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Towards a rook-theoretic model for ASEP

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Abstract

This thesis analyses the relation between rook covers over Ferrers boards and the Asymetric Exclusion Process (ASEP). A polynomial $f(q)$ defined over the set of all possible rook covers has been suggested to be identical to the polynomial that gives the probabilities of the stationary distribution of the ASEP.

In this thesis a draft is presented of a possible proof by induction of this claim, and parts of this induction are proved. Further results regarding $f_(q)$ that would follow from the main claim are also independently proved and a complete proof of the claim, invented by another author, is presented for the sake of completeness.

I den här uppsatsen undersöks förhållandet mellan tornplaceringar på Ferrersbräden och den asymetriska exklusionsprocessen (ASEP). Ett polynom $f_(q)$ över alla möjliga tornplaceringar har föreslagits vara ekvivalent med polynomet som ger sannolikheterna i den stationära fördelningen för ASEP.

Ett utkast till ett induktionsbevis av detta påstående presenteras i den här upsatsen. Vidare resultat kring $f(q)$ som skulle följa från detta huvudpåstående bevisas separat och ett mer utförligt bevis av huvudpåståendet skapat av en annan författare presenteras också.

Contents

Chapter 1

Introduction

1.1 Rooks, Boards and Inversions

This project considers a particular type of combinatorial objects, namely boards and rooks. A board B here consists of rows of possible positions, with the length of each row decreasing or staying the same counting from the top down, see Fig:1.1. Given a board, rooks can be placed on it in a way so that no two rooks occupy the same row or column, but so that all rows contain exactly one rook. On a board with n rows a set of n rooks positioned like described above will constitute a rook cover. The set of all possible rook covers over a board B will in this article be written as Ω_B.

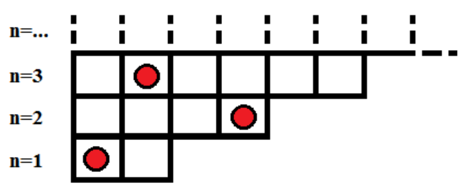

Figure 1.1: A board with an arbitrarily chosen rook cover.

Given a board with n rows, a rook cover can be written as a tuple of n integers, each element in the tuple showing in which column the rook in the corresponding row is placed. So given a board with $n = 3$ rows, the rook cover shown in Fig.1.1 can be expressed as $(1, 4, 2)$. The set of all possible positions on a board with n rows and k columns can be seen as a subset of the group of permutations S_k.

Given a board and a rook cover it is of interest to consider two measurements. The first one is the number k of *inversions*, i.e number of pairs of rooks were the one with the higher row index has the lower column index, for a rook cover. The other one is the tuple $\bar{X} = (X_1, X_2, X_3, ..., X_n)$ where X_i is the number of rooks in the i^{th} column with odd index.

These two quantities can be used to map a subset of all possible rook covers, $\Theta_B \subseteq \Omega_B$, to a polynomial, with the following sum over Θ_B.

$$f(q, X_1, X_2, X_3, \cdots, X_n) = \sum_{\Theta_B} q^k X_1^{p_1} X_2^{p_2} X_3^{p_3}, \cdots, X_n^{p_n} \tag{1.1}$$

A quick example

Fig:1.2 illustrates the possible rook covers for a specific board. The resulting polynomial for $\Theta_B = \Omega_B$ will, according to (1.1), be: $f(q, X_1, X_2, X_3, ..., X_n) = 1 + 2X_1 + X_2 + X_1X_2 + X_1q$.

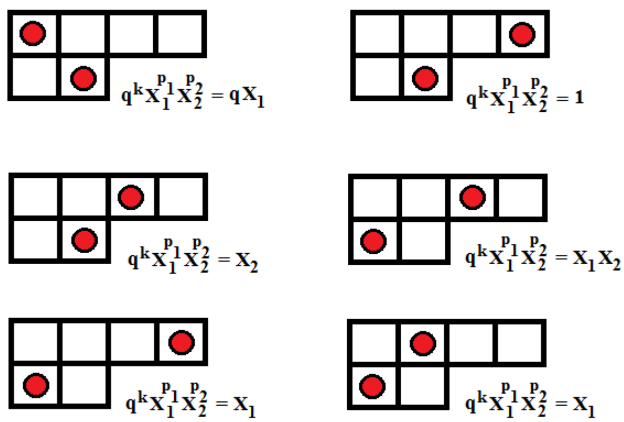

Figure 1.2: The six different possible rook covers for a board of size $n = 2$.

1.2 Relation to ASEP

In this article a certain kind of movement of particles on a line observed in statistical physics, the so called *Asymetric Simple Exclusion Process*, *ASEP*, will be studied from a combinatorial perspective.

Consider a line of possible positions for particles, where each position either is or isn't occupied. Over a number of discrete time steps the particles move left or right with probabilities q and $1 - q$, while new particles appear at the left end of the line and leave at the right end.

At any given time some of the positions are filled and some are not, each such state of the line of particles can be represented by a sequence of ones and zeroes $\bar{x} = (x_1, x_2, ..., x_n)$, referred to as a word. The main object of study in this article will be the relation between such a sequence or word and the set of filled and unfilled odd indexed columns of a rook cover mentioned above.

The sequence of states that the line of particles jumps between constitutes a Markov chain. The stationary distribution of this Markov chain is a function of $\bar{x}$, $P(\bar{x})$. It is shown in [1] that $P(\bar{x})$ multiplied by some constant obeys the recursive relation (1.2).

$$\begin{gathered} F(x_1, x_2, ..., x_{j-1}, 1, 0, x_{j+1}, ..., x_n) = \\ F(x_1, x_2, ..., x_{j-1}, 1, x_{j+1}, ..., x_n) + \\ F(x_1, x_2, ..., x_{j-1}, 0, x_{j+1}, ..., X_n) + \\ qF(x_1, x_2, ..., x_{j-1}, 0, 1, X_{j+1}, ..., X_n) \\ F(0, x_2, x_3, x_4, ..., x_n) = F(x_1, x_2, x_3, ..., x_n, 1) = F(x_1, x_2, x_3, ..., x_n) \\ F(0) = f(1) = 1 \end{gathered} \tag{1.2}$$

It has been suggested that $f(\bar{X})$ given by (1.1) also obeys the recursive relation (1.2), a suggestion formalized in *Conjecture 1.1.*

Conjecture 1.1:

- Given some board of size n, the polynomial f given by (1.1) will obey the recursive relation given by (1.2).

In this article attempts will be made to as far as possible prove this conjecture directly with rook theoretic arguments. There also exist indirect proofs of *Conjecture 1.1*, involving an intermediary, that will be presented in the interest of completeness.

The recursion will often conveniently be written as $f(u10v) = qf(u01v) + f(u1v) + f(u0v)$. u and v will often be used to indicate sub-words of some word, and ones and zeroes will sometimes be appended or prepend to words when a reference is to be made to the first or last element in the sequence. For example $(1u)$ is some word that starts with a one and $(v0)$ is some word that ends with a zero.

Introduce for future reference the notation $[k]$ to mean the polynomial $1+q+q^2+\cdots+q^{k-1}$ and with that notation in mind consider the polynomial $\hat{E}_{k,n}(q)$ defined by (1.3). It is shown in [1] that, given a (P)ASEP model with n sites that has converged towards its steady state, the probability of it containing exactly k particles is proportional to $\hat{E}_{k+1,n+1}(q)$.

$$\hat{E}_{k,n}(q) = q^{k-k^2} \sum_{i=0}^{k-1} (-1)^i [k-i]^n q^{ki-k} (\binom{n}{i} q^{k-i} + \binom{n}{i-1}) \tag{1.3}$$

Now for a board B, let $f_k(q)$ be the polynomial (1.1) over all covers for which exactly k odd-indexed columns contain a rook.

It has been proposed that $\hat{E}_{k+1,n+1}(q) = f_k(q)$ for a board of size n. This claim, although it would follow from *Conjecture 1.1* where that claim proved, will henceforth for now as a matter of convenience be formalized as *Conjecture 1.2.*

Conjecture 1.2:

- Given the polynomial $f_k(q)$ over a board of size n for some value k, $f_k(q) = E_{k+1,n+1}(q)$.

This article presents an attempt to show that this claim is plausible with simple direct rook-theoretic arguments by showing that $f_k(q)$ shares several important properties with $\hat{E}_{k+1,n+1}(q)$, such as degree, some coefficients etc.

1.3 Developments parallel to the writing of this article

Parallel to the writing of this article it came to the authors attention that an alternative, indirect method of proving *Conjecture 1.1* was being developed by [2]. This solution was not, at the time of writing, published in its entirety, but it did lead to the creation of an indirect proof of *Conjecture 1.1* by [3], using results from [4].

This indirect proof of *Conjecture 1.1* does not rely only on observations of rook covers over boards but also on using another class of combinatorial objects, so called *Motzkin paths*, as an intermediary. While it is the primary objective of this article to justify both *Conjecture 1.1* and *Conjecture 1.2* as far as possible using only direct rook-theoretic arguments, this indirect proof will nevertheless be presented in this article.

The proof using Motzkin paths as an intermediary is interesting both as background information for the sake of completeness and because some analogies to the results concerning Motzkin paths can be helpful when developing purely rook-theoretic results.

Chapter 2

Background

2.1 Reduction into simpler problems

In this chapter several methods of computing the polynomial (1.1) for different subsets of all possible rook covers are discussed.

Recall that (1.1) is a sum over a set $\Theta_B \subseteq \Omega_B$. This sum can be expressed as a sum of sums over several different subsets of Ω_B, i.e as $f(q) = \sum_{\Theta_B} q^k = \sum_{\Theta_1} q^k + \sum_{\Theta_2} q^k$. These subsets Θ_1, Θ_2 etc must obviously form an exact partition of Ω_B.

Several results presented in this article revolve around mappings from a set of covers Θ_B to another set of covers Θ'_B, where $f(q)$ is preserved. These mappings are often created using the above mentioned linearity by mapping subsets of Θ_B to subsets of Θ'_B, and showing how $f(q)$ is preserved for each such subset mapping.

Regarding edge cases

The recursive relation (1.2) contains expressions for two special cases, namely $f(0v) = f(v)$ and $f(u1) = f(u)$. In ASEP these two cases relate to the probability of a certain set of positions containing particles while the rightmost position is also filled or the leftmost position is not filled, and as part of *Conjecture 1.1* it is of interest to prove that these two relations also apply for the polynomial (1.1) over all covers forming the words $u1$ and $0v$.

Proof 2.1:
First observe the set of all covers Ω_u over the smaller board of size $n-1$. There is a bijection $\phi : \Omega_u \to \Omega_{u1}$ that also preserves the number of inversions. For each cover $C \in \Omega_u$, ϕ adds an extra row with a rook in the second position from the right. This is obviously the only place in which the rook corresponding to the 1 in $u1$ can be placed, and since it is located in the top row and has no rooks to its right it will not contribute to any inversions. See Fig.2.1 for a visualization.

Thus for every cover in Ω_u there is exactly one cover in Ω_{u1} with the same number

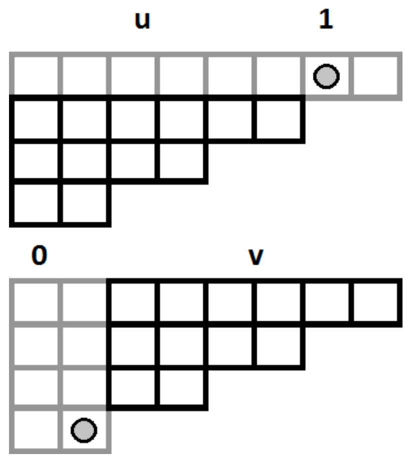

Figure 2.1: Visualization of the concepts behind the proof that the recursive relations $f(u1) = f(u)$ and $f(0v) = f(v)$ apply to rook covers over boards.

of inversions, and since $|\Omega_{u1}| = |\Omega_u|$ it can be concluded that $f(u1) = f(u)$.

A similar approach can be applied to the case of the word $0v$, i.e a bijection can be found between Ω_v and Ω_{0v} that conserves the number of inversions for each cover. This time the bijection is made by considering a larger board created by adding two columns of length n to the left edge of the smaller board of size $n-1$. A rook is then placed in the only position it can be, namely the right end of the new bottom row. This rook is part of no inversions as it is the leftmost rook of the cover and located in the bottom row, and so $f(0v) = f(v)$.

Ignoring inversions, $q = 1$

Consider the special case when $q = 1$. Since 1 raised to any power will always be 1, this case is equivalent to computing $f(X_1, X_2, X_3, ..., X_n)$ while ignoring the number of inversions, and thus $f(X_1, X_2, X_3, ..., X_n)$ will simply be the number of possible covers over the board for that word. The following proof has been suggested by [3] to show that in this special case $f(X_1, X_2, X_3, ..., X_n)$ obeys the recursive relation (1.2).

Proof 2.2:

- *Step 1:* This proof centres around the position of the rook corresponding to the 1 in $u10v$, from now on referred to as the *critical rook*. In accordance with the linearity of the problem discussed in this chapter, $f(X_1, X_2, X_3, ..., X_n)$ is here expressed as $f = f_1 + f_2$, where f_1 is the sum $\sum q^k$ over all rook covers where the critical rook is at the bottom of its column and f_2 is the corresponding sum over all covers where the critical rook is not at the bottom. It will now be shown that $f_1 = f(u1v) + f(u0v)$ and that $f_2 = qf(u01v)$, thereby fulfilling (1.2).
- *Step 2:* Consider all rook covers where the critical rook is not at the bottom of its column and can thus be moved two steps to the right. The 0 in $u10v$ means that the column two steps to the right of the critical rook is empty. For every cover

considered here, there is a corresponding cover where the critical rook is moved two steps to the right. Each such corresponding cover gives an equal contribution to $\sum q^k$ since $q = 1$. Thus it follows that $f_2 = f(u01v)$.

- *Step 3:* Now consider the rook covers for which the critical rook is at the bottom of its column, and the board obtained by simply removing the row where the critical rook is as well as both the columns containing the critical rook and the one two steps to the right.

  Obviously this gives a smaller board in which the words u and v are preserved but in which there is only one odd-indexed column between them. Each such rook cover over a smaller board will give the same contribution to $\sum q^k$ as the original cover it was made from. Since u and v are preserved over this smaller board, $f_2 = f(u0v)+f(u1v)$. The first term corresponds to the the cases where the column directly to the right of the critical rook is empty and the second term to the cases where it isn't.

2.2 Practically computing $f(q)$

The following recursive algorithm has been suggested by [3] for the computation of $f(q)$ over Ω_B for any Ferrers board, not necessarily just ASEP-boards. It is used in the computer aided mass computations discussed in Chapter 3.

Consider the subset of all possible rook covers that contain one rook placed in the rightmost column. This rook will only contribute to the number of inversions in that cover by being the lower part of an inversion, because it is in the rightmost column. Thus a rook in the rightmost column will contribute with exactly as many inversions as there are rows above it. If the rightmost column has odd index i, this rook will also contribute with a factor X_i.

$f(q, X_1, X_2, X_3, ..., X_n)$ can according to the previously discussed linearity be expressed as the sum of the polynomials for two subsets of all possible rook covers. Let one of these subsets be the set of all rook covers containing a rook in the rightmost column and the other subset be the set of all rook covers that don't contain a rook in that column.

The contributions to $f(q, X_1, X_2, X_3...X_n)$ of the covers with no rook in the rightmost column will be equal to the rook polynomial of the board gained by removing that particular column. The contribution from the covers with a rook in the rightmost column will be equal to $f(q, X_1, X_2, X_3...X_n)$ for the board gained by removing the rightmost column and the top row, multiplied with X_i if the index i of the rightmost column is odd and $[q]_k$ where k is the length of the rightmost column. Note that $[q]_k = 1+q+q^2+\cdots+q^{k-1}$.

Thus the following recursive relation is obtained:

$$F_{n_{1,1},n_{1,2},\ldots,n_{1,k_1},n_{2,1},n_{2,2},\ldots,n_{2,k_2},\ldots,n_{3,1}\ldots} =$$
$$X_i[q]_{k_1} F_{n_{1,1}-1,n_{1,2}-1,\ldots,n_{1,k_1-1}-1,n_{2,1},n_{2,2},\ldots,n_{2,k_2},\ldots,n_{3,1}\ldots} +$$
$$F_{n_{1,1}-1,n_{1,2}-1,\ldots,n_{1,k_1}-1,n_{2,1},n_{2,2},\ldots,n_{2,k_2},\ldots,n_{3,1}\ldots}$$

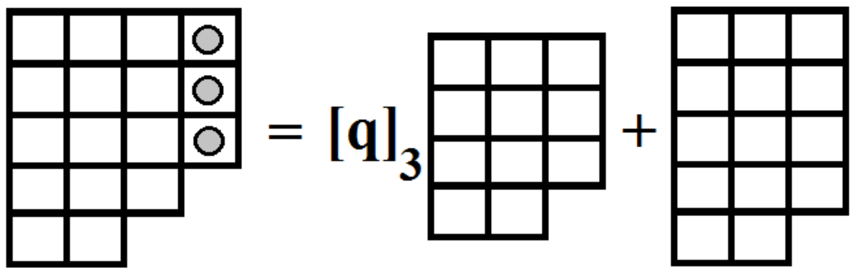

Figure 2.2: An illustration of the recursive algorithm used for practically computing $f(q)$.

2.3 Properties of $\hat{E}_{k,n}$

$\hat{E}_{k,n}$ is in [5] shown to have the following properties:

1. The term in $\hat{E}_{k,n}$ with the highest degree has coefficient 1.
2. $deg(\hat{E}_{k,n}) = (k-1)(n-1)$.
3. $\hat{E}_{k,n}(0) = N_{k,n} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

As an attempt to justify *Conjecture 1.2* by only using rook-theoretic arguments it will be shown that several of these properties of $\hat{E}_{k+1,n+1}(q)$ are shared by $f_k(q)$.

2.4 Using Motzkin paths as an intermediary

The following different, indirect, method of proving *Conjecture 1.1* has been suggested by [3]. This method uses so called *Motzkin paths* as intermediaries and first shows that there is an equivalence between sets of Motzkin paths and sets of rook covers. Furthermore, it is already shown in [4] that certain sets of Motzkin paths satisfy the criteria for being *basic weight functions*, and that basic weight functions obey the recursive relation 1.2.

A Motzkin path of length n is a sequence $p = (v_0, v_1, ..., v_n)$ of 2-tuples (x, y) of four different kinds: $u = (1,1)$, $d = (1,-1)$, $h = (1,0)$ and $\hat{h} = (1,0)$. The 2-tuples are considered to be coordinates of movements in the xy-plane, either diagonally up, diagonally down or to the right. There are two different kinds of steps to the right here, drawing the path as a graph one can consider the edges corresponding to h and $\bar{h}$ to be coloured in different colours. Furthermore, the numbers of u and of d in $p = (v_0, v_1, ..., v_n)$ have to be equal and for any sub-sequence $p' \subset p$ such that $p' = (v_0, v_1, ..., v_k)$, $k < n$, the number of d can't be larger than the number of u, thus restricting the path to one quadrant of the coordinate system.

A word $(w_1, w_2, ..., w_n)$, where $w_i \in (0, 1)$, can be constructed for a subsets of all Motzkin paths of length n, in a way similar to how $(X_1, X_2, ..., X_n)$ can be formulated for a set of rook covers of size n. In the case of a Motzkin paths, each u and $\bar{h}$ is mapped to a 1 and each h and d to a 0. The set of all Motzkin paths corresponding to a certain word $\bar{X}$ is written as $P(\bar{X})$.

Consider, as a quick example, the Motzkin path presented in Fig:2.3. This path, $p = (u, h, u, \bar{h}, d, d)$, would be mapped onto the word $(1, 0, 1, 1, 0, 0)$. Just like with rook covers, there is a one-to-many relationship between a word and a set of paths. The path $p_1 = (u, u, d, d)$ and $p_2 = (u, \bar{h}, h, d)$ both map to the word $(1, 1, 0, 0)$.

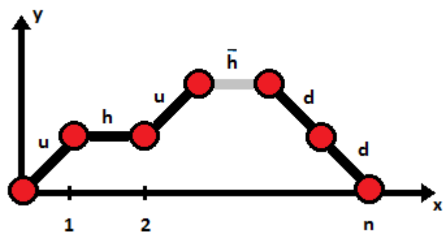

Figure 2.3: An example of a Motzkin path of length n. $p = (u, h, u, \bar{h}, d, d)$.

A polynomial $F_n(x, q)$ can then be formulated as:

$$F_n(X, q) = \sum_{P \in P_n} W(P)$$
$$W(P) = W(w_1)...W(w_n)$$
$$W(w_i) = [r + 1]_q x_i^{w_i}$$

r is here the *height* of w_i, the y-coordinate of the left node of w_i, i.e the height from which that step starts.

[3] has suggested a mapping $\Phi : \Omega_X \rightarrow P_X$ of the set of rook covers corresponding to a word $(X_1, X_2, ..., X_n)$ to the set of Motzkin paths corresponding to the equivalent word $(w_1, w_2, ..., w_n)$.

Let α_i be the sum of the number of rooks in the columns $2i-1$ and $2i$. Φ then assigns one Motzkin path to each rook cover depending on the cover's sequence $(\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ as follows:

1. $v_i \rightarrow u$ if $\alpha_i = 2$.

2. $v_i \rightarrow d$ if $\alpha_i = 0$.

3. $v_i \to h$ if $\alpha_i = 1$ and the rook is in the even-indexed column.

4. $v_i \to \bar{h}$ if $\alpha_i = 1$ and the rook is in the odd-indexed column.

[4] shows that the stationary distribution of THE PASEP is given by:

$$P(X) = \frac{W(X)}{Z_n} \tag{2.1}$$

$W(X)$ in (2.1) is a basic weight function while $\frac{1}{Z_n}$ is a normalization constant. [4] defines a *basic weight function* as a real-valued function $W(X)$ that satisfies:

1. $W(X) = 1, \quad X \in B_0$

2. $W(X) = \alpha W(0X)$

3. $W(X) = \beta W(X1)$

4. $W(u0v) + W(u1v) = \eta W(u10v) - qW(u01v)$

α, β, η are here transition intensities between certain states of the Markov chain corresponding to the asymmetric exclusion process. [4] goes on to introduce $f(X) = \sum g_{Y,X} P(Y)$, where $g_{Y,X}$ is the transition intensity from the particle configuration X to the particle configuration Y, and shows that $P(X) = \frac{W(X)}{Z_n}$ defines a stationary distribution by showing that $f(X) = 0$, using some combinatorial arguments and the fourth property of basic weight functions.

[4] also shows that other functions of $\bar{X}$ are basic weight functions, thereby showing that these functions also give the stationary distribution for the PASEP. In particular, it is shown that $\sum\limits_{P \in P_{\bar{X}}} W(P)$ over $P_{\bar{X}}$ is a basic weight function and thus gives the stationary distribution of the PASEP.

Finally, the circle is completed by using results from [3] to show that Φ is such that F_n for the set $P(\bar{X})$ equals (1.1) for $\Omega_{\bar{X}}$. This is done by considering $Y_P = \sum\limits_{C:P(C)=P} q^{invC}$ for a given Motzkin path P and the corresponding rook cover C.

[3] suggests considering the Ferrers board resulting from deleting all unocupied columns in the board of C, these being exactly defined by the steps in the Motzkin path P. This gives a board with row lengths $\alpha_1, \alpha_1 + \alpha_2$ etc. Results from[4] are referred to showing that for such a board:

$$Y_P = \prod_{i=1}^{n} [\alpha_1 + ... + \alpha_i - i + 1]_q$$

$\alpha_1 + ... + \alpha_i - i$ is recognized as the height r of the corresponding step in the Motzkin path. This can be realized by considering that for the mapping from rook covers to paths, $\alpha_j = 0$ for each step j down and $\alpha_k = 2$ for each step k up. The term $-i$ adjusts for this, making each step up add 1 to the sum and each step down subtract 1.

Having recognized $\alpha_1 + ... + \alpha_i - i$ as the height r, the expression becomes $Y_P = \prod_{i=1}^{n} [r+1]_q$, or the weight of the Motzkin path. Thus the rook cover and the path are equivalent, proving that (1.1) also obeys that recursive relation.

Chapter 3

Analysis

3.1 Conjecture 1.1

A draft of a proof by induction

Conjecture 1.1 can be proven through induction by both proving the conjecture for some special case and then proving that if it holds for some case in particular it holds for all in general. The first part of that is done here, for the special case when $v = \emptyset$. Given this restriction, proving (1.1) is equivalent to proving that $f(u10) = qf(u01) + f(u0) + f(u1)$, which can be proven directly and without intermediaries by observing the possible rook covers in Ω_{u10}.

This relation will be proved by utilizing the linearity of this problem mentioned in *Chapter 2*, i.e the set Ω_{u10} of covers over the board of size n for the word $(u10)$ will be partitioned into the four subsets Ω'_{u10}, Ω''_{u10}, Ω'''_{u10} and Ω''''_{u10}, and it will be shown that $f_{\Omega'_{u10} \cup \Omega''_{u10}}(u10) = qf(u01)$ and $f_{\Omega'''_{u10} \cup \Omega''''_{u10}}(u10) = f(u0) + f(u1)$.

The sets Ω'_{u10}, Ω''_{u10}, Ω'''_{u10} and Ω''''_{u10} are defined as follows:

- Ω'_{u10} is the set of all covers where the rook corresponding to the 1 in $(u10)$ is in the top position in its column and there is a rook in the rightmost position of the second row, counted from the top.

- Ω''_{u10} is the set of all covers where the rook corresponding to the 1 in $(u10)$ is in the bottom position in its column and the rook in the top row is not in any of the four rightmost positions.

- Ω'''_{u10} is the set of all covers where the rook corresponding to the 1 in $(u10)$ is in the top position in its column and there is no rook in the rightmost position of the second row, counted from the top.

- Ω''''_{u10} is the set of all covers where the rook corresponding to the 1 in $(u10)$ is in the bottom position in its column and the rook in the top row is in the last or third last position in that row.

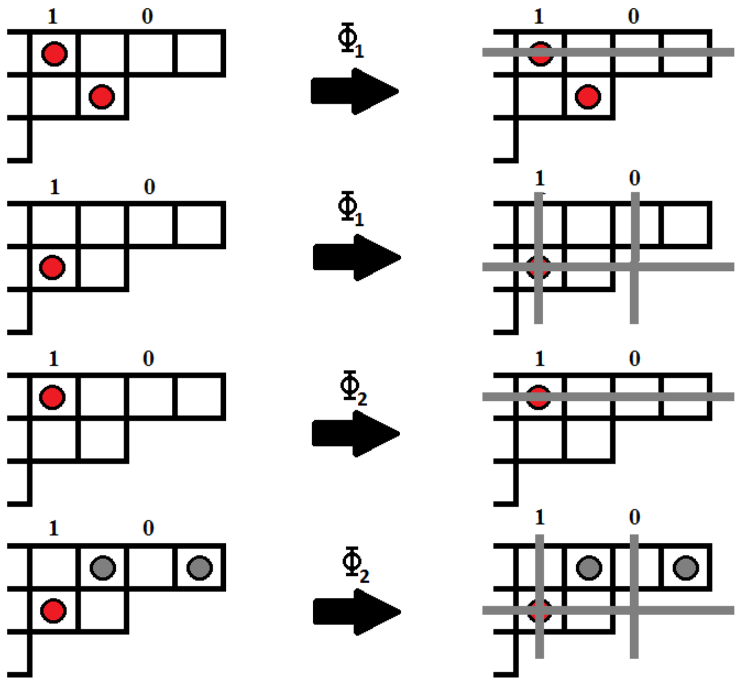

Figure 3.1: Visualizations of the rightmost ends of the covers in Ω_1, Ω_2, Ω_3 and Ω_4, and the bijections Φ_1 and Φ_2.

There is a bijection $\Phi_1 : \Omega'_{u10} \cup \Omega''_{u10} \to \Omega_{u0}$, where Ω_{u0} is the set of all covers over the board of size $n-1$ for the word $(u0)$.

Φ_1 maps the covers in Ω'_{u10} to the corresponding covers gained by simply removing the top row of each cover. This means that each cover in Ω'_{u10} is mapped to a unique cover in Ω_{u0} with exactly one less inversion.

Φ_1 maps the covers in Ω''_{u10} to the covers gained by removing the second row, counted from the top, and the columns corresponding to the 0 and 1 in $(u10)$. This removes the rook causing the 1, and since this rook is in the second row from the top and by definition of Ω''_{u10} there is a rook higher up to its left, all covers will be mapped to unique covers with one less inversion.

In both these cases the mapping is clearly and easily invertible, one can create a mapping back to the original set that is based on adding the removed rows and columns. Observe also that $\Phi_1(\Omega'_{u10}) \cup \Phi_1(\Omega''_{u10}) = \Omega_{u0}$. This is because $\Phi_1(\Omega'_{u10})$ is the set of all covers over the board of size $n-1$ for the word $(u0)$ that contain a rook in the rightmost position of the top row and $\Phi_1(\Omega''_{u10})$ is the set of all covers over the board of size $n-1$ that does not contain a rook in this position. Thus $\Phi_1(\Omega'_{u10})$ and $\Phi_1(\Omega''_{u10})$ complement each other in Ω_{u01} and form a perfect partition of Ω_{u01}.

From this it follows that $f_{\Omega'_{u10} \cup \Omega''_{u10}}(u10) = qf(u0)$. The existence of the bijection Φ_1 means that $|\Omega'_{u10} \cup \Omega''_{u10}| = |\Omega_{u0}|$, since for each cover in $\Omega'_{u10} \cup \Omega''_{u10}$ there is exactly one unique cover in Ω_{u0} and vice versa. The q in $qf(u0)$ is compensation for the fact that for every cover in $\Omega'_{u10} \cup \Omega''_{u10}$ there is a cover in Ω_{u0} with exactly one less inversion. Since it is know from *Chapter 2* that $f(u1) = f(u)$, it follows that $f_{\Omega'_{u10} \cup \Omega''_{u10}}(u10) = qf(u01)$.

The remaining step is now to show that $f_{\Omega'''_{u10} \cup \Omega''''_{u10}}(u10) = f(u0) + f(u1)$. This is done with a further bijection $\Phi_2 : \Omega'''_{u10} \cup \Omega''''_{u10} \to \Omega_{u0} \cup \Omega_{u1}$. This bijection maps all covers in Ω''' to covers in Ω_{u0} by removing the top row and all covers in Ω'''' by removing the second row from the top and the columns corresponding to the 1 and the 0 in the $(u10)$. All mappings are invertible by returning the removed rows, columns and rooks, in both cases a cover in one set is mapped onto a unique cover in the otehr set.

Φ_2 maps every cover in $\Omega'''_{u10} \cup \Omega''''_{u10}$ to a cover in $\Omega_{u0} \cup \Omega_{u1}$ with the same number of inversions. For Ω'''_{u10}, the rook in the top row has no rook that is both below it and to the right of it, hence its removal doesn't change the number of inversions. The removal of the row and the columns of covers in Ω''''_{u10} removes the rook in the second row from the top. The rook in the top row is by definition of Ω''''_{u10} to the left of the rook above it and to the right of all other rooks, hence removing no inversions.

$\Phi_2(\Omega'''_{u10})$ is the set of all covers in Ω_{u0} with no rook in the rightmost position of the top row, and $\Phi_2(\Omega''''_{u10})$ is the union of the set of all covers in Ω_{u1} and the set of all covers in Ω_{u0} where there is a rook in the rightmost position of the top row. Thus $\Phi_2(\Omega'''_{u10}) \cup \Phi_2(\Omega''''_{u10}) = \Omega_{u1} \cup \Omega_{u0}$. Because every cover in $\Omega'''_{u10} \cup \Omega''''_{u10}$ can be mapped onto a unique cover in $\Omega_{u1} \cup \Omega_{u0}$ and the other way around, $|\Phi_2(\Omega'''_{u10}) \cup \Phi_2(\Omega''''_{u10})| = |\Omega_{u1} \cup \Omega_{u0}|$, which together with the preservation of the number of inversions in Φ_2 for each cover means that $f_{\Omega'''_{u10} \cup \Omega''''_{u10}}(u10) = f(u1) + f(u0)$.

Combining these two results confirms that $f(u10) = qf(u01) + f(u0) + f(u1)$. ■

Given that the relation $f(u10v) = qf(u01v) + f(u1v) + f(u0v)$ holds for the word $v = \emptyset$, it holds in general if it can be further shown that $f(u10v0) = qf(u01v0) + f(u1v0) + f(u0v0)$ and $f(u10v1) = qf(u01v1) + f(u1v1) + f(u0v1)$. This is because every possible word v then can be constructed from $v = \emptyset$ by subsequently adding ones and zeros.

Proving that $f(u10v1) = qf(u01v1) + f(u1v1) + f(u0v1)$ holds if $f(u10v1) = qf(u01v1) + f(u1v1) + f(u0v1)$ holds is trivial, since it follows from the fact that, as already shown in Chapter 2, $f(u1) = f(u)$. The remaining dufficulty here is proving that $f(u10v0) = qf(u01v0) + f(u1v0) + f(u0v0)$ follows from $f(u10v) = qf(u01v) + f(u1v) + f(u0v)$, which would complete the induction.

3.2 Conjecture 1.2

Mass computation

A first step in evaluating *Conjecture 1.2* is through trial by exhaustion, i.e by computing (1.3) for larger and larger values of k and n and comparing the resulting polynomials with the corresponding polynomial computed for the corresponding board.

Consider first the case of having an (P)ASEP model with two positions, one of which contains a particle and one which doesn't. The corresponding value of $E_{k+1,n+1}$ will be:

$$E_{2,3}(q) = q^{-2}\sum_{i=0}^{2}(-1)^i[2-i]^3 q^{2i-2}(\binom{3}{i}q^{2-i} + \binom{2}{i-1}) =$$
$$= q^{-2}((1+q)^3 q^{-2}(\binom{3}{0}q^2 + \binom{3}{-1}) - (\binom{3}{1}q + \binom{3}{0})) = 3+q$$

There exists six possible covers over the corresponding board of size $n = 2$. Four of these covers contain exactly one rook in an odd-index column.

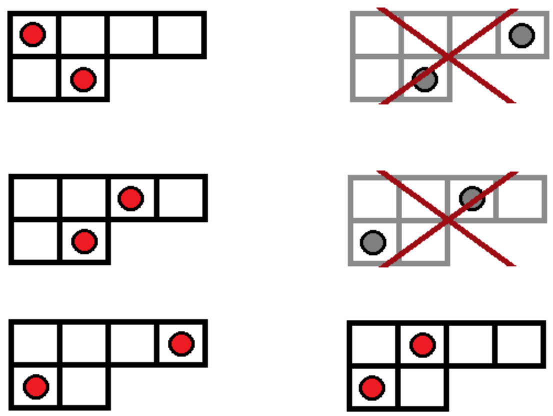

Figure 3.2: The rook covers for which the polynomial corresponds to $E_{2,3}(q)$.

$f(q) = \sum q^k$ over these four covers equals $3 + q$, showing that for a board of this size with this many filled odd columns, *Claim 2* holds true.

As part of the creation of this article, *Python* code has been written to automatize comparisons such as the one above and conduct them in large numbers. The value of $E_{k+1,n+1}(q)$ has in each case been explicitly computed using a library for symbolic operations and for each such case the polynomial for a board of size n with k filled odd columns has been computed using the recursive algorithm presented in Chapter 2.

Currently successful computations have be made showing equality in 65 cases, for boards up to size $n = 10$ and $0 \leq k \leq n$.

Shared properties with $\hat{E}_{k+1,n+1}(q)$

Considering *Conjecture 1.2*, it is of interest to show that for a given board of size n, the polynomial $f_k(q)$ shares the properties of $E_{k+1,n+1}(q)$ listed in *Chapter 2*. The following lemma will turn out to be useful when trying to show this.

Lemma 3.1:

- Let B be a board of arbitrary size n, and C be a rook cover over B such that no other rook cover over B has more inversions. Then C contains one rook in the upper left corner of B.

Proof of Lemma 3.1: Consider an arbitrary board B and an arbitrary rook cover C. C contains a rook in the top row since a rook cover by definition contains exactly one rook in each row. Either this rook is at the left end of that row or it is not. If it is the lemma holds true.

Otherwise consider the situation when that rook is moved all the way to the upper left corner. This position is open since we only have one rook per row. This may lead to two rooks occupying the leftmost column. If so, since there are two extra positions per row counting from the bottom up, the rest of the rooks in the cover can be shifted to the right to undo collisions along columns in such a way that the number of inversions is preserved among these rooks.

Once this has been done, the total number of inversions has increased by at least one, because there must now be one rook in the right position of the bottom row. Thus C either contains a rook in the top left corner of B or there is a rook cover with more inversions than C. ■

A rook cover over a board will be called *maximal* if there is no rook cover over the board with more inversions. Consequently a maximal k-cover is a rook cover over a board such that no other k-cover over the same board has more inversions.

Constructing a maximal k-cover

Lemma 3.1 can be used in the design of an algorithm that, given a board B and some value k, constructs a maximal k-cover. This algorithm goes as follows, given a board B of size n and a value k:

If $k \leq \frac{n}{2}$:

1. For the first k rows, place one rook in the 1^{st} column of the first row, one in the 3^{rd} column of the 2^{nd} row and so on until odd indexed column k.
2. For the $n-k$ bottom rows, place one rook at the rightmost end of each row.

else if $k > \frac{n}{2}$:

1. For the first $n-k$ rows from the top, place one rook in the 1^{st} column of the first row, one in the 3^{rd} column of the 2^{nd} row and so on until row $n-k$.
2. For the first k rows counting from the bottom, add one rook to the rightmost odd-indexed column if it is empty, otherwise to the rightmost even column.

The correctness of this algorithm in producing the maximal k-cover follows from *lemma 3.1*. The first k or $n-k$ rows will have rooks placed in the top left corner of the corresponding sub-board defined by removing all columns to the left of this rook and all rows above it. In accordance with *lemma 3.1* this maximizes the number of inversions in the cover over this sub-board. Not having rooks in these positions would lead to there being a cover with more inversions, and the remaining rooks are placed in the only positions they can be in.

Showing that the degree of $f_k(q)$ is equal to the degree of $E_{k+1,n+1}$

The first property of $E_{k+1,n+1}$ that will be shown to be shared by $f_k(q)$ is the degree. The number of inversions in the cover created with this algorithm will obviously be equal to the degree of $f_k(q)$ for that board and that value of k. This therefore has to be shown to be equal to the degree of $E_{k+1,n+1}$. To this end first recall from chapter 2 that the degree d_1 of $E_{k+1,n+1}$ is given in [5] to be:

$$d_1 = (k+1-1)(n+1-k-1) = k(n-k) = nk - k^2$$

The number d_2 of inversions in the cover produced by the algorithm in *Lemma 3.2*, and thus the degree of $f_k(q)$, will be shown to be:

$$d_2 = \sum_{i=1}^{k} n + 1 - 2i$$

The expression $d_2 = \sum_{i=1}^{k} n + 1 - 2i$ is motivated as follows. Consider the k sub-boards gained by considering first the whole board, then the board gained by removing the two leftmost columns and the top row, then the sub-boards gained by consequently repeating this operation. In each of these k cases there will be a rook in the top left corner that forms an inversion with each other remaining rook.

The first such rook forms an inversion with every other rook, thus $n-1$ inversions. Removing two columns removes the top left corner rook and the rook to the right in the bottom row. Thus the top left corner rook in the next sub-board forms an inversion with two less rooks and therefore forms $n-3$ inversions. The process can obviously be repeated k times, each time there will be two less inversions.

The rooks in the last $n-k$ rows are never the top left part of any inversion because they are according to the algorithm always placed at the right end of their respective rows.

The above recieved expression for d_2 can further be simplified as follows:

$$d_2 = \sum_{i=1}^{k} n + 1 - 2i = nk + k - 2\sum_{i=1}^{k} i =$$
$$= nk + k - 2\frac{k(k+1)}{2} = nk - k^2$$

For $k > \frac{n}{2}$:

$$d_3 = \sum_{i=1}^{n-k} n + 1 - 2i$$

The expression $d_2 = \sum_{i=1}^{n-k} n + 1 - 2i$ is motivated in an almost identical way to the corresponding expression for the case $k \leq \frac{n}{2}$. The only rooks that form the upper left element of inversions are the $n - k$ ones that are placed first. This follows from the simple fact that the algorithm places the k last rooks in the last or second last position of their respective rows, thus not having any rook to their right and below them. Hence the sum is only over the first $n - k$ rooks. For these rook, like in the case when $k < \frac{n}{2}$, the first forms an inversion with all rooks, the next with all rooks but two and so on.

d_3 can be simplified in a way similar to d_2:

$$\begin{aligned} d_3 &= \sum_{i=1}^{n-k} n + 1 - 2i = n(n-k) + (n-k) - 2\sum_{i=1}^{n-k} i = \\ &= n^2 - nk + n - k - 2\frac{(n-k)(n-k+1)}{2} = \\ &= n^2 - nk + n - k - n^2 + nk - n + nk - k^2 + k = nk - k^2 \end{aligned}$$

This means that $d_1 = d_2 = d_3 = nk - k^2$ and that the degree of $E_{k+1,n+1}$ equals the degree of $f_k(q)$.

Showing that the terms of $f_k(q)$ and of $E_{k+1,n+1}$ with highest degree both have coefficient 1

The coefficient of the highest degree term of $f_k(q)$ will now be shown to be 1 by showing that the rook cover produced by the algorithm for some board B and some value of k is the only maximal k-cover over that board and for the value k.

Consider the case when $k \leq \frac{n}{2}$. It has been showed that the maximal cover has one rook in the top left corner. Then by extension it has to have one in the rightmost position of the lowest row, because there has to be one rook in every row and that is the only position left.

This applies for the first k sub-boards gained by removing the two leftmost columns, according to *Lemma 3.1* there has to be a rook in the top left corner and then there is only one place the rook in the bottom corner can be in. Because there is no other position any other rook can be in without breaking the conditions of *Lemma 3.1*, there can be no other cover with the maximal number of inversions.

In the case of $k \leq \frac{n}{2}$ *Lemma 3.1* is also utilized. The $n - k$ first rooks have to be in the top left corners of their corresponding sub-boards and there is nowhere else the remaining rooks can be. If there is only one cover with this maximal number of inversions then the coefficient of the highest degree term of $f_q(k)$ has to be 1.

To conclude, it has now been shown that $f_k(q)$ shares the first two properties of $E_{k+1,n+1}$ presented in Chapter 2. The degree of $f_k(q)$ equals the degree of $E_{k+1,n+1}$ and the coefficient of the highest degree term of $f_k(q)$ equals the coefficient of the highest degree term of $E_{k+1,n+1}$, both being 1.

Chapter 4

Conclusions

This article has proved a number of results that are necessary for *Conjecture 1.1* and *Conjecture 1.2* to be true. It has also presented results by another author that prove *Conjecture 1.1* in its entirety, and thus by extension *Conjecture 1.2*.

In the case of *Conjecture 1.1* this article has presented results from other authors proving the conjecture for the special case when $q = 0$. Partially inspired by this result, the article has gone on to develop a draft for a proof by induction.

A base case has been proved for this induction, i.e this article has proved that $f(u10) = qf(u01) + f(u1) + f(u0)$.

With regards to *Conjecture 1.2* this article has first presented the result of mass computations that confirm *Conjecture 1.2* for low values of n and k. Then this article proceeds with showing that $f_k(q)$ shares a couple of properties with $E_{n+1,k+1}(q)$, properties which $f_k(q)$ and $E_{n+1,k+1}$ have to share if *Conjecture 1.2* holds. This article in particular proves that the degree of $f_k(q)$ is equal to the degree of $E_{n,k}(q)$ and that the coefficient of the highest-degree term of $f_k(q)$ is equal to 1.

This article also leaves open the possibility of constructing a definitive proof for *Conjecture 1.1* by completing the induction for which this article has provided a base case. As suggested further work it could perhaps be proven that $f(u10v0) = f(u01v0)+f(u1v0)+f(u0v0)$ to holds for each u and v where $f(u10v) = f(u01v) + f(u1v) + f(u0v)$.

Bibliography

[1] P.Corteel, L.K.Williams, "Tableaux combinatorics for the asymmetric exclusion process", *Advances in Applied Mathematics*, 24 October 2006.

[2] M.Jousat-Verges, "Stammering Tableaux"

[3] P.Bränden, personal communication, KTH, 2014.

[4] R.Brak, S.Corteel, J.Essam, R.Parviainen & A.Rechnitzer, "A Combinatorial Derivation of the PASEP Stationary State", 13 Nov 2006.

[5] L.K.Williams, "Enumeration of totally positive Grassman cells", *Advances in Mathematics*, 2005.

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