(I'(t))^2 > 0$. But this is equivalent to $\frac{d}{dt}\left(\frac{I'(t)}{I^{\alpha+1}(t)}\right) > 0$ which in turn implies $\frac{I'(t)}{I^{\alpha+1}(t)} > \frac{I'(0)}{I^{\alpha+1}(0)} =: \tilde{a}$ for all $t > 0$.

Integrating $I'(t) > \tilde{a} I^{\alpha+1}$ gives

$$\frac{1}{\alpha}\left(\frac{1}{I^\alpha(0)} - \frac{1}{I^\alpha(t)}\right) > \tilde{a}t \;\Rightarrow\; I^\alpha(t) > \frac{I^\alpha(0)}{1 - I^\alpha(0)\alpha\tilde{a}t} \to \infty$$

as $t \to \frac{1}{I^\alpha(0)\alpha\tilde{a}} = \frac{1}{A^\alpha\alpha\tilde{a}} =: \hat{t}$. This in turn implies that $\limsup_{t\to\hat{t}-} \|u\|_{L^2} = \infty$, showing that the solution cannot be globally in $C_t H^1$. Note also that (4.57) implies $\limsup_{t\to\hat{t}-} \|u\|_{L^{p+1}} = \infty$. $\square$

We present a refinement in the *critical case* which includes some positive energy data, and in particular establishes Theorem (4.2). So consider now equation (4.9), for which

$$E(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{2*}|u|^{2^*}\right) dx.$$

Proposition 4.26. *Let $u_0 \in H^1(\mathbb{R}^d)$ and $\delta_0 > 0$ such that*

$$E(u_0) < E(W) \;\; \textit{and} \;\; \|\nabla u_0\|_{L^2} \geq \|\nabla W\|_{L^2}. \tag{4.66}$$

Then the corresponding solution u to (4.9) blows up in finite time. That is, $T_{max}(u_0)$ (coming from the $\dot{H}^1$ local theory as in Theorem (4.6)*) is finite.*

Proof. We will give a sketch of the proof, which is largely a modification of the proof of the previous proposition.

By the Sobolev inequality (4.5),

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{2*}\int_{\mathbb{R}^d} |u|^{2*} dx \geq \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2*}\frac{\|W\|_{L^{2*}}^{2*}}{\|\nabla W\|_{L^2}^{2*}}\|\nabla u\|_{L^2}^{2*} \tag{4.67}$$

We define $f(y) := \frac{1}{2}y - \frac{1}{2*}C^{2*}y^{\frac{2*}{2}}$, $\quad C^{2*} = \frac{\|W\|_{L^{2*}}}{\|\nabla W\|_{L^2}} = \|\nabla W\|_{L^2}^{-\frac{2}{d}}$, so that by energy dissipation and (4.66),

$$f(\|\nabla u\|_{L^2}^2) \leq E(u) \leq E(u_0) < E(W). \tag{4.68}$$

It is straightforward to verify that $f(y)$ is concave for $y \geq 0$ and attains its maximum value $f(\|\nabla W\|_{L^2}^2) = E(W) = \frac{1}{d}\|\nabla W\|_{L^2}^2$ at $y = \|\nabla W\|_{L^2}^2$. Furthermore, it is strictly increasing on $[0, \|\nabla W\|_{L^2}^2]$ and strictly decreasing on $[\|\nabla W\|_{L^2}^2, +\infty)$. Denote the inverse function of f on $[\|\nabla W\|_{L^2}^2, +\infty)$ as

$$e = f^{-1} : (-\infty, E(W)] \to [\|\nabla W\|_{L^2}^2, +\infty),$$

strictly decreasing. By (4.68) and (4.66) then,

$$\|\nabla u(t)\|_{L^2}^2 \geq e(E(u(t))).$$

By the definitions of $K = K(u)$ and the energy $E = E(u)$

$$\begin{aligned} -K(u) &= -\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} |u|^{2^*} dx = \frac{2}{d-2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - 2^* E(u) \\ &\geq \frac{2}{d-2}\left(e(E) - dE\right) =: g(E). \end{aligned}$$

Note that $g(E(W)) = 0$ and for $E < E(W)$, $g(E) > 0$ and $g'(E) = \frac{2}{d-2}e'(E) - 2^* < -2^*$. Defining $I(t)$ as in (4.60):

$$I''(t) = -2K(u) \geq 2g(E(u)) > 0.$$

By the Fundamental Theorem of Calculus and the energy dissipation relation,

$$2g(E(u)) = 2g(E(u_0)) + 2\int_0^t |g'(E(u(s)))| \int_{\mathbb{R}^d} |u_t|^2 dx ds.$$

One can now repeat the proof of Proposition 4.25 replacing (4.63) by

$$I''(t) \geq 4(1+\delta)J(t) = 4(1+\delta)\left(2g(E(u_0)) + \int_0^t 2|g'(E(u(s)))| \int_{\mathbb{R}^d} |u_t|^2 dx ds\right) \tag{4.69}$$

Since $g(E(u_0)) > 0$, we can proceed exactly as in the proof of the previous Proposition to conclude that if $T_{max} = \infty$, then we must have $\limsup_{t \to \hat{t}-} \|u(t)\|_{L^2} = \infty$ for some $\hat{t} < \infty$, which by (4.61) implies $\limsup_{t \to \hat{t}-} \|u(t)\|_{L^{2*}} = \infty$, and so by Sobolev, $\limsup_{t \to \hat{t}-} \|\nabla u(t)\|_{L^2} = \infty$, contradicting $T_{max} < \infty$. $\square$

4.7 Directions for future research

In recent work, Collot, Merle and Raphael [27], provided a complete classification of solutions near the stationary solution W for $d \geq 7$. In particular, they proved that for $0 < \eta \ll 1$, if $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\|u_0 - W\|_{H^1} < \eta$, the corresponding solution $u \in C((0,T), \dot{H}^1 \cap \dot{H}^2)$ follows one of the three regimes: **(i)** *"Soliton"*: the solution is global and asymptotically attracted by a "solitary wave": there exist $(\lambda_\infty, z_\infty) \in \mathbb{R}^+_* \times \mathbb{R}^d$ such that

$$\lim_{t \to \infty} \left\| u(t, \cdot) - \frac{1}{\lambda_\infty^{\frac{d-2}{2}}} W\left(\frac{\cdot - z_\infty}{\lambda_\infty}\right) \right\|_{H^1} = 0,$$

(ii) *Dissipation*: the solution is global and dissipates to zero in both $\dot{H}^1$ and L^∞, or **(iii)** *Type I blow-up*: the solution blows-up in finite time $0 < T < +\infty$ in the ODE type I self-similar blow-up regime near the singularity. There exist solutions associated to each scenario. Moreover, the scenaria of dissipation and Type I blow up are stable in the energy topology. Note that Type II blow-up is ruled out in $d \geq 7$ near W.

In continuation of this work, there are several questions one can ask in order to help complete the picture. An interesting direction is provided by the following problem

Problem: Classify all solutions in a neighbourhood of W for dimensions $3-6$. Different behaviour is expected since we already know [118] that for $d = 4$ Type-II blow-up can occur.

For radial solutions, Merle and Matano have recently announced a soliton resolution result for global solutions and a soliton decomposition for blowing-up solutions, along with a two-soliton construction of a global solution in $d \geq 7$.

Problem: Can we establish such decompositions for general solutions, for any data? What about constructions of multi-solitons in lower dimensions?

The *non-radial case* of the soliton resolution is open but it is probably within the reach of recent technology. We expect that careful energy estimates will be able to provide the missing ingredient in the absence of the one-dimensional tools used in the proof of the radial case (in particular, the intersection number). New arguments are needed to compensate for the slow decay of W which is the main reason multi-solitonic solutions have not yet been constructed in lower dimensions.

Bibliography

[1] S. Angenent, J. Hulshof, Singularities at $t = \infty$ in equivariant harmonic map flow, Contemp. Math. 367, Geometric evolution equations, 1–15, Amer. Math. Soc., Providence, RI, 2005

[2] T.Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. (9), 55, 1976, 3, 269–296

[3] H. Bahouri, P. Gérard, High frequency approximation of solutions to critical nonlinear wave equations, Amer. J. Math. 121 (1999), 131-175

[4] Ball, J. M. Remarks on blow-up and nonexistence theorems for nonlinear evolution equations, Quart. J. Math. Oxford Ser. (2) 28 (1977), no. 112, 473–486

[5] I. Bejenaru, On Schrödinger maps, American Journal of Mathematics Vol. 130, No. 4 (Aug., 2008), p. 1033-1065

[6] M.Bertch, R. Van der Hout, J.Hulshof, Energy concentration for 2-dimensional radially symmetric equivariant Harmonic Map Heat Flows, Commun. Contemp. Math. 13, 675 (2011)

[7] I.Bejenaru, A.Ionescu, C.E. Kenig, D.Tataru, Equivariant Schrödinger Maps in two spatial dimensions, Duke Mathematical Journal 162, 1967-2025 (2013)

[8] I.Bejenaru, A.Ionescu, C.E. Kenig, D.Tataru, Equivariant Schrödinger Maps in two spatial dimensions: the H^2 target, preprint, arXiv:1212.2566

[9] G.J.B. Van den Berg, J. Hulshof, J. King, Formal asymptotics of bubbling in the harmonic map heat flow, SIAM J. Appl. Math. vol 63, no5, (2003) 1682-1717

[10] S. Blatt, M. Struwe, Well-posedness of the supercritical Lane-Emden heat flow in Morrey spaces, arXiv:1511.07685

[11] H.Brezis, T. Cazenave, A nonlinear heat equation with singular initial data. Journal D'Analyse Mathematique, 68(1), (1996), 277-304

[12] J. Bourgain, Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case. J. Amer. Math. Soc. 12 (1999), 145–171

[13] H. Brézis and J.-M. Coron, Convergence of solutions of H-systems or how to blow bubbles, Arch. Rational Mech. Anal. 89 (1985), 21-56

[14] A.Bulut, Maximizers for the Strichartz inequalities for the wave equation, Differential Integral Equations (2010) no. 11-12, 1035–1072

[15] N. Burq, F. Planchon, J. Stalker, and S. Tahvilder-Zadeh, Strichartz estimates for the Wave and Schrödinger equations with the inverse-square potential, J. Funct. Anal. 203 (2003), no. 2, 519 – 549.

[16] N. Burq, F. Planchon, J. Stalker, and S. Tahvilder-Zadeh, Strichartz estimates for the wave and Schrödinger equations potentials of critical decay, Indiana Univ. Math. J. 53 (2004), no. 6, 1665 – 1680

[17] T. Cazenave, Finite-time blowup for a family of nonlinear parabolic equations. Lecture notes prepared for VI ENAMA, Aracaju, Brazil, Nov. 7-9 2012

[18] T. Cazenave, F. Dickstein, F.B. Weissler, Global existence and blowup for sign-changing solutions of the nonlinear heat equation, J. Differential Equations 344 (2009), no. 7, 2669-2680

[19] T. Cazenave, F. Dickstein, F.B. Weissler, Non-regularity in Hölder and Sobolev spaces of solutions to the semilinear heat and Schrödinger equations, preprint, arXiv:1503.08954

[20] N.-H. Chang, J. Shatah, and K. Uhlenbeck, Schrödinger maps, Comm. Pure Appl. Math. 53 (2000), no. 5, 590 – 602

[21] T. Cazenave, F.B. Weissler, The Cauchy problem for the critical nonlinear Schrödinger equation in H^s, Nonlinear Anal., Theory, Methods and Applications. An International Multidisciplinary Journal. Series A: Theory and Methods, 14, 1990, 10, 807–836

[22] K.-C. Chang, W.-Y. Ding and R. Ye, Finite-time blow-up of the heat flow of harmonic maps from surfaces, J. Diff. Geom. 36 (1992), 507-515

[23] J.Colliander, M.Keel, G.Staffilani, H.Takaoka, T.Tao, Global well-posedness and scattering in the energy space for the critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Annals of Mathematics, 167 (2008), 767–865

[24] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$, Ann. Math. 167 (2008), 767–865

[25] C. Collot, Non radial type II blow up for the energy supercritical semilinear heat equation,preprint, arXiv:1604.02856

[26] C. Collot, P. Raphael, J. Szeftel, On the stability of type I blow up for the energy super critical heat equation,preprint, arXiv:1605.07337

[27] C. Collot, F. Merle, P. Raphael, Dynamics near the ground state for the energy critical nonlinear heat equation in large dimensions, preprint, arXiv:1604.08323

[28] C.Cortazar, M. del Pino, M. Musso, Green's function and infinite-time bubbling in the critical nonlinear heat equation, preprint, arXiv:1604.07117

[29] R.Côte, Instability of non-constant harmonic maps for the (1+2)-dimensional equivariant wave map system, International Mathematics Research Notices, 2005, no. 57, 3525-3549

[30] R.Côte, Soliton resolution for equivariant wave maps to the sphere, preprint, arXiv:1305.5325

[31] M. Creek, R. Donninger, W. Schlag, S. Snelson, Linear stability of the Skyrmion, preprint, arXiv: 1603.03662

[32] R.Côte, C. E.Kenig, F.Merle, Scattering below critical energy for the radial 4D Yang-Mills equation and for the 2D corotational wave map system, Communications in Mathematical Physics 284 (2008), no. 1, 203-225

[33] R.Côte, C.E.Kenig, A.Lawrie, W.Schlag, Characterization of large energy solutions of the equivariant wave map problem:I, preprint, arXiv:1209.3682

[34] R.Côte, C.E. Kenig, A.Lawrie, W.Schlag, Characterization of large energy solutions of the equivariant wave map problem:II, preprint, arXiv:1209.3684

[35] R.Côte, C.E. Kenig, A.Lawrie, W.Schlag, Profiles for the radial focusing 4d energy-critical wave equation, preprint, arXiv:1402.2307

[36] M. del Pino, M. Musso, J. Wei, Infinite time blow-up for the 3-dimensional energy critical heat equation, preprint, arXiv:1705.01672

[37] J.Davila, M. del Pino, J. Wei, Singularity formation for the two-dimensional harmonic map flow into, preprint, arXiv:1702.05801

[38] W.-Y. Ding, G. Tian, Energy identity for a class of approximate harmonic maps from surfaces, Comm. Anal. Geom. 3 (1995),543-554

[39] B. Dodson, P. Smith, A controlling norm for energy-critical Schrödinger maps, preprint, arXiv:1302.3879v1

[40] S.-Z. Du, On partial regularity of the borderline solution of semilinear parabolic equation with critical growth, Adv. Differential Equations 18 (2013), no. 1-2, 147–177

[41] T.Duyckaerts, C.E. Kenig, F. Merle, Universality of blow-up profile for small radial type II blow-up solutions of energy-critical wave equation, J. Eur. Math. Soc. 13 (2011), no. 3, 533-599

[42] T.Duyckaerts, C. E. Kenig, F. Merle, Profiles of bounded radial solutions of the focusing, energy-critical wave equation, Geometric and Functional Analysis. 22 (2012), no. 3, 639-698

[43] T.Duyckaerts, C.E. Kenig, F. Merle, Classification of radial solutions of the focusing, energy-critical wave equation,Cambridge Journal of Mathematics 1 (2013), no. 1, 75-144

[44] T.Duyckaerts, C.E. Kenig, F. Merle, Solutions of the focusing nonradial critical wave equation with the compactness property, preprint, arXiv:1402.0365

[45] T.Duyckaerts, C.E. Kenig, F. Merle, Concentration-compactness and universal profiles for the non-radial energy critical wave equation, preprint, arXiv:1510.01750

[46] T.Duyckaerts, H.Jia, C.E. Kenig, F. Merle, Soliton resolution along a sequence of times for the focusing energy critical wave equation preprint, arXiv:1601.01871v2

[47] J. Eells, J.H.Sampson, Harmonic mappings of Riemannian manifolds. Amer. J. Math. 86 (1964) 109-169

[48] W. Y. Ding and Y. D. Wang, Local Schrödinger flow into Kähler manifolds, Sci. China Ser. A 44 (2001), no. 11, 1446 – 1464

[49] L. Escauriaza, G. Seregin, V. *Š*ver*á*k, On backward uniqueness for parabolic equations. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 288(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 32):100–103, 272, 2002

[50] L. Escauriaza, G.A. Seregin, V. *Š*ver*á*k, Backward uniqueness for parabolic equations. Arch. Ration. Mech. Anal., 169(2):147–157, 2003

[51] L. Escauriaza, G. A. Seregin, and V. *Š*ver*á*k, $L^{3,\infty}$-solutions of Navier-Stokes equations and backward uniqueness. Uspekhi Mat. Nauk, 58(2(350)):3–44, 2003

[52] S.Filippas, M.A. Herrero, J.J.L. Vel*á*zquez, Fast blow-up mechanisms for sign-changing solutions of a semilinear parabolic equation with critical nonlinearity, y, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 456 (2000) 2957-2982

[53] V. Galaktionov, J. L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. 50 (1997), no. 1, 1-67

[54] I. Gallagher, Profile decomposition for solutions of the Navier-Stokes equations. Bull. Soc. Math. France, 129(2):285–316, 2001

[55] D.-A. Geba, K. Nakanishi, S. G. Rajeev., Global well-posedness and scattering for Skyrme wave maps. Commun. Pure Appl. Anal., 11(5):1923–1933, 2012

[56] I. Gallagher, D. Iftimie, F. Planchon, Non-explosion en temps grand et stabilit*é* de solutions globales des *é*quations de Navier-Stokes. C. R. Math. Acad. Sci. Paris, 334(4):289-292, 2002

[57] I. Gallagher, G.S.Koch, F. Planchon, A profile decomposition approach to the $L^\infty_t L^3_x$ Navier-Stokes regularity criterion. Math. Ann. 355, 4 (2013), 1527–1559

[58] I. Gallagher, G.S. Koch, F. Planchon, Blow-up of critical Besov norms at a potential Navier-Stokes singularity. Preprint, http://arxiv.org/abs/1407.4156

[59] P. Gérard, Description du défaut de compacité de l'injection de Sobolev, ESAIM Control Optim. Calc. Var. 3 (1998), 213-233

[60] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62, (1986) 186–212

[61] M.-H.Giga, Y. Giga, J. Saal, Nonlinear partial differential equations: Asymptotic behavior of solutions and self-similar solutions. Vol. 79. Springer Science & Business Media, 2010

[62] Y. Giga, R.V. Kohn, Asymptotically self-similar blowup of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985) 297–319

[63] Y. Giga, R.V. Kohn, Characterizing blowup using similarity variables, Indiana Univ. Math. J. 36 (1987) 1-40

[64] Y. Giga, R.V. Kohn, Nondegeneracy of blow up for semilinear heat equations, Comm. Pure Appl. Math. 42 (1989) 845–884

[65] J.Grotowski, J.Shatah, A note on Geometric heat flows in critical dimensions, Calculus of Variations (2007), 30, no 4, 499-512

[66] M.Guan, S. Gustafson, T.-P. Tsai, Global existence and blow-up for harmonic map heat flow, J. Diff. Equations 246 (2009),1-20

[67] M.Guan, S. Gustafson, K.Kang, T.-P. Tsai, Global questions for map evolution equations. Singularities in PDE and the calculus of variations, 61–74, CRM Proc. Lecture Notes, 44, Amer. Math. Soc., Providence, RI, 2008

[68] F. Gürsey, On the symmetries of strong and weak interactions, Nuovo Cimento (10), 16 (1960), 230–240

[69] F. Gürsey, On the structure and parity of weak interaction currents, Ann. Physics, 12 (1961), no. 1, 91–117.

[70] S. Gustafson, K. Kang, T.-P. Tsai, Schrödinger flow near harmonic maps, Comm. Pure Appl. Math. 60 (2007) no. 4, 463-499

[71] S. Gustafson, K. Kang, T.-P. Tsai, Asymptotic stability of harmonic maps under the Schrödinger flow, Duke Math. J. 145, No. 3 (2008), 537-583

[72] S. Gustafson, K. Nakanishi, T.-P. Tsai, Asymptotic stability, concentration, and oscillation in harmonic map heat-flow, Landau-Lifshitz, and Schrödinger maps on R^2, Comm. Math. Phys. 300, 205-242 (2010)

[73] S. Gustafson, E.Koo, Global well-posedness for 2D radial Schrödinger maps into the sphere, preprint, arXiv:1105.5659

[74] S. Gustafson, D. Roxanas, Global regularity and asymptotic convergence for the higher-degree $2d$ corotational harmonic map heat flow to $\mathbb{S}^2$ (preprint)

[75] S. Gustafson, D. Roxanas, Global, decaying solutions below the ground state for a critical heat equation (preprint)

[76] A. D. Ionescu and C. E. Kenig, Low-regularity Schrödinger maps, Differential Integral Equations 19 (2006), no. 11, 1271 – 1300

[77] J. Kato and H. Koch, Uniqueness of the modified Schrödinger map in $H^{3/4+\epsilon}(\mathbb{R}^2)$, Comm. Partial Differential Equations 32 (2007), no. 1-3, 415 – 429, available at arXiv:math/0508423

[78] C.E. Kenig, G.S. Koch. An alternative approach to the Navier-Stokes equations in critical spaces. Ann. I. H. Poincare – AN, 28(2):159 – 187, 2011

[79] C.E.Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case, Inventiones mathematicae (2006), 166, no 3, 645-675

[80] C.E.Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy critical focusing non-linear wave equation, Acta Mathematica (2008), 201, no 2, 147-212

[81] S. Keraani, On the defect of compactness for the Strichartz estimates for the Schrödinger equations. J. Diff. Eq. 175 (2001), 353-392

[82] R. Killip, M.Visan, The focusing energy-critical nonlinear Schrodinger equation in dimensions five and higher, Amer. J. Math. 132 (2010), 361-424

[83] R. Killip, M.Visan, Nonlinear Schrodinger Equations at Critical Regularity, “Evolution Equations”, 325-437, Clay Math. Proc., 17, Amer. Math. Soc., Providence, RI, 2013

[84] J. Ko, The construction of a partially regular solution to the Landau – Lifshitz - Gilbert equation in $\mathbb{R}^2$, Nonlinearity 18 (2005), no. 6, 2681 – 2714

[85] A. Kosevich, B. Ivanov, A. Kovalev, Magnetic solitons, Phys. Rep. 194 (1990) 117–238

[86] J. Krieger, W. Schlag, Concentration compactness for critical wave maps, EMS Monographs in Mathematics, European Mathematical Society, 2012

[87] J.Krieger, W.Schlag, D. Tataru, Renormalization and blow up for the critical Yang-Mills problem, Adv. Math. 221 (2009), no. 5, 1445-1521

[88] J.Krieger, W.Schlag, D. Tataru, Slow blow up solutions for the $H^1(\mathbb{R}^3)$ critical focusing semi-linear wave equation, Duke Math. J. 147 (2009), no. 1, 1-53

[89] L.D. Landau, E.M. Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies, Phys. Z. Sowietunion 8 (1935) 153; reproduced in: Collected Papers of L.D. Landau, Pergamon Press, New York, 1965, pp. 101–114.

[90] A. Lawrie, Conditional global existence and scattering for a semi-linear Skyrme equation with large data, Comm. Math. Phys.. 334 (2015) no. 2, 1025-1081

[91] A.Lawrie, S.-J. Oh, A refined threshold theorem for (1+2)-dimensional wave maps into surfaces, Comm. Math. Phys. 342 (2016) no. 3, 989-999

[92] A.Lawrie, S.-J. Oh, S. Shahshahani, Gap eigenvalues and asymptotic dynamics of geometric wave equations on hyperbolic space, preprint, arXiv:1502.00697v1

[93] A.Lawrie, S.-J. Oh, S. Shahshahani,Equivariant wave maps on the hyperbolic plane with large energy, preprint, arXiv:1505.03728v1

[94] B.W. Lee, Chiral dynamics, Cargèse lectures in physics, Vol. 5, Gordon and Breach, New York, 1972, pp. 1–117

[95] H. A. Levine, Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + F(u)$. Arch. Rational Mech. Anal. 51 (1973), 371–386

[96] D. Li. Global wellposedness of hedgehog solutions for the (3+1) Skyrme model. preprint, arXiv:1208.4977, 2012

[97] F. Lin, C. Wang, The analysis of harmonic maps and their heat flows. World Scientific, 2008

[98] P.-L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. I. Rev. Mat. Iberoamericana 1 (1985), 145-201

[99] M. Gell-Mann, M. Lévy, The axial vector current in beta decay, Nuovo Cimento (10) 16,(1960), 705–726

[100] H. Matano, F. Merle, Classification of type I and type II behaviors for a supercritical nonlinearheat equation. J. Funct. Anal. 256 (2009), no. 4, 992–1064

[101] H. Matano, F. Merle, On nonexistence of type II blowup for a supercritical nonlinear heat equation. Comm. Pure Appl. Math. 57 (2004), no. 11, 1494–1541

[102] H. Matano, F. Merle, Threshold and generic type I behaviors for a supercritical nonlinear heat equation. J. Funct. Anal. 261, no. 3, (2011), 716–748

[103] H. McGahagan, An approximation scheme for Schrödinger maps, Comm. Partial Differential Equations 32 (2007), no. 1-3, 375 – 400

[104] F. Merle, Solution of a nonlinear heat equation with arbitrarily given blow-up points, Communications on Pure and Applied Mathematics, 1992, 263-300

[105] F. Merle, L. Vega, Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in $2D$, Internat. Math. Res. Notices 8, (1998), 399–425

[106] F.Merle, P.Raphael, I. Rodnianski, Blow up dynamics for smooth equivariant solutions to the energy critical Schrödinger map, Comptes Rendus Mathématique, (2011), 349, no 5–6, 279–283

[107] F. Merle, H. Zaag, Uniform blow-up estimates for nonlinear heat equations and applications. IMS Conference on Differential Equations from Mechanics (Hong Kong, 1999)

[108] A. Nahmod, A. Stefanov, and K. Uhlenbeck, On Schrödinger maps, Comm. Pure Appl. Math. 56 (2003), no. 1, 114 – 151; Erratum, 57 (2004), no. 6, 833 – 839

[109] F. Planchon, Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$. Ann. Inst. H. Poincar*é* Anal. Non Lin*é*aire, 13 :319-336, 1996

[110] J.Qing, On singularities of the heat flow for harmonic maps from surfaces into spheres, Comm. Anal. Geom. 3 (1995), 297-315

[111] J.Qing, G. Tian, Bubbling of the heat flows for harmonic maps from surfaces, Comm. Pure Appl. Math. 50 (1997), 295-310

[112] P. Quitnner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhauser Advanced Texts, ISBN: 978-3-7643-8441-8 (2007)

[113] P.Raphael, I. Rodnianski, Stable blow up dynamics for the critical co-rotational wave maps and equivariant Yang-Mills problems, Publications mathématiques de l' IHES, 2012, 115, pp. 1-122

[114] P.Raphael, R.Schweyer, Stable blow up dynamics for the co-rotational energy critical harmonic heat flow, to appear in Comm. Pure App. Math

[115] P.Raphael, R.Schweyer, Quantized slow blow up dynamics for the co-rotational energy critical harmonic heat flow, preprint, arXiv:1301.1859

[116] I.Rodnianski, J. Sterbenz, On the formation of singularities in the critical O(3) sigma-model, Ann. of Math.(2010), 172, 187–242

[117] J. Sacks, K. Uhlenbeck, The existence of minimal immersions of 2-spheres. Ann. of Math. 113 (1981), 1-24

[118] R. Schweyer, Type II blow up for the four dimensional energy critical semi linear heat equation, J. Funct. Anal., 263 (2012), pp. 3922-3983

[119] J.Shatah, M. Struwe, Geometric wave equations. Courant Lecture Notes in Mathematics, 2, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1998

[120] P. Smith, Global regularity of critical Schrödinger maps: subthreshold dispersed energy, preprint,arXiv:1112.0251v3

[121] M.Struwe, A global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187, 511-517 (1984)

[122] M.Struwe, On the evolution of harmonic mappings of Riemannian surfaces, Comment. Math. Helv. (1985), 60, 558-581

[123] G. Talenti, Best constant in the Sobolev inequality, Ann. Mat. Pura Appl., (4) 110 1976, 353–372

[124] T. Tao, M. Visan, Stability of energy-critical nonlinear Schrödinger equations in high dimensions, Electron. J. Diff. Eqns. 118 (2005), 1–28. MR2174550

[125] P.Topping, Rigidity in the harmonic map heat flow, J. Differential Geometry (1997), 45, 593-610

[126] P.Topping, Winding behaviour of finite-time singularities of the harmonic map heat flow, Math. Zeit., 247 (2004) 279-302

[127] PJ.L. Vazquez and E. Zuazua, The Hardy inequality and the asymptotic behavior of the heat equation with an inverse-square potential, J. Funct. Anal. 173 (2000), no. 1, 103 – 153

[128] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p. Indiana Univ. Math. J, 29(1),(1980), 79-102.

[129] F. B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel Journal of Mathematics,(1981), Springer