Portfolio Optimization - Approaches to determining VaR and CVaR

A COMPARATIVE STUDY OF PORTFOLIO PERFORMANCES OVER TIME

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P A T R I K   B E R G M A N
V I K T O R   S O N E B Ä C K

Abstract

This thesis analyses portfolio optimization using the risk measures VaR and CVaR with two different underlying assumptions of probability distribution of returns; one being that portfolio returns are normal distributed and the other being a discrete distribution comprised of historical data. The models are run through numerous historical simulations on the OMXS30 with varying time period for historical data and rebalance frequencies. The resulting simulated returns as well as the CVaR outcomes are presented, compared and discussed in order to assess which model performs the best and under what circumstances.

Our key findings is that using a discrete, historical probability distribution for optimizing a portfolio with respect to CVaR, comprised of around 320 days worth of data and using a rebalancing frequency of 20 days performs the best with respect to total return and actual CVaR. This method manages to take the fat tails of the market return distribution into account and as such successfully avoids the larger market downturns.

The results of this thesis also indicate that historical VaR optimization is inferior to CVaR optimization. However due to lack of computational power this comparison is inconclusive.

Sammanfattning

Denna uppsats analyserar portföljoptimering med avseende på riskmåtten "Value at Risk" (VaR) och "Conditional Value at Risk" (CVaR) utifrån två olika antaganden om den underliggande sannolikhetsfördelning för hur avkastning ser ut. Den ena antar att portföljavkastningen följer en normalfördelning medan den andra skapar en diskret sannolikhetsfördelning direkt av de historiska aktiekurserna.

Modellerna analyseras med hjälp av en simulering på OMXS30 med varierande tidsspann för historiska data samt varierande ombalanseringsintervall. De resulterande avkastningarna och faktiska utfallen för CVaR presenteras, jämförs och diskuteras för att bestämma vilken metod som presterar bäst och under vilka förutsättningar.

Våra huvudsakliga resultat är den av våra fyra modeller som lyckas skapa den högsta avkastningen såväl som den lägsta risken är den som optimerar CVaR utifrån antagandet om en diskret historisk avkastningsfördelning. För att erhålla ett optimalt resultat verkar det vara bäst att använda sig av en tidshorisont på omkring 320 dagar av historiska data som underlag för optimeringen. Den historiska modellen för att förutsäga avkastningen lyckas även ta in den "sanna" avkastningsfördelningens tjocka svansar i beräkningen och lyckas i viss utsträckning också undvika stora marknadsnedgångar.

Resultaten indikerar att diskret, historisk VaR-optimering är underlägsen CVaR-optimering. På grund av bristande datorkraft är denna jämförelse dock ofullständig. Tidsåtgången för den historiska VaR-optimeringen är en stor nackdel för den modellen.

Innehåll

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R = return of the portfolio

R_k = return of the portfolio for day k

r_i = expected return of stock i

r_{ki} = return of stock i for day k

w_i = percentage of the capital invested in stock i

C = covariance matrix for the returns of the different stocks

α = confidence level

Φ^{-1} = the inverse cumulative distribution function for the standard normal distribution

1. Introduction

1.1 Background

Balancing risk against return is a crucial aspect of managing a financial portfolio. The expected return of a portfolio is a parameter that is rather straight forward to calculate and understand. Risk is a different matter and its meaning and concept is less obvious and less explicit. Since Markowitz introduced "Modern Portfolio Theory" (MPT) the most commonly used measure for risk amongst investors is the standard deviation (Markowitz, 1952). Over time and especially in the aftermath of a financial crisis, critical voices have been arguing that this measure falls short of taking many crucial points into account (Stoyanov, Rachev, & Fabozzi, 2013). One drawback is the way MPT models the return of a portfolio. The return of the portfolio is generally viewed as a stochastic variable belonging to a certain probability distribution. MPT assumes that this distribution is elliptical, usually the normal distribution. Several historical studies have shown that this assumption does not account for important abnormalities in the lower end of the probability distribution, namely that high losses are more likely to occur than the normal distribution would suggest (Fortin & Kuzmics, 2000) (J.P. Morgan Asset Management, 2009). This is mainly because of a factor known as tail dependence which implies that the covariances between stock-pairs are not the same during extraordinary market circumstances as they are during "normal" market conditions. This might lead to an underestimation of risk and mislead an investor into taking a riskier position than preferred (J.P. Morgan Asset Management, 2009). One way to circumvent this problem is to use a discrete probability function based directly on the historical stock returns instead of assuming that returns follow a certain probability distribution.

Another drawback of MPT is that the risk measure (standard deviation) focuses equally on profits and losses. A large profit carries as much weight as a large loss (Stoyanov, Rachev, & Fabozzi, 2013). When evaluating the risk of a portfolio the main interest is the potential losses and as a result several other risk measures have gained popularity in recent years. The most commonly known is Value at Risk (VaR), which was made popular by JP Morgan's RiskMetrics™ in 1994 and was included in BCBS's recommendations for banking regulation in Basel II, 2004. This measure focuses on the left end tail of the probability distribution and thus focuses on the risk for incurring heavy losses. Even though the concept of VaR is rather simple to grasp and that it in many aspects is a highly relevant risk measure, it is by no means perfect. Following the financial crisis of 2008 it has been criticized for some of its flaws (Einhorn, 2008) (Nocera, 2009). The greatest drawback is that VaR does not provide any information regarding how much one is expected to lose given that the "unlikely" scenarios of a large loss occur. A risk measure that does take this into account is "Conditional Value at Risk" (CVaR) which

is the expected loss given that the loss exceeds VaR. Aside from a superior description of risk, CVaR also possesses better mathematical properties than VaR as it is always sub-additive and the optimization problem when optimizing w.r.t CVaR has convex properties (Uryasev & Rockafellar, Optimization of conditional value-at-risk, 2000).

1.2 Purpose and aim

This thesis analyzes portfolio optimization with respect to CVaR and VaR using two different assumptions of underlying probability distributions of portfolio returns. The first model, the historical, is based on a discrete probability distribution created directly through historical data. The second, the analytical model, is based on the assumption of returns being normal distributed.

The main purpose of this study is to contribute to research and literature and to provide different types of investors, private of professional, with insights regarding portfolio management with the risk measures CVaR and VaR.

The results of this study provide information regarding the advantages and disadvantages for the different models as well as what effect the phenomenon of tail dependence of market returns has when looking at CVaR and VaR. This thesis also discusses the relevance of CVaR and VaR for different types of investors and the effects of fat tails in the lower end of the return distribution is analyzed with respect to how important it is for different investors to take these "abnormalities" into account.

The thesis seeks to answer the following questions:

- What is the effect of tail dependence of the market's historical returns when optimizing a portfolio with respect to VaR and CVaR?
- Is one approach (historical or analytical) to determining VaR and CVaR consistently better than the other, even when underlying assumptions are changed?
- What is the performance of the historical approach compared to the analytical approach during periods of economic crisis?
- Of what importance is the market's tail dependence for different kinds of investors and how should it be handled? What relevance does CVaR and VaR hold for different investors?

2. Theoretical background

This section serves to provide the mathematical theory on which this thesis is be based. The two different risk measures to be used are presented as well as how to model the portfolio return distribution.

2.1 Risk management

As touched upon in the introduction there are several ways for an investor to measure risk, none of them necessarily more correct than others. In this sub-section the two risk measures "Value at Risk" (VaR) and "Conditional Value at Risk" (CVaR) are explained.

2.1.1 Value at Risk - VaR

Value at Risk is a risk measure which describes the highest loss that is "likely" to occur. What "likely" means is that the losses with a certain confidence level do not exceed a certain magnitude. For example a VaR of $100 and a confidence interval of 0.95 implies that the portfolio loss, with a probability of 95%, does not exceed $100 over a certain time period.

The mathematical definition is as follows:

$$VaR_\alpha = \inf \{r \in \boldsymbol{R} : P(-R > r) \leq 1 - \alpha \}$$

where α is the confidence interval and R is a stochastic variable for the portfolio return. Since R denotes the return of the portfolio, $-R$ denotes the loss. VaR is discussed as a positive term, even though the return associated with the VaR level is most often negative.

There are several benefits in using VaR as a risk measure. Unlike standard deviation, the meaning of VaR as a risk measure is more comprehensible, even for persons who lack a deeper knowledge of mathematics. Furthermore VaR only focuses on losses and the opportunity of high profits does not affect VaR.

There are numerous drawbacks of VaR however. First and foremost, VaR does not take into account the magnitude of losses greater than VaR. For example, given a probability distribution of R that generates a VaR of $100 and a worst scenario for the loss on $500. If we would increase this "worst possible loss" to $1000 it would not affect the value of VaR but it would change the actual portfolio risk substantially. VaR is also not sub-additive which can lead to a portfolio VaR being higher than the sum of the components' individual VaR.

2.1.2 Conditional Value at Risk - CVaR

A risk measure that does take every scenario in the tail end of the return distribution into account is Conditional Value at Risk, also known as Expected Shortfall. CVaR informs the investor about the

expected value of the loss given that the outcome is in the bottom α-percentile. For example a CVaR of $100 with a confidence interval of 0.95 implies that the expected portfolio loss given that the portfolio loss falls in the bottom 5% is equal to $100.

The mathematical definition is as follows:

$$CVaR_\alpha = E[-R \mid R \leq -VaR_\alpha]$$

CVaR, unlike VaR, is sub additive which is preferable in a diversification aspect. The total risk of a portfolio is less than or equal to the sum of the risk of the individual assets.

2.2 Expected return

When predicting the future returns of a portfolio the portfolio return is assumed to be a stochastic variable belonging to some probability distribution. Which probability distribution to be used is determined beforehand and varies amongst investors and situation. Historical stock data is often used to calculate the parameters of the chosen distribution. The time horizon for historical data collection may vary from a month to several years.

The expected future return, r_i, for a certain stock is calculated using N number of historical closing prices as such:

$$r_i = \frac{1}{N} \sum_{k=1}^{N} r_{ki}$$

2.2.1 Analytically

In the classical Markowitz MPT the returns are assumed to belong to a normal distribution,

$R \in N(\mu, \sigma^2)$, where μ and σ^2 is calculated as follows:

$$\mu = \sum_{i=1}^{n} w_i * r_i , \qquad \sigma^2 = \sum_{i=1}^{n} w_i^2 * Var(r_i) + 2 * \sum_{i \leq i} \sum_{< j < n} w_i * w_j * Cov(r_i, r_j)$$

w_i is the percentage of the total capital invested in stock i and r_i is the expected return of stock i.

Or in matrix notation:

$$\mu = \bar{r} * \bar{w}, \qquad \sigma^2 = \bar{w}^T C \, \bar{w}$$

where C is the covariance matrix of the different stocks.

2.2.2 Historically

The return can also be assumed to have a discrete distribution where the different outcomes are denoted by how the portfolio would have performed the last N number of days.

If R_k is the return the portfolio would have had on day k, then $P(R = R_k) = \frac{1}{N}$, for $k = 1,2,\dots,N$

$R_k = \sum_{i=1}^{n} w_i * r_{ki}$, where r_{ki} is the return of asset i on day k and w_i is the portfolio weight. As can be seen the returns on the different days are linear functions of the portfolio weights.

If using the historical model when optimizing a portfolio with respect to a certain risk measure, the generated portfolio is the one that would have performed the best over the last N days.

2.3 Tail dependence of market returns

When determining the overall risk of a portfolio consisting of stocks with individual risk it is of interest for the investor to analyze how the stocks interact with one another. This is traditionally done by examining the correlation between stock-pairs and creating a correlation-matrix consisting of the correlation between each stock which may be a part of the portfolio (Markowitz, 1952). This relationship of how the stocks interact is not static and is subject to change over time. During normal market conditions one may assume the correlation to be quite indicative of the stocks' actual interaction. During periods of abnormal market movements such as a large decrease or increase of the market index these correlations between stocks no longer hold true (Fortin & Kuzmics, 2000). This leads to the conclusion that assuming that returns follow a normal distribution when analyzing portfolio risk does not hold true during large market movements. This is known as tail dependence of market returns. The effect of this is that large portfolio losses are more likely to occur than a normal distribution would suggest.

Looking at the two following tables of OMXS30 returns and FTSE100 returns compared to a normal distribution assumption of their returns the effect of tail dependence becomes quite obvious.

Table 1 shows the percentage of actual daily returns on OMXS30 index during the period 03012000-01012015 in comparison to what percentage a normal distribution would assume.

Lower tail-end of normal distribution	Percent actual returns
5.00	4.77
4.00	4.12
3.00	3.52
2.00	2.79
1.00	1.88

Table 1 - Tail dependence OMXS30

Table 2 shows the percentage of actual daily returns on FTSE100 index during the period 03012000-01012015 in comparison to what percentage a normal distribution would assume.

Lower tail-end of normal distribution	Percent actual returns
5.00	4.43
4.00	4.02
3.00	3.46
2.00	2.71
1.00	1.82

Table 2 - Tail dependence FTSE100

As can be seen from the above tables the return distributions has fat tails when looking at both OMXS30 and FTSE100. There are between 80-90% more returns in the lowest percentile than the assumption of normal distribution assumes.

3. Methodology

Four models of portfolio optimization based on two different ways of determining Conditional Value at Risk and Value at Risk are created. One model relies on determining CVaR and VaR using the analytical approach and the other uses the historical approach.

The models are run through numerous historical simulations with changes in underlying assumptions of parameters such as time horizon of past data and rebalancing interval and are then compared with each other.

There are a few academic papers regarding portfolio optimization based on CVaR, most notably "Optimization of conditional value-at-risk" by Rockafellar and Uryasev. These serve as the basis for the mathematical part of this thesis, with main focus on the one mentioned.

3.1 Optimization models and comparisons

The first two models determine CVaR and VaR analytically while the second two determine CVaR and VaR using the historical approach. All optimization models are subject to the same constraints and are based on the same historical data when simulated historically. The models are run through a historical simulation in order to gauge their performance over a certain time period. Rebalancing of the portfolios occur simultaneously and is based on a fixed time interval of 20 days or 40 days. The result of these simulations is a graphical comparison between the portfolios' performances under different constraints and assumptions.

3.1.1 Time horizon of returns and portfolio rebalancing

The simulation of the two models is run multiple times with different time horizons for determining returns and the different results are compared and analyzed. The period of rebalancing is also subject to change in order to determine how the models perform under varying circumstances. The length of this period will alternate between 20 days and 40 days chosen for practical reasons from an investor's perspective.

3.2 Data

The data needed for this thesis is the daily adjusted closing prices over the selected time periods during which the portfolios are simulated. The list of stocks to be considered in the portfolio optimizations is a mix between those with the largest turnover per 03-01-2000 (provided by SIX Financial Information) and those currently listed on OMXS30 with some exceptions due to unavailable data. A list of these stocks is included in the appendix. The optimal solution would be to update the list of stocks which would be part of OMXS30 every period so as to avoid any kind of survivorship bias but this is not something done in this thesis as it creates no significant problem

since the aim is to compare different models to each other where all models are based on the same data.

Data for each individual stock is retrieved from Yahoo Finance via Matlab. Preprocessing of the data as well as the final calculations, simulations and graphing all occur in Matlab as well.

Data is available from 03012000 up to present day with numerous data points missing for the different stocks due to situations such as trading halts. This will be accounted for by interpolation using the two closest data points.

3.3 Delimitations

This thesis only discusses portfolios consisting of stocks. The optimizations will occur during the time period 03012000 - 08042015.

This thesis looks at historical VaR optimization theoretically and simulates such a model historically. However, this cannot be done for a large time horizon of past returns with the computing power available for this thesis. The time horizon for the historical VaR optimization will therefore be limited to a maximum of 100 days of past returns. As a result the basis for analysis for the historical VaR optimization is limited.

The only confidence interval considered in the optimizations is $\alpha = 0.95$.

Taxes, risk-free rate and brokerage fees are all excluded from the models. As this thesis is a comparison of different optimization models this should not have great impact on the results.

4. Portfolio optimization

This section will construct the optimization problems; Value at Risk analytically and historically, and Conditional Value at Risk analytically and historically.

4.1 CVaR-optimization

The aim of the CVaR-optimization is to find the portfolio weights that would generate the lowest CVaR given that the returns are assumed to belong to some predetermined probability distribution. The two different types of return distributions to be considered in this thesis are the normal distribution, and a discrete distribution built on the historical portfolio performance as described in section 2.2.2.

4.1.1 Analytically

As mentioned above the analytical approach assumes the portfolio returns to be normal distributed. This section will construct the CVaR-optimization problem with this underlying assumption.

Given that the returns are normally distributed the expression for CVaR is:

$$CVaR_\alpha = E[-R \mid R \leq -VaR_\alpha] \text{ , } where \text{ } R \in N(\mu, \sigma^2)$$

The mean and variance of the distribution are functions of the portfolio weights, as described in (2.2.1).

If assuming normally distributed returns the $CVaR_\alpha$ of the portfolio is just a linear combination of the mean and the standard deviation:

$$CVaR_\alpha = -\sigma s_\alpha - \mu$$

The proof for this statement is found in the Appendix under *2. Mathematical derivation of analytical CVaR.*

Some values of the constant s_α given a confidence interval of α:

Confidence interval (α)	s_α
0.90	-1,7550
0.95	-2,0627
0.99	-2,6652

Table 3 – Some values of s_α

The optimization problem then becomes:

$$\begin{array}{cc} \min\limits_{\overline{w}} & CVaR_\alpha \\ s.t. & \overline{w} \in \Omega \end{array} = \begin{array}{cc} \min\limits_{\overline{w}} & -\mu - \sigma s_\alpha \\ s.t. & \overline{w} \in \Omega \end{array}$$

Ω is an arbitrarily chosen set that delimits the allowed portfolio weights, $\overline{w}$.

If $\mu(\overline{w})$ is concave, $\sigma(\overline{w})$ is convex and the set Ω is convex (which is the case for $CVaR$) then according to (Krokhmal, Palmquist, & Uryasev, 2004) this is equal to the following optimization problem:

$$\begin{array}{cc} \min\limits_{\overline{w}} & -\sigma s_\alpha \\ s.t. & \begin{matrix} \mu \geq c \\ \overline{w} \in \Omega \end{matrix} \end{array} = \begin{array}{cc} \min\limits_{\overline{w}} & \sigma \\ s.t. & \begin{matrix} \mu \geq c \\ \overline{w} \in \Omega \end{matrix} \end{array} = \begin{array}{cc} \min\limits_{\overline{w}} & \sigma^2 \\ s.t. & \begin{matrix} \mu \geq c \\ \overline{w} \in \Omega \end{matrix} \end{array} \quad \text{, where } c \text{ is an arbitrarily chosen constant.}$$

The reason for the lack of the minus sign in the second final problem is because s_α is always negative and thus the entire objective function becomes positive. If $c \to -\infty$ the constraint $\mu \geq c$ can be removed and the obtained value of $CVaR_\alpha$ is the smallest one for every possible combination of the portfolio weights that satisfy $\overline{w} \in \Omega$.

The optimization problem will thus be as follows:

$$\begin{array}{cc} \min\limits_{\overline{w}} & CVaR_\alpha \\ s.t. & \overline{w} \in \Omega \end{array} = \begin{array}{cc} \min\limits_{\overline{w}} & \sigma^2 \\ s.t. & \overline{w} \in \Omega \end{array} = \begin{array}{cc} \min\limits_{\overline{w}} & \overline{w}^T C \, \overline{w} \\ s.t. & \overline{w} \in \Omega \end{array}$$

If the portfolio constraints are linear the optimization problem will be on a quadratic form.

One interesting aspect of this result is that the confidence interval (α) will not affect the outcome of the optimization. Another interesting aspect is that this optimization problem minimizes the variance of the portfolio returns. This is equal to the classical Markowitz portfolio optimization and thus minimizing CVaR and minimizing variance is equivalent if the returns are assumed to be normally distributed.

4.1.2 Historically

When optimizing CVaR using the historical approach the portfolio weights are determined as the combination that would have generated the lowest CVaR over a time period of the last N days. The idea behind the CVaR-optimization was developed by (Uryasev & Rockafellar, Optimization of conditional value-at-risk, 2000) and states that:

$$\min\limits_{\overline{w}} CVaR_\alpha(\overline{w}) = \min\limits_{\overline{w}} F_\alpha(\overline{w}, u)$$

Where F_α is a help function that approximates $CVaR_\alpha$ with the introduced help variable u.

When having a discrete sample distribution of the returns the expression for F_α will be:

$$F_\alpha(\overline{w}, u) = u + \frac{1}{N(1-\alpha)} \sum_{k=1}^{N} [f(w_k, \lambda_k) - u]^+$$

This also according to (Uryasev & Rockafellar, Optimization of conditional value-at-risk, 2000).

The introduced help variable u actually represents the VaR_α of the problem. λ_k are slack variables and the function $f(w_k, \lambda_k)$ is the loss of the portfolio at day k. In our case the loss function will be:

$$f(w_k, \lambda_k) = -\overline{r}_k * \overline{w}$$

The reason why only the positive terms of the sum are included is because these are the terms representing losses greater than VaR_α.

The optimization problem using the historical approach for minimizing $CVaR_\alpha$ will look as follows:

$$\min_{\overline{w}} \; CVaR_\alpha \atop s.t. \; \overline{w} \in \Omega = \quad \begin{array}{c} \min_{w, \lambda, u} \; u + \frac{1}{N(1-\alpha)} \sum_{k=1}^{N} \lambda_k \\ s.t. \quad -\overline{r}_k * \overline{w} - \lambda_k - u \le 0 \\ \lambda_k \ge 0 \\ \overline{w} \in \Omega \end{array}$$

λ_k will be equal to zero for the days were the losses do not exceed u. For the rest of the days λ_k will correspond to the amount by which the losses exceed u. The objective function will as a result be the mean value of the losses greater than u.

4.2 VaR-optimization

The aim of the VaR-optimization is to find the portfolio weights that generate the lowest possible VaR at a confidence level of α. Most often finding the optimal VaR is a much more difficult task than finding the optimal CVaR. This is because VaR in the general case is a non-convex and sometimes discontinuous function, and in the discrete case has multiple local minimums (Uryasev, Var vs CVaR in Risk Management and Optimization, 2010).

4.2.1 Analytically

As in the case of the analytical CVaR-optimization it is assumed in this section that the portfolio returns are normally distributed, $R \in N(\mu, \sigma^2)$.

The expression for VaR at a confidence level of α will be:

$$VaR_\alpha = -\left(\mu + \sigma\phi^{-1}(1-\alpha)\right) = -\left(\mu - \sigma\phi^{-1}(\alpha)\right) = \sigma\phi^{-1}(\alpha) - \mu$$

Minimizing VaR will therefore generate the following optimization problem:

$$\begin{array}{l} \min_{\overline{w}} VaR_\alpha \\ s.t.\ \overline{w} \in \Omega \end{array} = \begin{array}{l} \min_{\overline{w}}\ \ \sigma\phi^{-1}(\alpha) - \mu \\ s.t.\ \ \ \overline{w} \in \Omega \end{array}$$

On a closer look this is very similar to the analytical CVaR-optimization problem. The only thing that differs is the coefficient of the standard deviation, $\phi^{-1}(\alpha)$ instead of $-s_\alpha$. If $\phi^{-1}(\alpha)$ belongs to the domain of $-s_\alpha$ the VaR optimization could be viewed as a CVaR optimization, but with a different confidence level. Since the coefficient $-s_\alpha$ does not affect the outcome of the CVaR optimization (it is not an existing factor in the final optimization problem) it logically follows that $\phi^{-1}(\alpha)$ will not affect the outcome of the VaR optimization. This provided that $\phi^{-1}(\alpha)$ belongs to the domain of $-s_\alpha$, that is $(0, \infty)$ which requires α to be greater than 0,5. If this requirement is not fulfilled the VaR optimization cannot be looked at as a CVaR optimization and the reasoning does not hold.

The optimization will consequently be:

$$\min VaR_\alpha \leftrightarrow \min CVaR_\alpha = \begin{array}{l} \min_{\overline{w}} \overline{w}^T C\ \overline{w} \\ s.t.\ \overline{w} \in \Omega \end{array}$$

As a result of this the portfolio generated from minimizing VaR will be equal to the portfolio generated by minimizing CVaR and as such also equal to minimizing the portfolio variance.

4.2.2 Historically

Minimizing the historical VaR implies that the n:st largest loss out of a total of N is to be minimized, where $n = \lceil N(1 - \alpha) \rceil$. (Mausser & Romanko, October 2014) suggest the following Mixed Integer Linear Program as a way of minimizing VaR_α.

$$\begin{array}{l} \min_{\overline{w}} VaR_\alpha \\ s.t.\ \overline{w} \in \Omega \end{array} = \begin{array}{l} \min_{\overline{w},\ \lambda,\ u}\ u \\ s.t.\ \ -\overline{r}_k\overline{w} - M\lambda_k - u \le 0 \\ \quad \sum_{k=1}^{N} \lambda_k \le N(1 - \alpha) \\ \quad \lambda_k \in \{0,1\} \\ \quad \overline{w} \in \Omega \end{array}$$

u is a variable representing VaR_α. As can be seen λ_k are binary variables which will be equal to one for the equations corresponding to the losses greater than VaR_α. M is a sufficiently large number that makes sure that $-\overline{r}_k\overline{w} - M\lambda_k \le u$ for every k.

4.3 Portfolio constraints

In the optimization problems above the portfolio weights, $\overline{w}$, are assumed to belong to some convex set Ω. This section will further specify this set and how to implement different portfolio constraints in the optimization problems.

One necessary constraint is that the portfolio weights cannot be larger than one:

$$\sum_{i=1}^{n} w_i \leq 1$$

If the inequality above is changed to an equality constraint one has to invest 100 % of capital in stocks.

If short selling is not allowed the lower bound of the weights is zero:

$$w_i \geq 0$$

If short selling is allowed then the variables x_i and y_i are introduced and w_i is replaced by these:

$$w_i = x_i - y_i, x_i \geq 0 \; y_i \geq 0$$

x_i is the percentage bought of stock i and y_i is the percentage sold of the same stock. The upper bounds of x_i and y_i limits the maximum amount one can buy/sell of the specific stocks.

If there is some limit for short selling, for example 30 % of the portfolio's value:

$$\sum_{i=1}^{n} y_i \leq 0.3$$

If the investor has some minimum requirement on the expected return of the portfolio:

$$\sum_{i=1}^{n} w_i * r_i \geq \text{"return requierment"}$$

4.4 Efficient frontiers

The efficient frontier is a graph of the tradeoff between return and risk where each point along the curve represents an optimal portfolio, i.e. the portfolio with the lowest estimated risk for a certain target return.

This frontier is calculated by setting a target return as a constraint in the optimization models and then calculating the optimal portfolio for this specific return. This is done for returns ranging from 0 to the highest possible return in the portfolio (the return given by investing 100 % of the portfolio into the single stock with the highest estimated return) with a certain number of steps, for example 1000. The result is an efficient frontier which shows how the portfolio risk increases as the set target return increases.

Along this curve there comes a certain point where this specific portfolio provides the best tradeoff between risk and return, that is $\frac{expected \; portfolio \; return}{portfolio \; CVaR}$ is as high as possible. If one were to remove the constraint that 100 % of the portfolio must be invested all the portfolios with lower target return than this "best trade-off return" would invest in this specific portfolio, but with a smaller percentage invested.

5. Results

This section consists of the results of simulating the models.

5.1 Efficient frontier

This section describes the efficient frontiers when optimizing with respect to historical or analytical CVaR. Historical VaR is excluded from this section due to the computational reasons mentioned in the methodology.

Each point along the frontier represents an optimal portfolio with the lowest possible CVaR for the given return. Historically and analytically optimized CVaR efficient frontier below:
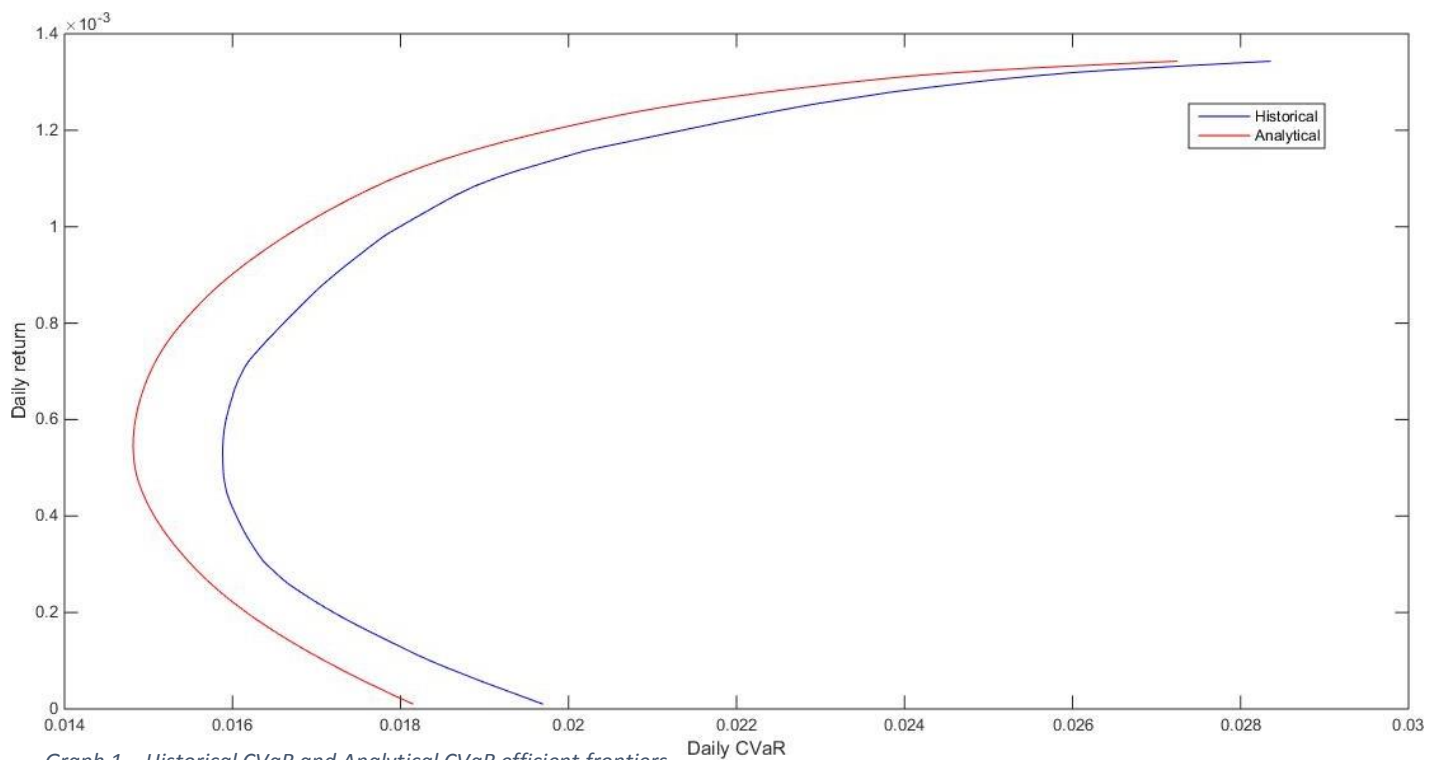
Graph 1 – Historical CVaR and Analytical CVaR efficient frontiers with 100 % of portfolio invested

As can be seen there is no significant differences between the two models, other than the fact that the historical efficient frontier is somewhat shifted to the right in the graph, meaning that the historical approach generally finds optimal portfolios with higher estimated CVaR. Note that the above efficient frontiers are based on each model's estimation of CVaR and not the result of historical simulations.

The following shows the efficient frontier when the constraint to spend 100 % of the portfolio's capital is removed.
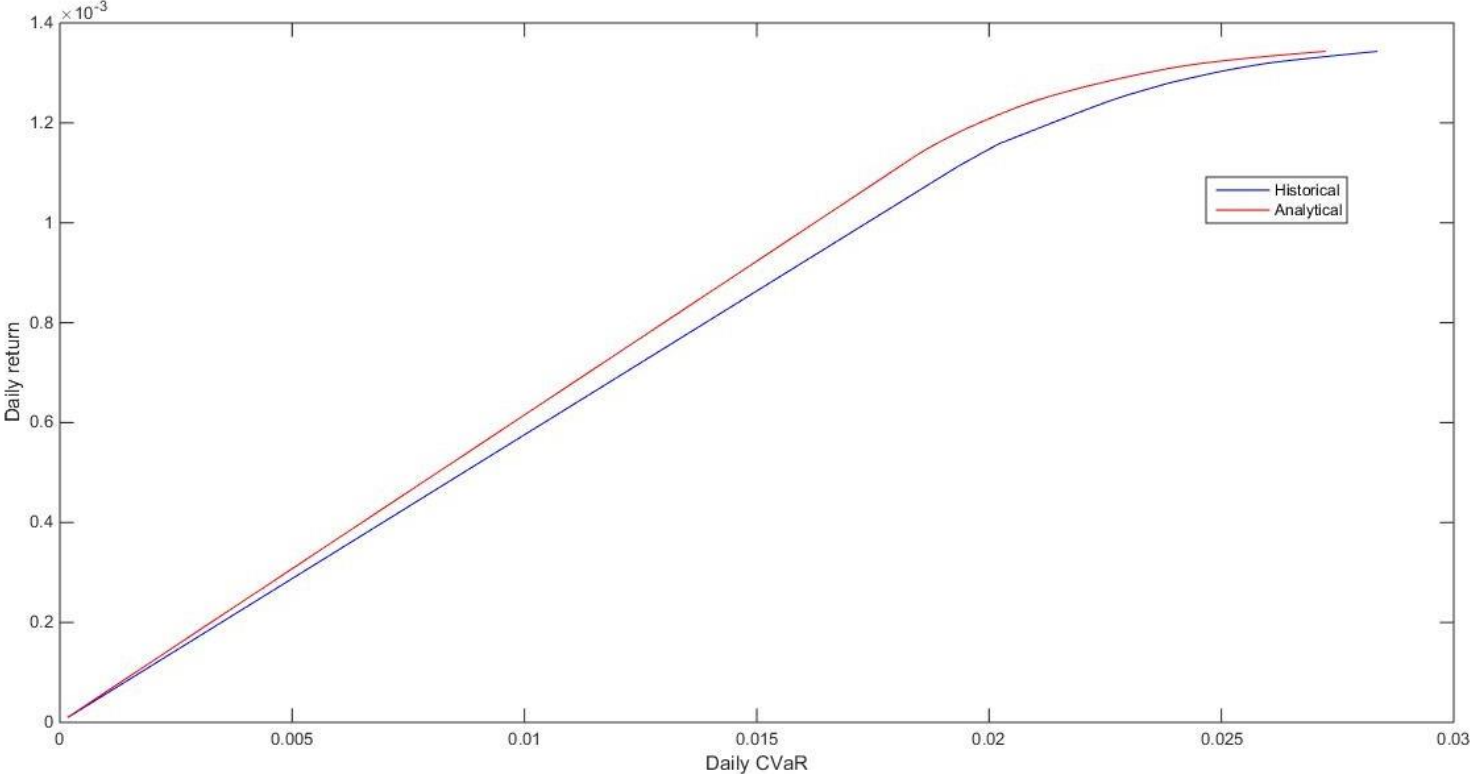
Graph 2 – Historical and Analytical CVaR efficient frontier with full investment constraint removed

The portfolios for the different optimal points along each curve starting from a CVaR around 0 up until a CVaR of around 2 % are all the same portfolio but with different percentages of the portfolio's capital invested. When all of the capital is invested in the portfolio with the maximum trade-off between return and CVaR, the portfolio allocations change in order to create a higher return which leads to a higher CVaR.

5.2 Historical CVaR versus historical VaR

This section is a comparison between optimizing a portfolio with respect to historical CVaR compared to optimizing with respect to historical VaR. Due to the VaR optimization being a mixed integer linear programming problem the calculations are considerably more time consuming than the CVaR optimization. The results in this section therefore only covers time horizons for past data for 3 different time periods, 40 days, 70 days and 100 days. It also only includes a rebalance frequency of 20 days.

The following table shows the resulting returns and their standard deviation of some historical simulations for the historical CVaR and the historical VaR. The rebalance frequency is with what time interval the portfolio is rebalanced. This time interval is also the number of different phases of rebalancing that may occur, for example starting at day 1, rebalancing day 21, 41 or starting at day 2 and rebalancing at day 22, 42 and so on. All these different phases are simulated. Time horizon is the number of days of past data on which the optimizations are based. Wins is the number of times the historical CVaR optimization for a given phase has higher total return than the historical VaR optimization, and vice versa. Mean is the mean, total return over the whole time period for all the different phases and SD is the standard deviation of these returns. For example, with time horizon for past returns set to 40 days the mean, total return at the end of the simulation would be 7.3791 times the initial investment. Starting date for the optimizations is 2000-05-25 with maximum time horizon backwards dating back to 2000-01-03. End date for the simulations is 2015-04-08

Rebalance frequency	Time horizon	Historical CVaR			Historical VaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	20	7.3791	1.6843	0	5.1162	1.8218
	70	17	5.5528	1.2353	3	3.9955	0.5766
	100	15	5.2617	0.6570	5	4.5793	0.6361

Table 4 – Historical CVaR versus Historical VaR, mean total return

The following table shows the resulting mean CVaR for all the phases as well as the standard deviation of these values.

Rebalance frequency	Time horizon	Historical CVaR			Historical VaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	14	10.07	1.60	6	10.93	1.34
	70	8	10.41	1.34	12	10.26	0.99
	100	11	10.09	1.38	9	10.27	1.11

Table 5 – Historical CVaR versus Historical VaR, mean 20-day CVaR

The historical CVaR optimization is shown in Table 4 to perform remarkably better than the VaR-minimization with respect to returns. The returns for CVaR are generally higher while the standard

deviation is about the same with the exception being for 70 days of past data. Table 5 gives some insight as to how the difference in CVaR might look but is all in all inconclusive given the limited number of simulations performed.

5.3 Historical simulation of portfolios

The two portfolio optimization models are run through a number of historical simulations with varying periods of rebalancing and varying time horizons for past returns. The results of these simulations are presented in tables as well as example graphs of performance. These graphs include the historical approach for determining CVaR, the analytical approach and the OMXS30 index. This section also includes a histogram of the portfolios' returns and actual CVaR and VaR over the time period in question.

Historical VaR optimization is excluded due to lack of computational power.

5.3.1 Historical simulation from 2001-06-26 to 2015-04-08

The following simulation is only one example of what the two models' returns, CVaR and VaR might be since it starts by creating an optimal portfolio at day 1, rebalancing day 21, 41, 61 and so on. By shifting the starting point with 1 day, the results will differ as the optimization now starts at day 2 and rebalances day 22, 42, 62 and so on.

When simulating a portfolio over the time period 2001-06-26 to 2015-04-08 with a period of rebalancing of 20 days and a time horizon for past returns of 320 days, the following graph is the resulting returns.
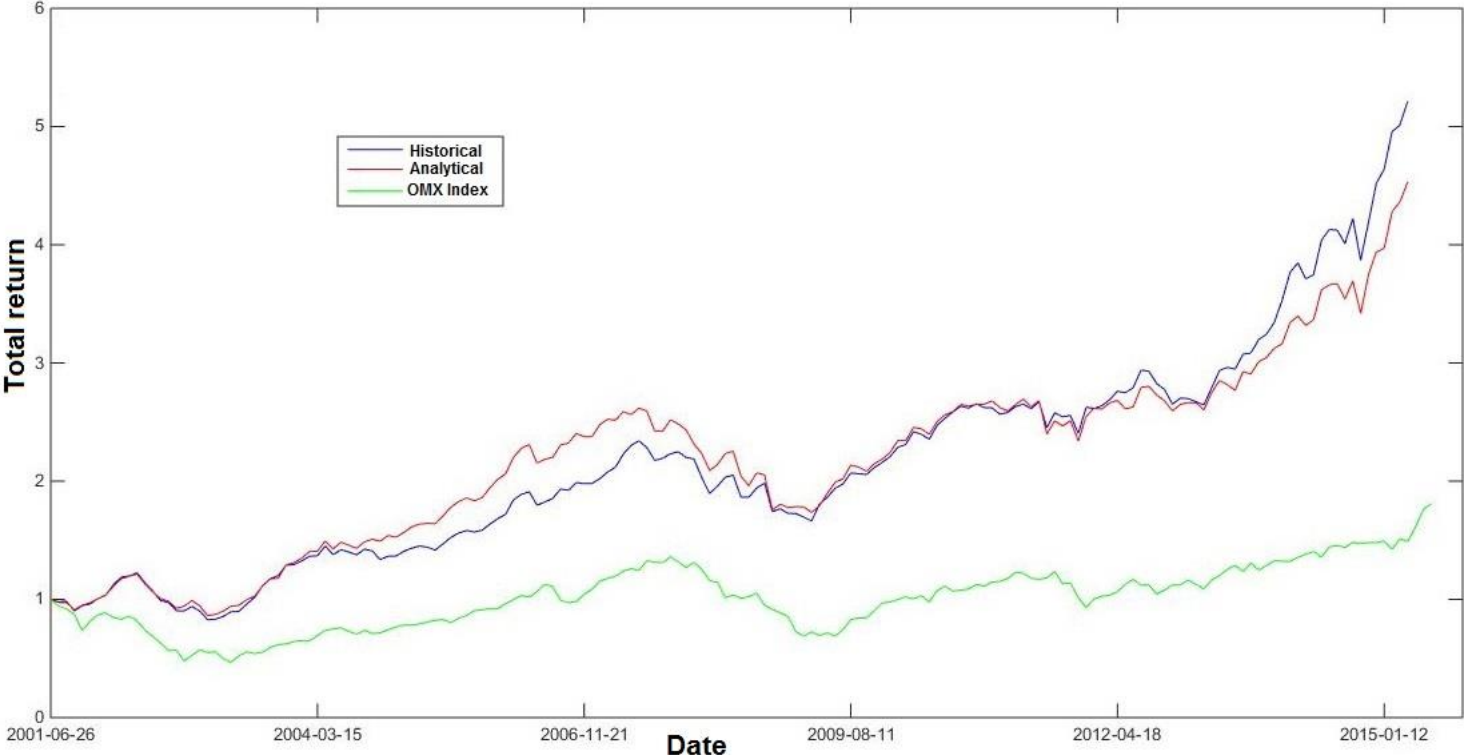
Graph 3 – Historical simulation 2001-06-26 to 2015-04-08

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Blue color represents the historical approach to minimizing portfolio CVaR while the red color represents the analytical approach to minimizing portfolio CVaR. Green is the OMX Index for the sake of comparison.

The following histogram shows the distribution of the two models' 20 day returns as well as the distribution of OMX Index returns over the same interval. The width of the bins is 0.05.
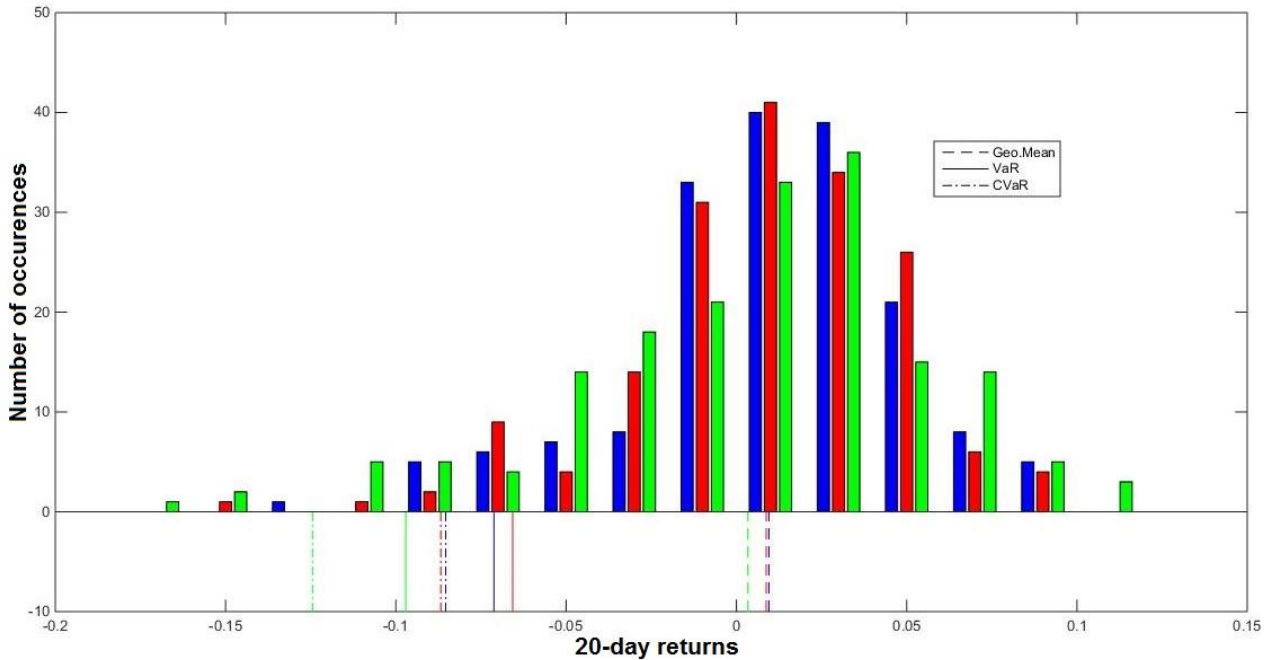
Graph 4 – Histogram of 20 day returns 2001-06-26 to 2015-04-08

The colors in this histogram represents the same as it does in the previous graph. Under the returns the CVaR, VaR and geometrical mean of monthly returns are found for the two models as well as for OMX Index. Here, CVaR is the mean of the lowest 5% of monthly returns which occurred during the simulation and VaR is the largest return of these 5%. This can be seen more clearly when zoomed in, as the next graph shows.
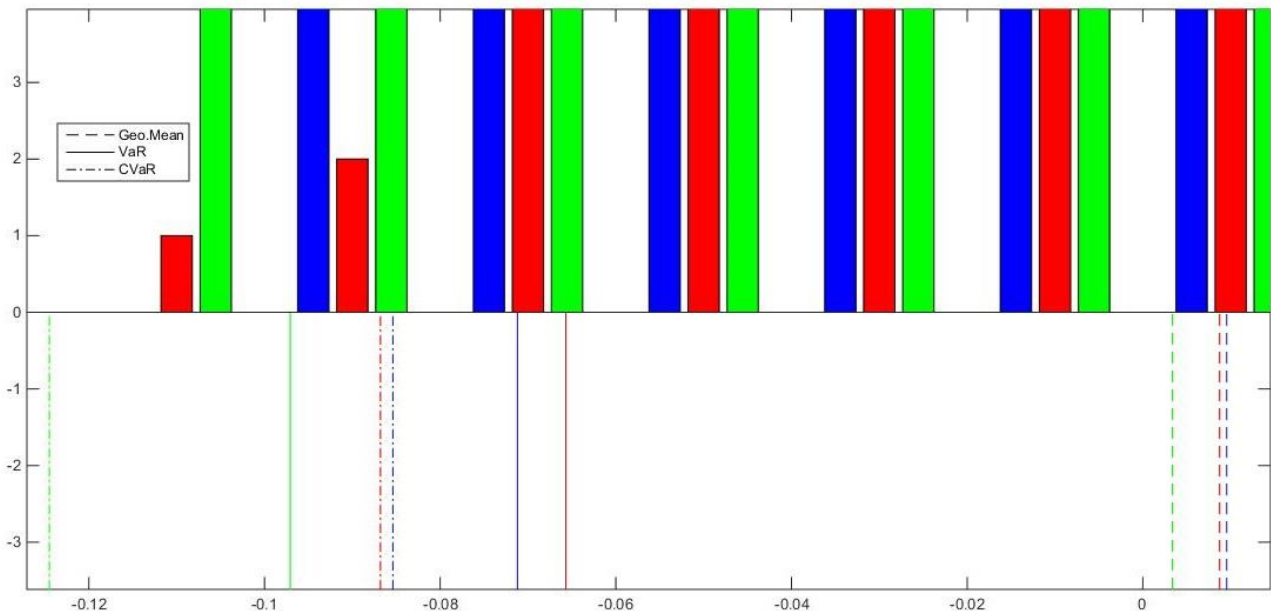
Graph 5 – Historical simulation VaR and CVaR outcome 2001-06-26 to 2015-04-08

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The historical VaR is larger than the analytical VaR but its CVaR is lower. OMX Index VaR and CVaR are much worse than both the analytical and the historical.

As stated earlier, the previous graphs are an example what the comparison between the two models may look like. The next two tables are comprised of more rigorous results.

The following table shows the resulting returns and their standard deviation of numerous historical simulations for the two optimization models with two different periods of rebalancing and 9 different time horizons for past return data. The rebalance frequency is with what time interval the portfolio is rebalanced. This time interval is also the number of different phases of rebalancing that may occur, for example starting at day 1, rebalancing day 21, 41 or starting at day 2 and rebalancing at day 22, 42 and so on. All these different phases are simulated. Time horizon is the number of days of past data on which the optimizations are based. Wins is the number of times the historical optimization is better than the analytical, and vice versa. Mean is the mean, final return over the whole time period for all the different phases and SD is the standard deviation of these returns. Starting date for the optimizations is 2001-07-10 with maximum time horizon backwards dating back to 2000-01-03. End date is 2015-04-08.

Rebalance frequency	Time horizon	Historical CVaR			Analytical CVaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	15	5.1568	1.1090	5	4.7148	0.6170
	80	0	3.1115	0.3553	20	3.6625	0.2049
	140	4	2.9741	0.4036	16	3.2532	0.1636
	200	8	3.4576	0.3703	12	3.6417	0.1824
	260	20	4.4173	0.2381	0	4.0313	0.1321
	320	20	5.0059	0.3181	0	4.1496	0.1420
	380	20	4.8243	0.2634	0	4.0994	0.1507
	440	14	4.1863	0.1162	6	4.1146	0.1495
	500	17	4.1132	0.1968	3	3.9065	0.1227
40 days	40	20	4.2442	0.9104	20	4.2276	0.5690
	80	14	2.9853	0.7170	26	3.1747	0.2000
	140	10	2.9223	0.2280	30	3.1167	0.3147
	200	29	3.5692	0.3015	11	3.3585	0.2590
	260	38	4.3316	0.5381	2	3.7699	0.3429
	320	40	4.8250	0.4525	0	3.9395	0.2537
	380	40	4.4773	0.3337	0	3.8619	0.2685
	440	27	3.8775	0.3342	13	3.7982	0.3186
	500	32	3.9043	0.2179	8	3.7257	0.2872

Table 6 - Historical CVaR versus Analytical CVaR, mean total return

The returns when minimizing CVaR using the analytical approach has a generally lower standard deviation as can be seen in Table 6 above. The standard deviation is considerably higher when only using 40 days' worth of past data than it is using a larger sample size. This holds true for the analytical approach as well as the historical. This standard deviation then decreases as the sample size of past data increases.

The highest achievable return when using a rebalancing period of 20 days is to use the historical method and looking at the past 40 days' returns. However, this method also provides a very large standard deviation in comparison to other options in the above table. The next highest return is by using the historical approach and using 320 days of past data which provides a marginally lower return with a significantly lower standard deviation. This method also outperforms all the analytical approaches with a much larger return but yet a reasonable standard deviation.

Using a rebalancing period of 40 days generally provides lower returns as well as higher standard deviations than using a rebalancing period of 20 days. Using 320 days and the historical approach provides the best return with a reasonable standard deviation when compared to other time horizons.

It can also be concluded that using a 20 day rebalancing period performs better than using a 40 day rebalancing period.

The next table shows the resulting, monthly CVaR and their standard deviation of numerous historical simulations for the two optimization models with two different periods of rebalancing and 9 different time horizons for past return data. Mean is the mean, monthly CVaR in percent for all the different phases and SD is the standard deviation of these values. Starting date for the optimizations is 2001-07-10 with maximum time horizon backwards dating back to 2000-01-03.

Rebalance frequency	Time horizon	Historical CVaR			Analytical CVaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	10	10.1899	1.6236	10	10.1528	1.2088
	80	9	10.4611	1.2981	11	10.2373	1.2225
	140	6	10.3968	1.3988	14	10.1088	1.2867
	200	5	10.1040	1.3366	15	9.8383	1.2339
	260	15	9.6343	1.0895	5	9.7371	1.1600
	320	17	9.3044	1.1364	3	9.7044	1.0601
	380	11	9.6516	1.0581	9	9.7915	1.0829
	440	18	9.4097	1.0166	2	9.7604	1.0846
	500	17	9.4851	0.9591	3	9.8134	1.0724
40 days	40	15	10.6321	1.3790	25	10.4690	1.4227
	80	6	10.7989	1.2297	34	10.3004	1.2402
	140	7	10.5635	1.3279	33	10.1629	1.2684
	200	16	9.9818	1.2065	24	9.8649	1.1950
	260	29	9.6069	1.0077	11	9.7986	1.1200
	320	30	9.4588	1.0641	10	9.7606	1.0492
	380	26	9.6680	1.0102	14	9.8568	1.0709
	440	34	9.5893	1.0062	6	9.8813	1.0632
	500	32	9.5494	0.9037	8	9.8746	1.0616

*Table 7 - Historical CVaR versus Analytical CVaR, mean 20-day CVaR*

Using 20 days as the rebalance frequency, the analytical approach of determining CVaR outperforms the historical approach both in terms of CVaR as well as with regards to standard deviation up until and including 200 days as the time horizon for past data. After this point, the historical CVaR has about the same or slightly lower standard deviation but with a consistently lower CVaR, where 320 days performs the best.

The above also holds true for using 40 days as the rebalancing frequency but with the difference that using 40 days generally returns a higher CVaR for the same time horizon of past returns.

5.3.2 Performance during the economic crisis of 2008
In this section, the two models are simulated historically to gauge their performance with respect to return, CVaR and VaR. The OMXS30 index is also included for the sake of comparison.

The graph below is the performance for the two portfolios with a rebalancing interval of 20 days and a time horizon for past returns of 320 days. The simulation starts 2006-12-19 and ends 2010-04-24.
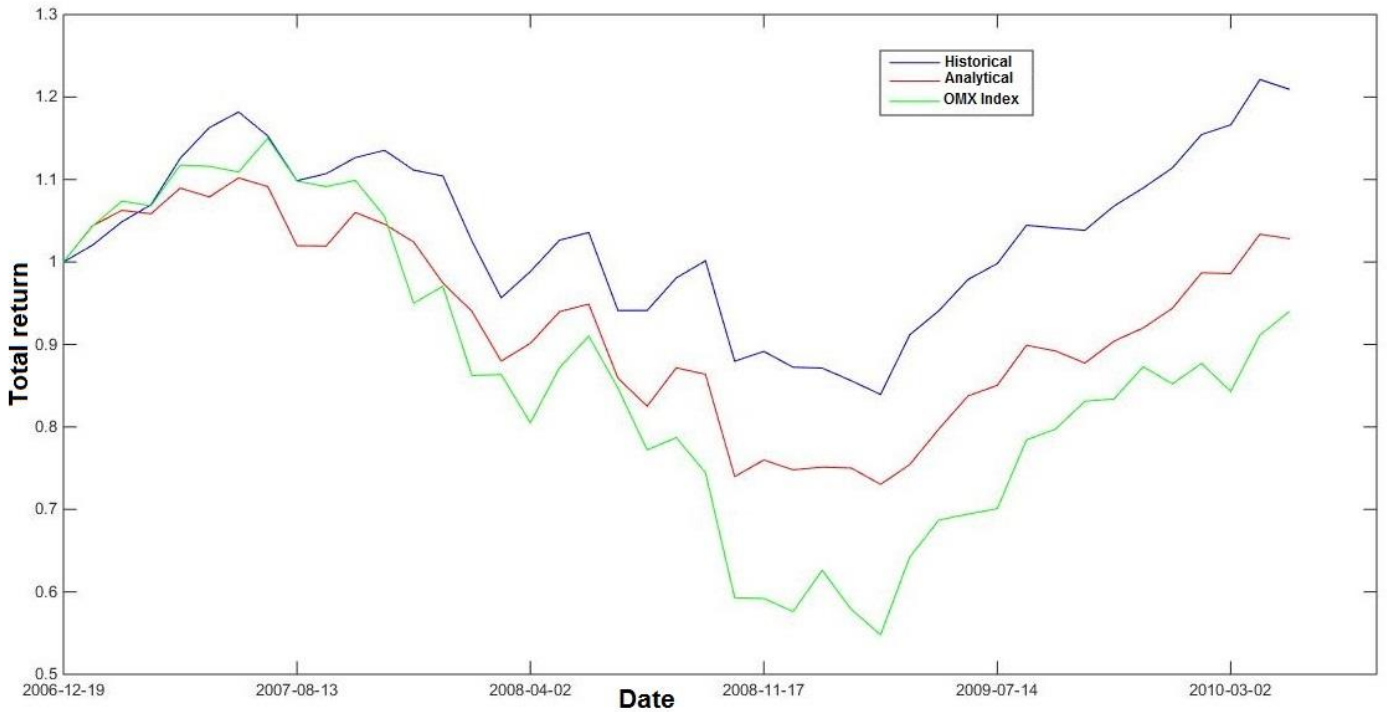
Graph 6 - Historical simulation 2006-12-19 to 2010-04-24

The color blue represents the historical approach of minimizing CVaR while red represents the analytical approach. Green is the performance of the OMX index. The historical approach can be seen to outperform both the analytical approach and the Index. Both the historical and the analytical models manage to not fall as much as the index does during the crisis.

The above described simulation gives the following histogram of returns for the different models and the index which provides more insight regarding the models' performances. The colors below represents the same as above.
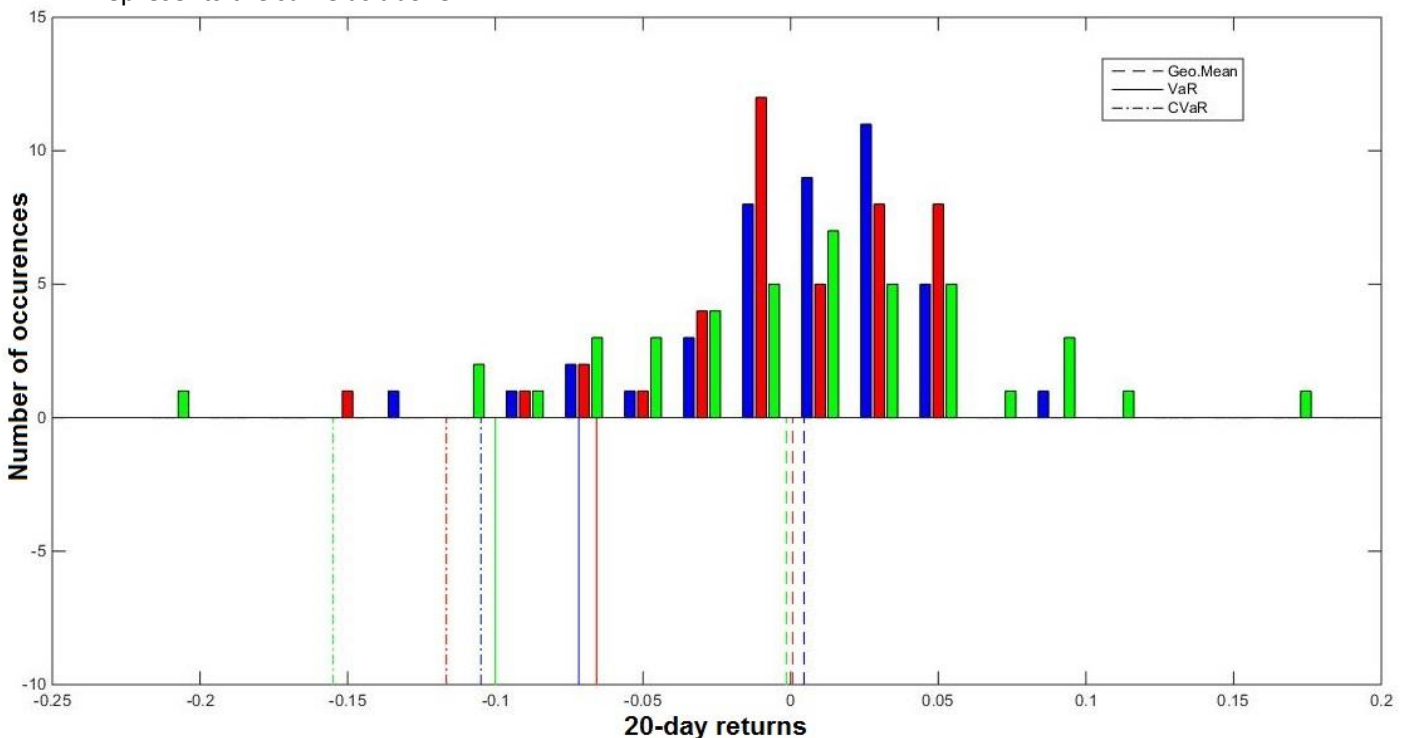
Graph 7 - Histogram of 20 day returns 2006-12-19 to 2010-04-29

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Both the analytical and the historical approach are seen to have their returns more centered together than the index. The index includes both the large losses as well as the large gains whereas the two models for determining CVaR manage to avoid the most extreme loss.
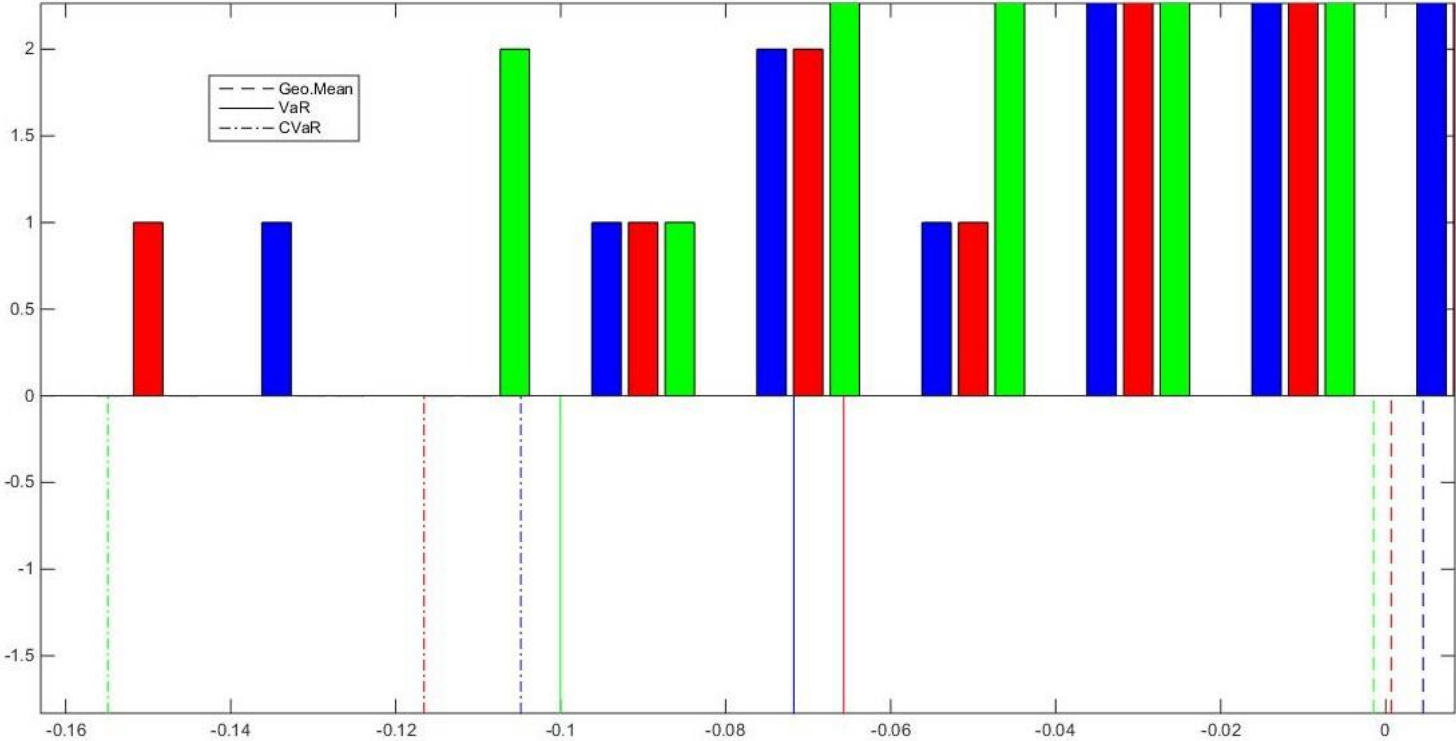
Under the returns in the histogram, the CVaR, VaR and geometrical mean of monthly returns are found for the two models as well as for OMXS30 Index. Here, CVaR is the mean of the lowest 5% of monthly returns which occurred during the simulation and VaR is the largest return of these 5%. This can be seen more clearly when zoomed in, as the next graph shows.

Graph 8 - Historical simulation VaR and CVaR outcome 2006-12-19 to 2010-04-24

OMX Index has the largest VaR as well as the largest CVaR. The historical VaR is larger than the analytical VaR, but the historical CVaR is considerably lower than the analytical.

The returns show that the historical approach has a larger mean return, followed by the analytical and lastly the index.

The above simulation is only one example of what the two models' returns, CVaR and VaR might be. The following two tables are summaries of numerous simulations with two different intervals of rebalancing as well as 9 different time horizons for past data on which the optimizations are based. The first table shows the mean total returns of all 20 or 40 phases over the time period 2006-11-07 to 2010-04-15 where SD is the standard deviation of these returns. For comparison, the return of OMXS30 index during this period was 0.9700, a decline of 3 %.

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Rebalance frequency	Time horizon	Historical CVaR			Analytical CVaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	15	1.2152	0.2304	5	1.0961	0.0598
	80	12	1.0540	0.0632	8	1.0396	0.0403
	140	8	0.9438	0.0679	12	0.9502	0.0143
	200	5	0.9076	0.0636	15	0.9651	0.0286
	260	20	1.1456	0.0552	0	1.0099	0.0365
	320	20	1.2246	0.0606	0	1.0391	0.0398
	380	20	1.2291	0.0411	0	1.0402	0.0476
	440	20	1.1774	0.0555	0	1.0477	0.0413
	500	14	1.0697	0.0345	6	1.0477	0.0363
40 days	40	34	1.1626	0.1884	6	0.9972	0.0706
	80	7	0.9517	0.0663	33	1.0047	0.0473
	140	4	0.8834	0.0594	36	0.9300	0.0304
	200	26	0.9722	0.0843	14	0.9394	0.0290
	260	40	1.1495	0.0824	0	0.9862	0.0258
	320	40	1.2416	0.0679	0	1.0143	0.0349
	380	40	1.2165	0.0744	0	1.0306	0.0449
	440	40	1.1413	0.0779	0	1.0184	0.0373
	500	36	1.0597	0.0481	4	1.0218	0.0370

Table 8 - Historical CVaR versus Analytical CVaR, mean total return

Using 20 days as the rebalance frequency, the historical approach outperforms the analytical for a 40 day time horizon and an 80 day time horizon. The standard deviation is much larger for the historical than the analytical though, when looking at a 40 day time horizon. For 140 and 200 days, the analytical performs slightly better than the historical, both with respect to portfolio returns and their standard deviation. Including and above 260 days, the historical approach has a slightly larger standard deviation of returns but with significantly larger returns.

Using a 40 day rebalancing frequency returns varying results when comparing the analytical method and the historical when looking at time horizons 40, 80 and 140. Above 140 days of past data, the historical method outperforms the analytical and the mean return for using 320 days as the time horizon is the largest and has a small standard deviation.

The next table shows the mean CVaR over the time period 2006-11-07 to 2010-04-15. This table shows the mean, monthly CVaR in percent for all the different phases and SD is the standard deviation of these values. For comparison, the actual outcome for monthly CVaR for the OMXS30 Index was 14.6345 %.

Rebalance frequency	Time horizon	Historical CVaR			Analytical CVaR		
		Wins	Mean	SD	Wins	Mean	SD
20 days	40	13	12.7602	2.5850	7	12.2647	1.9353
	80	12	12.9674	2.3884	8	13.0384	2.0123
	140	6	13.6322	1.9807	14	13.3325	1.7576
	200	0	14.0481	1.8383	20	13.0073	1.6429
	260	9	12.5165	1.8109	11	12.7762	1.5958
	320	18	11.7379	2.0963	2	12.7673	1.5792
	380	19	11.6013	1.8619	1	12.7890	1.5181
	440	20	11.3989	1.8668	0	12.7676	1.5867
	500	18	12.3209	1.4591	2	12.9071	1.5631
40 days	40	22	13.5652	1.6778	18	13.5144	2.1748
	80	13	13.7645	2.1656	27	13.3329	1.8876
	140	18	13.8846	1.8997	22	13.7303	1.7817
	200	10	13.7249	1.7931	30	13.2886	1.6221
	260	27	12.5528	1.7490	13	13.0678	1.5907
	320	36	12.0498	1.8689	4	12.9920	1.5500
	380	40	11.6904	1.7646	0	13.0849	1.5524
	440	39	12.0119	1.8167	1	13.1399	1.5946
	500	29	12.4840	1.5480	11	13.1120	1.5716

Table 9 - Historical CVaR versus Analytical CVaR, mean 20-day CVaR

Using a rebalancing frequency of 20 days, the historical approach generally returns a lower CVaR than the analytical method when using 260 days or more for the time horizon of past data. The standard deviations of these CVaR are lower for the analytical model, but the CVaRs themselves are lower for the historical. The lowest CVaR is received by using 440 days as time horizon which also returns a rather low standard deviation when compared to the other historical simulations.

When using a rebalancing frequency of 40 days, the result is largely as described above with the main difference being that a time horizon of 380 days returns the lowest CVaR as well as a rather low standard deviation.

6. Comparisons and discussion

This section will compare and discuss the results presented in section 5 from different perspectives and regarding different, relevant aspects.

6.1 Analytically versus historically

When looking at Table 6 over portfolio returns in section 5.3.1. the performances of both the analytical and historical approach change when varying the time horizon of the optimization method. This variation is more noticeable when using the historical CVaR-optimization compared with the analytical one. The analytical approach also has a lower variance when looking at the returns within one fixed group.

The conclusion that can be drawn from this is that the analytical optimization method produces more robust results compared to the historical method. This is a quite logical result since the optimal portfolio produced by the analytical method is generated with the help of a normal distribution, and the parameters of the distribution does not change very much if the number of days are increased from 260 to 320. The variance is also generally lower for the different time horizons as the changing of phase from 1, 21, 41 and so on to 2, 22, 42 and so on does not impact the choice of optimal portfolio very much when assuming a normal distribution. For the historical optimization on the other hand the individual daily returns might have a much larger impact on the resulting portfolio as only the lowest 5 percent of the daily returns are relevant and a day resulting in high losses carries more weight.

Since risk often is measured in terms of volatility one might be inclined to draw the conclusion that the analytical optimization method is better since it varies less. However looking at the actual returns of the two optimization methods the historical one almost always beats the analytical one when considering time periods greater than 260 trading days. When the time period is shorter the results differ much between the two approaches and it is difficult to draw any real conclusions. This thesis' hypothesis as to why the historical method outperforms the analytical when considering long time periods is that it accounts for the fat tails in the lower end of the distribution while the analytical method does not. For short time periods the historical method lacks enough data. For a certain number of data points, around 320 days for example, the standard deviation for the historical method is lowered, the returns higher and the CVaR lower. A too large sample size might include outdated data which are not indicative of future returns or stock correlations.

When observing how well the methods actually manage to minimize CVaR the results are the same as they are for the returns, the historical outperforming the analytical. In most of the cases the time

horizons that generated a better return for the historical method also generated a better CVaR. This is an interesting notation that for a certain time horizon one method produces a higher return as well as a lower risk (CVaR). In general the investor has to balance risk against returns when choosing optimization method but if one could have both it is clearly preferable.

The reason for this might depend on two different aspects. The first is that when having a large number of samples for the daily returns the historical method actually manages to describe the "true" return distribution in a superior way compared to the analytical. This includes describing tail-dependence and fat tails in the lower end of the distribution which can be a factor to why the historical method generates a better CVaR. The second aspect is that the analytical method minimizes the variance of the returns. This leads to that it tries to avoid big returns as well as avoid great losses. This holds true not only for the normal distribution but for all symmetrical distributions. The historical method does not penalize large returns at all since as long as the predicted CVaR is low it does not matter how the positive part of the distribution behaves.

The conclusion is thus that the historical method is superior to the analytical method if the time horizon for data collection is large enough, about 260 days or more. The robustness of the analytical method would suggest that it is to prefer when the time horizon of past return data is short, if one were to not have access to more data.

6.2 VaR versus CVaR

One big drawback of the historical VaR-optimization method is that it takes considerably longer to run compared to analytical VaR/CVaR (quadratic programming) or historical CVaR (linear programming). For example, if $a = 0,95$, $N = 320$ and the number of stocks is equal to 33, optimizing historical VaR takes roughly 3300 times longer than optimizing analytical VaR (using Matlab's intlinprog and quadprog functions).

As can be seen in Table 4 the historical CVaR – optimization is superior to the historical VaR-optimization when looking at the generated returns. But when looking at the performed CVaR in Table 5 the results are quite even. This result is rather interesting, the performed risk of the VaR-optimization does not exceed the CVaR-optimization, but the return for the VaR-optimization is significantly lower. However, this thesis lacks the computational power needed to better assess this situation or draw any real conclusions.

6.3 Different underlying assumptions

The length of the rebalancing interval has an effect on the portfolios' performances over time. As can be seen in tables 4 and 6 the returns for both the analytical CVaR-optimization and the historical CVaR-optimization are higher with a rebalancing interval of 20 days instead of 40.

When looking at the performed CVaR of the two methods it is found that CVaR in general is larger when the rebalancing interval is increased. This result is not very surprising since the shorter the rebalancing frequency, the more agile the portfolio allocation will be and the quicker the portfolio can react to new market information. For example the opportunity to capitalize on a running stock.

6.4 Performance during different market conditions

A highly relevant factor for the optimization methods is how well they perform during economic crises. It is during these periods that risk minimization is of the greatest importance, and if a risk optimization method is successful it manages to lessen the impact the crisis has on the portfolio. As can be seen in Graph 6 both the analytical and historical CVaR optimization methods clearly achieve this. However when analyzing Table 8 and Table 9 the historical CVaR-minimization method performs noticeably better compared to the analytical one for time horizons greater than 320 days, both regarding generated return as well as CVaR. This outcome is very similar to the outcome when analyzing the performances during "normal market conditions", that is from 2000 to 2015. The conclusion to be drawn from this is that for a long time horizon for data collection the historical CVaR-optimization is superior to the analytical. Another interesting aspect of Table 9 is that relatively long time horizons (about 400 days) seem to generate the best CVaR both for the analytical and the historical method. This result is a bit surprising because when using a long time horizon there are numerous data points for the days before the decline begins that weighs into the optimization model. One might think that the most relevant data comes from the days after the economic decline begins, and hence the data for the days of normal market conditions contains just unwanted biases when examining the portfolio performances during non-normal market conditions, such as the crisis of 2008. But apparently the stock data for normal market conditions also contain highly relevant information when putting together portfolios during economical declines.

7. CVaR/VaR-optimization and the effect of tail dependence for different types of investors

This section discusses the results from section 5 and 6 with respect to how it affects different types of investors on the market. It serves to answer the question:

- Of what importance is the market's tail dependence for different kinds of investors and how should it be handled? What investors have a use for CVaR and VaR as risk measurements?

7.1 Methodology

As a basis for the regulations which different types of funds are subject to, the U.S. Securities and Exchange Commission's rules and descriptions are used. The results from the previous part of this thesis are then discussed with respect to its relevance for the different types of investors together with theories from courses within corporate finance and economics and literary sources such as (Berk & DeMarzo, 2014) and (Markowitz, 1952).

The discussion regarding banks and the relevance of CVaR and VaR are largely based on the Basel accords Basel I and Basel II as well as sources discussing these such as (Latham & Watkins;, January 2011). The potential resulting complications of risk mismanagement are analyzed with (Berk & DeMarzo, 2014) as the principal source.

An interview with Tomas Sörensson is conducted to receive a broader view of different market actors and perspectives for which CVaR and VaR might hold relevance.

7.2 Theoretical framework

The discussion in this part is based on the classical investment assumption that investors have to balance return against risk. To reach a higher return the investor has to increase the risk of the portfolio. This is one of the key assumptions of the "Modern Portfolio Theory" (Markowitz, 1952). This theoretical statement is also supported by several empirical studies, for example (Berk & DeMarzo, 2014).

Furthermore the investor is assumed to be rational and risk averse. If two portfolios have the same expected return the investor chooses the one with the lowest risk. As a direct result of this the investor will always choose a portfolio along the efficient frontier. Which of these portfolios that is chosen depends on the preferences of the investor, whether a low risk or a high return is desired. The empirical findings presented earlier in this thesis show that the historical CVaR-optimization outperforms the analytical method. This in turn also means that the historical optimization outperforms portfolio optimization according to MPT, variance minimization. As such an investor

may achieve a better portfolio, lower risk and/or higher return, than when using the classical framework and the rational investor should hold a portfolio with minimized CVaR according to the historical model rather than minimized variance.

This entire thesis is built upon the widely established paradigm that stock prices from the past contains valuable information regarding how the stocks will perform in the future. For example how the stocks are correlated with one another and which stocks are in a positive or negative trend.

There are several conclusions that can be drawn from the earlier results of this thesis. Using the classical assumption of normal distribution of returns is shown to underestimate the likelihood of high losses due to fat tails in the actual distribution. Our historical approach better manages to gauge the actual portfolio risk and as such also avoids the larger losses better than the analytical model. A great drawback of assuming a symmetrical return distribution (such as the normal distribution) is that it minimizes large profits as well as large losses. The historical, discrete distribution does not penalize large profits and therefore manages to not only minimize the potential large losses but also maintain some of the larger gains.

Our results indicate that CVaR is superior to VaR from a practical as well as from a theoretical point of view. Optimizing with respect to CVaR takes considerably less computational power compared to VaR regarding the historical optimization. Furthermore VaR as a risk measure does not provide any information about the expected loss the portfolio is subject to if the unlikely scenario of exceeding VaR were to occur.

During normal periods in the economy, when there is no economic crisis, it is shown in section 5.2.1 that the portfolio optimization with respect to both historical as well as analytical CVaR very much outperforms the OMX index. The histogram displays the effects of the optimizations as the resulting CVaR and VaR for both the analytical and the historical approach are well below those for the index. It is also shown that the historical approach is superior if using a larger time sample of past returns.

Looking at section 5.2.2, the simulations during times of economic stress and crisis, there is a much more obvious effect of optimizing with respect to minimizing CVaR. These graphs and tables show how the CVaR methods manage to avoid incurring large portfolio losses as well as lessen the larger losses all together. The methods also create portfolios which largely outperforms the index, as can be seen from the tables in 5.2.2, with the historical approach being superior.

7.3 Discussion from the perspective of different investors

7.3.1 Banks

In 1973 the Basel Committee on Banking Supervision was created. This committee published a set of minimum capital requirements which they recommended banks to follow.

Basel I was incorporated into law in the G-10 countries in 1992. It has since been followed by Basel II and Basel III, both of which includes Value at Risk to assess market risk and stress testing of banks (Latham & Watkins;, January 2011). As previously stated in this thesis, using Value at Risk to assess potential losses might misguide the bank into being riskier than preferred as VaR gives no information regarding potential losses exceeding VaR. Complementing VaR with using CVaR would therefore be advisable as it provides valuable information regarding the actual risk of incurring a substantial loss. Such a loss for a bank, even if temporary, may result in serious problems and even lead to a bank defaulting due to liquidity problems or insolvencies. It may also result in the bank failing to meet the capital requirements which it is subject to and hence risk facing penalty fines (Lindquist, 2004). It is therefore of great importance that the risk measures used manage to determine the risk well and that the risk measure itself is not misguiding or is misunderstood by the user.

Ignoring the effects of the tail dependence on the market is shown to lead to a riskier portfolio and is therefore something that should be accounted for and incorporated into models for risk assessment. This thesis shows that this holds true for a portfolio consisting of stocks and it is reasonable to draw the conclusion that the same can be said for other markets experiencing fat tails in the return distribution.

During such periods as the economic crisis of 2008 it is of grave importance for banks to lessen their potential risk exposure in order to avoid financial complications. Thus it is vital to use the methods for determining risk which best approximates and avoids the large losses. During the crisis of 2008, the difference between the historical approach and the analytical becomes much more apparent than when looking at the larger time period used in 5.2.1. As such, if a bank were to assess their market risk using an inferior method, in the case of a stock portfolio the analytical one, they may be faced with larger losses than estimated.

7.3.2 Investment funds

Different kinds of investment funds are subject to different kinds of requirements, partly due to the fund structure itself but also due to a fund's own rules and regulations. This section serves to discuss the importance of using CVaR and VaR as risk measures when optimizing a portfolio from the perspective of fund managers.

For all kinds of investment funds there is a trade-off between risk and return (Sharpe, 1964). Some funds are by construction very volatile and subject to large losses as well as large profits whereas other funds are more conservative in their investing and seek to minimize their risk exposure to the best of their ability. The subject discussed in this thesis is best applicable to equity funds but might hold some value to other types of funds as well.

As shown in the results, minimizing CVaR using the historical approach (for a time horizon for past data of around 260 days and above) creates a better portfolio than using any other approach, both with respect to returns and CVaR. This method is therefore to prefer at all times over the analytical approach as it also manages to perform much better during periods of economic crisis. The investors using this model would experience fewer large losses, smaller losses overall and also a higher rate of return. Unlike the ordinary private saver the funds should not have any problem of implementing the models presented in this thesis. Neither the data handling nor setting up the optimization models should cause any difficulties.

An equity fund investing on the stock market should always be aware of the phenomenon of tail dependence of market returns. Using a historical approach for minimizing CVaR best avoids the large losses since it accounts more for the fat tails of the return distribution than assuming a normal distribution does. Ignoring the tails of the return distribution when creating a portfolio will lead to a misguided assessment of risk as well as a possible underperformance due to the occurrence of more and larger losses. As seen in Graph 5 and Graph 8 there is a much larger distance between VaR and CVaR for the analytical method than it is for the historical method. Using an analytical VaR would therefore give the investor the idea that the portfolio is less risky. The fact that the two risk measures are closer together as they are for the historical approach points to the conclusion that this approach is better at reducing the effect tail dependence has on the portfolio. To be able to neutralize the effect of tail dependence would be highly desirable for the funds since this minimizes the likelihood of large losses. This in turn might lead to that the persons who are investing in the fund feel safer and therefore are less likely to leave.

If a portfolio uses leverage when investing it has to maintain the ability to pay off its debts as well as the financial costs related to leveraging. Incurring large losses in such a position could result in financial complications and it might therefore be of relevance to track the portfolio CVaR using the appropriate model (in this case the historical) so as to best gauge and avoid unnecessary risk taking.

It is often not very attractive to invest in a fund which has shown to be subject to large losses from time to time even if its overall return is decent. Thus managing to create an almost equal return, or preferably better, whilst avoiding incurring large losses might be a much more preferable option for

funds directed towards private investors who might be rather risk-averse. From a marketing perspective it could be a good idea for funds to use the risk/return tradeoff as a way to differentiate themselves from competitors.

7.3.3 Private investors

A private investor is very rarely subject to the same or similar regulations or constraints as a fund when it comes to investing and the investor also often lack the same options as an investment fund such as leveraging a portfolio by loans or short selling. Lending money as a private investor is often more expensive and therefore the potential downside of leveraging becomes larger. If a private investor were to underestimate or ignore the large tails in the lower end of the return distribution whilst leveraging a portfolio the results might be serious financial complications. The same can be said for when the investor uses personal savings as the purpose of these might be to use during retirement. To fully implement the models presented in this thesis requires some knowledge in mathematics and programming which the ordinary saver not might be very familiar with. But the result in this thesis further supports the importance of diversification (divide the capital amongst multiple stocks to reduce the risk). This is something that the "ordinary investor" should be fully aware of.

A special type of investing is retirement saving. When there is a considerable time period remaining it is often better to invest in riskier portfolios or funds as these generally provide a higher return over time than more conservative ones (Markowitz, 1952) (Berk & DeMarzo, 2014). As the time for retirement comes closer however, this is not always the case. In order to have financial security it is not preferable to hold risky portfolios as this may lead to temporary large losses and a harsh economic situation for the individual. Therefore it is optimal to be able to avoid incurring large losses while still maintaining a decent return on the invested capital for the years to come. Tracking CVaR and using it as a subject for optimization is shown in 5.3.2 to avoid large losses during periods of economic turmoil which would give a retirement investor some form of safety from large economic downturns. A private investor using this method stands to make considerable gains while being somewhat shielded from large market declines such as those during a financial crisis. For the riskier investor, the best quotient between expected return and CVaR may be found along the efficient frontier and should lead to a more volatile portfolio but with higher long-term return.

8. Conclusion

The occurrence of fat tails, due to tail dependence, in the return distribution on the market is something to take into account when optimizing a portfolio. The discrete historical probability distribution based on historical data takes this into account better than the analytical method of approximating returns by a normal distribution for minimizing CVaR does. It also does not penalize large gains as symmetrical distributions do when minimizing CVaR. There are noticeable differences for the two models' performance over a longer time period such as 03-01-2000 to 08-04-2015 as well as during periods of economic crisis. An investor investing in stocks should be well aware of the fat tails of the return distribution, especially during periods of economic crisis, as ignoring them is shown to be unwise.

Banking institutions which by Basel II and Basel III or through internal regulations are subject to use VaR as a risk measure should also incorporate the usage of CVaR as it gives more accurate information of market risk regarding potential losses as well as holds better mathematical properties.

Private investors, given the effect of tail dependence, should be vary of holding soon-to-be-used pension funds in portfolios which are not optimized at least to some degree with respect to CVaR using a well-performing method such as the historical one. Were a crisis to occur while the capital is placed in a portfolio with high CVaR an investor risks losing a substantial amount.

Using a symmetrical distribution such as the normal distribution to determine CVaR and VaR not only minimizes the risk of large losses, but also does the same for the chance of large profits. Therefore, using a historical approach to create a discrete probability distribution not only generates a lower CVaR but also higher returns.

This thesis also concludes that using CVaR minimization based on a discrete probability distribution outperforms the classical portfolio optimization dictated by MPT since the analytical optimization of CVaR and VaR is equal to minimizing the portfolio variance. As such, an investor should use historical, discrete CVaR minimization instead of Markowitz's variance minimization to achieve lower risk and higher return.

9. References

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Appendix

1. List of stocks

ABB	HM B	Securitas B
Autoliv	Holmen B	Handelsbanken A
Alfa Laval	Investor B	Skanska B
Assa Abloy B	Kinnevik B	SKF B
Atlas Copco A	Lundin Petroleum	Stora Enso A
Atlas Copco B	MTG B	Swedbank A
Astra Zeneca	Nordea	Swedish Match
Boliden	Nokia	Tele2 B
Electrolux B	Sandvik	TeliaSonera
Ericsson B	SCA B	Trelleborg B
Getinge B	SEB A	Volvo B

2. Mathematical derivation of analytical CVaR, inspired by (Yamai & Yoshiba, January, 2002).

If a stochastic variable $\in N(0, \sigma^2)$, then

$$E[X \mid X \leq \beta] = \int_{-\infty}^{\beta} x * f_{X|X \leq \beta}(x) \, dx = \int_{-\infty}^{\beta} x \frac{f_X(x)}{P(X \leq \beta)} \, dx = \frac{1}{P(X \leq \beta)\sigma\sqrt{2\pi}} \int_{-\infty}^{\beta} x * e^{-\frac{x^2}{2\sigma^2}} \, dx =$$

$$= -\frac{\sigma}{P(X \leq \beta)\sqrt{2\pi}} * e^{-\frac{\beta^2}{2\sigma^2}}$$

If $\beta = \sigma\Phi^{-1}(1-\alpha)$ then:

$$-\frac{\sigma}{P(X \leq \beta)\sqrt{2\pi}} * e^{-\frac{\beta^2}{2\sigma^2}} = -\frac{\sigma}{(1-\alpha)\sqrt{2\pi}} * e^{-\frac{\left(\sigma\Phi^{-1}(1-\alpha)\right)^2}{2\sigma^2}} = -\frac{\sigma}{(1-\alpha)\sqrt{2\pi}} * e^{-\frac{\left(\Phi^{-1}(1-\alpha)\right)^2}{2}} =$$

$$= -\frac{\sigma}{(1-\alpha)\sqrt{2\pi}} * e^{-\frac{\left(\Phi^{-1}(\alpha)\right)^2}{2}}$$

If we set $s_\alpha = -\frac{1}{(1-\alpha)\sqrt{2\pi}} * e^{-\frac{\left(\Phi^{-1}(\alpha)\right)^2}{2}}$ then:

$$E[X \mid X \leq \sigma\Phi^{-1}(1-\alpha)] = \sigma s_\alpha$$

s_α is consequently a constant depending on the confidence level α. The codomain of s_α is the interval $(-\infty, 0)$.

If $R \in N(\mu, \sigma^2)$ then the analytical $CVaR_\alpha$ is computed as follows:

$$CVaR_\alpha = E[-R \mid R \leq -VaR_\alpha] = E - [R \mid R \leq \mu + \sigma\Phi^{-1}(1-\alpha)] =$$

$$= E[-R \mid R - \mu \leq \sigma\Phi^{-1}(1-\alpha)]$$

We do the substitution $R - \mu = \tilde{R}$ Then $\tilde{R} \in N(0, \sigma^2)$

$$= E[-R \mid R - \mu \leq \sigma \Phi^{-1}(1 - \alpha)] = E[-\tilde{R} - \mu \mid \tilde{R} \leq \sigma \Phi^{-1}(1 - \alpha)] =$$

$$= -E[\tilde{R} \mid \tilde{R} \leq \sigma \Phi^{-1}(1 - \alpha)] + E[-\mu \mid \tilde{R} \leq \sigma \Phi^{-1}(1 - \alpha)] = -E[\tilde{R} \mid \tilde{R} \leq \sigma \Phi^{-1}(1 - \alpha)] + (-\mu) =$$

$$= -\sigma s_\alpha - \mu$$

To conclude:

$$CVaR_\alpha = -\sigma s_\alpha - \mu$$