MINIMIZATION PROBLEMS INVOLVING POLYCONVEX INTEGRANDS

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MINIMIZATION PROBLEMS INVOLVING POLYCONVEX INTEGRANDS

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To

Baï & Yélognissè AWI

and to

Kadidjatou AWI.

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SUMMARY

This thesis is mainly concerned with problems in the areas of the Calculus of Variations and Partial Differential Equations (PDEs). The properties of the functional to minimize play an important role in the existence of minimizers of integral problems. We will introduce the important concepts of quasiconvexity and polyconvexity. Inspired by finite element methods from Numerical Analysis, we introduce a perturbed problem which has some surprising uniqueness properties. This thesis includes only a part of the article [2, Awi-Gangbo].

CHAPTER I

INTRODUCTION

This thesis is concerned with problems in the areas of the Calculus of Variations, Partial Differential Equations (PDEs) and their applications to Geometry, Physics and Material Science. In order to put this work in context and make our contribution more transparent, let us start by recalling the following basic principle of the Calculus Variations. Let X be a topological space and let $I : X \to \mathbb{R} \cup \{\infty\}$ be such that the sublevel sets $\{I \leq c\}$ for $c \in \mathbb{R}$ are precompact. Then $\inf_X I$ admits a minimizer provided that I is lower semicontinous. Therefore the properties of I with respect to the topology play a role in the existence of minimizers. In this thesis, we deal with a functional which does not satisfy properties usually needed to ensure existence of a minimizer. One of the most central notions in the Calculus of Variations is the notion of quasiconvexity. Under appropriate growth conditions on $L : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$; $\xi \mapsto L(\xi)$, the lower semicontinuity of the functional $u \mapsto \int_\Omega L(\nabla u)dx$ is equivalent to quasiconvexity of L. Quasiconvexity is the right notion to hope for existence of solutions in PDEs and the Calculus of Variations. A function $L : \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$; $\xi \mapsto L(\xi)$ is said to be polyconvex if it is a convex function of the minors of ξ, a sufficient condition for quasiconvexity of L. The latter class of functionals is the one encountered the most in elasticity theory. We will focus on nonlinear elasticity problems that involve polyconvex integrands.

1.1 *The general setting.*

In the Calculus of Variations one is often interested on finding solutions of integral problems of the form

$$\inf_{\mathbf{u}\in\mathscr{U}}\left\{\int_\Omega L(x,\mathbf{u}(x),\nabla\mathbf{u}(x))dx\right\},\tag{1}$$

with $L:\Omega\times\mathbb{R}^N\times\mathbb{R}^{N\times n}\to\mathbb{R}\cup\{+\infty\}$ a Carathéodory function (See for instance Definition 2.2.3 and [6, Dacorogna]). One also tries to characterize the minimizers of (1) in terms of the partial differential equations they satisfy. These partial differential equations are the so–called Euler–Lagrange Equations.

In most cases, these Euler–Lagrange Equations are difficult to identify and become a challenge when for instance L or its derivatives fail to satisfy the following growth conditions. Firstly a growth condition on L would be

$$|L(x,\mathbf{u},\xi)|\le\alpha_1(x)+\beta\left(|\mathbf{u}|^p+|\xi|^p\right).$$

Secondly, the function L would be such that for every $i=1,\cdots,N$ and every $\alpha=1,\cdots,n$ one has $L_{\mathbf{u}^i}:=\partial L/\partial\mathbf{u}^i$ and $L_{\xi^i_\alpha}:=\partial L/\partial\xi^i_\alpha$ are Carathéodory functions that satisfy for almost every $x\in\Omega$ and every $(\mathbf{u},\xi)\in\mathbb{R}^N\times\mathbb{R}^{N\times n}$

$$|D_{\mathbf{u}}L(x,\mathbf{u},\xi)|,|D_\xi L(x,\mathbf{u},\xi)|\le\alpha_1(x)+\beta\left(|\mathbf{u}|^{p-1}+|\xi|^{p-1}\right)$$

with $\beta\ge0$ and $\alpha_1\in L^{p/(p-1)}(\Omega)$ (See for instance Theorem 2.2.4). These Euler–Lagrange equations could provide a way to link the minimization problem (1) to the system of PDEs

$$\partial_t\mathbf{u}=\text{div}\,\left(\nabla_\xi L(x,\nabla\mathbf{u})\right)\quad\text{ in }(0,T)\times\Omega;\quad\mathbf{u}(0,\cdot)=\mathbf{u}_0\tag{2}$$

with $\mathbf{u}_0:\Omega\to\mathbb{R}^d$ belonging to $\mathscr{U}\subset L^p(\Omega)$. Assume that the map $\xi\mapsto L(x,\xi)$ is polyconvex. In this case, the existence of large time solutions to (2) remains an outstanding open problem unless $L(x,\cdot)$ is convex. An even more challenging problem is the hyperbolic type system of PDEs

$$\partial_{tt}u=\text{div}\,\left(\nabla_\xi L(x,\nabla\mathbf{u})\right).\tag{3}$$

Progress has been made in the two extreme cases $L = L_0$ and $L = L_1$ where $L_0 \equiv L_0(\xi)$ with L_0 convex and $L_1 \equiv H(\det \xi)$ with H strictly convex. Set $L_\lambda(\xi) = (1-\lambda)L_0(\xi) + \lambda L_1(\xi)$. When $\lambda = 0$, as just mentioned, (2) has global solution in time under mild conditions.

When $\lambda = 1$ the work of Evans-Gangbo-Savin [9] reduced (2) to a Porous Medium Type Equation and they obtained existence of a global solution. This equation when expressed in terms of $\rho = \det \nabla \mathbf{u}^{-1}$ reads off $\partial_t \rho = \operatorname{div} \rho \tilde{H}'(\rho) \nabla \rho$ with $\tilde{H}(t) = tH(t^{-1})$ for $t > 0$.

When L_1 is chosen appropriately, adapting the computation of [9, Evans-Gangbo-Savin], one shows that (3) is nothing but the celebrated Isentropic Euler Equation and so is a system of conservative laws. In this case, if $\rho = \det \nabla \mathbf{u}^{-1}$ represents the density of a fluid then $H(\rho)$ represents the internal energy and the pressure $P(\rho)$ is such that $P'(\rho) = \rho H''(\rho)$ (c.f. e.g. [16, Gangbo-Westdickenberg]). Indeed let $u : [0,1] \times \Omega \to \Lambda$. Recall first that if (X, Σ) and (Y, Σ') are two measurable spaces; μ is a measure on (X, Σ) and $T : X \to Y$ is a measurable map, then the push-forward of the mesure μ by the map T is the measure denoted $T\#\mu$ on (Y, Σ') defined by $T_{\#}\mu(B) = \mu(T^{-1}(B))$ for all $B \in \Sigma'$. If we set

$$\rho(t, u(t,x)) \det \nabla \mathbf{u}(t,x) = 1, \tag{4}$$

then $u(t,\cdot)$ pushes the measure $1_\Omega \mathcal{L}^d$ to the measure $\rho(t,\cdot)1_\Lambda \mathcal{L}^d$ if we assume in addition that u_t is a diffeomorphism. The equation $\partial_{tt} u = \operatorname{div} DL(\nabla \mathbf{u})$ is equivalent to

$$\partial_{tt} u = -H''(\rho(t,u)) \nabla \rho(t,u).$$

Thus

$$\partial_{tt} u(t, u^{-1}(t,y)) = -H''(\rho) \nabla \rho. \tag{5}$$

We have $u^{-1}(0,\cdot) : \Lambda \to \Omega$ and $u(t,\cdot) : \Omega \to \Lambda$. Set

$$y(t,x) = u(t, u^{-1}(0,x)) \text{ for all } (t,x) \in [0,1] \times \Lambda$$

and $y_t = y(t,\cdot) : \Lambda \to \Lambda$. We have by Equation (4) that the map u_0^{-1} pushes $\rho(0,\cdot)\mathcal{L}^d$ forward to $1_\Omega \mathcal{L}^d$ and the map u_t pushes $1_\Omega \mathcal{L}^d$ forward to $\rho(t,\cdot)\mathcal{L}^d$ and so

$$y(t,\cdot)_\# \rho(0,\cdot) = \rho(t,\cdot). \tag{6}$$

We have then

$$\partial_{tt} y_t = \partial_{tt} u_t \circ u_0^{-1} = \partial_{tt} u_t \circ u_t^{-1} \circ u_0^{-1} = \partial_{tt} u_t \circ u_t^{-1} \circ y_t.$$

Hence, by (5)

$$\partial_{tt} y_t = -\nabla(H'(\rho)) \circ y_t. \tag{7}$$

Let v be defined by $v = \partial_t y(t, y^{-1}(t,\cdot))$; in other words $v(t, y(t,\cdot)) = \partial_t y(t,\cdot)$. We have then by differentiation with respect to t, $\partial_t v(t,y) + \nabla v(t,y)\partial_t y = \partial_{tt} y$. This together with (7) yields

$$\partial_t v_t \circ y_t + \nabla(v_t \circ y_t) v_t \circ y_t = -\nabla\left(H'(\rho_t)\right) \circ y_t,$$

and so,

$$\partial_t v_t + \nabla v_t v_t = -\nabla(H'(\rho_t)). \tag{8}$$

By (6), one has

$$\partial_t \rho + \nabla(\rho v) = 0. \tag{9}$$

Combining (8) and (9) we obtain the Isentropic Euler Equation.

The cases of integrands L_λ with $0 < \lambda < 1$, of interest in our work, are the ones defying any standard theory of Partial Differential Equations. The first difficulties arise when we consider the approximation

$$\partial_t \mathbf{u}(kh,\cdot) \sim (\mathbf{u}_{k+1} - \mathbf{u}_k)/h.$$

The implicit Euler scheme of (2) is

$$\frac{\mathbf{u}_{k+1} - \mathbf{u}_k}{h} = \text{div}\left(\nabla_\xi L(x, \nabla \mathbf{u}_{k+1})\right). \tag{10}$$

If the functional

$$\mathcal{U} \ni \mathbf{u} \mapsto \frac{\|\mathbf{u}-\mathbf{u}_k\|^2_{L^2(\Omega)}}{2h} + \int_\Omega L(x, \nabla \mathbf{u}) dx \tag{11}$$

admits a minimizer $\mathbf{u}_{k+1}$; formally at least, the Euler-Lagrange equation of the minimization problem is (10). Typically, one requires the growth and coercivity conditions:

$$c_0(|\xi|^p - c_1) \leq L(x, \xi) \leq c_2(|\xi|^p + 1). \tag{12}$$

However those conditions cannot be satisfied by materials with stored energy satisfying:

$$\lim_{\det \xi \to 0} L(x, \xi) = \infty. \tag{13}$$

Because Inequality (12) fails, the current theory of the Calculus of Variations cannot be used to establish any connections between the equation (10) and minimizers of (11). In addition, no theory gives us any clues about the uniqueness of the minimizer and the Euler-Lagrange equations associated to (11).

1.2 Main results.

We focus in this study on a minimization problem involving Ogden functionals (see for instance [18] for a description of Ogden materials) of the form

$$L(x, \mathbf{u}, \xi) := f(\xi) + H(\det \xi)$$

that satisfy Equation (13). More precisely, we are interested in

$$\inf_{\mathbf{u} \in \mathcal{U}} \left\{ I(\mathbf{u}) := \int_\Omega \left(f(\nabla \mathbf{u}) + H(\det \nabla \mathbf{u}) + \frac{||\mathbf{u}_k - \mathbf{u}||^2}{2h} \right) dx \right\}. \tag{14}$$

One can convince oneself that from the technical point of view the level of difficulties in studying (14) is the same as studying

$$\inf_{\mathbf{u} \in \mathcal{U}} \left\{ I(\mathbf{u}) := \int_\Omega \left(f(\nabla \mathbf{u}) + H(\det \nabla \mathbf{u}) - F \cdot \mathbf{u} \right) dx \right\}. \tag{15}$$

When formulating a problem dual to (14), we were forced to introduce some generalization of the classical Legendre Transform called c-convex transform, while (15) involves only the classical Legendre Transform. For the sake of simplicity we have opted to keep our focus on (15).

Let us start with some heuristic discussions. Let

$$E[u,\beta] = \int_\Omega f(\nabla \mathbf{u}) + H(\beta) - F \cdot u$$

with "$\beta = \det \nabla \mathbf{u}$". When $u : \Omega \to \Lambda$ is smooth and invertible, $\beta = \det \nabla \mathbf{u}$ is equivalent to

$$\int_\Omega l(u(x))\beta(x)dx = \int_\Lambda l(y)dy; \tag{16}$$

for all $l \geq 0$ measurable. When u ceases to be smooth or one-to-one, Equation (16) may continue to have a meaning for $u \in W^{1,p}(\Omega, \mathbb{R}^d)$. In fact, we showed in [2, Awi-Gangbo] that $\beta := \frac{\det \nabla \mathbf{u}}{N_u(u)}$ satisfies (16), where $N_u(y)$ is the cardinality of $u^{-1}(\{y\})$. In general, the set of β satisfying (16) is a convex set which may be of cardinality bigger than 1. We denote it by $\det^* \nabla \mathbf{u}$. The elements of $\det^* \nabla \mathbf{u}$ of interest are of course the ones minimizing $\beta \mapsto \int_\Omega H(\beta)dx$ over $\det^* \nabla \mathbf{u}$. When H is strictly convex and the map $\det^* \nabla \mathbf{u} \ni \beta \mapsto \int_\Omega H(\beta)dx$ is not identically equal to ∞, its minimizer is unique and we denote it $\det^H \nabla \mathbf{u}$. Therefore the variational problem, formally at least is equivalent to

$$\inf_{\mathbf{u}} \left\{ I(\mathbf{u}) := \int_\Omega \big(f(\nabla \mathbf{u}) + H(\det^H \nabla \mathbf{u}) - \mathbf{F} \cdot \mathbf{u} \big) \, ; \mathbf{u} \in W^{1,p}(\Omega, \Lambda) \right\}. \tag{17}$$

Making all the above arguments rigorous is one of our tasks which requires a good amount of effort.

1.2.1 Main assumptions.

- Ω, Λ are bounded convex open sets of $\mathbb{R}^d$. Ω represents a reference configuration and Λ represents the region occupied by an elastic body at time $t > 0$.

- $p, q \in (1, \infty)$ and $p^{-1} + q^{-1} = 1$.

- The function $f \in C^1(\mathbb{R}^{d\times d})$ is strictly convex and such that for some $c > 0$ one has for all $\xi \in \mathbb{R}^{d\times d}$:

$$c^{-1}(|\xi|^p - 1) \leq f(\xi) \leq c(|\xi|^p + 1),$$

$$|\nabla f(\xi)| \leq c|\xi|^{p-1},$$

$$|\nabla f^*(\xi)| \leq c|\xi|^{q-1}.$$

- The function $H \in C^2(0,\infty)$ is strictly convex and satisfies

$$\lim_{t\to 0^+} H(t) = \lim_{t\to\infty} \frac{H(t)}{t} = \infty.$$

  We extend H by setting $H(t) = \infty$ for $t \leq 0$.

- The set $\mathcal{U}$ is defined to be the set $\{\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^{d\times d}) : \mathbf{u}(\Omega) = \Lambda\}$. This represents the set of admissible deformations of Ω into Λ. Remark that the map $\mathbf{u} : \Omega \to \Lambda$ represents the deformation field of an elastic body in the reference state Ω and I is the total elastic energy of the body under deformations.

- The map $F : \mathbb{R}^d \to \mathbb{R}^d$ is such that $F \in L^1(\Omega, \mathbb{R}^d)$. One can interpret F as a body force or a displacement.

1.2.2 Main results.

Motivated by finite element methods in Numerical Analysis and in order to contribute to the understanding of (15), we study the perturbed problems

$$\inf_{\mathbf{u}\in\mathcal{U}_0} \left\{ I_{S_\tau}(\mathbf{u}) := \int_\Omega \big(f(\nabla_{S_\tau}\mathbf{u}) + H(\det{}^H \nabla\mathbf{u}) - \mathbf{F}\cdot\mathbf{u} \big) \right\}. \tag{18}$$

Here we have made the following notations.

1. $\tau \in (0,1)$.

2. The set S_τ is a finite dimensional subspace of piecewise affine functions in $W_0^{1,q}(\Omega, \mathbb{R}^{d\times d})$.

3. The set $\mathcal{U}_0$ is defined to be the set of all $\mathbf{u} : \Omega \to \Lambda$ that are Borel functions.

4. For $\mathbf{u} \in \mathcal{U}_0$, define the set

$$\det{}^*\nabla\mathbf{u} = \left\{\beta \in L^1(\Omega) : \int_\Omega l(u(x))\beta(x)dx = \int_\Lambda l(y)dy, \ \forall l \in C_b(\mathbb{R}^d)\right\}.$$

If there exists $\beta_0 \in \det^* \nabla\mathbf{u}$ satisfying $\int_\Omega H(\beta_0(x))dx < \infty$, then $\det{}^H\nabla\mathbf{u}$ stands for the unique minimizer of

$$\inf_{\beta\in\det{}^*\nabla\mathbf{u}} \int_\Omega H(\beta)dx.$$

Otherwise we set $\int_\Omega H(\det{}^H\nabla\mathbf{u}) = \infty$.

5. For $\mathbf{u} \in \mathcal{U}_0$, $\nabla_{S_\tau}\mathbf{u}$ stands for the unique minimizer of

$$\inf \int_\Omega f(G)dx$$

over the set of all $G \in L^q(\Omega, \mathbb{R}^{d\times d})$ satisfying

$$\int_\Omega \langle \mathbf{u}, \operatorname{div}\psi\rangle = -\int_\Omega \langle G, \psi\rangle \ \forall\psi \in S_\tau.$$

Observe that (18) is not a finite dimensional approximation problem. Indeed, even though $\nabla_{\mathcal{S}_\tau}$ is a finite dimensional operator, $u \mapsto \det\nabla\mathbf{u}$ will remain an infinite dimensional operator. Despite the lack of compactness and convexity of the functionals to minimize in (18), we have proven the existence of a unique minimizer in (18) via the following sharp characterization:

Theorem 1.2.1 *Suppose F is non degenerate (i.e. if $N \subset \mathbb{R}^d$ has Lebesgue measure 0 then $F^{-1}(N)$ has Lebesgue measure 0). Problem (18) admits a unique minimizer $\mathbf{u}_0$ characterized by $\mathbf{u}_0 = \nabla k_0(F + \operatorname{div}\psi_0)$ where (k_0, l_0, ψ_0) is a maximizer of the dual problem*

$$\sup_{(k,l,\psi)\in\mathscr{A}_\tau} \left\{-J(k,l,\psi) := -\int_{\overline{\Omega}} f^*(\psi)dx - \int_{\overline{\Omega}} k(F + \operatorname{div}\psi) - \int_\Lambda l dx\right\}.$$

Here $\mathscr{A}_\tau$ stands for the set of all (k, l, ψ) satisfying $\psi \in S_\tau$; $k : \mathbb{R}^d \to \bar{\mathbb{R}}$ is Borel measurable; $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ is Borel measurable; $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(\mathbf{v}) + tl(\mathbf{u}) + H(t) \geq \mathbf{u} \cdot \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d.$$

Moreover, let S_0 be the set of all $\psi : \Omega \to \mathbb{R}^{d\times d}$ that are in $(L^q(\Omega))^{d\times d}$ and such that $\operatorname{div} \psi_{\mathbb{R}^d}$, the distributional divergence of the extension of ψ that takes the value 0 outside Ω, is a bounded Borel measure on $\bar{\Omega}$. Let $\mathcal{A}$ to be the set defined by replacing in the definition of $\mathcal{A}_\tau$ the set S_τ by the set S_0. Then the problem

$$\sup_{(k,l,\psi)\in\mathscr{A}} \left\{ -J(k, l, \psi) := -\int_{\overline{\Omega}} f^*(\psi)dx - \int_{\overline{\Omega}} k(F + \operatorname{div} \psi_{\mathbb{R}^d}) - \int_{\Lambda} l dx \right\} \tag{19}$$

admits a maximizer and if for all (k, l, ψ) maximizing $-J$ one has k differentiable at $F(x) + \operatorname{div} \psi(x)$ for almost every x in Ω, then the minimizer of Problem (15) is unique.

1.3 Plan of the thesis

This thesis is subdivided in 5 chapters followed by one appendix where we have collected definitions and tools that are useful. The content of the next chapters is as follows.

Chapter 2. This chapter contains the preliminaries. It recalls the essence of the direct methods in the Calculus of Variations, discusses some Numerical Analysis tools; the existence of homeomorphism between two convex bounded open sets and finally convex functions of measures.

Chapter 3. In chapter 3, we will discuss some variational problems involving polyconvex integrand.We present mostly existence results. We start by listing the main assumptions. We will introduce a notion of weak determinant and a notion of pseudo-projected gradient.We study the variational problem

$$\inf_{\mathbf{u}}\Big\{ I_*(\mathbf{u}) := \int_\Omega \big(f(\nabla \mathbf{u}) + H\big(\det \nabla \mathbf{u}\big) - \mathbf{F}\cdot \mathbf{u}\big)dx \;|\mathbf{u} : \bar{\Omega} \to \mathbb{R}^d;$$

$$\mathbf{u} \in W^{1,p}(\Omega, \Lambda);\;\; \mathbf{u}(\bar{\Omega}) = \bar{\Lambda};\; \det \nabla \mathbf{u} > 0\Big\}$$

in the case $p > d$. We next present existence result for the problem

$$\inf_{(\beta,\mathbf{u})} \left\{ I(\mathbf{u}) := \int_\Omega (f(\nabla \mathbf{u}) + H(\beta) - \mathbf{F}\cdot \mathbf{u})\, dx;\; \mathbf{u} \in W^{1,p}(\Omega, \Lambda);\; \beta \in \det{}^* \nabla \mathbf{u} \right\}.$$

Further, we discuss why a direct proof of existence is out of reach with the direct method of the calculus of variation in minimizing the functional

$$I_{\mathcal{S}}(\mathbf{u}) = \int_\Omega \big(f(\nabla_{\mathcal{S}} \mathbf{u}) + H(\det{}^H \nabla \mathbf{u}) - \mathbf{F}\cdot \mathbf{u}\big)\, dx.$$

To finish the chapter, we present a relaxed problem which is the minimization of the functional

$$\bar{I}(\gamma) = \int_C \big(f(\xi) + H(t) - \mathbf{F}(x)\cdot u\big)\gamma(dx, dt, du, d\xi).$$

over a set of measure Γ which is inspired by the Young Measures.

Chapter 4. This chapter discusses duality results, uniqueness and Euler–Lagrange equations of some of the problems introduced in Chapter 3. Mainly, we consider the following problem. Let S_0 be the set of all $\psi : \Omega \to \mathbb{R}^{d\times d}$ that are in $\left(L^q(\Omega)\right)^{d\times d}$ and such that $\operatorname{div} \psi_{\mathbb{R}^d}$, the distributional divergence of the extension of ψ that takes the value 0 outside Ω, is a bounded Borel measure on $\bar{\Omega}$. Let $\mathscr{A}_0$ stand for the set of all (k, l, ψ) satisfying $\psi \in S_0$; $k : \mathbb{R}^d \to \bar{\mathbb{R}}$ is Borel measurable; $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ is Borel measurable; $l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(\mathbf{v}) + tl(\mathbf{u}) + H(t) \geq \mathbf{u}\cdot \mathbf{v} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d.$$

We consider the problem

$$\sup_{(k,l,\psi)\in\mathscr{A}_0} \left\{ -J(k, l, \psi) := -\int_{\bar{\Omega}} f^*(\psi)dx - \int_{\bar{\Omega}} k(F + \operatorname{div} \psi_{\mathbb{R}^d}) - \int_\Lambda l dx \right\}.$$

We show that it admits a maximizer and discuss how it is related to Problem (15).

Chapter 5. In chapter five we discuss the limit case $H \equiv \chi_{\{1\}}$ which corresponds to $\beta = \det{}^H \nabla \mathbf{u} = 1$. We check that the main result of Chapter 4 is still true in this extreme case. We present duality, existence and uniqueness results.

1.4 Key Words

They are four key words of importance in this study:

1. Relaxation

2. Duality

3. Euler-Lagrange Equation and Polar Factorization

4. Drastic lack of compactness.

Relaxation. To achieve our goals, we relax the problem (15) to $\inf_{\gamma\in\Gamma} \bar{I}(\gamma)$ where

$$\bar{I}(\gamma) = \int_C \big(f(\xi) + H(t) - \mathbf{F}(x)\cdot u\big)\gamma(dx, dt, du, d\xi)$$

and the set C is defined by $C = \bar{\Omega} \times [0,\infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$.

The set Γ is the set of Radon measures on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ supported by C and satisfying for all $b \in C_b(\mathbb{R}^d)$; $l \in C_b(\mathbb{R}^d)$; and $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$ the conditions :

$$\begin{aligned}
\int_C b(x)\gamma(dx, dt, du, d\xi) &= \int_\Omega b\, dx;\\
\int_C tl(u)\gamma(dx, dt, du, d\xi) &= \int_\Lambda l dy;\\
\int_C \langle \xi, \psi(x)\rangle\gamma(dx, dt, du, d\xi) &= -\int_C \langle u, \operatorname{div}\psi(x)\rangle\,\gamma(dx, dt, du, d\xi);\\
\int_C f(\xi)\gamma(dx, dt, du, d\xi) &< \infty.
\end{aligned}$$

Let

$$\mathcal{U}_b = \{(\beta, \mathbf{u}) \mid \mathbf{u} \in W^{1,p}(\Omega, \Lambda), \mathbf{u}_\#\beta = 1_\Lambda \mathcal{L}^d\}$$

and suppose $(\mathbf{u}, \beta) \in \mathcal{U}_b$. Define the measure $\gamma^{(\mathbf{u},\beta)} = (id, \mathbf{u}, \beta, \nabla\mathbf{u})_\#(1_\Omega \mathcal{L}^d)$ on C. We have the embedding $\mathcal{U}_b \subset \Gamma$ which to $(\beta, \mathbf{u})$ associates $\gamma \equiv \gamma^{(\beta,\mathbf{u})}$.

Let $\gamma \in \Gamma$. One sees that if one defines D by $D = [0,\infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$. and $\Pi^1 : \Omega \times D; (x,u,t,\xi) \mapsto x$, then $\Pi^1 \# \gamma = 1_\Omega \mathcal{L}^d$. By the disintegration theorem (cf. Theorem A.3.16), there exists a family of probability measure $\{\gamma^x\}_{x\in\Omega}$ such that for all $L : C \to [0,\infty]$ measurable, one has

$$\int_C L(x,u,t,\xi)\gamma(dx,du,dt,d\xi) = \int_\Omega \left(\int_D L(x,u,t,\xi)\gamma^x(du,dt,d\xi)\right) dx.$$

For $x \in \Omega$, set $U_\gamma(x) = \int_D \xi\gamma^x(dt,du,d\xi)$ and set $\mathbf{u}_\gamma(x) = \int_D u\gamma^x(dt,du,d\xi)$. One shows that $U_\gamma \in L^p(\Omega,\mathbb{R}^{d\times d})$; $u_\gamma(x) \in \bar{\Lambda}$; $\nabla \mathbf{u}_\gamma = U_\gamma$ and $\mathbf{u}_\gamma \in W^{1,p}(\Omega,\mathbb{R}^d)$. Remark that for all $(\mathbf{u},\beta) \in \mathcal{U}_b$, one has $\bar{I}(\gamma^{(\mathbf{u},\beta)}) = I(\mathbf{u},\beta)$.

Duality. This is one of the tasks we successfully completed and will later better elaborate on.

Polar factorization. By Theorem 4.4.1, if $\mathbf{u}_1 \in W^{1,p}(\Omega,\Lambda)$ satisfies

$$\mathbf{F} + \operatorname{div}^a\big(Df(\nabla \mathbf{u}_1)^d_{\mathbb{R}}\big) \in \partial k^*(\mathbf{u}_1), \quad h'(\beta_1) + l(\mathbf{u}_1) = 0 \quad \mathcal{L}^d - \text{a.e.} \tag{20}$$

and

$$\mathbf{u}_1 \in \partial k^\infty\big(\operatorname{div}^s Df(\nabla \mathbf{u}_1)^d_{\mathbb{R}}\big) \quad g^s - \text{a.e.} \tag{21}$$

then $\mathbf{u}_1$ is the unique minimizer of I over $W^{1,p}(\Omega,\Lambda)$.

Consider the case $H \equiv \chi_{\{1\}}$. That is $H(t)$ equals ∞ everywhere except at $t = 1$ and $H(1) = 0$. Formally, $\mathbf{u}_1$ preserves Lebesgue measure and (20) can be interpreted as

$$\mathbf{F} = -\epsilon\triangle \mathbf{u}_1 + \nabla k^*(\mathbf{u}_1), \quad \mathcal{L}^d - \text{a.e.}.$$

When $\epsilon = 0$ we obtain the polar decomposition of $\mathbf{F}$ (cf. [3, Brenier] and [14, Gangbo]) and for $\epsilon > 0$ we obtain a variant of the polar decomposition where $\mathbf{u}_1$ is differentiable.

Lack of compactness. When investigating existence of minimizer in (18), the direct method fails since the set $\{\mathbf{u} \in \mathscr{U}_\mathcal{S} : \|\nabla_\mathcal{S}\mathbf{u}\|_{L^p(\Omega,\mathbb{R}^{d\times d})} \le c\}$ is not compact for any topology useful for the variational point of view. In fact the operator $\mathbf{u} \mapsto \nabla_\mathcal{S}\mathbf{u}$ behaves like a projection operator (See for instance Theorem 3.6.1).

CHAPTER II

PRELIMINARIES

In this chapter, we recall the essence of the direct methods in the Calculus of Variations, discuss some Numerical Analysis tools and the existence of homeomorphism between convex bounded open sets. This chapter finishes by discussing convex functions of measures.

2.1 An orientation preserving map.

Let Ω and Λ be two open bounded convex sets of $\mathbb{R}^d$. we present here a result that gives an homeomorphism $F : \Omega \to \Lambda$ such that there exists a positif real number α satisfying for a.e. $x \in \Omega$, $\det \nabla F(x) \in [\alpha^{-1}, \alpha]$.

First we consider the particular case $\Omega = B(0,1)$.

Lemma 2.1.1 *Consider a bounded convex open set $\Lambda \subset \mathbb{R}^d$. Let $r_\Lambda > 0$ such that $B(0, r_\Lambda) \subset \Lambda \subset B(0, r_\Lambda^{-1})$. Let ρ be the Minkowsky functional of Λ as defined in Definition A.1.7. Define $F : B(0,1) \subset \mathbb{R}^d \to \mathbb{R}^d$ and $G : \Lambda \to \mathbb{R}^d$ by*

$$F(x) = \begin{cases} 0, & \textit{if } x = 0 \\ \frac{x|x|}{\rho(x)}, & \textit{if } x \neq 0 \end{cases}, \quad G(y) = \begin{cases} 0, & \textit{if } y = 0 \\ \frac{y\rho(y)}{|y|}, & \textit{if } y \neq 0 \end{cases}.$$

Then F is an homeomorphism from $B(0,1)$ to Λ; G is an homeomorphism from Λ to $B(0,1)$; $F^{-1} = G$; F and G are differentiable a.e. and

$$r_\Lambda^d \leq \det \nabla F, \det \nabla G \leq r_\Lambda^{-d} \qquad a.e. \tag{22}$$

Next we present the general case which is obtained from Lemma 2.1.1 by change of variables.

Lemma 2.1.2 *Consider two bounded convex open sets* Ω *and* Λ *of* $\mathbb{R}^d$*. Then there exists an homeomorphism* F *from* Ω *to* Λ *that is differentiable a.e. and such that there exists a strictly positive real number* α *satisfying*

$$\alpha \leq \det \nabla F \leq \alpha^{-1}. \tag{23}$$

2.2 The direct methods in the Calculus of Variations

2.2.1 General settings

The essence of the direct methods follows from the following lemma.

Lemma 2.2.1 *Let* X *be a topological space. Consider a function* $f : X \to \bar{R}$ *such that* f *is lower semicontinous and there exists a nonempty level set* $\{x \in X : f(x) \leq c\}$ *that is sequentially relatively compact. Then there exists* $\bar{x} \in X$ *such that* $f(\bar{x}) = \inf_{x\in X} f(x)$.

Remark 2.2.2 *Let* $\mathbf{X}$ *be a reflexive normed vector space. Let a function* $f : \mathbf{X} \to \bar{R}$ *be weakly lower semicontinuous and such that* $\lim_{\|x\|\to\infty} |f(x)| = \infty$*. Then* f *admits a minimizer.*

2.2.2 Integral Problems

In Calculus of Variations one is often interested on finding solutions of integral problems of the form

$$\inf_{u\in\mathscr{U}} \mathrm{F}(u), \tag{24}$$

where :

$$\mathrm{F}(u) := \int_\Omega L(x, u(x), \nabla \mathbf{u}(x))dx$$

with $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ a Carathéodory function. We recall the definition of Carathéodory functions ([6, Dacorogna])

Definition 2.2.3 *The function* $L : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be a Carathéodory function if*

1. *For a.e. $x \in \Omega$, the map $\mathbb{R}^N \times \mathbb{R}^{N\times n} \ni (u,\xi) \mapsto L(x,\xi,\xi)$ is continuous.*

2. *For every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{N\times n}$, the map $\Omega \ni x \mapsto L(x,\xi,\xi)$ is measurable.*

One also tries to find a Partial Differential Equation called Euler-Lagrange Equation that a solution of (24) would satisfy. The following Theorem has many limitations as it can not be applied to an important class of problems appearing in Elasticity Theory. We state it just to indicate the state of the art in the Calculus of Variations. See for instance [6, Dacorogna] for more details.

Theorem 2.2.4 *Let $g : \mathbb{R}^{R\times N} \times \mathbb{R}^N \times \Omega$ be a Carathéodory function. Assume $g_{u^i} := \partial g/\partial u^i$ and $g_{\xi^i_\alpha} := \partial g/\partial \xi^i_\alpha$ are Carathéodory functions for every $i = 1,\cdots,N$, $\alpha = 1,\cdots,n$ and for almost every $x \in \Omega$, for every $(u,\xi) \in \mathbb{R}^{R\times N} \times \mathbb{R}^N$, one has*

$$|g(\xi,u,x)| \leq \alpha(x) + \beta\left(|u|^p + |\xi|^p\right) \tag{25}$$

and

$$|D_u g(x,u,\xi)|, |D_\xi g(x,u,\xi)| \leq \alpha_1(x) + \beta\left(|u|^{p-1} + |\xi|^{p-1}\right) \tag{26}$$

with $\beta \geq 0$ and $\alpha_1 \in L^{p/(p-1)}(\Omega)$.

Let $\bar{u}$ be a minimizer of

$$\inf\left\{\int_\Omega g(x,u,\nabla \mathbf{u})dx; u = u_0 + W_0^{1,p}(\Omega,\mathbb{R}^N)\right\} \tag{27}$$

Then

$$\int_\Omega \left(\langle D_\xi g(x,\bar{u},\nabla\bar{u}), \nabla\varphi\rangle + \langle D_u g(x,\bar{u},\nabla\bar{u}), \varphi\rangle\right) dx = 0 \quad \forall \varphi \in W_0^{1,p}(\Omega,\mathbb{R}^N). \tag{28}$$

Moreover, if $\bar{u}$ satisfies Equation (28) and the function $(u,\xi) \mapsto g(x,u,\xi)$ is convex for almost every $x \in \Omega$, then $\bar{u}$ is a solution of Problem (27).

Remark 2.2.5 *For $n = 1$ or $N = 1$, the condition "$\xi \mapsto f(x,u,\xi)$ is convex" is necessary to ensure lower semicontinuity of the map $W^{1,p}(\Omega,\mathbb{R}^N) \ni u \mapsto \int_\Omega f(x,u,\nabla\mathbf{u})$. But if $n, N > 1$, it is far from being necessary.*

We next turn our attention to vectorial problems (i.e. $n, N > 1$).

2.2.3 Integral Problems in the vectorial case

We first define the notions of Polyconvexity, Quasiconvexity and Rank one convexity of functions.

Definition 2.2.6 (Rank one convexity) *$f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is rank one convex if $f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda)f(\eta)$ whenever $\lambda \in [0,1]$, $\xi, \eta \in \mathbb{R}^{N\times n}$ with* $\mathrm{rank}\{\xi - \eta\} \leq 1$.

Definition 2.2.7 (Quasiconvexity) *A Borel measurable and locally bounded function $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ is said to be quasiconvex if $f(\xi) \leq \frac{1}{\mathrm{meas\ D}} \int_D f(\xi + \nabla\varphi(x))$ for every bounded open set $D \subset \mathbb{R}^n$, for every $\xi \in \mathbb{R}^{N\times n}$ and for every $\varphi \in W_0^{1,\infty}(D; \mathbb{R}^N)$.*

To define Polyconvexity, we need first to introduce some notations. For $n, N \in \mathbb{N}$, define $n \wedge N := \min\{n, N\}$;

$$\sigma(s) := \left(\begin{array}{c} N \\ s \end{array}\right)\left(\begin{array}{c} n \\ s \end{array}\right) = \frac{N!n!}{(s!)^2(N-s)!(n-s)!} \text{ and } \tau(n,N) := \sum_{s=1}^{n\wedge N} \sigma(s).$$

For a matrix $\xi \in \mathbb{R}^{N\times n}$, for $2 \leq s \leq n \wedge N$ define $\mathrm{adj_s}\ \xi$ to be the matrix of all $s \times s$ minors of ξ. Let $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ be defined by $T(\xi) := (\xi, \mathrm{adj_2}\ \xi, \cdots, \mathrm{adj_{n\wedge N}}\ \xi)$.

Examples. For $n = N = 2$, one has $\tau(2,2) = 5$, $T(\xi) = (\xi, \det \xi)$.
For $n = N = 3$, one has $\tau(2,2) = 19$, $T(\xi) = (\xi, \mathrm{adj}\ \xi, \det \xi)$.

We are now ready to define Polyconvexity.

Definition 2.2.8 (Polyconvexity) *A function $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ is said to be polyconvex if there exists $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ convex, such that $f(\xi) = F(T(\xi))$.*

Under the growth and coercivity conditions:

$$\begin{aligned} \alpha_1\|\xi\|^p + \beta_2\|u\|^q + \gamma_1(x) &\leq g(x,u,\xi) \\ g(x,u,\xi) &\leq \alpha_2\|\xi\|^p + \beta_2\|u\|^r + \gamma_2(x), \end{aligned}$$

if $\xi \mapsto g(x,u,\xi)$ is quasiconvex then Problem (27) has a solution.

2.3 Approximation of $W^{1,p}(\Omega)$ *functions*

Throughout this section, $\mathbb{R}^n$ for $n \in \mathbb{N}^*$ is endowed with the euclidean norm. Ω is a subset of $\mathbb{R}^n$ and its boundary is denoted $\partial\Omega$. The topological dual of a topological space E is denoted E'. We mean by domain a Lebesgue-measurable subset of $\mathbb{R}^n$ with nonempty interior.

2.3.1 Definitions

We recall the following definitions:

Definition 2.3.1 (Polyhedral set) *A set $\Omega \subset \mathbb{R}^n$ is said to be a polyhedral set if it is can be expressed as the intersection of a finite family of closed half-spaces or hyperplanes.*

We remind that a closed half-space is a set of the form $\{x \in \mathbb{R}^n, a \cdot (x - x_0) \leq 0\}$ where $a, x_0 \in \mathbb{R}^n$.

Definition 2.3.2 (Star-shaped domain) *A domain $\Omega \subset \mathbb{R}^n$ is said to be star-shaped with respect to a set B if for all $x \in \Omega$ the closed convex hull of $B \cup \{x\}$ is a subset of Ω.*

Definition 2.3.3 (Finite elements) *Let:*

1. *$K \subset \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary,*

2. *P be a finite-dimensional space of functions on K and*

3. *$\Sigma = \{\varphi_1, \varphi_2, ..., \varphi_k\}$ be a basis for P'.*

Then (K, P, Σ) is called a finite element, K the element domain, P the space of shape functions and Σ the set of nodal variables.

It is assumed that the nodal variables φ_i lie in the dual space of some larger function space, for instance, a Sobolev space.

Definition 2.3.4 (Subdivision) *A subdivision of a bounded domain* Ω *is a finite collection of element domains* $\{K_i, i = 1, \cdots, m\}$ *such that*

1. $K_i^\circ \cap K_j^\circ = \emptyset$ *if* $i \neq j$ *and*

2. $\cup_{i=1}^m K_i = \bar{\Omega}$.

Definition 2.3.5 (Local interpolant) *Let* (K, P, Σ) *be a finite element. Let*

$$\Sigma = \{\varphi_i, i = 1, \cdots, k\} \subset \left(C^l(K)\right)'$$

and $\{p_i\}_{i=1}^k$ *be a base of* P *associated to* Σ *(i.e.* $\varphi_i(p_j) = \delta_{ij}$ *for* $i, j \in \{1, \cdots, k\}$*). The local interpolant operator of* K *is*

$$\Pi^K : C^l(K) \to P, v \mapsto \sum_{i=1}^k \varphi_i(v) p_i.$$

Definition 2.3.6 (Global interpolant) *Let* Ω *be a bounded domain with a subdivision* $\mathscr{T}$*. Let each* $K \in \mathscr{T}$*, be equipped with a space of shape functions* P^K *and nodal variables* $\Sigma^K \subset \left(C^l(K)\right)'$*. For* $f \in C^l(\bar{\Omega})$*, the global interpolant* $\Pi_{\mathscr{T}}$ *is defined by:*

$$\Pi_{\mathscr{T}}(f)_{|K} = \Pi^K(f_{|K}).$$

We call $X_{\mathscr{T}}$ *the set* $\Pi_{\mathscr{T}}(C^l(\bar{\Omega}))$.

Definition 2.3.7 *Let* $F : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax + b$ *where* A *is a* $n \times n$ *non-degenerate matrix of real coefficients and* $b \in \mathbb{R}^n$.

1. *The pull-back of* $f : \mathbb{R}^n \to \mathbb{R}$ *by* F *is* $F^*(f) := f \circ F$.

2. *The push-forward by* F *of* $\varphi : S \to \mathbb{R}$ *where* S *is a space of functions defined from* $\mathbb{R}^n$ *to* R*, is defined for* $f \in S$ *by* $(F_*\varphi)(f) := \varphi(F^*(f)) = \varphi(f \circ F)$.

Definition 2.3.8 *Let (K, P, Σ) be a finite element. We say that a finite element $(\bar{K}, \bar{P}, \bar{\Sigma})$ is affine equivalent to (K, P, Σ) if there exists an affine transformation $F : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax + b$ with A a $n \times n$ non-degenerate matrix of real coefficients and $b \in \mathbb{R}^n$ such that $F(K) = \bar{K}$; $F^*(\bar{P}) = P$ and $F_*(\Sigma) = \bar{\Sigma}$.*

We then recall a particular type of finite element.

2.3.2 The n-simplex of type (1)

Define

$$P_k = \{p : p \text{ is a polynomial of degree less than or equal to } k \text{ on } \mathbb{R}^n\}.$$

For $U \subset \mathbb{R}^n$ we define $P_k(U) = \left\{p_{|U} : p \in P_k\right\}$.

Definition 2.3.9 (n-simplex) *A non-degenerate n-simplex is the convex hull K of $n+1$ points $a_j = (a_{ij})_{i=1}^n \in \mathbb{R}^n$ called the vertices such that the $n+1$ points are not contained in a hyperplane, i.e. the matrix*

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n+1} \\ a_{21} & a_{22} & \dots & a_{2,n+1} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,n+1} \\ 1 & 1 & \dots & 1 \end{pmatrix}$$

is regular.

The barycentric coordinates $\lambda_j(x), 1 \leq j \leq n+1$ of any $x \in \mathbb{R}^n$ with respect to the $n+1$ points a_j are the unique solutions of the linear system

$$\begin{cases} \sum_{j=1}^{n+1} a_{ij}\lambda_j(x) = x_i, \quad 1 \leq i \leq n \\ \sum_{j=1}^{n+1} \lambda_j(x) = 1 \end{cases}$$

The λ_j's are affine functions.

From now on we will simply say n-simplex for non-degenerate n-simplex.

Definition 2.3.10 (m-face of an n-simplex) *For $m \in \mathbb{N},\ 0 \leq m \leq n$ an m-face of an n-simplex K is any m-simplex whose vertices are also vertices of K. A $(n-1)$-face is simply called a face, a 1-face an edge or a side.*

Lemma 2.3.11 *Any polynomial of P_1 is uniquely determined by its values at the $n+1$ vertices of any n-simplex in $\mathbb{R}^n$.*
In fact $\lambda_j(a_i) = \delta_{ij}$ and $\forall p \in P_1,\ p = \sum_{j=1}^{n+1} p(a_j)\lambda_j$.

Definition 2.3.12 (n-simplex of type (1)) *An n-simplex of type (1) is a finite element (K, P, Σ) where K is a n-simplex of vertices $a_j = (a_{ij})_{i=1}^n \in \mathbb{R}^n$, P is $P_1(K)$ and $\Sigma = \{\varphi_i,\ 1 \leq i \leq n+1\}$ with $\varphi_i :\ C^0(K) \to \mathbb{R},\ f \mapsto f(a_i)$.*

Definition 2.3.13 (Assembly in triangulations) *Let Ω be a bounded polyhedral domain. Let $\{K_i\}_{i=1}^m$ be a subdivision of Ω into n-simplex. We say that $\{K_i\}_{i=1}^m$ is a triangulation of Ω if for any $i \in \{1, \cdots, m\}$ any face of K_i is either a subset of the boundary $\partial\Omega$ or a face of a n-simplex K_j in the subdivision such that $i \neq j$.*

We have the following proposition ([4, Brenner-Scott], prop 3.3.17).

Proposition 2.3.14 *Let $\mathscr{T}$ be a triangulation of a bounded polyhedral domain Ω with n-simplexes of type (1). It is possible to choose edge nodes for (K, P, Σ) , $K \in \mathscr{T}$ such that*

$$X_{\mathscr{T}} \subset C^0(\Omega) \cap W^{1,\infty}(\Omega).$$

2.3.3 Approximation of $W^{1,p}(\Omega)$ functions

The following theorem is a typical approximation error result in finite elements method that we recall for convenience. See for instance [4, Brenner-Scott] Theorem. 4.4.4 and 4.4.20 for more details.

Theorem 2.3.15 *Let $\{\tau_h : 0 < h \leq 1\}$, be a family of subdivisions of a polyhedral domain $\Omega \subset \mathbb{R}^n$ into finite element such that each element $K \in \tau_h$ is star-shaped with respect to some ball. Suppose this family is non-degenerate i.e.*

$$\max\{\text{diam}(K) : K \in \tau_h\} \leq h.\text{diam}(\Omega) \quad \forall 0 < h \leq 1 \tag{29}$$

and there exists $\rho > 0$ such that $\forall h \in (0, 1]$, $\forall K \in \tau_h$,

$$\text{diam}(B_K) \geq \rho.\text{diam}(K) \tag{30}$$

where B_K is the largest ball contained in K such that K is star-shaped with respect to B_K.
Suppose each $K \in \tau_h$, $0 < h \leq 1$ is associated with a finite element (K, P^K, Σ^K) affine-equivalent to a given finite element $(\hat{K}, \hat{P}, \hat{\Sigma})$ which we refer to as a reference element. We impose that for some $m \in \mathbb{N}^*$ and $l \in \mathbb{N}$:

1. $P_{m-1}(\hat{K}) \subset \hat{P} \subset W^{m,\infty}(\hat{K}^\circ)$
2. $\hat{\Sigma} \subset (C^l(\hat{K}))'$.

Suppose that $1 \leq p \leq \infty$, and either $m - l - \frac{n}{p} > 0$ when $p > 1$ or $m - l - n \geq 0$ when $p = 1$.

Then there exists a positive number C depending on the reference element, n, m, p and ρ such that for all $0 \leq s \leq m$, we have:

$$\left(\sum_{K \in \tau_h} \|v - \Pi_{\tau_h}(v)\|^p_{W^{s,p}(K)} \right)^{\frac{1}{p}} \leq Ch^{m-s}|v|_{W^{m,p}(\Omega)}, \quad \forall v \in W^{m,p}(\Omega), \text{ if } p < \infty;$$

and

$$\max_{K \in \tau_h} \|v - \Pi_{\tau_h}(v)\|_{W^{s,\infty}(K)} \leq Ch^{m-s}|v|_{W^{m,\infty}(\Omega)}, \quad \forall v \in W^{m,\infty}(\Omega) \text{ If } p = \infty.$$

The aim in the sequel is to establish the following proposition.

Proposition 2.3.16 *Let U be an open bounded domain of $\mathbb{R}^n$ with Lipschitz boundary. Let D be a polyhedral bounded domain containing U. For $h \in (0,1]$ let $\mathcal{T}_h$ be a triangulation of D into element domains T that are n-simplexes such that*

$$\max\{\text{diam}(T) : T \in \mathcal{T}_h\} \leq h.\text{diam}(U) \tag{31}$$

and there exists $\rho > 0$ such that $\forall h \in (0,1]$, $\forall T \in \mathcal{T}_h$

$$\text{diam}(B_T) \geq \rho \text{diam}(T), \tag{32}$$

where B_T is the largest ball contained in T such that T is star-shaped with respect to B_T. Define the space $\mathscr{S}_h$ by

$$\mathscr{S}_h = \left\{ v \in W^{1,\infty}(D^\circ) : v_{|T} \text{ is affine } \forall T \in \mathcal{T}_h \right\}.$$

Then for all $u \in W^{1,p}(U)$, $1 \leq p < \infty$, for all $\epsilon > 0$ and all $h_0 \in (0,1]$ there exists $h_\epsilon \in (0, h_0)$ such that $\|u - v_{h_\epsilon}\|_{W^{1,p}(U)} \leq \epsilon$

Proof. Let $u \in W^{1,p}(U)$. Let $\epsilon > 0$. As U open and $U \subset D$ we have $U \subset D^\circ$. Since ∂U is Lipschitz, we can extend u to $\bar{u} \in W^{1,p}(D^\circ)$. Since $C^\infty(D)$ is dense in $W^{1,p}(D^\circ)$, we can find $w \in C^\infty(D)$ such that $\|\bar{u} - w\|_{W^{1,p}(D^\circ)} \leq \frac{\epsilon}{2}$. $\mathcal{T}_h$ can be considered as a triangulation of D by an affine family of n-simplex of type (1). Those element domains are star-shaped with respect to some ball. Moreover, from theorem (2.3.14), the edge nodes may be chosen in such a way that

$$X_{\mathcal{T}_h} \subset C^0(D^\circ) \cap W^{1,\infty}(D^\circ).$$

Thanks to (31) and (32), we have assumption (29) and (30) in theorem (2.3.15). Take any reference element $(\bar{T}, \bar{P}, \bar{\Sigma})$ for this affine family.
Now, take $m = 2$, $l = 0$ and $q = \infty$. We have $\bar{\Sigma} \subset (C^0(\bar{T}))'$, $m - l - \frac{d}{q} > 0$ and

$$P_{m-1}(\bar{T}) = P_1(\bar{T}) = \bar{P} \subset W^{1,\infty}(\bar{T}).$$

Hence for $s = 1$, from theorem (2.3.15), there exists a constant C depending on $\bar{T}$, n and ρ such that

$$\max_{T \in \mathcal{T}_h} \|v - \Pi_{\mathcal{T}_h}(v)\|_{W^{1,\infty}(T)} \leq Ch|v|_{W^{2,\infty}(D^\circ)}, \quad \forall v \in W^{2,\infty}(D^\circ)$$

Thus, since $\Pi_{\mathcal{T}_h}(v) \in W^{1,\infty}(D^\circ)$,

$$\|v - \Pi_{\mathcal{T}_h}(v)\|_{W^{1,\infty}(D^\circ)} \leq Ch|v|_{W^{2,\infty}(D^\circ)}, \quad \forall v \in W^{2,\infty}(D^\circ)$$

and

$$\|w - \Pi_{\mathcal{T}_h}(w)\|_{W^{1,\infty}(D^\circ)} \leq Ch|w|_{W^{2,\infty}(D^\circ)}.$$

As D is bounded, there exists $\theta > 0$ such that

$$\|f\|_{W^{1,p}(D^\circ)} \leq \theta \|f\|_{W^{1,\infty}(D^\circ)} \quad \forall f \in W^{1,\infty}(D^\circ).$$

Choosing $h_\epsilon < h_0$ small enough, we get $\|w - \Pi_{\mathcal{T}_{h_\epsilon}}(w)\|_{W^{1,\infty}(D^\circ)} \leq \frac{\epsilon}{2\theta}$. Thus

$$\|w - \Pi_{\mathcal{T}_{h_\epsilon}}(w)\|_{W^{1,p}(D^\circ)} \leq \frac{\epsilon}{2}.$$

Take $v_{h_\epsilon} = \Pi_{\mathcal{T}_{h_\epsilon}}(w)$. One has $v_{h_\epsilon} \in W^{1,\infty}(D^\circ)$ and for $T \in \mathcal{T}_{h_\epsilon}$, $v_{h_\epsilon|T}$ is affine as a polynomial of degree one. So $v_{h_\epsilon} \in \mathscr{S}_{h_\epsilon}$ and $\|v_{h_\epsilon} - w\|_{W^{1,p}(D^\circ)} \leq \frac{\epsilon}{2}$. Hence

$$\|u - v_{h_\epsilon}\|_{W^{1,p}(U)} \leq \epsilon.$$

□

2.4 Convex functions of measures

Throughout this section, let $k : \mathbb{R}^d \to \mathbb{R}$ be a convex function satisfying

$$-a + b|u| \leq k(u) \leq c(1 + |u|) \tag{33}$$

with $a, b \in \mathbb{R}^d$ and $c > 0$. Call A_k the domain of k^*. $\mu = (\mu_1, ..., \mu_d)$ is a signed bounded Borel measure on $\mathbb{R}^d$ which has $\mu = \mu^s + hdx$ as Radon-Nykodym decomposition with respect to dx the Lebesgue measure. We assumed that $\int_{A_k} k^* dx < \infty$. We suppose that $0 \in int(\Lambda)$.

2.4.1 Elementary properties of k

Lemma 2.4.1 *Let $g : \mathbb{R}^d \to \mathbb{R}$ be defined by $g(x) = \alpha|x| + \beta$ with $\alpha, \beta \in \mathbb{R}$. Then $g^*(y) = -\beta + \chi_{B(0,\alpha)}(y)$ for all $y \in \mathbb{R}^d$ where we have made the convention $\chi_{B(0,\alpha)} \equiv \infty$ if $\alpha \leq 0$.*

Proof. For all $x, y \in \mathbb{R}^d$, one has $x \cdot y - g(x) = x \cdot y - \alpha|x| - \beta \leq |x|(|y| - \alpha) - \beta$.
If $|y| \leq \alpha$, then $x \cdot y - g(x) \leq -\beta = 0y - g(0)$. Hence $g^*(y) = -\beta + \chi_{B(0,\alpha)}(y)$.
If $|y| > \alpha$, then for $x = r\frac{y}{|y|}$, one has $x \cdot y - g(x) = r|y| - \beta - \alpha r = r(|y| - \alpha) - \beta \leq g^*(y)$.
Hence letting $r \to \infty$ we get $g^*(y) = \infty = -\beta + \chi_{B(0,\alpha)}(y)$.
Finally, $g^*(y) = -\beta + \chi_{B(0,\alpha)}(y)$ for all $y \in \mathbb{R}^d$.

□

Remark 2.4.2 *Thanks to Equation* (33) *and Lemma 2.4.1, one has*

$$-c + \chi_{B(0,c)}(y) \leq k^*(y) \leq a + \chi_{B(0,b)}(y); \quad \forall y \in \mathbb{R}^d. \tag{34}$$

In particular, $k^*(v) \leq a$ if $|v| < b$ and $k^*(v) = \infty$ if $|v| > c$. One deduces further that since $A_k \subset B(0, c)$, the function k is actually c-Lipschitz thanks to Lemma A.3.11.

In view of the growth condition (33) one has

$$b|y| \leq k_\infty(y) \leq c|y| \quad y \in \mathbb{R}^d. \tag{35}$$

In fact, let $v \in \mathbb{R}^d$. For $t > 0$,

$$\frac{-a + b|tv|}{t} \leq \frac{k(tv)}{t} \leq \frac{c(1 + |tv|)}{t}$$
$$\frac{-a}{t} + b|v| \leq \frac{k(tv)}{t} \leq \frac{c}{t} + c|v|.$$

Taking the limit when $t \to \infty$ gives the inequality as we have, thanks to Lemma A.1.5, $k_\infty(v) = \lim_{t\to\infty} \frac{k(tv)}{t}$.

2.4.2 Definition of convex function of measure

We starts with the following definition. We refer the reader to [22, Témam]

Definition 2.4.3 *We define*

$$\int_{\bar\Omega} k(\mu) := \int_{\bar\Omega} k(h)dx + \int_{\bar\Omega} k_\infty(\frac{d\mu^s}{d|\mu^s|})d|\mu^s|.$$

We will next work on giving a reformulation of this definition that highlights some lower semicontinuity properties. The first Lemma we will need is the following:

Lemma 2.4.4 *Let $\mathcal{B}$ be the set of all Borel measurable functions $v : \overline{\Omega} \to \mathbb{R}^d$. One has*

$$\sup_{v\in\mathcal{B}} \left\{ \int_{\bar\Omega} \mu v - \int_\Omega k^*(v)dx \right\} \leq \int_\Omega k(h)dx + \int_{\bar\Omega} k_\infty(\frac{d\mu^s}{d|\mu^s|})d|\mu^s|. \tag{36}$$

Proof

Let $v \in \mathbb{R}^d$. We have $k^*(v) \geq hv - k(h)$ and $-k^*(v) \leq -hv + k(h)$.

Thus for $v \in \mathcal{B}$, one has

$$\int_\Omega v\mu - \int_\Omega k^*(v)dx \leq \int_\Omega vhdx + \int_\Omega v\mu^s + \int_\Omega \left(-hv + k(h)\right)dx \leq \int_\Omega v\mu^s + \int_\Omega k(h)dx.$$

Remark next that

$$\sup_{v:\ \overline{\Omega}\to\mathbb{R}^d} \left\{ \int_{\bar\Omega} \mu v - \int_\Omega k^*(v)dx \right\} = \sup_{\substack{v:\overline{\Omega}\to\mathbb{R}^d \\ \int_\Omega k^*(v)dx<\infty}} \left\{ \int_{\bar\Omega} \mu v - \int_\Omega k^*(v)dx \right\} := A$$

and

$$A \leq \sup_{v:\overline{\Omega}\to\mathbb{R}^d} \left\{ \int_{\bar\Omega} \mu v - \int_\Omega k^*(v)dx : k^*(v(x)) < \infty \text{ for a.e.} x \in \Omega \right\}$$

Now for all $v : \overline{\Omega} \to \mathbb{R}^d$ satisfying $k^*(v(x)) < \infty$ for a.e. $x \in \Omega$, one has using the fact that

$$k_\infty(y) = \sup_{z\in \text{dom } k^* = A_k} z \cdot y; \quad \forall y \in \mathbb{R}^d,$$

$$\begin{aligned}\int_{\bar\Omega}\mu v-\int_\Omega k^*(v)dx &\le \int_{\bar\Omega} v\mu^s+\int_\Omega k(h)dx\\ &= \int_{\bar\Omega} v\frac{d\mu^s}{d|\mu^s|}d|\mu^s|+\int_\Omega k(h)dx\\ &\le \int_{\bar\Omega} k_\infty\left(\frac{d\mu^s}{d|\mu^s|}\right)d|\mu^s|+\int_\Omega k(h)dx.\end{aligned}$$

Taking the supremum we get Inequality (36).

□

The next Lemma will be used to prove the Proposition 2.4.6.

Lemma 2.4.5 *There exists a Borel function* $v_1:\mathbb{R}^d\to\mathbb{R}^d$ *such that* $\int_\Omega hv_1-k^*(v_1)$ *is finite and*

$$\int_\Omega (hv_1-k^*(v_1))=\int_\Omega k(h)dx.$$

Similarly, there exists a Borel function $v_2:\mathbb{R}^d\to\mathbb{R}^d$ *such that* $\int_\Omega hv_2-k^*(v_2)$ *is finite and*

$$\int_{\bar\Omega} v_2\mu^s=\int_{\bar\Omega} k_\infty(\frac{d\mu^s}{d|\mu^s|})d|\mu^s|.$$

Proof. Since k is convex and continuous, thanks to Corollary A.1.13 there exists a measurable map $s_1:\mathbb{R}^d\to\mathbb{R}^d$ such that for all $y\in\mathbb{R}^d$ one has $s_1(y)\in\partial k(y)$ and

$$k(y)=y\cdot s_1(y)-k^*(s_1(y)).$$

Hence the map $\mathbb{R}^d\ni x\mapsto s_1(h(x))$ may be used as v_1 since by the growth condition on k, one has that $\int_\Omega k(h)dx$ is a finite integral.

Furthermore, as k_∞ is convex and continuous, thanks to Corollary A.1.13 there exists a measurable map $s_2:\mathbb{R}^d\to\mathbb{R}^d$ such that for all $y\in\mathbb{R}^d$ one has $s_2(y)\in\partial k_\infty(y)$. Thanks to Lemma A.1.12, k_∞ being the support function of A_k by Corollary A.1.6, one has $s_2(y)\in A_k$ and $k_\infty(y)=y\cdot s_2(y)$. Hence

$$\begin{aligned}\int_{\overline\Omega} k_\infty\left(\frac{d\mu^s}{d|\mu^s|}\right)d|\mu^s| &= \int_{\overline\Omega}\left(\frac{d\mu^s}{d|\mu^s|}\right)\cdot s_2\left(\frac{d\mu^s}{d|\mu^s|}\right)d|\mu^s|\\ &= \int_{\overline\Omega} s_2\left(\frac{d\mu^s}{d|\mu^s|}\right)d\mu^s.\end{aligned}$$

Set $v_2 = s_2\left(\frac{d\mu^s}{d|\mu^s|}\right)$. Thanks to Remark 2.4.2, since $v_2(\bar{\Omega}) \subset A_k$, one has

$$\int_{\overline{\Omega}} (hv_2 - k^*(v_2))\, dx \geq \int_{\overline{\Omega}} (hv_2 - a) dx \geq -a\mathcal{L}^d(\Omega) - c\int_\Omega |h| dx > -\infty.$$

But

$$\int_{\overline{\Omega}} (hv_2 - k^*(v_2))\, dx \leq \int_{\overline{\Omega}} k(h) < \infty.$$

Thus $\int_{\overline{\Omega}} (hv_2 - k^*(v_2))\, dx$ is finite.

□

Proposition 2.4.6 *Let $\mathcal{B}$ be the set of all Borel measurable functions $v : \overline{\Omega} \to \mathbb{R}^d$. One has*

$$\int_{\bar{\Omega}} k(\mu) = \sup_{v \in \mathcal{B}} \left\{ \int_{\overline{\Omega}} \mu v - \int_\Omega k^*(v) dx \right\}. \tag{37}$$

Proof. Let $\delta > 0$. From Lemma 2.4.5, there exist Borel functions $v_1 : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\int_\Omega k(h) dx = \int_\Omega (hv_1 - k^*(v_1))\, dx$$

and v_2 such that $\int_\Omega k_\infty(\mu^s) = \int_\Omega v_2 \mu^s$ with

$$w = (v_2 - v_1)h - k^*(v_2) + k^*(v_1) \in L^1(\Omega).$$

We can find $\epsilon > 0$ such that for any set U measurable satisfying $\mathcal{L}^d(U) < \epsilon$, one has $|\int_U w dx| < \delta$. Since μ^s is singular we can find a closed set U satisfying $\mathcal{L}^d(U) < \epsilon$ on which μ^s is concentrated. Then $|\int_U w dx| < \delta$.

Define v by $v = v_2$ on U and $v = v_1$ on $\Omega \setminus U$. The function v is a Borel function. Set

$$a = \int_\Omega v\mu - \int_\Omega k^*(v) - \int_\Omega \{hv_1 - k^*(v_1)\} dx - \int_\Omega v_2 d\mu^s$$

and $V = \Omega \setminus U$. Keeping in mind that $\mu^s(V) = 0$, we have

$$\begin{aligned}
a &= \int_U v_2 h dx + \int_U v_2 \mu^s + \int_V v_1 h dx + \int_V v_1 \mu^s \\
&\quad - \int_U k^*(v_2) dx - \int_V k^*(v_1) dx - \int_U v_1 h dx - \int_V v_1 h dx \\
&\quad + \int_U k^*(v_1) dx + \int_V k^*(v_1) dx - \int_U v_2 \mu^s - \int_V v_2 \mu^s \\
&= \int_U v_2 h dx - \int_U F^*(v_2) dx - \int_U v_1 h dx + \int_U F^*(v_1) dx \\
&= \int_U w dx \\
&\geq -\delta.
\end{aligned}$$

As $a + \delta \geq 0$, So

$$\int_{\overline{\Omega}} v\mu - \int_{\overline{\Omega}} k^*(v) + \delta \geq \int_{\overline{\Omega}} \{hv_1 - k^*(v_1)\} dx + \int_{\overline{\Omega}} v_2 \mu^s = \int_{\overline{\Omega}} k(h) dx + \int_{\overline{\Omega}} k_\infty(\mu^s).$$

We then have that for all $\delta > 0$, there exists v_δ a Borel function such that

$$\int_{\overline{\Omega}} v_\delta \mu - \int_{\Omega} k^*(v_\delta) + \delta \geq \int_{\overline{\Omega}} k(h) dx + \int_{\Omega} k_\infty(\mu^s) \geq \int_{\overline{\Omega}} v_\delta \mu - k^*(v_\delta).$$

The last inequality is obtained from (36). We deduce that

$$\int_{\overline{\Omega}} k(\mu) = \int_{\overline{\Omega}} k(h) dx + \int_{\Omega} k_\infty(\mu^s) = \sup_{v \in \mathcal{B}} \left\{ \int_{\overline{\Omega}} \mu v - \int_{\Omega} k^*(v) dx \right\}.$$

□

Lemma 2.4.7 *Let $\mathcal{B}$ be the set of all Borel measurable functions $v : \overline{\Omega} \to \mathbb{R}^d$. One has*

$$S_C := \sup_{v \in C(\overline{\Omega}, \overline{A_k})} \left\{ \int_{\overline{\Omega}} \mu v - \int_{\Omega} k^*(v) dx \right\} = \sup_{v \in \mathcal{B}} \left\{ \int_{\overline{\Omega}} \mu v - \int_{\Omega} k^*(v) dx \right\} =: S_B. \quad (38)$$

Proof. Obviously we have $S_C \leq S_B$.

Remark next that the supremum in S_B may be taken only over the Borel functions v satisfying $v(\overline{\Omega}) \subset \bar{A}_k$; $\int_\Omega k^*(v) dx < \infty$ and $\int_{\overline{\Omega}} v\mu < \infty$. Set

$$T(w) := \int_{\overline{\Omega}} \mu w - \int_{\Omega} k^*(w) dx.$$

Let $\delta > 0$ and let us find a function $w \in C(\overline{\Omega}, \overline{A_k})$ such that $T(w) \geq T(v) - 2\delta$.

Let ρ stands for the Minkowsky functional of $\overline{A_k}$ as defined in Definition A.1.7. This functional is convex, homogenuous and $\overline{A_k} = \{x \in \mathbb{R}^d, \rho(x) \leq 1\}$. (See Lemma A.1.8).

We set $\bar{c}_n(t) := 1_{[0,1-2n^{-1})}(t) + (nt - n - 1)1_{[1-2n^{-1},1-n^{-1})}(t)$ for $n \in \mathbb{N}$ and $t \in [0,1]$. Define $c_n(x) = \bar{c}_n(\rho(x))$ for all $x \in \bar{\Omega}$ and all $n \in \mathbb{N}$. Define $v_n : \mathbb{R}^d \to \mathbb{R}^d$ to be $c_n v$ on Ω and 0 elsewhere. One has $\rho(v_n) < 1 - n^{-1}$.

Next, Remark that since k satisfies (33), $k^*(0)$ is finite thanks to (34). As k^* is convex, we have

$$k^*(c_n v) = k^*(c_n v + (1 - c_n)0) \leq c_n k^*(v) + (1 - c_n)k^*(0).$$

Thus $|k^*(c_n v)|$ is bounded by a Lebesgue integrable function on Ω. Applying the Lebesgue Dominated Convergence Theorem we get

$$\lim_{n\to\infty} \int_{\overline{\Omega}} c_n v\mu - \int_{\Omega} k^*(c_n v) = \int_{\overline{\Omega}} v\mu - \int_{\Omega} k^*(v).$$

This can be read

$$\lim_{n\to\infty} \int_{\Omega} v_n\mu - \int_{\Omega} k^*(v_n) = \int_{\Omega} v\mu - \int_{\Omega} k^*(v).$$

In the remaining of the proof, we fix n such that $I(v_n) \geq I(v) - \delta$.

Let $\epsilon > r > 0$. Define $\{\eta_r\}_r$ a family of standard mollifiers on $\mathbb{R}^d$ and set $w_r(x) = \eta_r * v_n(x)$.

One has:

$$\begin{aligned}
\rho(\eta_r * v_n(x)) &= \rho\left(\int_{\mathbb{R}^d} \eta_r(y)v_n(x-y)dy\right) \\
&\leq \int_{B_r(0)} \rho(v_n(x-y))\eta_r(y)dy \\
&\leq \int_{B_r(0)} (1 - n^{-1})\eta_r(y)dy \\
&\leq (1 - n^{-1}) \int_{B_r(0)} \eta_r(y)dy \\
&\leq (1 - n^{-1}),
\end{aligned}$$

where we have used the facts that $\eta_r(y)dy$ is a probability measure on $B_r(0)$ and ρ is a convex function. Thus $w_r(\bar{\Omega}) \subset \left\{y \in \overline{A_k} : \rho(y) \leq 1 - n^{-1}\right\}$. Next,

$$\begin{aligned} k^*(\eta_r * v_n(x)) &= k^*\left(\int_{B_r(0)} v_n(x-y)\eta_r(y)dy\right) \\ &\leq \int_{B_r(0)} k^*(v_n(x-y))\eta_r(y)dy \\ &= \left(\eta_r * k^*(v_n)\right)(x). \end{aligned}$$

Thus $\int_\Omega k^*(w_r(x)) \leq \int_\Omega \eta_r * k^*(v_n(x))$. As $k^*(v_n) \in L^1(\Omega)$, one has $\lim_{r\to 0}\int_{\Omega_r} \eta_r * k^*(v_n) = \int_\Omega k^*(v_n)$ and $\limsup_r \int_\Omega k^*(w_r(x)) \leq \int_\Omega k^*(v_n)$. As we also have $\lim_{r\to 0}\int_\Omega w_r\mu = \int_\Omega v_n\mu$, we deduce $\limsup_{r\to 0} I(w_r) \geq I(v_n)$. Thus we may find r_1 such that $I(w_{r_1}) \geq I(v_n) - \delta$ and $I(w_{r_1}) \geq I(v) - 2\delta$. We have $w_{r_1} \in C(\overline{\Omega}, \overline{A_k})$ and we may take $w = w_{r_1}$.

Finally, we deduce that $S_B = S_C$.

□

Remark 2.4.8 *1. In view of Proposition 2.4.6, we get that the map which to every bounded Borel measure μ associate $\int_{\overline{\Omega}} k(\mu)$ is convex and lower semicontinuous for the vague topology (the topology of convergence in distribution) of measures.*

2. From the proof of Proposition 2.4.6, it is apparent that for all $\delta > 0$ there exists $n \in \mathbb{N}^*$ and $w_\delta \in C(\bar{\Omega}, \overline{A_k})$ such that $w_\delta(\bar{\Omega}) \subset \left\{y \in \overline{A_k} : \rho(y) \leq 1 - n^{-1}\right\}$ and $\int_{\overline{\Omega}} k(\mu) \leq \int_{\overline{\Omega}} w_\delta\mu - \int_\Omega k^*(w_\delta)dx + \delta$.

2.4.3 A lower semicontinuity result

We will need the following Lemma.

Lemma 2.4.9 *Let $k_n : \mathbb{R}^d \to \mathbb{R}$ be convex functions which converge to k in $C_{loc}(\mathbb{R}^d)$. Suppose that $\{k_n^*\}_n$ converges to b in $C_{loc}(\Lambda)$ and $\sup_n \int_\Lambda |k_n^*| < \infty$. Then $b = k^*$.*

Proof. Let $\lambda \in (0,1)$. Consider ρ the Minkowsky functional associated to Λ. The set $\{\rho \leq \lambda\}$ is compact and contained in Λ.

Claim 1 $k^*(u) \leq b(u)$ for all $u \in \{\rho \leq \lambda\}$.

Let $u \in \{\rho \leq \lambda\}$ and take $\epsilon > 0$. Since k is convex and continuous, we can find v in $\mathbb{R}^d$ such that $k^*(u) = u \cdot v - k(v)$. We have:

$$\begin{aligned} u \cdot v - k(v) &= u \cdot v - k_n(v) + k_n(v) - k(v) \\ &\leq k_n^*(u) + k_n(v) - k(v) \\ &\leq b(u) + |k_n^*(u) - b(u)| + |k_n(v) - k(v)| \end{aligned}$$

Choose n big enough so that $|k_n^*(u) - b(u)| \leq \epsilon$ and $|k_n(v) - k(v)| \leq \epsilon$. With those inequalities, we obtain:

$$u \cdot v - k(v) \leq b(u) + \epsilon + \epsilon = b(u) + 2\epsilon,$$

and $k^*(u) \leq b(u) + 2\epsilon$. As the later is true for all $\epsilon > 0$, then $k^*(u) \leq b(u)$.

Claim 2 There exist a constant M not depending on n such that for all $n \in \mathbb{N}$ and all $u \in \{\rho \leq \lambda\}$, one has $\partial k_n^*(u) \subset B(0, M)$.

Using the fact that the sequence $\{k_n^*\}_{n=1}^{\infty}$ is a sequence of convex function satisfying $\sup_n \int_\Lambda |k_n^*| < \infty$ and the fact that $\{\rho \leq \lambda\}$ is a compact set of λ, one deduces that there exists a constant $M > 0$ such that for all $n \in \mathbb{N}$, k_n^* is M-Lipchitz on $\{\rho \leq \lambda\}$ and thus for all $u \in \{\rho \leq \lambda\}$, one has $\partial k_n^*(u) \subset B(0, M)$.

Claim 3 $k^*(u) \geq b(u)$ for all $u \in \{\rho \leq \lambda\}$.

For $u \in \{\rho \leq \lambda\}$ fixed, for all $n \in \mathbb{N}$ one has $\partial k_n^*(u) \neq \emptyset$ and we chose $v_n \in \partial k_n^*(u)$. Recall that thanks to Claim 2 $|v_n| \leq M$. This yield the existence of some subsequence $\{v_{n_s}\}_s$ of $\{v_n\}_n$ converging to some $v \in \mathbb{R}^d$. Set $V = \{v_{n_s}, s \in \mathbb{N}\} \cup \{v\}$. V is compact. The following hold:

$$\begin{aligned} k_{n_k}^*(u) + k_{n_k}^{**}(v_{n_s}) &= u \cdot v_{n_k} \\ k_{n_k}^*(u) + k_{n_k}(v_{n_k}) &= u \cdot v_{n_k}. \end{aligned}$$

Remark that

$$|k_{n_s}(v_{n_s}) - k(v)| \leq |k_{n_s}(v_{n_s}) - k(v_{n_s})| + |k(v_{n_s}) - k(v)|. \tag{39}$$

Since V is compact and $\{k_n\}_{n=1}^{\infty}$ converges to k in $C_{loc}(\Lambda)$, we have $\lim_{s\to\infty}\sup_{x\in V}|k_{n_s}(x) - k(x)| = 0$. Thus $\lim_{s\to\infty}|k_{n_s}(v_{n_s}) - k(v_{n_s})| = 0$. Moreover, using the continuity of k one deduces $\lim_{s\to\infty}|k(v_{n_s}) - k(v)| = 0$. Therefore, when $s \to 0$ the left hand side of Inequality (39) converges to 0. Hence $\lim_{s\to\infty} k_{n_s}(v_{n_s}) = k(v)$. Combining with $\lim_{s\to\infty} k_{n_s}^*(u) = b(u)$ and $\lim_{s\to\infty} u \cdot v_{n_s} = u \cdot v$, we get $b(u) + k(v) = u \cdot v$ and

$$b(u) = u \cdot v - k(v) \leq k^*(u) \ \forall u \in \{\rho \leq \lambda\}.$$

Claim 4 $k^*(u) = b(u)$ for all $u \in \{\rho \leq \lambda\}$.

This is a consequence of Claim 1 and claim 3.

Claim 5 $k^* = b$.

For all $u \in \Lambda$, there exists $\lambda \in (0,1)$ such that $u \in \{\rho \leq \lambda\}$. Thanks to Claim 4 we have $k^*(u) = b(u)$. Thus $k^* = b$.

□

Lemma 2.4.10 *Suppose the sequence of bounded measures $\{\mu_n\}_{n=1}^{\infty}$ converges in measure to μ. Suppose that $\{k_n\}_{n=1}^{\infty}$ a sequence of a-Lipchitz functions from $\mathbb{R}^d$ to $\mathbb{R}$ converges uniformly locally to k. Suppose* $\sup_n \int_\Lambda |k_n^*| < \infty$ *and* $\operatorname{dom} k_n^* \subset \bar{\Lambda}$. *Then*

$$\int_{\overline{\Omega}} k(\mu) \leq \liminf_{n\to\infty} \int_{\overline{\Omega}} k_n(\mu_n).$$

Proof. We use the fact that for all $n \in \mathbb{N}$, k_n^* is convex and $\sup_n \int_\Lambda |k_n^*| < \infty$ to deduce that a subsequence of $\{k_n^*\}_{n=1}^{\infty}$ that we denote $\{k_{m_n}^*\}_{n=1}^{\infty}$ that converges locally uniformly to some $b : \Lambda \to \mathbb{R}^d$ on Λ. Using Lemma 2.4.9, we deduce that $b = k^*$ on Λ and in fact $\{k_n^*\}_{n=1}^{\infty}$ converges locally uniformly to k^* on Λ.

Let $\delta > 0$. Remark first that $\overline{\text{dom } k_n^*} = \bar{\Lambda}$. Let ρ be the Minkowsky functional of Λ. Thanks to Remark 2.4.8, there exists $m \in \mathbb{N}^*$ and $w_\delta \in C(\bar{\Omega}, \overline{A_k})$ such that $w_\delta(\bar{\Omega}) \subset \{ y \in \overline{\Lambda} : \rho(y) \leq 1 - m^{-1} \}$ and $\int_{\overline{\Omega}} k(\mu) \leq \int_{\overline{\Omega}} w_\delta \mu - \int_\Omega k^*(w_\delta) dx + \delta$.
We use one more time the fact that for all $n \in \mathbb{N}$, k_n^* is convex and $\sup_n \int_\Lambda |k_n^*| < \infty$ to deduce that there exists $M > 0$ such that for all $u \in w_\delta(\Omega)$ and all $n \in \mathbb{N}^*$, one has $|k_n^*(u)| \leq M$. We use next the fact that $\{k_n^*\}_{n=1}^\infty$ converges uniformly to k^* on $\{ y \in \overline{\Lambda} : \rho(y) \leq 1 - m^{-1} \}$; to deduce that

$$\int_{\overline{\Omega}} w_\delta \mu - \int_\Omega k^*(w_\delta) dx = \lim_{n \to \infty} \left(\int_{\overline{\Omega}} w_\delta \mu_n - \int_\Omega k_n^*(w_\delta) dx \right) \leq \liminf \int_{\overline{\Omega}} k_n(\mu_n).$$

Therefore $\int_{\overline{\Omega}} k(\mu) \leq \liminf \int_{\overline{\Omega}} k_n(\mu_n) + \delta$. And letting δ go to 0, one gets $\int_{\overline{\Omega}} k(\mu) \leq \liminf \int_{\overline{\Omega}} k_n(\mu_n)$.

□

CHAPTER III

THE PRIMAL PROBLEMS

In this chapter, we describe some variational problems involving polyconvex integrand. We discuss mostly existence results. We start by listing the main assumptions.

3.1 Main assumptions

Throughout the chapter, Ω and Λ will denote bounded convex open sets of $\mathbb{R}^d$. We assume $\mathcal{L}^d(\Omega) = 1$ and $diam(\Lambda) \leq r^*$. Consider $f \in C^1(\mathbb{R}^{d\times d})$ strictly convex such that for some $c_1,\ c_2,\ c_3 \in \mathbb{R}^*_+$, one has:

$$c_1(|\xi|^p - 1) \leq f(\xi) \leq c_2(|\xi|^p + 1); \tag{40}$$

$$|\nabla f(\xi)| \leq c_3|\xi|^{p-1} \text{ and} \tag{41}$$

$$|\nabla f^*(\xi)| \leq c_3|\xi|^{q-1} \tag{42}$$

where $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$. Let $H \in C^2(0,\infty)$ be strictly convex such that

$$\lim_{t\to 0^+} H(t) = \lim_{t\to\infty} \frac{H(t)}{t} = \infty. \tag{43}$$

We extend H to the whole $\mathbb{R}$ by setting $H(t) = \infty$ for $t \leq 0$. Let $F : \mathbb{R}^d \to \mathbb{R}^d$ such that $F \in L^1(\Omega)$.

We gather in the next section some properties of the function H.

3.1.1 Elementary properties of H

3.1.1.1 Properties of the derivative of H

Lemma 3.1.1 *1. The function $H' : (0,\infty) \to \mathbb{R}$ is bijective, strictly increasing and continuous.*

2. *Moreover, if C is a bounded set of $\mathbb{R}$ then there exists $m \in \mathbb{N}\backslash\{0\}$, $(H')^{-1}(C) \subset [m^{-1}, m]$.*

Proof. **(i)** ***Claim 1 :*** The function $H' : (0,\infty) \to \mathbb{R}$ is continuous and strictly increasing.

Proof. The function $H : (0,\infty) \to \mathbb{R}$ is convex and C^1 so $H' : (0,\infty) \to \mathbb{R}$ is continuous and non decreasing. Assume that for $t_1 < t_2$ we have $H'(t_1) = H'(t_2) =: a$. Since H' is continuous and non decreasing, $H'(s) = a$ for all $s \in [t_1, t_2]$. We have

$$H(t) = H(t_1) + \int_{t_1}^{t} adu = H(t_1) + (t - t_1)a, \ \forall t \in [t_1, t_2].$$

Thus

$$H(t_2) = H(t_1) + (t_2 - t_1)a; \text{ and } H\left(\frac{t_2 + t_1}{2}\right) = H(t_1) + a\frac{t_2 - t_1}{2}.$$

Hence

$$H\left(\frac{t_2 + t_1}{2}\right) = \frac{1}{2}\left(H(t_1) + H(t_2)\right).$$

This contradicts the fact that H is strictly convex. Therefore H' is strictly increasing.

Claim 2 : One has $\lim_{t\to 0} H'(t) = -\infty$.

Proof. Assume $H'(t) \geq a$ for some $a \in \mathbb{R}$ and for all $t \in (0,\infty)$. We have for $0 < t < 1$

$$H(1) - H(t) = \int_t^1 H'(u)du \geq \int_t^1 adu \geq a(1-t)$$

and so

$$\lim_{t\to 0^+} -H(t) \geq -H(1) + \lim_{t\to 0^+} a(1-t) = -H(1) + a.$$

But, exploiting Equation (43), one gets $\lim_{t\to 0^+} H(1) - H(t) = -\infty$. This is a contradiction and so,

$$\lim_{t\to 0^+} H'(t) = -\infty. \tag{44}$$

Claim 3 : One has $\lim_{t\to\infty} H'(t) = \infty$.

Proof. In the same manner, as in the proof of Claim 2, assume $H'(t) \leq b$ for some

$b \in \mathbb{R}$ and for all $t \in (0, \infty)$. We have for $1 < t$:

$$H(t) - H(1) = \int_1^t H'(t)du \leq \int_1^t bdu \leq b(t-1).$$

But

$$\lim_{t\to\infty} \frac{H(t) - H(1)}{t} = \infty \text{ and } \lim_{t\to\infty} \frac{b(t-1)}{t} = b.$$

This is a contradiction and so,

$$\lim_{t\to\infty} H'(t) = \infty. \tag{45}$$

Finally, using (44), (45) and the fact that $H' : (0, \infty) \to \mathbb{R}$ is continuous and strictly increasing we deduce that $H' : (0, \infty) \to \mathbb{R}$ is bijective, continuous and strictly increasing.

(ii) Let C be a bounded set. Assume there exists $\{t_n\}_{n=1}^{\infty} \subset (H')^{-1}(C)$ such that $\lim_{n\to\infty} t_n = 0^+$. Since $H'(t_n) \in C$ and $\lim_{n\to\infty} H'(t_n) = -\infty$, C is unbounded. This is a contradiction and so, there exists $a > 0$ such that $a < t$ for all $t \in (H')^{-1}(C)$.
Assume there exists $\{t_n\}_{n=1}^{\infty} \subset (H')^{-1}(C)$ such that $\lim_{n\to\infty} t_n = \infty$. Since $H'(t_n) \in C$ and $\lim_{n\to\infty} H'(t_n) = \infty$, C is unbounded. This is a contradiction and so, there exists $b > 0$ such that $t < b$ for all $t \in (H')^{-1}(C)$.
Finally, since there exists $a, b > 0$ such that $a < t < b$ for all $t \in (H')^{-1}(C)$, there exists $m \in \mathbb{N}$, $m \neq 0$ such that $(H')^{-1}(C) \subset [m^{-1}, m]$.

□

3.1.2 Properties of the Legendre transform of H

Lemma 3.1.2 *We have:*

1. *The Legendre transform H^* of H is a strictly increasing bijection from $\mathbb{R}$ to $\mathbb{R}$.*

2. *For every $s \in \mathbb{R}$ there exists a unique $t_s > 0$ such that $H^*(s) = st_s - H(t_s)$. Moreover, t_s is the unique solution of the equation $s = H'(t)$ for $t \in (0, \infty)$.*

3. Let $g : \mathbb{R} \to \bar{\mathbb{R}}$ be defined by $g(s) = \alpha s - \beta H^*(s)$, with $\alpha, \beta > 0$. Then

$$\lim_{s \to -\infty} g(s) = \lim_{s \to \infty} g(s) = -\infty.$$

Proof. ***Claim 1 :*** The function H^* is finite.

Proof. We have

$$H^*(s) = \sup_{t \in \mathbb{R}} \{ts - H(t)\} = \sup_{t > 0} \{ts - H(t)\}$$

since $H(t) = \infty$ for $t \leq 0$. The function $H_s : (0, \infty) \to \mathbb{R}, t \mapsto st - H(t)$ is continuous. We have $\lim_{t \to 0^+} H_s(t) = \lim_{t \to \infty} H_s(t) = -\infty$ and thus H_s admits a maximun on $(0, \infty)$ and $H^*(s)$ is finite.

Claim 2 : Point 2 holds.

Proof. Let $s \in \mathbb{R}$. We have shown there exists a $t_s > 0$ such that

$$H^*(s) = H_s(t_s) = st_s - H(t_s).$$

Moreover, as H_s is differentiable, its minimizer t_s satisfies the equation $s = H'(t)$ that has a unique solution for $t \in (0, \infty)$.

Claim 3 : H^* is strictly increasing

Proof. Assume $s_1 < s_2$. We have :

$$\begin{aligned} ts_1 - H(t) &< ts_2 - H(t) \text{ for } t > 0 \\ \sup_{t>0} \{ts_1 - H(t)\} &\leq \sup_{t>0} \{ts_2 - H(t)\} \\ H^*(s_1) &\leq H^*(s_2) \end{aligned}$$

Assume $H^*(s_1) = H^*(s_2)$. We have

$$s_1 t_1 - H(t_1) = H^*(s_2) \geq s_2 t_1 - H(t_1).$$

Thus $s_1 \geq s_2$ which is a contradiction. We deduce that H^* is strictly increasing.

Claim 4 : One has $\lim_{t \to \infty} H^*(s) = \infty$ and $\lim_{t \to -\infty} H^*(s) = -\infty$.

Proof. Assume that for $m \in \mathbb{R}$, one has $m < H^*(s), \forall s \in \mathbb{R}$. For all $n \in \mathbb{N}$ there exists $x_n \in \mathbb{R}^+$ such that

$$m \leq -nx_n - H(x_n) = H^*(-n).$$

Now no subsequences of $\{x_n\}_{n=1}^{\infty}$ go to 0 since the right hand side will go to $-\infty$. In the same way no subsequences of $\{x_n\}_{n=1}^{\infty}$ go to ∞ since the right hand side will go to $-\infty$. Thus x_n is bounded away from 0 and we may find a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converging to some $\bar{x} \in (0, \infty)$. By taking the limit for $k \to \infty$ we get the right hand side to be $-\infty$. Thus we can not find m such that $m < H^*(s), \forall s \in \mathbb{R}$. Then $\lim_{s \to -\infty} H^*(s) = -\infty$.

Assume next that for $M \in \mathbb{R}$, one has $H^*(s) \leq M, \forall s \in \mathbb{R}$. For all $t \in \mathbb{R}^+, s \in \mathbb{R}$, it holds that $st - H(t) \leq M$. When $s \to \infty$, the left hand side goes to ∞, leading to a contradiction. Thus $\lim_{t \to \infty} H^*(s) = \infty$.

Claim 5 : Point 1 holds.

Proof. H^* is convex finite thus continuous. Adding Claim 1, 3 and 4, H^* is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

Claim 6 : Point 3 holds.

Proof. Let $t > 0$. For $s \in \mathbb{R}$, one has

$$\begin{aligned} H^*(s) + H(t) \geq& ts \\ H(t) - ts \geq& -H^*(s) \\ \beta H(t) - \beta ts \geq& -\beta H^*(s) \\ \alpha s + \beta H(t) - \beta ts \geq& \alpha s - H^*(s)\beta \\ \beta H(t) + s(\alpha - t\beta) \geq& g(s). \end{aligned}$$

If $t < \frac{\alpha}{\beta}$, then $\alpha - t\beta > 0$ and $\lim_{s \to -\infty} g(s) = -\infty$.

Similarly, if $t > \frac{\alpha}{\beta}$, then $\alpha - t\beta < 0$ and $\lim_{s \to \infty} g(s) = -\infty$.

$\square$

3.2 A notion of determinant in a weak sense

We begin this section with the following definition.

Definition 3.2.1 *Let $u : \Omega \to \Lambda$ be a Borel map. We say that $\beta : \Omega \to (0, \infty)$ is a weak determinant of u if*

$$\int_{\Omega} l(u(x))\beta(x)dx = \int_{\Lambda} l(y)dy; \quad \forall l \in C_b(\mathbb{R}^d). \tag{46}$$

Remark that if $u \in C^1(\Omega, \Lambda)$ is bijective, the change of variable formula (cf Theorem A.3.2) gives

$$\int_{\Omega} l(u(x))|\det \nabla \mathbf{u}(x)|dx = \int_{\Lambda} l(y)dy; \quad \forall l \in C_b(\mathbb{R}^d).$$

This inspires the notation $\det{}^*\nabla\mathbf{u}$ for the set of all weak determinant of u. So

$$\det{}^*\nabla\mathbf{u} = \{\beta : \Omega \to [0, \infty) \text{ measurable } : \text{ Equation (46) holds}\}. \tag{47}$$

Remark 3.2.2 1. *Let $u : \Omega \to \Lambda$ be measurable and suppose that there exists a compact $K \subset \Lambda$ such that $u(\Omega) \subset K$ up to a set of zero Lebesgue measure. Then the set $\det^* \nabla\mathbf{u}$ is empty.*

2. *For $u : \Omega \to \Lambda$ a Borel map. Then the set $\det^* \nabla\mathbf{u}$ is convex. This can be seen by exploiting Equation (46). The same equation for $l \equiv 1$ shows that $\det^* \nabla\mathbf{u}$ is a subset of the sphere of radius $\mathcal{L}^d(\Lambda)$ with respect to the $L^1(\Omega, \mathbb{R}^d)$ norm.*

When $\mathbf{u} \in W^{1,r}(\bar{\Omega}, \bar{\Lambda})$ for $r > d$, the following Lemma allows us to fully characterize the set $\det{}^*\nabla\mathbf{u}$.

Lemma 3.2.3 *Suppose $\mathbf{u} \in W^{1,r}(\bar{\Omega}, \bar{\Lambda})$; $r > d$; $\mathbf{u}(\Omega) = \Lambda$ and $Z_{\mathbf{u}}$, the set of $x \in \Omega$ such that $\det \nabla\mathbf{u}(x) = 0$ has zero Lebesgue measure. Then a Borel function $\beta : \Omega \to (0, \infty)$ belongs to $\det^* \nabla\mathbf{u}$ if and only if for almost every $y \in \Lambda$, one has*

$$\sum_{x \in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla\mathbf{u}(x)|} = 1. \tag{48}$$

Proof. We first use Sard's Theorem for Sobolev functions (cf. e.g. [13]) to infer that the set $\mathbf{u}(Z_{\mathbf{u}})$ is a set of null Lebesgue measure. Let $l \in C(\mathbb{R}^d)$ and $\beta : \Omega \to (0, \infty)$ a Borel function. Then we have

$$\int_\Omega l(\mathbf{u}(x))\beta(x)dx = \int_\Omega l(\mathbf{u}(x))\frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|}|\det \nabla \mathbf{u}(x)|dx$$

Now, by the area formula (c.f. Theorem A.3.2) we get

$$\int_\Omega l(\mathbf{u}(x))\frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|}|\det \nabla \mathbf{u}(x)|dx = \int_\Lambda l(y)\Big(\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|}\Big)dy.$$

Hence β satisfies (46) if and only if for a.e. $y \in \Lambda$, one has

$$\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|} = 1.$$

□

Remark 3.2.4 *Under the hypothesis of Lemma 3.2.3, one gets in particular that* $|\det \nabla \mathbf{u}|/N_{\mathbf{u}}(\mathbf{u}) \in \det^* \nabla \mathbf{u}$.

Here is an application of Lemma 3.2.3. Consider $\Omega = (0, 1)$ and $\Lambda = (-1, 1)$. Let $u : (0, 1) \to (0, 1)$ be defined by

$$u(x) = 4x\mathbf{1}_{(0,\frac{1}{4})}(x) + (-4x + 2)\mathbf{1}_{[\frac{1}{4},\frac{3}{4}]}(x) + (4x - 4)\mathbf{1}_{(\frac{3}{4},1)}(x).$$

Then $|\det \nabla u| = 4$. Let $\beta : (0, 1) \to (0, \infty)$. Now if $y \in [0, 1)$, one has

$$\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|} = 4^{-1}(\beta(4^{-1}y) + \beta(-4^{-1}(y - 2))).$$

If $y \in [-1, 0)$, one has

$$\sum_{x\in \mathbf{u}^{-1}(y)} \frac{\beta(x)}{|\det \nabla \mathbf{u}(x)|} = 4^{-1}(\beta(4^{-1}(y + 4)) + \beta(-4^{-1}(y - 2))).$$

So $\beta \in \det{}^*\nabla u$ if and only if

$$\begin{cases} \beta\left(\frac{y+4}{4}\right) + \beta\left(\frac{-y+2}{4}\right) = 4 \text{ for } y \in (-1, 0) \\ \beta\left(\frac{y}{4}\right) + \beta\left(\frac{-y+2}{4}\right) = 4 \text{ for } y \in (0, 1) \end{cases} \tag{49}$$

Take $a, b \in [0,4]$. A family of functions that satisfies Equation (49) is

$$\beta_{a,b} = a\mathbf{1}_{(0,\frac{1}{4})} + (4-a)\mathbf{1}_{(\frac{1}{4},\frac{1}{2})} + b\mathbf{1}_{(\frac{1}{2},\frac{3}{4})} + (4-b)\mathbf{1}_{(\frac{3}{4},1)} \tag{50}$$

So $\beta_{a,b} \in \det^* \nabla \mathbf{u}$ for all $a, b \in [0,4]$. In particular $\beta_{2,2} \equiv 2 \in \det^* \nabla \mathbf{u}$.

3.3 The notation $\det^H \nabla \mathbf{u}$.

Lemma 3.3.1 *Let $u : \Omega \to \Lambda$ a Borel map such that the set there exists $\beta_0 \in \det^* \nabla \mathbf{u}$ satisfying $\int_\Omega H(\beta_0(x))dx < \infty$. Then the problem*

$$\inf_{\beta \in \det^* \nabla \mathbf{u}} \int_\Omega H(\beta(x))dx \tag{51}$$

admits a unique minimizer that we will denote $\det^H \nabla \mathbf{u}$.

Proof. The set $\det^* \nabla \mathbf{u}$ is strongly closed in $L^1(\Omega)$ and thus weakly closed in $L^1(\Omega)$ since it is convex. A minimizing sequence of (51) is weakly compact in $L^1(\Omega)$ thanks to the growth condition (43). Hence, using the convexity and lower semicontinuity of H we get a minimizer for (51) (c.f. Lemma A.3.13). Uniqueness follows from the convexity of $\det^* \nabla \mathbf{u}$ and the strict convexity of H.

□

3.4 A first variational problem

In this section we will show the following Lemma.

Lemma 3.4.1 *If $p > d$, the variational problem*

$$\inf_{\mathbf{u}} \Big\{ I_*(\mathbf{u}) := \int_\Omega \big(f(\nabla \mathbf{u}) + H\big(\det \nabla \mathbf{u}\big) - \mathbf{F} \cdot \mathbf{u}\big)dx \;|\mathbf{u} : \bar{\Omega} \to \mathbb{R}^d; \\ \mathbf{u} \in W^{1,p}(\Omega, \Lambda);\;\; \mathbf{u}(\bar{\Omega}) = \bar{\Lambda};\; \det \nabla \mathbf{u} > 0 \Big\}$$

has a minimizer u_*.

First we will need the following lemma.

3.4.1 An auxiliary lemma

Lemma 3.4.2 *Let Ω, Λ be convex open sets of $\mathbb{R}^d$. Let $u_n : \bar{\Omega} \to \bar{\Lambda}$ be continuous, satisfying $u_n(\bar{\Omega}) = \bar{\Lambda}$ and converging uniformly to $u : \bar{\Omega} \to \mathbb{R}^d$. Then $u(\bar{\Omega}) = \bar{\Lambda}$.*

Proof. Let $y \in \bar{\Lambda}$. For all $n \in \mathbb{N}$, let $x_n \in \bar{\Omega}$ such that $u_n(x_n) = y$. Assume without lost of generality (having $\bar{\Omega}$ compact) that $\{x_n\}_{n=1}^{\infty}$ converges to $x \in \bar{\Omega}$. One has

$$\begin{aligned}|u(x) - u_n(x_n)| \leq & |u(x) - u(x_n)| + |u_n(x_n) - u(x_n)| \\ \leq & |u(x) - u(x_n)| + |u_n - u|_{L^\infty(\bar{\Omega})}.\end{aligned}$$

Since $\{u_n\}_{n=1}^{\infty}$ converges strongly in $C(\bar{\Omega}, \mathbb{R}^d)$ to u; we deduce that

$$y = \lim_{n \to \infty} u_n(x_n) = u(x).$$

Hence $y \in \bar{\Lambda}$. One deduces that $\bar{\Lambda} \subset u(\bar{\Omega})$.

Let $x \in \bar{\Omega}$. Since for all $n \in \mathbb{N}$ one has $u_n(x) \in \bar{\Lambda}$ and we have the pointwise convergence of $\{u_n\}_{n=1}^{\infty}$ to u, we get that $u(x) \in \bar{\Lambda}$. Hence $u(\bar{\Omega}) \subset \bar{\Lambda}$.

Finally $u(\bar{\Omega}) = \bar{\Lambda}$.

$\square$

3.4.2 Proof of Lemma 3.4.1

Let $\mathbf{u}_0 \in W^{1,p}(\Omega, \Lambda)$ such that $\mathbf{u}_0(\bar{\Omega}) = \bar{\Lambda}$ and $\det \nabla \mathbf{u}_0 > 0$ as given by Lemma 2.1.2. Set

$$c_0 := \int_\Omega \big(f(\nabla \mathbf{u}_0) + H\big(\det \nabla \mathbf{u}_0\big) - \mathbf{F} \cdot \mathbf{u}_0\big) dx.$$

Let $\{\mathbf{u}_n\}_{n=1}^{\infty}$ be a minimizing sequence satisfying $I_*(\mathbf{u}_n) \leq c_0$. It follows that

$$\begin{aligned}\int_\Omega \big(f(\nabla \mathbf{u}_n) + H\big(\det \nabla \mathbf{u}_n\big) - \mathbf{F} \cdot \mathbf{u}_n\big) dx \leq c_0 \\ \int_\Omega \big(f(\nabla \mathbf{u}_n) + H\big(\det \nabla \mathbf{u}_n\big) \leq c_0 + \int_\Omega \mathbf{F} \cdot \mathbf{u}_n \\ \leq c_0 + \int_\Omega |\mathbf{F}| r^* := c(c_0, \mathbf{F})\end{aligned}$$

So $\int_\Omega f(\nabla \mathbf{u}_n) \leq \min H + c(c_0, \mathbf{F})$ and hence $\|\nabla \mathbf{u}_n\|_{L^p(\Omega,\mathbb{R}^d)}$ is bounded.

Exploiting the fact that $\mathbf{u}_n(x) \in \Lambda$, we have that $\|u_n\|_{L^1(\Omega)}$ is bounded by r^* and so a subsequence $\{u_{n_k}\}_{k=1}^\infty$ of $\{u_n\}_{n=1}^\infty$ converges weakly in $W^{1,p}(\Omega, \Lambda)$ to some $\mathbf{u}$. We deduce that $\{\det \nabla \mathbf{u}_n\}_{n=1}^\infty$ converges weakly to $\det \nabla \mathbf{u}$ in $L^{\frac{p}{d}}(\Omega)$ and so in $L^1(\Omega)$ (c.f. [10, Evans]). This leads to

$$\int_\Omega H(\det \nabla \mathbf{u}) \leq \lim_{k\to\infty} \int_\Omega H(\det \nabla \mathbf{u}_{n_k}),$$

Moreover, as $\{u_{n_k}\}_{k=1}^\infty$ converges weakly in $W^{1,p}(\Omega, \Lambda)$ $\mathbf{u}$ and f is convex, we have:

$$\int_\Omega \big(f(\nabla \mathbf{u}_n) - \mathbf{F}\cdot \mathbf{u}_n\big)dx \leq \liminf_{k\to\infty} \int_\Omega \big(f(\nabla \mathbf{u}_{n_k}) - \mathbf{F}\cdot \mathbf{u}_{n_k}\big)dx.$$

So $I_*(\mathbf{u}) \leq \liminf_{k\to\infty} I_*(\mathbf{u}_{n_k})$. We show next that $\det \nabla \mathbf{u} > 0$.
Exploiting the fact that $I_*(\mathbf{u}) < \infty$ we deduce that the set $\{\det \nabla \mathbf{u} \leq 0\}$ has zero Lebesgue measure and thus $\det \nabla \mathbf{u} > 0$.

It remains to show that $\mathbf{u}(\bar{\Omega}) = \bar{\Lambda}$. Since $p > d$, we may suppose that $\{u_{n_k}\}_{k=1}^\infty$ converges strongly to u in $C(\bar{\Omega}, \mathbb{R}^d)$. Thanks to Lemma 3.4.2, one has $\mathbf{u}(\bar{\Omega}) = \bar{\Lambda}$.

□

3.5 A Second Variational Problem

Definition 3.5.1 *The set $\mathcal{U}_b$ will stand for the set of pairs $(\beta, \mathbf{u})$ such that $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$ and $\beta : \Omega \to (0, \infty)$ is a Borel function satisfying $\beta \in \det^* \nabla \mathbf{u}$.*

Lemma 3.5.2 *The problem*

$$\inf_{(\beta,\mathbf{u})\in\mathcal{U}_b} \left\{ I(\mathbf{u}, \beta) := \int_\Omega (f(\nabla \mathbf{u}) + H(\beta) - \mathbf{F}\cdot \mathbf{u}) \right\} \tag{52}$$

admits a minimum.

Proof. Thanks to Lemma 2.1.2, there exists $\mathbf{u}_0 \in W^{1,p}(\Omega, \Lambda)$ an homeomorphism such that $\mathbf{u}_0(\Omega) = \Lambda$ and $\det \nabla \mathbf{u} > 0$. Set $\beta_0 = \det \nabla \mathbf{u}_0$. One has $\beta_0 \in \det^* \nabla \mathbf{u}_0$. Let

$$c_0 := \int_\Omega \big(f(\nabla \mathbf{u}_0) + H\big(\beta_0\big) - \mathbf{F}\cdot \mathbf{u}_0\big)dx.$$

Let $\{\mathbf{u}_n\}_{n=1}^{\infty}$ be a minimizing sequence satisfying $I(\mathbf{u}_n) \leq c_0$. It holds that

$$\int_\Omega \big(f(\nabla \mathbf{u}_n) + H\big(\beta_n\big) - \mathbf{F} \cdot \mathbf{u}_n\big) dx \leq c_0$$

$$\int_\Omega \big(f(\nabla \mathbf{u}_n) + H\big(\beta_n\big) \leq c_0 + \int_\Omega \mathbf{F} \cdot \mathbf{u}_n$$

$$\leq c_0 + \int_\Omega |\mathbf{F}| r^* := c(c_0, \mathbf{F}).$$

So $\int_\Omega f(\nabla \mathbf{u}_n) \leq -\min H + c(c_0, \mathbf{F})$ and $\|\nabla \mathbf{u}_n\|_{L^p(\Omega,\mathbb{R})}$ is bounded.

We also have $\int_\Omega H\big(\beta_n\big) \leq c(c_0, \mathbf{F})$ since $f \geq 0$. We deduce that

$$\int_\Omega H\big(\beta_n\big) - \min H \leq c(c_0, \mathbf{F}) - \min H.$$

We have

$$\lim_{t \to \infty} \frac{H(t) - \min H}{t} = \lim \frac{H(t)}{t} = \infty.$$

Hence by the Delavalle-Poussin criterion, we may assume without lost of generality that $\{\beta_n\}_{n=1}^{\infty}$ converges weakly to β in $L^1(\Omega)$.

Exploiting the fact that $\mathbf{u}_n(x) \in \Lambda$, we have that $\|u_n\|_{L^1(\Omega)}$ is bounded by r^* and so we may assume without lost of generality that $\{u_n\}_{n=1}^{\infty}$ converges weakly in $W^{1,p}(\Omega, \Lambda)$ to some $\mathbf{u}$. We may suppose in addition that $\{u_n\}_{n=1}^{\infty}$ converges a.e. to some u using the compact embedding of $W^{1,p}(\Omega, \mathbb{R}^d)$ into $L^1(\Omega, \mathbb{R}^d)$.

$\{\beta_n\}_{n=1}^{\infty}$ converges weakly to β in $L^1(\Omega)$ and H is convex and lower semicontinous. This leads to

$$\int_\Omega H(\beta) \leq \lim_{n \to \infty} \int_\Omega H(\beta_n).$$

Moreover, as $\{u_n\}_{n=1}^{\infty}$ converges weakly in $W^{1,p}(\Omega, \Lambda)$ $\mathbf{u}$ and f is convex, we have:

$$\int_\Omega \big(f(\nabla \mathbf{u}_n) - \mathbf{F} \cdot \mathbf{u}_n\big) dx \leq \liminf_{k \to \infty} \int_\Omega \big(f(\nabla \mathbf{u}_{n_k}) - \mathbf{F} \cdot \mathbf{u}_{n_k}\big) dx.$$

So

$$I(\mathbf{u}, \beta) \leq \liminf_{k \to \infty} I(\mathbf{u}_n, \beta_n).$$

It remains to show that $(\mathbf{u}, \beta) \in \mathcal{U}_b$.

Let $l \in C_b(\mathbb{R}^d)$. Then $\{l(u_n)\}_n$ converges a.e. to $l(\mathbf{u})$. Since $\{\beta_n\}_{n=1}^{\infty}$ converges weakly to β, thanks to Lemma A.3.14, one has

$$\lim_{n\to\infty} \int_{\Omega} l(u_n)\beta_n dx = \int_{\Omega} l(\mathbf{u})\beta dx.$$

But for all $n \in \mathbb{N}^*$, one has $\int_{\Omega} l(u_n)\beta_n dx = \int_{\Lambda} l(y)dy$. Thus $\int_{\Omega} l(\mathbf{u})\beta dx = \int_{\Lambda} l(y)dy$ and $(\mathbf{u}, \beta) \in \mathcal{U}_b$.

Finally, $I(\mathbf{u}, \beta) = \min_{(\beta,\mathbf{u})\in\mathcal{U}_b} \left\{ I(\mathbf{u}) := \int_{\Omega} (f(\nabla\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u}) \right\}$.

□

Remark 3.5.3 *In fact, every minimizing sequence of Problem* (52)*, has a subsequence converging strongly to a minimizer.*

Proof. Take a minimizing sequence of problem (52). We have shown in the proof of Lemma 3.5.2 that we can extract a minimizing subsequence $(u_n, \beta_n)_{n\in\mathbb{N}}$ converging weakly to some $(\mathbf{u}, \beta) \in \mathcal{U}_b$ that is a minimizer. Since $\{u_n\}_{n=1}^{\infty}$ converges weakly to u in $L^1(\Omega)$, one has

$$\begin{aligned}\lim_{n\to\infty} \int_{\Omega} f(\nabla\mathbf{u}_n) + H(\beta_n) &= \lim_{n\to\infty} \int_{\Omega} f(\nabla\mathbf{u}_n) + H(\beta_n) - F\cdot u_n + F\cdot u \\ &= \int_{\Omega} f(\nabla\mathbf{u}) + H(\beta).\end{aligned}$$

Using the strict convexity of the map

$$\mathbb{R}^{d\times d} \times \mathbb{R} \ni (\xi, t) \mapsto f(\xi) + H(t)$$

and the fact that

$$\lim_{n\to\infty} \int_{\Omega} f(\nabla\mathbf{u}_n) + H(\beta_n) = \int_{\Omega} f(\nabla\mathbf{u}) + H(\beta),$$

we deduce (thanks to Lemma A.2.12) that $\{\nabla\mathbf{u}_n\}_{n=1}^{\infty}$ converges strongly to $\nabla\mathbf{u}$ and $\{\beta_n\}_{n=1}^{\infty}$ converges strongly to β. We already had the strong convergence of $\{\mathbf{u}_n\}_{n=1}^{\infty}$ to

$\mathbf{u}$ in $L^p(\Omega, \mathbb{R}^d)$ by the the compact embedding of $W^{1,p}(\Omega, \Lambda)$ into $L^p(\Omega, \Lambda)$. Therefore, $\{\mathbf{u}_n\}_{n=1}^{\infty}$ converges strongly to $\mathbf{u}$ in $W^{1,p}(\Omega, \mathbb{R}^d)$ and $\{\beta_n\}_{n=1}^{\infty}$ converges strongly to β in $L^1(\Omega)$.

□

One has also that

$$\inf_{(\beta,\mathbf{u})\in\mathcal{U}_b} I(\mathbf{u},\beta) = \inf_{\mathbf{u}\in\mathcal{U}} \left(\inf_{\beta\in\det{}^*\nabla\mathbf{u}} I(\mathbf{u},\beta) \right).$$

But

$$\begin{aligned} \inf_{\beta\in\det{}^*\nabla\mathbf{u}} I(\mathbf{u},\beta) &= \inf_{\beta\in\det{}^*\nabla\mathbf{u}} \left\{ \int_\Omega (f(\nabla\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u})\, dx \right\} \\ &= \int_\Omega \big(f(\nabla\mathbf{u}) + H(\det{}^H\nabla\mathbf{u}) - \mathbf{F}\cdot\mathbf{u} \big)\, dx. \end{aligned}$$

Hence

$$\inf_{(\beta,\mathbf{u})\in\mathcal{U}_b} I(\mathbf{u},\beta) = \inf_{\mathbf{u}\in\mathcal{U}} \left\{ \int_\Omega \big(f(\nabla\mathbf{u}) + H(\det{}^H\nabla\mathbf{u}) - \mathbf{F}\cdot\mathbf{u} \big)\, dx \right\}.$$

3.6 A Perturbed problem

We will first need to define pseudo-projected gradients.

3.6.1 A discrete gradient method

Throughout this section, the set $\mathcal{S}$ is a finite dimensional subspace of piecewise affine functions in $W_0^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$. For $\psi \in \mathcal{S}$ of the form

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_d \end{pmatrix}, \text{ set } \mathrm{div}\psi = \begin{pmatrix} \mathrm{div}\psi_1 \\ \mathrm{div}\psi_2 \\ \vdots \\ \mathrm{div}\psi_d \end{pmatrix}.$$

Call

$$\mathcal{U}_0 = \{\mathbf{u} : \Omega \to \Lambda |\ \text{Borel map}\}.$$

Theorem 3.6.1 *Let* $\mathbf{u} \in \mathcal{U}_0$*. Define*

$$\mathcal{G}(\mathbf{u}) = \left\{ G \in L^p\left(\Omega,\ \mathbb{R}^{d\times d}\right) \mid \int_\Omega \langle \mathbf{u}, \mathrm{div}\psi\rangle = -\int_\Omega \langle G, \psi\rangle\ dx, \forall \psi \in \mathcal{S} \right\}. \tag{53}$$

Then there exists a unique $G_0 \in \mathcal{G}(\mathbf{u})$ *which satisfies* $G_0 = \nabla f^*(\psi_0)$ *for some* $\psi_0 \in \mathcal{S}$*. In fact* G_0 *is the unique minimizer of*

$$\inf_{G\in\mathcal{G}(\mathbf{u})} \left\{ \int_\Omega f(G)dx \right\}. \tag{54}$$

We will denote G_0 *by* $\nabla_{\mathcal{S}}\mathbf{u}$*.*

Proof. ***Claim 1 :*** Problem (54) admits a unique minimizer G_0.

Let $\mathbf{u} \in \mathcal{U}_0$. Consider the map $T : \mathcal{S} \to \mathbb{R};\ \psi \mapsto \int_\Omega \mathbf{u}\cdot \mathrm{div}\psi\ dx$. T is linear on $\mathscr{S}$ which is a finite dimensional linear space. Hence T is continuous on $\mathcal{S}$. By the Hahn-Banach's Theorem we can extend it to a linear functional $\bar{T}$ on the all $L^q\left(\Omega,\ \mathbb{R}^{d\times d}\right)$ with the same norm. Hence by the Riesz Representation Theorem for linear functional in $L^q\left(\Omega,\ \mathbb{R}^{d\times d}\right)$ there exists $G \in L^p\left(\Omega,\ \mathbb{R}^{d\times d}\right)$ such that for all $\psi \in L^q\left(\Omega,\ \mathbb{R}^{d\times d}\right)$, one has $\bar{T}(\psi) = \int_\Omega \langle G, \psi\rangle\, dx$. Taking its restriction to $\mathcal{S}$ we get that

$$\int_\Omega \langle \mathbf{u}, \mathrm{div}\psi\rangle\ dx = -\int_\Omega \langle G, \psi\rangle\ dx, \forall \psi \in \mathcal{S}$$

and $\mathcal{G}(\mathbf{u})$ is nonempty.

The set $\mathcal{G}(\mathbf{u})$ is non empty, convex and weakly closed in $L^p(\Omega, \mathbb{R}^{d\times d})$. In addition the map $f : \mathbb{R}^{d\times d} \to \mathbb{R}$ is strictly convex and satisfies the growth condition (40). Then $\inf_{G\in\mathcal{G}(\mathbf{u})} \left\{\int_\Omega f(G)dx\right\}$ admits a unique minimizer $G_0 \in \mathcal{G}(\mathbf{u})$.

Claim 2 : One has $\nabla f^*(\psi_0) = G_0$ for some $\psi_0 \in \mathcal{S}$.

Define the annihilator set of $\mathcal{S}$ by

$$\mathcal{S}^\perp = \left\{ \varphi \in L^p\left(\Omega,\ \mathbb{R}^{d\times d}\right) \middle| \int_\Omega \langle \varphi, \psi\rangle\ dx = 0\ \forall \psi \in \mathcal{S} \right\} \tag{55}$$

and define

$$^\perp(\mathcal{S}^\perp) = \left\{ \psi \in L^q\left(\Omega,\ \mathbb{R}^{d\times d}\right) \middle| \int_\Omega \langle \varphi, \psi\rangle\ dx = 0\ \forall \psi \in \mathcal{S}^\perp \right\} \tag{56}$$

As $\mathcal{S}$ is closed in $L^q\left(\Omega,\ \mathbb{R}^{d\times d}\right)$, we have ${}^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$ (cf. [5] Proposition 1.9 p. 9).
Let $G \in \mathcal{S}^{\perp}$. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}; t \mapsto \int_\Omega f(G_0 + tG)\ dx$. It attains its minimum at $t = 0$. By the Lebesgue Dominated Convergence, for $|t| < 1$, as

$$f(G_0 + tG) \leq c_1(|G_0 + tG|^p + 1) \leq c_1 2^p(|G_0|^p + |G|^p + 1),$$

$G_0, G \in L^p(\Omega)$ and Ω is bounded we have

$$\varphi'(t) = \int_\Omega \frac{d}{dt} f(G_0 + tG)\ dx = \int_\Omega \langle G, \nabla f(G_0 + tG)\rangle\ dx.$$

Hence $\varphi'(0) = \int_\Omega \langle G, \nabla f(G_0)\rangle\ dx$. So

$$\int_\Omega \langle G, \nabla f(G_0)\rangle\ dx = 0 \qquad \forall G \in \mathcal{S}^{\perp}.$$

Therefore $\nabla f(G_0) \in {}^{\perp}\left(\mathcal{S}^{\perp}\right) = \mathcal{S}$. Set $\psi_0 = \nabla f(G_0)$. We then have $\nabla f^*(\psi_0) = G_0$ and $\psi_0 \in \mathcal{S}$.

Claim 3 : If $G_1 \in \mathcal{G}(\mathbf{u})$ is such that $\nabla f^*(\psi_0) = G_1$ for some $\psi_0 \in \mathcal{S}$ then G_1 is the minimizer of Problem (54).

Let $G \in \mathcal{G}(\mathbf{u})$. One has $G - G_1 \in \mathcal{S}^{\perp}$ thanks to Equation (53). One has $\psi_0 \in \mathcal{S}$ so $\int_\Omega \langle G - G_1, \psi_0\rangle\, dx = 0$. Since $\psi_0 = \nabla f(G_1)$ and f is strictly convex, one has

$$0 = \int_\Omega \langle G - G_1, \nabla f(G_1)\rangle \leq \int_\Omega f(G) - \int_\Omega f(G_1),$$

With equality if and only if $G = G_1$. Hence G_1 is the unique minimizer of Problem (54).

□

3.6.2 A minimization problem with the pseudo-projected gradient.

For $u \in \mathcal{U}_0$, we define

$$I_{\mathcal{S}}(\mathbf{u}) = \int_\Omega \left(f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\det{}^H\nabla\mathbf{u}) - \mathbf{F}\cdot\mathbf{u}\right) dx$$

when $\det{}^*\nabla \mathbf{u} \neq \emptyset$. Otherwise, we set

$$I_{\mathcal{S}}(\mathbf{u}) = \infty.$$

We would like to study

$$\inf_{\mathbf{u}\in\mathcal{U}_0} I_{\mathcal{S}}(\mathbf{u}) \tag{57}$$

which is

$$\inf_{(\mathbf{u},\beta)\in\mathcal{U}_1} \int_\Omega (f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u})\, dx, \tag{58}$$

where $\mathcal{U}_1 = \{(\mathbf{u},\beta) : \mathbf{u}\in\mathcal{U}_0; \beta \in \det{}^*\nabla\mathbf{u}\}$. We have the following Lemma.

Lemma 3.6.2 *From every minimizing sequence of Problem* (58)*, one can extract a subsequence* $\{(\mathbf{u}_n,\beta_n)\}_{n\in\mathbb{N}} \subset \mathcal{U}_b$ *such that there exist* $\mathbf{u}\in\mathcal{U}$ *and* $\beta \in L^1(\Omega)$ *and*

1. $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ *converges weakly in* $L^\infty(\Omega,\mathbb{R}^d)$ *to* $\mathbf{u}$.

2. $\{\beta_n\}_{n\in\mathbb{N}}$ *converges weakly in* $L^1(\Omega)$ *to* β.

3. $\{\nabla_S\mathbf{u}_n\}_{n\in\mathbb{N}}$ *converges strongly in* $L^p(\Omega,\mathbb{R}^{d\times d})$ *and a.e. to* $\nabla_S\mathbf{u}$.

Proof. The results of the Lemma follow from the growth conditions on f and H, the fact that $F\in L^1(\Omega,\mathbb{R}^d)$, $|\mathbf{u}| < r^*$ and $\mathcal{S}$ is a finite dimensional linear space.

□

Remark that despite Lemma 3.6.2, nothing can be directly said about existence of minimizers in Problem (58) since nothing a priori tells us that $(\mathbf{u},\beta)$ found in Lemma 3.6.2 will satisfy $\beta\in\det{}^*\nabla\mathbf{u}$.

We have replaced the functional in (52) whose properties favor a direct proof of existence of minimizers by a worse functional from the direct methods of the Calculus of Variations point of view. This daring approach will surprisingly be rewarding as we will not only establish an existence and uniqueness result but we will also obtain the Euler–Lagrange Equations characterizing the minimizer.

3.7 A Relaxed Problem

3.7.1 The set over which to minimize.

Let $C = \bar{\Omega} \times [0,\infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$. Set $D = [0,\infty) \times \bar{\Lambda} \times \mathbb{R}^{d\times d}$. We define $\Pi^1 : C \to \bar{\Omega}; (x,t,u,\xi) \mapsto x$. Define also $\Pi^2 : C \to \bar{\Lambda}; (x,t,u,\xi) \mapsto u$ and finally, define $\Pi^3 : C \to [0,\infty); (x,t,u,\xi) \mapsto t$. Let Γ be the set of Radon measures on $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d\times d}$ supported by C and satisfying the conditions :

$$\int_C f(\xi)\gamma(dx,dt,du,d\xi) < \infty; \tag{59}$$

$$\Pi^1_{\#}(\gamma) = 1_\Omega \mathcal{L}^d; \tag{60}$$

$$\Pi^3_{\#}(\Pi^2\gamma) = 1_\Lambda \mathcal{L}^d \tag{61}$$

and for all $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$

$$\int_C \langle \xi, \psi(x)\rangle \gamma(dx,dt,du,d\xi) = -\int_C \langle u, \operatorname{div}\psi(x)\rangle\, \gamma(dx,dt,du,d\xi). \tag{62}$$

Let $(\mathbf{u},\beta) \in \mathcal{U}_b$. Define a measure $\gamma^{(\mathbf{u},\beta)}$ on C by $\gamma^{(\mathbf{u},\beta)} = (Id \times \beta \times \mathbf{u} \times \nabla\mathbf{u})_{\#}(1_\Omega \mathcal{L}^d)$.

Lemma 3.7.1 *For $(\mathbf{u},\beta) \in \mathcal{U}_b$, the measure $\gamma^{(\mathbf{u},\beta)}$ belongs to Γ.*

Proof. One has

$$\Pi^1_{\#}(\gamma^{(\mathbf{u},\beta)}) = \Pi^1_{\#}((Id \times \beta \times \mathbf{u} \times \nabla\mathbf{u})_{\#}(1_\Omega\mathcal{L}^d)) = Id_{\#}(1_\Omega\mathcal{L}^d) = 1_\Omega\mathcal{L}^d.$$

Let $l \in C_b(\mathbb{R}^d)$

$$\int_C t l(\mathbf{u})\gamma^{(\mathbf{u},\beta)}(dx,dt,du,d\xi) = \int_\Omega \beta(x) l(\mathbf{u}(x))dx = \int_\Lambda l dy.$$

Let $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$

$$\begin{aligned}\int_C \langle \xi, \psi(x)\rangle \gamma^{(\mathbf{u},\beta)}(dx,dt,du,d\xi) &= \int_\Omega \langle \nabla\mathbf{u}(x), \psi(x)\rangle\, dx \\ &= -\int_\Omega \langle \mathbf{u}, \cdot\operatorname{div}\psi\rangle\, dx \\ &= -\int_C \langle u, \operatorname{div}\psi(x)\rangle\, \gamma^{(\mathbf{u},\beta)}(dx,dt,du,d\xi).\end{aligned}$$

Next

$$\int_C f(\xi)\gamma^{(\mathbf{u},\beta)}(dx,dt,du,d\xi) = \int_\Omega f(\nabla\mathbf{u}(x))dx \le c_1 \int_\Omega (|\nabla\mathbf{u}(x)|^p + 1)\, dx < \infty$$

since $\mathbf{u} \in W^{1,p}(\Omega, \mathbb{R}^d)$ and $\mathcal{L}^d(\Omega) < \infty$.

In summary, the measure $\gamma^{(\mathbf{u},\beta)}$ belongs to Γ.

$\square$

We have the embedding $\mathcal{U}_b = \{(\beta, \mathbf{u}) \mid \mathbf{u} \in W^{1,p}(\Omega, \Lambda), \mathbf{u}_{\#}\beta = \chi_\Lambda\} \subset \Gamma$, which to $(\beta, \mathbf{u})$ associates $\gamma \equiv \gamma^{(\beta,\mathbf{u})}$.

Let $\gamma \in \Gamma$. Equation (60) tells us that $\Pi^1\#\gamma = 1_\Omega\mathcal{L}^d$. By the disintegration theorem (cf. Theorem A.3.16), there exists a family of probability measure $\{\gamma^x\}_{x\in\Omega}$ such that for all $L : C \to [0,\infty]$ measurable, one has

$$\int_C L(x,u,t,\xi)\gamma(dx,du,dt,d\xi) = \int_\Omega \left(\int_D L(x,u,t,\xi)\gamma^x(du,dt,d\xi)\right) dx.$$

For $x \in \Omega$, set

$$U_\gamma(x) = \int_D \xi\gamma^x(dt,du,d\xi), \tag{63}$$

$$\mathbf{u}_\gamma(x) = \int_D u\gamma^x(dt,du,d\xi). \tag{64}$$

Using Jensen's inequality, one has

$$\int_\Omega f(U_\gamma(x))dx \le \int_\Omega \Big(\int_D f(\xi)\gamma^x(dt,du,d\xi)\Big)dx = \int_C f(\xi)\gamma(dx,dt,du,d\xi) < \infty.$$

Thus the growth condition (40) on f implies that $U_\gamma \in L^p(\Omega, \mathbb{R}^{d\times d})$. Moreover, the fact that the support of γ in the u variables is contained in the convex set $\bar{\Lambda}$ yields that for a.e. $x \in \Omega$, one has $u_\gamma(x) \in \bar{\Lambda}$. In fact, by Jensen's inequality, the map ρ_Λ standing for the Minkowsky function of Λ (cf. Definition A.1.7), one has

$$\rho_\Lambda(\mathbf{u}_\gamma) \le \int_D \rho_\Lambda(\mathbf{u})\gamma^x(dt,du,d\xi) \le \int_D \gamma^x(dt,du,d\xi) = 1.$$

By Equation (62), for all $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$, it holds

$$\begin{aligned}
\int_\Omega \langle U_\gamma(x), \psi(x)\rangle\, dx &= \int_\Omega \left(\int_D \langle \xi, \psi(x)\rangle\, \gamma^x(dt, du, d\xi) \right) dx \\
&= \int_C \langle \xi, \psi(x)\rangle \gamma(dx, dt, du, d\xi) \\
&= -\int_C \langle u, \operatorname{div} \psi(x)\rangle\, \gamma(dx, dt, du, d\xi) \\
&= -\int_\Omega \left(\int_D \langle u, \operatorname{div} \psi(x)\rangle\, \gamma^x(dt, du, d\xi) \right) dx \\
&= -\int_\Omega \langle \mathbf{u}_\gamma, \operatorname{div} \psi(x)\rangle\, dx.
\end{aligned}$$

Hence $\nabla \mathbf{u}_\gamma = U_\gamma$ and $\mathbf{u}_\gamma \in W^{1,p}(\Omega, \mathbb{R}^d)$.

3.7.2 The functional to minimize.

We define $\bar{I}$ on Γ by

$$\bar{I}(\gamma) = \int_C \big(f(\xi) + H(t) - \mathbf{F}(x)\cdot u\big)\gamma(dx, dt, du, d\xi).$$

Remark that for all $(\mathbf{u}, \beta) \in \mathcal{U}_b$, its holds that $\bar{I}(\gamma^{(\mathbf{u},\beta)}) = I(\mathbf{u}, \beta)$.

Lemma 3.7.2 *The sublevel sets of $\bar{I}$ are compact for the narrow topology on Γ.*

Proof. Consider a sequence $\{\gamma_n\}_{n=0}^\infty \subset \Gamma$ such that for all $n \in \mathbb{N}$

$$\int_C \big(f(\xi) + H(t) - \mathbf{F}(x)\cdot u\big)\gamma_n(dx, dt, du, d\xi) < c.$$

Then, for all $n \in \mathbb{N}$

$$\int_C \big(f(\xi) + H(t) - \min H - \min f\big)\gamma(dx, dt, du, d\xi) < -\min H - \min f + r^* \|F\|_{L^1(\Omega, \mathbb{R}^d)} := A.$$

Define $\varphi : C \to [0, \infty]$ by

$$\varphi(x, t, u, \xi) = f(\xi) + H(t) - \min H - \min f.$$

Then the sublevel sets $\{(x, t, u, \xi) \in C : \varphi(x, t, u, \xi) < a\}$ are compact in C and

$$\sup_n \int_C \varphi(x, t, u, \xi)\gamma_n \le A < \infty.$$

Hence the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is tight thanks to Lemma A.2.6. By Prokorov's theorem, a subsequence that we still denote $\{\gamma_n\}_{n=0}^{\infty}$ converges weakly to some measure γ. We show next that $\gamma \in \Gamma$.

Let $b \in C_b(\mathbb{R}^d)$ then by the definition of the weak convergence of measures, one has

$$\int_C b(x)\gamma(dx, dt, du, d\xi) = \int_\Omega b\, dx$$

and Equation (60) is satisfied for γ.

Let $l \in C_b(\mathbb{R}^d)$. Suppose $l \not\equiv 0$. The map $C \ni (x, t, u, \xi) \mapsto tl(\mathbf{u})$ is continuous. Moreover $|tl(\mathbf{u})| \leq |l|_\infty |t|$;

$$\sup_n \int_C (H(|t|) - \min H) < \infty;$$

one has $h - \min H \geq 0$ and

$$\lim_{t\to\infty} \frac{H(t) - \min H}{t} = \infty.$$

Thus thanks to Lemma A.2.2 and Lemma A.2.3 one gets

$$\lim_{n\to\infty} \int_C tl(\mathbf{u}) d\gamma_n = \int_C tl(\mathbf{u}) d\gamma.$$

Having for all $n \in \mathbb{N}$ $\int_C tl(\mathbf{u}) d\gamma_n = \int_\Lambda l dy$, one deduces that $\int_C tl(\mathbf{u}) d\gamma = \int_\Lambda l dy$. Remark that the last equation is trivially true for $l \equiv 0$ and Equation (61) holds for γ.

Let $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$. Suppose $\psi \not\equiv 0$. The map $C \ni (x, t, u, \xi) \mapsto \langle \xi, \psi(x)\rangle$ is continuous. Moreover $|\langle \xi, \psi(x)\rangle| \leq |\psi|_\infty |\xi|$; $\sup_n \int_C (|\xi|^p) < \infty$ and $\lim_{t\to\infty} \frac{|t|^p}{t} = \infty$. Thus thanks to Lemma A.2.2 and Lemma A.2.3 one gets

$$\lim_{n\to\infty} \int_C \langle \xi, \psi(x)\rangle\, d\gamma_n = \int_C \langle \xi, \psi(x)\rangle\, d\gamma.$$

In addition, the map $C \ni (x, t, u, \xi) \mapsto \langle u, \operatorname{div} \psi(x)\rangle$ is continuous bounded. Hence, by the definition of the weak convergence of measures, one has

$$\lim_{n\to\infty} \int_C \langle u, \operatorname{div} \psi(x)\rangle\, d\gamma_n = \int_C \langle u, \operatorname{div} \psi(x)\rangle\, d\gamma.$$

Having for all $n \in \mathbb{N}$ $\int_C \langle \xi, \psi(x) \rangle \, d\gamma_n = -\int_C \langle u, \operatorname{div} \psi(x) \rangle \, d\gamma_n$, one deduces that

$$\int_C \langle \xi, \psi(x) \rangle \, d\gamma = -\int_C \langle u, \operatorname{div} \psi(x) \rangle \, d\gamma.$$

Remark that the last equation is trivialy true for $\psi \equiv 0$ and Equation (62) holds for γ.

The map $C \ni (x, t, u, \xi) \mapsto f(\xi)$ is continuous and bounded below. Using Lemma A.2.2,

$$\int_C f(\xi)\gamma(dx, dt, du, d\xi) < \liminf \int_C f(\xi)\gamma_n(dx, dt, du, d\xi) < \infty$$

and Equation 59 is satisfied for γ.

□

We will further need the following Lemma.

Lemma 3.7.3 *Suppose $\{\gamma_n\}_{n=1}^{\infty}$ converges narrowly to γ. Then*

$$\lim_{m \to \infty} \int_C (F(x) \cdot u) \, \gamma_m = \int_C (F(x) \cdot u) \, \gamma$$

Proof. Let $\epsilon > 0$. There exists a continuous and bounded function $F_\epsilon : \Omega \to \mathbb{R}^d$ such that $\int_\Omega |F_\epsilon(x) - F(x)| dx < \epsilon$. One has

$$\begin{aligned}\int_C (F(x) \cdot u) \, \gamma_m - \int_C (F(x) \cdot u) \, \gamma = &\int_C (F(x) \cdot u) \, \gamma_m - \int_C (F_\epsilon(x) \cdot u) \, \gamma_m \\ &+ \int_C (F_\epsilon(x) \cdot u) \, \gamma_m - \int_C (F_\epsilon(x) \cdot u) \, \gamma \\ &+ \int_C (F_\epsilon(x) \cdot u) \, \gamma - \int_C (F(x) \cdot u) \, \gamma\end{aligned}$$

Now

$$\begin{aligned}\left| \int_C (F(x) \cdot u) \, \gamma_m - \int_C (F_\epsilon(x) \cdot u) \, \gamma_m \right| \leq &\int_C |F(x) - F_\epsilon(x)| \, r^* d\gamma_m \\ =& r^* \int_\Omega |F(x) - F_\epsilon(x)| \, dx \\ \leq& \epsilon r^*.\end{aligned}$$

Next, the map $C \ni (x,u,t,\xi) \mapsto F_\epsilon(x) \cdot u$ is continuous bounded. Hence, as $\{\gamma_n\}_{n=1}^\infty$ converges narrowly to γ, it holds that there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, one has

$$\left| \int_C (F_\epsilon(x) \cdot u)\, \gamma_m - \int_C (F_\epsilon(x) \cdot u)\, \gamma \right| \leq \epsilon.$$

Last,

$$\begin{aligned} \left| \int_C (F(x) \cdot u)\, \gamma - \int_C (F_\epsilon(x) \cdot u)\, \gamma \right| \leq & \int_C |F(x) - F_\epsilon(x)|\, r^* d\gamma \\ =& r^* \int_\Omega |F(x) - F_\epsilon(x)|\, dx \\ \leq & \epsilon r^*. \end{aligned}$$

Therefore

$$\lim_{m\to\infty} \int_C (F(x) \cdot u)\, \gamma_m = \int_C (F(x) \cdot u)\, \gamma.$$

□

We finish this section by proving the following existence of minimizer result.

Lemma 3.7.4 *The functional $\bar{I}$ achieves its minimum over Γ.*

Proof. Thanks to Lemma 2.1.2, there exists $\mathbf{u}_0 \in W^{1,p}(\Omega, \Lambda)$ an homeomorphism such that $\mathbf{u}_0(\Omega) = \Lambda$ and $\det \nabla \mathbf{u} > 0$. Set $\beta_0 = \det \nabla \mathbf{u}_0$. One has $\beta_0 \in \det^* \nabla \mathbf{u}_0$. Set $\gamma_0 = \gamma^{(\mathbf{u}_0, \beta_0)}$. Thanks to Lemma 3.7.1, one has $\gamma_0 \in \Gamma$. Set $c_0 = \bar{I}(\gamma_0)$. The sublevel set $\{\gamma \in \Gamma : \bar{I}(\gamma) \leq c_0\}$ is compact for the weak topology thanks to Lemma 3.7.2. Hence a minimizing sequence $\{\gamma_n\}_{n=1}^\infty \subset \{\gamma \in \Gamma : \bar{I}(\gamma) \leq c_0\}$ of $\bar{I}$ converges weakly to some $\bar{\gamma} \in \{\gamma \in \Gamma : \bar{I}(\gamma) < c_0\}$. Using Lemma A.2.2 and the fact that the function $C \ni (x,u,t,\xi) \mapsto f(\xi) + H(t)$ is lower semicontinuous and bounded below, one gets

$$\int_C f(\xi) + H(t) d\bar{\gamma} \leq \liminf_{n\to\infty} \int_C f(\xi) + H(t) d\gamma_n.$$

Moreover, using Lemma 3.7.3 one has $\lim_{m\to\infty} \int_C (F(x) \cdot u)\, \gamma_m = \int_C (F(x) \cdot u)\, \gamma$. Thus

$$\inf_{\gamma\in\Gamma} \bar{I}(\gamma) \leq \bar{I}(\bar{\gamma}) \leq \liminf_{n\to\infty} \bar{I}(\gamma_n) = \inf_{\gamma\in\Gamma} \bar{I}(\gamma),$$

and $\inf_{\gamma\in\Gamma} \bar{I}(\gamma) = \bar{I}(\bar{\gamma})$.

$\square$

CHAPTER IV

A DUALITY APPROACH

In this chapter we study problems that are dual to some of the problems studied in Chapter 3. For $\psi \in (L^q(\Omega))^{d\times d}$ we define $\operatorname{div}\psi_{\mathbb{R}^d}$ to be the distributional divergence of $\bar{\psi}$, the extension of ψ that takes the value 0 outside Ω. That is

$$\forall \varphi \in \left(C_c^\infty(\mathbb{R}^d)\right)^d, \int_{\mathbb{R}^d} \operatorname{div}\psi_{\mathbb{R}^d} \cdot \varphi = -\int_\Omega \psi \cdot \nabla\varphi,$$

or

$$\forall \varphi \in \left(C_c^\infty(\mathbb{R}^d)\right)^d, \int_{\overline{\Omega}} \operatorname{div}\psi_{\mathbb{R}^d} \cdot \varphi = -\int_\Omega \psi \cdot \nabla\varphi. \tag{65}$$

As $\operatorname{div}\psi_{\mathbb{R}^d}$ and ψ have compact support, Equation (65) holds for all $\varphi \in C^1(\mathbb{R}^d, \mathbb{R}^d)$.

Let $\mathscr{A}_0$ be the set of (k, l, ψ) such that $k : \mathbb{R}^d \to \bar{\mathbb{R}}$ Borel measurable, $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ Borel measurable,

$$k(\mathbf{v}) + tl(\mathbf{u}) + H(t) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d;\ \forall t > 0. \tag{66}$$

$\psi : \Omega \to \mathbb{R}^{d\times d}$, $\quad \psi \in (L^q(\Omega))^{d\times d}$ and $\operatorname{div}\psi_{\mathbb{R}^d}$ is a bounded Borel measure on $\bar{\Omega}$.

We suppose there exists $M_0 > 0$ such that $\|F\|_{L^1(\Omega,\mathbb{R}^{d\times d})} \leq M_0$.

The aim of this chapter is to study

$$\sup_{(k,l,\psi)\in\mathscr{A}_0} \left\{ -J(k,l,\psi) := -\int_{\overline{\Omega}} f^*(\psi)dx - \int_{\overline{\Omega}} k(\mathbf{F} + \operatorname{div}\psi_{\mathbb{R}^d}) - \int_\Lambda l dx \right\}. \tag{67}$$

4.1 An auxiliary problem

Let $S \subset (L^q(\Omega))^{d\times d}$ be a closed subspace such that for each element $\psi \in S$; the distributional derivative of $\bar{\psi}$ is a bounded measure concentrated on $\bar{\Omega}$.

Define $\mathcal{C}$ to be the set of couples (k, l) where $k : \mathbb{R}^d \to \mathbb{R}$ is Borel measurable, $l : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is Borel measurable and finite at least at one point and satisfy

$l \equiv \infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$

$$k(\mathbf{v}) + tl(\mathbf{u}) + H(t) \geq u \cdot v \quad \forall u, v \in \mathbb{R}^d; t > 0. \tag{68}$$

Let $\mathscr{A}$ be the set of (k, l, ψ) such that $(k, l) \in \mathcal{C}$ and $\psi \in S$. Consider:

$$\sup_{(k,l,\psi)\in\mathscr{A}} -J(k, l, \psi) := -\int_{\overline{\Omega}} f^*(\psi) - k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d}) - \int_\Lambda ldx. \tag{69}$$

4.1.1 Basic regularity properties of maximizers

Definition 4.1.1 *We define for* $l : \mathbb{R}^d \to (-\infty, \infty]$

$$l^{\#}(v) = \sup_{u\in\mathbb{R}^d, t>0} \{u \cdot v - l(\mathbf{u})t - H(t)\}.$$

For $k : \mathbb{R}^d \to [-\infty, \infty]$ *we define*

$$k_{\#}(\mathbf{u}) = -\inf_{v\in\mathbb{R}^d, t>0} \left\{ \frac{k(\mathbf{v}) + H(t) - u \cdot v}{t} \right\} = \sup_{v\in\mathbb{R}^d, t>0} \left\{ \frac{u \cdot v - k(\mathbf{v}) - H(t)}{t} \right\}.$$

We have

$$\begin{aligned} l^{\#}(v) &= \sup_{u\in\bar{\Lambda};\, t>0} \{u \cdot v - tl(u) - H(t)\} \\ &= \sup_{t>0} \sup_{u\in\bar{\Lambda}} \{u \cdot v - tl(u) - H(t)\} \\ &= \sup_{t>0} \{tl^*(\frac{v}{t}) - H(t)\}. \end{aligned}$$

But also

$$\begin{aligned} l^{\#}(v) &= \sup_{u\in\bar{\Lambda};\, t>0} \{u \cdot v - tl(u) - H(t)\} \\ &= \sup_{u\in\bar{\Lambda}} \sup_{t>0} \{u \cdot v - tl(u) - H(t)\} \\ &= \sup_{u\in\bar{\Lambda}} \{u \cdot v + H(-l(u))\} \\ &= [-H^*(-l)]^*(v). \end{aligned}$$

In a similar way,

$$\begin{aligned} k_{\#}(u) &= \sup_{v\in\mathbb{R}^d, t>0} \frac{u\cdot v - H(t) - k(v)}{t} \\ &= \sup_{t>0} \frac{k^*(u) - H(t)}{t}. \end{aligned}$$

Now, thanks to Lemma 3.1.2, H^* is invertible. Set $y = (H^*)^{-1}(-k^*(u))$. There exists $s > 0$ such that $H(s) + H(y) = sy$. Then for all $t > 0$ one has

$$\frac{H(t) + H^*(y)}{t} \geq y = \frac{H(s) + H^*(y)}{s}.$$

Thus

$$\sup_{t>0} \frac{k^*(u) - H(t)}{t} = -y = -(H^*)^{-1}(-k^*(u)).$$

We deduce that $k_{\#}(u) = -(H^*)^{-1}(-k^*(u))$.

Remark 4.1.2 *We have* $\left((l^{\#})_{\#}\right)^{\#} = l^{\#}$ *and* $\left((k_{\#})^{\#}\right)_{\#} = k_{\#}$.

Proof. We have by definition of $l^{\#}(v)$ that $l^{\#}(v) \geq -l(\mathbf{u})t - H(t) + u\cdot v$ for any $u \in \mathbb{R}^d$ and $t > 0$. So $(l^{\#}(v) + H(t) - u\cdot v)t^{-1} \geq -l(\mathbf{u})$ and

$$\inf_{v\in\mathbb{R}^d, t>0} \left\{ \frac{l^{\#}(v) + H(t) - u\cdot v}{t} \right\} \geq -l(\mathbf{u}).$$

That is $\left(l^{\#}\right)_{\#}(\mathbf{u}) \leq l(\mathbf{u})$. Therefore, $\left(H^{\#}\right)_{\#}^{\#}(v) \geq H^{\#}(v)$.

We also have

$$-\left(l^{\#}\right)_{\#}(\mathbf{u}) = \inf_{v\in\mathbb{R}^d, t>0} \left\{ \frac{l^{\#}(v) + H(t) - u\cdot v}{t} \right\} \leq \frac{l^{\#}(v) + H(t) - u\cdot v}{t}$$

for all $v \in \mathbb{R}^d$ and $t > 0$. So $-t\left(l^{\#}\right)_{\#}(\mathbf{u}) \leq l^{\#}(v) + H(t) - u\cdot v$. We deduce $-t\left(l^{\#}\right)_{\#}(\mathbf{u}) - H(t) + u\cdot v \leq l^{\#}(v)$ and $\left(l^{\#}\right)_{\#}^{\#}(v) \leq l^{\#}(v)$.

Therefore, $\left(l^{\#}\right)_{\#}^{\#}(v) = l^{\#}(v)$.

Next, we have

$$-k_{\#}(\mathbf{u}) = \inf_{v\in\mathbb{R}^d, t>0} \left\{ \frac{k(\mathbf{v}) + H(t) - u\cdot v}{t} \right\}$$

and hence $-tk_{\#}(\mathbf{u}) \leq k(\mathbf{v}) + H(t) - u \cdot v$ for any $v \in \mathbb{R}^d, t > 0$. Next $-tk_{\#}(\mathbf{u}) - H(t) + u \cdot v \leq k(\mathbf{v})$ for any $v \in \mathbb{R}, t > 0$. So $(k_{\#})^{\#}(\mathbf{u}) \leq k(\mathbf{u})$ and $\left((k_{\#})^{\#}\right)_{\#}(\mathbf{u}) \geq k_{\#}$. Furthermore we have for all $v \in \mathbb{R}, t > 0$

$$(k_{\#})^{\#}(\mathbf{u}) \geq -t(k_{\#})(\mathbf{u}) - H(t) + u \cdot v.$$

Hence

$$\frac{(k_{\#})^{\#}(\mathbf{u}) + H(t) - u \cdot v}{t} \geq -k_{\#}(\mathbf{u}).$$

That is $\left((k_{\#})^{\#}\right)_{\#}(\mathbf{u}) \geq -k_{\#}(\mathbf{u})$. Therefore, $\left((k_{\#})^{\#}\right)_{\#}(\mathbf{u}) = k_{\#}(\mathbf{u})$.

□

Remark 4.1.3 *If $(k, l, \psi) \in \mathscr{A}$, then $(l^{\#}, l, \psi) \in \mathscr{A}$ and $(k, (k)_{\#}, \psi) \in \mathscr{A}$. On has $J(l^{\#}, l, \psi) \leq J(k, l, \psi)$ and $J(k, (k)_{\#}, \psi) \leq J(k, l, \psi)$. Furthermore, one has*

$$\sup_{\mathscr{A}} -J(k, l, \psi) = \sup_{\mathscr{A}'} -J(k, l, \psi)$$

where $\mathscr{A}'$ is the subset of $\mathscr{A}$ whose elements (k, l, ψ) are such that $l = k_{\#}$ and $k = l^{\#}$.

Proof. Since

$$l^{\#}(v) = \sup_{u \in \mathbb{R}^d, t > 0} \{-tl(\mathbf{u}) - H(t) + u \cdot v\}$$

and $k(\mathbf{v}) \geq -tl(\mathbf{u}) - H(t) + u \cdot v$ for all $u, v \in \mathbb{R}^d$ and $t > 0$, we have $k \geq l^{\#}$. Next, $l^{\#}$ satisfies the relation $l^{\#}(v) \geq -tl(\mathbf{u}) - H(t) + u \cdot v$. Thus $(l^{\#}, l, \psi) \in \mathscr{A}$ and clearly $J(k, l, \psi) \geq J(l^{\#}, l, \psi)$.

Furthermore, since

$$-(k)_{\#}(\mathbf{u}) = \inf_{v \in \mathbb{R}^d, t > 0} \{\frac{k(\mathbf{v}) + H(t) - u \cdot v}{t}\} \geq -l(\mathbf{u}),$$

we get $k_{\#}(\mathbf{u}) \leq l(\mathbf{u})$. Moreover $k_{\#}$ satisfies the relation $k(\mathbf{v}) \geq -tk_{\#}(\mathbf{u}) - H(t) + u \cdot v$. We get them $(k, k_{\#}, \psi) \in \mathscr{A}$ and $J(k, l, \psi) \geq J(k, k_{\#}, \psi)$.

Let $(k, l, \psi) \in \mathscr{A}$. We have $(k, k_{\#}, \psi) \in \mathscr{A}$ and $((k_{\#})^{\#}, k_{\#}, \psi) \in \mathscr{A}$. Set $l_0 = k_{\#}$ and $k_0 = (k_{\#})^{\#}$. One has

$$(k_0)_{\#} = ((k_{\#})^{\#})_{\#} = k_{\#} = l_0$$

thanks to Remark 4.1.2 and $(l_0)^{\#} = k_0$. Thus $(k_0, l_0, \psi) \in \mathscr{A}'$ and $J(k_0, l_0, \psi) \leq J(k, l, \psi)$. So

$$\sup_{\mathscr{A}} -J(k, l, \psi) = \sup_{\mathscr{A}'} -J(k, l, \psi).$$

□

4.1.2 Coercivity properties of J

Lemma 4.1.4 *Consider a sequence $\{(k_n, l_n, \psi_n)\}_n \subset \mathscr{A}'$ such that for some $C \in \mathbb{R}$ one has $J(k_n, l_n, \psi_n) \leq C$. Then there exist some constants α_C and β_C such that for all $n \in \mathbb{N}$,*

$$\alpha_C \leq \inf_{u \in \Lambda} l_n(\mathbf{u}) \leq \beta_C. \tag{70}$$

Moreover, this constants depends only on Ω, Λ, H and any constant M_0 satisfying $\|F\|_{L^1(\Omega, \mathbb{R}^d)} \leq M_0$.

Proof. Define for all $n \in \mathbb{N}$ $s_n := \sup_{u \in \bar{\Lambda}} -l_n(\mathbf{u}) = -\inf_{u \in \bar{\Lambda}} l_n(\mathbf{u})$. As $\overline{\Lambda}$ is bounded, l_n is lower semicontinuous, we can find some $u_n \in \overline{\Lambda}$ such that $l_n(u_n) = s_n$ Now, for all $v \in \mathbb{R}^d$, $u \in \Lambda$, $t > 0$, one has

$$k_n(v) \geq -t l_n(\mathbf{u}) - H(t) + u \cdot v.$$

Thus the following successively hold:

$$\begin{aligned} k_n(v) \geq & -t l_n(u_n) - H(t) + u_n \cdot v \quad \forall t > 0 \\ k_n(v) \geq & t s_n - H(t) + u_n \cdot v \quad \forall t > 0 \\ k_n(v) \geq & H^*(s_n) + u_n \cdot v. \end{aligned}$$

Using the last Inequality, one gets

$$\begin{aligned} \int_{\overline{\Omega}} k_n(F + \operatorname{div}(\psi_n)_{\mathbb{R}^d}) \geq & \int_{\overline{\Omega}} H^*(s_n) + u_n \cdot (F + \operatorname{div}(\psi_n)_{\mathbb{R}^d}) \\ \geq & \mathcal{L}^d(\Omega) H^*(s_n) + \int_{\overline{\Omega}} u_n \cdot F + \int_{\overline{\Omega}} u_n \cdot \operatorname{div}(\psi_n)_{\mathbb{R}^d} \end{aligned}$$

But

$$\int_{\Omega} u_n \cdot F \geq -\int_{\Omega} |u_n|.|F| \geq -|u_n|.\|F\|_{L^1(\Omega)} \geq -r^* \|F\|_{L^1(\Omega)}.$$

Furthermore

$$\int_{\Omega} u_n \cdot \mathrm{div}\,(\psi_n)_{\mathbb{R}^d} = -\int_{\Omega} \nabla(u_n) \cdot \psi_n = 0$$

since u_n is a constant. Hence:

$$\int_{\Omega} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) \geq \mathcal{L}^d(\Omega) H^*(s_n) - r^* \|F\|_{L^1(\Omega)}. \tag{71}$$

Thanks to Inequality (42), f^* is bounded below so there exists $A_f \in \mathbb{R}$ such that

$$\int_{\Omega} f^*(\psi_n) dx \geq A_f. \tag{72}$$

We also have $\int_{\Lambda} l_n dx \geq -\mathcal{L}^d(\Lambda) s_n$. Then

$$\begin{aligned} J(k_n, l_n, \psi_n) &= \int_{\Omega} f^*(\psi_n) dx + \int_{\Omega} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) + \int_{\Lambda} l_n dx \\ &\geq A_f + \mathcal{L}^d(\Omega) H^*(s_n) - \mathcal{L}^d(\Lambda) s_n - r^* \|F\|_{L^1(\Omega)}, \end{aligned}$$

and

$$-C \leq -A_f - \mathcal{L}^d(\Omega) H^*(s_n) + \mathcal{L}^d(\Lambda) s_n + r.\|F\|_{L^1(\Omega)}$$

Thanks to Lemma 3.1.2, the sequence $\{s_n\}_{n=0}^{\infty}$ must be bounded, and this is equivalent to the boundedness of $\inf_{u \in \Lambda} l_n(\mathbf{u})$. Thus there exist some constants α_C and β_C such that for all $n \in \mathbb{N}$,

$$\alpha_C \leq \inf_{u \in \Lambda} l_n(\mathbf{u}) \leq \beta_C.$$

□

Lemma 4.1.5 *Consider a sequence $\{(k_n, l_n, \psi_n)\}_n \subset \mathscr{A}'$ such that for some $C \in \mathbb{R}$ one has $J(k_n, l_n, \psi_n) \leq C$. There exists $M, b \in \mathbb{R}$; $a > 0$ such that for all $n \in \mathbb{N}$:*

1. *Lip $k_n \leq r^*\ \Lambda$. Moreover there exists $e > 0$ such that for all $n \in \mathbb{N}$ and for all $v \in \mathbb{R}^d$, one has $k(\mathbf{v}) \leq r^*|v| + e$.*

2. *One has $k_n(0)$, $\int_{\Lambda} |l_n(\mathbf{u})|$, $\int_{\Omega} |\psi_n|^q$, $\int_{\Omega} |\mathrm{div}\,(\psi_n)_{\mathbb{R}^d}| < M$.*

3. *One has $k_n(v) \geq a|v| + b$.*

Proof. (i) **Let us show that there exists** $M_1 \in \mathbb{R}$ **such that for all** $n \in \mathbb{N}$**,** $k_n(0) < M_1$.

We have $k_n(0) = \sup_{u \in \mathbb{R}^d, t>0}\{-tl_n(\mathbf{u}) - H(t)\}$. Let

$$s_0 := \sup_n \left(\sup_{u\in\Lambda} -l_n(\mathbf{u}) \right) := \sup_n (s_n).$$

Using (70) we have $-\beta_C \leq s_n \leq -\alpha_C, \quad \forall n \in \mathbb{N}$ for some reals α_C and β_C. Thus s_0 is finite. We have next

$$\sup_{u\in\mathbb{R}^d, t>0} \{-tl_n(\mathbf{u}) - H(t)\} \leq \sup_t \{ts_0 - H(t)\} = H^*(s_0) < \infty,$$

where we got the last inequality from Lemma 3.1.2. Thus there exists $M_1 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $k_n(0) < M_1$.

(ii) **Let us show that** $Lip\ k_n \leq r^*$**.**

We have $l_n(\mathbf{u}) \geq \alpha_C$ and thus

$$\begin{aligned} k_n(v) = l_n^{\#}(v) &= \sup_{u\in\bar{\Lambda},\, t>0} \{u \cdot v - l(\mathbf{u})t - H(t)\} \\ &\leq \sup_{u\in\bar{\Lambda},\, t>0} \{u \cdot v - \alpha_C t - H(t)\} \\ &\leq r^*|v| + H^*(-\alpha_C). \end{aligned}$$

Let $v_1, v_2 \in \mathbb{R}^d$. Let $\epsilon > 0$.

There exists some u_1, t_1 such that $k_n(v_1) - \epsilon \leq -l_n(u_1)t_1 - H(t_1) + u_1 v_1$. But $-l_n(u_1)t_1 - H(t_1) + u_1 v_2 \leq k(v_2)$. Thus $k_n(v_1) - k_n(v_2) \leq u_1 v_1 - u_1 v_2 + \epsilon$. So $k_n(v_1) - k_n(v_2) \leq |v_1 - v_2| r^* + \epsilon$.

Taking ϵ going to 0 we get $k_n(v_1) - k_n(v_2) \leq |v_1 - v_2| r^*$. The inequality $k_n(v_1) - k_n(v_2) \leq |v_1 - v_2| r^*$ occurs when we switch v_1 and v_2.

Thus $Lip\ k_0 \leq r^*$.

(iii) Let us show that there exists $M_2 \in \mathbb{R}$ **such that for all** $n \in \mathbb{N}$**,** $\int_\Lambda |l_n(\mathbf{u})| < M_2$**.**

Using inequality (70) we have $-\beta_C \leq s_n \leq -\alpha_C, \quad \forall n \in \mathbb{N}$. Since inequality (71) gives:

$$\int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) \geq \mathcal{L}^d(\Omega) H^*(s_n) - r^* \|F\|_{L^1(\Omega)},$$

One has:

$$\int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) \geq \mathcal{L}^d(\Omega) H^*(-\beta_C) - r^* \|F\|_{L^1(\Omega)} := A_k$$

as H^* is nondecreasing by Lemma 3.1.2. Thus

$$\begin{aligned} C \geq J(k_n, l_n, \psi_n) &= \int_{\overline{\Omega}} f^*(\psi_n) dx + \int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) + \int_{\Lambda} l_n dx \\ &\geq A_f + A_k + \int_{\Lambda} l_n dx, \end{aligned}$$

and $\int_\Lambda l_n dx \leq C - (A_f + A_k)$. Moreover

$$\int_{\Lambda} l_n dx \geq \int_{\Omega} \inf_{u \in \Omega} l_n(\mathbf{u}) dx \geq \alpha_C \mathcal{L}^d(\Lambda).$$

Thus there exists $M_2 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\int_\Lambda |l_n(\mathbf{u})| < M_2$.

(iv) **Let us show that there exists $M_3 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\int_\Omega |\psi_n|^q < M_3$.**

One has:

$$\begin{aligned} C \geq J(k_n, l_n, \psi_n) &= \int_{\overline{\Omega}} f^*(\psi_n) dx + \int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) + \int_{\Lambda} l_n dx \\ &\geq -M_2 + A_k + \int_{\overline{\Omega}} f^*(\psi_n) dx \\ &\geq -M_2 + A_k + \int_{\overline{\Omega}} c_1(|\psi_n|^q) dx - c_1 \mathcal{L}^d(\Omega), \end{aligned}$$

where we have used inequality (42). Hence, there exists $M_3 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\int_\Omega |\psi_n|^q < M_3$.

(v) **Let us prove that there exists $b \in \mathbb{R}$, $a > 0$ such that for all $n \in \mathbb{N}$, $v \in \mathbb{R}^d$, $k_n(v) \geq a|v| + b$.**

Since Λ is open and contains the origin, there exists $a > 0$ such that $B(0, 4a) \subset \Lambda$. Suppose $|v| \neq 0$. We have :

$$k_n(v) \geq -l_n(-a\frac{v}{|v|})f - H(1) + v \cdot a\frac{v}{|v|} \geq l_n(-a\frac{v}{|v|}) - H(1) + a|v|.$$

Since l_n is finite on Λ we can find a constant c_d depending only on d such that

$$\sup_{B(0,2a)} |l_n| \leq \frac{c_d}{\mathcal{L}^d(B(0,4a))} \int_{B(0,4a)} |l_n| \quad \forall n \in \mathbb{N}$$

(See. for instance [11, Evans-Gariepy]). Thus

$$\sup_{S(0,a)} |l_n| \leq \frac{c_d}{\mathcal{L}^d(B(0,4a))} \int_{\Lambda} |l_n| \leq \frac{c_d}{\mathcal{L}^d(B(0,4a))} M_2.$$

So $k_n(v) \geq b + a|v|$ where $b = H(1) - \frac{c_d}{\mathcal{L}^d(B(0,4a))} M_2$. If $v = 0$, we have $k_n(0) \geq -l_n(0) - H(1) - 0$ and $k_n(0) \geq b = a|0| + b$. Thus there exists $b \in \mathbb{R}$, $a > 0$ such that for all $n \in \mathbb{N}$, $v \in \mathbb{R}^d$, $k_n(v) \geq a|v| + b$.

(vi) **Let us show that there exists $M_4 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\int_\Omega |\mathrm{div}\,(\psi_n)_{\mathbb{R}^d}| < M_4$.**

We have:

$$\begin{aligned} C \geq J(k_n, l_n, \psi_n) &= \int_{\overline{\Omega}} f^*(\psi_n) dx + \int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) + \int_{\Lambda} l_n dx \\ &\geq A_f + \alpha_C \mathcal{L}^d(\Lambda) + \int_{\overline{\Omega}} k_n(F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}) \\ &\geq A_f + \alpha_C \mathcal{L}^d(\Lambda) + \int_{\overline{\Omega}} a|F + \mathrm{div}\,(\psi_n)_{\mathbb{R}^d}| + b \\ &\geq A_f + \alpha_C \mathcal{L}^d(\Lambda) + b\mathcal{L}^d(\Omega) - a \int_{\overline{\Omega}} |F| dx + \int_{\overline{\Omega}} a|\mathrm{div}\,(\psi_n)_{\mathbb{R}^d}|. \end{aligned}$$

Thus there exists $M_4 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $\int_\Omega |\mathrm{div}\,(\psi_n)_{\mathbb{R}^d}| < M_4$.

□

Lemma 4.1.6 *Consider a sequence $\{(k_n, l_n, \psi_n)\}_n \subset \mathscr{A}'$ such that the consequences of Lemma 4.1.5 hold. Then*

1. *There exists a function $k : \mathbb{R}^d \to \mathbb{R}$ and a subsequence of $\{k_n\}_{n=1}^\infty$ that converges to k locally uniformly.*

2. *There exists a function $l : \Lambda \to \mathbb{R}$ and a subsequence of $\{l_n\}_{n=1}^\infty$ that converges to l locally uniformly.*

3. *There exists* $\psi \in (L^p(\Omega))^{d\times d}$ *and a bounded Borel measure* ξ *such that a subsequence of* $\{\psi_{n_m}\}_m$ *converges weakly to* ψ *in* $(L^p(\Omega))^{d\times d}$ *and* $\{\operatorname{div}(\psi_{n_m})_{|\mathbb{R}^d}\}_m$ *converges weakly to* ξ *(in measures) and* $\operatorname{div}\psi_{\mathbb{R}^d} = \xi$.

Proof. **Proof of (i)**. Let B_N denote the open ball centered at the origin of radius N. $\{k_n\}_{n=1}^{\infty}$ is bounded pointwise. Indeed, there exists $M_1 \in \mathbb{R}$ such that for all $n \in \mathbb{N}$, $k_n(0) < M_1$ and for all $n \in \mathbb{N}$, *Lip* $k_n \leq r^*$. Then for all $n \in \mathbb{N}$ and $v \in B_N$, we have:

$$|k_n(v)| \leq |k_n(0)| + (r^*)|v| \leq M_1 + (r^*)|v|.$$

Next $\{k_n\}_{n=1}^{\infty}$ is equicontinuous. To see this, let $\epsilon > 0$ and take $\delta = \frac{\epsilon}{r^*}$. For $u, v \in \mathbb{R}^d$ such that $|u - v| \leq \delta$ we have:

$$|k_n(\mathbf{u}) - k_n(v)| \leq (r^*)|u - v| \leq (r^*)\frac{\epsilon}{r^*} = \epsilon.$$

Using Ascoli-Arzela's theorem, there exists a continuous function $\bar{k}$, and a subsequence $\{k_{n_m}\}_{m=1}^{\infty}$ such that $k_{n_m} \to \bar{k}$ in $\mathbb{R}^d$ locally uniformly.

Proof of (ii). Let

$$K_m = \{x \in \Lambda : d(x, \partial\Lambda) \geq \frac{1}{m}\}.$$

Assume that $m \geq m_0$ where m_0 is chosen in a way that K_{m_0} is nonempty. K_m is compact. Since for all $n \in \mathbb{N}$, l_n is convex and $\int_\Lambda |l_n| < M_2$, there exists a constant c depending only on d, K_m and M_2 such that $\|l_n\|_{W^{1,\infty}(K_m)} \leq c$. Since $W^{1,\infty}(K_m)$ is embedded in $C^{0,1}(K_m)$ we get that the $\mathrm{Lip}_{K_m}(l_n)$ is bounded say by some $M > 0$ and so $\{l_n\}_{n=1}^{\infty}$ is equicontinuous on K_m. The sequence $\{l_n\}_{n=1}^{\infty}$ is also equi-bounded on K_m thus using Ascoli-Arzela's theorem, there exists a subsequence of $\{l_n\}_{n=1}^{\infty}$ that converges uniformly to some function $l^{(m)}$ on K_m.

For $m = m_0 + 1$, construct a subsequence $\{l_{m_0+1,n}\}_n$ of $\{l_n\}_{n=1}^{\infty}$ such that $l_{m_0+1,n} \to l^{(m_0+1)}$ in K_{m_0+1}.

For $m = m_0+2$, construct a subsequence $\{l_{m_0+2,n}\}_n$ of $\{l_{m_0+1,n}\}_n$ such that $l_{m_0+2,n} \to$

$l^{(m_0+2)}$ in K_{m_0+2}.

Proceed that way for all $m > m_0$ and construct a subsequence $\{l_{m,n}\}_n$ of $\{l_{m-1,n}\}$ such that $l_{m,n} \to l^{(m)}$ in K_m.

Remark that if $m' \geq m$, $K_m \subset K_{m'}$ and since $l_{m,n} \to l^{(m)}$ in K_m uniformly and $l_{m',n} \to l^{(m')}$ in $K_{(m')}$ uniformly, we have $l^{(m')} = l^{(m)}$ on K_m.

Now let $x \in \Lambda$. Let m_1 such that $x \in K_{m_1}$. For $m > m_1$, $l_{(m)}(x) = l_{(m_1)}(x)$. So $\{l_{(m)}(x)\}$ converges. Thus $\{l_{(m)}\}$ converges pointwise to some function l and therefore $l = l^{(m)}$ on K_m.

Thus there exists a function $l : \Lambda \to \mathbb{R}$ such that $l_{n,n} \to l$ locally uniformly.

Proof of (iii). Using Lemma 4.1.5, there exists $\psi \in (L^p(\Omega))^{d\times d}$ and a bounded Borel measure ξ such that a subsequence $\{\psi_{n_m}\}_m$ of $\{\psi_n\}_n$ converges weakly to ψ in $(L^p(\Omega))^{d\times d}$ and $\{\operatorname{div}(\psi_{n_m})_{|\mathbb{R}^d}\}_m$ converges weakly to ξ (in measures). Let us prove $\operatorname{div}\psi_{\mathbb{R}^d} = \xi$.

For all $n \in \mathbb{N}$ and all $\varphi \in \left(C_c^\infty(\mathbb{R}^d)\right)^d$ we have:

$$\int_{\overline{\Omega}} \varphi \cdot \operatorname{div}(\psi_n)_{\mathbb{R}^d} = -\int_\Omega \psi_n \cdot \nabla\varphi.$$

Since $\{\psi_{n_m}\}_m$ converges weakly to ψ in $(L^q(\Omega))^{d\times d}$, we have

$$\lim_{m\to\infty} -\int_\Omega \psi_{n_m} \cdot \nabla\varphi = -\int_\Omega \psi \cdot \nabla\varphi.$$

Since $\{\operatorname{div}(\psi_{n_m})_{|\mathbb{R}^d}\}_m$ converges weakly to ξ, we have

$$\lim_{m\to\infty} \int_{\overline{\Omega}} \varphi \cdot \operatorname{div}(\psi_{n_m})_{|\mathbb{R}^d} = \int_{\overline{\Omega}} \varphi \cdot \xi.$$

Hence $\int_{\overline{\Omega}} \varphi \cdot \xi = -\int_\Omega \psi \cdot \nabla\varphi$. Thus $\operatorname{div}\psi = \xi$.

□

Lemma 4.1.7 *Consider a sequence* $\{(k_n, l_n, \psi_n)\}_n \subset \mathscr{A}'$ *such that for some* $C \in \mathbb{R}$*, one has* $J(k_n, l_n, \psi_n) \leq C$*. Then there exist* $(k, l, \psi) \in \mathscr{A}'$*; a subsequence of* $\{(k_n, l_n, \psi_n)\}_n$ *denoted* $\{(k_{n_m}, l_{n_m}, \psi_{n_m})\}_m$ *such that* $J(k, l, \psi) \leq \liminf_m J(k_{n_m}, l_{n_m}, \psi_{n_m})$.

Proof. Thanks to Lemma(4.1.6), there exists a subsequence of $\{(k_n, l_n, \psi_n)\}_n$ that we denote again $\{(k_{n_m}, l_{n_m}, \psi_{n_m})\}_m$ such that for some function $k : \mathbb{R}^d \to \mathbb{R}$, $\{k_{n_m}\}_{m=1}^{\infty}$ converges to k locally uniformly; for some function $l : \Lambda \to \mathbb{R}$, $\{l_{n_m}\}_{m=1}^{\infty}$ converges to l locally uniformly; for some $\psi \in (L^p(\Omega))^{d\times d}$ and a bounded Borel measure ξ, $\{\psi_{n_m}\}_m$ converges weakly to ψ in $(L^p(\Omega))^{d\times d}$ and $\{\operatorname{div}(\psi_{n_m})_{|\mathbb{R}^d}\}_m$ converges weakly to ξ in measures and $\operatorname{div}\psi_{\mathbb{R}^d} = \xi$. Remark that since S is closed, we have $\psi \in S$. We also have $(k, l) \in \mathcal{C}$.

Let us prove $J(k, l, \psi) \leq \liminf_m J(k_{n_m}, l_{n_m} \psi_{n_m})$.

We recall that

$$J(k_{n_m}, l_{n_m}, \psi_{n_m}) = \int_{\Omega} f^*(\psi_{n_m})dx + \int_{\Omega} k_{n_m}(F + \operatorname{div}\psi_{n_m}) + \int_{\Lambda} l_{n_m} dx.$$

We have $k^*_{n_m}(v) \geq -k_{n_m}(0)$ and thanks to Lemma 4.1.5 , there exists $M_1 > 0$ such that for all $m \in \mathbb{N}^*$, one has $-k_{n_m}(0) \geq -M$. It holds also that for all $m \in \mathbb{N}^*$, since $(k_{n_m})_{\#} = l_{n_m}$, we have $k^*_{n_m}(\mathbf{u}) + H(1) \leq l_{n_m}(\mathbf{u})$ for all $u \in \mathbb{R}^d$. Using Lemma 4.1.5 one more time, there exists $M_2 > 0$ such that for all $m \in \mathbb{N}^*$ one has $\int_{\Lambda} |l_{n_m}|dx < M_2$. We deduce that there exists $M_3 > 0$ such that for all $m \in \mathbb{N}^*$, $\overline{\text{dom } k^*_{n_m}} = \bar{\Lambda}$ and $\int_{\Omega} |k^*_{n_m}|dx < M$. We are now in position to use Lemma 2.4.10 to deduce that

$$\int_{\Omega} k(\operatorname{div}\psi_{\mathbb{R}^d} + \mathbf{F}) \leq \liminf_m \int_{\Omega} k_n(\operatorname{div}\psi_{n_m} + \mathbf{F}).$$

Furthermore, Since f^* is convex and finite, the functional

$$(L^p(\Omega))^{d\times d} \ni u \mapsto \int_{\Omega} f^*(\mathbf{u})dx$$

is is weakly lower semicontinuous. Hence:

$$\int_{\Omega} f^*(\psi) \leq \liminf_m \int_{\Omega} f^*(\psi_{n_m}).$$

Now, using Lemma (4.1.4), there exists $\alpha_C \in \mathbb{R}$ such that $\alpha_C \leq l_{n_m}(\mathbf{u})$ for all

$u \in \Lambda$ and $n \in \mathbb{N}$, we may apply Fatou's lemma to $l_{n_m} - \alpha_C$ and get

$$
\begin{aligned}
\int_\Lambda \liminf_m (l_{n_m}(\mathbf{u}) - \alpha_C) &\leq \liminf_m \int_\Lambda (l_{n_m}(\mathbf{u}) - \alpha_C) \\
\int_\Lambda l(\mathbf{u}) - \int_\Lambda \alpha_C &\leq \liminf_m \int_\Lambda l_{n_m}(\mathbf{u}) - \int_\Lambda \alpha_C \\
\int_\Lambda l(\mathbf{u}) &\leq \liminf_m \int_\Lambda l_{n_m}(\mathbf{u}).
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
J(k, l, \psi) &= \int_{\overline{\Omega}} f^*(\psi)dx + \int_{\overline{\Omega}} k(F + \operatorname{div}\psi) + \int_\Lambda l dx \\
&\leq \liminf_m \left[\int_{\overline{\Omega}} f^*(\psi_{n_m})dx + \int_{\overline{\Omega}} k_{n_m}(F + \operatorname{div}(\psi_n)_{\mathbb{R}^d}) + \int_\Lambda l_{n_m} dx \right] \\
&= \liminf_m J(k_{n_m}, l_{n_m}, \psi_{n_m}).
\end{aligned}
$$

We have $(k, k_\#, \psi) \in \mathscr{A}$ and $((k)_\#{}^\#, k_\#, \psi) \in \mathscr{A}$. Set $\bar{l} = k_\#$ and $\bar{k} = (k)_\#{}^\#$. One has $\bar{k}_\# = (k_\#{}^\#)_\# = k_\# = \bar{l}$ and $\bar{l}^\# = \bar{k}$. Thus $(\bar{k}, \bar{l}, \psi) \in \mathscr{A}'$ and $J(\bar{k}, \bar{l}, \psi) \leq J(k, l, \psi)$. Set $\bar{\psi} = \psi$. $(\bar{k}, \bar{l}, \bar{\psi}) \in \mathscr{A}'$ and $-J(\bar{k}, \bar{l}, \bar{\psi}) \leq -J(k, l, \psi) \leq \liminf_m J(k_{n_m}, l_{n_m}, \psi_{n_m})$.

□

4.1.3 An existence result

Proposition 4.1.8 *There exists $(\bar{k}, \bar{l}, \bar{\psi}) \subset \mathscr{A}'$ such that*

$$
-J(\bar{k}, \bar{l}, \bar{\psi}) = \sup_{\mathscr{A}} -J(k, l, \psi)
$$

Proof. Remark that for all $u \in \bar{\Lambda}$ and $v \in \mathbb{R}^d$, $u \cdot v \leq r^*|v|$ and for all $t > 0$, $0 \leq t + H(t) + H^*(-1)$. Thus for all $u \in \bar{\Lambda}$, $v \in \mathbb{R}^d$ and for all $t > 0$

$$
u \cdot v \leq t + H(t) + H^*(-1) + r^*|v|,
$$

and if we take $l_0 = 1 + \chi_{\bar{\Lambda}}$, $k_0(v) = H^*(-1) + r^*|v|$ for all $v \in \mathbb{R}^d$ and $\psi_0 = 0$ then $(k_0, l_0, \psi_0) \in \mathscr{A}$. Set $J(k_0, l_0, \psi_0) = C$. Thanks to Remark (4.1.3), we can find a sequence $\{(k_n, l_n, \psi_n)\}_n \subset \mathscr{A}'$ such that $J(k_n, l_n, \psi_n) \leq C$ and $\lim_{n\to\infty} -J(k_n, l_n, \psi_n) =$

$\sup_{\mathscr{A}'} -J(k,l,\psi)$. We next use Lemma 4.1.7 to deduce that there exist $(\bar{k},\bar{l},\bar{\psi}) \in \mathscr{A}'$ and a subsequence of $\{(k_n,l_n,\psi_n)\}_n$ denoted $\{(k_{n_m},l_{n_m},\psi_{n_m})\}_m$ such that

$$J(\bar{k},\bar{l},\bar{\psi}) \leq \liminf_m J(k_{n_m},l_{n_m},\psi_{n_m}) = \sup_{\mathscr{A}'} -J(k,l,\psi).$$

Thus $J(\bar{k},\bar{l},\bar{\psi}) = \sup_{\mathscr{A}'} -J(k,l,\psi)$.

4.1.4 Additional results

Lemma 4.1.9 *Suppose a lower semicontinuous function $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ is such that $\inf_{\bar{\Lambda}} l \geq \alpha$ and l is finite on $\bar{\Lambda}$, $l \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and $k = l^{\#}$. For $v \in \mathbb{R}^d$ there exists (u_0,t_0) such that $u_0 \in \bar{\Lambda}$, $t_0 > 0$ and*

$$k(\mathbf{v}) = -t_0 l(u_0) - H(t_0) - u_0 \cdot v.$$

Proof. Let $v \in \mathbb{R}^d$. We have $k(\mathbf{v}) = \sup_{u\in\bar{\Lambda},t>0} \{-tl(\mathbf{u}) - H(t) + u \cdot v\}$. Consider $(u_n,t_n) \in \bar{\Lambda} \times (0,\infty)$ such that $\lim_{n\to\infty} \{-t_n l_n(\mathbf{u}) - H(t_n) - u_n \cdot v\} = k(\mathbf{v})$. One has

$$-t_n l(u_n) - H(t_n) - u_n \cdot v \leq -t_n\alpha - H(t_n) + |v|.r^*.$$

Assume that a subsequence of $\{t_n\}_{n=1}^\infty$ called $\{t_{n_m}\}_{m=1}^\infty$ converges to ∞. In that case,

$$\lim_{m\to\infty} -t_{n_m}\alpha - H(t_{n_m}) + |v|.r^* = -\infty$$

which contradict the fact that $k(\mathbf{v}) \geq l(0) - H(1)$.

In the same manner, no subsequences of $\{t_n\}_{n=1}^\infty$ go to 0^+. Thus t_n stays in a closed bounded interval of $(0,\infty)$. Since $u_n \in \bar{\Lambda}$ we may then find a subsequence $\{(u_{n_m},t_{n_m})\}_m$ of $\{(u_n,t_n)\}_n$ converging to $(u_0,t_0) \in \bar{\Lambda} \times (0,\infty)$. Using the continuity of H and the fact that l is lower semicontinuous ,

$$k(\mathbf{v}) = \lim_{m\to\infty} -t_{n_m} l(u_{n_m}) - H(t_{n_m}) - u_{n_m} \cdot v \leq -t_0 l(u_0) - H(t_0) - u_0 \cdot v \leq k(\mathbf{v}).$$

Thus $-t_0 l(u_0) - H(t_0) - u_0 \cdot v = k(\mathbf{v})$.

$\square$

Lemma 4.1.10 *Suppose a lower semicontinuous function* $l_0 : \mathbb{R}^d \to \bar{\mathbb{R}}$ *is such that* $\inf_{\bar{\Lambda}} l_0 \geq \alpha$ *and* l_0 *is finite on* Λ, $l_0 \equiv +\infty$ *on* $\mathbb{R}^d \setminus \bar{\Lambda}$ *and* $k = l_0{}^{\#}$. *For all* $v \in \mathbb{R}^d$ *such that* k *is differentiable at* v:

1. *There exist unique* $u_0 \in \bar{\Lambda}$, $t_0 > 0$ *such that*

$$k(v) = -t_0 l(u_0) - H(t_0) - u_0 \cdot v.$$

   In addition, $u_0 = \nabla k(v)$ *and* $H'(t_0) + l(u_0) = 0$.

2. *Moreover, let* $l \in C_b(\mathbb{R}^d)$ *and let* $1 \geq \epsilon > 0$. *Define* $l_\epsilon = l_0 + \epsilon l$ *and* $k_\epsilon(v) = (l_\epsilon)^{\#}$.

   (a) *There exist a constant* M *independent of* v *and* ϵ *such that ,*

$$\left| \frac{k_\epsilon(v) - k(v)}{\epsilon} \right| \leq M.$$

   (b) *We have*

$$\lim_{\epsilon \to 0} \frac{k_\epsilon(v) - k(v)}{\epsilon} = -t_0 l(u_0).$$

Proof. **(i)** Thanks to Lemma (4.1.9) there exists (u_0, t_0) such that $u_0 \in \bar{\Lambda}$, $t_0 > 0$ and $k(v) = -t_0 l(u_0) - H(t_0) - u_0 \cdot v$. Remark that we have for $u \in \bar{\Lambda}$ and $t > 0$,

$$\begin{cases} k(v) + t l_0(u) + H(t) - u \cdot v & \geq & 0 \\ k(v) + t_0 . l_0(u_0) + H(t_0) - u_0 \cdot v & = & 0 \end{cases}$$

we get taking the partial derivative with respect to t of $k(v) + t l_0(u) + H(t) - u \cdot v$:

$$l_0(u_0) + H'(t_0) = 0. \tag{73}$$

Since k is differentiable at v, taking the partial derivative with respect to v of $k(v) + t l_0(u) + H(t) - u \cdot v$, we get :

$$\nabla k(v) = u_0. \tag{74}$$

Equality (74) tells us that u_0 is uniquely defined. As u_0 is uniquely defined, Equality (73) tells us that t_0 is uniquely defined as $H' : (0, \infty) \to \mathbb{R}$ is a bijection (Lemma

(3.1.1)). Thus there exist unique $u_0 \in \bar{\Lambda}$, $t_0 > 0$ such that

$$k(v) = -t_0 l(u_0) - H(t_0) - u_0 \cdot v.$$

(ii.a) Set $k_0 = k$. Remarking that for all $\epsilon \geq 0$, $\inf_{\bar{\Lambda}} l_\epsilon \geq \alpha - |l|_\infty$, we use Part (i) to deduce that for all $v \in \mathbb{R}^d$, there exists unique $(u_\epsilon, t_\epsilon) \in \bar{\Lambda} \times (0, \infty)$ such that

$$k_\epsilon(v) = \sup_{u \in \bar{\Lambda}, t > 0} -f l_\epsilon(u) - H(t) + u \cdot v = -t_\epsilon l_\epsilon(u_\epsilon) - H(t_\epsilon) + u_\epsilon \cdot v,$$

$u_\epsilon \in \partial k_\epsilon(v)$ and $t_\epsilon = (H')^{-1}(-l_\epsilon(u_\epsilon))$.

We have $-l_\epsilon(u_\epsilon) \leq -\alpha + |l|_\infty$. Hence there exists M_1 depending only on α and $|l|_\infty$ such that for all $\epsilon > 0$, one has $t_\epsilon \leq M_1$.

We have:

$$\begin{aligned} k_\epsilon(v) &= -t_\epsilon l_\epsilon(u_\epsilon) - H(t_\epsilon) + u_\epsilon \cdot v \\ &= -t_\epsilon \bar{l}(u_\epsilon) - H(t_\epsilon) + u_\epsilon \cdot v - \epsilon t_\epsilon l(u_\epsilon) \\ &\leq k(v) - \epsilon t_\epsilon l(u_\epsilon) \end{aligned}$$

So for $\epsilon > 0$

$$\frac{k_\epsilon(v) - k(v)}{\epsilon} \leq -t_\epsilon l(u_\epsilon) \tag{75}$$

and

$$\frac{k_\epsilon(v) - k(v)}{\epsilon} \leq |l|_\infty M_1. \tag{76}$$

We also have

$$\begin{aligned} k(v) &= -t_0 . \bar{l}(u_0) - H(t_0) + u_0 \cdot v \\ &= -t_0 \bar{l}(u_0) - \epsilon t_0 l(u_0) - H(t_0) + u_0 \cdot v + \epsilon t_0 l(u_0) \\ &= -t_0 l_\epsilon(u_0) - H(t_0) + u_0 \cdot v + \epsilon t_0 l(u_0) \\ &\leq k_\epsilon(v) + \epsilon t_0 l(u_0). \end{aligned}$$

Thus

$$-t_0 l(u_0) \leq \frac{k_\epsilon(v) - k(v)}{\epsilon}. \tag{77}$$

Using inequalities (76) and (77) we get: $\left|\frac{k_\epsilon(v)-k(v)}{\epsilon}\right| \leq |l|_\infty M_1$.

(ii.b) Using (75) we have:

$$\overline{\lim_{\epsilon\to 0}} \frac{k_\epsilon(v) - k(v)}{\epsilon} \leq \overline{\lim_{\epsilon}} -t_\epsilon l(u_\epsilon). \tag{78}$$

There exists a sequence $(u_{\epsilon_n}, f_{\epsilon_n})$ with $\epsilon_n \to 0$ such that

$$\overline{\lim_{\epsilon}} -t_\epsilon l(u_\epsilon) = \lim_{n\to\infty} -f_{\epsilon_n} l(u_{\epsilon_n}).$$

As $t_{\epsilon_n} \in (0, M_1]$ and $u_{\epsilon_n} \in \bar{\Lambda}$, we may find a subsequence $\{(t_{\epsilon_{n_\nu}}, u_{\epsilon_{n_\nu}})\}_\nu$ such that

$$(f_{\epsilon_{n_\nu}}, u_{\epsilon_{n_\nu}}) \xrightarrow{\nu} (\tilde{t}, \tilde{u}) \in (0, M_1] \times \bar{\Lambda}.$$

Remark that we exclude the possibility $\tilde{t} = 0$ since as $k_\epsilon(v)$ is bounded uniformly in ϵ, no subsequence of $\{t_\epsilon\}_\epsilon$ goes to 0. Thanks to **(ii. a)**, $\lim_{\epsilon\to 0^+} k_\epsilon(v) = k(v)$. Using the continuity of H and the fact that l is lower semicontinuous we get

$$k(v) \leq \lim_{\nu\to\infty} -t_{\epsilon_{n_\nu}} l(u_{\epsilon_{n_\nu}}) - H(t_{\epsilon_{n_\nu}}) + u_{\epsilon_{n_\nu}} \cdot v \leq -\tilde{f} l(\tilde{u}) - H(\tilde{f}) + \tilde{u} \cdot v \leq k(v).$$

Thus $k(v) = -\tilde{f} l(\tilde{u}) - H(\tilde{f}) + \tilde{u} \cdot v$ and using **(i)**, we have $u_0 = \tilde{u}$ and $t_0 = \tilde{f}$. Hence:

$$\overline{\lim_{\epsilon}} -t_\epsilon l(u_\epsilon) = \lim_{n\to\infty} -t_{\epsilon_n} l(u_{\epsilon_n}) = \lim_{\nu\to\infty} -t_{\epsilon_{n_\nu}} l(u_{\epsilon_{n_\nu}}) = -t_0 l(u_0).$$

We deduce that

$$\overline{\lim_{\epsilon\to 0}} \frac{k_\epsilon(v) - k(v)}{\epsilon} \leq -t_0 l(u_0). \tag{79}$$

Using (77) we get

$$-t_0 l(u_0) \leq \underline{\lim_{\epsilon\to 0^+}} \frac{k_\epsilon(v) - k(v)}{\epsilon}. \tag{80}$$

Finally, combining (79) and (80) we get

$$\lim_{\epsilon\to 0} \frac{k_\epsilon(v) - k(v)}{\epsilon} = -t_0 l(u_0).$$

□

Lemma 4.1.11 *Let $\gamma : \Omega \to \mathbb{R}^d$ be a piece-wise constant function and $\lambda : \Omega \to \mathbb{R}^d$ be a non degenerate function i.e. $\mathcal{L}^d(\lambda^{-1}(N)) = 0$ whenever $N \subset \mathbb{R}^d$ and $\mathcal{L}^d(N) = 0$. Then $\gamma + \lambda$ is non degenerate.*

Proof. Let $\{\Omega_i\}_{i\in\infty}$ with I countable be a partition of Ω such that γ is constant on Ω_i and takes on that set the value γ_i. We have $\gamma(x) = \sum_{i\in I}\gamma_i 1_{\Omega_i}, \quad \forall x \in \Omega$. Let $N \subset \mathbb{R}^d$ such that $\mathcal{L}^d(N) = 0$.

$$\begin{aligned}
(\gamma+\lambda)^{-1}(N) &= \{x \in \Omega,\ \gamma(x)+\lambda(x) \in N\} \\
&= \bigcup_{i\in I}\Omega_i \cap \{x \in \Omega,\ \gamma(x)+\lambda(x) \in N\} \\
&= \bigcup_{i\in I}\Omega_i \cap \{x \in \Omega,\ \gamma_i+\lambda(x) \in N\} \\
&= \bigcup_{i\in I}\Omega_i \cap \lambda^{-1}(N-\gamma_i)
\end{aligned}$$

Thus

$$\mathcal{L}^d((\gamma+\lambda)^{-1}(N)) \leq \sum_{i\in I}\mathcal{L}^d(\Omega_i \cap \lambda^{-1}(N-\gamma_i)) \leq \sum_{i\in I}\mathcal{L}^d(\lambda^{-1}(N-\gamma_i)).$$

But, as $\mathcal{L}^d(N-\gamma_i) = \mathcal{L}^d(N) = 0$ and λ is non degenerate, we have $\mathcal{L}^d(\lambda^{-1}(N-\gamma_i)) = 0$. Thus $\mathcal{L}^d((\gamma+\lambda)^{-1}(N)) \leq 0$ and $\mathcal{L}^d((\gamma+\lambda)^{-1}(N)) = 0$. Hence $\gamma+\lambda$ is non degenerate.

$\square$

4.2 A duality result for problem (58)

In this section we suppose that $\mathbf{F}$ is non degenerate and the set $\mathcal{S}$ is a finite dimensional subspace of piecewise affine functions in $W_0^{1,\infty}(\Omega,\mathbb{R}^{d\times d})$. Suppose $(\bar{k},\bar{l},\bar{\psi}) \in \mathscr{A}'$ is a maximizer of $\sup_{(k,l,\psi)\in\mathscr{A}} -J(k,l,\psi)$ as given by Proposition 4.1.8. Our goal in this section is to make a link between Problem (69) and problem (58).

Let $\psi \in \mathcal{S}$. Remark first that $\operatorname{div}\psi_{_{\mathbb{R}^d}} = \operatorname{div}\psi 1_\Omega\mathcal{L}^d$ and if $(k,l) \in \mathcal{C}$, then $\int_{\overline{\Omega}} k(\mathbf{F} + \operatorname{div}\psi_{_{\mathbb{R}^d}}) = \int_\Omega k(\mathbf{F}+\operatorname{div}\psi)dx$. As $\operatorname{div}\bar{\psi}$ is constant piecewise and F is non degenerate, we have $\operatorname{div}\bar{\psi}+F$ that is non degenerate thanks to Lemma (4.1.11). As $\bar{k}$ is convex, $\bar{k}$ is differentiable a.e. on Ω. So there exists a set $N \subset \mathbb{R}^d$ with $\mathcal{L}^d(N) = 0$ such that $\bar{k}$ is differentiable for all $v \in \Omega\setminus N$. Set $N_0 = (\operatorname{div}\bar{\psi}+F)^{-1}(N)$. We have $\mathcal{L}^d(N_0) = 0$ as $\operatorname{div}\bar{\psi}+F$ is non degenerate. So for $x \in \Omega\setminus N_0$, $(\operatorname{div}\bar{\psi}+F)(x) \notin N$ and $\bar{k}$ is differentiable at $(\operatorname{div}\bar{\psi}+F)(x)$. Thus for almost every $x\in\Omega$, $\bar{k}$ is differentiable at $\operatorname{div}\bar{\psi}(x)+F(x)$. So, thanks to Lemma 4.1.10 there exist $\Omega_1\subset\Omega$ with $\mathcal{L}^d(\Omega\setminus\Omega_1)=0$ and unique measurable functions $t_0:\Omega_1\to(0,\infty)$ and $u_0:\Omega_1\to\bar{\Lambda}$ such that for all $x\in\Omega_1$

$$\bar{k}(\operatorname{div}\bar{\psi}(x)+F(x))=-t_0(x)\bar{l}(u_0(x))-H(t_0(x))-u_0(x)\cdot\big(\operatorname{div}\bar{\psi}(x)\big)\,;$$

$u_0(x)=\nabla\bar{k}(\operatorname{div}\bar{\psi}(x)+F(x))$ and $t_0(x)=(H')^{-1}(\bar{l}(u_0(x)))$.

4.2.1 Differential of $J(\cdot,\cdot,\psi)$ along a special curve

In this subsection we show that performing variations in the l variable in problem $\sup_{(k,l,\psi)\in\mathscr{A}}-J(k,l,\psi)$ gives the following result.

Lemma 4.2.1 *We have $t_0\in\det^*\nabla\mathbf{u}_0$.*

Proof. Let $l\in C_b(\mathbb{R}^d)$ and let $1\geq\epsilon>0$. Define $l_\epsilon=\bar{l}+\epsilon l$ and $k_\epsilon=(l_\epsilon)^{\#}$. Since for all $x\in\Omega_1$, $\bar{k}$ is differentiable at $\operatorname{div}\bar{\psi}(x)+F(x)$ we have by Lemma 4.1.10 that there exist a constant M independent of x and ϵ such that, for all $x\in\Omega_1$:

$$\left|\frac{k_\epsilon-k}{\epsilon}\big(\operatorname{div}\bar{\psi}(x)+F(x)\big)\right|\leq M. \tag{81}$$

and

$$\lim_{\epsilon\to0}\frac{k_\epsilon-k}{\epsilon}\big(\operatorname{div}\bar{\psi}(x)+F(x)\big)=-t_0(x)l(u_0(x)). \tag{82}$$

Furthermore, as $-J(\bar{k},\bar{l},\bar{\psi})=\sup_{\mathscr{A}}-J(k,l,\psi)$, we have

$$\begin{aligned}
J(\bar{k},\bar{l},\bar{\psi}) &\leq J(k_\epsilon,l_\epsilon,\bar{\psi})\\
\int_{\overline{\Omega}}\bar{k}(\operatorname{div}\psi+F)+\int_\Lambda\bar{l}dx &\leq \int_{\overline{\Omega}}k_\epsilon(F+\operatorname{div}\psi_{_{\mathbb{R}^d}})+\int_\Lambda l_\epsilon dx\\
\int_{\overline{\Omega}}\big(\bar{k}-k_\epsilon\big)(\operatorname{div}\psi+F) &\leq \int_\Lambda\big(l_\epsilon-\bar{l}\big)\,dx\\
\int_{\overline{\Omega}}\big(\bar{k}-k_\epsilon\big)(\operatorname{div}\psi+F) &\leq \epsilon\int_\Lambda ldx\\
\int_{\overline{\Omega}}\left(\frac{\bar{k}-k_\epsilon}{\epsilon}\right)(\operatorname{div}\psi+F) &\leq \int_\Lambda ldx.
\end{aligned}$$

As $\mathcal{L}^d(\Omega\backslash\Omega_1) = 0$, (81) and (82) holds for almost every $x \in \Omega$ and using the Lebesgue Dominated convergence Theorem, we get

$$\lim_{\epsilon\to 0^+}\int_{\overline{\Omega}}\left(\frac{\bar{k}-k_\epsilon}{\epsilon}\right)(\operatorname{div}\psi + F) = \int_{\overline{\Omega}} t_0(x)l(u_0(x))dx.$$

So

$$\int_{\overline{\Omega}} t_0(x)l(u_0(x))dx. \leq \int_\Lambda l dx \tag{83}$$

Remark that Inequality (83) is still true when l is replaced by $-l$ and reads

$$\int_{\overline{\Omega}} -t_0(x)l(u_0(x))dx \leq -\int_\Lambda l dx,$$

that is

$$\int_{\overline{\Omega}} t_0(x)l(u_0(x))dx \geq \int_\Lambda l dx. \tag{84}$$

Next Inequalities (83) and (84) imply

$$\int_{\overline{\Omega}} t_0(x)l(u_0(x))dx = \int_\Lambda l dx. \tag{85}$$

Since Equality (85) holds for all $l \in C_b(\mathbb{R}^d)$ we deduce that $t_0 \in \det^* \nabla \mathbf{u}_0$.

□

4.2.2 Differential of $J(k, l, \cdot)$ along a special curve

In this subsection we show that performing variations in the ψ variable in problem $\sup_{(k,l,\psi)\in\mathscr{A}} -J(k, l, \psi)$ gives the following result.

Lemma 4.2.2 *We have* $\nabla f^*(\bar{\psi}) = \nabla_{\mathcal{S}} u_0$.

Proof. Recall that

$$\mathcal{G}(u_0) = \left\{G \in L^p\left(\Omega,\ \mathbb{R}^{d\times d}\right) \mid \int_\Omega \langle u_0, \operatorname{div}\psi\rangle = -\int_\Omega \langle G, \psi\rangle\, dx, \forall\psi \in \mathcal{S}\right\}$$

and from Theorem 3.6.1, $\nabla_{\mathcal{S}} u_0$ is the unique $G \in \mathcal{G}$ such that $G = \nabla f^*(\psi_0)$ for some $\psi_0 \in \mathcal{S}$.

Let $\psi \in \mathcal{S}$ and $\epsilon \in (0,1)$. Define $\psi_\epsilon = \bar{\psi} + \epsilon\psi$. One has $J(\bar{k},\bar{l},\bar{\psi}) \leq J(\bar{k},\bar{l},\psi_\epsilon)$. Hence

$$\int_\Omega f^*(\bar{\psi}) + \int_\Omega \bar{k}(F + \operatorname{div}\bar{\psi})dx \leq \int_\Omega f^*(\bar{\psi}_\epsilon) + \int_\Omega \bar{k}(F + \operatorname{div}\bar{\psi}_\epsilon)dx$$

$$\int_\Omega \frac{f^*(\bar{\psi}) - f^*(\bar{\psi} + \epsilon\psi)}{\epsilon} \leq \int_\Omega \frac{-\bar{k}(F + \operatorname{div}\bar{\psi}) + \bar{k}(F + \operatorname{div}\bar{\psi} + \epsilon \operatorname{div}\psi)}{\epsilon} dx.$$

First, we use the fact that f^* is differentiable every where, the growth condition (42) on f^*, the fact that $\psi, \bar{\psi} \in \mathcal{S} \subset L^q(\Omega, \mathbb{R}^{d\times d})$ and the Lebesgue Dominated Convergence Theorem to get

$$\lim_{\epsilon\to 0^+} \int_\Omega \frac{f^*(\bar{\psi}) - f^*(\bar{\psi} + \epsilon\psi)}{\epsilon} = \int_\Omega \nabla f^*(\bar{\psi}) \cdot \psi.$$

Next, we use the fact that k is differentiable at $F(x) + \operatorname{div}\bar{\psi}(x)$ for a.e. $x \in \Omega$, the fact that there exists $e > 0$ such that for all $v \in \mathbb{R}^d$, $k(v) \leq r^*|v| + e$ (this is given by Lemma 4.1.5), the fact that $F, \operatorname{div}\bar{\psi} \in L^1(\Omega, \mathbb{R}^d)$ and the Lebesgue Dominated Convergence Theorem to get

$$\lim_{\epsilon\to 0^+} \int_\Omega \frac{-\bar{k}(F + \operatorname{div}\bar{\psi}) + \bar{k}(F + \operatorname{div}\bar{\psi} + \epsilon \operatorname{div}\psi)}{\epsilon} dx = -\int_\Omega \operatorname{div}\psi \cdot \nabla\bar{k}(F + \operatorname{div}\bar{\psi})dx.$$

Hence $\int_\Omega \nabla f^*(\bar{\psi}) \cdot \psi \leq -\int_\Omega \operatorname{div}\psi \cdot \nabla\bar{k}(F + \operatorname{div}\bar{\psi})dx$ and since we could replace ψ by $-\psi$ we deduce that $\int_\Omega \nabla f^*(\bar{\psi}) \cdot \psi = -\int_\Omega \operatorname{div}\psi \cdot \nabla\bar{k}(F + \operatorname{div}\bar{\psi})dx$. But we set $u_0(x) = \nabla\bar{k}(\operatorname{div}\bar{\psi}(x) + F(x))$. Hence

$$\int_\Omega \nabla f^*(\bar{\psi}) \cdot \psi = -\int_\Omega \operatorname{div}\psi \cdot u_0 dx$$

and $\nabla f^*(\bar{\psi}) \in \mathscr{G}(u_0)$. In addition, since $\bar{\psi} \in \mathcal{S}$, we deduce that $\nabla_{\mathcal{S}} u_0 = \nabla f^*(\bar{\psi})$.

□

4.2.3 Duality, existence and uniqueness result

Theorem 4.2.3 *We have* $-J(\bar{k},\bar{l},\bar{\psi}) = \int_\Omega \left(f(\nabla_{\mathcal{S}} u_0) + H(t_0) - \mathbf{F}\cdot u_0\right) dx$. *Moreover*

$$\sup_{(k,l,\psi)\in\mathscr{A}} -J(k,l,\psi) = \inf_{(\mathbf{u},\beta)\in\mathcal{U}_b} \int_\Omega \left(f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u}\right) dx$$

and the problem $\inf_{(\mathbf{u},\beta)\in\mathcal{U}_b} \int_\Omega \left(f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u}\right) dx$ *admit a unique minimizer characterized by* $u_0(x) = \nabla\bar{k}(\operatorname{div}\bar{\psi}(x) + F(x))$ *and* $t_0(x) = (H')^{-1}(\bar{l}(u_0(x)))$ *for a.e.* $x \in \Omega$.

Proof. Recall that for a.e. $x \in \Omega$, one has

$$\bar{k}(\operatorname{div} \bar{\psi}(x) + F(x)) = -t_0(x).\bar{l}(u_0(x)) - H(t_0(x)) - u_0(x) \cdot \left(\operatorname{div} \bar{\psi}(x) + F(x)\right).$$

Hence

$$\begin{aligned}
\int_{\overline{\Omega}} \bar{k}\left(\operatorname{div} \bar{\psi} + F\right) dx &= \int_{\overline{\Omega}} \left(-t_0 \bar{l}(u_0) - H(t_0) + u_0 \cdot \left(\operatorname{div} \bar{\psi} + F\right)\right) dx \\
&= -\int_{\Lambda} l dy + \int_{\Omega} \left(-H(t_0) + u_0 \cdot F\right) dx - \int_{\Omega} \nabla f^*(\bar{\psi}) \cdot \bar{\psi} \\
&= -\int_{\Lambda} l dy + \int_{\Omega} \left(-H(t_0) + u_0 \cdot F\right) dx - \int_{\Omega} f(\nabla f^*(\bar{\psi})) - f^*(\bar{\psi}) \\
&= -\int_{\Lambda} l dy - \int_{\Omega} f^*(\bar{\psi}) - \int_{\Omega} \left(f(\nabla_{\mathcal{S}} u_0) + H(t_0) - u_0 \cdot F\right) dx,
\end{aligned}$$

where we have exploited the fact that $t_0 \in \det{}^* \nabla \mathbf{u}_0$ (Lemma 4.2.1) and $\nabla_{\mathcal{S}} u_0 = \nabla f^*(\bar{\psi})$ (Lemma 4.2.2). This shows that $-J(\bar{k}, \bar{l}, \bar{\psi}) = \int_{\Omega} \left(f(\nabla_{\mathcal{S}} u_0) + H(t_0) - \mathbf{F} \cdot u_0\right) dx$. Let now $(k, l, \psi) \in \mathscr{A}$ and $(\mathbf{u}, \beta) \in \mathcal{U}_b$. One has

$$\int_{\Omega} k(F + \operatorname{div} \psi) dx \geq \int_{\Omega} \left(\mathbf{u} \cdot (F + \operatorname{div} \psi) - \beta l(\mathbf{u}) - H(\beta)\right) dx$$

with equality if and only if for a.e. $x \in \Omega$, one has $u(x) = \nabla k(F(x) + \operatorname{div} \psi(x))$ and $\beta(x) = (H')^{-1}(l(u(x)))$. Using $\beta \in \det{}^* \nabla \mathbf{u}$ and $\nabla_{\mathcal{S}} \mathbf{u} \in \mathcal{G}(\mathbf{u})$, one gets

$$\begin{aligned}
\int_{\Omega} k(F + \operatorname{div} \psi) dx &\geq \int_{\Omega} (-\nabla_{\mathcal{S}} \mathbf{u} \cdot \psi - H(\beta) + \mathbf{u} \cdot F) dx - \int_{\Lambda} l dy \\
&\geq \int_{\Omega} (-f\left(\nabla_{\mathcal{S}} \mathbf{u}\right) - f^*(\psi) - H(\beta) + \mathbf{u} \cdot F) dx - \int_{\Lambda} l dy
\end{aligned}$$

and the last inequality is strict unless $\nabla_{\mathcal{S}} \mathbf{u} = \nabla f^*(\psi)$. We deduce

$$-\int_{\Omega} \left(f^*(\psi) + k(F + \operatorname{div} \psi) dx\right) dx + \int_{\Lambda} l dy \leq \int_{\Omega} (f\left(\nabla_{\mathcal{S}} \mathbf{u}\right) + H(\beta) - \mathbf{u} \cdot F) dx$$

and the equality is strict unless for a.e. $x \in \Omega$, one has $u(x) = \nabla k(F(x) + \operatorname{div} \psi(x))$; $\beta(x) = (H')^{-1}(l(u(x)))$ and $\nabla_{\mathcal{S}} \mathbf{u} = \nabla f^*(\psi)$. Combining with

$$-J(\bar{k}, \bar{l}, \bar{\psi}) = \int_{\Omega} \left(f(\nabla_{\mathcal{S}} u_0) + H(t_0) - \mathbf{F} \cdot u_0\right) dx$$

one deduces that

$$\sup_{(k,l,\psi)\in\mathscr{A}} -J(k,l,\psi) = \inf_{(\mathbf{u},\beta)\in\mathcal{U}_b} \int_\Omega \left(f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u}\right) dx$$

and the problem $\inf_{(\mathbf{u},\beta)\in\mathcal{U}_b} \int_\Omega \left(f(\nabla_{\mathcal{S}}\mathbf{u}) + H(\beta) - \mathbf{F}\cdot\mathbf{u}\right) dx$ admit a unique minimizer characterized by $u_0(x) = \nabla\bar{k}(\operatorname{div}\bar{\psi}(x) + F(x))$ and $t_0(x) = (H')^{-1}(\bar{l}(u_0(x)))$ for a.e. $x \in \Omega$.

□

4.3 A duality result for the relaxed variational problem

4.3.1 Half way to duality

4.3.1.1 The case F non degenerate

In this section, we consider an increasing family $\{S_n\}_{n\in\mathbb{N}^*}$ of finite dimensional linear subspace of $W_0^{1,\infty}(\Omega,\mathbb{R}^{d\times d})$ consisting of functions affine piecewise such that for all $\psi \in W_0^{1,\infty}(\Omega,\mathbb{R}^{d\times d})$, one can find a sequence $\psi_n \in S_n$ for all $n \in \mathbb{N}^*$ satisfying $\lim_{n\to\infty} \|\psi-\psi_n\|_{W_0^{1,\infty}(\Omega,\mathbb{R}^{d\times d})} = 0$. Such family may be provided by Proposition 2.3.16. Thanks to Lemma 2.1.2, there exists $\mathbf{u}_0 \in W^{1,p}(\Omega,\Lambda)$ an homeomorphism such that $\mathbf{u}_0(\Omega) = \Lambda$ and $\det\nabla\mathbf{u} > 0$. Set $\beta_0 = \det\nabla\mathbf{u}_0$. One has $\beta_0 \in \det^*\nabla\mathbf{u}_0$. Let

$$c_0 := \int_\Omega \big(f(\nabla\mathbf{u}_0) + H\big(\beta_0\big) - \mathbf{F}\cdot\mathbf{u}_0\big)dx.$$

Since $\mathbf{u}_0 \in W^{1,p}(\Omega,\Lambda)$, one has $\nabla\mathbf{u}_0 \in \mathscr{G}(\mathbf{u}_0)$ and thus $\int_\Omega f(\nabla\mathbf{u}_0)dx > \int_\Omega f(\nabla_{S_e}\mathbf{u}_0)dx$. Hence $I_{S_e}(\mathbf{u}_0,\beta_0) \le c_0$.

Now for every $e \in \mathbb{N}^*$, thanks to Theorem 4.2.3, there exists unique $(\mathbf{u}_e,\beta_e) \in \mathscr{U}_b$ such that $I_{s_e}(\mathbf{u}_e,\beta_e) = \min_{(\mathbf{u},\beta)\in\mathscr{U}_b} I_{s_e}(\mathbf{u},\beta)$. In particular, $I_{S_e}(\mathbf{u}_e,\beta_e) \le c_0$ and we deduce that

$$\int_\Omega \left(f(\nabla_{S_e}\mathbf{u}_0) + H(\beta_e)\right) dx \le c_0 + r^*|\mathbf{F}|_{L^1(\Omega,\mathbb{R}^d)}; \quad \forall e \in (0,1). \tag{86}$$

Define for every $e \in \mathbb{N}^*$ a measure γ_e on $C = \bar{\Omega}\times D = \bar{\Omega}\times\bar{\Lambda}\times[0,\infty)\times\mathbb{R}^{d\times d}$ by

$$\int_C L(x,u,t,\xi)\gamma_e(dx,du,dt,d\xi) = \int_\Omega L(x,\mathbf{u}_e(x),\beta_e(x),\nabla_{S_e}\mathbf{u}(x))dx$$

for all L measurable positive. Consider the function $\varphi : C \to [0, \infty)$ defined by $\varphi(x, u, t, \xi) = f(\xi) + H(t) - \min f - \min H$. Thanks to the growth condition (40) on f and (43) on H, one deduces that for all $\alpha \in \mathbb{R}$, the sublevel sets $\{\varphi \leq \alpha\}$ are compact in C. Next, for all $e \in \mathbb{N}^*$

$$\begin{aligned}\int_C \varphi(x, u, t, \xi)\gamma_e(dx, du, dt, d\xi) =& \int_\Omega (f(\nabla_{S_e}\mathbf{u}_e) + H(\beta_e) - \min f - \min H)\, dx \\ \leq& c_0 + r^*|\mathbf{F}|_{L^1(\Omega,\mathbb{R}^d)} - \min f - \min H.\end{aligned}$$

Hence $\{\gamma_e\}_e$ is tight (by Lemma A.2.6) and we may find a subsequence $\{\gamma_{e_n}\}_{n=1}^\infty$ converging weakly to a measure $\bar{\gamma}$ on C (by Prokorov's Theorem).

Lemma 4.3.1 *One has* $\bar{\gamma} \in \Gamma$.

Proof. ***Claim 1 :*** $\int_E b(x)d\bar{\gamma}(dx, du, dt, d\xi) = \int_\Omega b(x)dx,\ \forall b \in C_b(\mathbb{R}^d)$.

We have for $b \in C_b(\mathbb{R}^d)$, by definition of the weak convergence

$$\int_C b(x)d\bar{\gamma}(dx, du, dt, d\xi) = \lim_{n\to 0} \int_C b(x)d\gamma_{e_n}(dx, dt, du, d\xi) = \int_\Omega b(x)dx,\ \forall b \in C_b(\mathbb{R}^d).$$

Thus

$$\int_C b(x)d\bar{\gamma}(dx, du, dt, d\xi) = \int_\Omega b(x)dx\ \forall b \in C_b(\mathbb{R}^d) \tag{87}$$

Claim 2 : We have $\int_C f(\xi)d\bar{\gamma} \leq \infty$.

Since the map $C \ni (x, u, t, \xi) \mapsto f(\xi)$ is lower semicontinuous and bounded below, we have by Lemma A.2.2

$$\int_C f(\xi)d\bar{\gamma} \leq \int_C f(\xi)d\gamma_{e_n} \leq c_0 + r^*\|\mathbf{F}\|_{L^1(\Omega,\mathbb{R}^d)} - \min h < \infty.$$

Claim 3 : One has $\int_C tl(\mathbf{u})d\bar{\gamma}(dx, du, dt, d\xi) = \int_\Lambda l dy,\ \forall l \in C_c(\mathbb{R}^d)$.

Let $l \in C_b(\mathbb{R}^d)$. The map $C \ni (x, t, u, \xi) \mapsto tl(\mathbf{u})$ is continuous. Moreover $|tl(\mathbf{u})| \leq |l|_\infty|t|$;

$$\begin{aligned}\sup_n \int_C (H(|t|) - \min H)\, d\gamma_{e_n} =& \sup_n \int_\Omega (H(\beta_{e_n}) - \min H)\, dx \\ \leq& c_0 + r^*\|\mathbf{F}\|_{L^1(\Omega,\mathbb{R}^d)} - \min h - \min f < \infty;\end{aligned}$$

one has $H - \min H \geq 0$ and

$$\lim_{t\to\infty} \frac{H(t) - \min H}{t} = \infty.$$

Thus thanks to Lemma A.2.2 and Lemma A.2.3 one gets $\lim_{n\to\infty} \int_C tl(\mathbf{u})d\gamma_{e_n} = \int_C tl(\mathbf{u})d\bar{\gamma}$. Having for all $n \in \mathbb{N}$ $\int_C tl(\mathbf{u})d\bar{\gamma}_{e_n} = \int_\Lambda ldy$, one deduces that $\int_C tl(\mathbf{u})d\bar{\gamma} = \int_\Lambda ldy$.

Claim 4 : For all $\psi \in C_c^\infty(\Omega, \mathbb{R}^{d\times d})$ one has $\int_C \langle \xi, \psi(x)\rangle \, d\bar{\gamma} = -\int_C \langle u, \mathrm{div}\psi(x)\rangle \, d\bar{\gamma}$. First Remark that the map $C \ni (x, u, t, \xi) \mapsto \langle \xi, \psi(x)\rangle$ is continuous, for all $(x, u, t, \xi) \in C$, $|\langle \xi, \psi(x)\rangle| \leq |\psi|_\infty |\xi|$; $\lim_{t\to\infty} \frac{t^p}{t} = \infty$ and $\sup_n \int_C |\xi|^p d\gamma_{e_n} < \infty$. Hence

$$\lim_{n\to\infty} \int_C \langle \xi, \psi(x)\rangle \, d\gamma_{e_n} = \int_C \langle \xi, \psi(x)\rangle \, d\bar{\gamma}. \tag{88}$$

Similarly, the map $C \ni (x, u, t, \xi) \mapsto \langle u, \mathrm{div}\psi(x)\rangle$ is continuous, for all $(x, u, t, \xi) \in C$, $|\langle u, \mathrm{div}\psi(x)\rangle| \leq |\mathrm{div}\,\psi|_\infty |u|$; $\lim_{t\to\infty} \frac{t^2}{t} = \infty$ and $\sup_n \int_C |u|^2 d\gamma_{e_n} < \infty$. Hence

$$\lim_{n\to\infty} \int_C \langle u, \mathrm{div}\psi(x)\rangle \, d\gamma_{e_n} = \int_C \langle u, \mathrm{div}\psi(x)\rangle \, d\bar{\gamma}. \tag{89}$$

Remark that

$$\begin{aligned}
&\left|\int_C \langle u, \mathrm{div}\psi(x)\rangle \, d\bar{\gamma} + \int_C \langle \xi, \psi(x)\rangle \, d\bar{\gamma}\right| \\
\leq &\left|\int_C \langle u, \mathrm{div}\psi\rangle \, d\bar{\gamma} - \int_C \langle u, \mathrm{div}\psi\rangle \, d\gamma_{e_n}\right| + \left|\int_C \langle u, \mathrm{div}\psi\rangle \, d\gamma_{e_n} - \int_C \langle u, \mathrm{div}\psi^{e_m}\rangle \, d\gamma_{e_n}\right| \\
&+ \left|\int_C \langle u, \mathrm{div}\psi\rangle \, d\gamma_{e_n} + \int_C \langle \xi, \psi^{e_m}\rangle \, d\gamma_{e_n}\right| \\
&+ \left|-\int_C \langle \xi, \psi^{e_m}\rangle \, d\gamma_{e_n} + \int_C \langle \xi, \psi\rangle \, d\gamma_{e_n}\right| + \left|-\int_C \langle \xi, \psi\rangle \, d\gamma_{e_n} + \int_C \langle \xi, \psi\rangle \, d\bar{\gamma}\right|. \\
:= &a_1 + a_2 + a_3 + a_4 + a_5.
\end{aligned}$$

Thanks to Equations (89) and (88) we can find $N_1 \in \mathbb{N}$ such that $n \geq N_1$ implies $a_1, a_5 < \epsilon$. We chose such n. Next, it holds:

$$\begin{aligned}
a_2 &\leq \int_\Omega |u_{e_n} \mathrm{div}\,(\psi - \psi^{e_m})| \, dx \leq r^* \mathcal{L}^d(\Omega) |\mathrm{div}\,(\psi - \psi^{e_m})|_\infty \\
a_4 &\leq \int_\Omega \left|\nabla_{S_{e_n}} u_{e_n} (\psi - \psi^{e_m})\right| dx \leq |\psi - \psi^{e_m}|_\infty \left(\mathcal{L}^d(\Omega)\right)^{\frac{1}{q}} \|\nabla_{S_{e_n}} u_{e_n}\|_{L^p(\Omega, \mathbb{R}^{d\times d})}
\end{aligned}$$

Since there exists $M > 0$ such that for all $n \in \mathbb{N}^*$ one has $\|\nabla_{S_{e_n}} u_{e_n}\|_{L^p(\Omega,\mathbb{R}^{d\times d})} < M$, we may find $N_2 > N_1$ such that for all $m \geq N_2$ one has $a_1, a_4 < \epsilon$. We chose such m. Remark that since $m > n$, one has $S_{e_n} \subset S_{e_m}$; $\int_\Omega \nabla_{S_{e_n}} u_{e_n} \cdot \psi^{e_m} dx = -\int_\Omega u_{e_n} \operatorname{div} \psi^{e_m} dx$ and $a_3 = 0$. Thus $\left|\int_C \langle u, \operatorname{div}\psi(x)\rangle \, d\bar{\gamma} + \int_C \langle \xi, \psi(x)\rangle \, d\bar{\gamma}\right| < 4\epsilon$. Since ϵ has been chosen to be arbitrary, we deduce that $\int_C \langle u, \operatorname{div}\psi(x)\rangle \, d\bar{\gamma} = -\int_C \langle \xi, \psi(x)\rangle \, d\bar{\gamma}$.

Lemma 4.3.2 *One has*

$$\int_C (f(\xi) + H(t) - F(x) \cdot u) \, d\bar{\gamma} \leq \liminf_n \int_\Omega \left(f(\nabla_{S_{e_n}} \mathbf{u}_{e_n}) + H(\beta_{e_n}) - \mathbf{F}(x) \cdot \mathbf{u}_{e_n}\right) dx.$$

Proof. Since $\{\gamma_{e_n}\}_{n=1}^\infty$ converges weakly to $\bar{\gamma}$ on C, we use Lemma 3.7.3 to deduce $\int_C F(x) \cdot u d\bar{\gamma} = \lim_{n\to\infty} \int_C F(x) \cdot u d\gamma_{e_n}$. We next use the lower semicontinuity of the map $C \ni (x, u, t, \xi) \mapsto f(\xi) + H(t)$ and its boundedness below to infer thanks to Lemma A.2.2 that $\int_C (f(\xi) + H(t)) \, d\bar{\gamma} \leq \liminf_n \int_C (f(\xi) + H(t)) \, d\gamma_{e_n}$. Hence

$$\begin{aligned}\int_C (f(\xi) + H(t) - F(x) \cdot u) \, d\bar{\gamma} &\leq \liminf_n \int_C (f(\xi) + H(t) - F(x) \cdot u) \, d\gamma_{e_n} \\ &= \liminf_n \int_\Omega \left(f(\nabla_{S_{e_n}} \mathbf{u}_{e_n}) + H(\beta_{e_n}) - \mathbf{F}(x) \cdot \mathbf{u}_{e_n}\right) dx.\end{aligned}$$

□

We summarize this subsubsection as follow:

Remark 4.3.3 *If F is nondegenerate, then there exists $(\bar{k}, \bar{l}, \bar{\psi}) \in \mathscr{A}'$ such that $\int_C (f(\xi) + H(t) - F(x) \cdot u) \, d\bar{\gamma} \leq -J(\bar{k}, \bar{l}, \bar{\psi})$*

Proof. From Theorem 4.2.3, we infer that there exists $\{(k_n, l_n)\}_n \subset \mathcal{C}$ and $\{\psi_n\}_{n=1}^\infty$ such that for all $n \in \mathbb{N}^*$, one has $\psi_n \in S_{e_n}$ and

$$I_{s_{e_n}}(\mathbf{u}_{e_n}, \beta_{e_n}) = \min_{(\mathbf{u},\beta)\in\mathscr{U}_b} I_{S_{e_n}}(\mathbf{u}, \beta) = -J(k_n, l_n, \psi_n) = \sup_{(k,l,\psi)\in\mathscr{A}_n} -J(k, l, \psi)$$

where $\mathscr{A}_n = \mathcal{C} \times S_{e_n}$. Exploiting Lemma 4.1.7, we can find $(\bar{k}, \bar{l}, \bar{\psi}) \in \mathscr{A}$ such that $\liminf -J(k_n, l_n, \psi_n) \leq -J(\bar{k}, \bar{l}, \bar{\psi})$. Combining with Lemma 4.3.2, we deduce $\int_C (f(\xi) + H(t) - F(x) \cdot u) \, d\bar{\gamma} \leq -J(\bar{k}, \bar{l}, \bar{\psi})$.

□

4.3.1.2 The case $F \in L^1(\Omega)$

Lemma 4.3.4 *Let $A \subset \mathbb{R}^d$ be Lebesgue measurable and bounded. Then for all $\epsilon > 0$ there exists $g \in L^\infty(A, \mathbb{R}^d)$ non degenerate such that $\|g\|_{L^1(A,\mathbb{R}^d)} \leq \epsilon$.*

Proof. Let $r > 0$ be such that $A \subset B(0, r)$. Consider for $t > 0$ the map $g : A \to \mathbb{R}^d;\ x \mapsto g_t(x) = tx$. Then $\|g_t\|_{L^\infty(A)} \leq tr$. Moreover $\int_A |g_t| dx \leq tr\mathcal{L}^d(A)$. Choose $t < (r\mathcal{L}^d(A))^{-1}\epsilon$, one has $\|g\|_{L^1(A,\mathbb{R}^d)} \leq \epsilon$. Let $N \subset \mathbb{R}^d$ be such that $\mathcal{L}^d(N) = 0$. Then $g_t^{-1}(N) = t^{-1}N \cap A$ and hence $\mathcal{L}^d(g_t^{-1}(N)) = 0$. Thus g is non degenerate.

□

Lemma 4.3.5 *Let $\mathbf{F} \in L^1(\Omega, \mathbb{R}^d)$. Then there exists a sequence $\{\mathbf{F}_n\}_n \subset L^\infty(\Omega, \mathbb{R}^d)$ such that $\mathbf{F}_n$ is non degenerate and* $\lim_{n\to\infty} \|\mathbf{F} - \mathbf{F}_n\|_{L^1} = 0$.

Proof. Let $n \in \mathbb{N}^*$. As the set of simple functions (finite linear combination of characteristic functions) is dense in $L^1(\Omega, \mathbb{R}^d)$, we can find $\mathbf{H}_n \in L^1(\Omega, \mathbb{R}^d)$ such that

$$\|\mathbf{F} - \mathbf{H}_n\|_{L^1} \leq \frac{1}{2n} \text{ and } \mathbf{H}_n = \sum_{i=0}^{N(n)} \mathbf{f}_i \chi_{A_i},$$

where the A_i are measurable disjoint subsets of Ω.

Thanks to Lemma (4.3.4), we may find $\mathbf{G}_n \in L^\infty(\Omega, \mathbb{R}^d)$ a non-degenerate function such that $\|\mathbf{G}_n\|_{L^1(\Omega,\mathbb{R}^d)} \leq \frac{1}{2n}$. Moreover, since $\mathbf{H}_n$ has a countable range and $\mathbf{G}_n$ is non-degenerate, thanks to Lemma 4.1.11 we deduce that $\mathbf{F}_n := \mathbf{G}_n + \mathbf{H}_n$ is non-degenerate. Furthermore,

$$\|\mathbf{F} - \mathbf{F}_n\|_{L^1(\Omega,\mathbb{R}^d)} \leq \|\mathbf{F} - \mathbf{H}_n\|_{L^1(\Omega,\mathbb{R}^d)} + \|\mathbf{G}_n\|_{L^1(\Omega,\mathbb{R}^d)} \leq \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$

Thus $\lim_{n\to\infty} \|\mathbf{F} - \mathbf{F}_n\|_{L^1(\Omega,\mathbb{R}^d)} = 0$. Remarking that by $\mathbf{G}_n \in L^\infty(\Omega, \mathbb{R}^d)$ and $\mathbf{H}_n \in L^\infty(\Omega, \mathbb{R}^d)$, one deduces that $\mathbf{F}_n = \mathbf{G}_n + \mathbf{H}_n \in L^\infty(\Omega, \mathbb{R}^d)$.

□

Lemma 4.3.6 *Assume that* $\{F_\nu\}_{\nu=1}^{\infty} \in L^\infty(\Omega, \mathbb{R}^d)$ *converges to* F *in* $L^1(\Omega, \mathbb{R}^d)$ *and* $\{\gamma_\nu\}_{\nu=1}^{\infty} \in \Gamma$ *converges narrowly to* γ*. Then*

$$\int_C (-F(x) \cdot u)\, \gamma(dx, dt, du, d\xi) = \lim_{\nu \to \infty} \int_C (-F_\nu(x) \cdot u)\, \gamma_\nu(dx, dt, du, d\xi).$$

Proof. One has:

$$\begin{aligned}
&\left| \int_C \mathbf{F}_\nu(x) \cdot u d\gamma_\nu - \int_C \mathbf{F}(x) \cdot u d\gamma \right| \\
&= \left| \int_C \mathbf{F}_\nu(x) \cdot u d\gamma_\nu - \int_C \mathbf{F}(x) \cdot u d\gamma_\nu + \int_C \mathbf{F}(x) \cdot u d\gamma_\nu - \int_C \mathbf{F}(x) \cdot u d\gamma \right| \\
&\leq \left| \int_C (\mathbf{F}_\nu(x) \cdot u - \mathbf{F}(x) \cdot u)\, d\gamma_\nu \right| + \left| \int_C \mathbf{F}(x) \cdot u d\gamma_\nu - \int_C \mathbf{F}(x) \cdot u d\gamma \right| \\
&\leq \int_C |\mathbf{F}_\nu(x) \cdot u - \mathbf{F}(x) \cdot u|\, d\gamma_\nu + \left| \int_C \mathbf{F}(x) \cdot u d\gamma_\nu - \int_C \mathbf{F}(x) \cdot u d\gamma \right|.
\end{aligned}$$

Claim 1 $\lim_{\nu \to \infty} \int_C |\mathbf{F}_\nu(x) \cdot u - \mathbf{F}(x) \cdot u|\, d\gamma_\nu = 0.$

One has:

$$\begin{aligned}
\int_C |\mathbf{F}_\nu(x) \cdot u - \mathbf{F}(x) \cdot u|\, d\gamma_\nu &\leq \int_C |\mathbf{F}_\nu(x) - \mathbf{F}(x)|\, r^* d\gamma_\nu \text{ since } u \in \bar{\Lambda} \subset B(0, r^*) \\
&= r^* \int_\Omega |\mathbf{F}_\nu(x) - \mathbf{F}(x)|\, dx.
\end{aligned}$$

In addition to that, we exploit the fact that $\{F_\nu\}_{\nu=1}^{\infty}$ converges to F in $L^1(\Omega, \mathbb{R}^d)$ to finish the proof of Claim 1.

Claim 2 : $\lim_{\nu \to \infty} \int_C F(x) \cdot u d\gamma_\nu = \int_C F(x) \cdot u d\gamma.$

This is basically a consequence of Lemma 3.7.3.

Finally Combining Claim 1 and 2, we finish the proof of the lemma 4.3.6.

Lemma 4.3.7 *Assume that* $\{F_\nu\}_{\nu=1}^{\infty} \in L^\infty(\Omega, \mathbb{R}^d)$ *converges to* $F \in L^1(\Omega, \mathbb{R}^d)$ *and* $\{\gamma_\nu\}_{\nu=1}^{\infty} \in \Gamma$ *converges narrowly to* γ*. Then*

$$I_F(\gamma) \leq \liminf_{\nu \to \infty} I_{F_\nu}(\gamma_\nu).$$

Proof. Since $\gamma_\nu \to \gamma$ narrowly and the map $C \ni (x,u,t,\xi) \mapsto f(\xi) + H(t)$ is lower semicontinuous and bounded below, we get thanks to Lemma A.2.2

$$\int_C f(\xi) + H(t) d\gamma \leq \liminf_{\nu \to \infty} \int_C f(\xi) + H(t).$$

This in addition to Lemma 4.3.6 yields

$$I_F(\gamma) \leq \liminf_{\nu \to \infty} I_{F_\nu}(\gamma_\nu).$$

□

Lemma 4.3.8 *For all $F \in L^1(\Omega, \mathbb{R}^d)$ set*

$$J_F(k,l,\psi) = \int_{\bar{\Omega}} \left(f^*(\psi) dx + k \left(\operatorname{div} \psi^d_{\mathbb{R}} + \mathbf{F}\mathcal{L}^d \right) \right) + \int_\Lambda l dy.$$

and

$$I_F(\gamma) = \int_C \left(f(\xi) + H(t) - F(x) \cdot u \right) \gamma(dx, dt, du, d\xi)$$

Then there exists $\gamma \in \Gamma$ and $(k,l,\psi) \in \mathscr{A}_0$ such that $-J_F(k,l,\psi) \geq \bar{I}_F(\gamma)$.

Proof. Thanks to Lemma 4.3.5, we can find $\{\mathbf{F}_n\}_{n=1}^\infty \subset L^\infty(\omega, \mathbb{R}^d)$ a sequence of nondegenerate functions converging to $\mathbf{F}$ in $L^1(\omega, \mathbb{R}^d)$. Thanks to Lemma 4.3.3, for all $n \in \mathbb{N}^*$, there exist $\gamma_n \in \Gamma$ and $(k_n, l_n, \psi_n) \in \mathscr{A}'$ such that

$$\int_C \left(f(\xi) + H(t) - F_n(x) \cdot u \right) d\gamma_n \leq -J(k_n, l_n, \psi_n).$$

We may further assume that there exists $C \in \mathbb{R}$ such that for all $n \in \mathbb{N}^*$, one has $\bar{I}_{F_n}(\gamma_n) \leq C$ and $-J_{F_n}(\bar{k}_n, \bar{l}_n, \bar{\psi}_n) \leq C$. Using the boundedness of $\{\|F_n\|\}_n$ and an adapted version of Lemma 4.1.7 we may find a subsequence of $\{(k_n, l_n, \psi_n)\}_n$ still denoted $\{(k_n, l_n, \psi_n)\}_n$ and $(\bar{k}, \bar{l}, \bar{\psi}) \in \mathscr{A}_0$ such that $J_F(\bar{k}, \bar{l}, \bar{\psi}) \leq \liminf_n J_{F_n}(k_n, l_n, \psi_n)$. Similarly, using the boundedness of $\{\|F_n\|\}_n$, we may find a subsequence of $\{\gamma_n\}_{n=1}^\infty$ still denoted $\{\gamma_n\}_{n=1}^\infty$ that converges weakly to some $\bar{\gamma} \in \Gamma$. We use Lemma 4.3.7 to deduce $\bar{I}_F(\bar{\gamma}) \leq \liminf_n \bar{I}_{F_n}(\gamma_n)$.

Finally, $\bar{I}_F(\bar{\gamma}) \leq -J_F(\bar{k}, \bar{l}, \bar{\psi})$.

□

4.3.2 The full duality result

We have the following result

Lemma 4.3.9

$$\bar{I}(\bar{\gamma}) = \inf_{\gamma\in\Gamma} \bar{I}(\gamma) = \sup_{(k,l,\psi)\in\mathscr{A}_0} -J(k,l,\psi) = -J(\bar{k},\bar{l},\bar{\psi}).$$

Moreover if for $\gamma \in \Gamma$ *and* $(k,l,\psi) \in \mathscr{A}_0$ *one has* $\bar{I}(\gamma) = -J(k,l,\psi)$ *then* $\nabla \mathbf{u}_\gamma = \nabla f^*(\psi)$*.*

Proof. Thanks to Lemma 4.3.8, it suffices to show that for all $(k,l,\psi) \in \mathscr{A}$ and all $\gamma \in \Gamma$ one has $\bar{I}(\gamma) \geq -J(k,l,\psi)$. Recall first that defining $\mathbf{u}_\gamma$ by Equation (64), one has $\mathbf{u}_\gamma \in W^{1,p}(\Omega,\Lambda)$. Recall that for $\psi \in S$ there exists a Borel map $\mathbf{u}_\psi$ depending on ψ and $\mathbf{u}_\gamma$ such that $\mathbf{u}_\gamma = u_\psi$ $\mathcal{L}^d$ a.e. and

$$\int_\Omega \langle \nabla \mathbf{u}_\gamma, \psi\rangle\, dx = -\int_{\overline{\Omega}} \mathbf{u}_\psi \cdot \operatorname{div} \psi_{\mathbb{R}^d} = \int_C \langle \xi, \psi(x)\rangle\, d\gamma. \tag{90}$$

Next, using Jensen's inequality

$$\int_{\bar{\Omega}} k^*(\mathbf{u}_\gamma) dx \leq \int_C k^*(\mathbf{u})\gamma(dx,dt,du,d\xi) \leq \int_C \big(tl(\mathbf{u}) + H(t)\big)\gamma(dx,dt,du,d\xi). \tag{91}$$

Combining Equations (90) and (91), one has:

$$\begin{aligned}
\int_{\overline{\Omega}} k(\mathbf{F} + \operatorname{div}\psi_{\mathbb{R}^d}) &\geq \int_{\overline{\Omega}} \mathbf{u}_\psi \cdot (\mathbf{F} + \operatorname{div}\psi_{\mathbb{R}^d}) - \int_{\overline{\Omega}} k^*(\mathbf{u}_\psi) dx \\
&= \int_\Omega \mathbf{u}_\gamma \cdot \mathbf{F} dx + \int_{\overline{\Omega}} \mathbf{u}_\psi \cdot \operatorname{div}\psi_{\mathbb{R}^d} - \int_\Omega k^*(\mathbf{u}_\gamma) dx \\
&= \int_C (u\cdot \mathbf{F}(x) - \langle \xi, \psi(x)\rangle)\, d\gamma - \int_\Omega k^*(\mathbf{u}_\gamma) dx \\
&\geq \int_C (u\cdot \mathbf{F}(x) - \langle \xi, \psi(x)\rangle)\, d\gamma - \int_C \big(tl(\mathbf{u}) + H(t)\big) d\gamma \\
&= -\int_\Lambda l(y)dy + \int_C (-H(t) + u\cdot\mathbf{F}(x) - \langle \xi, \psi(x)\rangle)\, d\gamma \\
&> -\int_\Lambda l(y)dy + \int_C (-H(t) + u\cdot\mathbf{F}(x) - f(\xi) - f^*(\psi(x)))\, d\gamma
\end{aligned}$$

unless $\xi = \nabla f^*(\psi(x))$ γ-a.e. Hence

$$\int_{\overline{\Omega}} k(\mathbf{F} + \operatorname{div} \psi_{\mathbb{R}^d}) + \int_{\Lambda} l(y)dy + \int_{\Omega} f^*(\psi(x)) \geq \int_{C} (-H(t) + u \cdot \mathbf{F}(x) - f(\xi))\, d\gamma$$

and thus $\bar{I}(\gamma) > -J(k, l, \psi)$ unless $\xi = \nabla f^*(\psi(x))$ γ-a.e.

□

4.4 Sufficient conditions for uniqueness

In this section we turn our attention back to Problem (52). Recall The set $\mathcal{U}_b$ stands for the set of pairs $(\beta, \mathbf{u})$ such that $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$ and $\beta : \Omega \to (0, \infty)$ is a Borel function satisfying $\beta \in \det^* \nabla \mathbf{u}$. The problem at hand is

$$\inf_{(\beta,\mathbf{u})\in\mathcal{U}_b} \left\{ I(\mathbf{u}) := \int_{\Omega} (f(\nabla \mathbf{u}) + H(\beta) - \mathbf{F} \cdot \mathbf{u}) \right\} \tag{92}$$

Throughout the section; for $\psi \in \mathcal{S}_0$, we denote $\operatorname{div}^s \psi_{\mathbb{R}^d}$ (resp. $\operatorname{div}^a \psi_{\mathbb{R}^d}$) the singular (resp. absolutely continuous) part of $\operatorname{div} \psi_{\mathbb{R}^d}$ with respect to the Lebesgue measure and set $g^s_\psi = |\operatorname{div}^s \psi_{\mathbb{R}^d}|$ and $b^s_\psi = \frac{d(\operatorname{div}^s \psi_{\mathbb{R}^d})}{dg^s_\psi}$.

Theorem 4.4.1 *Suppose $(k, l, \psi) \in \mathscr{A}_0$ and $k = l^{\#}$ and k is differentiable at $\mathbf{F}(x) + \operatorname{div}^a \psi_{\mathbb{R}^d}(x)$ for almost every $x \in \Omega$. Suppose $\mathbf{u} \in W^{1,p}(\Omega, \Lambda)$, $\beta \in \det^* \nabla \mathbf{u}$, satisfies:*

$$\nabla \mathbf{u} = Df^*(\psi), \quad \mathbf{u} = \nabla k(\mathbf{F} + \operatorname{div}^a \psi_{\mathbb{R}^d}), \quad H'(\beta) + l(\mathbf{u}) = 0 \quad \mathcal{L}^d - a.e. \tag{93}$$

and

$$\mathbf{u}_\psi \in \partial k^\infty(b^s_\psi) \quad g^s - a.e. \tag{94}$$

Then $\mathbf{u}$ is the unique minimizer of I over $W^{1,p}(\Omega, \Lambda)$.

Proof. We use Lemma 4.1.10 to deduce from (93) that

$$k(\mathbf{F} + \operatorname{div}^a \psi) + \beta l(\mathbf{u}) + H(\beta) = \mathbf{u} \cdot (\mathbf{F} + \operatorname{div}^a \psi)$$

and so,

$$\int_{\Omega} \big(k(\mathbf{F} + \operatorname{div}^a \psi) + \beta l(\mathbf{u}) + h(\beta)\big) dx = \int_{\Omega} \mathbf{u} \cdot (\mathbf{F} + \operatorname{div}^a \psi) dx. \tag{95}$$

Next, since for all $v \in \mathbb{R}^d$ and $y \in \partial k_\infty(v)$ one has $k_\infty(v) = v \cdot y$ (Lemma A.1.6 and Lemma A.1.12), we use (93) to get

$$\int_{\bar{\Omega}} k^\infty(\operatorname{div}^s \psi) = \int_{\bar{\Omega}} k^\infty\Big(\frac{d(\operatorname{div}^s \psi_{\mathbb{R}^d})}{dg^s_\psi}\Big)dg^s_\psi = \int_{\bar{\Omega}} \mathbf{u}_\psi \cdot \frac{d(\operatorname{div}^s \psi_{\mathbb{R}^d})}{dg^s_\psi} dg^s_\psi = \int_{\bar{\Omega}} \mathbf{u}_\psi \cdot \operatorname{div}^s \psi.$$

Hence

$$\int_{\bar{\Omega}} k^\infty(\operatorname{div}^s \psi) = \int_{\bar{\Omega}} \mathbf{u}_\psi \cdot \operatorname{div}^s \psi. \tag{96}$$

We combine the definition of $\mathbf{u}_\psi$, (95), (96) and the fact that $\beta \in \det^* \nabla \mathbf{u}$ to deduce

$$\int_{\bar{\Omega}} k(\mathbf{F} + \operatorname{div} \psi) + \int_\Omega h(\beta)dx + \int_\Lambda l dy = \int_\Omega \mathbf{u}_\psi \cdot (\mathbf{F}\mathcal{L}^d + \operatorname{div} \psi_{\mathbb{R}^d}) = \int_\Omega \mathbf{u} \cdot \mathbf{F} dx - \int_\Omega \langle \nabla \mathbf{u}, \psi \rangle dx.$$

Next from (93), one has $\nabla \mathbf{u} = Df^*(\psi)$ and thus

$$\int_{\bar{\Omega}} k(\mathbf{F} + \operatorname{div} \psi) + \int_\Omega h(\beta)dx + \int_\Lambda l dy = \int_\Omega \mathbf{u} \cdot \mathbf{F} dx - \int_\Omega \big(f^*(\psi) + f(\nabla \mathbf{u})\big)dx,$$

and therefore

$$-J(k, l, \psi) = \int_\Omega \big(f(\nabla \mathbf{u}) + h(\beta) - \mathbf{u} \cdot \mathbf{F}\big)dx \geq I(\mathbf{u}) = \bar{I}(\gamma^{(\mathbf{u},\beta)}) \geq -J(k, l, \psi).$$

We deduce that $(\mathbf{u}, \beta)$ is a minimizer of I over $\mathcal{U}_b$ and $\gamma^{(\mathbf{u},\beta)}$ minimizes $\bar{I}$ over Γ. Let $(\mathbf{u}_1, \beta_1)$ be a minimizer of I over $\mathcal{U}_b$. Then $\gamma^{(\mathbf{u}_1,\beta_1)}$ minimizes $\bar{I}$ over Γ. We use furthermore Lemma 4.3.9 to get that $\nabla \mathbf{u}_1 = \nabla f^*(\psi) = \nabla \mathbf{u}$. Since in addition $\mathbf{u}$ and $\mathbf{u}_1$ have range Λ up to a set of 0 Lebesgue measure we deduce that $\mathbf{u} = \mathbf{u}_1$ a.e.

$\square$

CHAPTER V

MINIMIZATION WITH INCOMPRESSIBLE MATERIALS

In this chapter we turn our attention to some problems with the limit case $H = \chi_{\{1\}}$. This corresponds to the case where $\beta = \det{}^H \nabla \mathbf{u} = 1$. We check that many of the arguments in the previous chapters can be adapted to such a singular H.

5.1 Settings

Throughout this chapter Ω and Λ stand for two open and bounded convex sets of $\mathbb{R}^d$. We suppose that $\mathcal{L}^d(\Omega) = \mathcal{L}^d(\Lambda)$. Let $\mathcal{S}$ be a finite dimensional subspace of $W^{1,\infty}(\Omega, \mathbb{R}^{d\times d})$ consisting of function that are affine piecewise. Let $v_0 \in W^{1,2}(\Omega, \mathbb{R}^{d\times d})$ and let $g \in L^2(\partial\Omega, \mathbb{R}^d, \mathcal{H}^{d-1})$ be its trace. Let

$$\mathcal{U} := \Big\{ \mathbf{u} : \mathbf{\Omega} \to \mathbf{\Lambda} \, ; \mathbf{u} \text{ is a Borel map} \Big\}.$$

In the next Lemma we define a pseudo-projected gradient of $\mathbf{u} \in \mathcal{U}$ analogue to the one defined in Theorem 3.6.1.

Lemma 5.1.1 *Let $\mathbf{u} \in \mathcal{U}$. Let $\mathscr{G}(\mathbf{u})$ be the set of $G \in L^2(\Omega, \mathbb{R}^{d\times d})$ satisfying the relation*

$$\int_\Omega \mathbf{u} \mathrm{div}\, \psi \, \mathbf{dx} = -\int_{\mathbf{\Omega}} \langle \mathbf{G}, \psi \rangle \mathbf{dx} + \int_{\partial\mathbf{\Omega}} \mathbf{g}(\psi \cdot \nu) \mathbf{d}\mathcal{H}^{\mathbf{d-1}} \qquad \forall \psi \in \mathcal{S}. \tag{97}$$

There exists a unique $\nabla_{\mathcal{S}} \mathbf{u} \in \mathbf{L^2}(\mathbf{\Omega}, \mathbb{R}^{\mathbf{d\times d}})$ that minimizes $\int_\Omega |G|^2/2dx$ over $\mathscr{G}(\mathbf{u})$. In fact $\nabla_{\mathcal{S}} \mathbf{u}$ is also the unique $G \in \mathscr{G}(\mathbf{u})$ satisfying $G \in \mathcal{S}$.

Proof. First remark that the functional

$$\mathcal{S} \ni \psi \mapsto -\int_\Omega \mathbf{u} \mathrm{div}\, \psi \, \mathbf{dx} + \int_{\partial\mathbf{\Omega}} \mathbf{g}(\psi \cdot \nu) \mathbf{d}\mathcal{H}^{\mathbf{d-1}}$$

is linear over a finite dimensional space. When we endow $\mathcal{S}$ with the $L^2(\Omega, \mathbb{R}^{d\times d})$ norm, by Riesz representation theorem, there exists a unique $G_0 \in \mathcal{S}$ such that

$$\int_\Omega \mathbf{u} \mathrm{div}\, \psi\, \mathbf{dx} = -\int_{\mathbf{\Omega}} \langle \mathbf{G_0}, \psi \rangle \mathbf{dx} + \int_{\partial\mathbf{\Omega}} \mathbf{g}(\psi \cdot \nu) \mathbf{d}\mathcal{H}^{\mathbf{d-1}} \qquad \forall \psi \in \mathcal{S}.$$

It follows that $\mathscr{G}(\mathbf{u})$ is nonempty and since in addition it is a convex set, there exists a unique $G_1 \in \mathscr{G}(\mathbf{u})$ that minimizes $\int_\Omega |G|^2/2dx$ over $\mathscr{G}(\mathbf{u})$. Let

$$\mathcal{S}^\perp = \{G \in L^2(\Omega, \mathbb{R}^{d\times d}) : \int_\Omega \langle G, \psi \rangle \, dx = 0; \, \forall \psi \in \mathcal{S}\}.$$

Remark that since G_1 and G_0 satisfy Equation 97, one has $G_1 - G_0 \in \mathcal{S}^\perp$. Next we have

$$\int_\Omega |G_1|^2/2dx = \int_\Omega |G_1 - G_0|^2/2dx + \int_\Omega |G_0|^2/2dx.$$

Hence $\int_\Omega |G_1|^2/2dx \geq \int_\Omega |G_0|^2/2dx$. Since $G_0 \in \mathscr{G}(\mathbf{u})$, we conclude $G_1 = G_0$.

□

Remark 5.1.2 *If* $\mathbf{u} \in W^{1,2}(\Omega, \mathbb{R}^{d\times d})$ *and we replace* $\mathcal{S}$ *by* $W^{1,2}(\Omega, \mathbb{R}^{d\times d})$ *in Equation* (97)*, we get that* $\mathbf{g}$ *is the trace of* $\mathbf{u}$ *on* $\partial\Omega$*.*

Let $\mathscr{H}$ be the set of all $u \in \mathcal{U}$ satisfying:

$$\int_\Omega l(\mathbf{u}(x))dx = \int_\Lambda l(y)dy; \quad \forall l \in C_b(\mathbb{R}^d). \tag{98}$$

$\mathscr{H}$ is the well studied set of measure preserving map. Let $\mathbf{F} \in L^1(\Omega, \mathbb{R}^d)$. We consider the problem

$$\inf_{u\in\mathscr{H}} \left\{ I(\mathbf{u}) := \int_\Omega \frac{|\nabla_{\mathcal{S}} u|^2}{2} - F \cdot u \right\}. \tag{99}$$

5.2 *Dual problem*

Call $\mathcal{C}$ the set of Borel measurable functions $k : \mathbb{R}^d \to \bar{\mathbb{R}}$ and $l : \mathbb{R}^d \to \bar{\mathbb{R}}$ that are proper and such that $l \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Lambda}$ and

$$k(\mathbf{v}) + l(\mathbf{u}) \geq \mathbf{u} \cdot \mathbf{v}, \quad \forall u, v \in \mathbb{R}^d. \tag{100}$$

Remark that if $(k,l) \in \mathcal{C}$, then k and l are bounded below. Define

$$J(k,l,\psi) := \int_\Omega \left(k(F + \operatorname{div}\psi) + \frac{|\psi|^2}{2} \right) dx - \int_{\partial\Omega} \mathbf{g}(\psi \cdot \nu) d\mathcal{H}^{d-1} + \int_\Lambda l(y)dy.$$

5.3 *The functional $-J$ achieves it maximum over $\mathcal{C} \times \mathcal{S}$*

5.3.1 A regularity result on maximizers

Let $(k,l) \in \mathcal{C}$. Then $(k,k^*) \in \mathcal{C}$ and $J(k,l) \geq J(k,k^*)$. Similarly, $(l^*,l) \in \mathcal{C}$ and $J(k,l) \geq J(l^*,l)$. Furthermore for all $a \in \mathbb{R}$, if $(k,l) \in C$, $k^* = l$, $l^* = k$, then $(k+a, l-a) \in \mathcal{C}$. Moreover, $(k+a)^* = l-a$, $(l-a)^* = k+a$ and $J(k,l) = J(k+a, l-a)$ (indeed we have $\mathcal{L}^d(\Omega) = \mathcal{L}^d(\Lambda)$). Hence we may assume without lost of generality that the maximization is performed over the set

$$\mathcal{C}' = \{(k,l) \in C : k^* = l, l^* = k, l(0) = 0\}.$$

We have proved the following Lemma:

Lemma 5.3.1 *One has:*

$$\sup_{(k,l)\in\mathcal{C},\psi\in\mathcal{S}} -J(k,l,\psi) = \sup_{(k,l)\in\mathcal{C}',\psi\in\mathcal{S}} -J(k,l,\psi).$$

Remark 5.3.2 *Remark that if $(k,l) \in \mathcal{C}$, then l^* and k^* corresponds respectively to $l_\#$ and $k^\#$ as in Definition 4.1.1.*

5.3.2 A lower bound for J

Let $u \in \Omega$. It holds that

$$\begin{aligned}
\int_\Omega k(F + \operatorname{div}\psi)dx \geq& \int_\Omega (u \cdot (\operatorname{div}\psi + F) - l(\mathbf{u})) \\
&= - \int_\Omega \langle 0, \psi \rangle \, dx + \int_{\partial\Omega} u(\psi \cdot \nu) d\mathcal{H}^{d-1} + \int_\Omega u \cdot F dx - \int_\Omega l(\mathbf{u})dx \\
&= \int_{\partial\Omega} u(\psi \cdot \nu) d\mathcal{H}^{d-1} + \int_\Omega u \cdot F dx - \int_\Lambda l(\mathbf{u})dx \\
&\geq - r^* \|F\|_{L^1(\Omega,\mathbb{R}^d)} + \int_{\partial\Omega} u(\psi \cdot \nu) d\mathcal{H}^{d-1} - \int_\Lambda l(\mathbf{u})dx.
\end{aligned}$$

Hence

$$J(k,l,\psi) \geq -r^*\|F\|_{L^1(\Omega,\mathbb{R}^d)} + \int_\Omega \frac{|\psi|^2}{2}dx + \int_{\partial\Omega}(u-\mathbf{g})(\psi\cdot\nu)d\mathcal{H}^{d-1} + \int_\Lambda (l(y) - l(\mathbf{u}))\,dy. \tag{101}$$

Let us define for $u \in \Omega$ and $\psi \in \mathcal{S}$

$$A(u,\psi) := \int_\Omega \frac{|\psi|^2}{2}dx + \int_{\partial\Omega}(u-\mathbf{g})(\psi\cdot\nu)d\mathcal{H}^{d-1}.$$

5.3.3 A minorant of A.

In this sequel we prove the following Lemma:

Lemma 5.3.3 *There exists $m \in \mathbb{R}$ such that for all $u \in \Omega$ and all $\psi \in \mathcal{S}$, one has*

$$\int_\Omega \frac{|\psi|^2}{2}dx + \int_{\partial\Omega}(u-\mathbf{g})(\psi\cdot\nu)d\mathcal{H}^{d-1} \geq m. \tag{102}$$

Proof. We have $u \in \Omega$ and thus $|u|$ is bounded. Remark that for $u \in \Omega$ fixed, the functional $A^u : \mathcal{S} \to \mathbb{R}$ defined by $A^u(\psi) := A(u,\psi)$ is a quadratic form defined on a finite dimensional space. Hence we can find a uniform lower bound which is an affine function of $|u|$. Since Ω is bounded, we can find $m \in \mathbb{R}$ such that for all $u \in \Omega$ and all $\psi \in \mathcal{S}$, one has

$$\int_\Omega \frac{|\psi|^2}{2}dx + \int_{\partial\Omega}(u-\mathbf{g})(\psi\cdot\nu)d\mathcal{H}^{d-1} \geq m.$$

□

5.3.4 Sub–level sets of A.

The aim of this sequel is to establish the following Lemma

Lemma 5.3.4 *For $c \in \mathbb{R}$ the sub–level set $S_c := \{\psi \in \mathcal{S} : A(u,\psi) \leq c \ \forall u \in \Omega\}$ is compact.*

Proof. Since $\mathcal{S}$ has a finite dimension, it is enough to show that S_c is a closed and bounded set. Set for all $\psi \in \mathcal{S}$:

$$L_u(\psi) = \int_{\partial\Omega} (u - \mathbf{g})(\psi \cdot \nu) d\mathcal{H}^{d-1}.$$

Since $u \in \Omega$ is bounded, there exists C depending only on $\mathcal{S}$ and r^* such that $\|L_u\| \leq C$ for all $u \in \Omega$. Suppose $\|\psi\|_{L^2(\Omega,\mathbb{R}^{d\times d})} > 1$.

$$\begin{aligned} \|\psi\|^2_{L^2(\Omega,\mathbb{R}^{d\times d})} + L_u(\psi) &\leq c \\ \|\psi\|_{L^2(\Omega,\mathbb{R}^{d\times d})} - C \leq c(\|\psi\|_{L^2(\Omega,\mathbb{R}^{d\times d})})^{-1} &\leq c \\ \|\psi\|_{L^2(\Omega,\mathbb{R}^{d\times d})} &\leq C + c \end{aligned}$$

Thus we deduce that $\|\psi\|_{L^2(\Omega,\mathbb{R}^{d\times d})} \leq \max(C + c, 1)$. Hence S_c is bounded. The closure of S_c comes from the continuity of A. Thus S_c is compact.

□

5.3.5 Restriction to $\mathcal{C}' \times \mathcal{S}$ of Sub–level sets of J.

For $r > 0$, set $\Omega_r = \{x \in \Omega : d(x, \partial\Omega) > r\}$. Let $r_0 > 0$ be small enough so that $B(0, r_0) \subset \Omega_{r_0}$. Let $(k, l) \in \mathcal{C}'$ and $\psi \in \mathcal{S}$ such that $J(k, l, \psi) \leq c$. Having $l(0) = 0$, one gets $k(\mathbf{v}) \geq 0 \cdot v - l(0) = 0$, and we deduce that $k(\mathbf{v}) \geq 0$ for all $v \in \mathbb{R}^d$. We have $\inf l > -\infty$. Let $u \in \Omega$.

Thanks to Equations (101) and (102) we have

$$J(k, l, \psi) \geq - r^* \|F\|_{L^1(\Omega,\mathbb{R}^d)} + \int_\Omega \frac{|\psi|^2}{2} dx + \int_{\partial\Omega} (u - \mathbf{g})(\psi \cdot \nu) d\mathcal{H}^{d-1} + \int_\Lambda (l(y) - l(\mathbf{u}))\, dy$$

and there exists $m \in \mathbb{R}$ such that for all $\psi \in \mathcal{S}$ and all $u \in \Omega$,

$$\int_\Omega \frac{|\psi|^2}{2} dx + \int_{\partial\Omega} (u - \mathbf{g})(\psi \cdot \nu) d\mathcal{H}^{d-1} \geq m. \tag{103}$$

Hence $J(k, l, \psi) \leq c$ implies that

$$c \geq - r^* \|F\|_{L^1(\Omega,\mathbb{R}^d)} + m + \int_\Lambda (l(y) - l(\mathbf{u}))\, dy$$

$$\alpha := c + r^* \|F\|_{L^1(\Omega)} - m \geq \int_\Omega l(x) - l(\mathbf{u}).$$

Since the later inequality is true for all $u \in \Omega$ we have $\alpha \geq \int_\Omega l(x) - \inf_\Omega l$. Thanks to Lemma A.3.12, for $r \leq r_0$ there exists a real number $C(\alpha, \Omega_r, \Omega)$ depending only on c, Ω_r and Ω such that

$$\sup_{x \in \Omega_r} |l - \inf l|, \text{Lip } (l - \inf l)_{|\Omega_r} \leq C(\alpha, \Omega_r, \Omega).$$

We also have Lip $l_{|\Omega_r} \leq C(\alpha, \Omega_r, \Omega)$. The convex function l is bounded in Ω_{r_0} that contains the origin. Hence $\partial l(0)$ is non empty and moreover, letting $y \in \partial l(0)$, we have $|y| \leq C(\alpha, \Omega_{r_0}, \Omega)$ (c.f. Lemmas A.3.12 and A.3.10). Thus for $u \in \Omega$,

$$l(\mathbf{u}) \geq l(0) + y \cdot u = y \cdot u \geq -r^* C(\alpha, \Omega_{r_0}, \Omega).$$

Having $k = l^*$, we get

$$k(\mathbf{v}) = \sup_{u \in \bar{\Omega}} \mathbf{u} \cdot \mathbf{v} - l(\mathbf{u}) \leq r^* |v| + r^* C(\alpha, \Omega_{r_0}, \Omega).$$

We exploit also the fact that $k = l^*$ and properties of l to deduce that (c.f. Lemma A.3.11) Lip $k \leq r^*$. Hence we have for all $0 < r < r_0$, $v \in \mathbb{R}^d$ and $u \in \Omega$:

$$\begin{aligned} l(0) =&0; \qquad l(\mathbf{u}) \geq C(\alpha, \Omega_{r_0}, \Omega); \qquad \text{Lip } l_{|\Omega_r} \leq C(\alpha, \Omega_r, \Omega); \\ \text{Lip } k \leq& r^*; \qquad 0 \leq k(\mathbf{v}) \leq r^* |v| + r^* C(\alpha, \Omega_{r_0}, \Omega). \end{aligned}$$

Using Equation (101) again and the fact that $\int_\Lambda l(y) - \inf_\Omega l \geq 0$, We get that $J(k, l, \psi) \leq c$ implies that

$$\begin{aligned} c \geq& - r^* \|F\|_{L^1(\Omega, \mathbb{R}^d)} + A(u, \psi) \\ c + r^* \|F\|_{L^1(\Omega)} \geq& A(u, \psi). \end{aligned}$$

Hence ψ belongs to a compact set. We deduce that there exists (k_0, l_0, ψ_0) with $(k_0, l_0) \in \mathcal{C}$ and $\psi \in \mathcal{S}$ such that

$$-J(k_0, l_0, \psi_0) = \sup_{(k,l) \in \mathcal{C}', \psi \in \mathcal{S}} -J(k, l, \psi) = \sup_{(k,l) \in \mathcal{C}, \psi \in \mathcal{S}} -J(k, l, \psi).$$

Thus we have the following Lemma.

Lemma 5.3.5 *There exists* $(k_0, l_0, \psi_0) \in \mathcal{C} \times \mathcal{S}$ *such that*

$$-J(k_0, l_0, \psi_0) = \sup_{(k,l)\in\mathcal{C},\psi\in\mathcal{S}} -J(k, l, \psi).$$

5.4 A duality result

5.4.1 Variations of $J(\cdot,\cdot,\psi_0)$ along a special curve.

Take $\bar{l} \in C_c(\mathbb{R}^d)$. For $\epsilon > 0$, define $l_\epsilon = l_0 + \epsilon \bar{l}$. Define $k_\epsilon = (l_\epsilon)^*$. Assume k_0 is differentiable at v. There exists (Thanks to Lemma A.3.9) $T_0(v) \in \bar{\Omega}$ such that

$$k_0(v) + l_0(T_0(v)) = T_0(v) \cdot v.$$

For all $\epsilon > 0$, (Thanks to Lemma A.3.9) there exists $T_\epsilon(v) \in \bar{\Omega}$ such that

$$k_\epsilon(v) + l_0(T_\epsilon(v)) + \epsilon \bar{l}(T_\epsilon(v)) = T_\epsilon(v) \cdot v. \tag{104}$$

Thus we have

$$\begin{aligned} k_\epsilon(v) \leq& - \epsilon \bar{l}(T_\epsilon(v)) + T_\epsilon(v) \cdot v - l_0(T_\epsilon(v)) \\ k_\epsilon(v) \leq& - \epsilon \bar{l}(T_\epsilon(v)) + k_0(v) \\ k_\epsilon(v) - k_0(v) \leq& - \epsilon \bar{l}(T_\epsilon(v)). \end{aligned}$$

In the same fashion,

$$\begin{aligned} k_\epsilon(v) =& T_0(v) \cdot v - l_0(T_0(v)) - \epsilon \bar{l}(T_0(v)) + \epsilon \bar{l}(T_0(v)) \\ k_0(v) \leq& \epsilon \bar{l}(T_0(v)) + k_\epsilon(v) \\ k_\epsilon(v) - k_0(v) \geq& - \epsilon \bar{l}(T_0(v)). \end{aligned}$$

We deduce

$$\bar{l}(T_\epsilon(v)) \leq -\frac{k_\epsilon(v) - k_0(v)}{\epsilon} \leq \bar{l}(T_0(v)), \tag{105}$$

from which it follows that

$$\left| \frac{k_\epsilon(v) - k_0(v)}{\epsilon} \right| \leq \|\bar{l}\|_{L^\infty(\mathbb{R}^d)} \tag{106}$$

$\lim_{\epsilon\to 0} k_\epsilon(v) = k_0(v)$. We exploit the later equality, the lower semicontinuity of l_0 and Equation (104) to deduce that

$$k_0(v) + l_0(\underline{\lim}_{\epsilon} T_\epsilon(v)) \leq \underline{\lim}_{\epsilon} T_\epsilon(v) \cdot v.$$

It follows that $\underline{\lim}_\epsilon T_\epsilon(v)$ belongs to $\partial k(\mathbf{v})$ which has a unique element $T_0(v)$ since k_0 is differentiable at v. Thus $\underline{\lim}_\epsilon T_\epsilon(v) = T_0(v)$. We use the later fact and the continuity of $\bar{l}$ in Equation (105) to deduce

$$\bar{l}(T_0(v)) \leq \underline{\lim}_{\epsilon\to 0} \bar{l}(T_\epsilon(v)) \leq \underline{\lim}_{\epsilon\to 0} -\frac{k_\epsilon(v) - k_0(v)}{\epsilon} \leq \overline{\lim}_{\epsilon\to 0} -\frac{k_\epsilon(v) - k_0(v)}{\epsilon} \leq \bar{l}(T_0(v)),$$

and thus

$$\lim_{\epsilon\to 0^+} -\frac{k_\epsilon(v) - k_0(v)}{\epsilon} = \bar{l}(T_0(v)), \tag{107}$$

Combining equation (107) with Equation (106), since for almost all $x \in \Omega$, k is differentiable at $F(x) + \operatorname{div}\psi(x)$:

$$\lim_{\epsilon\to 0^+} \int_\Omega \left(\frac{k_0(F + \operatorname{div}\psi_0) - k_\epsilon(F + \operatorname{div}\psi_\epsilon)}{\epsilon} \right) dx = \int_\Omega \bar{l}(T(F + \operatorname{div}\psi_0)). \tag{108}$$

Set $u_0 := T(F + \operatorname{div}\psi_0)$. One has:

$$\begin{aligned} &\lim_{\epsilon\to 0^+} \frac{J(k_0, l_0, \psi_0) - J(k_\epsilon, l_0, \psi_0)}{\epsilon} \\ &= -\int_\Omega \bar{l} + \lim_{\epsilon\to 0^+} \int_\Omega \left(\frac{k_0(F + \operatorname{div}\psi_0) - k_\epsilon(F + \operatorname{div}\psi_\epsilon)}{\epsilon} \right) dx \\ &= -\int_\Omega \bar{l} + \int_\Omega \bar{l}(u_0). \end{aligned}$$

As for all $\epsilon > 0$, having $\frac{J(k_0,l_0,\psi_0) - J(k_\epsilon,l_0,\psi_0)}{\epsilon} \leq 0$, we get for all $\bar{l} \in C_c(\mathbb{R}^d)$ $0 \geq -\int_\Omega \bar{l} + \int_\Omega \bar{l}(u_0)$. Replacing $\bar{l}$ by $-\bar{l}$, we get $0 \leq -\int_\Omega \bar{l} + \int_\Omega \bar{l}(u_0)$ and thus $\int_\Omega \bar{l}(u_0)dx = \int_\Omega \bar{l}$. Thus $(u_0)_\#(\chi_\Omega \mathcal{L}^d) = \chi_\Omega \mathcal{L}^d$.

5.4.2 Variations of $J(k_0, l_0, \cdot)$ along a special curve.

Let $\epsilon \in (0, 1)$. Let $\bar{\psi} \in C_c(\mathbb{R}^{d\times d})$. Set $u_0 = \nabla k_0(F + \operatorname{div}\psi_0)$. Set $\psi_\epsilon = \psi + \epsilon\bar{\psi}$.

The term in k_0. Since k_0 is r^*-Lipchitz, we have

$$\left|\frac{k_0(F+\operatorname{div}\psi_0)-k_0(F+\operatorname{div}\psi_\epsilon)}{\epsilon}\right| \leq r^*|\operatorname{div}\bar{\psi}|. \tag{109}$$

Moreover, at almost every $x \in \Omega$ (as F is non degenerate and the range of ψ_0 is countable) $k_0(F+\operatorname{div}\psi_0)$ is differentiable and one has

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{k_0(F+\operatorname{div}\psi_0)-k_0(F+\operatorname{div}\psi_\epsilon)}{\epsilon} &= -\nabla k_0(F+\operatorname{div}\psi_0)\cdot\operatorname{div}\bar{\psi}\\ &= -u_0\cdot\operatorname{div}\bar{\psi}.\end{aligned}$$

Combining with Equation (109), by the Lebesgue Dominated Convergence Theorem, we get:

$$\lim_{\epsilon\to 0}\int_\Omega\left(\frac{k_0(F+\operatorname{div}\psi_0)-k_0(F+\operatorname{div}\psi_\epsilon)}{\epsilon}\right)dx = -\int_\Omega u_0\cdot\operatorname{div}\bar{\psi}.$$

The term in ψ_0. We have

$$\frac{|\psi_0|^2}{2}-\frac{|\psi_0+\epsilon\bar{\psi}|^2}{2} = -\epsilon\left\langle\psi_0,\bar{\psi}\right\rangle-\epsilon^2\frac{|\bar{\psi}|^2}{2}.$$

Hence

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\int_\Omega\frac{|\psi_0|^2}{2}-\int_\Omega\frac{|\psi_0+\epsilon\bar{\psi}|^2}{2}\right) = -\int_\Omega\left\langle\psi_0,\bar{\psi}\right\rangle.$$

We also have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(-\int_{\partial\Omega}\mathbf{g}(\psi_0\cdot\nu)d\mathcal{H}^{d-1}+\int_{\partial\Omega}\mathbf{g}((\psi_0+\epsilon\bar{\psi})\cdot\nu)d\mathcal{H}^{d-1}\right) = \int_{\partial\Omega}\mathbf{g}(\bar{\psi}\cdot\nu)d\mathcal{H}^{d-1}.$$

Wrapping up. One has

$$\lim_{\epsilon\to 0^+}\frac{J(k_0,l_0,\psi_0)-J(k_0,l_0,\psi_\epsilon)}{\epsilon} = -\int_\Omega u_0\cdot\operatorname{div}\bar{\psi}+\bar{\psi}\cdot\psi_0+\int_{\partial\Omega}\mathbf{g}(\bar{\psi}\cdot\nu)d\mathcal{H}^{d-1}.$$

Having for all $\epsilon \in (0,1)$

$$\frac{J(k_0,l_0,\psi_0)-J(k_0,l_0,\psi_\epsilon)}{\epsilon} \geq 0,$$

We deduce

$$0 \geq -\int_\Omega u_0\cdot\operatorname{div}\bar{\psi}+\bar{\psi}\cdot\psi_0+\int_{\partial\Omega}\mathbf{g}(\bar{\psi}\cdot\nu)d\mathcal{H}^{d-1}.$$

Replacing $\bar{\psi}$ by $-\bar{\psi}$, one deduces that

$$0 \leq -\int_{\Omega} u_0 \cdot \operatorname{div} \bar{\psi} + \bar{\psi} \cdot \psi_0 + \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1}.$$

and thus

$$-\int_{\Omega} u_0 \cdot \operatorname{div} \bar{\psi} = \int_{\Omega} \bar{\psi} \cdot \psi_0 - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1}.$$

Moreover, as $\psi_0 \in \mathcal{S}$ we have $\psi_0 = \nabla_{\mathcal{S}} u_0$.

5.4.3 A duality result

Suppose F is non degenerate . Let $(k, l) \in \mathcal{C}$ and $\psi \in \mathcal{S}$. Let also $\mathbf{u} \in \mathscr{H}$. One has

$$\begin{aligned} J(k, l, \psi) &= \int_{\Omega} \left(k(F + \operatorname{div} \psi) + \frac{|\psi|^2}{2} \right) dx + \int_{\Lambda} l(y) dy - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1} \\ &\geq \int_{\Omega} \left(\mathbf{u}(x) \cdot (F + \operatorname{div} \psi) - l(u(x)) + \frac{|\psi|^2}{2} \right) + \int_{\Lambda} l(y) dy - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1} \\ &=: A_1 \end{aligned}$$

with equality if and only if $u = \nabla k(F + \operatorname{div} \psi)$ for a.e. $x \in \Omega$. Next, Using $\mathbf{u} \in \mathcal{H}$, we get:

$$\begin{aligned} A_1 &= \int_{\Omega} \left(\mathbf{u}(x) \cdot (F + \operatorname{div} \psi) + \frac{|\psi|^2}{2} \right) - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1} \\ & \int_{\Omega} \mathbf{u}(x) \cdot F(x) + \int_{\Omega} \left(\mathbf{u}(x) \cdot \operatorname{div} \psi) + \frac{|\psi|^2}{2} \right) - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1}. \end{aligned}$$

Using the definition of $\nabla_{\mathcal{S}} \mathbf{u}$, we get

$$\begin{aligned} A_1 &= \int_{\Omega} \mathbf{u}(x) \cdot F(x) - \int_{\Omega} \nabla_{\mathcal{S}} \mathbf{u} \cdot \psi + \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1} + \int_{\Omega} \frac{|\psi|^2}{2} - \int_{\partial\Omega} \mathbf{g}(\bar{\psi} \cdot \nu) d\mathcal{H}^{d-1} \\ &= \int_{\Omega} \mathbf{u}(x) \cdot F(x) - \int_{\Omega} \nabla_{\mathcal{S}} \mathbf{u} \cdot \psi + \int_{\Omega} \frac{|\psi|^2}{2} \\ &\geq \int_{\Omega} \left(\mathbf{u} \cdot F(x) - \frac{|\nabla_{\mathcal{S}} \mathbf{u}|^2}{2} \right) dx \end{aligned}$$

with equality if and only if $\nabla_{\mathcal{S}} \mathbf{u}(x) = \psi(x)$ a.e. Hence we have shown the following Lemma

Lemma 5.4.1 *Suppose F is non degenerate . Let $(k,l) \in \mathcal{C}$ and $\psi \in \mathcal{S}$. Let also $\mathbf{u} \in \mathscr{H}$. Then*

$$-J(k,l,\psi) \leq \bar{I}(\mathbf{u})$$

with equality if and only if one has $\nabla_{\mathcal{S}}\mathbf{u}(x) = \psi(x)$ and $u(x) = \nabla k(F(x) + \operatorname{div}\psi(x))$ a.e. in Ω.

As a consequence we have the following Theorem:

Theorem 5.4.2 *Suppose F is non degenerate. Let $(k_0, l_0, \psi_0) \in \mathcal{C} \times \mathcal{S}$ be such that*

$$-J(k_0, l_0, \psi_0) = \sup_{(k,l)\in\mathcal{C},\, \psi\in\mathcal{S}} -J(k,l,\psi).$$

The problem

$$\inf_{u\in\mathscr{H}} \int_\Omega \frac{|\nabla_{\mathcal{S}}\mathbf{u}|^2}{2} - F\cdot\mathbf{u}$$

admits a unique minimizer $\mathbf{u}$ satisfying

$$\begin{aligned}\mathbf{u}(x) =& \nabla k_0(F(x) + div\psi_0(x))\\ \nabla_{\mathcal{S}}\mathbf{u}(x) =& \psi_0(x).\end{aligned}$$

APPENDIX A

USEFUL RESULTS AND DEFINITIONS

A.1 Convex analysis tools

We start this section by recalling the basic definitions in convex analysis. Classical references are [20, Rockafellar], [6, Dacorogna] and [8, Ekeland- Témam].

Definition A.1.1

- *A set $A \subset \mathbb{R}^d$ is convex whenever for all $x, y \in A$ and all $t \in [0,1]$ one has $tx + (1-t)y \in A$.*

- *Let Ω be a convex set. A function $f : \Omega \to \bar{\mathbb{R}}$ is said to be convex if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall t \in [0,1],\ \forall x, y \in \Omega,$$

  whenever the right hand side of the inequality is well defined.

- *A function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is said to be lower semi-continuous (or closed) if whenever $\lim_{n\to\infty} x_n = \bar{x}$ in $\mathbb{R}^d$, one has $f(\bar{x}) \leq \liminf_{n\to\infty} f(x_n)$.*

- *The domain of the function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is the set*

$$\operatorname{dom} f = \{x \in \mathbb{R}^d : f(x) < \infty\}.$$

- *Let A be a subset of $\mathbb{R}^d$. The characteristic function of A is the function defined on $\mathbb{R}^d$ by $\chi_A(x) = 0$ if $x \in A$ and $\chi_A(x) = \infty$ if $x \notin A$.*

- *The epigraph of the function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is the set*

$$\operatorname{epi} f = \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : f(x) \leq t\}.$$

Legendre-Fenchel transform.

Definition A.1.2 *One calls the Legendre-Fenchel transform of the function* $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ *the function denoted* f^* *defined for all* $y \in \mathbb{R}^d$ *by* $f^*(y) = \sup_{x \in \mathbb{R}^d}\{x \cdot y - f(x)\}$. *Similarly, one defines* f^{**} *to be the Legendre-Fenchel transform of the function* f^* *so that for all* $x \in \mathbb{R}^d$, $f^{**}(x) = \sup_{y \in \mathbb{R}^d}\{x \cdot y - f^*(y)\}$.

Subdifferentials.

Definition A.1.3 *One says that* $y \in \mathbb{R}^d$ *is a subgradient of* f *at the point* x *if*

$$f(x) \geq f(z) + y \cdot (x - z), \quad \forall z \in \mathbb{R}^d.$$

The set of all subgradients of f *at* x *is called the subdifferential of* f *at* x. *It is denoted* $\partial f(x)$.

Remark that $y \in \partial f(x)$ means either of the following

(1) $y \cdot z - f(z)$ achieves its maximum at x.

(2) $y \cdot x - f(x) = f^*(y)$.

Recession functions.

Definition A.1.4 *Let* $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ *be a convex function. The function* $f_\infty : \mathbb{R}^d \to \bar{\mathbb{R}}$ *defined for all* $y \in \mathbb{R}^d$ *by* $f_\infty(y) = \sup_{x \in \mathrm{dom}\ f}\{f(x + y) - f(x)\}$ *is called the recession function of* f.

We turn next our attention to recession functions of closed convex functions.

Lemma A.1.5 *Let* $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ *be a convex function. If* f *is closed and* dom $f \neq \emptyset$, *then for any* $x \in$ dom f,

$$f_\infty(y) = \sup_{\lambda > 0}\{\frac{f(x + \lambda y) - f(x)}{\lambda}\} = \lim_{\lambda \to \infty}\{\frac{f(x + \lambda y) - f(x)}{\lambda}\}, \quad \forall y \in \mathbb{R}^d.$$

Lemma A.1.6 *Assume* $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ *is proper, convex and closed. Then*

$$f_\infty(x) = \sup_{x^* \in \mathrm{dom}\ f^*} x^* \cdot x.$$

Minkowsky functional. Throughout this paragraph, $\Lambda \subset \mathbb{R}^d$ is an open bounded convex set and we assume that there exist $r, R > 0$ such that $B(0,r) \subset \Lambda \subset B(0,R)$.

Definition A.1.7 *One defines the Minkowsky functional (or gauge) of Λ for all $x \in \mathbb{R}^d$ by $\rho_\Lambda(x) = \inf\{t > 0 : x \in t\Lambda\}$.*

The following Lemma gives the main properties of the Minkowsky functional.

Lemma A.1.8 *We have*

1. *For all $x \in \mathbb{R}^d$ $R^{-1}|x| \le \rho_\Lambda(x) \le r^{-1}|x|$. In particular $\rho_\Lambda(x) = 0$ if and only if $x = 0$.*

2. *For all $x \in \mathbb{R}^d$, $\rho_\Lambda(x) = \inf\left\{t > 0 : x \in t\bar{\Lambda}\right\}$.*

3. *For all $x \in \mathbb{R}^d$ such that $x \neq 0$, one has $\frac{x}{\rho_\Lambda(x)} \in \bar{\Lambda}$. Moreover, $\bar{\Lambda} = \left\{x \in \mathbb{R}^d : \rho_\Lambda(x) \le 1\right\}$, $\Lambda = \left\{x \in \mathbb{R}^d : \rho_\Lambda(x) < 1\right\}$ and $\partial\Lambda = \left\{x \in \mathbb{R}^d : \rho_\Lambda(x) = 1\right\}$.*

4. *The function ρ_Λ is semi-linear, i.e*

   (a) $\rho_\Lambda(x+y) \le \rho_\Lambda(x) + \rho_\Lambda(y)$, for all $x, y \in \mathbb{R}^d$;

   (b) $\rho_\Lambda(tx) = t\rho_\Lambda(x)$ for all $x \in \mathbb{R}^d$, for all $t \ge 0$.

   Moreover, ρ_Λ is convex and continuous.

Lemma A.1.9 *For $x \in \mathbb{R}^d$, one has:*

1. *$w \in \partial\rho_\Lambda(x) \Rightarrow \rho_\Lambda(x) = x \cdot w$.*

2. *ρ_Λ is differentiable almost every where and for a.e. x, one has:*

$$|\nabla\rho_\Lambda(x)| \in [R^{-1}, r^{-1}].$$

Support function of a closed convex set Throughout this paragraph, let $\Lambda \subset \mathbb{R}^d$ be open and convex. We assume that there exist $r,\ R > 0$ such that $B(0,r) \subset \Lambda \subset B(0,R)$. Set $g = \chi_{\bar{\Lambda}}$, the characteristic function of $\bar{\Lambda}$. Define the function $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ by

$$f(v) = \sup_{u \in \bar{\Lambda}} \mathbf{u} \cdot \mathbf{v}. \tag{110}$$

Definition A.1.10 *The function f defined by Equation 110 is called the support function of $\bar{\Lambda}$.*

Lemma A.1.11 (Some properties of f)

1. *For all $v \in \mathbb{R}^d$ there exists $u \in \bar{\Lambda}$ such that $f(v) = \mathbf{u} \cdot \mathbf{v}$. In particular,* $\operatorname{dom} f = \mathbb{R}^d$.
2. *$f(v) = 0$ if and only if $v = 0$.*
3. *For all $v \in \mathbb{R}^d$, $f(v) \geq 0$.*
4. *For all $0 \neq v \in \mathbb{R}^d$ we have*

$$\left\{u \in \bar{\Lambda} : f(v) = \mathbf{u} \cdot \mathbf{v}\right\} = \left\{u \in \partial\bar{\Lambda} : f(v) = \mathbf{u} \cdot \mathbf{v}\right\}.$$

Lemma A.1.12 *For all $v \in \mathbb{R}^d$, one has*

$$\partial f(v) = \left\{u \in \bar{\Lambda} : f(v) = \mathbf{u} \cdot \mathbf{v}\right\} = \left\{u \in \partial\Lambda : f(v) = \mathbf{u} \cdot \mathbf{v}\right\}.$$

Measurable selection. We will need the following Lemma.

Lemma A.1.13 *Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be a convex lower-semicontinuous function and let O be a non-empty open set of int*$(\operatorname{dom} f)$*. Then there exists a measurable function $S : O \to \mathbb{R}^d$ such that $S(x) \in \partial f(x)$ for all $x \in O$.*

See for e.g. [21, Rockafellar-Wets], for more information.

A.2 Tools from measure theory

A.2.1 Weak convergence of measure

Good references for this topics include [7, Dellacherie, C. and Meyer] and [1, Ambrosio-Gigli-Savaré].

Definition A.2.1 (weak convergence of measures) *A sequence of bounded measures $\{\mu_n\}_{n=1}^{\infty}$ on X is said to converge weakly to $\mu \in \mathcal{M}_b(X)$ if and only if*

$$\lim_{n\to\infty} \int_X \varphi d\mu_n = \int_X \varphi d\mu, \quad \forall \varphi \in C_b(X). \tag{111}$$

We write $\mu_n \rightharpoonup \mu$.

Here follows a proposition that allows us to get a result similar to Equation (111) under weaker conditions.

Lemma A.2.2 *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of probability measures on X weakly converging to μ.*

1. *Let f be a continuous function on X such that $\lim_{a\to\infty} \sup_n \int_{|f|>a} |f| d\mu_n = 0$. Then we have $\lim_{n\to\infty} \int_X f d\mu_n = \int_X f d\mu$.*

2. *Let g be a bounded below lower semicontinuous function on X. Then we have*

$$\lim_{n\to\infty} \int_X g d\mu_n \geq \int_X g d\mu.$$

We also have

Lemma A.2.3 *Let $\{\mu_n\}_{n=1}^{\infty}$ be a sequence of measures on X. Suppose $f : X \to \bar{R}$ and $g : X \to \bar{R}$ satisfy $|f| \leq \alpha |g|$ for some constant $\alpha \geq 0$. Assume in addition that there exists $\varphi : [0,\infty) \to [0,\infty)$ measurable such that $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$ and $\sup_n \int_X \varphi(|g|)\mu_n < \infty$. Then*

$$\lim_{a\to\infty} \left(\sup_n \int_{\{|f|>a\}} |f| \mu_n \right) = 0.$$

In the remaining of the sequel X denotes a metric space. Let $\mathcal{M}_b(X)$ be the space of bounded measures on X.

Definition A.2.4 (Polish space) *A polish space is a separable topological space which has a compatible metric that is complete.*

Definition A.2.5 (Tightness of a family of probability measures) *A sequence $\{\mu_n\}_{n=1}^{\infty}$ of $\mathcal{M}_b(X)$ is said to be tight if for every $\epsilon > 0$, there exists K_ϵ, a compact set of X such that :*

$$\sup_n \mu_n(X \setminus K_\epsilon) \leq \epsilon.$$

Lemma A.2.6 *A sequence $\{\mu_n\}_{n=1}^{\infty}$ of $\mathcal{M}_b(X)$ is tight if there exists a function $\varphi : X \to [0, \infty]$ whose sublevels $\{x \in X : \varphi(x) \leq c\}$ are compact in X such that*

$$\sup_n \int_X \varphi(x) d\mu_n < \infty.$$

Theorem A.2.7 (Prokhorov) *Let X be a Polish space. Then a family of probability measures on X is relatively compact (has a subsequence that converges weakly) if and only if it is tight.*

A.2.2 Parametrized measures

We begin this subsection by giving definitions of some useful spaces.

Definition A.2.8 *Let $\Omega \subset \mathbb{R}^d$ and let X be a Banach space with norm $\|\cdot\|$ and dual X'.*

1. *A function $f : \Omega \to X$ is said to be simple if f can be writen in the form $f(x) = \sum_{i=1}^{m} u_i 1_{E_i}$ for $E_i \subset \Omega$ measurable and $u_i \in X$.*

2. *A function $f : \Omega \to X$ is said to be strongly measurable if f is the a.e. limit of a sequence of simple functions $\{f_n\}_{n=1}^{\infty}$.*

3. *One defines the space*

$$L^p(\Omega, X) = \left\{ f | f : \Omega \to X;\ f \text{ is strongly measurable and } \int_\Omega \|f(x)\|^p dx < \infty \right\}.$$

The space $L^p(\Omega, X)$ *is endowed with the norm* $\|f\|_{L^p(\Omega,X)} = \int_\Omega \|f(x)\|^p dx$.

4. *A function* $g : \Omega \to X'$ *is said to be weakly star measurable if for every* $u \in X$, *the map* $g_u : \Omega \to \mathbb{R}; x \mapsto \langle g(x), u\rangle$ *is measurable. The set of weakly star measurable functions from* Ω *to* X' *will be denoted* $L^0_{w^*}(\Omega, X')$.

5. *One defines the space*

$$L^p_{w^*}(\Omega, X') = \left\{ g | g \in L^0_{w^*}(\Omega, X');\ \|g(\cdot)\|_{X'} \in L^0(\Omega),\ \int_\Omega \|g(x)\|^p_{X'} dx < \infty \right\}.$$

The space $L^p_{w^*}(\Omega, X')$ *is endowed with the norm* $\|g\|_{L^p_{w^*}(\Omega,X')} = \int_\Omega \|g(x)\|^p_{X'} dx$.

The next theorem gives the essence of parametrized measures. We refer the reader to [19, Pedregal].

Theorem A.2.9 *Let the sequence* $\{u_n\}_{n=1}^\infty \subset L^p(\Omega, \mathbb{R}^d)$ *be such that*

$$\lim_{M\to\infty} \sup_n \mathcal{L}^d\{x \in \Omega : |u_n(x)| \geq M\} = 0. \tag{112}$$

then there exists a subsequence $\{u_{n_k}\}_{k=1}^\infty$ *of* $\{u_n\}_{n=1}^\infty$ *and* $\mu \in L^\infty_{w^*}(\Omega, M_b(\mathbb{R}^d))$ *such that for a.e.* $x \in \Omega$, μ_x *is a probability measure on* $\mathbb{R}^d$ *and whenever* $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ *is Caratheodory function such that* $\{\psi(\cdot, u_{n_k}(\cdot))\}_n$ *is uniformly integrable, then*

$$\psi(\cdot, u_{n_k}(\cdot)) \rightharpoonup \bar{\psi} \text{ in } L^1(\Omega, \mathbb{R}^d) \text{ with } \psi(x) = \int_{\mathbb{R}^d} \psi(x, \lambda) d\mu_x. \tag{113}$$

Corollary A.2.10 *Assume that the sequence* $\{u_n\}_{n=1}^\infty \subset L^p(\Omega, \mathbb{R}^d)$ *is such that* $u_n \rightharpoonup u$ *in* $L^1(\Omega, \mathbb{R}^d)$. *Then a subsequence of* $\{u_n\}_{n=1}^\infty$ *generates a parametrized measure* μ. *Moreover, for a.e.* $x \in \Omega$, *one has:*

$$u(x) = \int_{\mathbb{R}^d} \lambda d\mu_x. \tag{114}$$

Corollary A.2.11 *Assume that the sequence* $\{u_n\}_{n=1}^{\infty} \subset L^p(\Omega, \mathbb{R}^d)$ *converges weakly to* u *in* $L^p(\Omega, \mathbb{R}^d)$ *and* $\{u_n\}_{n=1}^{\infty}$ *generates a parametrized measure* μ*. Assume that for a.e.* $x \in \Omega$*, one has:* $\mu_x = \delta_{u(x)}$*. Then* $\{u_n\}_{n=1}^{\infty}$ *converges strongly to* u *in* $L^p(\Omega, \mathbb{R}^d)$*.*

The next lemma proves to be useful.

Lemma A.2.12 *Assume that* $\{u_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ *converges weakly to* u*. Assume that* $f : \mathbb{R}^d \to \mathbb{R}^d$ *is strictly convex and*

$$\lim_{n\to\infty} \int_{\Omega} f(u_n) = \int_{\Omega} f(u).$$

Then $\{u_n\}_{n=1}^{\infty}$ *converges strongly to* u *in* $L^p(\Omega)$*.*

Proof. It is enough to show that every subsequence has a subsequence converging strongly to u. Consider a subsequence of $\{u_n\}_{n=1}^{\infty}$ again denoted $\{u_n\}_{n=1}^{\infty}$. Since $\{u_n\}_{n=1}^{\infty} \subset L^p(\Omega)$ converges weakly to u, thank to Corollary A.2.10, a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ of $\{u_n\}_{n=k}^{\infty}$ generates a Young measure μ that satisfies $u(x) = \int_{\mathbb{R}^d} \lambda d\mu_x$ for a.e $x \in \Omega$.

Thanks to A.2.11 , it is enough to show that $\mu_x = \delta_{u(x)}$ for a.e. $x \in \Omega$. The following holds

$$\begin{aligned}
\int_{\Omega} f(u(x))dx &= \lim_{k\to\infty} \int_{\Omega} f(u_{n_k}(x))dx \\
&= \int_{\Omega} \left(\int_{\mathbb{R}^d} f(\lambda) d\mu_x \right) dx \\
&\geq \int_{\Omega} f\left(\int_{\mathbb{R}^d} \lambda d\mu_x \right) dx \\
&= \int_{\Omega} f(u(x))dx.
\end{aligned}$$

One deduces that

$$\int_{\Omega} \left(\int_{\mathbb{R}^d} f(\lambda) d\mu_x - f\left(\int_{\mathbb{R}^d} \lambda d\mu_x \right) \right) = 0.$$

Exploiting Jensen Inequality and the fact that for a.e. $x \in \Omega$, one has

$$\int_{\mathbb{R}^d} f(\lambda) d\mu_x - f\left(\int_{\mathbb{R}^d} \lambda d\mu_x \right) \geq 0,$$

one obtains that for a.e. $x \in \Omega$, one has

$$\int_{\mathbb{R}^d} f(\lambda) d\mu_x - f\left(\int_{\mathbb{R}^d} \lambda d\mu_x\right) = 0.$$

Using Jensen's inequality one more time together with the strict convexity of f yields that for a.e. $x \in \Omega$, we have μ_x is a Dirac measure. Since $u(x) = \int_{\mathbb{R}^d} \lambda d\mu_x$, we get $\mu_x = \delta_{u(x)}$ for a.e. $x \in \Omega$. □

A.3 Other results

A.3.1 Change of variable Formula

In this subsection we assume that $\Omega \subset \mathbb{R}^d$ is an open set and $f \in W^{1,1}_{loc}(\Omega, \mathbb{R}^{d'})$. We set $J_f(x) := \det(Df(x))$. For a non zero positive integer k, $J_f^k(x)$ denotes a matrix whose elements are the determinant of the k-dimensional sub-matrices of $Df(x)$. For a subset E of Ω and $y \in \mathbb{R}^{d'}$, $N(f, y, E)$ denotes the cardinality of the set $E \cap f^{-1}(\{y\})$. The proof of the results in this subsection can be found in [17, Maly]. We refer the reader also to [13, Fonseca-Gangbo] and [11, Evans-Gariepy]

A.3.1.1 Change of variable via the area formula

Definition A.3.1 (Area Formula) *Assume $d' \geq d$. One says that the area formula holds for f if for all measurable set $E \subset \Omega$, one has that the function $\mathbb{R}^{d'} \ni y \mapsto N(f, y, E)$ is $\mathcal{H}^d$-measurable and*

$$\int_E |J_f(x)| dx = \int_{\mathbb{R}^{d'}} N(f, y, E) d\mathcal{H}^d(y). \tag{115}$$

Theorem A.3.2 *Assume $d' \geq d$ and the area formula holds for f. If $u : \Omega \to \mathbb{R}$ is measurable and $E \subset \Omega$ is measurable, then*

$$\int_E u(x)|J_f(x)| dx = \int_{\mathbb{R}^d} \left(\sum_{x \in E \cap f^{-1}(\{y\})} u(x) \right) dy, \tag{116}$$

provided that either $u \geq 0$ or the left hand side is well defined.

Theorem A.3.3 *If $p > d$, $d' \geq d$ and $f \in W^{1,p}(\Omega, \mathbb{R}^{d'})$, then the area formula holds.*

A.3.1.2 Change of variable via the coarea formula

Definition A.3.4 (Coarea Formula) *Assume $d' \leq d$. One says that the coarea formula holds for f if for all measurable set $E \subset \Omega$, one has that the function $\mathbb{R}^{d'} \ni y \mapsto \mathcal{H}^{d-d'}(E \cap f^{-1}(\{y\}))$ is measurable and*

$$\int_E |J_f^{d'}(x)|dx = \int_{\mathbb{R}^{d'}} \mathcal{H}^{d-d'}(E \cap f^{-1}(\{y\}))dy. \tag{117}$$

Theorem A.3.5 *Assume $d' \leq d$ and the coarea formula holds for f. If $u : \Omega \to \mathbb{R}$ is measurable and $E \subset \Omega$ is measurable, then*

$$\int_E u(x)|J_f(x)|dx = \int_{\mathbb{R}^d} \left(\int_{E \cap f^{-1}(\{y\})} u(x) d\mathcal{H}^{d-d'}(x) \right) dy, \tag{118}$$

provided that either $u \geq 0$ or the left hand side is well defined.

Theorem A.3.6 *If $p > d$, $d' \leq d$ and $f \in W^{1,p}(\Omega, \mathbb{R}^{d'})$, then the Coarea formula holds.*

A.3.2 Ascoli-Arzela theorem

In this subsection we recall the definitions of equicontinuity and the Ascoli-Arzela's theorem. A proof of the later can be found for instance in [12, Folland].

Definition A.3.7 (Equicontinuty) *A family of functions $\{f_i\}_{i \in I}$ defined on $\mathbb{R}^d$ is said to be uniformly equicontinuous if for all $\epsilon > 0$, on can find $\delta(\epsilon) > 0$ such that for all $x, y \in \mathbb{R}^d$ satisfying $|x - y| < \delta(\epsilon)$ and all $i \in I$, one has*

$$|f_i(x) - f_i(y)| < \epsilon.$$

Theorem A.3.8 (Ascoli-Arzela) *Let $\{f_n\}_{n=1}^{\infty}$ be a family of real valued continuous functions on $\mathbb{R}^d$ that are uniformly equicontinuous and uniformly bounded. Then there exists a continuous function f and a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ that converges uniformly to f on every compact sets.*

A.3.3 Results on weak convergence, convexity and Lipchitz functions

Lemma A.3.9 *Let Ω be bounded set. Let $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ be lower semicontinuous. Assume that $f \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Omega}$. Then for every $v_0 \in \mathbb{R}^d$ there exists $u_0 \in \bar{\Omega}$ such that $f^*(v_0) = u_0 \cdot v_0 - f(u_0)$.*

Proof. Let $v_0 \in \mathbb{R}^d$. As $f \equiv +\infty$ on $\mathbb{R}^d \setminus \bar{\Omega}$,

$$f^*(v_0) = \sup_{u \in \mathbb{R}^d} \{\mathbf{u} \cdot \mathbf{v}_0 - f(u)\} = \sup_{u \in \bar{\Omega}} \{\mathbf{u} \cdot \mathbf{v}_0 - f(u)\}.$$

Consider a maximazing sequence $\{u_n\}_{n=1}^{\infty}$ of $\sup_{u \in \bar{\Omega}} \{\mathbf{u} \cdot \mathbf{v}_0 - f(u)\}$. We assume without lost of generality that $\{u_n\}_{n=1}^{\infty}$ converges to some u_0 in $\bar{\Omega}$. Then, as f lower semicontinuous,

$$f^*(v_0) = \liminf_n u_n \cdot v_0 - f(u_n) \leq u_0 \cdot v_0 - f(u_0) \leq f^*(v_0).$$

Thus $f^*(v_0) = u_0 \cdot v_0 - f(u_0)$.

Lemma A.3.10 *Let $L, R > 0$. Assume $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is convex and $L-$Lipchitz on $B(x_0, R)$. Then $\partial f(x_0)$ is nonempty and for all $y \in \partial f(x_0)$, we have $|y| \leq L$.*

Proof. The nonemptyness of $\partial f(x_0)$ follows from the convexity and the boundedness of f on $B(x_0, R)$. Next for $y \in \partial f(x_0)$

$$L|y| = L|y + x_0 - x_0| \geq f(y + x_0) - f(x_0) \geq y \cdot (y + x_0 - x_0) = |y|^2.$$

We deduce that $|y| \leq L$.

□

Lemma A.3.11 *Assume $f : \mathbb{R}^d \to \bar{\mathbb{R}}$ is proper, bounded below and $f \equiv +\infty$ on $\mathbb{R}^d \setminus B(0, r)$. Then f^* is a $r-$Lipchitz function.*

Proof. Call $m := \inf f$. There exists $x_0 \in B(0,r)$ such that $f(x_0) \in \mathbb{R}$. Let $y \in \mathbb{R}^d$. It holds that

$$-f(x_0) - r|y| \leq -f(x_0) + x_0 \cdot y \leq f^*(y) \leq r|y| - m.$$

This implies that $f^*(y)$ is always a real number.

Let $y_1, y_2 \in \mathbb{R}^d$ and $\epsilon > 0$. As f^* is a real valued function, there exists $x_1 \in B(0,r)$ such that $f^*(y_1) - \epsilon \leq x_1 \cdot y_1 - f(x_1)$. furthermore, $x_1 \cdot y_2 - f(x_1) \leq f^*(y_2)$. Thus

$$f^*(y_1) - f^*(y_2) \leq x_1 \cdot (y_1 - y_2) + \epsilon \leq r|y_1 - y_2| + \epsilon.$$

Having $f^*(y_1) - f^*(y_2) \leq r|y_1 - y_2| + \epsilon$ for all ϵ, we deduce

$$f^*(y_1) - f^*(y_2) \leq r|y_1 - y_2|.$$

Similarly, one proves that $f^*(y_2) - f^*(y_1) \leq r|y_1 - y_2|$ and deduce

$$|f^*(y_1) - f^*(y_2)| \leq r|y_1 - y_2|,$$

which shows that f^* is a $r-$Lipchitz function

□

The next Lemma may be drawn from [11, Evans-Gariepy]

Lemma A.3.12 *Let $\Omega \subset \mathbb{R}^d$ be a an open set. Let K be a compact set contained in Ω and let $\alpha > 0$. Then there exists a real number $C(\alpha, K, \Omega)$ depending only on α, K and Ω such that for all $f : \Omega \to \bar{\mathbb{R}}$ convex satisfying $\int_\Omega |f| \leq \alpha$, we have*

$$\begin{aligned} \sup_{x \in K} |f(x)| &\leq C(\alpha, K, \Omega); \\ ess \sup_{x \in K} |\nabla f(x)| &\leq C(\alpha, K, \Omega); \\ Lip f_{|K} &\leq C(\alpha, K, \Omega). \end{aligned}$$

Lemma A.3.13 *If $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is convex, then the functional $F : L^p(\Omega) \to \bar{\mathbb{R}}$ defined by*

$$F(u) = \int_\Omega f(u)dx$$

is weakly lower semicontinuous. In particular, F is strongly lower semicontinuous.

Lemma A.3.14 *Assume $\Omega \subset \mathbb{R}^d$ is is a finite measure measurable set. Consider a sequence $\{u_n\}_{n=1}^\infty$ such that $|u_n| \leq C$ for all n. Assume the sequence $\{v_n\}_{n=1}^\infty$ converges weakly to v in $L^1(\Omega)$. Moreover, assume that $u_n \to u$ a.e. Then*

$$\lim_{n\to\infty} \int_\Omega u_n \cdot v_n = \int_\Omega u \cdot v.$$

Proof. Set $M := \sup_n |v_n|_{L^1(\Omega)}$. Take $\delta > 0$. As $\{v_n\}_{n=1}^\infty$ converges weakly to v, there exists $\epsilon > 0$ such that for all measurable set E,

$$|E| < \epsilon \Rightarrow \int_E |v_n| < \delta, \forall n.$$

As $|\Omega| < \infty$, by Ergorov theorem, there exists a compact set K_ϵ such that $|\Omega \backslash K_\epsilon| < \epsilon$ and $u_n \to u$ uniformly on K_ϵ.

Let

$$I_n := \int_\Omega (u_n - u)v_n = \int_{\Omega\backslash K_\epsilon} (u_n - u)v_n + \int_{K_\epsilon} (u_n - u)v_n =: a_n + b_n$$

One has $|a_n| \leq \int_{\Omega\backslash K_\epsilon} 2C|v_n| \leq 2C\delta$. Moreover

$$\begin{aligned}|b_n| \leq &\sup_{K_\epsilon} |u_n - u| \int_{K_\epsilon} |v_n| \\ \leq &M \sup_{K_\epsilon} |u_n - u|.\end{aligned}$$

As $u_n \to u$ uniformly on K_ϵ, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, one has $\sup_{K_\epsilon} |u_n - u| < \delta$. Hence for all $n > N_1$, one has $|I_n| \leq 2C\delta + M\delta$.

As $\{u_n\}_{n=1}^\infty \subset L^\infty(\Omega)$ and $\{v_n\}_{n=1}^\infty$ converges weakly to v, setting $II_n = \int_\Omega u(v - v_n)$, there exists $N_2 > N_1$ such that $n > N_2$ implies $|II_n| < \delta$. Having $\int_\Omega u_n \cdot v_n - \int_\Omega u \cdot v = I_n + II_n$, one gets the result.

A.3.4 Disintegration Theorem

The proof of the next Theorem may be found in [7, Dellacherie, C. and Meyer]. We refer the reader also to [15, Gangbo]

Theorem A.3.15 (Disintegration Theorem) *Let (X, d_1) be a separable complete metric space and let μ be a finite Borel measure on $(X, \mathscr{B}(X))$. Let (Y, d_2) be a separable metric space and let ν be a finite measure on $(Y, \mathscr{B}(Y))$. Suppose $T : X \to Y$ is measurable and satisfies $T\#\mu << \nu$. Then there exists a family of finite measures $\{\mu_y\}_{y \in Y}$ on X unique up to a ν-negligible set such that*

1. *For ν-a.e $y \in Y$, one has $\mu_y(\{x \in X : T(x) \neq y\}) = 0$.*
2. *If $f : X \to [0, \infty)$ and $g : Y \to [0, \infty)$ are measurable, then*
   (a) *The map $Y \ni y \mapsto \int_X f(x)g(y)\mu_y(dx)$ is measurable.*
   (b) *One has:*

$$\int_X f(x)g(T(x))\mu(dx) = \int_Y \left(\int_X f(x)g(y)\mu_y(dx) \right) \nu(dy).$$

If in addition $T\#\mu = \nu$, then for ν-a.e $y \in Y$, the measure μ_y is a probability measure.

We have the following application.

Theorem A.3.16 *Let X, Y be two separable metric spaces and let μ be a Borel measure on $X \times Y$ and suppose that the map $\Pi : X \times Y \to X : (x, y) \mapsto x$ pushes μ forward to a measure σ on X. Then there exists a familly $\{\mu_x\}_{x \in \Omega}$ of Borel probability measure unique σ a.e. such that for all $f : X \times Y \to [0, \infty]$ measurable, one has that the map $X \ni x \mapsto \int_Y f(x, y)\mu_x(dy)$ is measurable and*

$$\int_{X \times Y} f(x, y)\mu(dx, dy) = \int_X \left(\int_Y f(x, y)\mu_x(dy) \right) \sigma(dx).$$

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