SMALL-TIME ASYMPTOTICS OF CALL PRICES AND IMPLIED VOLATILITIES FOR EXPONENTIAL LÉVY MODELS

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To my wife, Jamie, and my parents, Jane and Charles.

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SUMMARY

We derive call-price and implied volatility asymptotic expansions, in time to maturity, for a selection of exponential Lévy models. We consider asset-price models whose log returns structure is a Lévy process, i.e. processes of the form $(L_t + \sigma W_t)_{t \geq 0}$, where $L = (L_t)_{t \geq 0}$ is a pure-jump Lévy process in the domain of attraction of a stable random variable, where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion independent of L, and where $\sigma \geq 0$.

Call-price asymptotics for in-the-money (ITM) and out-of-the-money (OTM) options are extensively covered in the literature; however, at-the-money (ATM) call-price asymptotics under exponential Lévy models are relatively new.

In this thesis, we consider two main problems. First, under some relatively minor assumptions, we prove the first-order call-price and implied volatility asymptotics when L is a very general Lévy model. More precisely, when L is in the domain of attraction of a stable random variable. Second, we reconsider the case where L is a CGMY process. In this case, we use the Lipton-Lewis formula to derive second-order call-price asymptotics. We also correct and reprove a first-order asymptotic result that appears in the literature.

For the first problem, when $\sigma = 0$, new orders of convergence are discovered which show a much richer structure than was previously considered. Concretely, we show that in this case the rate of convergence can be of the form $t^{1/\alpha}\ell(t)$ where ℓ is a slowly varying function. We also give an example of a Lévy model exhibiting this new type of behavior where ℓ is not asymptotically constant.

When $\sigma \neq 0$, we show that the Brownian component is the dominant term in the asymptotic expansion of the call-price. Under more general conditions on L (even

removing the requirement of L to be in the domain of attraction of a stable random variable), the first-order call-price asymptotics is shown to be of the order $\sqrt{t}$.

For the second problem where we consider the CGMY process, call-price asymptotics are already known to third order. Up until now, the only tools available for proving the second and third-order asymptotics were measure transformation techniques that involved very technical estimations. In the last chapter, we give a new method that relies on the Lipton-Lewis (LL) formula. Using this formula guarantees that we can estimate the call-price asymptotics using only the characteristic function of the Lévy process. While this method does not provide a less technical approach, it is novel and is promising for obtaining second-order call-price asymptotics and beyond for ATM options in a more general class of Lévy processes.

CHAPTER I

INTRODUCTION

The popularity of the Black-Scholes model belies its ability to describe true market dynamics. The weaknesses of the Black-Scholes model are well-known and well documented. Empirically, implied volatility is not constant across strikes, as is assumed in the Black-Scholes model, and log-returns are not normally distributed.

Consider the foreign exchange (FX) market. Empirically, there is a premium attached to both out-of-the-money (OTM) puts and calls. Recall that FX options (calls and puts) are contracts that confer on the holder the right but not the obligation to exchange one currency for another at a predefined exchange rate (called the *strike*) at a certain date in the future (called the *expiration date*). Additional information about the mechanics and conventions for FX options can be found in [11]. The premium attached to OTM calls and puts implies higher implied volatilities for OTM puts and calls than for at-the-money (ATM) puts and calls. The higher implied volatilities gives a convex shape to the volatility surface, and this is referred to as the volatility smile (see [18] and [11]).

The Black-Scholes model naturally underestimates the risk that exists in the market and tends to produce option prices that are too low. The so-called tail events are not given adequate weight in the Black-Scholes market; fundamentally, this is due to the exponentially small tails of the normal distribution.

There are a few natural alternatives to the traditional Black-Scholes model, and let us consider some of them. In the Black-Scholes model, one assumes that the asset model $S = (S_t)_{t \geq 0}$ satisfies the stochastic differential equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1.1}$$

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where μ and $\sigma > 0$ are constants and $(W_t)_{t \geq 0}$ is a standard Brownian motion. Solving (1.1) gives for each $t > 0$

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}. \tag{1.2}$$

We know the assumption that σ is constant is empirically incorrect, so we might look at changing it.

First, we could make the volatility structure richer (albeit still deterministic). This idea was introduced by Dupire in his work on local volatility models in [18]. In local volatility models, we assume that $\sigma := \sigma(S, t)$ and $\mu := \mu(t)$. That is, we assume that the current volatility levels are a deterministic function of both the current asset price level S and the current time t,

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(S, t)\, dW_t.$$

While local volatility models lend themselves to simple computations and recover market volatilities exactly, they predict volatility dynamics that can be completely contrary to observed phenomenon (see e.g. [19], [18], [32]).

Incorporating randomness into the volatility component is a more realistic way to inject more realistic smile dynamics into our model. These are the *stochastic volatility models*. Stochastic volatility models are most easily described as a system of stochastic differential equations, e.g.

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t^{(1)}$$

$$d\sigma = \theta(\sigma, t)\, dt + \upsilon(\sigma, t)\, dW_t^{(2)},$$

where $\left(W_t^{(1)}\right)_{t \geq 0}$ and $\left(W_t^{(2)}\right)_{t \geq 0}$ are standard Brownian motions with correlation structure given via

$$\mathbb{E}\left[W_t^{(1)} W_t^{(2)}\right] = \rho t.$$

A classic and widely-used stochastic volatility model is the Stochastic Alpha, Beta,

Rho (SABR) model which has stochastic differential equation

$$dF_t = \sigma_t F_t^\beta dW_t^{(1)}$$

$$d\sigma_t = \alpha \sigma_t dW_t^{(2)},$$

where $0 \leq \beta \leq 1$, $\alpha \geq 0$, and $W^{(1)}$ and $W^{(2)}$ are as above. While stochastic volatility models are capable of very closely reproducing dynamics similar to those observed in the market, they don't reproduce the observed smile values exactly. In practice, they can also require difficult computation and analysis.

Local volatility and stochastic volatility models are outside of the scope of this thesis, so we leave further discussion of their advantages and disadvantages to the interested reader.

Yet another way to change the Black-Scholes model, and the one that we are chiefly concerned with in this thesis, is to add a jump component to the Black-Scholes stochastic differential equation (1.1). We will do so by considering a Lévy process $X = (X_t)_{t \geq 0}$, which as is well-known can be represented as an independent sum

$$X_t = bt + \sigma W_t + L_t,$$

where $b \in \mathbb{R}$, $\sigma \geq 0$, $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, and $L = (L_t)_{t \geq 0}$ is a pure-jump Lévy process. The corresponding asset price process $S = (S_t)_{t \geq 0}$ is then given via

$$S_t = S_0 e^{X_t} = S_0 e^{bt + \sigma W_t + L_t}.$$

There are several advantages and disadvantages of using this model. These are discussed nicely in [12], and we recount some of their discussion here.

Most of the advantages of using Lévy-based models revolve around return structure and volatility dynamics. First, Lévy models can provide the heavy tails that are observed in the markets while retaining stationarity. This alleviates the light-tail disadvantage in the traditional Black-Scholes model. Next, Lévy models do generate

implied volatility skews. These volatility surfaces also steepen as time to maturity decreases, another empirical phenomenon observed from the market. Finally, under Lévy models the market is incomplete. So, for example, options cannot be perfectly replicated using only the underlying asset.

There are some significant drawbacks to using Lévy based models. Perhaps the most severe disadvantage is the nonexistence of closed-form option-pricing formulas except in the simplest of cases. We must rely on asymptotics for price and volatility behavior. This limits us to either near-expiry or long-term options or even extreme strike regimes. Also, there is no clear hedging or replication strategy. Often, we must solve very difficult portfolio optimization problems or deal with very rough, no-arbitrage bounds on prices.

Different techniques are required depending on whether the options considered are at-the-money (ATM) or not ATM (or non-ATM). By at-the-money, we roughly mean that the strike price equals the current asset price. In FX markets, there are two different conventions for ATM, at-the-money forward and delta-neutral straddle. At-the-money forward means that the ATM strike is the current forward price (as opposed to the spot price). Delta-neutral straddle is defined as the strike that gives a straddle with net zero delta (a straddle is transaction where one purchases a call and a put at the same strike price). We choose to ignore these conventions here for clarity, though the interested reader can consult [11] for additional information.

Previously, option prices under Lévy models close to expiration have received a great deal of attention. The first significant work was done with options that are not ATM in 2002 by Boyarchenko and Levendorksii (see [6]). While important, these cases are not the focus of this thesis. For a good background on the non-ATM case, see the works of Figueroa-López and Forde in [22] and Tankov in [52].

After the developments in non-ATM, some attention was given to the ATM case. Some of the earliest work was done concurrently in 2010 by Tankov, Figueroa-López

and Forde, and Muhle-Karbe and Nutz in [52], [22], and [44], respectively. We discuss next some of their results. In what follows, we use d with a subscript (e.g. d_1) to denotes a generic (but known) constant that might change from formula to formula.

Tankov obtained the asymptotic behavior in a few cases. In the finite variation case (i.e. $\int_{|x|\leq 1} |x|\,\nu(dx) < \infty$, where ν is the Lévy measure of the Lévy process $(X_t)_{t\geq 0}$), he shows that the ATM call-option price function, denoted by C, has asymptotic behavior

$$C(t) = S_0 d_1 t + o(t),$$

as $t \downarrow 0$, where $d_1 = \max\left(\int (e^x - 1)_+ \nu(dx), \int (1 - e^x)_+ \nu(dx)\right)$. He also obtained ATM call-price asymptotics for a stable-like case where he assumes that the process has the characteristic exponent

$$i\gamma u - |u|^\alpha f(u),$$

with $1 < \alpha < 2$, $\gamma \in \mathbb{R}$, and where f is a continuous bounded function such that

$$\lim_{u\to\pm\infty} f(u) = c_\pm,$$

with $0 < c_\pm < \infty$. Under these assumptions, the first-order call-price asymptotics are shown to be of the form

$$C(t) = S_0 d_1 t^{1/\alpha} + o\left(t^{1/\alpha}\right),$$

as $t \downarrow 0$, where d_1 is an explicit constant depending only on α. Finally, for Lévy processes with finite second moment, the ATM call-price asymptotics are given by

$$C(t) = S_0 d_1 \sqrt{t} + o\left(\sqrt{t}\right),$$

as $t \to 0$. Muhle-Karbe and Nutz studied the asymptotics for a wide class of Lévy processes, for example, for Lévy processes having Lévy measure with stable-like small jumps, i.e. the Lévy measure is of the form

$$\left(\frac{f(x)}{|x|^{1+\alpha_-}}\mathbb{1}_{(-\infty,0)}(x) + \frac{f(x)}{x^{1+\alpha_+}}\mathbb{1}_{(0,\infty)}(x)\right)dx,$$

where $f \geq 0$ is a Borel function such that

$$\lim_{x \downarrow 0} f(x) = f_+ \quad \text{and} \quad \lim_{x \uparrow 0} f(x) = f_-,$$

with also $f(x) - f_+ = O(x)$ as $x \downarrow 0$ and $f(x) - f_- = O(x)$ as $x \uparrow 0$. Under these assumptions, they proved that if $\alpha := \alpha_+ \vee \alpha_- \in (1,2)$, then the call price has asymptotics

$$C(t) = d_1 t^{1/\alpha} + o\left(t^{1/\alpha}\right),$$

as $t \downarrow 0$, where $d_1 := d_1\left(\alpha_\pm, f_\pm\right)$. Moreover, if $\alpha_+ = \alpha_- = 1$ and $f_+ = f_-$, then the call-price asymptotics are

$$C(t) = \frac{1}{2}\left(f_+ + f_-\right) t \left|\log t\right| + o\left(t \left|\log t\right|\right),$$

as $t \downarrow 0$.

Figueroa-López and Forde found the first-order ATM asymptotics for CGMY processes in [22]. CGMY processes are Lévy processes with characteristic triplet $(b, 0, \nu)$ where ν is the Lévy measure

$$\nu\left(dx\right) = \left(\frac{Ce^{Gx}}{|x|^{1+Y}} \mathbb{1}_{(-\infty,0)}(x) + \frac{Ce^{-Mx}}{x^{1+Y}} \mathbb{1}_{(0,\infty)}(x)\right) dx,$$

where $Y \in (0,2)$, $M > 1$, and $G, C > 0$. For this process, without Brownian component, the call-price asymptotics are

$$C(t) = d_1 t^{1/Y} + o(t^{1/Y}),$$

where $d_1 := d_1\left(C, Y\right)$. In the case where an independent Brownian component is added to the CGMY process, the asymptotics take the familiar form

$$C(t) = \frac{\sigma}{\sqrt{2\pi}} \sqrt{t} + o\left(\sqrt{t}\right).$$

Figueroa-López, Houdré, and Gong in [23] expanded the first-order asymptotics up to third-order asymptotics for the CGMY process, both with and without Brownian

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component. In the absence of a Brownian component, they showed that

$$C(t) = d_1 t^{1/Y} + d_2 t + \begin{cases} d_{31} t^{2-\frac{1}{Y}} + o\left(t^{2-\frac{1}{Y}}\right), & \text{if } 1 < Y \le \frac{3}{2} \\ d_{32} t^{2/Y} + o\left(t^{2/Y}\right), & \text{if } \frac{3}{2} \le Y < 2. \end{cases}$$

In the case with nonzero Brownian component, the third-order asymptotics become

$$C(t) = d_1 \sqrt{t} + d_2 t^{\frac{3-Y}{2}} + \begin{cases} d_{31} t + o\left(t\right), & \text{if } 1 < Y \le \frac{3}{2} \\ d_{32} t^{\frac{5}{2}-Y} + o\left(t^{\frac{5}{2}-Y}\right), & \text{if } \frac{3}{2} \le Y < 2. \end{cases}$$

Finally, Figueroa-López, Gong, and Houdré in [24] obtained second-order asymptotics for a class of "tempered" Lévy processes, i.e. Lévy processes with Lévy measure

$$s(x) = |x|^{-Y-1} q(x),$$

where $1 < Y < 2$ and where q satisfied certain decay conditions at the origin (along with several other technical conditions). In this case, the second-order call-price asymptotics satisfy

$$C(t) = d_1 t^{1/Y} + d_2 t + o(t),$$

without Brownian component and

$$C(t) = d_1 \sqrt{t} + d_2 t^{\frac{3-Y}{2}} + o\left(t^{\frac{3-Y}{2}}\right),$$

with Brownian component.

In this thesis, we obtain two main results concerning call-price asymptotics. Previously, for Lévy processes $(X_t)_{t \ge 0}$ such that

$$\mathbb{E} e^{sX_t} < \infty,$$

for some $s > 1$, the only known first-order rate of covergence was $t^{1/\alpha}$, for some $1 < \alpha < 2$. We show that in a more general class of Lévy processes, it is possible to have different rates of convergence. Namely, if ℓ is a slowly varying function at

infinity, then the rate of convergence in the first-order can be of the form $t^{1/\alpha}\ell\left(1/t\right)$. This order of convergence was not previously exhibited.

Next, since a Lévy process is completely and uniquely described by its characteristic function, it seeems that some justification for second-order results in the CGMY case should follow from Fourier arguments; however, obtaining second-order asymptotics, even if only formally, using only the characteristic function has so far resisted discovery. We show that the second-order terms appear naturally from the characteristic function.

This thesis is divided into six chapters. The current chapter contains the introduction. Chapter II gives a brief overview of Lévy processes and exponential Lévy models. Chapter III covers stable domains of attraction and regular variation in order to consider classes of Lévy processes more general than those currently considered in the literature. Chapter IV gives the first-order behavior of this general class of Lévy processes and considers a specific example of this new behavior. Chapter V discusses the second-order CGMY result mentioned earlier including a discussion of the Lipton-Lewis option pricing formula. Finally, in Chapter VI we make conclusions and summarize our work.

CHAPTER II

EXPONENTIAL LÉVY MODELS

In order to properly study the market dynamics of exponential Lévy processes, we first need a strong understanding of Lévy processes themselves. In this section, we define Lévy processes and introduce some of their basic properties. We discuss the two main approaches to Lévy processes, namely via the Lévy-Khintchine representation and via the Lévy-Itô decomposition. We then discuss measure transformations for Lévy processes. Finally, we present the notion of exponential Lévy models and cover Carr and Madan's pricing formula.

2.1 Lévy Processes

We start with some background on Lévy processes following the expositions in [50], [12], and [2]. Throughout, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Definition 2.1. A stochastic process $(X_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *Lévy process* if the following conditions are satisfied.

(i) $X_0 = 0$ a.s.

(ii) For any $n \in \mathbb{N}$ and any increasing sequence $0 \leq t_0 < t_1 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \cdots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments).

(iii) For any $s, t \geq 0$, the distribution of $X_{s+t} - X_s$ does not depend on s (stationary increments).

(iv) For every $\varepsilon > 0$ and $t \geq 0$, we have $\lim_{s \to 0} \mathbb{P}\left(|X_{t+s} - X_t| > \varepsilon\right) = 0$ (stochastic continuity).

(v) There exists $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$, $X_t(\omega)$ is right-continuous for $t \geq 0$ and has left limits for $t > 0$.

We also recall a few basic definitions.

Definition 2.2. A *filtration* on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection of σ-fields, usually denoted $(\mathcal{F}_t)_{t \geq 0}$, such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F},$$

for every $0 \leq s \leq t$. A σ-field is *right continuous* if for every $t \geq 0$,

$$\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s.$$

Definition 2.3. Let $\mathcal{B}$ denote the Borel σ-field on $\mathbb{R}^d$ and let X be a random variable on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) The *σ-field generated by X*, denoted $\sigma(X)$, is defined as

$$\sigma(X) = \left\{ X^{-1}(D) : D \in \mathcal{B} \right\},$$

where $X^{-1}(D) = \{\omega \in S : X(\omega) \in D\}$.

(ii) If $(X_t)_{t \geq 0}$ is a stochastic process on the same probability space, then for $t \geq 0$

$$\sigma(X_s, s \geq t)$$

is the smallest σ-field such that each X_s is measurable for $s \geq t$, and we call it the *σ-field generated by* $(X_s)_{s \geq t}$.

In certain circumstances, we may consider the same stochastic process under different probability measures, and we find it useful to use notation to distinguish these two processes. In particular, if we consider a stochastic process $(X_t)_{t \geq 0}$ on two different probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, then we denote the first process as $\left((X_t)_{t \geq 0}, \mathbb{P}_1 \right)$ and the second as $\left((X_t)_{t \geq 0}, \mathbb{P}_2 \right)$.

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We can view Lévy processes in a different way by identifying processes with their sample paths that are right-continuous with left limits, i.e. cádlág functions. More concretely, let $\mathbb{D}\left([0,\infty),\mathbb{R}^d\right)$ be the space of right-continuous functions with left limits from $[0,\infty)$ into $\mathbb{R}^d$. We let $(X_t)_{t\geq 0}$ be the canonical process $X_t(\omega) = \omega(t)$ defined on $\Omega = \mathbb{D}\left([0,\infty),\mathbb{R}^d\right)$ with the σ-field $\mathcal{F}_{\mathbb{D}} = \sigma\left(X_s, s \geq 0\right)$ and the right-continuous filtration $\mathcal{F}_t = \cap_{s>t}\sigma\left(X_u, u \leq s\right)$. The Lévy process $\left((X_t)_{t\geq 0}, \mathbb{P}\right)$ induces a probability measure $\mathbb{P}^{\mathbb{D}}$ on $(\mathbb{D}, \mathcal{F}_{\mathbb{D}})$ such that $\left((X_t)_{t\geq 0}, \mathbb{P}\right)$ is identical in law to $\left((X_t)_{t\geq 0}, \mathbb{P}^{\mathbb{D}}\right)$. For what follows, we use $\left((X_t)_{t\geq 0}, \mathbb{P}\right)$ to refer to the Lévy process where $\mathbb{P}$ is a probability measure on $(\mathbb{D}, \mathcal{F}_{\mathbb{D}})$. The interested reader can find Chapter 4, Section 19 of [50] for more information on the natural topology and metric associated with $(\mathbb{D}, \mathcal{F}_{\mathbb{D}})$.

To further understand Lévy processes, we also need to introduce the notion of infinite divisibility. For a probability distribution μ, let μ^{*n} denote the n-fold convolution of μ with itself, where the convolution of two Borel measures μ_1 and μ_2 on $\mathbb{R}^d$ is defined for each Borel set $D \subset \mathbb{R}^d$ as

$$(\mu_1 * \mu_2)(D) := \int \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{1}_D(x+y)\,\mu_1(dx)\mu_2(dy).$$

We now define infinite divisibility.

Definition 2.4. A probability measure μ on $\mathbb{R}^d$ is *infinitely divisible* if for every $n \in \mathbb{N}$, there exists a probability measure μ_n on $\mathbb{R}^d$ such that $\mu = \mu_n^{*n}$.

We give a probabilistic definition of infinite divisibility. Namely, let X be a random variable on our probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.5. A random variable X is infinitely divisible if for each $n \in \mathbb{N}$, there exist independent and identically distributed (i.i.d.) random variables $X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}$ such that

$$X \stackrel{\mathcal{L}}{=} X_1^{(n)} + \cdots + X_n^{(n)}$$

where $\stackrel{\mathcal{L}}{=}$ denotes equality in law.

It is easy to see that if $(X_t)_{t\geq 0}$ is a Lévy process, then X_t is infinitely divisible for each $t \geq 0$ since

$$X_t \overset{\mathcal{L}}{=} \sum_{i=0}^{n-1} \left(X_{\frac{t(i+1)}{n}} - X_{\frac{ti}{n}} \right).$$

The converse is also true. Specifically, we have the following relationship.

Proposition 2.6. *Let μ be an infinitely divisible distribution. Then, there exists a Lévy process $(X_t)_{t\geq 0}$ such that the distribution of X_1 is μ. Conversely, if $(X_t)_{t\geq 0}$ is a Lévy process, then X_t is infinitely divisible for each $t \geq 0$.*

There are a few examples of well-known and foundational Lévy processes that will serve as building blocks for the Lévy-Itô decomposition and our intuitive understanding of Lévy processes:

(i) A Poisson process $(N_t)_{t\geq 0}$ with intensity $\lambda > 0$, i.e for each $t \geq 0$, $N_t \overset{\mathcal{L}}{=}$ Poisson(λt).

(ii) A compound Poisson process $(L_t)_{t\geq 0}$ where for each $t \geq 0$, $L_t = \sum_{k=1}^{N_t} Y_k$ where $(Y_k)_{k\geq 1}$ is a sequence of i.i.d. random variables (representing the distribution of the jump sizes) and $N = (N)_{t\geq 0}$ is a Poisson process with intensity $\lambda > 0$ independent of $(Y_k)_{k\geq 1}$.

(iii) A Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}$ with drift $\mu \in \mathbb{R}$ and variance σ^2, i.e. for each $t \geq 0$, $B_t \overset{\mathcal{L}}{=} \mathcal{N}(\mu t, \sigma^2 t)$.

2.1.1 Lévy-Khintchine Representation and Lévy-Itô Decomposition

There are two main tools at our disposal for characterizing and understanding Lévy processes: the Lévy-Khintchine representation and the Lévy-Itô decomposition. In this thesis, we mainly use the Lévy-Khintchine representation, but we will briefly use the Lévy-Itô decomposition to view the Lévy-Khintchine representation in a probabilistic way. We denote the inner product on $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$, and $\hat{\mu}$ is the Fourier

transform of a probability measure μ on $\mathbb{R}^d$. For the results in this thesis, we often only need results for Lévy processes on $\mathbb{R}$; however, for the sake of completeness and whenever sufficiently straightforward, we will state results on $\mathbb{R}^d$.

Theorem 2.7. *(i) If μ is an infinitely divisible probability measure on $\mathbb{R}^d$, then for any $s \in \mathbb{R}^d$*

$$\hat{\mu}(s) = \int_{\mathbb{R}^d} e^{i\langle s,x \rangle} \mu(dx)$$

$$= \exp\left(i\langle b,s \rangle - \frac{1}{2}\langle s, \Sigma s \rangle + \int_{\mathbb{R}^d} \left(e^{i\langle s,x \rangle} - 1 - i\langle s,x \rangle \mathbb{1}_{\{|x|\leq 1\}} \right) \nu(dx) \right),$$

(2.1)

where $b \in \mathbb{R}^d$, Σ is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a Borel measure on $\mathbb{R}^d$ satisfying

$$\nu(\{0\}) = 0 \quad and \quad \int_{\mathbb{R}^d} \left(|x|^2 \wedge 1 \right) \nu(dx) < \infty.$$

(2.2)

(ii) The representation of $\hat{\mu}$ above by b, Σ, and ν is unique.

(iii) Conversely, if $b \in \mathbb{R}^d$, Σ is a symmetric nonnegative-definite $d \times d$ matrix, and ν is a measure satisfying (2.2), then there exists an infinitely divisible distribution whose characteristic function is given by (2.1).

Definition 2.8. The triplet (b, Σ, ν) is called the *generating triplet* of μ, b is called the *drift*, Σ is the *Gaussian covariance matrix*, and ν is the *Lévy measure*.

A small corollary to Theorem 2.7 (e.g. Corollary 8.3 in [50]) gives that if $(X_t)_{t \geq 0}$ is a Lévy process, then it has generating triplet $(tb, t\Sigma, t\nu)$ where $b \in \mathbb{R}^d$, Σ is a Gaussian covariance matrix (i.e. Σ is symmetric and positive-semidefinite), and ν is a Lévy measure. That is, for each $t \geq 0$, X_t has characteristic function

$$\hat{\mu}_t(s) = \mathbb{E}\left[e^{isX_t} \right]$$

$$= \exp\left(it\langle b,s \rangle - \frac{1}{2}t\langle s, \Sigma s \rangle + t\int_{\mathbb{R}^d} \left(e^{i\langle s,x \rangle} - 1 - i\langle s,x \rangle \mathbb{1}_{\{|x|\leq 1\}} \right) \nu(dx) \right).$$

(2.3)

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Equation (2.1) can be interpreted probabilistically via the Lévy-Itô decomposition. We can view a Lévy process $(X_t)_{t \geq 0}$ as the following sum of independent components

$$X \overset{\mathcal{L}}{=} W + Y + L,$$

where $W = (W_t)_{t \geq 0}$ is a Brownian motion with drift, $Y = (Y_t)_{t \geq 0}$ is a compound Poisson process, and $L = (L_t)_{t \geq 0}$ is a square integrable, pure jump martingale. Roughly, the process Y includes all large jumps and the process L includes all of the small jumps. We can relate each piece of the Lévy-Itô decomposition to a corresponding piece of the Lévy-Khintchine representation. Recalling (2.3), for each $t > 0$, the random variable W_t has Fourier transform

$$\exp\left(it \langle b, s \rangle - \frac{1}{2} t \langle s, \Sigma s \rangle \right),$$

the random variable Y_t has Fourier transform

$$\exp\left(t \int_{|x| > 1} \left(e^{i \langle s, x \rangle} - 1\right) \nu\,(dx)\right),$$

and the random variable L_t has Fourier transform

$$\exp\left(t \int_{|x| \leq 1} \left(e^{i \langle s, x \rangle} - 1 - i \langle s, x \rangle\right) \nu\,(dx)\right).$$

More detail can be found in Chapter 4 of [50].

2.1.2 Densities of Lévy Processes

Lévy processes are simultaneously convenient and elusive. While they possess densities under very mild conditions, these distributions lack simple representations (e.g. see [50] or [3]). We start with a result from [50] and [12].

Proposition 2.9. *Let $(X_t)_{t \geq 0}$ be a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ with Lévy triplet (b, σ^2, ν).*

(i) *If $\sigma^2 > 0$ or $\nu(\mathbb{R}) = \infty$, then for each $t > 0$, X_t has a continuous density $p(t, \cdot)$ on $\mathbb{R}$.*

(ii) Suppose that the Lévy measure is such that, for some $\alpha \in (0,2)$,

$$\liminf_{r\downarrow 0} \frac{\int_{-r}^{r} x^2 \nu(dx)}{r^{2-\alpha}} > 0.$$

Then for each $t > 0$, X_t has a density $p(t,\cdot) \in C_0^\infty(\mathbb{R})$.

2.1.3 Moments of Lévy Processes

Lévy processes can have many different distributional properties. Some of these properties are time-dependent and some are not. For example, the absolute continuity of the distribution of a Lévy process on $\mathbb{R}$ is a time-dependent property; however, whether or not the distribution of a Lévy process has point masses is not a time-dependent property. The existence of certain moments of a Lévy process turns out not to be time-dependent (see Section 23 in [50]).

This section follows Chapter 24 in [50]. A few definitions are in order.

Definition 2.10. Let g be a nonnegative measurable function on $\mathbb{R}^d$. The *g-moment* of a measure μ on $\mathbb{R}^d$ is $\int g(x)\mu(dx)$. Similarly, if X is an $\mathbb{R}^d$-valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, we call $\mathbb{E}\left[g\left(X\right)\right]$ the *g-moment* of X.

Our main interest is when a g-moment is finite or not. The results in [50] on g-moments of Lévy processes depend on a couple of function properties.

Definition 2.11.   (i) A function on $\mathbb{R}^d$ is *locally bounded* if it is bounded on every compact subset of $\mathbb{R}^d$.

(ii) A nonnegative function g on $\mathbb{R}^d$ is *submultiplicative* if there exists a positive constant a such that

$$g(x + y) \le ag(x)g(y),$$

for every $x, y \in \mathbb{R}^d$.

A large list of submultiplicative functions is available in [50], but we list a few of the most used ones here for convenience.

Proposition 2.12. *Let $\alpha > 0$ and $\gamma \geq 0$. Then the following functions are submultiplicative:*

(i) $g(x) = |x|^{\alpha}$,

(ii) $g(x) = (0 \vee \log(|x|))^{\alpha}$,

(iii) *and* $g(x) = |x|^{\gamma} e^{\alpha |x|}$.

For functions that are both locally bounded and submultiplicative, the finiteness of the g-moment is not a time-dependent property.

Theorem 2.13. *Let g be a locally bounded, submultiplicative function and let $(X_t)_{t \geq 0}$ be a Lévy process on $\mathbb{R}^d$ with Lévy measure ν. Then, for any $t > 0$*

$$\mathbb{E}\left[g\left(X_t\right)\right] < \infty \quad \text{if and only if} \quad \int_{|x|>1} g(x)\nu(dx) < \infty.$$

We can now use our examples of submultiplicative functions to state a corollary.

Corollary 2.14. *Let $X = (X_t)_{t \geq 0}$ be a Lévy process. For the functions listed in Proposition 2.12, the g-moments of X are either finite or infinite for all $t > 0$, i.e. the g-moment property is not time dependent for the functions in Proposition 2.12.*

There are g-moments whose finiteness does depend on time (e.g. for $g(x) = \left(1 \wedge |x|^{-\alpha}\right) e^{|x|}$, $\alpha > 0$, see remark 25.9 in [50]); however, the finiteness of the g-moments studied in this work are not time dependent. Indeed, we explicitly mention a special case of Proposition 2.12.

Corollary 2.15. *Let $(X_t)_{t \geq 0}$ be a Lévy process with Lévy measure ν and let $\alpha > 0$. Then,*

$$\mathbb{E}e^{\alpha X_t} < \infty \quad \text{for every } t \geq 0 \quad \text{if and only if} \quad \int_{|x|>1} e^{\alpha x}\nu(dx) < \infty.$$

2.2 Stable Lévy Processes

In this section, we develop the necessary machinery for working with stable random variables and stable Lévy processes. Stable Lévy processes are natural to consider for two important reasons. First, certain stable Lévy processes possess self-similarity in a way similar to Brownian motion. For a Brownian motion $(W_t)_{t \geq 0}$ on $\mathbb{R}^d$ and any constant $c > 0$, we have

$$(W_{ct})_{t \geq 0} \overset{\mathcal{L}}{=} \left(c^{\frac{1}{2}} W_t \right)_{t \geq 0}.$$

Similarly, for a wide subclass of stable Lévy processes $(L_t)_{t \geq 0}$, there exists $\alpha \in (0, 2]$ such that

$$(L_{ct})_{t \geq 0} \overset{\mathcal{L}}{=} \left(c^{\frac{1}{\alpha}} L_t \right)_{t \geq 0},$$

for every $c > 0$. Second, stable distributions are natural attractors, i.e. if $X_1, X_2, \ldots$ are i.i.d. random variables in $\mathbb{R}^d$ and if there exist nonrandom sequences $A_n \in \mathbb{R}^d$ and $B_n > 0$ such that

$$\frac{X_1 + X_2 + \cdots + X_n - A_n}{B_n},$$

converges in law to some random variable Z, then Z is necessarily stable (see I.6 of [53]). In particular, if X_1 has finite variance, then the limiting distribution is Gaussian.

There are many different ways to introduce stable random variables and stable Lévy processes. We proceed using characteristic functions and following the development in [49], [50], and [36]. We start with some definitions and then move on to representations and properties, concentrating on stable random variables and stable Lévy processes on $\mathbb{R}$ with finite mean but infinite variance.

Definition 2.16. An infinitely divisible probability measure μ on $\mathbb{R}^d$ is called *stable* if for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(z)^a = \hat{\mu}(bz) \, e^{i \langle c, z \rangle}.$$

It is called *strictly stable* if for any $a > 0$, there exists $b > 0$ such that

$$\hat{\mu}(z)^a = \hat{\mu}(bz).$$

With this definition in hand, we can give an alternate definition in terms of the characteristic function. We will restrict our attention to stable random variables and stable Lévy processes on $\mathbb{R}$ though most results are valid in $\mathbb{R}^d$ as well. Stable random variables are characterized by four parameters: $\alpha, \sigma, \beta,$ and η.

Definition 2.17. (i) The random variable X is said to have a *stable distribution* if there exist parameters $\alpha \in (0, 2]$, $\sigma \geq 0$, $-1 \leq \beta \leq 1$, and $\eta \in \mathbb{R}$ such that X has characteristic function

$$
\begin{aligned}
\hat{\mu}_X(s) &= \mathbb{E}\left[e^{isX}\right] \\
&= \begin{cases} \exp\left(-\sigma^\alpha |s|^\alpha \left(1 - i\beta(\operatorname{sgn} s)\tan\left(\frac{\pi\alpha}{2}\right)\right) + i\eta s\right), & \text{if } \alpha \neq 1 \\ \exp\left(-\sigma |s| \left(1 + i\beta\frac{2}{\pi}(\operatorname{sgn} s)\ln s\right) + i\eta s\right), & \text{if } \alpha = 1, \end{cases}
\end{aligned}
\tag{2.4}
$$

and in this case, we write $X \sim S_\alpha(\sigma, \beta, \eta)$.

(ii) A Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}$ with $X_1 \sim S_\alpha(\sigma, \beta, \eta)$ is called a *stable Lévy process*.

We will always assume that the stable variable is nondegenerate, that is we will assume that X is not distributed as $S_\alpha(0, 0, \eta)$. The parameter α is the *index of stability*, and it determines the rate of decay of the tail probabilities of the stable random variable. The parameter η is the *shift parameter*, and it determines the location of the distribution on the real line; indeed, if $X \sim S_\alpha(\sigma, \beta, \eta)$ and $a \in \mathbb{R}$, then the random variable $X + a \sim S_\alpha(\sigma, \beta, \eta + a)$. The parameter β is the *skewness parameter*, and it determines the level of asymmetry in the Lévy measure of X. Specifically, any stable random variable on $\mathbb{R}$ has Lévy measure

$$\nu(dx) = \frac{1}{|x|^{1+\alpha}}\left(C_+ \mathbb{1}_{\{x>0\}} + C_- \mathbb{1}_{\{x<0\}}\right) dx, \tag{2.5}$$

where α is the index of stability, $C_+, C_- \geq 0$, and the parameter β is

$$\beta = \frac{C_+ - C_-}{C_+ + C_-},$$

whenever $C_+ + C_- > 0$. Finally, the parameter σ is called the *scale parameter*.

There are many examples of stable random variables, and we give some of the more fundamental examples here.

Example 2.18. The following are stable distributions:

(i) Gaussian distribution $S_2(\sigma, 0, \eta) = \mathcal{N}(\mu, 2\sigma^2)$

(ii) Lévy distribution $S_{\frac{1}{2}}(\sigma, 1, \eta)$ with density

$$\left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x - \eta)^{3/2}} \exp\left(-\frac{\sigma}{2(x - \eta)}\right) \mathbb{1}_{\{x > \eta\}}.$$

(iii) Cauchy distribution $S_1(\sigma, 0, \eta)$ with density

$$\frac{1}{\pi\sigma\left(1 + \left(\frac{x - \eta}{\sigma}\right)^2\right)}.$$

(iv) Symmetric α-stable $S_\alpha(\sigma, 0, 0)$ has characteristic function

$$\exp\left(-\sigma^\alpha |z|^\alpha\right).$$

In the general case of stable distributions on $\mathbb{R}^d$, we can generalize (2.5) following Proposition 3.15 in [12].

Proposition 2.19. *An $\mathbb{R}^d$-valued random variable Z is α-stable with $0 < \alpha < 2$ if and only if it is infinitely divisible with characteristic triplet $(b, 0, \nu)$ and there is a finite measure λ on $\mathbb{S}^{d-1}$ (unit sphere of $\mathbb{R}^d$), called the spectral measure of Z, such that for every Borel set $D \subset \mathbb{R}^d$,*

$$\nu(D) = \int_{\mathbb{S}^{d-1}} \int_0^\infty \mathbb{1}_D(r\xi) \frac{dr}{r^{1+\alpha}} \lambda(d\xi).$$

We now give several results concerning stable random variables.

Proposition 2.20. *Let $X \sim S_\alpha(\sigma, \beta, \eta)$ with $0 < \alpha < 2$. Then*

$$\mathbb{E} \, |X|^p < \infty,$$

for any $0 < p < \alpha$, and

$$\mathbb{E} \, |X|^\alpha = \infty.$$

Throughout this work, we are most interested in processes which satisfy certain moment conditions. In particular, we will require the processes and random variables to have finite mean. In the stable case, Proposition 2.20 shows that we need to restrict our attention to $\alpha > 1$. Also, $\alpha = 2$ corresponds to the Gaussian case, and overall we need $\alpha \leq 2$. So, our focus is on stable random variables with $1 < \alpha < 2$.

Given this restriction, it is helpful to have some characterization of how the tails of stable random variables behave (see Property 1.2.15 in [49]).

Proposition 2.21. *Let $X \sim S_\alpha(\sigma, \beta, \eta)$ with $1 < \alpha < 2$. Then*

$$\lim_{x \to \infty} x^\alpha \mathbb{P}\left(X > x\right) = C_\alpha \left(\frac{1 + \beta}{2}\right) \sigma^\alpha,$$
$$\lim_{x \to \infty} x^\alpha \mathbb{P}\left(X < -x\right) = C_\alpha \left(\frac{1 - \beta}{2}\right) \sigma^\alpha,$$

where

$$C_\alpha = \frac{1 - \alpha}{\Gamma\left(2 - \alpha\right) \cos\left(\pi\alpha/2\right)},$$

and Γ is Euler's Gamma function.

2.3   Measure Transformations

For the next section, we now assume we have a filtration $(\mathcal{F}_t)_{t \geq 0}$ along with our probability space $(\Omega, \mathcal{F}, \mathbb{P})$, so called a *filtered probaility space* and denoted $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0}\right)$. Quite reasonably, we might wonder when two probability measures are equivalent

(note that the terminology *mutually absolutely continuous* is sometimes used in place of *equivalent*). We answer this question, following closely the results in [50] and [12].

Recall that two probability measures μ_1 and μ_2 on a common measurable space $(\Omega, \mathcal{F})$ are *equivalent* if the collections $\{D \in \mathcal{F} : \mu_1(D) = 0\}$ and $\{D \in \mathcal{F} : \mu_2(D) = 0\}$ coincide. In this case we write $\mu_1 \approx \mu_2$ and we denote the Radon-Nikodým derivative of μ_2 with respect to μ_1 by $d\mu_2/d\mu_1$ and vice versa. We are now in a position to state the main theorem of interest for us on measure transformations (see section 33 [50]). For each $t > 0$,

Theorem 2.22. *Let* $\left((X_t)_{t \geq 0}, \mathbb{P}\right)$ *and* $\left((X_t)_{t \geq 0}, \mathbb{P}^*\right)$ *be two Lévy processes on the filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}\right)$ *with characteristic triplets* (b, σ^2, ν) *and* $(b^*, (\sigma^*)^2, \nu^*)$, *respectively. Then the following statements are equivalent.*

(i) $\mathbb{P}|_{\mathcal{F}_t} \approx \mathbb{P}^*|_{\mathcal{F}_t}$ *for all* $t > 0$, *where we use* $\mathbb{P}|_{\mathcal{F}_t}$ *to denote the probability measure* $\mathbb{P}$ *restricted to the* σ-*field* $\mathcal{F}_t$.

(ii) *The generating triplets satisfy*

$$\sigma^2 = (\sigma^*)^2$$

$$\nu \approx \nu^*$$

with the function φ *defined by* $d\nu^*/d\nu = e^{\varphi(x)}$ *satisfying*

$$\int_{-\infty}^{\infty} \left(e^{\varphi(x)/2} - 1\right)^2 \nu(dx) < \infty. \tag{2.6}$$

If $\sigma = 0$, *then we must also have*

$$b^* - b - \int_{-1}^{1} x(\nu^* - \nu)(dx) = 0.$$

Furthermore, when the above hold, then

$$\frac{d\,\mathbb{P}^*|_{\mathcal{F}_t}}{d\,\mathbb{P}|_{\mathcal{F}_t}} = e^{U_t}$$

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where

$$U_t = \eta X_t^c - \frac{\eta^2 \sigma^2 t}{2} - \eta b t$$

$$+ \lim_{\varepsilon \downarrow 0} \left(\sum_{s \le t, |\Delta X_s| > \varepsilon} \varphi\left(\Delta X_s\right) - t \int_{|x| > \varepsilon} \left(e^{\varphi(x)} - 1\right) \nu(dx) \right);$$

$\Delta X_t := X_t - X_{t-}$ *for* $t > 0$; $X_t^c = b + \sigma W_t$, *where* $(W_t)_{t \ge 0}$ *is a standard Brownian motion, is the continuous part of* X_t; *and* η *is such that*

$$b^* - b - \int_{-1}^{1} x \left(\nu^* - \nu\right)(dx) = \sigma^2 \eta.$$

Finally, the process $U = (U_t)_{t \ge 0}$ *is again a Lévy process with characteristic triplet* (b_U, σ_U, ν_U) *where* $\sigma_U = \sigma^2 \eta^2$, $\nu_U = \nu \circ \varphi^{-1}$, *and* $b_U = -\frac{1}{2}\sigma\eta^2 - \int_{-\infty}^{\infty} \left(e^y - 1 - y \mathbb{1}_{\{|y| \le 1\}}\right) (\nu \circ \varphi^{-1})(dy)$.

A very important special case of this theorem centers around the measure transform where $\varphi(x) = ax$ with $a \ne 0$. Following Example 33.14 in [50], if

$$\int_{ax > 1} e^{ax} \nu(dx) < \infty,$$

(from Theorem 2.13 we know this is equivalent to $\mathbb{E}e^{aX_t} < \infty$, where the expectation is taken with respect to $\mathbb{P}$), then the measure $\mathbb{P}^*$, defined as

$$\mathbb{P}^*(D) = \mathbb{E}e^{X_t}\mathbb{1}_D, \tag{2.7}$$

for any Borel set D, is well-defined.

2.4   Exponential Lévy Models

Our goal in this section is to develop the theory required to consider an asset whose dynamics behave like $\left(e^{X_t}\right)_{t \ge 0}$ where $X = (X_t)_{t \ge 0}$ is a Lévy process. To this end, we will need a couple of conditions to hold. First, in order for $\left(e^{X_t}\right)_{t \ge 0}$ to be well-defined, we must have an exponential moment condition for the Lévy measure. Second, in order to avoid arbitrage, we need the existence of an equivalent martingale measure.

To simplify, we will enforce conditions on the characteristic triplet of $(X_t)_{t\geq0}$ so that $\left(e^{X_t}\right)_{t\geq0}$ is a martingale under $\mathbb{P}$. More generally, we have Proposition 8.20 in [12].

Proposition 2.23. *Let $(X_t)_{t\geq0}$ be a Lévy process with Lévy triplet (b, σ^2, ν) satisfying*

$$\int_{|y|\geq1} e^y \nu(dy) < \infty. \tag{2.8}$$

Then the process $(Y_t)_{t\geq0} = \left(e^{X_t}\right)_{t\geq0}$ is well-defined. Moreover, it is a martingale with respect to its own filtration if and only if

$$b + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} \left(e^y - 1 - y\mathbb{1}_{\{|y|\leq1\}}\right) \nu(dy) = 0. \tag{2.9}$$

Note that the Proposition 2.23 implies that $\mathbb{E}e^{X_t} = 1$ for all $t \geq 0$. We are now in a position to make the asset-price dynamics precise. For simplicity, we will always assume that interest rates and the dividend yield are both 0. We let $X = (X_t)_{t\geq0}$ be a Lévy process with characteristic triplet (b, σ^2, ν) satisfying (2.8) and (2.9), let $S_0 > 0$, and define the stock-price process $(S_t)_{t\geq0} = \left(S_0 e^{X_t}\right)_{t\geq0}$.

Under this model, the process $S = (S_t)_{t\geq0}$ is a martingale with respect to the filtration of X, i.e. the filtration $(\mathcal{F}_t)_{t\geq0}$ defined, for each $t > 0$, by $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Thus, the market is arbitrage-free (since the given measure is an equivalent martingale measure), and we can use this probability measure to estimate option prices. Recall that if an asset has dynamics $(S_t)_{t\geq0}$ under $\mathbb{P}$, an equivalent martingale measure, then the call-option price can be found using

$$C(t, K) = \mathbb{E}_{\mathbb{P}}\left[(S_T - K)_+ \middle| \mathcal{F}_t\right],$$

where T is the expiration time, t is the current time, and K is the strike price.

Letting $\tau = T - t$ be the time to expiration and $k = \log(K/S_0)$ be the *moneyness* (note, the moneyness is 0 for at-the-money options), the Markov property for $(S_t)_{t\geq0}$ and some simple substitutions lead to the call option price

$$C(\tau, k) = S_0 \mathbb{E}\left(e^{X_\tau} - e^k\right)_+.$$

Notationally, it will be convenient to consider the normalized call-price function and to use t to represent the time to maturity:

$$c(t, k) = \frac{C(t, k)}{S_0}.$$

We are interested in the behavior of the function c as $t \downarrow 0$ while k remains fixed.

2.4.1 Carr and Madan's Option-Pricing Formula

The asymptotics of at-the-money option prices and implied volatility are the main objects of study in this manuscript. We are looking to develop asymptotics for the function

$$c(t, 0) = \mathbb{E} \left(e^{X_t} - 1 \right)_+.$$

To this end, we use a slightly more convenient representation of the function c due to Carr and Madan (see [10] and [22]). We use the transformed probability measure $\mathbb{P}^*$ defined in (2.7), which is given, for all Borel sets $D \subset \mathbb{R}$, by

$$\mathbb{P}^* (D) = \mathbb{E} e^{X_t} \mathbb{1}_D.$$

Carr and Madan showed the following.

Theorem 2.24. *Under $\mathbb{P}^*$, let E be a mean 1 exponential random variable that is independent of $(X_t)_{t \geq 0}$. Then,*

$$\frac{1}{S_0} \mathbb{E} \left(S_t - K \right)_+ = \mathbb{P}^* \left(X_t - E > \log \left(\frac{K}{S_0} \right) \right). \tag{2.10}$$

Corollary 2.25. *The normalized, at-the-money European call option price has representation*

$$
\begin{aligned}
c(t, 0) &= \frac{1}{S_0} \mathbb{E} \left(S_t - K \right)_+ \\
&= \int_0^\infty e^{-x} \mathbb{P}^* \left(X_t \geq x \right) dx.
\end{aligned}
\tag{2.11}
$$

These last two results will help us in finding the first-order asymptotic behavior of $c(t, 0)$ as $t \downarrow 0$.

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CHAPTER III

STABLE DOMAINS OF ATTRACTION AND REGULAR VARIATION

Stable random variables and their respective domains of attraction are central to results previously discussed in the literature on short-time asymptotics for at-the-money call-option prices, although lying beneath the surface (see e.g. [22], [23], [24], [52], and [44]). The Lévy processes considered by those authors satisfy the relationship

$$\frac{X_t}{t^{1/\alpha}} \Rightarrow Z \tag{3.1}$$

as $t \downarrow 0$ where $\alpha \in (0, 2)$, where Z is an α-stable random variable, and where "$\Rightarrow$" denotes convergence in distribution. This property is indispensable for our results on small time-asymptotics of more general Lévy processes.

In this chapter we review some results of Mason and Maller [42] and Grabchak [30] that characterize when (3.1) holds for general Lévy processes. We also look at the relationship of expression (3.1) to regularly varying functions. Finally, we present some results on the connection between regular variation and concentration inequalities for Lévy processes.

3.1 Regular Variation

The study of domains of attraction of stable random variables in continuous time processes requires knowledge of regular variation and slow variation. We follow closely the work in [4] for the results on regular variation and the work in [42] and [30] for the relationship of regular variation to stable domains of attraction.

3.1.1 Slow Variation

First, we recall the definition of a slowly varying function and give some preliminary results.

Definition 3.1. A nonnegative function ℓ defined on some neighborhood $[M, \infty)$ of infinity $(M > 0)$ is *slowly varying at* ∞ (in Karamata's sense) if for every $\lambda > 0$, we have

$$\lim_{x \to \infty} \frac{\ell(\lambda x)}{\ell(x)} = 1. \tag{3.2}$$

Theorem 3.2. *(Uniform Convergence Theorem) If ℓ is slowly varying, then (3.2) holds uniformly on each λ-compact set in $(0, \infty)$.*

Theorem 3.3. *(Representation Theorem) A function ℓ is slowly varying if and only if it can be written in the form*

$$\ell(x) = h(x) \exp\left(\int_a^x \varepsilon(u) \frac{du}{u} \right),$$

for $x \geq a$ where $a > 0$, $h \geq 0$ is measurable and $\lim_{x \to \infty} h(x) = h \in (0, \infty)$, and ε is such that $\lim_{x \to \infty} \varepsilon(x) = 0$.

There are many example of slowly varying functions that possess very different behavior. For example, logarithms and constants are both slowly varying functions. So, slowly varying functions need not be bounded (although they are always locally bounded). Perhaps even more surprising, there exist slowly varying functions such that

$$\liminf_{x \to \infty} \ell(x) = 0 \quad \text{and} \quad \limsup_{x \to \infty} \ell(x) = \infty,$$

(the function

$$\exp\left((\log x)^{1/3} \cos\left((\log x)^{1/3} \right) \right),$$

is an example). We end our discussion of slow variation with some basic facts (again from [4]).

Proposition 3.4. *Let ℓ, ℓ_1, and ℓ_2 be slowly varying functions. Then:*

(i) *The function ℓ^α is slowly varying for every $\alpha \in \mathbb{R}$.*

(ii) *The functions $\ell_1 \ell_2$, $\ell_1 + \ell_2$, and (if $\ell_2(x) \to \infty$ as $x \to \infty$) $\ell_1 \circ \ell_2$ are all slowly varying.*

(iii) *For any $\alpha > 0$,*

$$x^\alpha \ell(x) \to \infty \quad and \quad x^{-\alpha} \ell(x) \to 0, \tag{3.3}$$

as $x \to \infty$.

It is important to note that slowly varying functions are not closed under linear combinations, as we can see with the function $f(x) = \ln(x+1) - \ln(x)$.

3.1.2 Regular Variation

We now turn to the notion of regular variation and give a couple of equivalent definitions.

Definition 3.5. Let f be a positive, measurable function. We say that f is *regularly varying* at ∞ if any one of the following conditions holds.

(i) The limit

$$\lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = g(\lambda) \in (0, \infty), \tag{3.4}$$

exists for every $\lambda \in (0, \infty)$.

(ii) The limit (3.4) exists for every $\lambda \in S$ where S is either a subset of $(0, \infty)$ having positive measure or a dense subset of $(0, \infty)$.

(iii) The limit (3.4) exists and equals $g(\lambda) = \lambda^\rho$ for some $\rho \in \mathbb{R}$.

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(iv) The function f has representation

$$f(x) = x^\rho \ell(x), \tag{3.5}$$

where $\rho \in \mathbb{R}$ and ℓ is slowly varying at ∞.

If any of the above hold, we write $f \in RV_\rho^\infty$ where ρ is the real found in (3.5). We use the notation $\ell \in RV_0^\infty$ if the function ℓ is slowly varying at ∞.

The regular variation property of a function defines how f behaves near ∞. That is, f can be arbitrarily defined on a finite interval $[0, M]$ where $M > 0$. Nevertheless, we will assume that f is locally bounded on any subset of $[0, \infty)$. For our purposes, we will also be concerned with functions that are regularly varying at the origin.

Definition 3.6. A function f is *regularly varying at* 0 (from the right) with index ρ if $f\left(\frac{1}{\cdot}\right) \in RV_{-\rho}^\infty$. We denote this by writing $f \in RV_\rho^0$.

Combining (3.5) with the representation theorem for slowly varying functions gives the general form for a regularly varying function. That is, f is regularly varying at ∞ if and only if it has representation

$$f(x) = x^\rho h(x) \exp\left(\int_a^x \varepsilon(u) \frac{du}{u}\right), \tag{3.6}$$

for $x \geq a$ where $\rho \in \mathbb{R}$, h is a function with a positive, finite limit at ∞, and ε is a function such that $\lim_{x\to\infty} \varepsilon(x) = 0$.

The following corollary is clear.

Corollary 3.7. *Let f be regularly varying with index $\rho \in \mathbb{R}$ at ∞. Then,*

$$\lim_{x\to\infty} f(x) = \begin{cases} \infty, & \text{if } \rho > 0 \\ 0, & \text{if } \rho < 0. \end{cases}$$

There are many important results concerning regularly varying functions that we will need to exploit. These results can be found in [4], and we list them here without proof.

Theorem 3.8. *(Potter's Theorem)*

(i) *If ℓ is slowly varying, then for any given constants $A > 1$ and $\delta > 0$ there exists*

$x_0 = x_0(A, \delta)$ *such that*

$$\frac{\ell(y)}{\ell(x)} \le A \max\left\{ \left(\frac{y}{x}\right)^{\delta}, \left(\frac{y}{x}\right)^{-\delta} \right\},$$

for all $x, y \ge x_0$.

(ii) *If $f \in RV_\rho^\infty$, then for any given $A > 1$ and $\delta > 0$ there exists $x_0 = x_0(A, \delta)$*

such that

$$\frac{\ell(y)}{\ell(x)} \le A \max\left\{ \left(\frac{y}{x}\right)^{\rho+\delta}, \left(\frac{y}{x}\right)^{\rho-\delta} \right\},$$

for all $x, y \ge x_0$.

Theorem 3.9. *(Karamata's Theorem)*

(i) *(Direct Half) If ℓ is slowly varying, x_0 is such that ℓ is locally bounded on*

$[x_0, \infty)$, *and $\alpha > -1$, then*

$$\lim_{x \to \infty} \frac{\int_{x_0}^{x} u^\alpha \ell(u) du}{x^{\alpha+1} \ell(x)} = \frac{1}{(\alpha+1)}.$$

(ii) *(Converse Half) Let f be positive and locally integrable on $[x_0, \infty)$.*

  • *If for some $\zeta > -(\rho + 1)$,*

$$\lim_{x \to \infty} \frac{x^{\zeta+1} f(x)}{\int_{x_0}^{x} u^\zeta f(u) du} = \zeta + \rho + 1,$$

  then f varies regularly with index ρ.

  • *If for some $\zeta < -(\rho + 1)$ we have*

$$\lim_{x \to \infty} \frac{x^{\zeta+1} f(x)}{\int_{x}^{\infty} u^\zeta f(u) du} = -(\zeta + \rho + 1),$$

  then f varies regularly with index ρ.

Theorem 3.10. *If $f \in RV_\alpha^\infty$ with $\alpha > 0$, then there exists $g \in RV_{1/\alpha}^\infty$ such that*

$$\lim_{x \to \infty} \frac{f(g(x))}{x} = \lim_{x \to \infty} \frac{g(f(x))}{x} = 1,$$

i.e. f and g are asymptotically invertible.

Finally, we ponder whether or not absolutely continuous functions that are regularly varying have a derivative that is also regularly varying. For this, we only need to assume monotonicity of the derivative in some neighborhood of ∞. Again, we state the result as in [4].

Theorem 3.11. *(Monotone Density Theorem) Let $H(x) = \int_0^x h(u)du$ where $h : [0, \infty) \to \mathbb{R}$. If $H(x) \sim cx^\rho \ell(x)$ as $x \to \infty$ where $c, \rho \in \mathbb{R}$ and $\ell \in RV_0^\infty$ and if there exists $x_0 > 0$ such that $h(x)$ is monotone on (x_0, ∞), then*

$$\lim_{x \to \infty} \frac{h(x)}{c\rho x^{\rho-1}\ell(x)} = 1.$$

Here, we used the notation that for functions f and g defined on some neighborhood of $a \in \mathbb{R}$, $f(x) \sim g(x)$ as $x \to a$ if and only if

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1.$$

It is important to note that Theorem 3.11 does *not* imply that H or h is regularly varying, as the quantities c and $c\rho$ are potentially negative; however, if $c > 0$, then $H \in RV_\rho^\infty$, but we still do not necessarily have $h \in RV_{\rho-1}^\infty$.

Most of the concepts in this section are easily adapted for regular variation at 0. These results are straightforward and will be omitted.

3.2   Stable Domains of Attraction and Normal Attraction

In this section, we develop the notions of domains of attraction in terms of discrete random processes and then extend our consideration to continuous random processes. First, we introduce some notation and recall concepts. Following [36], we observe the following results.

Theorem 3.12. *Let* $(X_n)_{n\geq 1}$ *be a sequence of i.i.d random variables such that, as* $n \to \infty$,

$$\frac{\sum_{k=1}^{n} X_k - A_n}{B_n} \Rightarrow Z, \tag{3.7}$$

where $(A_n)_{n\geq 1}$ *and* $(B_n)_{n\geq 1}$ *are two sequences of reals with* $\lim_{n\to\infty} B_n = \infty$. *Then* Z *is* α*-stable for some* $\alpha \in (0,2]$. *Moreover,*

$$B_n = n^{1/\alpha}\ell(n),$$

where ℓ *is a slowly varying function at infinity.*

After Theorem 3.12, we can define stable domains of attraction. It turns out that there are some more nuanced ways to look at the convergence that depend on the behavior of the sequence $(B_n)_{n\geq 1}$. So, we will require two definitions.

Definition 3.13. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d random variables satisfying

$$\frac{\sum_{k=1}^{n} X_k - A_n}{B_n} \Rightarrow Z, \tag{3.8}$$

as $n \to \infty$, where Z is a normal random variable and where $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of reals with $\lim_{n\to\infty} B_n = \infty$. Then X is said to be in the *domain of attraction* of a normal distribution.

Definition 3.14. Let $X, X_1, X_2, \dots$ be a sequence of i.i.d random variables satisfying

$$\frac{\sum_{k=1}^{n} X_k - A_n}{B_n} \Rightarrow Z_\alpha, \tag{3.9}$$

as $n \to \infty$, where Z_α is an α-stable random variable and where $(A_n)_{n\geq 1}$ and $(B_n)_{n\geq 1}$ are two sequences of reals with $\lim_{n\to\infty} B_n = \infty$. Then X is said to be in the *domain of attraction* of Z_α. Furthermore, if $B_n = \kappa n^{1/\alpha}$ where $\kappa > 0$ is a constant, then we say that X is in the *domain of normal attraction* of Z_α.

Intuitively, the form of B_n for $n \geq 1$ is the sole difference between a domain of normal attraction and a domain of attraction, i.e. in the former case $B_n = \kappa n^{1/\alpha}$ for some positive constant κ and in the latter $B_n = n^{1/\alpha} \ell(n)$ where ℓ is slowly varying at ∞. For more information about the subtleties of these definitions, we refer the reader to Chapter 3 of [45].

We can also look at domains of attraction from the perspective of distribution functions. In this case, we have a few notable results. Note, if a random variable X has distribution function F, then F is said to be in the domain of attraction of an α-stable distribution whenever X is in the domain of attraction of an α-stable distribution.

Theorem 3.15. *A distribution function F belongs to the domain of attraction of an α-stable distribution with $\alpha \in (0, 2)$ if and only if*

$$F(x) = (C_- + o(1)) |x|^{-\alpha} h(|x|) \quad as \ x \to -\infty, \tag{3.10}$$

and

$$1 - F(x) = (C_+ + o(1)) x^{-\alpha} h(x) \quad as \ x \to \infty, \tag{3.11}$$

for some constants $C_\pm \geq 0$ and some function h slowly varying at ∞.

Theorem 3.15 also applies to distribution function in the domain of normal attraction of and α-stable distribution; however, the constants $C_\pm$ are different.

For completeness, we also consider the situation where a distribution function belongs to the domain of attraction of a normal distribution.

Theorem 3.16. *A distribution function F belongs to the domain of attraction of a normal distribution if and only if*

$$\int_{|x| \geq y} dF(x) = o\left(\frac{1}{y^2} \int_{|x| < y} x^2 dF(x)\right), \tag{3.12}$$

as $y \to \infty$.

We consider some examples from the previous two theorems.

Example 3.17. (i) If X is a stable random variable, then X is in its own domain of attraction. While clear, we can use Proposition 2.21 to see that both (3.10) and (3.11) hold.

(ii) Let X be a mean 1 exponential random variable. Then, X is in the domain of attraction of a normal distribution. Indeed

$$\frac{y^2 \int_y^\infty dF(x)}{\int_0^y x^2 dF(x)} = \frac{y^2 e^{-y}}{e^{-y}\left(-2 + 2e^y - y\left(2 + y\right)\right)}$$

$$= \frac{y^2}{-2 + 2e^y - y\left(2 + y\right)} \to 0,$$

as $y \to \infty$.

(iii) Let X be a Pareto distribution with scale parameter $x_0 > 0$ and shape $0 < \alpha < 2$, i.e. X has cumulative distribution function

$$F\left(x\right) = \begin{cases} 1 - \left(\frac{x_0}{x}\right)^\alpha, & \text{if } x \geq x_0 \\ 0, & \text{if } x < x_0. \end{cases}$$

Then X is in the domain of attraction of an α-stable random variable.

We might ponder when an expression such as (3.7) has an analog for continuous-time processes. Moreover, since we are concerned with short-time asymptotics, we are interested in this property as $t \to 0$. Concretely, if $(X_t)_{t \geq 0}$ is a stochastic process and if there exist functions $B_t > 0$ and A_t with $B_t \to 0$ as $t \to 0$ such that

$$\frac{X_t - A_t}{B_t} \Rightarrow Z,$$

is Z then necessarily stable?

Clearly this cannot be true for all processes as can easily be seen by considering $(X_t)_{t \geq 0} = (tY)_{t \geq 0}$ where Y is not stable; however, when $(X_t)_{t \geq 0}$ is a real-valued Lévy process, the random variable Z must be stable. So, we find the conditions under which such functions A_t and B_t exist. Some preliminary results are required first.

3.3 Lévy Processes in Stable DOA and DNA

As we alluded to earlier, the domain-of-attraction notions can be carried over to the continuous setting. Indeed we have the following definitions.

Definition 3.18. A stochastic process $(X_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is in the *domain of attraction* (DOA) of a stable random variable Z at $a \in \{0, \infty\}$ if there exist functions $B_t > 0$ and $A_t \in \mathbb{R}$ such that

$$\frac{X_t - A_t}{B_t} \Rightarrow Z, \tag{3.13}$$

as $t \to a$.

So far, we have not placed any restrictions on the form of the scaling function B. As seen later, as for discrete processes, B must be regularly varying. This leads to an extension of the definition from the discrete process case.

Definition 3.19. A stochastic process $(X_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is in the *domain of normal attraction* (DNA) of a stable random variable Z at $a \in \{0, \infty\}$ if $(X_t)_{t \geq 0}$ is in the DOA of Z at $a \in \{0, \infty\}$ with scaling function B satisfying $\lim_{t \to a} B_t / \kappa t^\rho = 1$ for some $\rho, \kappa > 0$.

The collection of processes that are in the domain of normal attraction of a stable random variable Z form a proper subset of the collection of processes that are in the domain of attraction of Z (as we will see with an example later). Indeed, for processes in the DOA of a stable random variable, the scaling function $B_t = t^{1/\alpha} \ell(t)$ where ℓ is slowly varying at a, whereas for processes in the DNA of a stable random variable, the scaling function $B_t = \kappa t^{1/\alpha}$.

We are now in a position to develop the relationship between Lévy processes and stable domains of attraction.

3.4 Regular Variation and Lévy Processes

Again, we will be in the setting where $(X_t)_{t \geq 0}$ is a real-valued Lévy process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with characteristic triplet (b, σ^2, ν) where $b \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a Lévy measure. The results of this section come from [42] and [30]. While approaching the problems from two different perspectives, [42] and [30] arrive at very similar results.

First, let us introduce some notation. For $x > 0$, let

$$\gamma(x) = \gamma_+(x) + \gamma_-(x) := \nu(\{y > x\}) + \nu(\{y < -x\}), \tag{3.14}$$

and

$$V(x) := \int_{|y| \leq x} y^2 \nu(dy). \tag{3.15}$$

We are mainly interested in conditions under which

$$\frac{X_t - A_t}{B_t} \tag{3.16}$$

converges in distribution to an α-stable distribution as $t \to 0$ for some $\alpha \in (0, 2)$ where $A : [0, \infty) \to \mathbb{R}$ and $B : (0, \infty) \to (0, \infty)$ are functions with $\lim_{t \to 0} B_t = 0$. If the Lévy process has finite second moment, then the Central Limit Theorem (or the Lévy-Khintchine formula) gives

$$\frac{X_t - t\mathbb{E}X_1}{\sqrt{t}} \Rightarrow \mathcal{N}(0, \sigma^2),$$

as $t \to 0$.

We first present the result from [42] as it requires less machinery.

Theorem 3.20. *The following are equivalent:*

(i) there exist deterministic functions A and B with $B_t > 0$ and $\lim_{t \to 0} B_t = 0$ such that

$$\frac{X_t - A_t}{B_t} \Rightarrow Z$$

as $t \to 0$, where Z is an a.s. finite, nondegenerate random variable (in fact, Z is necessarily an α-stable random variable with $\alpha \in (0, 2]$);

(ii) (a) the function $\gamma \in RV^0_{-\alpha}$ with $\alpha \in (0, 2)$ and the limits

$$\lim_{x \downarrow 0} \frac{\gamma_{\pm}(x)}{\gamma(x)}, \tag{3.17}$$

exist or (b) V is slowly varying at 0.

Note that in (ii) above, exactly one of cases (a) and (b) holds. The case where V is slowly varying at 0 corresponds exactly to the case where the central limit theorem applies with $B_t = \sqrt{t}$ and $Z \sim \mathcal{N}(0, \sigma^2)$. Also, although the statement of the theorem does not require it, we can choose the function B to be monotone decreasing. Finally, the result shows that the weak convergence of expressions like (3.16) is necessarily to a stable or normal distribution if $(X_t)_{t \geq 0}$ is a Lévy process. In fact, we can say more about the functions A and B.

We will need two more functions that are defined for $x > 0$ as

$$\mu_x = b - \int_{x \leq |y| \leq 1} y\nu(dy), \tag{3.18}$$

and

$$U(x) = 2 \int_0^x y\gamma(y)dy. \tag{3.19}$$

In [42] Maller and Mason show that one can take

$$B_t = \inf \left\{ 0 < x \leq 1 : x^{-2}U(x) \leq \frac{1}{t} \right\} \quad \text{and} \quad A_t = t\mu_{b_t}.$$

In [30], Grabchak uses the asymptotic inversion formula for regularly varying functions to show that B_t is regularly varying, giving a more exact formula for B_t (this formula is given in Theorem 3.23). In fact, the result in [30] is equivalent to the one in [42] in one dimension. We next state the result from [30] in full generality. This requires some preliminary definitions.

Definition 3.21. Let $\rho \leq 0$, $a \in \{0, \infty\}$, and let ν be a Lévy measure on $\mathbb{R}^d \backslash \{0\}$ with $\nu \neq 0$. If $a = \infty$ assume further that ν has unbounded support. The measure ν is said to be *regularly varying at a with index ρ* if there is a finite, nonzero Borel measure σ on $\mathbb{S}^{d-1}$ such that for any $t > 0$ and any $D \in \mathcal{B}\left(\mathbb{S}^{d-1}\right)$ with $\sigma(\partial D) = 0$

$$\lim_{r \to a} \frac{\nu\left(|x| > rt, \frac{x}{|x|} \in D\right)}{\nu(|x| > r)} = t^\rho \frac{\sigma(D)}{\sigma\left(\mathbb{S}^{d-1}\right)}. \tag{3.20}$$

When this holds, we write $\nu \in RV_\rho^a(\sigma)$ and when ν is a measure on $\mathbb{R} \backslash \{0\}$ we write $\nu \in RV_\rho^a$.

If restricted to Lévy measures on $\mathbb{R}$ and regular variation at 0, Definition 3.21 is equivalent to the condition *(ii)(a)* of Theorem 3.20. First, assume that *(ii)(a)* of Theorem 3.20 holds. For $D = \mathbb{S}^0 = \{-1, 1\}$, then (3.20) holds true from the regular variation property of γ. For $D = \{1\}$ we have

$$\lim_{r \to 0} \frac{\nu(x > rt)}{\nu(|x| > r)} = \lim_{r \to 0} \frac{\nu(x > rt)}{\nu(|x| > rt)} \cdot \frac{\nu(|x| > rt)}{\nu(|x| > r)} \tag{3.21}$$

$$= pt^\rho,$$

and letting $\sigma(\{1\}) / \sigma(\mathbb{S}^0) = p$ gives (3.20). A similar argument can be made for $D = \{-1\}$.

Conversely, assume that (3.20) holds. Choosing $D = \mathbb{S}^0$ gives $\gamma \in RV_\rho^0$, while choosing $D = \{1\}$ gives

$$
\begin{aligned}
\lim_{x \downarrow 0} \frac{\gamma_+(x)}{\gamma(x)} &= \lim_{r \downarrow 0} \frac{\gamma_+(rt)}{\gamma(rt)} \\
&= \lim_{r \downarrow 0} \frac{\nu(z > rt)}{\nu(|z| > rt)} \\
&= \lim_{r \downarrow 0} \frac{\nu(z > rt)}{\nu(|z| > r)} \cdot \frac{\nu(|z| > r)}{\nu(|z| > rt)} \\
&= \frac{\sigma(\{1\})}{\sigma(\mathbb{S}^0)} t^\rho t^{-\rho} \\
&= \frac{\sigma(\{1\})}{\sigma(\mathbb{S}^0)}.
\end{aligned}
$$

A similar argument can be made for the limit involving the negative tail. So, indeed, the two notions are equivalent.

We are now in a position to state the main result from [30].

Theorem 3.22. *Fix $\alpha \in (0, 2)$ and let σ be a finite nonzero Borel measure on $\mathbb{S}^{d-1}$. Let $(X_t)_{t \geq 0}$ be a Lévy process with triplet $(b, 0, \nu)$. There exist functions $B > 0$ and A, as well as an α-stable random variable Z with spectral measure σ such that*

$$\frac{X_t - A_t}{B_t} \Rightarrow Z$$

as $t \downarrow 0$ if and only if the Lévy measure $\nu \in RV^0_{-\alpha}$.

Grabchak's argument relies on finding a Lévy process $X^0 = (X^0_t)_{t \geq 0}$ such that the behavior of X^0 as $t \to \infty$ is equivalent in some sense to the behavior of $X = (X_t)_{t \geq 0}$ as $t \downarrow 0$. Specifically, let ν^0 be defined, for any Borel set D, by

$$\nu^0 (D) = \int_{\mathbb{R}^d} \mathbb{1}_D \left(\frac{x}{|x|^2} \right) |x|^2 \nu(dx)$$

and let $b^0 = b$ where $(b, 0, \nu)$ is the Lévy triplet of X. Then, ν^0 is a well-defined Lévy measure, and defining $X^0 = (X^0_t)_{t \geq 0}$ as the Lévy process with triple $(b^0, 0, \nu^0)$ gives us the Lévy process we are looking for. We can be more precise with a theorem from [30].

Theorem 3.23. *Fix $\alpha \in (0, 2)$ and let $X = (X_t)_{t \geq 0}$ and $X^0 = (X^0_t)_{t \geq 0}$ be Lévy processes with Lévy triplets $(b, 0, \nu)$ and $(b^0, 0, \nu^0)$, respectively. Then there exist functions A and B, as well as an α-stable random variable Z with spectral measure σ such that*

$$\frac{X_t - A_t}{B_t} \Rightarrow Z,$$

as $t \downarrow 0$, if and only if there exist functions ξ and C such that

$$\frac{X^0_t - \xi_t}{C_t} \Rightarrow Z^0,$$

as $t \to \infty$, where Z^0 is a $(2-\alpha)$-stable random variable with spectral measure σ. Moreover, when this holds we have

$$B_t \sim \left(\frac{1}{t} h^{-1} \left(\frac{1}{t} \right) \right)^{-1/2},$$

as $t \downarrow 0$, for any invertible function h satisfying

$$h(t) \sim \frac{1}{t} C_t^2,$$

as $t \to \infty$.

For our results, we will tend to use the conditions from part *(ii)(a)* of Theorem 3.20 since they are easier to state.

For the one-dimensional case, we see that the quantities

$$\lim_{x \downarrow 0} \frac{\gamma_{\pm}(x)}{\gamma(x)},$$

directly define the measure σ. This allows us to get an exact representation for the α-stable random variable involved in the convergence. Mainly, we have the following result.

Lemma 3.24. *Suppose $(X_t)_{t \geq 0}$ is a Lévy process satisfying (3.13) so that the limits in (3.17) exist and are equal to $p_{\pm} \geq 0$ respectively, where $p_+ + p_- = 1$. Then, the characteristic function of Z is given, for any $u \in \mathbb{R}$, by*

$$\varphi_Z(s) = \exp \left(- c_\alpha |s|^\alpha \left(1 - i \left(p_+ - p_- \right) \operatorname{sgn}(s) \tan (\pi \alpha / 2) \right) \right), \qquad (3.22)$$

where $c_\alpha > 0$ is a constant.

3.5 Connection to Concentration Inequalities

One important quantity in the estimation of call option prices in exponential Lévy models is the expression $\mathbb{P}(X_t \geq y)$ where $y \geq 0$ and $(X_t)_{t \geq 0}$ is a Lévy process on $\mathbb{R}$ with triplet $(b, 0, \nu)$. For estimating these tail quantities, we need concentration

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inequalities similar to those found in [7] and [33]. The next lemma provides the estimation we need. For now, we will allow γ to be any nonnegative function, but our future application of this lemma will use the definition of γ in (3.14); however, for the statement of the next theorem we do require the function V defined in (3.15) and μ_y defined in (3.18).

Lemma 3.25. *Let $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ be such that for all $R > 0$,*

(i) $\int_{|x|>R} \nu\,(dx) \leq \gamma\,(R)$,

(ii) and there exists $C > 0$ independent of R such that $V(R) \leq CR^2 \gamma\,(R)$.

Then, for every $y > 0$ and for every $0 < t < y/4 \left(\mu_{y/4}\right)_+$ (with $y/0 = \infty$),

$$\mathbb{P}\left(X_t \geq y\right) \leq \left(1 + Ce^2\right) t\gamma\left(\frac{y}{4}\right).$$

Proof. In the traditional manner (e.g. see [7] or [34]), we break $X = (X_t)_{t \geq 0}$ up into two parts, $X^\varepsilon = (X_t^\varepsilon)_{t \geq 0}$ which consists of all jumps smaller than ε and $\widetilde{X}^\varepsilon = \left(\widetilde{X}_t^\varepsilon\right)_{t \geq 0}$ consisting of all jumps larger than ε. For each $t > 0$, we can represent X_t as

$$X_t = bt + \int_0^t \int_{|x| \leq 1} x\,(\mu - \bar{\mu})\,(dx, ds) + \int_0^t \int_{|x| \geq 1} x\mu(dx, ds) \qquad (3.23)$$

where μ is a Poisson random measure on $\mathbb{R} \backslash \{0\}$ with mean measure $\bar{\mu}(dx, dt) = \nu(dx)dt$. Let $f_\varepsilon(x) = \mathbb{1}_{[-\varepsilon, \varepsilon]}$ and $\bar{f}_\varepsilon = 1 - f_\varepsilon$. We can define the processes for each $t > 0$ by

$$\widetilde{X}_t^\varepsilon = \int_0^t \int_{\mathbb{R}} x\bar{f}_\varepsilon(x)\mu(dx, ds) \quad \text{and} \quad X_t^\varepsilon = X_t - \widetilde{X}_t^\varepsilon. \qquad (3.24)$$

The process $\widetilde{X}^\varepsilon$ is a compound Poisson process with intensity $\lambda_\varepsilon = \int \bar{f}_\varepsilon(x)\nu(dx)$ and jump distribution

$$\frac{\bar{f}_\varepsilon(x)\nu(dx)}{\lambda_\varepsilon},$$

and X^ε is a Lévy process with characteristic triplet $(b_\varepsilon, 0, f_\varepsilon\nu)$ where

$$b_\varepsilon = b - \int_{|x| \leq 1} x\bar{f}_\varepsilon(x)\nu(dx).$$

We will need the fact that $\mathbb{E}X_t^\varepsilon = t\mu_\varepsilon$ where μ_ε is defined by (3.18). For a fixed $y > 0$, we have

$$
\begin{aligned}
\mathbb{P}\left(X_t \geq y\right) &\leq \mathbb{P}\left(X_t^\varepsilon \geq y/2\right) + \mathbb{P}\left(\widetilde{X}_t^\varepsilon \geq y/2\right) \\
&\leq \mathbb{P}\left(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon \geq y/2 - \mathbb{E}X_t^\varepsilon\right) + \mathbb{P}\left(\widetilde{X}_t^\varepsilon \neq 0\right) \\
&\leq \mathbb{P}\left(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon \geq y/2 - \mathbb{E}X_t^\varepsilon\right) + t\gamma(\varepsilon) \\
&= \mathbb{P}\left(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon \geq y/2 - t\mu_\varepsilon\right) + t\gamma(\varepsilon).
\end{aligned} \tag{3.25}
$$

Using a general concentration inequality (e.g. Corollary 1 in [33]), we obtain for $z > 0$

$$
\begin{aligned}
\mathbb{P}\left(X_t^\varepsilon - \mathbb{E}X_t^\varepsilon \geq z\right) &\leq \exp\left[\frac{z}{\varepsilon} - \left(\frac{z}{\varepsilon} + \frac{tV_\varepsilon}{\varepsilon^2}\right)\log\left(1 + \frac{\varepsilon z}{tV_\varepsilon}\right)\right] \\
&\leq \exp\left[\frac{z}{\varepsilon} - \frac{z}{\varepsilon}\log\left(1 + \frac{\varepsilon z}{Ct\varepsilon^2\gamma(\varepsilon)}\right)\right] \\
&= \exp\left[\frac{z}{\varepsilon} - \frac{z}{\varepsilon}\log\left(1 + \frac{z}{Ct\varepsilon\gamma(\varepsilon)}\right)\right] \\
&= \frac{\exp\left(\frac{z}{\varepsilon}\right)}{\left(1 + \frac{z}{Ct\varepsilon\gamma(\varepsilon)}\right)^{z/\varepsilon}}.
\end{aligned} \tag{3.26}
$$

We now need to choose ε in such a way that both terms in (3.25) are of the same order. We choose $\varepsilon = y/4$ and we consider two cases. First, consider the case where $\mu_{y/4} \geq 0$ and further assume that $0 < t < y/4\mu_{y/4}$ (equivalently $y/2 - t\mu_{y/4} > y/4$). Then,

$$
\begin{aligned}
\mathbb{P}\left(X_t^{y/4} - \mathbb{E}X_t^{y/4} \geq y/2 - t\mu_{y/4}\right) &\leq \mathbb{P}\left(X_t^{y/4} - \mathbb{E}X_t^{y/4} \geq y/4\right) \\
&= \frac{\exp\left(\frac{y/4}{y/4}\right)}{\left(1 + \frac{y/4}{Ct(y/4)\gamma(y/4)}\right)^{\frac{y/4}{y/4}}} \\
&= \frac{e}{\left(1 + \frac{1}{Ct\gamma(y/4)}\right)} \\
&\leq Ce^2 t\gamma\left(\frac{y}{4}\right).
\end{aligned} \tag{3.27}
$$

Next, consider the case where $\mu_{y/4} < 0$. Then, for all $t > 0$,

$$\mathbb{P}\left(X_t^{y/4} - \mathbb{E}X_t^{y/4} \geq y/2 - t\mu_{y/4}\right) \leq \mathbb{P}\left(X_t^{y/4} - \mathbb{E}X_t^{y/4} \geq y/2\right)$$

$$\leq \frac{\exp\left(\frac{y/2}{y/4}\right)}{\left(1 + \frac{y/2}{Ct(y/4)\gamma(y/4)}\right)^{\frac{y/2}{y/4}}}$$

$$= \frac{e^2}{\left(1 + \frac{2}{Ct\gamma(y/4)}\right)^2}$$

$$= \frac{e^2}{\left(1 + \frac{2}{Ct\gamma(y/4)}\right)}$$

$$\leq Ce^2 t\gamma\left(\frac{y}{4}\right). \tag{3.28}$$

Notice that the terms in (3.27) and (3.28) are the same. Combining this result with (3.25) gives

$$\mathbb{P}\left(X_t \geq y\right) \leq Ce^2 t\gamma\left(\frac{y}{4}\right) + t\gamma\left(\frac{y}{4}\right) = (1 + Ce^2)t\gamma\left(\frac{y}{4}\right),$$

for all $y > 0$ and $0 < t < y/4(\mu_{y/4})_+$. $\qquad\square$

We will usually apply Lemma 3.25 when γ is defined by (3.14) (and so condition *(i)* is trivially satisfied). It is shown in [4] (Chapter 8.1), [21] (VII.9), and [42] that condition *(ii)* in Lemma 3.25 is satisfied automatically for small values of R whenever $(X_t)_{t\geq 0}$ is in the DOA of an α-stable random variable. We can simplify things further by showing that the condition *(ii)* is also naturally satisfied for compact intervals of $\mathbb{R}_+$.

Proposition 3.26. *Let $(X_t)_{t\geq 0}$ be a Lévy process in the domain of attraction of an α-stable random variable with $\alpha \in (1,2)$. Then for any $0 < x < y < \infty$,*

$$\sup_{x\leq R\leq y} \frac{V(R)}{R^2\gamma(R)} < \infty.$$

Proof. By Theorem 3.20, $\gamma(R) = R^{-\alpha}\psi(R)$ where ψ is slowly varying and $\alpha \in (1, 2)$. Integration by parts gives

$$0 < V(z) = -z^2\gamma(z) + 2\int_0^z \xi\gamma(\xi)d\xi$$
$$= -z^2\gamma(z) + 2\int_0^z \xi^{1-\alpha}\psi(\xi)\,d\xi \tag{3.29}$$

which is well-defined since $1 - \alpha \in (-1, 0)$. So,

$$\frac{V(z)}{z^2\gamma(z)} = \frac{-z^2\gamma(z) + 2\int_0^z \xi^{1-\alpha}\psi(\xi)\,d\xi}{z^2\gamma(z)}$$
$$= -1 + \frac{2\int_0^z \xi^{1-\alpha}\psi(\xi)\,d\xi}{z^{1-\alpha}\psi(z)}.$$

The numerator is continuous and the denominator is piecewise continuous and bounded away from 0 in any compact interval of $\mathbb{R}_+$ not including 0 (the function γ is nonincreasing on $(0, \infty)$ so it can only have jump discontinuities). Thus, the supremum is bounded and the result follows. $\quad\square$

So, the condition *(ii)* only needs to be verified for R sufficiently large when $(X_t)_{t\geq 0}$ is in the domain of attraction of a stable random variable. We state the full result in a proposition.

Proposition 3.27. *Let $(X_t)_{t\geq 0}$ be a Lévy process in the domain of attraction of an α-stable random variable, $0 < \alpha < 2$. Further, let there exist $R_0 > 0$ and $C > 0$ possibly depending on R_0 such that for all $R > R_0$, $V(R) \leq CR^2\gamma(R)$ where V and γ are defined in (3.15) and (3.14), respectively. Then, for every $y > 0$ and for every $0 < t < y/4\left(\mu_{y/4}\right)_+$ (with $y/0 = \infty$),*

$$\mathbb{P}(X_t \geq y) \leq \left(1 + Ce^2\right)t\gamma\left(\frac{y}{4}\right).$$

We now give some examples of the application of Theorem 3.27 given in [7].

Example 3.28.   (i) Consider the Lévy process $(X_t)_{t\geq 0}$ where X_1 has triplet $(0, 0, \nu)$ with

$$\nu(dx) = \frac{|\log|x||}{x^2}\mathbb{1}_{\{x\neq 0\}}dx. \tag{3.30}$$

Then, for $R > 1$,

$$\gamma(R) = \int_{|x|>R} \nu(dx)$$
$$= 2\left(\frac{1+\log R}{R}\right),$$

and

$$V(R) = \int_{|x|\leq R} x^2 \nu(dx)$$
$$= 2(2 - R + R\log R)$$
$$\leq 2(R + R\log R)$$
$$= R^2\gamma(R).$$

Similarly, we can compute, for $0 < R < 1$,

$$\gamma(R) = 4 - \frac{2}{R} - \frac{2\log R}{R},$$

and

$$V(R) = 2R(1 - \log R)$$
$$\leq 3R^2\gamma(R).$$

Additionally, the Lévy measure is symmetric and so $\mu_\varepsilon = 0$, for every $\varepsilon > 0$.
Therefore, from the concentration inequalities, we obtain for all $t > 0$ and $y > 0$,

$$\mathbb{P}(X_t \geq y) \leq (1 + 3e^2)\, t\gamma\left(\frac{y}{4}\right),$$

where

$$\gamma(R) = \begin{cases} \frac{4R-2-2\log R}{R}, & \text{if } 0 < R < 1 \\[2mm] \frac{2+2\log R}{R}, & \text{if } R \geq 1. \end{cases}$$

(ii) Consider the Lévy process $(X_t)_{t\geq 0}$ where X_1 has triplet $(0, 0, \nu)$ with

$$\nu(dx) = \frac{e^{-1/x^2}}{x^2}\mathbb{1}_{\{x\neq 0\}}dx.$$

For $R > 0$, we have

$$\gamma(R) = \int_{|x|>R} \nu(dx)$$

$$= 2 \int_0^{1/R} e^{-u^2} du$$

$$\leq \frac{2}{R},$$

and

$$V(R) = \int_{|x|\leq R} x^2 \nu(dx)$$

$$= 2 \int_{1/R}^\infty \frac{e^{-u^2}}{u^2} du$$

$$= 2Re^{-1/R^2} - \int_{1/R}^\infty 4e^{-u^2} du$$

$$\leq R^2 \gamma(R).$$

As in the previous example, $\mu_\varepsilon = 0$ from the symmetry of the Lévy measure, and therefore for all $t > 0$ and $y > 0$

$$\mathbb{P}(X_t \geq y) \leq \frac{8t}{y}.$$

CHAPTER IV

FIRST-ORDER RESULTS

As mentioned in the introduction, there are many processes for which the first-order dynamics are well-understood. For the processes considered (e.g. CGMY, tempered stable, etc.) without Brownian component, the asymptotic behavior takes the form

$$C(t,0) = \kappa t^{1/\alpha} + o\left(t^{1/\alpha}\right)$$

whenever $1 < \alpha < 2$ and $\kappa > 0$ is a constant which depends on the distribution of some α-stable random variable Z. These results rely on the convergence of $X_t/t^{1/\alpha}$ to an α-stable random variable. In this chapter, we extend these first order results to a more general class of Lévy processes. In fact, for this more general class of Lévy processes with no Brownian component, we can have quite different behavior. In particular, for the normalized call-price formula c and in the case $1 < \alpha < 2$,

$$c(t,0) = \kappa t^{1/\alpha}\psi(t) + o\left(t^{1/\alpha}\psi(t)\right), \tag{4.1}$$

where ψ is a slowly varying function at 0 and $\kappa > 0$ again is a positive constant depending on some α-stable random variable. The previous cases covered where ψ is a positive constant. Furthermore, we show that when a Brownian component is included, the expected behavior still occurs, i.e. the normalized ATM call price satisfies

$$c(t,0) = \kappa\sqrt{t} + o\left(\sqrt{t}\right).$$

After the introductory material and proofs, we give an example of a Lévy process that satisfies (4.1) where ψ is nonconstant.

4.1 Share Measure and DOA

In what follows, $(X_t)_{t \geq 0}$ is a Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with characteristic triplet $(b, 0, \nu)$. We will also assume that $\mathbb{E}\left[\,|X_t|\,e^{X_t}\right] < \infty$. This last assumption is equivalent to $\mathbb{E}\left[\,|X_1|\,e^{X_1}\right] < \infty$ or

$$\int_{|x|>1} |x|\,e^x \nu(dx) < \infty, \tag{4.2}$$

by Theorem 2.13. Note that we clearly have a finite exponential moment. Finally, we suppose that ν has a density with respect to Lebesgue measure. That is there exists a function $\xi \geq 0$, called the Lévy density, such that for any Borel set $D \subset \mathbb{R}$,

$$\nu(D) = \int_D \xi(x)dx.$$

We will also need the probability measure $\mathbb{P}^*$ defined in (2.7) via

$$\mathbb{P}^*(D) = \mathbb{E}\left[e^{X_t}\mathbb{1}_D\right],$$

for each Borel set $D \subset \mathbb{R}$. Note that $\mathbb{P}^*$ is well-defined by (4.2) and the work done in Example 33.14 in [50]. Under $\mathbb{P}^*$, the process $(X_t)_{t \geq 0}$ is again a Lévy process with triplet $(b^*, 0, \nu^*)$ where

$$\nu^*(dx) = e^x \nu(dx) = e^x \xi(x)dx, \tag{4.3}$$

and

$$b^* = b + \int_{|x| \leq 1} (\nu^* - \nu)(dx) = b + \int_{|x| \leq 1} (e^x - 1)\,\xi(x)dx. \tag{4.4}$$

Throughout this chapter, we will define quantities under both $\mathbb{P}$ and $\mathbb{P}^*$, and we use the star notation to mean the associated quantity under $\mathbb{P}^*$. For example, we will define the mean μ_ε of the Lévy random variable X_1^ε (jumps truncated at ε) as

$$\mu_\varepsilon = b - \int_{|x| \leq 1} x\mathbb{1}_{\{|x| > \varepsilon\}}\nu(dx) + \int_{|x| \geq 1} x\mathbb{1}_{\{|x| \leq \varepsilon\}}\nu(dx),$$

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and will define μ_ε^* as

$$\mu_\varepsilon^* = b^* - \int_{|x|\leq 1} x\mathbb{1}_{\{|x|>\varepsilon\}}\nu^*(dx) + \int_{|x|\geq 1} x\mathbb{1}_{\{|x|\leq\varepsilon\}}\nu^*(dx).$$

Similarly, whereas the function $\gamma(x) = \int_{|y|>x}\nu(dy)$, the associated $*$ function is defined as $\gamma^*(x) = \int_{|y|>x}\nu^*(dy)$.

We now define a few more functions and constants that we will need in this chapter. For $x > 0$, set

$$\xi_S(x) := \xi(x) + \xi(-x), \tag{4.5}$$

while the associated quantity under $\mathbb{P}^*$ is

$$\xi_S^*(x) := e^x\xi(x) + e^{-x}\xi(-x). \tag{4.6}$$

We are interested in the quantities

$$\sup_{0<\eta<\infty} |\mu_\eta| \quad \text{and} \quad \sup_{0<\eta<\infty} \left|\mu_\eta^*\right|,$$

which we will need to be finite. Note that

$$\sup_{1\leq\eta<\infty} |\mu_\eta| = \sup_{1\leq\eta<\infty} \left|b + \int_{1\leq|x|\leq\eta} x\nu(dx)\right| \leq |b| + \int_{1\leq|x|<\infty} |x|\,\nu(dx) < \infty,$$

and

$$\sup_{1\leq\eta<\infty} \left|\mu_\eta^*\right| = \sup_{1\leq\eta<\infty} \left|b^* + \int_{1\leq|x|\leq\eta} x\nu^*(dx)\right| \leq |b^*| + \int_{1\leq|x|<\infty} |x|\,\nu^*(dx) < \infty,$$

since the Lévy process $(X_t)_{t\geq 0}$ has finite first moment under both $\mathbb{P}$ and $\mathbb{P}^*$. So, we need only consider when the quantities

$$\bar\mu = \sup_{0<\eta\leq 1} |\mu_\eta| \quad \text{and} \quad \bar\mu^* = \sup_{0<\eta\leq 1} \left|\mu_\eta^*\right|,$$

are finite (e.g. when ν is symmetric, see [22]).

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4.2 Regular Variation Under the Share Measure

The share measure, defined in (2.7), is an integral part of the analysis of ATM call-price asymptotics. Note that the Carr-Madan formula (2.11) is expressed with respect to the share measure $\mathbb{P}^*$. It is therefore natural to ask what properties are preserved when we consider our Lévy process $(X_t)_{t\geq 0}$ under the share measure $\mathbb{P}^*$.

In this section, we show that regular variation of the Lévy measure ν is preserved under $\mathbb{P}^*$. Intuitively, this preservation of regular variation stems from that fact that the regular variation property of ν depends on the behavior of ν near the origin. The transformed measure has very similar behavior very close to the origin ($e^x \approx 1$ for x close to 0).

Lemma 4.1. *If ν is regularly varying of index α at 0, then ν^* is also regularly varying of index α at 0.*

Proof. Suppose that ν is regularly varying of index α at 0, i.e.

$$\lim_{r\to 0} \frac{\nu\left(|x| > rt\right)}{\nu\left(|x| > r\right)} = t^{-\alpha},$$

for all $t > 0$. Note that this is equivalent to saying

$$\lim_{r\to 0} \frac{\int_{rt}^{\infty} \xi_S(x)dx}{\int_r^{\infty} \xi_S(x)dx} = t^{-\alpha}.$$

First, we show that for any fixed $\delta > 0$ and $t > 0$ we also have

$$\lim_{r\to 0} \frac{\int_{rt}^{\delta} \xi_S(x)dx}{\int_r^{\delta} \xi_S(x)dx} = t^{-\alpha}.$$

Recall that $\gamma(x) = \int_x^{\infty} \xi_S(z)dz \to \infty$ as $x \to 0$ by the representation theorem for regularly varying functions (see (3.6)). So for fixed $\delta > 0$,

$$
\begin{aligned}
\lim_{r\to 0} \frac{\int_{rt}^{\delta} \xi_S(x)dx}{\int_r^{\delta} \xi_S(x)dx} &= \lim_{r\to 0} \frac{\int_{rt}^{\infty} \xi_S(x)dx - \int_{\delta}^{\infty} \xi_S(x)dx}{\int_r^{\infty} \xi_S(x)dx - \int_{\delta}^{\infty} \xi_S(x)dx} \\
&= \lim_{r\to 0} \frac{\int_{rt}^{\infty} \xi_S(x)dx \left(1 - \frac{\int_{\delta}^{\infty} \xi_S(x)dx}{\int_{rt}^{\infty} \xi_S(x)dx}\right)}{\int_r^{\infty} \xi_S(x)dx \left(1 - \frac{\int_{\delta}^{\infty} \xi_S(x)dx}{\int_r^{\infty} \xi_S(x)dx}\right)} \\
&= t^{-\alpha}. && (4.7)
\end{aligned}
$$

Continuing, we let $0 < \varepsilon < 1$ and choose $\delta > 0$ such that

$$1 - \varepsilon \le e^x \le 1 + \varepsilon,$$

for all $x \in (-\delta, \delta)$. Now, recalling (4.6), we have

$$
\begin{aligned}
\frac{\nu^* \left(|x| > rt \right)}{\nu^* \left(|x| > r \right)} &= \frac{\int_{|x|>rt} e^x \xi(x) dx}{\int_{|x|>r} e^x \xi(x) dx} \\
&= \frac{\int_{rt}^\infty \left(e^x \xi(x) + e^{-x} \xi(-x) \right) dx}{\int_r^\infty \left(e^x \xi(x) + e^{-x} \xi(-x) \right) dx} \\
&= \frac{\int_{rt}^\delta \xi_S^*(x) dx + C_\delta^*}{\int_r^\delta \xi_S^*(x) dx + C_\delta^*},
\end{aligned}
\tag{4.8}
$$

where $C_\delta^* = \int_\delta^\infty \xi_S^*(x) dx < \infty$. For $0 < x < \delta$, define $\gamma_\delta^*(x) = \int_x^\delta \xi_S^*(x) dx$ and $\gamma_\delta(x) = \int_x^\delta \xi_S(x) dx$ as the truncated tail functions. Note that $\gamma_\delta(x) \to \infty$ as $x \to 0$ by the representation theorem for regularly varying functions. We estimate (4.8) as

$$
\begin{aligned}
\frac{\int_{rt}^\delta \xi_S^*(x) dx + C_\delta^*}{\int_r^\delta \xi_S^*(x) dx + C_\delta^*} &\le \frac{\int_{rt}^\delta e^x \xi_S(x) dx + C_\delta^*}{\int_r^\delta e^{-x} \xi_S(x) dx + C_\delta^*} \\
&\le \frac{(1+\varepsilon) \int_{rt}^\delta \xi_S(x) dx + C_\delta^*}{(1-\varepsilon) \int_r^\delta \xi_S(x) dx + C_\delta^*} \\
&= \frac{\gamma_\delta(rt)}{\gamma_\delta(r)} \left(\frac{1 + \varepsilon + \frac{C_\delta^*}{\gamma_\delta(rt)}}{1 - \varepsilon + \frac{C_\delta^*}{\gamma_\delta(r)}} \right),
\end{aligned}
\tag{4.9}
$$

and from (4.9) obtain

$$\limsup_{r \to 0} \frac{\nu^* \left(|x| > rt \right)}{\nu^* \left(|x| > r \right)} \le t^{-\alpha} \frac{1 + \varepsilon}{1 - \varepsilon}.$$

We estimate (4.8) from below

$$
\begin{aligned}
\frac{\int_{rt}^\delta \xi_S^*(x) dx + C_\delta^*}{\int_r^\delta \xi_S^*(x) dx + C_\delta^*} &\ge \frac{\int_{rt}^\delta e^{-x} \xi_S(x) dx + C_\delta^*}{\int_r^\delta e^x \xi_S(x) dx + C_\delta^*} \\
&\ge \frac{\gamma_\delta(rt)}{\gamma_\delta(r)} \left(\frac{1 - \varepsilon + \frac{C_\delta^*}{\gamma_\delta(rt)}}{1 + \varepsilon + \frac{C_\delta^*}{\gamma_\delta(r)}} \right),
\end{aligned}
\tag{4.10}
$$

and from (4.10) obtain

$$\liminf_{r \to 0} \frac{\nu^* \left(|x| > rt \right)}{\nu^* \left(|x| > r \right)} \ge t^{-\alpha} \frac{1 - \varepsilon}{1 + \varepsilon}.$$

Thus,

$$t^{-\alpha}\frac{1-\varepsilon}{1+\varepsilon} \le \liminf_{r\to 0}\frac{\nu^*\left(|x|>rt\right)}{\nu^*\left(|x|>r\right)} \le \limsup_{r\to 0}\frac{\nu^*\left(|x|>rt\right)}{\nu^*\left(|x|>r\right)} \le t^{-\alpha}\frac{1+\varepsilon}{1-\varepsilon},$$

and letting $\varepsilon \to 0$ gives the result.

Next, we need to show existence of the limits

$$\lim_{x\to 0}\frac{\gamma_\pm^*(x)}{\gamma^*(x)},$$

given the existence of the limits

$$\lim_{x\to 0}\frac{\gamma_\pm(x)}{\gamma(x)}.$$

Assume that $\lim_{x\to 0}\gamma_+(x)/\gamma(x) = p$ and $\lim_{x\to 0}\gamma_-(x)/\gamma(x) = q$. By an argument similar to the one developed in the beginning of this proof, the limits can also be computed via

$$\lim_{x\to 0}\frac{\nu\left(x<y<\delta\right)}{\nu\left(x<|y|<\delta\right)} = p,$$

and

$$\lim_{x\to 0}\frac{\nu\left(-\delta<y<-x\right)}{\nu\left(x<|y|<\delta\right)} = q,$$

where $\delta > 0$. We now show that the same limits hold for γ^*.

First, we treat the case where $0 < p < 1$ (note that $p + q = 1$). In this case, $0 < q < 1$, and so

$$\lim_{x\to 0}\gamma_+^*(x) = \lim_{x\to 0}\gamma_-^*(x) = \infty,$$

since $\lim_{x\to 0}\gamma^*(x) = \infty$. Again, let $\varepsilon > 0$ and choose $\delta > 0$ such that $1-\varepsilon \le e^x \le 1+\varepsilon$ for all $x \in (-\delta, \delta)$. Continuing

$$
\begin{aligned}
\frac{\nu^*\left(y>x\right)}{\nu^*\left(|y|>x\right)} &= \frac{\int_x^\infty \xi^*(y)dy}{\int_x^\infty \xi_S^*(y)dy}\\
&= \frac{\int_x^\infty e^y\xi(y)dy}{\int_x^\infty \xi_S^*(y)dy}\\
&= \frac{\int_x^\delta e^y\xi(y)dy + \int_\delta^\infty e^y\xi(y)dy}{\int_x^\delta \xi_S^*(y)dy + \int_\delta^\infty \xi_S^*(y)dy}\\
&= \frac{\int_x^\delta e^y\xi(y)dy + D_\delta^*}{\int_x^\delta \xi_S^*(y)dy + C_\delta^*},
\end{aligned}
\tag{4.11}
$$

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where $D_\delta^* = \int_\delta^\infty e^y \xi(y) dy$ and, again, $C_\delta^* = \int_\delta^\infty \xi_S^*(y) dy$ are constants depending only on δ. Estimating (4.11) above by

$$
\begin{aligned}
\frac{\int_x^\delta e^y \xi(y) dy + D_\delta^*}{\int_x^\delta \xi_S^*(y) dy + C_\delta^*} &\leq \frac{\int_x^\delta e^y \xi(y) dy + D_\delta^*}{\int_x^\delta e^{-y} \xi_S(y) dy + C_\delta^*} \\
&\leq \frac{(1+\varepsilon) \int_x^\delta \xi(y) dy + D_\delta^*}{(1-\varepsilon) \int_x^\delta \xi_S(y) dy + C_\delta^*} \\
&= \frac{\int_x^\delta \xi(y) dy}{\int_x^\delta \xi_S(y) dy} \left(\frac{1 + \varepsilon + \frac{D_\delta^*}{\int_x^\delta \xi(y) dy}}{1 - \varepsilon + \frac{C_\delta^*}{\int_x^\delta \xi_S(y) dy}} \right),
\end{aligned} \tag{4.12}
$$

which, taking the limsup, gives

$$
\limsup_{x \to 0} \frac{\nu^*(y > x)}{\nu^*(|y| > x)} \leq p \left(\frac{1+\varepsilon}{1-\varepsilon} \right),
$$

(note that $\lim_{x \to 0} \int_x^\delta \xi(y) dy = \lim_{x \to 0} \int_x^\delta \xi_S(y) dy = \infty$ also for every $\delta > x$). We estimate (4.11) from below as

$$
\begin{aligned}
\frac{\int_x^\delta e^y \xi(y) dy + D_\delta^*}{\int_x^\delta \xi_S^*(y) dy + C_\delta^*} &\geq \frac{\int_x^\delta e^y \xi(y) dy + D_\delta^*}{\int_x^\delta e^y \xi_S(y) dy + C_\delta^*} \\
&\geq \frac{\int_x^\delta \xi(y) dy + D_\delta^*}{(1+\varepsilon) \int_x^\delta \xi_S(y) dy + C_\delta^*} \\
&= \frac{\int_x^\delta \xi(y) dy}{\int_x^\delta \xi_S(y) dy} \left(\frac{1 + \frac{D_\delta^*}{\int_x^\delta \xi(y) dy}}{1 + \varepsilon + \frac{C_\delta^*}{\int_x^\delta \xi_S(y) dy}} \right),
\end{aligned} \tag{4.13}
$$

and taking the liminf gives

$$
\liminf_{x \to 0} \frac{\nu^*(y > x)}{\nu^*(|y| > x)} \geq p \left(\frac{1}{1+\varepsilon} \right).
$$

Combining these estimates, we obtain

$$
p \left(\frac{1}{1+\varepsilon} \right) \leq \liminf_{x \to 0} \frac{\nu^*(y > x)}{\nu^*(|y| > x)} \leq \limsup_{x \to 0} \frac{\nu^*(y > x)}{\nu^*(|y| > x)} \leq p \left(\frac{1+\varepsilon}{1-\varepsilon} \right),
$$

and letting $\varepsilon \to 0$ gives the first limit. An identical argument shows that

$$
\lim_{x \to 0} \frac{\nu^*(y < -x)}{\nu^*(|y| > x)} = q.
$$

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We now deal with the remaining cases, i.e. $p = 0$ and $p = 1$, and assume without loss of generality that $p = 0$. This implies that

$$\lim_{x \to 0} \frac{\gamma_+(x)}{\gamma(x)} = 0,$$

and

$$\lim_{x \to 0} \frac{\gamma_-(x)}{\gamma(x)} = 1,$$

which in turn implies that $\lim_{x \to 0} \gamma_-(x) = \infty$. There are two distinct possibilites for γ_+, either

$$\lim_{x \to 0} \gamma_+(x) < \infty, \tag{4.14}$$

or

$$\lim_{x \to 0} \gamma_+(x) = \infty. \tag{4.15}$$

If (4.15) holds true, then both tails have infinite mass and a proof similar to the one for $0 < p < 1$ gives the result. In the case (4.14),

$$\lim_{x \to 0} \frac{\gamma_+^*(x)}{\gamma^*(x)} = 0,$$

since $\gamma^*(x) \to \infty$ as $x \to 0$. Indeed, we estimate

$$\begin{aligned}
\gamma^*(x) &= \int_x^\infty \xi_S^*(y) dy \\
&= \int_x^\delta \xi_S^*(y) dy + C_\delta^* \\
&\geq \int_x^\delta e^{-y} \xi_S(y) dy + C_\delta^* \\
&\geq (1 - \varepsilon) \int_x^\delta \xi_S(y) dy + C_\delta^* \to \infty,
\end{aligned}$$

as $x \to 0$. Thus,

$$\lim_{x \to 0} \frac{\gamma_-^*(x)}{\gamma^*(x)} = \lim_{x \to 0} \frac{\gamma^*(x) - \gamma_+^*(x)}{\gamma^*(x)} = 1.$$

$\square$

As a major consequence of the previous result, the constants p and q in the limit remain unchanged under the measure transform. This fact, combined with Lemma 3.24, give the following result.

Lemma 4.2. *Let* (3.13) *hold with respect to* $\mathbb{P}$ *and hence with respect to* $\mathbb{P}^*$, *where* Z *is an* α*-stable random variable. Then* Z *has the same representation under both* $\mathbb{P}$ *and* $\mathbb{P}^*$. *That is, the parameters of the stable distribution* Z *are the same under both probability measures* $\mathbb{P}$ *and* $\mathbb{P}^*$.

Next, we show that the finiteness of the constant $\bar{\mu}$ is also a property that survives the share measure transformation. This quantity will be important for Theorem 4.5, which is one of the main new results of this thesis.

Lemma 4.3. *The quantity* $\bar{\mu} < \infty$ *if and only if* $\bar{\mu}^* < \infty$.

Proof. First, we assume $\bar{\mu} < \infty$. Observe that $\sup_{0<\eta\leq 1} |\mu_\eta| < \infty$ implies that

$$\sup \mu_\eta < \infty \quad \text{and} \quad \inf \mu_\eta > -\infty.$$

Thus,

$$-\infty < \inf_{0<\eta\leq 1} \left(b - \int_{\eta<|y|\leq 1} y\nu(dy) \right)$$

$$= b + \inf_{0<\eta\leq 1} \left(-\int_{\eta<|y|\leq 1} y\nu(dy) \right)$$

$$= b - \sup_{0<\eta\leq 1} \int_{\eta<|y|\leq 1} y\nu(dy),$$

which implies $\sup_{0<\eta\leq 1} \int_{\eta<|y|\leq 1} y\nu(dy) < \infty$. A similar argument implies that

$$\inf_{0<\eta\leq 1} \int_{\eta<|y|\leq 1} y\nu(dy) > -\infty,$$

and so we know

$$\sup_{0<\eta\leq 1} \left| \int_{\eta<|y|\leq 1} y\nu(dy) \right| < \infty.$$

For fixed $0 < \eta \leq 1$,

$$
\begin{aligned}
\left|\mu_\eta^*\right| &\leq |b| + \left|\int_{\eta < |y| \leq 1} y\nu^*(dy)\right| \\
&= |b| + \left|\int_{\eta < |y| \leq 1} ye^y \nu(dy)\right| \\
&\leq |b| + e\left|\int_{\eta < |y| \leq 1} y\nu(dy)\right| \\
&\leq |b| + e \sup_{0 < \eta \leq 1} \left|\int_{\eta < |y| \leq 1} y\nu(dy)\right| < \infty.
\end{aligned}
$$

Taking the supremum gives the implication.

The converse can be proven by noting the inequality

$$
\left|\int_{\eta < |y| \leq 1} ye^y \nu(dy)\right| \geq \frac{1}{e}\left|\int_{\eta < |y| \leq 1} y\nu(dy)\right|,
$$

and taking the supremum since the left-hand side is bounded when $\eta \to 0$ by our assumption. $\qquad\square$

4.3   Properties of the Rate of Convergence

Previous results on ATM call prices (e.g. [22], [52], and [44]) considered only the case $B_t = \kappa t^{1/\alpha}$ whenever $1 < \alpha < 2$, where $\kappa > 0$ is a constant. The results of this section show more general asymptotics for ATM call option prices.

In [21], [30], and [43], the authors show that the rate function $B \in RV_{-1/\alpha}^0$ whenever the convergence is towards an α-stable random variable. In this section, we aim to further restrict what kind of behavior B can exhibit when (3.13) is satisfied. Throughout the section, we use the notation $\beta_t := 1/B_t$ for convenience.

Assume that (3.13) holds for the Lévy process $(X_t)_{t \geq 0}$ under the measure $\mathbb{P}$ (and hence also under $\mathbb{P}^*$). Thus, the Lévy measures ν and ν^* are regularly varying with index $\alpha > 0$ at 0, and we further assume that $\alpha \in (1, 2)$. Since γ^* is regularly varying at 0 of order $-\alpha$, it has representation

$$
\gamma^*(x) = x^{-\alpha}\ell\left(\frac{1}{x}\right), \tag{4.16}
$$

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for all $x > 0$, where ℓ is slowly varying at ∞ (equivalently, we can take $\gamma^*(x) = x^{-\alpha}\psi(x)$ where ψ is slowly varying at 0). We deal with γ^* rather than γ as most of our calculations will be done with the quantities under the measure $\mathbb{P}^*$.

The representation for γ^* is derived in the following way. First, note that $\gamma^*(\cdot) \in RV_{-\alpha}^0$ if and only if $\gamma\left(\frac{1}{\cdot}\right) \in RV_\alpha^\infty$, and then the representation theorem for regularly varying functions gives (4.16).

Some observations about (4.16) are in order. First, ℓ has asymptotically controlled behavior near ∞ due to its slow variation; however, we cannot specify any properties of ℓ near 0. There are really only two properties governing the behavior of ℓ near 0. First, γ^* is nonincreasing (this follows naturally from its definition). Second, there is some control exerted on γ^* from its relation to ν^* and the requirement that $\int_{-1}^1 x^2 \nu^*(dx) < \infty$.

We can simplify how we look at (3.13) because we do not need the additive correction term. In [42], the authors show that A_t can be taken to be $O(t)$ (recall that $\alpha \in (1,2)$). So, under $\mathbb{P}^*$

$$\beta_t(X_t - A_t) \Rightarrow Z,$$

as $t \to 0$. Recall that $\beta_t \in RV_{1/\alpha}^0$ so that, again by the representation theorem,

$$\beta_t = t^{-1/\alpha}\zeta(1/t),$$

as $t \to 0$, where ζ is slowly varying at ∞. Also, for some absolute constant $C > 0$ we have

$$|A_t\beta_t| \le Ct\beta_t$$
$$= Ct^{1-1/\alpha}\zeta(1/t).$$

The function $s^{1/\alpha-1}\zeta(s)$ is regularly varying at ∞ with index $1/\alpha - 1 < 0$. Standard results (e.g. Proposition 1.3.6(v) in [4]) imply that $s^{1/\alpha-1}\zeta(s) \to 0$ as $s \to \infty$, which

shows that $A_t\beta_t \to 0$ as $t \to 0$. So, we need only look at the convergence $\beta_t X_t \Rightarrow Z$, as $t \to 0$ under $\mathbb{P}^*$.

The next theorem states the asymptotic behavior of β_t as $t \to 0$ under a small technical condition.

Theorem 4.4. *Let there exist $x_0 > 0$ such that $x^2 \xi_S^*(x)$ is monotone (increasing or decreasing) for $0 < x < x_0$. Then the function β satisfies*

$$\lim_{t \to 0} t\beta_t^\alpha \ell(\beta_t) = \Lambda,$$

where Λ is a positive numerical constant.

Proof. By the previous argument, we can ignore the $\beta_t A_t$ term and know

$$\beta_t X_t \Rightarrow Z, \tag{4.17}$$

as $t \to 0$, under $\mathbb{P}^*$ where Z is an α-stable random variable. Recall that [30] and [4] both give the representation

$$\varphi_Z(u) = \exp\left(-c_\alpha |u|^\alpha (1 - i(p_+ - p_-)\operatorname{sgn}(u)\tan(\pi\alpha/2))\right), \tag{4.18}$$

where $p_\pm$ are defined in Lemma 3.24. We need further information concerning the slowly varying part of γ^*. So, we examine the characteristic functions of both $\beta_t X_t$ and Z, which we know must be equal when $t \to 0$ by (4.17).

First, we will need to determine the behavior of $\xi_S^*(x)$. To this end, we will use the Monotone Density Theorem (Theorem 3.11). We know $\gamma^*(x) = \int_x^\infty \xi_S^*(x)dx = x^{-\alpha}\ell(1/x)$, for $x > 0$. Hence,

$$\begin{aligned}
x^{-\alpha}\ell(x) &= \gamma^*(1/x) \\
&= \int_{1/x}^\infty \xi_S^*(y)dy \\
&= -\int_x^0 \xi_S^*(1/u)\frac{du}{u^2} \\
&= \int_0^x \xi_S^*(1/u)\frac{du}{u^2} \\
&= \int_0^x s(u)du, \tag{4.19}
\end{aligned}$$

where $s(u) := \xi_S^*(1/u)\, u^{-2}$. Note that $x^2 \xi_S^*(x) = s(1/x)$ and $x^2 \xi_S^*$ is monotone for x close enough to 0 (i.e. $s(u)$ is monotone for u large enough). Now, we use the Monotone Density Theorem to get that $s(x) \sim \alpha x^{\alpha-1} \ell(x)$, as $x \to \infty$. That is, for y close to 0, we have $s(1/y) = y^2 \xi_S^*(y) \sim \alpha y^{-\alpha+1} \ell(1/y)$ which implies $\xi_S^*(y) \sim \alpha y^{-\alpha-1} \ell(1/y)$ for y positive and near 0. Now, the exponent of the characteristic function of $\beta_t X_t$ is given by

$$\log\left(\mathbb{E}^* e^{iu\beta_t X_t}\right) = t \int_{-\infty}^{\infty} \left(\exp\left(iu\beta_t y\right) - 1 - iu\beta_t y\right) \xi^*(y) dy. \tag{4.20}$$

Fix any $0 < \varepsilon < 1$ and let $w_0(\varepsilon) > 0$ be such that

$$(1 - \varepsilon)\alpha x^{-\alpha-1}\ell(1/x) \le \xi_S^*(x) \le (1 + \varepsilon)\alpha x^{-\alpha-1}\ell(1/x), \tag{4.21}$$

for all $0 < x < w_0$. The real part of (4.20) converges to $-c_\alpha |u|^\alpha$ where $c_\alpha > 0$ as $t \to 0$ (again see [30], [4], and [42]). First, we need to rewrite (4.20) in a nicer form. Let $g(u,y) = \exp\left(iu\beta_t y\right) - 1 - iu\beta_t y$ and rewrite

$$\begin{aligned}
\log\left(\mathbb{E}^* e^{iu\beta_t X_t}\right) &= t \int_{-\infty}^{\infty} g(u,y)\xi^*(y)dy \\
&= t \int_{0}^{\infty} \left(g(u,y)\xi^*(y) + \overline{g(u,y)}\xi^*(-y)\right) dy.
\end{aligned} \tag{4.22}$$

Note that the real part of g is $\Re(g(u,y)) = \cos\left(u\beta_t y\right) - 1$, and the real part of (4.22) is

$$\begin{aligned}
\Re\left(\log\left(\mathbb{E}^* e^{iu\beta_t X_t}\right)\right) &= t \int_{0}^{\infty} \Re g(u,y)\left(\xi^*(y) + \xi^*(-y)\right) dy \\
&= t \int_{0}^{\infty} \left(\cos\left(u\beta_t y\right) - 1\right)\xi_S^*(y)dy \\
&= \frac{t}{|u|\,\beta_t} \int_{0}^{\infty} \left(\cos\left(\mathrm{sgn}\left(u\right)w\right) - 1\right)\xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) dw \\
&= \frac{t}{|u|\,\beta_t} \int_{0}^{\infty} \left(\cos\left(w\right) - 1\right)\xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) dw.
\end{aligned} \tag{4.23}$$

Continuing, we break (4.23) into two parts by writing for $L > 0$

$$\Re\left(\log\left(\mathbb{E}^* e^{iu\beta_t X_t}\right)\right) = \frac{t}{|u|\,\beta_t} \int_0^\infty \left(\cos\left(w\right) - 1\right) \xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) dw$$

$$= \frac{t}{|u|\,\beta_t} \int_0^\infty \left(\cos\left(w\right) - 1\right) \xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) \mathbb{1}_{\{\frac{w}{|u|\beta_t} \leq L\}} dw \qquad (4.24)$$

$$+ \frac{t}{|u|\,\beta_t} \int_0^\infty \left(\cos\left(w\right) - 1\right) \xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) \mathbb{1}_{\{\frac{w}{|u|\beta_t} > L\}} dw. \qquad (4.25)$$

It is easy to see that (4.25) goes to 0, as $t \to 0$, since

$$\frac{t}{|u|\,\beta_t} \left| \int_0^\infty \left(\cos\left(w\right) - 1\right) \xi_S^*\left(\frac{w}{|u|\,\beta_t}\right) \mathbb{1}_{\{\frac{w}{|u|\beta_t} > L\}} dw \right|$$

$$= t \left| \int_0^\infty \left(\cos\left(u\beta_t z\right) - 1\right) \xi_S^*(z) \mathbb{1}_{\{z \geq L\}} dz \right|$$

$$\leq 2t \int_L^\infty \xi_S^*(z) dz.$$

Now, we estimate (4.24) to get the desired result. First, we show a preliminary result of use later. Namely, we show that there exists $M > 0$ with $1/M \leq w_0$ such that

$$\int_0^\infty \frac{\left(\cos\left(w\right) - 1\right)}{w^{1+\alpha}} \frac{\ell\left(\frac{|u|\beta_t}{w}\right)}{\ell\left(\beta_t\right)} \mathbb{1}_{\{\frac{w}{|u|\beta_t} \leq \frac{1}{M}\}} dw \to \int_0^\infty \frac{\left(\cos\left(w\right) - 1\right)}{w^{1+\alpha}} dw < \infty, \qquad (4.26)$$

as $t \to 0$. Recall that ℓ slowly varying implies that $\ell(\lambda x)/\ell(x) \to 1$, for any $\lambda > 0$, as $x \to \infty$. Letting $x = \beta_t$ and $\lambda = |u|/w$ implies that

$$\frac{\ell\left(\frac{|u|\beta_t}{w}\right)}{\ell(\beta_t)} \to 1,$$

as $t \to 0$. We recall the Potter bounds from Theorem 3.8 for ℓ, that is for any $A > 1$ and $\delta > 0$ there exists $M > 0$ such that for $x, y \geq M$,

$$\frac{\ell(y)}{\ell(x)} \leq A\left(\left(\frac{y}{x}\right)^\delta \vee \left(\frac{y}{x}\right)^{-\delta}\right).$$

Choose $A = 2$ and $\delta > 0$ such that $\alpha + 1 \pm \delta \in (2, 3)$ and let $M_0 > 0$ be the M in the

above statement for the given A and δ. Note if $w \leq |u|\,\beta_t w_0$ and $1 \geq w_0 M_0$ then

$$
(\cos w - 1)w^{-\alpha-1}\frac{\ell\left(\frac{|u|\,\beta_t}{w}\right)}{\ell(\beta_t)}\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq w_0\right\}}
$$

$$
\leq (\cos w - 1)w^{-\alpha-1}A\max\left\{\left(\frac{|u|}{w}\right)^{\delta},\left(\frac{|u|}{w}\right)^{-\delta}\right\}.
$$

$$
\leq A(\cos w - 1)\left(|u|^{\delta}\,w^{-\alpha-1-\delta} + |u|^{-\delta}\,w^{-\alpha-1+\delta}\right). \tag{4.27}
$$

In (4.27) the first term is integrable on $[0,\infty)$ since $\alpha+1+\delta < 3$, and so is the second term since $\alpha+1-\delta > 2$. Applying Lebesgue's Dominated Convergence Theorem gives (4.26). If $w_0 M_0 > 1$, then we apply similar arguments on the set $\{|u|\,\beta_t \geq wM_0\}$. In either case, there exists $M > 0$ such that $1/M \leq w_0$ and (4.26) holds. Equation (4.25) converging to 0 as $t \to 0$ implies (as stated before) that (4.24) converges to $-c\,|u|^{\alpha}$ as $t \to 0$. If we let $L = \min\left(\frac{1}{M_0}, w_0\right)$, then we are in a position to analyze the conditions under which (4.24) converges to $-c_{\alpha}\,|u|^{\alpha}$. Using (4.21) we obtain

$$
\frac{t}{|u|\,\beta_t}\int_0^{\infty}(\cos w - 1)\,\xi_S^*\left(\frac{w}{|u|\,\beta_t}\right)\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw
$$

$$
\leq \frac{t(1+\varepsilon)}{|u|\,\beta_t}\int_0^{\infty}(\cos w - 1)\left(\frac{w}{|u|\,\beta_t}\right)^{-\alpha-1}\ell\left(\frac{|u|\,\beta_t}{w}\right)\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw
$$

$$
= (1+\varepsilon)\,|u|^{\alpha}\,t\beta_t^{\alpha}\int_0^{\infty}\frac{(\cos w - 1)}{w^{\alpha+1}}\ell\left(\frac{|u|\,\beta_t}{w}\right)\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw
$$

$$
= (1+\varepsilon)\,|u|^{\alpha}\,t\beta_t^{\alpha}\ell(\beta_t)\int_0^{\infty}\frac{(\cos w - 1)}{w^{\alpha+1}}\frac{\ell\left(\frac{|u|\beta_t}{w}\right)}{\ell(\beta_t)}\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw. \tag{4.28}
$$

Similarly, we obtain the lower bound

$$
\frac{t}{|u|\,\beta_t}\int_0^{\infty}(\cos w - 1)\,\xi^*\left(\frac{w}{|u|\,\beta_t}\right)\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw
$$

$$
\geq (1-\varepsilon)\,|u|^{\alpha}\,t\beta_t^{\alpha}\ell(\beta_t)\int_0^{\infty}\frac{(\cos w - 1)}{w^{\alpha+1}}\frac{\ell\left(\frac{|u|\beta_t}{w}\right)}{\ell(\beta_t)}\mathbb{1}_{\left\{\frac{w}{|u|\beta_t}\leq L\right\}}dw. \tag{4.29}
$$

Dividing each side of both (4.28) and (4.29) by $t\beta_t^{\alpha}\ell(\beta_t)$, letting $-\varsigma = \int_0^{\infty}(\cos(w) - 1)w^{-\alpha-1}dw$, and letting $t \to 0$ gives

$$
-(1+\varepsilon)\varsigma\,|u|^{\alpha} \leq \frac{-c_{\alpha}\,|u|^{\alpha}}{\lim_{t\to 0}t\beta_t^{\alpha}\ell(\beta_t)} \leq -(1-\varepsilon)\varsigma\,|u|^{\alpha}.
$$

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Letting $\varepsilon \to 0$ implies that

$$\lim_{t \to 0} t \beta_t^\alpha \ell(\beta_t) = \Lambda \in (0, \infty), \tag{4.30}$$

where $\Lambda = c_\alpha/\varsigma$ (note that $\varsigma > 0$, for $1 < \alpha < 2$). In fact, e.g. see [50],

$$\varsigma = \frac{\pi}{2\Gamma(1+\alpha)\sin\left(\frac{\pi\alpha}{2}\right)},$$

where Γ here is Euler's Gamma function. □

4.4 First Order European Call Price and Implied Volatility Asymptotics

We now move on to show our main first order results. We keep all of the assumptions made to date in this chapter, keeping also the constants from the previous argument (e.g. M_0 and w_0).

More specifically, we make the following assumptions:

(A1) ν is regularly varying of order $-\alpha$ at 0 with $\alpha \in (1, 2)$.

(A2) There exist $C > 0$ and $x_1 > 0$ such that $\int_{|y| \leq x} y^2 e^y \xi(y) \, dy \leq Cx^2 \int_{|y| > x} e^y \xi(y) \, dy$ for all $x \geq x_1$.

(A3) There exists x_0 such that $x^2 \xi_S(x)$ is monotone (either increasing or decreasing) for $0 < x < x_0$.

Note that all of these assumptions are requirements on the Lévy measure ν and not on the transformed Lévy measure ν^* (although we can state them in terms of ν^* as well).

Although the assumptions seem strict and technical, we discuss their purpose and meaning to show that they are not overly cumbersome. The most important and indispensable assumption by far is (A1). Under this assumption, the Lévy process, when properly scaled by some B_t, converges to a stable random variable. This assumption is essential; however, assumptions (A2) and (A3) are less so.

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Assumption (A2) guarantees that we can apply the concentration inequality from Chapter 3. We already saw that (A2) holds for Lévy processes in the DOA of a stable random variable, when x is small. Therefore, we need only show that the inequality also holds as $x \to \infty$. It might be that assumption (A2) is always satisfied for Lévy processes that satisfy assumption (A1).

Assumption (A3) seems to be the least important. It is used to obtain the asymptotic form of the Lévy measure close to 0 given the expression for γ^* (e.g., see (4.19)). We believe that this restriction is unnecessary since processes in the DOA of stable random variables are, in some sense, very close to stable random variables near $t = 0$.

4.4.1 Without Brownian Component

Our first result shows that the function $\beta_t = 1/B_t$ determines the first order asymptotics of call option prices in the absence of a Brownian component (i.e. $(X_t)_{t \geq 0}$ has characteristic triplet $(b, 0, \nu)$). Recall, that the asset-price dynamics are defined by the process $S = (S_t)_{t \geq 0}$ where

$$S_t = S_0 e^{X_t},$$

for each $t \geq 0$. We have the following.

Theorem 4.5. *Along with the conditions (A1)–(A3), assume that*

(i) $\bar{\mu} < \infty$,

(ii) and there exists $R_0 > 0$ such that

$$\int_{|y| > R_0} \gamma^* (y) \, dy < \infty.$$

Then an ATM European call option has asymptotic expansion

$$\mathbb{E} \left(S_t - S_0 \right)_+ = \left(S_0 \mathbb{E}^* Z_+ \right) B_t + o \left(B_t \right), \tag{4.31}$$

as $t \to 0$, where Z is the α-stable random variable from (4.17).

One quick remark is in order. Note that if ℓ is also slowly varying at 0, then conditions $(A2)$ and (ii) are automatically satisfied. In this case, the function γ^* is also regularly varying at ∞ and so the representation results (3.6) and Karamata's theorem (Theorem 3.9) show (ii) (alternatively, we could use a Potter's bound argument to obtain that the integral is finite for all $x > 0$).

Proof. Recalling (2.11) and $\beta_t = 1/B_t$ and using M_0 from the Potter bound argument in Theorem 4.4, we obtain

$$
\begin{aligned}
\frac{c(t,0)}{B_t} &= \frac{1}{B_t} \int_0^\infty e^{-x} \mathbb{P}^* \left(X_t \geq x \right) dx \\
&= \int_0^\infty e^{-B_t u} \mathbb{P}^* \left(X_t \geq B_t u \right) du \\
&= \int_0^\infty e^{-B_t u} \mathbb{P}^* \left(X_t \geq B_t u \right) \left(\mathbb{1}_{\left\{ \frac{1}{M_0} \geq \frac{B_t u}{4} \geq t\bar{\mu}^* \right\}} + \mathbb{1}_{\left\{ \frac{B_t u}{4} > \frac{1}{M_0} \right\}} + \mathbb{1}_{\left\{ \frac{B_t u}{4} < t\bar{\mu}^* \right\}} \right) du
\end{aligned}
$$

$$(4.32)$$

$$
:= \int_0^\infty \left(A_1(t,u) + A_2(t,u) + A_3(t,u) \right) du, \tag{4.33}
$$

where t is so small that $t\bar{\mu}^* < 1/M_0$. In what follows, we will write $A_i(t) := A_i(t,u)$ for $i = 1, 2, 3$.

First, we note that the integral of $A_3(t)$ can be estimated as

$$
\int_0^\infty e^{-B_t u} \mathbb{P}^* \left(X_t \geq B_t u \right) \mathbb{1}_{\left\{ \frac{B_t u}{4} < t\bar{\mu}^* \right\}} du \leq \int_0^\infty \mathbb{1}_{\left\{ u < 4\bar{\mu}^* t \beta_t \right\}} du
$$

$$
= 4\bar{\mu}^* t\beta_t \to 0, \tag{4.34}
$$

as $t \to 0$ since $t\beta_t^\alpha \ell(\beta_t) \sim \Lambda$. Therefore, we only need to deal with the integral of $A_1(t)$ and $A_2(t)$. Using $\{B_t u > 4t\bar{\mu}^*\} \subseteq \{B_t u > 4t\mu_{B_t u/4}\}$ and the estimate from Lemma 3.27, for some constant $C > 0$ and for any

$$
u \in \mathcal{I} \subset \{B_t u > 4t\bar{\mu}^*\} \tag{4.35}
$$

with $\mathcal{I}$ measurable and $t > 0$ fixed,

$$
\begin{aligned}
\mathbb{P}^* \left(X_t \geq B_t u \right) &\leq \left[\left(1 + Ce^2 \right) t \gamma^* \left(\frac{B_t u}{4} \right) \vee 1 \right] \\
&= \left[\left(1 + Ce^2 \right) t \gamma^* \left(\frac{u}{4\beta_t} \right) \vee 1 \right] \\
&\leq \left[\left(\left(1 + Ce^2 \right) t \left(\frac{u}{4\beta_t} \right)^{-\alpha} \ell \left(\frac{4\beta_t}{u} \right) \right) \vee 1 \right] \\
&= \left[\kappa t \beta_t^\alpha u^{-\alpha} \ell \left(\frac{4\beta_t}{u} \right) \vee 1 \right],
\end{aligned}
\tag{4.36}
$$

where $\kappa > 0$ is a collection of all the constants. In what follows, we use κ to represent a positive constant whose value might change from line to line.

Recall from our Potter bound argument in Theorem 4.4 that $\alpha \pm \delta \in (1, 2)$. We also choose $t_0 > 0$ such that for all $0 < t < t_0$,

$$
\frac{\Lambda}{2\ell(\beta_t)} < t\beta_t^\alpha < \frac{3\Lambda}{2\ell(\beta_t)}.
$$

Continuing (4.36) for $0 < t < t_0$ and $u \in \mathcal{I}$,

$$
\mathbb{P}^* \left(X_t \geq B_t u \right) \leq \kappa u^{-\alpha} \frac{\ell \left(\frac{4\beta_t}{u} \right)}{\ell(\beta_t)} \vee 1.
$$

First, we show that the integral of $A_2(t)$ goes to 0 as $t \to 0$. Indeed, choosing $\mathcal{I} = \{4\beta_t < M_0 u\}$ in (4.35) and changing variables gives

$$
\begin{aligned}
\int_0^\infty A_2(t) du &\leq \int_0^\infty \left[\kappa u^{-\alpha} \frac{\ell \left(\frac{4\beta_t}{u} \right)}{\ell(\beta_t)} \vee 1 \right] \mathbb{1}_{\left\{ \frac{4\beta_t}{u} < M_0 \right\}} du \\
&\leq \frac{1}{4\beta_t} \int_0^\infty \kappa u^{2-\alpha} \frac{\ell \left(\frac{4\beta_t}{u} \right)}{\ell(\beta_t)} \mathbb{1}_{\left\{ \frac{4\beta_t}{u} < M_0 \right\}} 4\beta_t \frac{du}{u^2} \\
&= \frac{1}{4\beta_t} \int_0^{M_0} \kappa \left(\frac{4\beta_t}{w} \right)^{2-\alpha} \frac{\ell(w)}{\ell(\beta_t)} dw \\
&= \frac{\kappa}{\beta_t^{\alpha-1} \ell(\beta_t)} \int_0^{M_0} \frac{\ell(w)}{w^{2-\alpha}} dw \\
&= \frac{\kappa}{\beta_t^{\alpha-1} \ell(\beta_t)} \int_{1/M_0}^\infty \frac{\ell \left(\frac{1}{z} \right)}{z^\alpha} dz \\
&= \frac{\kappa}{\beta_t^{\alpha-1} \ell(\beta_t)} \int_{1/M_0}^\infty \gamma^*(z) dz \to 0,
\end{aligned}
\tag{4.37}
$$

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as $t \to 0$, since the integral is finite and $t\beta_t^\alpha \ell(\beta_t) \sim \Lambda$ as $t \to 0$. In order to estimate $A_1(t)$, we use (4.36) and Potter bounds from Theorem 4.4. Choosing $\mathcal{I} = \left\{ M_0 \le \frac{4\beta_t}{u} < \frac{1}{t\bar{\mu}^*} \right\}$ and for $u \in \mathcal{I}$ and any $t > 0$,

$$A_1(t, u) \le \left[\kappa u^{-\alpha} \frac{\ell\left(\frac{4\beta_t}{u}\right)}{\ell(\beta_t)} \vee 1 \right] \mathbb{1}_{\{M_0 \le \frac{4\beta_t}{u} < \frac{1}{t\bar{\mu}^*}\}}$$

$$\le \max\left\{ \kappa 4^\delta u^{-\alpha-\delta} \vee 1, \kappa 4^{-\delta} u^{-\alpha+\delta} \vee 1 \right\} \in L^1([0, \infty)), \qquad (4.38)$$

so that we are able to apply Lebesgue's Dominated Convergence theorem to

$$\int_0^\infty A_2(t, u)\, du = \int_0^\infty e^{-B_t u} \mathbb{P}^*(X_t \ge B_t u)\, \mathbb{1}_{\{\frac{1}{M_0} \ge \frac{B_t u}{4} > t\bar{\mu}^*\}}\, du.$$

Combining this fact with (4.37) and (4.38) gives

$$\lim_{t \to 0} \frac{c(t, 0)}{B_t} = \lim_{t \to 0} \int_0^\infty A_2(t, u)\, du$$

$$= \int_0^\infty \mathbb{P}^*(Z \ge u)\, du$$

$$= \mathbb{E}^* Z_+,$$

which can be rewritten as

$$\lim_{t \to 0} \mathbb{E}(S_t - S_0)_+ = S_0 B_t \mathbb{E}^* Z_+ + o(B_t),$$

proving the theorem. □

We next obtain ATM implied-volatility asymptotics close to expiration. In fact, we show that the implied volatility for ATM options, as approaching expiration, collapses at the rate $B_t/\sqrt{t}$.

Corollary 4.6. *Under the assumptions of Theorem 4.5, $\hat{\sigma}$, the implied volatility of an ATM call option is such that*

$$\hat{\sigma}(t) = \sqrt{2\pi} \frac{B_t}{\sqrt{t}} \mathbb{E}^* Z_+ + o\left(\frac{B_t}{\sqrt{t}}\right), \qquad (4.39)$$

as $t \to 0$.

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Proof. We proceed as in the proof of Proposition 3.7 in [23]. We know that the Black-Scholes call price asymptotics are $S_0 c_{BS}$, where

$$c_{BS}(t, \sigma) = \frac{\sigma}{\sqrt{2\pi}}\sqrt{t} + o\left(\sqrt{t}\right), \tag{4.40}$$

as $t \to 0$, since $c_{BS} = N\left(\sigma\sqrt{t}\right)$ where

$$N(\theta) := \int_0^\theta \Phi'\left(\frac{u}{2}\right) du = \frac{1}{\sqrt{2\pi}}\int_0^\theta \exp\left(-\frac{u^2}{8}\right) du,$$

where Φ is the standard normal cumulative distribution function, and where N has asymptotic behavior

$$N(\theta) = \frac{1}{\sqrt{2\pi}}\theta + o(\theta),$$

as $\theta \to 0$. We need an expression similar to (4.40) where the constant σ is replaced by the implied volatility function $\hat{\sigma}(t)$. Now, $\hat{\sigma}(t) \to 0$ as $t \to 0$, so a substitution in c_{BS} gives

$$\begin{aligned}
c_{BS}(t, \hat{\sigma}(t)) &= \frac{\hat{\sigma}(t)}{\sqrt{2\pi}}\sqrt{t} + o\left(\hat{\sigma}(t)\sqrt{t}\right) \\
&= \frac{\hat{\sigma}(t)}{\sqrt{2\pi}}\sqrt{t} + o\left(\sqrt{t}\right), \tag{4.41}
\end{aligned}$$

as $t \to 0$, since $\hat{\sigma}(t) = o(1)$ as $t \to 0$. Equating (4.41) with (4.31), leads to

$$\frac{\hat{\sigma}(t)}{\sqrt{2\pi}}\sqrt{t} \sim B_t \mathbb{E}^* Z_+,$$

as $t \to 0$, i.e.

$$\hat{\sigma}(t) \sim \sqrt{2\pi}\frac{B_t}{\sqrt{t}}\mathbb{E}^* Z_+,$$

as $t \to 0$, giving the result. $\qquad\square$

4.4.2 With Brownian Component

We now add an independent Brownian component to the Lévy process. In this case, a new proof technique significantly reduces the complexity of deriving the first order dynamics of more general Lévy processes with Brownian component. We find, as in previous results (e.g. [22], [44], [23], [24]), that the order of convergence is $\sqrt{t}$.

Define a Lévy process $X = (X_t)_{t \geq 0}$ for every $t \geq 0$ via $X_t = \sigma W_t + L_t$, where $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, $\sigma > 0$, and $L = (L_t)_{t \geq 0}$ is the pure-jump process with Lévy triplet $(b, 0, \nu)$, independent of W, where ν is the same as in the non-Brownian case; however, now our drift term b must satisfy

$$b + \frac{\sigma^2}{2} + \int_{\mathbb{R}} \left(e^x - 1 - x \mathbb{1}_{\{|x| \leq 1\}} \right) \nu \, (dx) = 0,$$

so that $\left(e^{X_t} \right)_{t \geq 0}$ still satisfies the martingale condition. We also need a different definition for μ_ε to account for the Brownian component, namely

$$\mu_\varepsilon = b + \frac{\sigma^2}{2} - \int_{|x| \leq 1} z \mathbb{1}_{\{|x| > \varepsilon\}} \nu \, (dx) + \int_{|x| \geq 1} z \mathbb{1}_{\{|x| \leq \varepsilon\}} \nu \, (dx).$$

As in [50] and using the martingale condition $\mathbb{E} e^{X_t} = 1$, we define a probability measure $\mathbb{P}^*$ such that $\mathbb{P}^* (B) = \mathbb{E} e^{X_t} \mathbb{1}_B$ for every Borel set B. Under this probability measure, the process $X = (X_t)_{t \geq 0}$ is again a Lévy process with triplet (b^*, σ^*, ν^*) where

$$b^* = b + \int_{|x| \leq 1} x \left(e^x - 1 \right) \nu(dx) + \sigma^2,$$

$\sigma^* = \sigma$, and $\nu^*(dx) = e^x \nu \, (dx)$. The processes L and W are still independent under $\mathbb{P}^*$.

In order to prove our result, we prove a basic convergence theorem.

Lemma 4.7. *Let (S, Σ, μ) be a measure space. Let $f, g : S \times [0, \infty) \to [0, \infty)$ and $h : S \to [0, \infty)$ be measurable and such that $f(\cdot, t), g(\cdot, t), h \in L^1(S)$ for almost every $t \geq 0$. Also, suppose*

(C1) $f(s, t) \to \bar{f}(s) \in L^1(S)$ as $t \to 0$,

(C2) $f(s, t) \leq h(s) + g(s, t)$ for μ-a.e. $s \in S$ and almost every $t \geq 0$,

(C3) $g(s, t) \to 0$ as $t \to 0$ μ-a.e $s \in S$,

(C4) $\int_S g(s, t) \mu(ds) \to 0$ as $t \to 0$.

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Then

$$\lim_{t \to 0} \int_S f(s,t)\mu(ds) = \int_S \bar{f}(s)\mu(ds).$$

Proof. First, we choose a sequence $(t_n)_{n \geq 1}$ such that $t_n \to 0$, as $n \to \infty$, and

$$\lim_{n \to \infty} \int_S f(s,t_n)\,\mu(ds) = \liminf_{t \to 0} \int_S f(s,t)\,\mu(ds). \tag{4.42}$$

We apply Fatou's lemma, since $f \geq 0$, to obtain

$$\begin{aligned}
\int_S \bar{f}(s)\mu(ds) &= \int_S \liminf_{t \to 0} f(s,t)\mu(ds) \\
&\leq \int_S \liminf_{n \to \infty} f(s,t_n)\mu(ds) \\
&\leq \liminf_{n \to \infty} \int_S f(s,t_n)\mu(ds) \\
&= \liminf_{t \to 0} \int_S f(s,t)\mu(ds). \tag{4.43}
\end{aligned}$$

Next, we choose another sequence $\left(t'_n\right)_{n \geq 1}$ such that $t'_n \to 0$, as $n \to \infty$, and

$$\lim_{n \to \infty} \int_S f\left(s,t'_n\right)\mu(ds) = \limsup_{t \to 0} \int_S f(s,t)\,\mu(ds). \tag{4.44}$$

So, (C2) implies $h(s) + g(s,t) - f(s,t) \geq 0$, and applying Fatou's lemma again,

$$\begin{aligned}
\int_S \left(h(s) - \bar{f}(s)\right)\mu(ds) &= \int_S \liminf_{t \to 0} \left(h(s) + g(s,t) - f(s,t)\right)\mu(ds) \\
&\leq \int_S \liminf_{n \to \infty} \left(h(s) + g\left(s,t'_n\right) - f\left(s,t'_n\right)\right)\mu(ds) \\
&\leq \liminf_{n \to \infty} \int_S \left(h(s) + g\left(s,t'_n\right) - f\left(s,t'_n\right)\right)\mu(ds) \\
&= \int_S h(s)\mu(ds) + \liminf_{n \to \infty} \left(-\int_S f\left(s,t'_n\right)\mu(ds)\right) \\
&= \int_S h(s)\mu(ds) - \limsup_{n \to \infty} \int_S f(s,t'_n)\mu(ds) \\
&= \int_S h(s)\mu(ds) - \limsup_{t \to 0} \int_S f(s,t)\mu(ds).
\end{aligned}$$

Note that the limsup in the last line is finite due to (C2). Canceling the h term, we obtain

$$\limsup_{t \to 0} \int_S f(s,t)\mu(ds) \leq \int_S \bar{f}(s)\mu(ds). \tag{4.45}$$

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Combining (4.43) and (4.45), we have

$$\int_S \bar{f}(s)\mu(ds) \leq \liminf_{t \to 0} \int_S f(s,t)\mu(ds) \leq \limsup_{t \to 0} \int_S f(s,t)\mu(ds) \leq \int_S \bar{f}(s)\mu(ds),$$

which proves the result. □

In order to show the new result, we make the same assumptions as in Theorem 4.5. These assumptions are only assumptions on the jump part L. We present the basic proof of the first-order call-price asymptotics with Brownian part, and later, we present a more general result and simplify the proof significantly. We include the basic proof to demonstrate the techniques used and for completeness.

Theorem 4.8. *Let* $(L_t)_{t \geq 0}$ *satisfy the hypotheses of Theorem 4.5. Then, in case* $X_t \stackrel{\mathcal{L}}{=} \sigma W_t + L_t$ *for every* $t \geq 0$ *with* $\sigma > 0$,

$$\mathbb{E}(S_t - S_0)_+ = S_0 \sigma \sqrt{t} \mathbb{E}^*(W_1^*)_+ + o\left(\sqrt{t}\right),$$

as $t \to 0$.

Proof. We make use of the fact that

$$\lim_{t \to 0} \mathbb{E}^*\left[\exp\left(iu\frac{X_t}{\sqrt{t}}\right)\right] = \exp\left(-\frac{1}{2}(\sigma^*)^2 u^2\right), \tag{4.46}$$

and

$$\lim_{t \to 0} \mathbb{P}^*\left(\frac{X_t}{\sqrt{t}} \geq x\right) = \mathbb{P}^*\left(\sigma W_1 \geq x\right). \tag{4.47}$$

Continuing,

$$\frac{1}{\sqrt{t}}\mathbb{E}(S_t - S_0)_+ = \frac{1}{\sqrt{t}} \int_0^\infty e^x \mathbb{P}^*\left(X_t \geq x\right) dx$$

$$= \int_0^\infty e^{-\sqrt{t}u} \mathbb{P}^*\left(X_t \geq u\sqrt{t}\right) du. \tag{4.48}$$

Note that

$$e^{-\sqrt{t}u}\mathbb{P}^*\left(X_t \geq u\sqrt{t}\right) \leq \mathbb{P}^*\left(\sigma W_t \geq \frac{\sqrt{t}u}{2}\right) + \mathbb{P}^*\left(L_t \geq \frac{\sqrt{t}u}{2}\right)$$

$$= \mathbb{P}^*\left(\sigma W_1 \geq \frac{u}{2}\right) + \mathbb{P}^*\left(L_t \geq \frac{\sqrt{t}u}{2}\right). \tag{4.49}$$

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We next use Lemma 4.7 with

$$f(u,t) = e^{-\sqrt{tu}} \mathbb{P}^* \left(X_t \geq u\sqrt{t} \right),$$

$$\bar{f}(u) = \mathbb{P}^* \left(\sigma W_1 \geq u \right),$$

$$h(u) = \mathbb{P}^* \left(\sigma W_1 \geq u/2 \right),$$

and $g(u,t) = \mathbb{P}^* \left(L_t \geq \sqrt{tu}/2 \right)$. It is easy to see that conditions (C1) – (C2) of the lemma are satisfied. We need to show that (C3) and (C4) also hold. It is not too difficult to see that (C3) holds from (4.46) and the independence of W and L under $\mathbb{P}^*$. We now show that (C4) is satisfied as well. Once done, we immediately have the result by applying Lemma 4.7 to (4.48). First note

$$\int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{tu}}{2} \right) du = \int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{tu}}{2} \right) \left(\mathbb{1}_{\{\frac{\sqrt{tu}}{8} \geq t\bar{\mu}^*\}} + \mathbb{1}_{\{\frac{\sqrt{tu}}{8} < t\bar{\mu}^*\}} \right) du$$

$$:= D_1(t) + D_2(t).$$

It is easy to see that $D_2(t) \to 0$ as $t \to 0$ since

$$D_2(t) \leq \int_0^\infty \mathbb{1}_{\{\frac{\sqrt{tu}}{8} < t\bar{\mu}^*\}} du = 8\sqrt{t}\bar{\mu}^*.$$

We now show $\lim_{t \to 0} D_1(t) = 0$ by making use of Lemma 3.27 and Potter's bounds. To do so, we need to break up $D_1(t)$ into several pieces. Choose $A > 1$ and $\delta > 0$ such that $\alpha \pm \delta \in (1,2)$ and let $M_0 > 0$ be such that the Potter bounds hold for ℓ on $[M_0, \infty)$. We have

$$D_1(t) = \int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{tu}}{2} \right) \left(\mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\bar{\mu}^*}\}} + \mathbb{1}_{\{\frac{8}{\sqrt{tu}} < M_0\}} \right) du$$

$$:= D_{11}(t) + D_{12}(t),$$

and we will apply the Potter bounds to $D_{11}(t)$ in order to use a Dominated Convergence Theorem argument. We are concerned with the limit as $t \to 0$, so there is no loss of generality in assuming that t is so small that $8 > M_0\sqrt{t}$. We proceed by

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estimating (and using κ as a general positive coefficient that can change from line to line)

$$
\begin{aligned}
D_{11}(t) &\leq \int_0^\infty \left((1 + Ce^2) t \gamma^* \left(\frac{\sqrt{tu}}{8} \right) \vee 1 \right) \mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\tilde{\mu}^*}\}} du \\
&= \int_0^\infty \left(\kappa t \left(\frac{\sqrt{tu}}{8} \right)^{-\alpha} \ell \left(\frac{8}{\sqrt{tu}} \right) \vee 1 \right) \mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\tilde{\mu}^*}\}} du \\
&= \int_0^\infty \left(\kappa t^{1-\alpha/2} u^{-\alpha} \ell \left(\frac{8}{\sqrt{tu}} \right) \vee 1 \right) \mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\tilde{\mu}^*}\}} du.
\end{aligned}
$$

We have

$$
\begin{aligned}
\left(\kappa t^{1-\alpha/2} u^{-\alpha} \ell \left(\frac{8}{\sqrt{tu}} \right) \vee 1 \right) &\mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\tilde{\mu}^*}\}} \\
&= \left(\kappa t^{1-\alpha/2} u^{-\alpha} \frac{\ell\left(\frac{8}{\sqrt{tu}} \right)}{\ell\left(\frac{8}{\sqrt{t}} \right)} \ell \left(\frac{8}{\sqrt{t}} \right) \vee 1 \right) \mathbb{1}_{\{M_0 \leq \frac{8}{\sqrt{tu}} \leq \frac{1}{t\tilde{\mu}^*}\}} \\
&\leq \kappa t^{1-\alpha/2} \ell \left(\frac{8}{\sqrt{t}} \right) A \max \left(u^{-\alpha-\delta}, u^{-\alpha+\delta} \right) \vee 1. \qquad (4.50)
\end{aligned}
$$

Now, letting $\beta_t = 8/\sqrt{t}$,

$$
\begin{aligned}
t^{\left(1 - \frac{\alpha}{2}\right)} \ell \left(\frac{8}{\sqrt{t}} \right) &= 8^{2-\alpha} \left(\frac{8}{\sqrt{t}} \right)^{2\left(\frac{\alpha}{2}-1\right)} \ell(\beta_t) \\
&= 8^{2-\alpha} \beta_t^{\alpha-2} \ell(\beta_t) \to 0,
\end{aligned}
$$

since $\alpha - 2 < 0$ and $\beta_t \to \infty$ as $t \to 0$. Thus, there exists t_0 such that $t^{(1-\alpha/2)} \ell(8/\sqrt{t}) \leq 1$ for all $0 \leq t < t_0$. So, (4.50) is bounded by

$$
\kappa \max \left(u^{-\alpha-\delta}, u^{-\alpha+\delta} \right) \vee 1 \in L^1([0, \infty)).
$$

Thus, we can apply Lebesgue's Dominated Convergence Theorem to $D_{11}(t)$ which gives $D_{11}(t) \to 0$, as $t \to 0$, since $\mathbb{P}^* \left(L_t \geq \frac{\sqrt{tu}}{2} \right) \to 0$, as $t \to 0$, for $u > 0$. We now

consider $D_{12}(t)$ and estimate

$$
\begin{aligned}
D_{12}(t) &= \int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{tu}}{2} \right) \mathbb{1}_{\left\{ \frac{8}{\sqrt{tu}} < M_0 \right\}} du \\
&\leq \kappa t^{1-\alpha/2} \int_0^\infty u^{-\alpha} \ell \left(\frac{8}{\sqrt{tu}} \right) \mathbb{1}_{\left\{ \frac{8}{\sqrt{tu}} < M_0 \right\}} du \\
&= \kappa t^{1-\alpha/2} \int_0^\infty \left(\frac{8w}{\sqrt{t}} \right)^{-\alpha} \ell \left(\frac{1}{w} \right) \mathbb{1}_{\left\{ \frac{1}{w} < M_0 \right\}} \frac{8dw}{t^{1/2}} \\
&= \kappa t^{1/2} \int_{1/M_0}^\infty w^{-\alpha} \ell(1/w) dw \\
&= \kappa t^{1/2} \int_{1/M_0}^\infty \gamma^*(z) dz \to 0,
\end{aligned}
$$

since the last integral is finite. Therefore, $\int_0^\infty \mathbb{P}^* \left(L_t^* \geq \frac{\sqrt{tu}}{2} \right) du \to 0$ as $t \to 0$. Applying Lemma 4.7 gives the result. $\qquad\square$

We now simplify Theorem 4.8 to give a result that is more generally useful. The intuitive idea for the result is that when a Lévy process has a nonzero Brownian component, the Brownian dynamics will always govern the first-order asymptotics.

Theorem 4.9. *Let* $L = (L_t)_{t \geq 0}$ *be a Lévy process with triplet* $(b, 0, \nu)$ *such that* $\mathbb{E}\left[|L_1| e^{L_1} \right] < \infty$ *and let* $S_t = S_0 e^{L_t}$. *Let there exist* $B_t > 0$ *with* $B_t \to 0$ *as* $t \to 0$, $\alpha \in (1, 2)$, *and a probability measure* $\mathbb{P}^*$ *such that*

$$
\frac{1}{B_t} \mathbb{E} \left(S_t - S_0 \right)_+ \to \mathbb{E}^* Z_+,
$$

and

$$
\mathbb{P}^* \left(L_t \geq B_t u \right) \to \mathbb{P}^* \left(Z \geq u \right),
$$

for every $u \geq 0$, *where* Z *is an* α-*stable random variable under* $\mathbb{P}^*$. *For* $t \geq 0$, *let* $(X_t)_{t \geq 0}$ *be given by* $X_t = \sigma W_t + L_t$ *where* $W = (W_t)_{t \geq 0}$ *is a Brownian motion independent of* L *under* $\mathbb{P}^*$. *If*

$$
\frac{B_t}{\sqrt{t}} \to 0,
$$

as $t \to 0$, and if $(S_t)_{t \geq 0} = \left(S_0 e^{X_t} \right)_{t \geq 0}$ is a martingale with respect to its own filtration, then

$$c(t,0) = \sigma \mathbb{E}^* (W_1)_+ \sqrt{t} + o\left(\sqrt{t} \right), \tag{4.51}$$

as $t \to 0$.

Proof. Note we still have (4.46) and (4.47). Continuing as in the proof of Theorem 4.8,

$$\begin{aligned}
\frac{1}{\sqrt{t}} c(t,0) &= \frac{1}{\sqrt{t}} \int_0^\infty e^x \mathbb{P}^* \left(X_t \geq x \right) dx \\
&= \int_0^\infty e^{-\sqrt{t}u} \mathbb{P}^* \left(X_t \geq u\sqrt{t} \right) du.
\end{aligned} \tag{4.52}$$

and

$$\begin{aligned}
e^{-\sqrt{t}u} \mathbb{P}^* \left(X_t \geq u\sqrt{t} \right) &\leq \mathbb{P}^* \left(\sigma W_t \geq \frac{\sqrt{t}u}{2} \right) + \mathbb{P}^* \left(L_t \geq \frac{\sqrt{t}u}{2} \right) \\
&= \mathbb{P}^* \left(\sigma W_1 \geq \frac{u}{2} \right) + \mathbb{P}^* \left(L_t \geq \frac{\sqrt{t}u}{2} \right).
\end{aligned}$$

We aim to use Lemma 4.7 with

$$f(u,t) = e^{-\sqrt{t}u} \mathbb{P}^* \left(X_t \geq u\sqrt{t} \right),$$

$$\bar{f}(u) = \mathbb{P}^* \left(\sigma W_1 \geq u \right),$$

$$h(u) = \mathbb{P}^* \left(\sigma W_1 \geq u/2 \right),$$

and $g(u,t) = \mathbb{P}^* \left(L_t \geq \sqrt{t}u/2 \right)$. It is easy to see that conditions (C1)-(C2) of the lemma are satisfied. We need to show (C3) and (C4) hold. It is not too difficult to see that (C3) holds from (4.46) and the independence of W and L under $\mathbb{P}^*$. We now show that (C4) is satisfied as well. Namely, we prove that

$$\int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{t}u}{2} \right) du \to 0,$$

as $t \to 0$. Once we show this, we immediately have the result by applying Lemma 4.7 to (4.52). Note that there exists $t_0 > 0$ such that $B_t \leq \sqrt{t}$ for all $0 \leq t \leq t_0$. For $0 \leq t \leq t_0$, we have

$$\mathbb{P}^* \left(L_t \geq \sqrt{t}u \right) \leq \mathbb{P}^* \left(L_t \geq B_t u \right).$$

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for every $u \geq 0$.

To simplify notation, let $F(t,u) = \mathbb{P}^* \left(L_t \geq \sqrt{t}u/2 \right)$, $G(t,u) = \mathbb{P}^* \left(L_t \geq B_t u/2 \right)$, and $\bar{G}(u) = \mathbb{P}^* \left(Z \geq u \right)$. Note that $\int_0^\infty G(t,u)du \to \int_0^\infty \bar{G}(u)du$ as $t \to 0$. It is clear that $0 \leq \liminf_{t \to 0} \int_0^\infty F(t,u)du$. Now, $G(t,u) - F(t,u) \geq 0$, so we can apply Fatou's lemma to get

$$\int_0^\infty \bar{G}(u)du \leq \liminf_{t \to 0} \int_0^\infty \left(G(t,u) - F(t,u) \right) du$$
$$= \liminf_{t \to 0} \left(\int_0^\infty G(t,u)du - \int_0^\infty F(t,u)du \right)$$
$$= \int_0^\infty \bar{G}(u)du - \limsup_{t \to 0} \int_0^\infty F(t,u)du.$$

Canceling terms gives $\limsup_{t \to 0} \int_0^\infty F(t,u)du \leq 0$. Thus,

$$0 \leq \liminf_{t \to 0} \int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{t}u}{2} \right) du \leq \limsup_{t \to 0} \int_0^\infty \mathbb{P}^* \left(L_t \geq \frac{\sqrt{t}u}{2} \right) du \leq 0,$$

and therefore (C4) is satisfied, proving the result. $\qquad \square$

In this general setting, we can again find the asymptotics of the implied volatility function close to expiration.

Corollary 4.10. *Under the hypotheses of Theorem 4.9, $\hat{\sigma}$, the implied volatility is such that*

$$\hat{\sigma}(t) = \sigma\sqrt{2\pi}\mathbb{E}^* \left(W_1 \right)_+ + o\left(1 \right),$$

as $t \to 0$.

Proof. We proceed exactly as in the proof of Corollary 4.6. We are now comparing (4.40) with (4.51) multiplied by S_0. So,

$$\frac{S_0\hat{\sigma}(t)}{\sqrt{2\pi}}\sqrt{t} \sim S_0\sigma\sqrt{t}\mathbb{E}^* \left(W_1 \right)_+,$$

as $t \to 0$. This implies that

$$\hat{\sigma}(t) \sim \sigma\sqrt{2\pi}\mathbb{E}^* \left(W_1 \right)_+,$$

as $t \to 0$. $\qquad \square$

4.5 *Example of New Rate of Convergence*

So far we have shown that new first-order dynamics are possible; however, that would be relatively useless without some concrete example. In this section we present a simple example that shows some of this new behavior. That is, this example satisfies all the assumptions required, and we find the rate of convergence for the call option price and implied volatility.

Consider the Lévy measure

$$
\nu\,(dx) = \begin{cases} 0, & x < -1 \text{ or } x = 0 \\ x^{-\alpha-1}e^{-x}\,|\ln|x||\,dx, & x \geq -1 \text{ and } x \neq 0, \end{cases} \tag{4.53}
$$

with $\alpha \in (1,2)$. We next consider the Lévy process $(X_t)_{t \geq 0}$ with triplet $(b, 0, \nu)$ where we choose b such that the martingale condition is satisfied.

Note that under the measure transform, the new Lévy measure becomes

$$
\nu^*\,(dx) = \begin{cases} 0, & x < -1 \text{ or } x = 0, \\ x^{-\alpha-1}\,|\ln|x||\,dx, & x \geq -1 \text{ and } x \neq 0, \end{cases} \tag{4.54}
$$

and b^* is defined by (4.4). In order to show that Theorem 4.5 holds, we show directly the properties needed on ν^*. While this is not materially different from showing the assumptions on ν, it will save us a considerable amount of calculation (dealing with the exponential part) and is enough to give the call price asymptotics. That is, we need to show the following statements

(D1) The function γ^* is regularly varying of order $-\alpha$ at 0 where $\alpha \in (1,2)$.

(D2) There exist $C > 0$ and $x_1 > 0$ such that $\int_{|y|\leq x} y^2\nu^*(dy) \leq Cx^2 \int_{|y|>x} \nu^*(dy)$ for all $x > x_1$.

(D3) $\bar{\mu}^* < \infty$.

(D4) There exists $x_0 > 0$ such that $x^2 \left(e^x \nu(dx) + e^{-x} \nu(-dx) \right)$ is monotone (either increasing or decreasing) for $0 < x < x_0$.

(D5) There exists $R_0 > 0$ such that $\int_{R_0}^{\infty} \gamma^*(z)\, dz < \infty$.

We compute some quantities directly.

Proposition 4.11. *The function γ^* has the representation*

$$\gamma^*(x) = \begin{cases} \frac{2}{\alpha} x^{-\alpha} \left(\ln\left(\frac{1}{x}\right) - \frac{1}{\alpha} \right) + \frac{3}{\alpha^2}, & 0 < x < 1 \\ \frac{1}{\alpha^2}, & x = 1 \\ \frac{1}{\alpha} x^{-\alpha} \left(\ln x + \frac{1}{\alpha} \right), & x > 1, \end{cases} \tag{4.55}$$

and is regularly varying of order $-\alpha$ at both 0 and ∞.

Proof. The calculation of γ^* is a simple calculus exercise. It is also clear that γ^* is regularly varying at ∞ by the representation theorem and the fact that $\ln(x) + a$ is slowly varying at ∞ for all $a \geq 0$. To see that γ^* is regularly varying at 0, we show that

$$g(x) := \ln\left(\frac{1}{x}\right) - \frac{1}{\alpha} + \frac{3}{\alpha} x^{\alpha},$$

is slowly varying at 0. Note that for $\lambda > 0$

$$\frac{g(\lambda x)}{g(x)} = \frac{-\ln \lambda - \ln x - \frac{1}{\alpha} + \frac{3}{\alpha} \lambda^{\alpha} x^{\alpha}}{-\ln x - \frac{1}{\alpha} + \frac{3}{\alpha} x^{\alpha}},$$

and factoring out $-\ln(x)$ and letting $x \to 0$ gives limit of 1. Thus, γ^* is regularly varying at 0. $\square$

Note that we have already shown (D1), (D2), and (D5). The requirement (D2) is satisfied since γ^* is regularly varying at both 0 and ∞, and the requirement (D5) is satisfied simply by the form of γ^* for large x and by the fact that $\alpha \in (1, 2)$. The requirement (D4) also holds simply because the Lévy measure is symmetric about 0 for $|x| < 1$. So, the expression in (D4) becomes, for $0 < x < 1$,

$$2x^{1-\alpha} \ln\left(\frac{1}{x}\right),$$

which is easily seen to be monotone.

It is worth noting that it is not really necessary to show (D4). The purpose of assumption (D4) in the original theorem was to get the form of the Lévy measure around the origin. Since here we directly have the form of the Lévy measure about the origin, this requirement is superfluous.

It only remains to show that $\bar{\mu}^* < \infty$. From the symmetry of the Lévy measure, we know that

$$\mu_\varepsilon^* = b,$$

for every $0 < \varepsilon < 1$. Thus, (D3) also holds.

All the hypotheses of the original theorem are now satisfied, and we proceed to determine the dynamics of β_t as $t \to 0$. Again, $B_t = 1/\beta_t$ can be expressed as

$$t^{1/\alpha} \psi \left(\frac{1}{t} \right),$$

for $t > 0$, where ψ is a slowly varying function at ∞. Further, there exists $\Lambda > 0$ such that

$$t \beta_t^\alpha \ell \left(\beta_t \right) \to \Lambda,$$

as $t \to 0$, where ℓ is the slowly varying part of γ^* near 0.

The functions ℓ and β are defined up to asymptotic equivalence, so we use asymptotic versions that are simpler to manipulate. For example, instead of considering

$$\ell \left(\frac{1}{x} \right) = - \ln x - \frac{1}{\alpha} + \frac{3}{\alpha^2} x^\alpha,$$

for x close to 0, we can consider the asymptotically equivalent

$$\ell^\# (x) = - \ln x.$$

So, we find β_t which satisfies the relationship

$$t \beta_t^\alpha \ln \beta_t \to \Lambda,$$

which is equivalent to

$$t\beta_t^\alpha \ln \beta_t^\alpha \to \alpha\Lambda. \tag{4.56}$$

Writing $f(x) = x \log x$, we rewrite (4.56) as $tf(\beta_t^\alpha) \to \alpha\Lambda$ as $t \to 0$. Furthermore, we have that $\beta_t^\alpha \sim f^{-1}\left(\frac{\alpha\Lambda}{t}\right)$, as $t \to 0$ (f has an inverse for x large enough and β_t^α grows large as $t \to 0$). We need asymptotics for $f^{-1}(x)$ as $x \to \infty$. An inverse for f will be a function g such that $g(x) \log g(x) = x$. Due to the increasing and unbounded nature of f for large x, we know that g must be positive for x large enough and we therefore make the substitution $w(x) = \log g(x)$ to get the equation $w(x)e^{w(x)} = x$. This is the well know Lambert W function (here, we use lower-case w instead of W as to not confuse the function with our Brownian motion process); that is, we have $f^{-1}(x) = g(x) = e^{w(x)} = \frac{x}{w(x)}$ (the last equality follows by the very definition of the function w). It is known that $w(x) \sim \log x$ as $x \to \infty$ (see e.g. [14]). So, we obtain

$$f^{-1}(x) = \frac{x}{w(x)} \sim \frac{x}{\log x},$$

as $x \to \infty$. Continuing, we now know

$$\beta_t^\alpha \sim \frac{\frac{\alpha\Lambda}{t}}{\log\left(\frac{\alpha\Lambda}{t}\right)},$$

hence

$$t^{-1}\psi(1/t)^\alpha \sim \frac{\frac{\alpha\Lambda}{t}}{\log\left(\frac{\alpha\Lambda}{t}\right)},$$

and

$$\psi(1/t)^\alpha \sim \frac{\alpha\Lambda}{\log\left(\frac{\alpha\Lambda}{t}\right)}.$$

Finally, we arrive at

$$B_t = \left(\frac{\alpha\Lambda t}{\log\left(\frac{\alpha\Lambda}{t}\right)}\right)^{\frac{1}{\alpha}}.$$

Ignoring the constants, the rate of convergence in the first order is

$$\left(\frac{t}{\log(1/t)}\right)^{\frac{1}{\alpha}}.$$

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Note that we could get even more interesting behavior by introducing further slowly varying function behavior near the origin (e.g. $\cos x$). We compare the traditional rate $t^{1/\alpha}$ to this example's rate in the following figures.
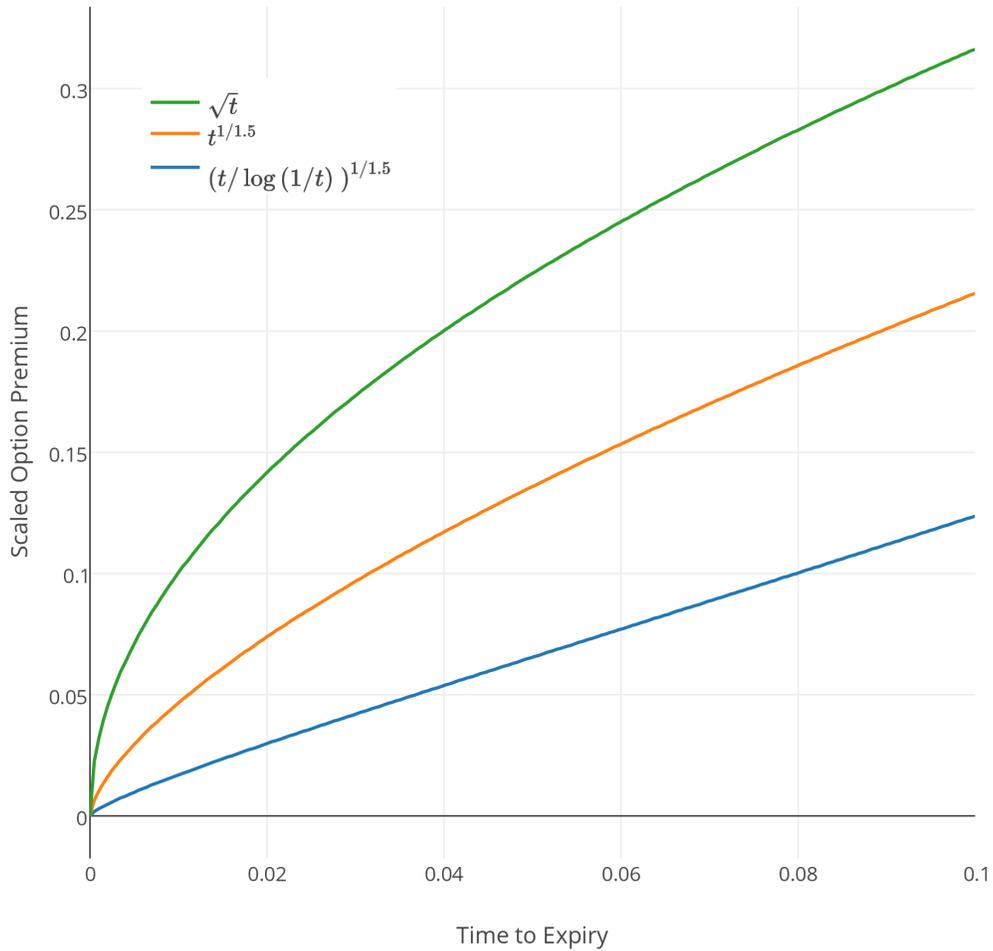

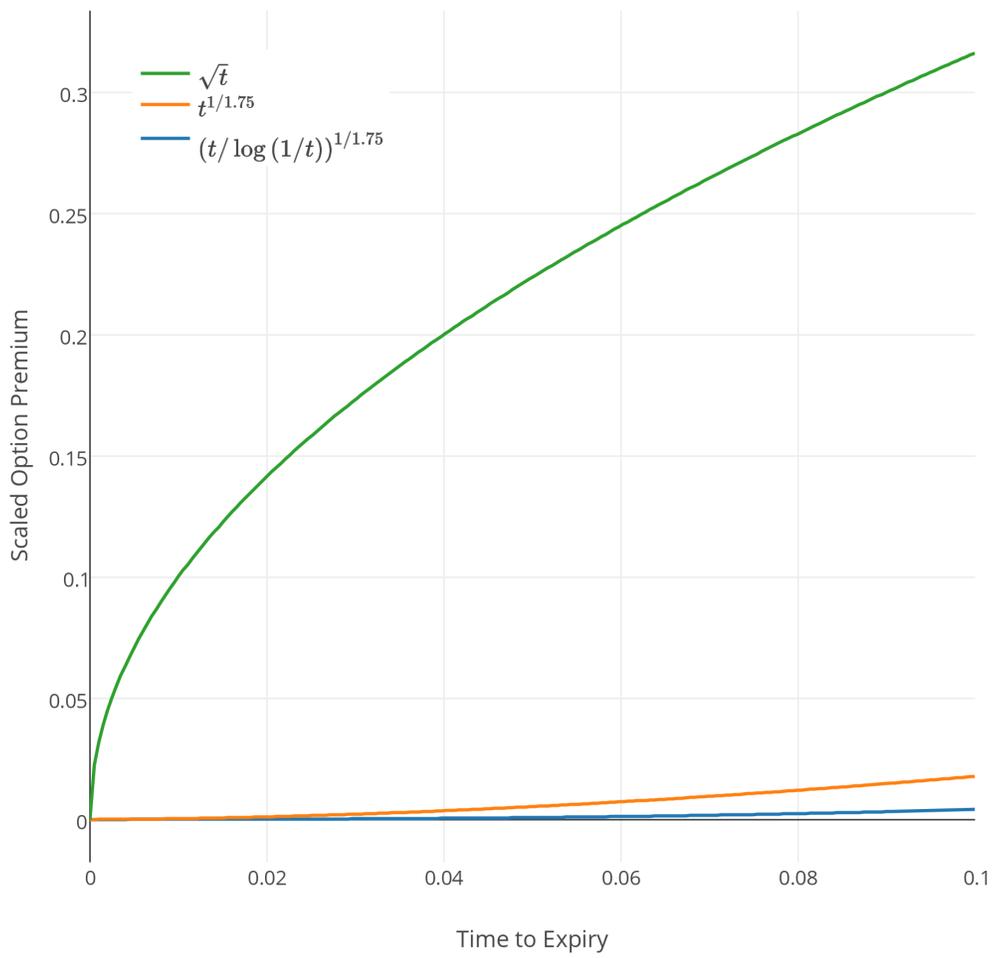Figure 1: Option premium decay rate comparison for $\alpha = 1.5$

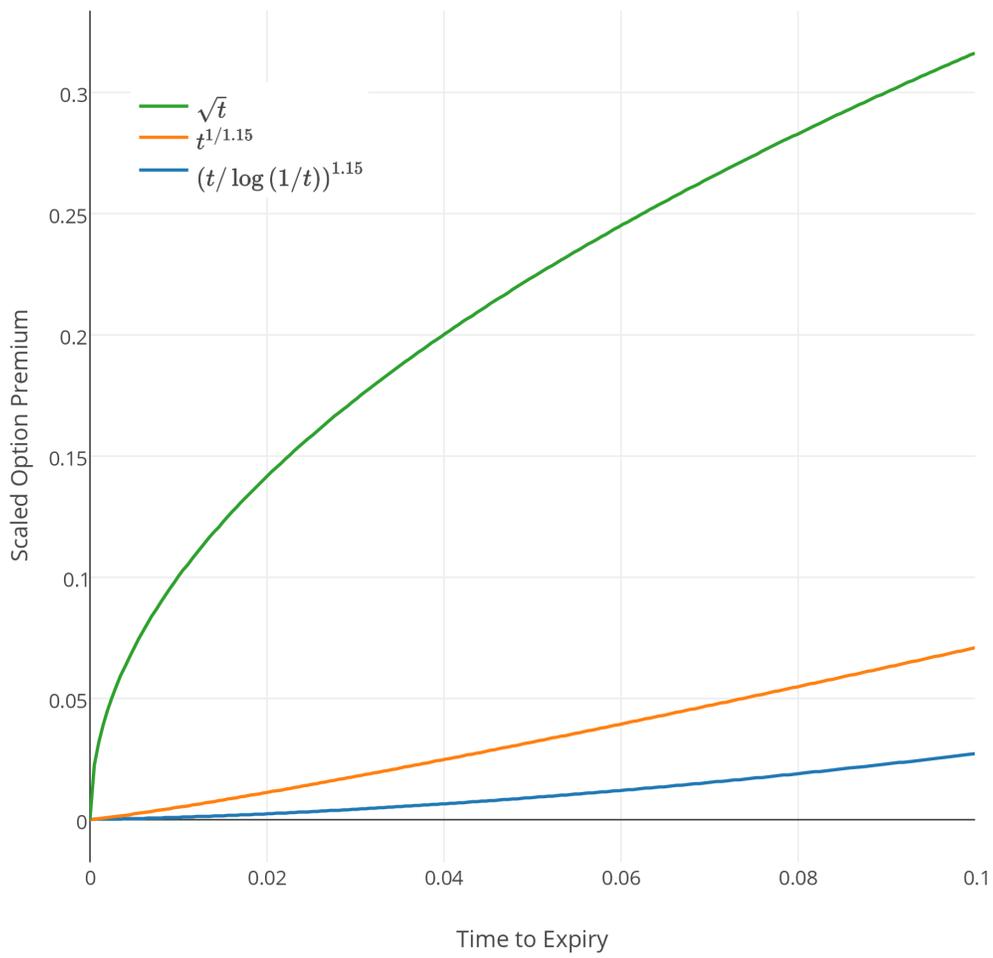Figure 2: Option premium decay rate comparison for $\alpha = 1.75$

Figure 3: Option premium decay rate comparison for $\alpha = 1.15$

CHAPTER V

SECOND ORDER RESULTS

Results containing second-order ATM call-price and implied volatility asymptotics were discovered relatively recently for exponential Lévy models. Most of the difficulties in these asymptotics arise from the unexpected order of the second-order correction term. In the subclass tempered stable processes, Figueroa-López, Gong, and Houdré proved in [24] that the second order term was t under certain technical conditions.

Indeed, these results are very technical and rely on the existence of measure transformation techniques. In more general settings, such measure transformations may not be available or may not even exist. Heuristic expansions up to the second order exist, but are only available when a measure transformation of a tempered stable process to a stable process exists.

The Lévy-Khintchine representation guarantees that Lévy processes are completely characterized by their Fourier transforms. Quite naturally, we might ask if it is possible to describe the second-order ATM call-price asymptotics using only the characteristic exponent of the Lévy process, even if only heuristically. In this chapter, we give an affirmative answer to that question by looking at the CGMY model whose asymptotics are known up to third order.

Let us take a quick detour and consider intuitively why higher-order asymptotic expansions are so difficult to develop, even formally. Consider the third-order asymptotic results for the CGMY model (shown in [23])

$$c(t) = d_1 t^{1/Y} + d_2 t + \begin{cases} d_{31} t^{2-\frac{1}{Y}} + o\left(t^{2-\frac{1}{Y}}\right), & \text{if } 1 < Y \le \frac{3}{2} \\ d_{32} t^{2/Y} + o\left(t^{2/Y}\right), & \text{if } \frac{3}{2} \le Y < 2, \end{cases} \tag{5.1}$$

as $t \to 0$.

Here, we can already see that any heuristic expansion will have to account for this third-order behavior. Even if we restrict to second-order, we still need to account for the fact that there is no apparent pattern between the first and second order, further diminishing the hope that we could find a formal expansion for the second order.

Finally, if we look at the basic expression for the call-option price

$$\mathbb{E}\left[\left(\exp\left(X_t\right) - 1\right)_+\right],$$

then (5.1) appears at odds with what we expect from expansions of exponential functions. Clearly the distribution of X_t and the lack of smoothness of the function x_+ at $x = 0$ play major roles in these asymptotic expansions.

Now that we have explored why these expansions are so counterintuitive, we turn our attention to justifying, and showing precisely when possible, the second-order expansions for the CGMY process. First, we introduce the exponential CGMY model and briefly discuss its properties. Next, we exhibit the Lipton-Lewis option pricing formula and derive the first-order asymptotics, correcting the proof found in [1]. Finally, we use the Lipton-Lewis formula to obtain heuristic, second-order asymptotics for ATM call options.

5.1 Revisiting the CGMY Process

In [8] Carr, Geman, Madan, and Yor introduced the appropriately named CGMY process. The CGMY process is a real-valued Lévy process with triplet $(b, 0, \nu)$, where $b \in \mathbb{R}$ arbitrary, and where ν is given by

$$\nu(dx) = \left(\frac{Ce^{-G|x|}}{|x|^{1+Y}}\mathbb{1}_{x<0} + \frac{Ce^{-Mx}}{x^{1+Y}}\mathbb{1}_{x>0}\right)dx, \tag{5.2}$$

with $C > 0$, $M, G \geq 0$, and $Y < 2$. Intuitively, the CGMY process can be thought of as a stable-like process where larger jumps are much less likely. This intuition comes from the Lévy measure (5.2), which looks like the Lévy measure of a stable random

variable, save for the inclusion of the exponential damping terms. These exponential damping terms serve to decrease the intensity of the jumps when $|x|$ is large. We restrict our attention to CGMY processes where $1 < Y < 2$.

From Proposition 2.23, equation (5.2), and

$$\int_1^\infty e^x C \frac{e^{-Mx}}{x^{1+Y}} dx < \infty,$$

when $M \geq 1$, we find condition (2.8) is satisfied and $\left(e^{X_t}\right)_{t \geq 0}$ is a well-defined exponential Lévy model when $(X_t)_{t \geq 0}$ is a CGMY process with $M \geq 1$ and $1 < Y < 2$. Additionally, we assume $M > 1$ which implies that e^{X_t} has finite moments of all orders for $t \geq 0$.

In addition to having finite moments of all orders (under the condition $M > 1$), CGMY processes have simple closed-form characteristic functions. It is given below as presented in Proposition 4.2 in [12]. In what follows, we say $X = (X_t)_{t \geq 0}$ is a CGMY process whenever X is a Lévy process where X_1 has Lévy triplet $(b, 0, \nu)$, $b \in \mathbb{R}$, ν is given by (5.2), and $1 < Y < 2$.

Proposition 5.1. *If $(X_t)_{t \geq 0}$ is a CGMY process, then its characteristic exponent is given by, for $u \in \mathbb{R}$,*

$$\Psi\left(u\right) = t^{-1} \log\left(\mathbb{E}\left[e^{iuX_t}\right]\right) = iu\tilde{b} + C\Gamma\left(-Y\right)\left((M - iu)^Y + (G + iu)^Y - M^Y - G^Y\right),$$

$$(5.3)$$

where $\tilde{b} = b + \int_{|x|>1} x\nu(dx)$. If

$$\tilde{b} = -C\Gamma\left(-Y\right)\left((M - 1)^Y + (G + 1)^Y - M^Y - G^Y\right), \qquad (5.4)$$

then $(S_t)_{t \geq 0} = \left(S_0 e^{X_t}\right)_{t \geq 0}$ is a martingale with respect to its own filtration.

Proof. First, $Y > 1$ implies that $\int_{|x|>1} |x|\,\nu(dx) < \infty$ by 2.13, so that $\tilde{b}$ is well-defined.

From the Lévy-Khintchine representation,

$$
\begin{aligned}
\Psi\left(u\right) &= iub + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux \mathbb{1}_{\{|x| \leq 1\}}\right) \nu(dx) \\
&= iub + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux\right) \nu(dx) + iu \int_{|x|>1} x\nu(dx) \\
&= iu\tilde{b} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux\right) \nu(dx).
\end{aligned}
\tag{5.5}
$$

Evaluating the integral in (5.5) is a standard exercise and can be found, for example, in Proposition 4.2 of [12]. Finally, $\tilde{b}$ as in (5.4) guarantees that $\Psi\left(-i\right) = 1$, implying that (2.9) is satisfied. $\qquad\square$

From here and on, we will suppose that $\tilde{b}$ is given by (5.4) so that the exponential CGMY process $(S_t)_{t \geq 0} = \left(S_0 e^{X_t}\right)_{t \geq 0}$ is an exponential Lévy asset model in the sense of Section 2.4.

From Proposition 5.1, simple calculations lead to convergence results for CGMY processes, shown in [22].

Proposition 5.2. *If $(X_t)_{t \geq 0}$ is a CGMY process, then*

$$
\frac{X_t}{t^{1/Y}} \Rightarrow Z,
$$

as $t \to 0$, where Z is a Y-stable, symmetric random variable with characteristic function, given for any $u \in \mathbb{R}$, by

$$
\phi_Z\left(u\right) = \mathbb{E}\left[e^{iuZ}\right] = \exp\left(-2C\Gamma\left(-Y\right)\left|\cos\left(\frac{Y\pi}{2}\right)\right| |u|^Y\right),
\tag{5.6}
$$

where Γ is Euler's gamma function.

Remark 5.3. Note that $\Gamma\left(x\right) > 0$, for $-2 < x < -1$, and so in (5.6), the exponent is negative for all $u \neq 0$.

Proof. First,

$$\log\left(\mathbb{E}\left[e^{iu\frac{X_t}{t^{1/Y}}}\right]\right) = t\Psi\left(ut^{-1/Y}\right)$$

$$= iut^{-1/Y}\tilde{b}t + tC\Gamma\left(-Y\right)\left[\left(M - iut^{-1/Y}\right)^Y + \left(G + iut^{-1/Y}\right)^Y\right.$$

$$\left. - M^Y - G^Y\right]$$

$$= iu\tilde{b}t^{1-1/Y} + C\Gamma\left(-Y\right)\left[\left(Mt^{1/Y} - iu\right)^Y + \left(Gt^{1/Y} + iu\right)^Y\right.$$

$$\left. - M^Y t - G^Y t\right].$$

$$(5.7)$$

Therefore, taking the limit in (5.7) gives

$$\lim_{t \to 0}\log\left(\mathbb{E}\left[e^{iu\frac{X_t}{t^{1/Y}}}\right]\right) = C\Gamma\left(-Y\right)\left((-i)^Y + i^Y\right)|u|^Y$$

$$= -2C\Gamma\left(-Y\right)\left|\cos\left(\frac{Y\pi}{2}\right)\right||u|^Y.$$

$$\square$$

One important consequence of Proposition 5.2 is that the stable random variable Z has very different moment properties from the original CGMY process. The Y-stable random variable Z has finite moments up to, but not including, order Y. In particular, the random variable Z has infinite variance since $Y < 2$. This is very different from the underlying CGMY process which has finite moments of all orders.

5.2 Lipton-Lewis Formula and First Order Results

In order to use the characteristic function to analyze the first and second-order asymptotics, we need a reliable formula that represents the call price as a function of the characteristic function. For this, we turn to the Lipton-Lewis (LL) formula. For a good discussion of the LL formula and its many applications, we refer the reader to Andersen and Lipton [1], which we follow here.

Below, we introduce the LL formula and demonstrate how it can be used to obtain first-order asymptotics for the CGMY process, agreeing with the expansion found in

[22]. First, we impose conditions on the characteristic function necessary to validate the LL result. Specifically, the characteristic function has to be well defined as a function of a complex variable in a certain domain in $\mathbb{C}$. (As usual, for $z \in \mathbb{C}$, $\Re z$ is its real part, $\Im z$ its imaginary part, and $\bar{z}$ its complex conjugate.)

Proposition 5.4. *Let $(X_t)_{t \geq 0}$ be a Lévy process where X_1 has Lévy triplet $(b, 0, \nu)$, $b \in \mathbb{R}$. Let the Lévy measure satisfy*

$$\int_{|x|>1} e^x \nu(dx) < \infty,$$

and let $\phi_t(z) = \mathbb{E}\left[e^{izX_t} \right]$ exist in the complex strip

$$\mathcal{S} = \{ z \in \mathbb{C} : -1 \leq \Im z \leq 0 \},$$

with $\phi_t(-i) = 1$, then the process $S = (S_t)_{t \geq 0} = \left(S_0 e^{X_t} \right)_{t \geq 0}$ is an arbitrage-free, well-defined exponential Lévy model.

Proof. In order for S to be an arbitrage-free, well-defined exponential Lévy model, we need (2.9) to hold and we need X_t to have finite exponential moment for $t \geq 0$. Since ϕ is well defined at $-i$, we know that X_t has finite exponential moment as

$$\mathbb{E}e^{X_t} = \phi_t(-i) = 1. \tag{5.8}$$

Finally, equation (5.8) implies

$$b + \int_{\mathbb{R}} \left(e^x - 1 - x \mathbb{1}_{\{|x| \leq 1\}} \right) \nu(dx) = 0,$$

which is (2.9). $\qquad\square$

Recall that we use $k = \log(S/K)$ to refer to *moneyness* where K is the strike price of the option and S is the asset price. We present the following result discovered independently by Lipton and Lewis (see Proposition 5.1 [1], Theorem 3.5 and formula (3.11) in [39], or formula (3) in [41]).

Theorem 5.5. *Let Ψ be the characteristic exponent of the Lévy process $(X_t)_{t\geq0}$ driving the exponential Lévy asset model $(S_t)_{t\geq0} = \left(S_0 e^{X_t}\right)_{t\geq0}$. Let Ψ exist in a domain of $\mathbb{C}$ containing $\mathcal{S}$, then the normalized call price is given by*

$$c(t,k) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{t\Psi\left(u-\frac{i}{2}\right)}}{u^2 + \frac{1}{4}} e^{k\left(iu-\frac{1}{2}\right)} du. \tag{5.9}$$

Setting $k = 0$ and using $\int_{-\infty}^{\infty} \left(u^2 + 1/4\right)^{-1} du = 2\pi$ gives the following ATM call-option pricing formula.

Corollary 5.6. *Under the hypotheses of Theorem 5.5, an ATM call option has normalized price*

$$c(t,0) = \frac{1}{2\pi}\Re\left(\int_{-\infty}^{\infty} \frac{1 - e^{t\Psi\left(u-\frac{i}{2}\right)}}{u^2 + \frac{1}{4}} du\right)$$

$$= \frac{1}{\pi}\Re\left(\int_{0}^{\infty} \frac{1 - e^{t\Psi\left(u-\frac{i}{2}\right)}}{u^2 + \frac{1}{4}} du\right). \tag{5.10}$$

We now use Theorem 5.5 and its corollary to demonstrate how we might go about obtaining call-price asymptotics. The proof below is a corrected version of the proof given in [1]. We pursue the same line of argument as Andersen and Lipton; however, in showing the authors' results precisely, we were not able to verify their estimation of the integral when the interval of integration is restricted to $[0, \varepsilon)$, some $\varepsilon > 0$.

Theorem 5.7. *If $(X_t)_{t\geq0}$ is a CGMY process with $M > 1$ and $(S_t)_{t\geq0} = \left(S_0 e^{X_t}\right)_{t\geq0}$ is a martingale, then the first-order normalized call-price can be represented as*

$$c(t,0) = d_1 t^{1/Y} + o\left(t^{1/Y}\right), \tag{5.11}$$

as $t \to 0$, with d_1 given by

$$d_1 = \frac{1}{\pi}\Re\left(\int_{0}^{\infty} \frac{1 - \exp\left(\theta_0(u)\right)}{u^2} du\right) = \frac{1}{\pi}\Gamma\left(1 - \frac{1}{Y}\right)\left(2C\Gamma\left(-Y\right)\left|\cos\left(\frac{\pi Y}{2}\right)\right|\right)^{1/Y}, \tag{5.12}$$

where

$$\theta_0\left(u\right) := C\Gamma\left(-Y\right)\left(\left(-i\right)^Y + i^Y\right)\left|u\right|^Y. \tag{5.13}$$

Before proving Theorem 5.7, we state a result that will help us in our proof.

Proposition 5.8. *Let $(X_t)_{t\geq0}$ be a CGMY process with $M > 1$ and $(S_t)_{t\geq0} = \left(S_0 e^{X_t}\right)_{t\geq0}$ be a martingale. Then the normalized call-price function has representation*

$$c(t,0) = t^{1/Y}\mathcal{L}(t), \tag{5.14}$$

where

$$\mathcal{L}(t) = \frac{1}{\pi}\Re\left(\int_0^\infty \frac{1 - \exp\left(\theta(t,v)\right)}{v^2 + \frac{1}{4}t^{2/Y}}dv\right), \tag{5.15}$$

where

$$\theta(t,v) = iv\tilde{b}t^{1-1/Y} + C\Gamma(-Y)\left[\left(\left(M - \frac{1}{2}\right)t^{1/Y} - iv\right)^Y + \left(\left(G + \frac{1}{2}\right)t^{1/Y} + iv\right)^Y\right.$$

$$\left. - M^Y t - G^Y t\right]$$

$$= iv\tilde{b}t^{1-1/Y} + \kappa t + C\Gamma(-Y)\left(\left(\widetilde{M}t^{1/Y} - iv\right)^Y + \left(\widetilde{G}t^{1/Y} + iv\right)^Y\right), \tag{5.16}$$

where $\tilde{b}$ is given by (5.4), and where $\kappa = -M^Y - G^Y$. Moreover, the real part of (5.16) has representation

$$r(t,v) := \Re(\theta(t,v))$$

$$= \kappa t + C\Gamma(-Y)\left[\left(\widetilde{M}^2 t^{2/Y} + v^2\right)^{Y/2}\cos\left(Y\arctan\left(-\frac{v}{\widetilde{M}t^{1/Y}}\right)\right)\right.$$

$$\left. + \left(\widetilde{G}^2 t^{2/Y} + v^2\right)^{Y/2}\cos\left(Y\arctan\left(\frac{v}{\widetilde{G}t^{1/Y}}\right)\right)\right]. \tag{5.17}$$

Proof. The identities (5.14) and (5.15) follow immediately by making the substitution $u = vt^{1/Y}$ in equation (5.10). For (5.17), use the polar coordinate representation of complex numbers to rewrite

$$\left(\widetilde{M}t^{1/Y} - iu\right)^Y$$

and

$$\left(\widetilde{G}t^{1/Y} + iu\right)^Y.$$

$\square$

Remark 5.9.   (i) Note that $\theta(t,v) \to \theta_0(v)$ as $t \to 0$ for every $v \geq 0$, and θ_0 is a real-valued function as

$$(-i)^Y + (i)^Y = -2 \left| \cos\left(\frac{Y\pi}{2} \right) \right|.$$

(ii) We will use certain substitutions frequently. To that end, note that

$$\theta\left(t, t^{1/Y} v\right) = t\psi(v), \tag{5.18}$$

where

$$\psi(v) = iv\tilde{b} + \kappa + C\Gamma(-Y)\left(\left(\widetilde{M} - iv\right)^Y + \left(\widetilde{G} + iv\right)^Y\right), \tag{5.19}$$

and

$$\theta_0\left(t^{1/Y} v\right) = t\theta_0(v). \tag{5.20}$$

(iii) Using (5.17) and (5.18) leads to

$$\Re(\psi(v)) = \frac{r\left(t, t^{1/Y} v\right)}{t}$$

$$= \kappa + C\Gamma(-Y)\left[\left(\widetilde{M}^2 + v^2\right)^{Y/2} \cos\left(Y \arctan\left(-\frac{v}{\widetilde{M}}\right)\right)\right.$$

$$\left. + \left(\widetilde{G}^2 + v^2\right)^{Y/2} \cos\left(Y \arctan\left(\frac{v}{\widetilde{G}}\right)\right)\right]. \tag{5.21}$$

Observe, $\Re(\psi(v)) \sim -2C\Gamma(-Y)\left|\cos(Y\pi/2)\right| v^Y$ as $v \to \infty$ where the coefficient is negative so that $\exp(\Re(\psi(v)))$ is bounded by 1 for $v \geq 0$.

Proof of Theorem 5.7. We proceed as in [1] by considering the first order of $\mathcal{L}$. In order to prove the result, we break $\mathcal{L}$ into two parts: one where the integration is restricted to the interval $[0, \varepsilon]$ and the other where the integration is restricted to (ε, ∞), with $\varepsilon > 0$ small. We denote these two parts as $\mathcal{L}_0^\varepsilon$ and $\mathcal{L}_\varepsilon^\infty$, respectively. In what follows, we will expand the integrand of $\mathcal{L}_0^\varepsilon$ around the origin, and we will apply Lebesgue's Dominated Convergence Theorem to $\mathcal{L}_\varepsilon^\infty$. We use η to represent a positive constant whose value can change from line to line in the remaining work.

To first order, we have

$$\mathcal{L}_0^\varepsilon(t) = \frac{1}{\pi} \Re \left(\int_0^\varepsilon \frac{1 - \exp\left(\theta(t,v)\right)}{v^2 + \frac{1}{4}t^{2/Y}} dv \right)$$

$$= \frac{1}{\pi} \Re \left(\int_0^\varepsilon \frac{-\theta(t,v) + D(t,v)}{v^2 + \frac{1}{4}t^{2/Y}} dv \right)$$

$$= -\frac{1}{\pi} \int_0^\varepsilon \frac{\Re(\theta(t,v))}{v^2 + \frac{1}{4}t^{2/Y}} dv + \frac{1}{\pi} \int_0^\varepsilon \frac{\Re(D(t,v))}{v^2 + \frac{1}{4}t^{2/Y}} dv, \tag{5.22}$$

where $D(t,v) = O\left(\theta(t,v)^2\right)$ as $t, v \to 0$.

Here is where our proof differs from the one presented in [1]: the authors claim that (5.22) is $O(\varepsilon)$ after letting $t \to 0$ and the authors ignore the remainder term involving D. We were not able to verify this claim that (5.22) is $O(\varepsilon)$ as $t \to 0$, and further we obtain $O\left(\varepsilon^{Y-1}\right)$ after letting $t \to 0$.

First, we show that the remainder term is $O\left(\varepsilon^{2Y-1}\right)$ as $t \to 0$. We estimate, for some constant $\eta > 0$ whose value might change from line to line, and make the substitution $v = t^{1/Y}w$, to obtain

$$\int_0^\varepsilon \left| \frac{\Re(D(t,v))}{v^2 + \frac{1}{4}t^{2/y}} \right| dv \le \int_0^\varepsilon \frac{\eta \left| \theta(t,v) \right|^2}{v^2 + \frac{1}{4}t^{2/y}} dv$$

$$= \eta t^{-1/Y} \int_0^{\varepsilon t^{-1/Y}} \frac{\left| \theta\left(t, t^{1/Y}w\right) \right|^2}{w^2 + \frac{1}{4}} dw$$

$$= \eta t^{-1/Y} \int_0^{\varepsilon t^{-1/Y}} \frac{t^2 \left| \psi(w) \right|^2}{w^2 + \frac{1}{4}} dw$$

$$\le \eta t^{2-1/Y} \int_0^{\varepsilon t^{-1/Y}} \frac{\left(1 \vee w^{2Y}\right)}{w^2 + \frac{1}{4}} dw$$

$$\le \eta t^{2-1/Y} \int_0^1 \frac{1}{w^2 + \frac{1}{4}} dw + \eta t^{2-1/Y} \int_1^{\varepsilon t^{-1/Y}} \frac{w^{2Y}}{w^2} dw$$

$$\le \eta t^{2-1/Y} + \eta t^{2-1/Y} \int_1^{\varepsilon t^{-1/Y}} w^{2Y-2} dw$$

$$= \eta t^{2-1/Y} + \frac{\eta}{2Y-1} \varepsilon^{2Y-1}, \tag{5.23}$$

which is $O\left(\varepsilon^{2Y-1}\right)$ (and hence $o\left(\varepsilon^{Y-1}\right)$) as $t \to 0$.

Continuing the estimation of (5.22) and again using the substitution $v = t^{1/Y}w$,

$$\left| \int_0^\varepsilon \frac{\Re(\theta(t,v))}{v^2 + \frac{1}{4}t^{2/Y}}dv \right| = \left| \int_0^{\varepsilon t^{-1/Y}} \frac{\Re\left(\theta(t,t^{1/Y}w)\right)}{t^{2/Y}w^2 + \frac{1}{4}t^{2/Y}}t^{1/Y}dw \right|$$

$$= t^{-1/Y}\left| \int_0^{\varepsilon t^{-1/Y}} \frac{\Re(t\psi(w))}{w^2 + \frac{1}{4}}dw \right|$$

$$\leq t^{1-1/Y}\int_0^{\varepsilon t^{-1/Y}} \frac{|\Re(\psi(w))|}{w^2 + \frac{1}{4}}dw, \tag{5.24}$$

where ψ is defined in (5.19). Here, we can use (5.21) to estimate the real part of ψ as

$$t^{1-1/Y}\int_0^{\varepsilon t^{-1/Y}} \frac{|\Re(\psi(w))|}{w^2 + \frac{1}{4}}dw \leq t^{1-1/Y}\int_0^{\varepsilon t^{-1/Y}} \frac{\kappa + \eta\left(w^Y \vee 1\right)}{w^2 + \frac{1}{4}}dw. \tag{5.25}$$

Splitting the integral, it is clear that the first part can be bounded by

$$t^{1-1/Y}\int_0^\infty \frac{\kappa}{w^2 + \frac{1}{4}}dw = t^{1-1/Y}\kappa\pi \to 0, \tag{5.26}$$

as $t \to 0$. For the second part, split up the integral into two further parts: one integrating on the interval $[0,1]$ and one integrating on the interval $\left(1, \varepsilon t^{-1/Y}\right)$. We assume that t is small enough such that $\varepsilon t^{-1/Y} > 1$. On the interval $[0,1]$, the integral can be estimated similarly to (5.26), so we only consider the interval $\left(1, \varepsilon t^{-1/Y}\right)$. Then,

$$t^{1-1/Y}\int_1^{\varepsilon t^{-1/Y}} \frac{\eta w^Y}{w^2 + \frac{1}{4}} \leq t^{1-1/Y}\int_1^{\varepsilon t^{-1/Y}} \frac{\eta w^Y}{w^2}$$

$$= \frac{\eta}{Y-1}t^{1-1/Y}\left(\left(\varepsilon t^{-1/Y}\right)^{Y-1} - 1\right)$$

$$= \frac{\eta}{Y-1}\varepsilon^{Y-1} - \frac{\eta}{Y-1}t^{1-1/Y}. \tag{5.27}$$

Combining (5.26), (5.23), and (5.27) proves that (5.22) is $O\left(\varepsilon^{Y-1}\right)$ as $t \to 0$.

For the integral $\mathcal{L}_\varepsilon^\infty$, our proof is much simpler. Here, $|\exp\theta(t,v)|$ is bounded above by a constant, say by η. We estimate the integrand of $\mathcal{L}_\varepsilon^\infty$ as

$$\left| \frac{1 - \Re(\exp(\theta(t,v)))}{v^2 + \frac{1}{4}t^{2/Y}} \right| \leq \frac{1+\eta}{v^2} \in L^1\left(\varepsilon, \infty\right),$$

where η possibly depends on ε. Thus, we apply Lebesgue's Dominated Convergence Theorem to obtain that

$$\lim_{t \to 0} \mathcal{L}_\varepsilon^\infty (t) = \int_\varepsilon^\infty \frac{1 - \exp(\theta_0(v))}{v^2} dv. \tag{5.28}$$

Finally, we have

$$\limsup_{t \to 0} \left| \mathcal{L}(t) - \int_0^\infty \frac{1 - \exp(\theta_0(v))}{v^2} dv \right|$$

$$= \limsup_{t \to 0} \left| \mathcal{L}_0^\varepsilon (t) + \mathcal{L}_\varepsilon^\infty (t) - \int_0^\infty \frac{1 - \exp(\theta_0(v))}{v^2} dv \right|$$

$$\leq \limsup_{t \to 0} \left| \mathcal{L}_0^\varepsilon (t) \right| + \left| \int_0^\varepsilon \frac{1 - \exp(\theta_0(v))}{v^2} dv \right|$$

$$\leq \eta \varepsilon^{Y-1} + \left| \int_0^\varepsilon \frac{1 - \exp(\theta_0(v))}{v^2} dv \right|, \tag{5.29}$$

and (5.29) converges to 0 as $\varepsilon \to 0$. $\qquad \square$

Now that we have established the first order results for CGMY processes using this method, we move on to the second order.

5.3 CGMY Second Order Results

In this section, we obtain heuristically second order call-price asymptotics for exponential CGMY processes using only asymptotic expansions involving the characteristic function, and we show how such expansions can help obtain asymptotics for more general exponential Lévy models.

We now turn to the main result of this chapter.

Theorem 5.10. *Let $(X_t)_{t \geq 0}$ be a CGMY process with $M > 1$ and let $(S_t)_{t \geq 0} = \left(S_0 e^{X_t}\right)_{t \geq 0}$ be a martingale, then the second-order normalized call-price can be represented as*

$$c(t, 0) = d_1 t^{1/Y} + d_2(\varepsilon) t + o(t), \tag{5.30}$$

as $t \to 0$, *where* d_1 *is as in Theorem 5.7 and* $d_2(\varepsilon)$ *is defined for* $\varepsilon > 0$ *as*

$$d_2(\varepsilon) = \frac{1}{\pi} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{w^2 \left(\theta_0(w) - \Re(\psi(w))\right) - \frac{1}{4}\Re(\psi(w))}{w^4} dw$$

$$+ \frac{1}{\pi} \int_0^{\varepsilon^{-1/Y}} \frac{\left(w^2 + \frac{1}{4}\right)\theta_0(w) - w^2\Re(\psi(w))}{w^2 \left(w^2 + \frac{1}{4}\right)} dw. \tag{5.31}$$

Proof. We obtain (5.30) by showing that the function

$$R(t) := \frac{c(t,0)}{t^{1/Y}} - \mathcal{L}(0), \tag{5.32}$$

is of order $t^{1-1/Y}$. In terms of $\mathcal{L}$, write

$$R(t) = \frac{1}{\pi}\Re\left(\int_0^{\infty} \frac{1 - \exp(\theta(t,v))}{v^2 + \frac{1}{4}t^{2/Y}} dv\right) - \frac{1}{\pi}\int_0^{\infty} \frac{1 - \exp(\theta_0(v))}{v^2} dv$$

$$= \frac{1}{\pi}\Re\left(\int_0^{\infty} \frac{1 - \exp(\theta(t,v))}{v^2 + \frac{1}{4}t^{2/Y}} dv - \int_0^{\infty} \frac{1 - \exp(\theta_0(v))}{v^2} dv\right). \tag{5.33}$$

In (5.33), temporarily ignore the $1/\pi$ and real part and just consider the two integrals.

Fix $\varepsilon > 0$ and consider the two regions $\left\{0 < t/v^Y < \varepsilon\right\}$ and $\left\{t/v^Y \geq \varepsilon\right\}$. Splitting (5.33), we obtain

$$R(t) = \left(\int_{(t/\varepsilon)^{1/Y}}^{\infty} + \int_0^{(t/\varepsilon)^{1/Y}}\right)\left(\frac{1 - \exp(\theta(t,v))}{v^2 + \frac{1}{4}t^{2/Y}} - \frac{1 - \exp(\theta_0(v))}{v^2}\right) dv$$

$$= A_1(t,\varepsilon) + A_2(t,\varepsilon). \tag{5.34}$$

First, we evaluate A_2 by combining fractions and making the substitution $v = t^{1/Y}w$,

$$A_2(t,\varepsilon) = \int_0^{(t/\varepsilon)^{1/Y}}\left(\frac{v^2\left(1 - \exp(\theta(t,v))\right) - \left(v^2 + \frac{1}{4}t^{2/Y}\right)\left(1 - \exp(\theta_0(v))\right)}{v^2\left(v^2 + \frac{1}{4}t^{2/Y}\right)}\right) dv$$

$$= t^{-1/Y}\int_0^{\varepsilon^{-1/Y}} \frac{w^2\left(1 - \exp\left(\theta\left(t, t^{1/Y}w\right)\right)\right) - \left(w^2 + \frac{1}{4}\right)\left(1 - \exp\left(\theta_0\left(t^{1/Y}w\right)\right)\right)}{w^2\left(w^2 + \frac{1}{4}\right)} dw$$

$$= t^{-1/Y}\int_0^{\varepsilon^{-1/Y}}\left(\frac{w^2\left(1 - \exp(t\psi(w))\right) - \left(w^2 + \frac{1}{4}\right)\left(1 - \exp(t\theta_0(w))\right)}{w^2\left(w^2 + \frac{1}{4}\right)}\right) dw. \tag{5.35}$$

Notice that for w close to 0 and t close to 0, we can formally expand the numerator to first order and arrive at

$$t^{1-1/Y} \int_0^{\varepsilon^{-1/Y}} \left(\frac{\left(w^2 + \frac{1}{4}\right)\theta_0\left(w\right) - w^2\psi\left(w\right)}{w^2\left(w^2 + \frac{1}{4}\right)} \right) dw. \tag{5.36}$$

The integral in (5.36), considered on its own, is in fact well-defined after taking real parts, as $\Re\psi\left(w\right) \sim w^Y$ for w small. Using Lebesgue's Dominated Convergence Theorem, we can show (5.36) holds precisely, i.e.

$$\lim_{t\to 0} \frac{\Re(A_2\left(t,\varepsilon\right))}{t^{1-1/Y}} = \int_0^{\varepsilon^{-1/Y}} \left(\frac{\left(w^2 + \frac{1}{4}\right)\theta_0\left(w\right) - w^2\Re(\psi\left(w\right))}{w^2\left(w^2 + \frac{1}{4}\right)} \right) dw. \tag{5.37}$$

We start with the representation (5.35) and compute

$$\frac{A_2\left(t,\varepsilon\right)}{t^{1-1/Y}} = t^{-1} \int_0^{\varepsilon^{-1/Y}} \left(\frac{w^2\left(1 - \exp\left(t\psi\left(w\right)\right)\right) - \left(w^2 + \frac{1}{4}\right)\left(1 - \exp\left(t\theta_0\left(w\right)\right)\right)}{w^2\left(w^2 + \frac{1}{4}\right)} \right) dw$$

$$= t^{-1} \int_0^{\varepsilon^{-1/Y}} \left(\frac{1 - \exp\left(t\psi\left(w\right)\right)}{\left(w^2 + \frac{1}{4}\right)} \right) dw - t^{-1} \int_0^{\varepsilon^{-1/Y}} \left(\frac{1 - \exp\left(t\theta_0\left(w\right)\right)}{w^2} \right) dw$$

$$= A_{21}\left(t,\varepsilon\right) - A_{22}\left(t,\varepsilon\right). \tag{5.38}$$

First, we estimate the integrand of A_{22} as

$$\left| \frac{1}{t}\Re\left(\frac{1 - \exp\left(t\theta_0\left(w\right)\right)}{w^2} \right) \right| \leq \frac{1}{t}\frac{\left|1 - \exp\left(t\theta_0\left(w\right)\right)\right|}{w^2}$$

$$\leq \frac{2\left|\theta_0\left(w\right)\right|}{w^2}$$

$$= \frac{2\eta w^Y}{w^2}, \tag{5.39}$$

where $\eta > 0$ and (5.39) is in $L^1\left[0,\varepsilon^{-1/Y}\right]$. Noting that, for every $v \geq 0$,

$$\lim_{t\to 0} \frac{1 - \exp\left(t\theta_0\left(w\right)\right)}{t} = -\theta_0\left(w\right),$$

Lebesgue's Dominated Convergence Theorem gives

$$\lim_{t\to 0} \frac{A_{22}\left(t,\varepsilon\right)}{t^{1-1/Y}} = -\int_0^{\varepsilon^{-1/Y}} \frac{\theta_0\left(w\right)}{w^2} dw.$$

We can apply a similar argument to A_{21} with the one exception being that our bounding function is now

$$\frac{2\left|\psi\left(w\right)\right|}{w^2 + \frac{1}{4}},$$

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where $|\psi|$ and $1/\left(w^2+1/4\right)$ are bounded on $\left[0, \varepsilon^{-1/Y}\right]$. Again, using Lebesgue's Dominated Convergence Theorem and recombining the results gives

$$\lim_{t\to 0} \frac{A_2\left(t,\varepsilon\right)}{t^{1-1/Y}} = \int_0^{\varepsilon^{-1/Y}} \left(\frac{\left(w^2+\frac{1}{4}\right)\theta_0\left(w\right) - w^2\psi\left(w\right)}{w^2\left(w^2+\frac{1}{4}\right)}\right) dw.$$

In the case of A_1, we are only able to show the expansion holds heuristically. After showing this, we discuss why making the expansion of A_1 precise is so difficult and suggest possible remedies for this difficulty.

We compute

$$
\begin{aligned}
A_1\left(t,\varepsilon\right) &= \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta\left(t,v\right)\right)}{v^2 + \frac{1}{4}t^{2/Y}} dv - \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta_0\left(v\right)\right)}{v^2} dv \\
&= \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta\left(t,v\right)\right)}{v^2} \left(\frac{1}{1+\frac{1}{4}\left(\frac{t}{v^Y}\right)^{2/Y}}\right) dv - \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta_0\left(v\right)\right)}{v^2} dv \\
&= \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta\left(t,v\right)\right)}{v^2} \left(1 - \frac{1}{4}\left(\frac{t}{v^Y}\right)^{2/Y} + D\left(t,v\right)\right) dv \\
&\quad - \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta_0\left(v\right)\right)}{v^2} dv,
\end{aligned}
$$

where D is the error of the estimation and so $D(t,v) = O\left(\left(t/v^Y\right)^{4/Y}\right)$, as $t/v^Y \to 0$. Simplifying, we write

$$
\begin{aligned}
A_1\left(t,\varepsilon\right) &= \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \left(\frac{\exp\left(\theta_0\left(v\right)\right) - \exp\left(\theta\left(t,v\right)\right)}{v^2}\right) dv \\
&\quad - \frac{1}{4}t^{2/Y} \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta\left(t,v\right)\right)}{v^4} dv + \int_{\left(t/\varepsilon\right)^{1/Y}}^{\infty} \frac{1-\exp\left(\theta\left(t,v\right)\right)}{v^2} D\left(t,v\right) dv \\
&= A_{11}\left(t,\varepsilon\right) + A_{12}\left(t,\varepsilon\right) + A_{13}\left(t,\varepsilon\right).
\end{aligned}
\tag{5.40}
$$

We begin by proving precisely that the error term A_{13} can be safely ignored. To this end, we verify

$$\limsup_{t\to 0} \left|\frac{\Re(A_{13}\left(t,\varepsilon\right))}{t^{1-1/Y}}\right| = 0. \tag{5.41}$$

Again, the substitution $v = t^{1/Y} w$ leads to

$$
\begin{aligned}
\left| \frac{\Re(A_{13}(t, \varepsilon))}{t^{1-1/Y}} \right| &\leq \frac{1}{t^{1-1/Y}} \int_{(t/\varepsilon)^{1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(\theta(t, v)))}{v^2} D(t, v) \right| dv \\
&\leq \frac{t^{4/Y}}{t^{1-1/Y}} \int_{(t/\varepsilon)^{1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(\theta(t, v)))}{v^2 v^4} \right| dv \\
&= t^{5/Y - 1} \int_{(t/\varepsilon)^{1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(\theta(t, v)))}{v^6} \right| dv \\
&\leq t^{5/Y - 1} \int_{(t/\varepsilon)^{1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(\theta(t, v)))}{v^2} \right| dv \\
&= t^{5/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(\theta(t, t^{1/Y} w)))}{t^{2/Y} w^2} \right| t^{1/Y} dw \\
&= t^{4/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \left| \frac{1 - \Re(\exp(t\psi(w)))}{w^2} \right| dw \\
&\leq t^{4/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{1 + |\Re(\exp(t\psi(w)))|}{w^2} dw \\
&\leq t^{4/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{1 + \exp(t\Re(\psi(w)))}{w^2} dw \\
&\leq t^{4/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{\eta(\varepsilon)}{w^2} dw, \quad\quad\quad\quad (5.42)
\end{aligned}
$$

where $\eta(\varepsilon) > 0$ possibly depends on ε and the integral in (5.42) is finite. Noting that $4/Y - 1 \in (1, 3)$, we find

$$
\limsup_{t \to 0} \left| \frac{\Re(A_{13}(t, \varepsilon))}{t^{1-1/Y}} \right| \leq \limsup_{t \to 0} t^{4/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{\eta(\varepsilon)}{w^2} dw
$$

$$
= 0. \quad\quad\quad\quad (5.43)
$$

Next, we consider A_{11} and A_{12}, albeit only formally. For A_{12},

$$
\begin{aligned}
\frac{A_{12}(t, \varepsilon)}{t^{1-1/Y}} &= -\frac{1}{4} t^{3/Y - 1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{1 - \exp(\theta(t, t^{1/Y} w))}{t^{4/Y} w^4} t^{1/Y} dw \\
&= -\frac{1}{4} t^{-1} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{1 - \exp(t\psi(w))}{w^4} dw. \quad\quad\quad\quad (5.44)
\end{aligned}
$$

Now, when taking the limit in (5.44), if we can exchange the limit and integral, then (5.44) becomes

$$
\lim_{t \to 0} \frac{\Re(A_{12}(t, \varepsilon))}{t^{1-1/Y}} = -\frac{1}{4} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{\Re(\psi(w))}{w^4} dw. \quad\quad\quad\quad (5.45)
$$

Observe that (5.45) is well-defined as $|\Re(\psi(w))| \leq 2C\Gamma(-Y)|\cos(Y\pi/2)|w^Y$, for all w large enough.

The jump from (5.44) to (5.45) is difficult to make precise due to the lack of uniform bounds in w and t when w is large and t is small for

$$\frac{|1 - \exp(t\psi(w))|}{t}, \tag{5.46}$$

or

$$\frac{|1 - \Re(\exp(t\psi(w)))|}{t}. \tag{5.47}$$

Another complication is that

$$\Re(\exp(t\psi(w))), \tag{5.48}$$

is more difficult to work with than the real part of ψ. Indeed, (5.48) depends on the imaginary part of ψ and has closed form expression

$$\Re(\exp(t\psi(w))) = \exp(t\Re(\psi(w)))\cos(t\Im(\psi(w))), \tag{5.49}$$

where the real part in the exponent is given by (5.21) and

$$\Im(\psi(w)) = \tilde{b}w + C\Gamma(-Y)\left[\left(\widetilde{M}^2 + w^2\right)^{\frac{Y}{2}}\sin\left(Y\arctan\left(-\frac{w}{\widetilde{M}}\right)\right)\right.$$
$$\left. + \left(\widetilde{G}^2 + w^2\right)^{\frac{Y}{2}}\sin\left(Y\arctan\left(\frac{w}{\widetilde{G}}\right)\right)\right]. \tag{5.50}$$

Finally, we turn our attention to the most difficult term A_{11}. Using the same substitution as in A_2,

$$\frac{A_{11}(t,\varepsilon)}{t^{1-1/Y}} = \frac{1}{t^{1-1/Y}}\int_{(t/\varepsilon)^{1/Y}}^{\infty}\left(\frac{\exp(\theta_0(v)) - \exp(\theta(t,v))}{v^2}\right)dv$$
$$= \frac{1}{t^{1-1/Y}}\int_{\varepsilon^{-1/Y}}^{\infty}\left(\frac{\exp(\theta_0(t^{1/Y}w)) - \exp(\theta(t,t^{1/Y}w))}{t^{2/Y}w^2}\right)t^{1/Y}dw$$
$$= t^{-1}\int_{\varepsilon^{-1/Y}}^{\infty}\left(\frac{\exp(t\theta_0(w)) - \exp(t\psi(w))}{w^2}\right)dw. \tag{5.51}$$

As before, when taking the limit in (5.51), if we were able to interchange the limit and the integral, then (5.51) would have the formal representation

$$\lim_{t \to 0} \frac{\Re(A_{11}(t, \varepsilon))}{t^{1-1/Y}} = \int_{\varepsilon^{-1/Y}}^{\infty} \frac{\Re(\theta_0(w) - \psi(w))}{w^2} dw. \tag{5.52}$$

It is not immediately clear that (5.52) is well-defined. In fact, a cursory glance at the integral indicates that the numerator acts like w^Y for large w and so the integral is potentially not well-defined. Luckily, this is not the case, and there is enough cancellation of the leading orders as w gets large to give that the numerator is of order w^{Y-1}. Thus, the integral is well-defined since $Y - 3 \in (-2, -1)$, as our next proposition indicates. $\qquad \square$

Proposition 5.11. *There exists $w_0 > 0$ and $\eta(w_0) > 0$ such that for all $w > w_0$*

$$|\Re(\theta_0(w) - \psi(w))| \leq \eta w^{Y-1}. \tag{5.53}$$

Proof. Instead of considering the whole expression $\Re(\theta_0(v) - \psi(w))$, we consider

$$\Re\left((B - iu)^Y - (-iu)^Y\right). \tag{5.54}$$

By showing that the absolute value of (5.54) is bounded above by a constant times w^{Y-1}, we will have proved the result since both terms in the original expression are of the form (5.54). In particular, we show

$$\left| \frac{\Re\left((B - iu)^Y - (-iu)^Y\right)}{u^{Y-1}} \right| \leq \eta, \tag{5.55}$$

where $\eta > 0$.

First,

$$\Re\left((-iu)^Y\right) = \Re\left(e^{-i\frac{Y\pi}{2}} u^Y\right)$$
$$= u^Y \cos\left(\frac{Y\pi}{2}\right), \tag{5.56}$$

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and

$$\Re\left((B - iu)^Y\right) = \left(B^2 + u^2\right)^{Y/2} \cos\left(Y \arctan\left(\frac{-u}{B}\right)\right)$$

$$= u^Y \left(\frac{B^2}{u^2} + 1\right)^{Y/2} \cos\left(Y \left(\frac{\pi}{2} - \arctan\frac{-B}{u}\right)\right). \tag{5.57}$$

Evaluating the left-hand side of (5.55) using (5.56) and (5.57) gives

$$g(u) := \frac{\Re\left((B - iu)^Y - (-iu)^Y\right)}{u^{Y-1}}$$

$$= u\left(\left(\frac{B^2}{u^2} + 1\right)^{Y/2} \cos\left(Y\left(\frac{\pi}{2} - \arctan\frac{-B}{u}\right)\right) - \cos\left(\frac{Y\pi}{2}\right)\right). \tag{5.58}$$

To estimate the behavior of the function g as $u \to \infty$, make the substitution $u = 1/v$ and evaluate how g behaves as $v \to 0$. Continuing,

$$g(v) = \frac{\left(B^2 v^2 + 1\right)^{Y/2} \cos\left(Y\left(\frac{\pi}{2} - \arctan(-Bv)\right)\right) - \cos\left(\frac{Y\pi}{2}\right)}{v}, \tag{5.59}$$

and expanding (5.59) around $v = 0$ gives

$$g(v) = -BY \sin\left(\frac{Y\pi}{2}\right) + o(1). \tag{5.60}$$

Thus, g behaves like a constant plus a term that goes to 0 as $v \to 0$, or rather $u \to \infty$. Taking absolute values gives (5.55). $\qquad\square$

Remark 5.12. At first glance, it might seem odd that the coefficient has some dependence on ε. While atypical, the dependence of coefficients on a parameter was observed in a similar setting in [26]. In particular, consider Remark 3.3 and the work in the appendix there. We would like to take the limit as $\varepsilon \to 0$. Indeed, in the limit, the coefficient $d_2(\varepsilon)$ is well-defined. The first integral in (5.31) satisfies

$$\frac{1}{\pi} \int_{\varepsilon^{-1/Y}}^{\infty} \frac{w^2\left(\theta_0(w) - \Re(\psi(w))\right) - \frac{1}{4}\Re(\psi(w))}{w^4} dw \to 0,$$

as $\varepsilon \to 0$. The second integral in (5.31) was already shown to be well-defined, the only question remaining is if the integrand is integrable on e.g. $[1, \infty)$. A similar

argument shows that this is true as

$$\frac{1}{\pi} \int_0^{\varepsilon^{-1/Y}} \frac{\left(w^2 + \frac{1}{4}\right) \theta_0\left(w\right) - w^2 \Re(\psi\left(w\right))}{w^2 \left(w^2 + \frac{1}{4}\right)} dw$$

$$\to \frac{1}{\pi} \int_0^{\infty} \frac{\left(w^2 + \frac{1}{4}\right) \theta_0\left(w\right) - w^2 \Re(\psi\left(w\right))}{w^2 \left(w^2 + \frac{1}{4}\right)} dw < \infty, \qquad (5.61)$$

when $\varepsilon \to 0$, since

$$\left| \frac{\left(w^2 + \frac{1}{4}\right) \theta_0\left(w\right) - w^2 \Re(\psi\left(w\right))}{w^2 \left(w^2 + \frac{1}{4}\right)} \right| = \left| \frac{w^2 \left(\theta_0\left(w\right) - \Re(\psi\left(w\right))\right) + \frac{1}{4}\theta_0\left(w\right)}{w^2 \left(w^2 + \frac{1}{4}\right)} \right|$$

$$\leq \frac{w^2 \left|\theta_0\left(w\right) - \Re(\psi\left(w\right))\right| + \frac{1}{4} \left|\theta_0\left(w\right)\right|}{w^4}$$

$$\leq \frac{\eta w^2 w^{Y-1} + \frac{1}{4} w^Y}{w^4}$$

$$= \eta w^{Y-3} + \frac{1}{4} w^{Y-4}, \qquad (5.62)$$

for some $\eta > 0$, and (5.62) is integrable on $[1, \infty)$ as $Y - 3 \in (-2, -1)$.

We now discuss why A_1, and in particular A_{11}, appears to be so difficult to estimate precisely. First, notice that sharp estimation of the integrand of A_{11} is very similar to sharp estimation for A_{12}. In fact, we can split the difference of exponentials in A_2 by considering

$$\frac{1 - \exp\left(t\theta_0\left(w\right)\right)}{t},$$

and

$$\frac{1 - \Re(\exp\left(\psi\left(w\right)\right))}{t},$$

separately. In this way, we see that the estimate for A_{11} not too dissimilar from the one we need for A_{12}; however, the denominator is very important. For A_{12} the denominator is w^4 while for A_{11} the denominator is w^2. Thus, we must have much sharper estimates for the exponential terms in A_{11} as $w \to \infty$ since the decay of $1/w^2$ is much slower than the decay of $1/w^4$.

Second, from the sharpness of the inequalities needed to prove that the right-hand side of (5.52) is well-defined, we can see that any kind of argument involving

Lebesgue's Dominated Convergence Theorem will also need sharp inequalities. It is very likely that the inequalities required will be needed on expressions more akin to (5.47) than to (5.46). The added complexity of the real part of an exponential expression rather than just the real part of the exponent adds difficulty to our estimations.

Finally, we examine how the method of this section might be more widely applicable to general exponential Lévy processes. At first glance, it might appear that, from the very technical nature of our arguments, these methods might not be extensible to more general exponential Lévy processes. However, we do not believe this is the case for a variety of reasons: almost all of the technical arguments that we made were in order to show that integrals were well-defined or to make precise the argument that an integral converged, e.g. in the right-hand side of (5.52) or (5.37). Nevertheless, showing the convergence of A_1 and A_2 formally only required some simple properties of the characteristic function, specifically (5.18) and (5.20). These properties hold more generally for certain exponential Lévy processes. For example, any Lévy process that is in the domain of attraction of a stable random variable will satisfy (5.20).

CHAPTER VI

CONCLUSION

In this manuscript, we studied the small-time asymptotics of at-the-money call-option prices and implied volatility surfaces under exponential Lévy models of two varieties: a subclass of pure-jump Lévy processes that are in the domain of attraction of a stable random variable, both with and without adding an independent Brownian component, and the CGMY process. For the former without Brownian component, we assumed only that the tails of the Lévy measure were regularly varying and satisfied a moment condition, among some other small technical assumptions. For the CGMY process, we applied new techniques to derive the second-order small-time ATM call-price asymptotics.

In Chapters II and III, we reviewed Lévy processes and stable domains of attraction, developing the preliminary results needed for our theorems (e.g. concentration inequalities).

In Chapter IV, we obtained first-order ATM call-price and implied volatility asymptotics for those Lévy processes in the domain of attraction of stable random variables with minor technical restrictions. For those processes without a Brownian component, new first order rates of convergence were uncovered. To this end, we demonstrated that regular variation of the tails of Lévy measures is preserved under certain measure transformations; we proved that, for $(X_t)_{t \geq 0}$ in the subclass of Lévy processes considered, both $(X_t)_{t \geq 0}$ and $\left(e^{X_t}\right)_{t \geq 0}$ are in the domain of attraction of the same stable random variable; and we exhibited the possible orders of convergence of Lévy processes.

When combining the pure-jump Lévy process with an independent Brownian component, we were able to show that the first order ATM call-price asymptotics are still of order $\sqrt{t}$. That is, the Brownian component is still the dominating term in regards to the first-order asymptotic expansion. Moreover, we exhibited this property for a wider class of pure-jump Lévy processes than was previously considered in the literature.

Finally, we considered a model that gives first-order ATM call-price dynamics that have not been shown before, studying an asset model whose first-order call-price asymptotics are of order

$$\left(\frac{t}{\log\left(1/t\right)}\right)^{1/\alpha}$$

where $\alpha \in (1, 2)$.

In Chapter V, we revisited the exponential CGMY model as an asset model. Under this model, we derived the (already known) first and second-order ATM call-price asymptotics in a novel way. Specifically, we corroborated the second-order asymptotics only using the characteristic function of the CGMY process via the Lipton-Lewis formula.

All in all, while general, the extension of first order asymptotics to a wider class of Lévy processes might only be an academic exercise. While interesting, more complicated models would almost certainly not be used in practice for a variety of reasons. First, very short-term options are a small, very illiquid, part of developed capital markets, often considered exotic products (e.g. crash cliquet options which are strips of forward-starting ATM call options with very short expiration, sometimes a single day). Next, given the difficulty of pricing under these exponential Lévy models farther away from expiration, these models could only be used for very short-term pricing (e.g. less than one week to expiration). Finally, even if we specified more explicit models (like the example given in Section 4.5), calibration under these models would likely be difficult and would require the development of new statistical techniques.

The heuristic expansion given in Chapter IV give some hope that formal expansions might exist for further higher orders. Moreover, extending the techniques used for the second-order expansion to a more general process than the CGMY process is very likely to succeed. For example, we could consider the class of tempered stable models discussed in [24].

Finally, there are several directions that future work could follow. We could examine the second-order ATM call-price asymptotics of asset models whose log return structure is as in Chapter IV. Some work is already done along these lines, e.g. [24], where the authors considered generalized tempered stable models; however, there are no second-order ATM call-price asymptotics for processes where the first order rate of convergence is $t^{1/\alpha}\ell(t)$ where $1 < \alpha < 2$ and where ℓ is slowly varying at 0 and nonconstant. The second-order call-price asymptotics for the toy model given in Chapter IV (and other models where the slowly varying part of the Lévy tail is not asymptotically constant) would be a good starting point.

Reiterating what we mentioned previously, the tools and methods used in Chapter V could potentially provide an alternate method for second-order call-price asymptotics of models where measure transformation methods are perhaps infeasible or impossible. We might even be able to extend the results in Chapters IV and V to obtain asymptotics of "close-to-the-money" options (as is considered in [24] and [25]). In this case, instead of letting time to maturity go to 0 linearly, we choose some function k_t that goes to 0 at a different rate. Indeed, doing so would require finding an expansion for the full Lipton-Lewis formula given in (5.9) and not the simplified version for ATM options found in (5.10).

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