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Universidade Estadual Paulista

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Type-II Defects in Integrable Classical Field Theories

ALEXIS ROA AGUIRRE

Orientador

Prof. Dr. ABRAHAM HIRSZ ZIMMERMAN

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Resumo

Nesta tese discutimos as propriedades de integrabilidade das teorias de campo clássicas em duas dimensões na presença de descontinuidades ou defeitos tipo-II, principalmente usando a linguagem do formalismo do espalhamento inverso. Um método geral para calcular a função geradora de um conjunto infinito de grandezas conservadas modificadas para qualquer equação de campo integrável é apresentado, uma vez que seus respectivos problemas lineares associados são dados e suas correspondentes matrizes do defeito são calculadas. O método é aplicado no cálculo das contribuições dos defeitos para a energia e o momento para vários modelos e mostramos a relação entre as condições de defeito integráveis e suas respectivas transformações de Bäcklund para cada modelo.

Palavras Chaves: Sistemas Integráveis; Simetrias; Leis de Conservação; Solitons.

Áreas do conhecimento: Teoria de Campos; Física Matemática.

Abstract

In this thesis we discuss the integrability properties of two-dimensional classical field theories in the presence of discontinuities or type-II defects, mainly using the language of the inverse scattering approach. We present a general method to compute the generating function of an infinite set of modified conserved quantities for any integrable field equation given their associated linear problems and computing their corresponding defect matrices. We apply this method to derive in particular defect contributions to the energy and momentum for several models and show the relationship between the integrable defect conditions and the Bäcklund transformations for each model.

Keywords: Integrable systems; Symmetries; Conservation Laws; Solitons.

Areas of knowledge: Field Theory; Mathematical Physics.

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CHAPTER 1

Introduction

This thesis aims to investigate integrable classical field theories in $(1 + 1)$ -dimensional space-time with internal discontinuities which will be referred to as defects. Two dimensional integrable field theories possessing soliton solutions constitute, besides their intrinsic beauty, an interesting laboratory to test ideas and develop exact methods on non-perturbative aspects in many areas of physics. In fact, during the twentieth century the original purpose and implications of the idea of integrability have experienced considerable progress. Besides the construction of solutions for nonlinear evolution equations in two dimensions, a wide variety of modern mathematical methods, such as the dressing method and classical and quantum inverse scattering method, have been developed.

It is well-known that integrable systems are physical systems for which the equations of motion can (in principle) be solved exactly. A remarkable feature of this particular systems is the fact that they possess soliton solutions. These solutions are localized non-linear waves which evolve in time preserving its shape and velocity even after scattering. This kind of particle-like behaviour is directly related with the existence of a sufficient number (infinity in the case of field theories) of conserved quantities or integrals of motion, such that all the relevant quantities of the model can be computed. At this point it is worth pointing out that in this thesis we will use the definition of “integrability” as the existence of both a constructive way of finding solutions and an infinite number of conserved quantities.

In the quantum version of this situation, the existence of an infinite number of conserved charges that are mutually commuting constrain severely the scattering processes implying in the factorizability of the S-matrix [1]. Such theories are commonly called exactly soluble. Indeed, the study of integrable field theories on restricted domains was firstly explored within the quantum framework by Ghoshal and Zamolodchikov in [2], where they considered the sine-Gordon theory defined on the half-line $(-\infty, 0)$. They found that the integrability of

the bulk theory, that means those which are defined on the infinite real line $(-\infty, \infty)$, could be preserved in the boundary theory by choosing carefully the boundary condition.

After that, there was also a great interest in the situation of integrable models in the presence of defects (or impurities). At this point, a defect is introduced as an internal boundary condition linking a field theory in the left region ($x < 0$) with a (not necessarily the same) field theory in the right region by a set of suitable defect conditions at the position of the defect $x = 0$. In a pioneering work by Delfino et.al.[3, 4], it was pointing out that the factorization of the S-matrix with non-trivial scattering are incompatible with both reflection and transmission. In fact, some years later it was shown in the case of sine-Gordon model that this compatibility requires the defect to be purely transmitting [5]. Another point of view of this situation was also provided recently in [6]. Despite the study of these quantum aspects related to the presence of defects into integrable models deserves special attention, in this thesis we will focus mainly on the classical aspects of the integrable defects.

From a Lagrangian point of view, it was noticed some years ago that several integrable field theories can accommodate defects without spoil the integrability properties of the bulk theory [7, 8]. In this framework, the usual variational principle from a local Lagrangian density located at some fixed point, reveals frozen Bäcklund transformations [9] as the defect conditions for the fields. In addition, it turns out that these kind of defect conditions allow not only the conservation of the energy but also the conservation of the momentum which have been suitable modified after including defect contributions. Moreover, their integrability is provided by the existence of a modified Lax pair involving a limit procedure, but in general it was only checked explicitly for a few conserved charges. As a novel feature of most of these models is that only physical fields, namely the fields present in the original bulk Lagrangian density, were present in the defect description and therefore they were called type-I defects [10]. However, it was noticed that not all the possible relativistic integrable models could be accommodate within this framework and then it was proposed a generalization by allowing a defect to have its own degree of freedom, and after that they are called type-II defects [10]. Many examples were also discussed in [10] like sine/sinh-Gordon, Liouville, massive free field, and the $a_2^{(2)}$ affine Toda model also known as Tzitzéica, Bullogh-Dodd or Zhiber-Shabat-Mikhailov equation. In supersymmetric extensions of sine-Gordon model [11, 12] those auxiliary boundary fields, which correspond to the degree of freedom of the defect itself, also appear naturally.

In spite of the success of showing the conservation of the energy and momentum in all of these models, the corresponding Lax pair approach for describing the type-II defects was necessary. In order to make progress and fill this gap, in this thesis we provided the Lax formalism where the type-II defect conditions corresponding to frozen type-II Bäcklund transformations are encoded in a well-known defect matrix [13]. This matrix provided an elegant way to compute the modified conserved quantities, ensuring integrability of the defect theories. In particular, we will present the type-II defect matrices for the sine-Gordon, Tzitzéica-Bullough-Dodd[14], and for the massive Thirring models [15], which constitute probably one of the most important contributions of this thesis to the program of integrable defects.

This thesis will be organized as follows:

- In the next chapter, we will introduce the main ideas of type-II defects in integrable field theories. Firstly, by using a Lagrangian description we will derive some conditions that significantly constrain the form of the defect potentials. Subsequently, the Lax approach is used to construct an infinite set of conservation laws in the bulk theory and then these results are showed to be useful to derive a general formula for computing the defect contributions to the modified conserved quantities for any given $m \times m$ associated linear problem.
- In chapter 3, we will use the tools introduced in chapter 2 in the analysis of the sine-Gordon (sG) model with type-I and type-II defects. We derive the defect matrices for the model and then compute the respective defect contributions to the energy and momentum.
- In chapter 4, we present the analysis of the Tzitzéica-Bullough-Dodd (TBD) model with type-II defects. We derive the defect matrix and recovered the Bäcklund transformation derived before by Lagrangian methods. Consequently, we establish the integrability of the defect theory by computing explicitly the corresponding modified conserved energy and momentum.
- In chapter 5, we will discuss the integrability properties of the Grassmannian massive Thirring (GMT) model in the presence of the type-II defects, which are related with its Bäcklund transformation. Firstly, we present some basic aspects of the bulk theory and construct the conservation laws using the method proposed in chapter 3. Then, we provide the local Lagrangian density for describing the defect theory and derive the defect conditions preserving the modified momentum. Finally, we construct the defect matrix to ensure that the defects does not spoil the integrability.
- In chapter 6, we will present the bosonic version of the massive Thirring model in the presence of the type-II defects. The motivation for this model is the possibility of study the interaction of Thirring solitons with defect. We will apply the inverse scattering method to derive its defect matrix. From this, we derive the Bäcklund transformation for the Bosonic massive Thirring (BMT), which seems not to have been reported elsewhere in the literature before [14]. The integrability of defect model is ensured by computing explicitly the defect contributions to the modified energy and momentum as well.
- In the last chapter, we conclude with some final remarks and comments on future directions which emerged from the work contained here.
- In the appendices, we have collected useful information of the algebraic content used in thesis and we present sketchily boundary theories starting from the defect theories.
- The bibliography presented in this thesis helps to provide reference to some essential works and is by no means rigorously complete.

CHAPTER 2

Integrable defects

In this chapter we introduce the basics ideas of defects in integrable field theories. Systems with defects are in some sense more realistic and play an important role of any physical theory. Usually the introduction of this kind of discontinuities spoils the integrability or solvability of the models in the bulk. For example, introducing δ -impurities in classical integrable field theories may have interesting effects in the behaviour of solitons but as a consequence the integrability is lost [16].

However, some years ago it was pointing out that several integrable field theories permit defects which are able to preserve the property of classical integrability from a Lagrangian point of view. In this description, models like free fields, sine/sinh-Gordon, Liouville models and $a_n^{(1)}$ were considered and the conserved momentum required for the integrability of the whole system were found [7, 8]. In addition, the defect conditions emerged naturally as the Bäcklund transformation of the model located at the defect position, and then they were named type-I defects. Despite the success of this approach, it was noticed that not all the possible relativistic integrable models could be accommodate within this framework and then it was proposed a generalization by allowing a defect to have its own degree of freedom, and after that they were named type-II defects [10].

It is the purpose of this chapter to show precisely that the classical integrability can be preserved after introduction of type-II defects, by deriving firstly some constraints that determine the explicit form of the defect potentials. Subsequently, the Lax approach is used to construct an infinite set of conservation laws in the bulk theory and then these results are showed to be useful to derive a general formula for computing the defect contributions to the modified conserved quantities for all orders.

2.1 General setting

We will start our discussion on integrable defects from the Lagrangian point of view following the main ideas of the pioneering work [7]. The defect is introduced as an internal boundary in a selected point of the x -axis, say $x_0 = 0$, linking a field theory defined in the region $x < 0$ with a (not necessarily the same) field theory in the region $x > 0$,

$$\begin{array}{c} \text{---} \bullet \text{---} \\ -\infty \qquad \Phi_1(x, t) \qquad x = 0 \qquad \Phi_2(x, t) \qquad +\infty \end{array}$$

where the fields on either side of the defect can interact to each other by a set of defect conditions given at $x = 0$. This kind of study was also performed in early works [17], however in different contexts.

The starting point for the whole construction is to define a Lagrangian density for a general theory $\Phi_1 = (\phi_1, \psi_1, \bar{\psi}_1)$ in the region $x < 0$ describing bosonic ϕ and fermionic* $\psi, \bar{\psi}$ fields, and correspondingly $\Phi_2 = (\phi_2, \psi_2, \bar{\psi}_2)$ in the region $x > 0$, and a contribution in $x = 0$ describing the defect,

$$\mathcal{L} = \theta(-x)\mathcal{L}_1 + \theta(x)\mathcal{L}_2 + \delta(x)\mathcal{L}_D, \quad (2.1)$$

with

$$\mathcal{L}_p = \frac{1}{2} (\partial_x \phi_p)^2 - \frac{1}{2} (\partial_t \phi_p)^2 + \bar{\psi}_p (\partial_t - \partial_x) \bar{\psi}_p + \psi_p (\partial_t + \partial_x) \psi_p + V_p(\phi_p) + W_p(\phi_p, \psi_p, \bar{\psi}_p), \quad (2.2)$$

$$\begin{aligned} \mathcal{L}_D = & \frac{1}{2} (\phi_2 \partial_t \phi_1 - \phi_1 \partial_t \phi_2) - \Lambda \partial_t (\phi_1 - \phi_2) + \partial_t \Lambda (\phi_1 - \phi_2) + B_0(\phi_1, \phi_2, \Lambda) - \psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2 \\ & + 2f \partial_t f + B_1(\phi_1, \phi_2, \Lambda, \psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2, f), \end{aligned} \quad (2.3)$$

where $p = 1, 2$ and $\Lambda(t)$ and $f(t)$ are two bosonic and fermionic auxiliary fields respectively. The bulk potential are given by V_p and W_p , and B_0 and B_1 are the corresponding defect potentials which depend on the bulk fields valued at the point $x = 0$. The auxiliary fields play the role of Lagrange multipliers and the defects potentials also depends on them. For the sake of classification, the case without extra degree of freedom received the name of type-I defects ($\Lambda = 0 = f$), otherwise we will call it type-II defects.

Besides the standard fields equations in the bulk regions[†],

$$\begin{aligned} \partial_t^2 \phi_p - \partial_x^2 \phi_p &= -\frac{\partial V_p}{\partial \phi_p} - \frac{\partial W_p}{\partial \phi_p}, \\ (\partial_t + \partial_x) \psi_p &= -\frac{1}{2} \frac{\partial W_p}{\partial \psi_p}, \\ (\partial_t - \partial_x) \bar{\psi}_p &= -\frac{1}{2} \frac{\partial W_p}{\partial \bar{\psi}_p}, \quad p = 1, 2 \end{aligned} \quad (2.4)$$

*In this presentation we consider Majorana fields, however in chapter 5 we will deal with Dirac fields.

[†]Where by notation fermionic derivatives act on the left.

we have the defect conditions at $x = 0$,

$$\partial_x \phi_1 - \partial_t \phi_2 + 2\partial_t \Lambda = -\frac{\partial}{\partial \phi_1} (B_0 + B_1), \quad (2.5)$$

$$\partial_x \phi_2 - \partial_t \phi_1 + 2\partial_t \Lambda = \frac{\partial}{\partial \phi_2} (B_0 + B_1), \quad (2.6)$$

$$\psi_1 + \psi_2 = \frac{\partial B_1}{\partial \psi_1} = -\frac{\partial B_1}{\partial \psi_2}, \quad (2.7)$$

$$\bar{\psi}_1 - \bar{\psi}_2 = -\frac{\partial B_1}{\partial \bar{\psi}_1} = -\frac{\partial B_1}{\partial \bar{\psi}_2}, \quad (2.8)$$

$$\partial_t f = -\frac{1}{4} \frac{\partial B_1}{\partial f}, \quad (2.9)$$

$$\partial_t (\phi_1 - \phi_2) = \frac{1}{2} \frac{\partial}{\partial \Lambda} (B_0 + B_1). \quad (2.10)$$

Notice that these defect conditions are not invariant under a parity transformation, say $\Phi_1 \longleftrightarrow \Phi_2$. On the other hand, since the time translation invariance has not been violated, the total energy is expected to be conserved including a defect contribution. Then, for the energy we have

$$\begin{aligned} E &= \int_{-\infty}^0 dx \left[\frac{1}{2} (\partial_x \phi_1)^2 + \frac{1}{2} (\partial_t \phi_1)^2 - \bar{\psi}_1 \partial_x \bar{\psi}_1 + \psi_1 \partial_x \psi_1 + V_1 + W_1 \right] \\ &+ \int_0^{\infty} dx \left[\frac{1}{2} (\partial_x \phi_2)^2 + \frac{1}{2} (\partial_t \phi_2)^2 - \bar{\psi}_2 \partial_x \bar{\psi}_2 + \psi_2 \partial_x \psi_2 + V_2 + W_2 \right]. \end{aligned} \quad (2.11)$$

Now by taking its time-derivative, we get

$$\frac{dE}{dt} = \left[(\partial_x \phi_1) (\partial_t \phi_1) + \psi_1 \partial_t \psi_1 - \bar{\psi}_1 \partial_t \bar{\psi}_1 - (\partial_x \phi_2) (\partial_t \phi_2) - \psi_2 \partial_t \psi_2 + \bar{\psi}_2 \partial_t \bar{\psi}_2 \right]_{x=0}, \quad (2.12)$$

and using the boundary conditions (2.5)–(2.10) we easily find that the modified conserved quantity includes a defect contribution given by the following combination,

$$\mathcal{E} = E + \left[(B_0 + B_1) - \bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2 \right]_{x=0}. \quad (2.13)$$

without any constraints on the form of the defect potentials B_0 and B_1 . However, from the above Lagrangian density it is clear that the presence of the defect breaks explicitly the bulk translational invariance and in principle the conservation of momentum should be violated. An interesting fact is that it is also possible to add a defect term to the bulk momentum in order to conserve it by a suitable choice of the defect potentials B_0 and B_1 . To do that, let us consider the time-derivative of the momentum,

$$\frac{dP}{dt} = \frac{d}{dt} \left[\int_{-\infty}^0 dx (\partial_t \phi_1 \partial_x \phi_1 - \bar{\psi}_1 \partial_x \bar{\psi}_1 - \psi_1 \partial_x \psi_1) + \int_0^{+\infty} dx (\partial_t \phi_2 \partial_x \phi_2 - \bar{\psi}_2 \partial_x \bar{\psi}_2 - \psi_2 \partial_x \psi_2) \right] \quad (2.14)$$

and using the field equations, we get

$$\begin{aligned} \frac{dP}{dt} = & \left[\frac{1}{2}(\partial_x \phi_1)^2 + \frac{1}{2}(\partial_t \phi_1)^2 - \bar{\psi}_1 \partial_t \bar{\psi}_1 - \psi_1 \partial_t \psi_1 - V_1 - W_1 \right. \\ & \left. - \frac{1}{2}(\partial_x \phi_2)^2 - \frac{1}{2}(\partial_t \phi_2)^2 + \bar{\psi}_2 \partial_t \bar{\psi}_2 + \psi_2 \partial_t \psi_2 + V_2 + W_2 \right]_{x=0}. \end{aligned} \quad (2.15)$$

Now, by using the defect conditions, we obtain

$$\begin{aligned} \frac{dP}{dt} = & -(\partial_t \phi_2) \frac{\partial}{\partial \phi_1} (B_0 + B_1) - (\partial_t \phi_1) \frac{\partial}{\partial \phi_2} (B_0 + B_1) + \left(\frac{\partial B_0}{\partial \phi_1} \right) \left(\frac{\partial B_1}{\partial \phi_1} \right) - \left(\frac{\partial B_0}{\partial \phi_2} \right) \left(\frac{\partial B_1}{\partial \phi_2} \right) \\ & + 2(\partial_t \Lambda) \left(\frac{\partial}{\partial \phi_1} (B_0 + B_1) + \frac{\partial}{\partial \phi_2} (B_0 + B_1) + \frac{1}{2} \frac{\partial}{\partial \Lambda} (B_0 + B_1) \right) \\ & + \frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] - (\partial_t \bar{\psi}_1) \frac{\partial B_1}{\partial \bar{\psi}_1} - (\partial_t \bar{\psi}_2) \frac{\partial B_1}{\partial \bar{\psi}_2} + (\partial_t \psi_1) \frac{\partial B_1}{\partial \psi_1} + (\partial_t \psi_2) \frac{\partial B_1}{\partial \psi_2} \\ & + (V_2 - V_1) + (W_2 - W_1) + \partial_t (\bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2), \end{aligned} \quad (2.16)$$

where we have assumed that

$$\left(\frac{\partial B_1}{\partial \phi_1} \right)^2 = \left(\frac{\partial B_1}{\partial \phi_2} \right)^2 = 0. \quad (2.17)$$

Analysing the expression (2.16) to obtain the sufficient conditions for the conservation of momentum should be quite complicated, so we can consider some simpler cases.

Firstly, for a purely bosonic type-I defect, (2.16) reduces to,

$$\frac{dP}{dt} = -(\partial_t \phi_2) \frac{\partial B_0}{\partial \phi_1} - (\partial_t \phi_1) \frac{\partial B_0}{\partial \phi_2} + \frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] + (V_2 - V_1), \quad (2.18)$$

and then it is natural to require that the term without time-derivatives vanishes and the other terms should be a total time-derivative leading to the following equations,

$$\frac{\partial^2 B_0}{\partial \phi_1^2} = \frac{\partial^2 B_0}{\partial \phi_2^2}, \quad \left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 = 2(V_1 - V_2), \quad (2.19)$$

which can be solved by the ansatz $B_0 = B_0^+(\phi_+) + B_0^-(\phi_-)$, where $\phi_{\pm} = (\phi_1 \pm \phi_2)/2$. By substituting this form into the second relation, we get

$$\frac{\partial^3 B_0^{\pm}}{\partial \phi_{\pm}^3} = \zeta^2 \frac{\partial B_0^{\pm}}{\partial \phi_{\pm}}. \quad (2.20)$$

This is exactly the constraints for the defect potential originally derived in the pioneering work of Bowcock et.al.[7]. In this case, the modified conserved momentum containing the bulk and defect contributions is given by $\mathcal{P} = P + (B_0^+ - B_0^-)|_{x=0}$. An important point to note is that the possible defect potentials are quite limited by the form of the bulk ones. In particular, the sine/sinh-Gordon[‡], Liouville, massive and massless free fields can be enclosed

[‡]In the next chapter we will discuss this case in more detail.

within this setting. The simplest case would be to take both free and massive bulk theories,

$$V_p = \frac{m^2 \phi_p^2}{2}, \quad B_0 = \frac{m\sigma}{4} (\phi_1 + \phi_2)^2 + \frac{m}{4\sigma} (\phi_1 - \phi_2)^2, \quad (2.21)$$

where σ is a free parameter. Then, we easily find that the modified conserved energy and momentum are given by,

$$\mathcal{E} = E + \frac{m\sigma}{4} (\phi_1 + \phi_2)^2 + \frac{m}{4\sigma} (\phi_1 - \phi_2)^2, \quad (2.22)$$

$$\mathcal{P} = P + \frac{m\sigma}{4} (\phi_1 + \phi_2)^2 - \frac{m}{4\sigma} (\phi_1 - \phi_2)^2. \quad (2.23)$$

Now, for a purely bosonic type-II defect ($\Lambda \neq 0$) the expression (2.16) becomes,

$$\begin{aligned} \frac{dP}{dt} = & -(\partial_t \phi_2) \frac{\partial B_0}{\partial \phi_1} - (\partial_t \phi_1) \frac{\partial B_0}{\partial \phi_2} + 2(\partial_t \Lambda) \left(\frac{\partial B_0}{\partial \phi_1} + \frac{\partial B_0}{\partial \phi_2} + \frac{1}{2} \frac{\partial B_0}{\partial \Lambda} \right) \\ & + \frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] + (V_2 - V_1), \end{aligned} \quad (2.24)$$

so now the last term does not need to vanish, and in fact it can be assumed that exist some function $F_0(\phi_1, \phi_2, \Lambda)$ such that,

$$\frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] + (V_2 - V_1) = \frac{1}{2} F_0(\phi_1, \phi_2, \Lambda) \frac{\partial B_0}{\partial \Lambda}. \quad (2.25)$$

Now, by demanding that the time-derivative of the momentum can be written as a total derivative, requires

$$\frac{\partial^2 B_0}{\partial \phi_+ \partial \phi_-} = -\frac{\partial F_0}{\partial \phi_+}, \quad \frac{\partial^2 B_0}{\partial \phi_+ \partial \Lambda} = -\frac{\partial^2 B_0}{\partial \phi_+^2}, \quad \frac{\partial F_0}{\partial \Lambda} = -\frac{\partial F_0}{\partial \phi_+}, \quad (2.26)$$

from where it is immediately concluded that,

$$B_0 = \Pi_0(\phi_+ - \Lambda, \phi_-) + \Xi_0(\phi_-, \Lambda), \quad F_0(\phi_+, \phi_-, \Lambda) = -\frac{\partial \Pi_0}{\partial \phi_-}, \quad (2.27)$$

where the function Π_0 depends on ϕ_- and $\phi_+ - \Lambda$, Ξ_0 depends on ϕ_- and Λ , and they satisfy the following relation,

$$\left(\frac{\partial \Pi_0}{\partial \phi_-} \right) \left(\frac{\partial \Xi_0}{\partial \Lambda} \right) - \left(\frac{\partial \Pi_0}{\partial \Lambda} \right) \left(\frac{\partial \Xi_0}{\partial \phi_-} \right) = 2(V_1 - V_2), \quad (2.28)$$

where, the left-hand side has the form of a Poisson bracket of Π_0 and Ξ_0 in terms of the conjugate variables (ϕ_-, Λ) . Of course the corresponding modified conserved momentum can be written as $\mathcal{P} = P + (\Pi_0 - \Xi_0)$. The relation (2.28) imposes strong constraints on the form

of the Π_0 , Ξ_0 and V_1, V_2 as well, because all the Λ -dependence contained in the left-hand side must be totally cancel out. It was also shown in [10] that these type-II defects can be enclosed within the sine/sinh-Gordon, Liouville, massive and massless free fields, and in general the untwisted $a_n^{(1)}$ [18] and the twisted $a_2^{(2)\S}$ affine Toda models. For example, an adequate choice for the massive free fields is,

$$\Pi_0 = m \left(\frac{(\phi_+ - \Lambda)^2}{\alpha} + \beta \phi_-^2 \right), \quad \Xi_0 = m \left(\frac{\Lambda^2}{\beta} + \alpha \phi_-^2 \right), \quad (2.29)$$

where α and β are two free parameters. Notice, the type-I defect can be recovered as a limit case by eliminating the auxiliary field Λ .

In [11] it was explored the case of a general theory with bosonic and fermionic case for $\Lambda = 0$ but $f \neq 0$. In this case (2.16) takes the form

$$\begin{aligned} \frac{dP}{dt} = & -(\partial_t \phi_2) \frac{\partial}{\partial \phi_1} (B_0 + B_1) - (\partial_t \phi_1) \frac{\partial}{\partial \phi_2} (B_0 + B_1) + \left(\frac{\partial B_0}{\partial \phi_1} \right) \left(\frac{\partial B_1}{\partial \phi_1} \right) - \left(\frac{\partial B_0}{\partial \phi_2} \right) \left(\frac{\partial B_1}{\partial \phi_2} \right) \\ & + \frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] - (\partial_t \bar{\psi}_1) \frac{\partial B_1}{\partial \bar{\psi}_1} - (\partial_t \bar{\psi}_2) \frac{\partial B_1}{\partial \bar{\psi}_2} + (\partial_t \psi_1) \frac{\partial B_1}{\partial \psi_1} + (\partial_t \psi_2) \frac{\partial B_1}{\partial \psi_2} \\ & + (V_2 - V_1) + (W_2 - W_1) + \partial_t (\bar{\psi}_1 \bar{\psi}_2 - \psi_1 \psi_2). \end{aligned} \quad (2.30)$$

Following the same reasoning and assuming that the following conditions

$$\begin{aligned} \frac{\partial^2 B_0}{\partial \phi_+ \partial \phi_-} = 0, \quad \frac{\partial^2 B_1}{\partial \phi_+ \partial \phi_-} = 0, \quad \frac{\partial^2 B_1}{\partial \bar{\psi}_+ \partial \psi_-} = 0, \\ \left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 = 2(V_1 - V_2), \\ \left(\frac{\partial B_1^+}{\partial f} \right) \left(\frac{\partial B_1^-}{\partial f} \right) + \left(\frac{\partial B_0^+}{\partial \phi_+} \right) \left(\frac{\partial B_1^-}{\partial \phi_-} \right) + \left(\frac{\partial B_0^-}{\partial \phi_-} \right) \left(\frac{\partial B_1^+}{\partial \phi_+} \right) = 2(W_1 - W_2), \end{aligned} \quad (2.31)$$

are satisfied, we then find that the modified conserved momentum is given by the combination,

$$\mathcal{P} = P + [(B_0^+ - B_0^-) + (B_1^+ - B_1^-) + \psi_1 \psi_2 - \bar{\psi}_1 \bar{\psi}_2], \quad (2.32)$$

where it has been used the decompositions

$$B = B_0^+(\phi_+) + B_0^-(\phi_-), \quad B_1 = B_1^+(\phi_+, \bar{\psi}_+, f) + B_1^-(\phi_-, \psi_-, f) \quad (2.33)$$

^{\S}This model is also known as Tzitzéica, Bullough-Dodd or Zhiber-Shabat-Mikhailov model, and it will be studied in more detail in chapter 4.

after introducing new variables $\phi_{\pm} = (\phi_1 \pm \phi_2)/2$, $\psi_{\pm} = (\psi_1 \pm \psi_2)/2$, and $\bar{\psi}_{\pm} = (\bar{\psi}_1 \pm \bar{\psi}_2)/2$. The $N = 1$ super sinh-Gordon model fits into this scheme given the following potentials,

$$V_p = 4m^2 \cosh(2\phi_p), \quad W_p = 8m\bar{\psi}_p\psi_p \cosh(\phi_p), \quad (2.34)$$

$$B_0 = 2m \left[\sigma \cosh(2\phi_+) + \frac{1}{\sigma} \cosh(2\phi_-) \right], \quad (2.35)$$

$$B_1 = 4\sqrt{2m} \left[\sqrt{\sigma} \bar{\psi}_+ \cosh \phi_+ + \frac{1}{\sqrt{\sigma}} \psi_- \cosh \phi_- \right] f. \quad (2.36)$$

Now, returning to the general case in (2.16) we now propose the following ansatz for the decomposition of the defect potentials,

$$B_0 = \Pi_0(\phi_+ - \Lambda, \phi_-) + \Xi_0(\phi_-, \Lambda), \quad (2.37)$$

$$B_1 = B_1^+(\phi_+ - \Lambda, \bar{\psi}_+, f) + B_1^-(\phi_- + \Lambda, \psi_-, f), \quad (2.38)$$

such that

$$\frac{1}{2} \left[\left(\frac{\partial B_0}{\partial \phi_1} \right)^2 - \left(\frac{\partial B_0}{\partial \phi_2} \right)^2 \right] + (V_2 - V_1) = -\frac{1}{2} \left(\frac{\partial \Pi_0}{\partial \phi_-} \right) \left(\frac{\partial B_0}{\partial \Lambda} \right). \quad (2.39)$$

Then, after some manipulations we found that the modified conserved momentum can be written as,

$$\mathcal{P} = P + [(\Pi_0 - \Xi_0) + (B_1^+ - B_1^-) + \psi_1\psi_2 - \bar{\psi}_1\bar{\psi}_2] \Big|_{x=0}, \quad (2.40)$$

if the following set of conditions are satisfied,

$$\begin{aligned} \frac{\partial^2 B_1}{\partial \phi_+ \partial \phi_-} &= 0, & \frac{\partial^2 B_1}{\partial \bar{\psi}_+ \partial \psi_-} &= 0, \\ \left(\frac{\partial \Pi_0}{\partial \phi_-} \right) \left(\frac{\partial \Xi_0}{\partial \Lambda} \right) - \left(\frac{\partial \Pi_0}{\partial \Lambda} \right) \left(\frac{\partial \Xi_0}{\partial \phi_-} \right) &= 2(V_1 - V_2), \\ \left(\frac{\partial B_1^+}{\partial f} \right) \left(\frac{\partial B_1^-}{\partial f} \right) + \left(\frac{\partial \Pi_0}{\partial \phi_+} \right) \left(\frac{\partial B_1^-}{\partial \phi_-} \right) + \left(\frac{\partial B_0}{\partial \phi_-} \right) \left(\frac{\partial B_1^+}{\partial \phi_+} \right) &= 2(W_1 - W_2). \end{aligned} \quad (2.41)$$

It is worth noting that the general set of conditions (2.41) represents a generalized framework for determining defects potentials. Despite most of the known models that allow type-I and type-II defects preserving the energy and momentum can be derived from the above setting, this does not exhaust all possibilities. Here, we had only consider one bosonic and one fermionic extra degree of freedom, but there is no reason in principle for not considering more than one of such fields. The generalisation to multicomponent fields is rather straightforward.

In addition, there is no a supersymmetric relation between the fields Λ and f , even though it should be interesting to investigate possible supersymmetric extensions of integrable models with these type-II defects. Finally, it is worth pointing out that the conservation of the momentum is a sufficient condition to determine the explicit form of the defect potentials, but is not sufficient to establish integrability of the theory. For such a claim we need to construct an infinite number of modified conserved quantity and it is done in the next section using the Lax approach.

2.2 Lax representation and conservation laws

In this section we will discuss one of the most important modern topics in mathematical physics, namely the Lax formulation. Historically, this approach has been extremely useful in order to construct infinite set of independent conserved quantities for some integrable evolution equations. Such equations can be formulated as a compatibility condition of an associated linear auxiliary problem as follows,

$$\partial_t \Psi(x, t; \lambda) = V(x, t; \lambda) \Psi(x, t; \lambda), \quad (2.42)$$

$$\partial_x \Psi(x, t; \lambda) = U(x, t; \lambda) \Psi(x, t; \lambda), \quad (2.43)$$

where $\Psi(x, t; \lambda)$ is in general an m -dimensional vector, λ is a spectral parameter, and $U(x, t; \lambda), V(x, t; \lambda)$ are $(m \times m)$ matrices, which usually are named Lax pair or Lax connections. Then, from the compatibility condition

$$(\partial_x \partial_t - \partial_t \partial_x) \Psi(x, t; \lambda) = 0, \quad (2.44)$$

we obtain the zero-curvature condition or Lax-Zakharov-Shabat equation,

$$\partial_t U - \partial_x V + [U, V] = 0, \quad (2.45)$$

which gives the corresponding equations of motion for the integrable model. Now, let us show how to construct a generating function for the infinite set of conservation laws. Firstly, for every auxiliary field component Ψ_j with $j = 1, \dots, m$, we can define a set of $(m - 1)$ auxiliary functions $\Gamma_{ij} = \Psi_i \Psi_j^{-1}$ with $i \neq j$. Then, considering the linear system (2.42) and (2.43), it is not so difficult to identify the j -th conservation equation,

$$\partial_t \left[U_{jj} + \sum_{i \neq j} U_{ji} \Gamma_{ij} \right] = \partial_x \left[V_{jj} + \sum_{i \neq j} V_{ji} \Gamma_{ij} \right], \quad (2.46)$$

where each auxiliary functions Γ_{ij} satisfy coupled Riccati equations for the x -part,

$$\partial_x \Gamma_{ij} = (U_{ij} - U_{jj} \Gamma_{ij}) + \sum_{k \neq j} \left[U_{ik} - \Gamma_{ij} U_{jk} \right] \Gamma_{kj}, \quad (2.47)$$

and respectively for the t -part,

$$\partial_t \Gamma_{ij} = (V_{ij} - V_{jj} \Gamma_{ij}) + \sum_{k \neq j} \left[V_{ik} - \Gamma_{ij} V_{jk} \right] \Gamma_{kj}, \quad (2.48)$$

where without loss of generality we have assumed that Ψ_j is a commuting field and for now we postpone the discussion about the case of anticommuting fields. Now, by considering solutions that vanish rapidly as $|x| \rightarrow \infty$, we found that the corresponding j -th generating function of the conserved quantities reads,

$$\mathbb{I}_j = \int_{-\infty}^{\infty} dx \left[U_{jj} + \sum_{i \neq j} U_{ji} \Gamma_{ij} \right]. \quad (2.49)$$

A wide group of integrable nonlinear evolution equations can be formulated using this approach, among which the most of known examples correspond to the particular case $m = 2$, e.g, the nonlinear Schrödinger equation (NLS), Korteweg-de Vries (KdV) and the modified KdV equation (mKdV), Liouville equation, and sine/sinh-Gordon. For a more complete review of these cases see for example [19].

It is worth noting that if the respective analytic properties of the solutions are considered, we can expand the functions Γ_{ij} in positive and negative powers of the spectral parameter λ and then solve (2.47) and (2.48) recursively for each coefficient. This immediately provides an expansion of the j -th generating function \mathbb{I}_j in powers of λ , obtaining in this way an infinite set of conserved quantities. For the usual energy and momentum integrals of motion, commonly also derived from the Lagrangian formalism through variational principle, turn out to be in general linear combinations of these set of conserved quantities \mathbb{I}_j , by taking into account coefficients for the expansions in both positive and negative powers of λ . However, these sets of conserved quantities are not functionally independent in the bulk theory because not all of the auxiliary fields Γ_{ij} are. Although, it seems that in principle there is no need to consider all the conservation laws to derive the apparently overdetermined sets of conserved quantities, we will show in different models supporting type-II integrable defects that the most general form for the defect potentials is obtained by considering all the conservation laws. To make it clearer, in the following section we will derive the formula for obtaining the modified conserved quantities which helps us to compute integrable defect potentials.

2.3 Modified integrals of motion

In this section, we construct the infinite sets of modified conserved quantities in the presence of defects using the Lax pair approach. Firstly, let us suppose that we have two different configurations, namely two column-vector functions $\tilde{\Psi}$ and Ψ corresponding to solutions of auxiliary linear problems described by Lax pairs (\tilde{U}, \tilde{V}) , and (U, V) respectively. Let us now introduce a matrix polynomial $K(x, t; \lambda)$ of the spectral parameter λ connecting the two configurations, namely,

$$\tilde{\Psi}(x, t; \lambda) = K(x, t; \lambda) \Psi(x, t; \lambda), \quad (2.50)$$

where K , commonly named the defect matrix, satisfies differential equations corresponding to a gauge transformation [20] as follows,

$$\partial_t K = \tilde{V}K - KV, \quad \partial_x K = \tilde{U}K - UV. \quad (2.51)$$

This matrix is expected to generate the auto-Bäcklund transformations of each model, and consequently the corresponding defect conditions when the transformation (2.50) is considered in the point of the defect, say $x = 0$. A simple classification of these defect matrices was performed and several examples corresponding to the $m = 2$ linear problem were examined by choosing a very simple form for this matrix [13].

Let us now consider a defect placed at $x = 0$, then the generating functions (2.49) in the presence of the defect take the following form

$$\mathcal{J}_j = \int_{-\infty}^0 dx \left[\tilde{U}_{jj} + \sum_{k \neq j} \tilde{U}_{jk} \tilde{\Gamma}_{kj} \right] + \int_0^{\infty} dx \left[U_{jj} + \sum_{k \neq j} U_{jk} \Gamma_{kj} \right]. \quad (2.52)$$

Hence, taking the time derivative and using the conservation equation (2.46), we get

$$\frac{d\mathcal{J}_j}{dt} = \left[\tilde{V}_{jj} + \sum_{i \neq j} \tilde{V}_{ji} \tilde{\Gamma}_{ij} \right] \Big|_{x=0} - \left[V_{jj} + \sum_{i \neq j} V_{ji} \Gamma_{ij} \right] \Big|_{x=0}. \quad (2.53)$$

Then, it is not difficult to show from (2.50) that the sets of auxiliary functions $\tilde{\Gamma}_{ij}$ and Γ_{ij} satisfy the relation,

$$\tilde{\Gamma}_{ij} = \frac{K_{ij} + \sum_{k \neq j} K_{ik} \Gamma_{kj}}{K_{jj} + \sum_{k \neq j} K_{jk} \Gamma_{kj}}. \quad (2.54)$$

Inserting the above relation in (2.53), one gets

$$\frac{dJ_j}{dt} = \frac{\left(\tilde{V}_{jj} - V_{jj} - \sum_{i \neq j} V_{ji} \Gamma_{ij}\right) \left(K_{jj} + \sum_{k \neq j} K_{jk}\right) + \sum_{i \neq j} \tilde{V}_{ji} K_{ij} + \sum_{i, k \neq j} \tilde{V}_{ji} K_{ik} \Gamma_{kj}}{K_{jj} + \sum_{k \neq j} K_{jk} \Gamma_{kj}}. \quad (2.55)$$

Finally, we consider the equations (2.48) and (2.51) to obtain,

$$\frac{d}{dt} [J_j + D_j] = 0, \quad (2.56)$$

where the defect contribution to the j -th generating function of infinite conserved quantities is given exactly by

$$D_j = -\ln \left[K_{jj} + \sum_{k \neq j} K_{jk} \Gamma_{kj} \right] \Big|_{x=0}. \quad (2.57)$$

This formula was proposed in [13] to be valid for $m = 2$ and shown for any value of m in [15]. Notice that expansions in powers of λ provides the defect contributions to the modified conserved quantities at all orders for every $m \times m$ associated linear problem. In particular, it will be shown that the modified energy and momentum contributions can be computed from certain linear combinations of the first coefficients⁽¹⁾ of the set of conserved quantities $D_j^{(1)}$, taking into account all the possible conservation laws, i.e., for $j = 1, \dots, m$. It is worth noting that an alternative approach was also provided in [21] to prove the classical and quantum integrability in the case of sine-Gordon model with defects, by using the monodromy matrix language, by using a matrix Bäcklund transformation and a matrix Riccati equation.

Before going into the study of the different specific models, we will make some comments about the relation between the formula presented above and standard results of Liouville integrability in the framework of the r -matrix.

2.4 Liouville integrability

So far, an infinite set of independent modified conserved quantities arising from the defect contributions have been systematically constructed through a general formula derived from a variant of the classical inverse scattering method, which are from our point of view sufficient for these kind of defects to be regarded as integrable. However, we have not made any comment about the question of the involutivity of such quantities (required to discuss complete integrability in the sense of Liouville) yet. In this section we present some ideas in that direction considering a general setting.

Certainly, the Hamiltonian formulation of the classical inverse scattering method, which is essentially based on the concept of a classical r -matrix [22], is perhaps the most elegant and convenient framework to discuss involutivity. Let us start with the main aspects of the method in order to discuss this issue in the bulk. In the inverse scattering method the construction of the action-angle variables depends basically on the entries of the monodromy matrix $\tau(\lambda) = T(\infty, -\infty; \lambda)$, where

$$T(x, y; \lambda) = P \exp \left\{ \int_y^x U(z; \lambda) dz \right\}, \quad (2.58)$$

is the transition matrix, $U(x; \lambda)$ is the x -part of the Lax (2.43) at a given time, and P being the path ordering. This $m \times m$ matrix $T(x, y, \lambda)$ is the solution on the interval $[y, x]$ of the following auxiliary problem at a given time,

$$(\partial_x - U(x; \lambda))T(x, y; \lambda) = 0, \quad T(x, x; \lambda) = \mathbf{I}_m. \quad (2.59)$$

As it was noticed in [22, 23], the existence of the classical r -matrix, an $m^2 \times m^2$ matrix which satisfies the relation

$$\{U(x; \lambda_1) \otimes U(y; \lambda_2)\} = \delta(x - y) \left[r(\lambda_1, \lambda_2), U(x; \lambda_1) \otimes \mathbf{I}_m + \mathbf{I}_m \otimes U(y; \lambda_2) \right]. \quad (2.60)$$

Here $\{A \otimes B\}$ denotes the $m^2 \times m^2$ matrix whose elements are given by the Poisson brackets $\{A \otimes B\}_{ij,kl} = \{a_{ij}, b_{kl}\}$. Then, from (2.60) we can write down the Poisson brackets between matrix elements of the transition matrix in the following form,

$$\{T(x, y; \lambda_1) \otimes T(x, y; \lambda_2)\} = \left[r(\lambda_1, \lambda_2), T(x, y; \lambda_1) \otimes T(x, y; \lambda_2) \right], \quad (2.61)$$

from which it is derived that logarithm of the traces of the monodromy matrix commute for different values of the spectral parameter, namely

$$\{ \ln \tau(\lambda_1), \ln \tau(\lambda_2) \} = 0. \quad (2.62)$$

Expanding (2.62) with respect to λ_1 and λ_2 , we get the involutivity of the conserved quantities $\{\mathbf{I}_j^{(n)}\}$, which means that $\tau(\lambda)$ is the generating functional for the integrals of motion.

Now, let us discuss how the classical r -matrix approach is modified by including jump-defect (or point like-defect) in the system. As it was noticed in [21] and more recently in [24], the description of an integrable defect in the r -matrix approach requires to introduce a modified transition matrix,

$$\mathcal{T}(x, y; \lambda) = T(x, 0^+; \lambda) K^{-1}(0; \lambda) \tilde{T}(0^-, y; \lambda), \quad (2.63)$$

which is a combined bulk-defect transition matrix, where $T(x, 0^+; \lambda)$ and $\widetilde{T}(0^-, y; \lambda)$ are the bulk transition matrices corresponding to $x > 0$ and $x < 0$ respectively, and $K(\lambda) \equiv K(0; \lambda)$ is the defect matrix whose entries are evaluated in the single point $x = 0$. The key point in order to show Liouville integrability is to require that the defect matrix satisfies the Poisson algebra (2.65), namely,

$$\{K^{-1}(\lambda_1) \otimes K^{-1}(\lambda_2)\} = \left[r(\lambda_1, \lambda_2), K^{-1}(\lambda_1) \otimes K^{-1}(\lambda_2) \right], \quad (2.64)$$

where $r(\lambda_1, \lambda_2)$ is the same classical r -matrix for the bulk transition matrices. Hence, the above requirement is a sufficient condition to obtain the important result,

$$\{\mathcal{T}(x, y; \lambda_1) \otimes \mathcal{T}(x, y; \lambda_2)\} = \left[r(\lambda_1, \lambda_2), \mathcal{T}(x, y; \lambda_1) \otimes \mathcal{T}(x, y; \lambda_2) \right], \quad (2.65)$$

which guarantees the existence of the infinite set of modified conserved quantities. Similar to the bulk theory, the explicit form of these integrals of motion can be extracted by introducing the following representation for the bulk transition matrix [25],

$$T(x, y; \lambda) = (1 + W(x; \lambda)) e^{Z(x, y; \lambda)} (1 + W(y; \lambda))^{-1}, \quad (2.66)$$

where $W(x; \lambda)$ is an off-diagonal and $Z(x, y; \lambda)$ a diagonal matrix. Then, the logarithm of the trace of the modified monodromy matrix (2.63) is the generating function of the modified conserved quantities, where the defect contributions in an appropriate expansion in λ , read [21, 24]:

$$D(\lambda) = \ln \left[\left(1 + W(0^+, \lambda) \right)^{-1} K^{-1}(\lambda) \left(1 + \widetilde{W}(0^-, \lambda) \right) \right]_{ii}, \quad (2.67)$$

where the subscript ii denotes the leading term coming from the trace of the modified monodromy matrix for the given expansion. At first sight, it seems not to exist a direct relationship between the above result and the generating function (2.57) what we have derived in last section. However, note that $W(x; \lambda)$ satisfy a matrix Riccati equation similar to (2.47), which permits us to derive recursively its coefficients in an asymptotic series expansion as $\lambda \rightarrow \infty$ and $\lambda \rightarrow 0$, and to demonstrate order by order that the results are completely equivalent. This specific analysis deserves more attention than we could give at this moment and we point out that it is not a goal of this thesis to go forward in this approach.

However, it is worth mentioning that the approach we adopt will use essentially an on-shell defect matrix which implies that its entries have non-vanishing Poisson brackets with the bulk monodromy matrices elements. This fact has already been outlined in [10] for the Hamiltonian formulation of the type-II defects in the sine-Gordon and Tzitzéica models,

where the defect conditions appear as a set of second class constraints on the fields, which induces a slight modification of the canonical Poisson brackets. This issue indeed can be solved by working firstly with the off-shell defect matrix to compute the Poisson brackets and then derive the constraints as consistency conditions in constructing the time-like operator in the Lax pair such that the zero curvature condition provides the same equations of motion as the ones coming from the Hamiltonian evolution derived via Poisson brackets as it has claimed by Avan and Doikou recently in [24].

Summarising, in this chapter we have presented the Lagrangian and the Lax pair approach to integrable defects. We have derived a set of conditions to determine the form of the defect potentials from the Lagrangian point of view. Additionally, we have provided a formula to derive all the defect contributions to the modified conserved quantities in order to ensure integrability. In the following chapters we will examine several models and in particular we will provide their corresponding defect matrices for computing such conserved quantities.

The sine-Gordon model

The sine-Gordon (sG) model provides the simplest example of a two-dimensional classical and quantum* integrable field theory, which like many other integrable models exhibits soliton solutions that carry topological charge. The study of this model has a quite wide literature and a lot of exact results are known about it.

It was noted [7] almost a decade ago that, the sine-Gordon model permits type-I integrable defects such that the defects conditions are its standard Bäcklund transformation [9], however being frozen at the defect location. Although the defect condition explicitly breaks the translational invariance, it was shown that the momentum is conserved once it has been suitably modified and the behaviour of soliton solutions passing through the defect was also extensively studied [7]. In addition, the extension of these ideas to the quantum sine-Gordon was investigated by computing and analysing the corresponding transmission matrix both by solving the Defect Yang-Baxter Equation [27] and using the representation framework [28, 18].

Some years later [10] this setting was generalized to include degree of freedom in the defect locations and consequently it was noticed that in fact the sine-Gordon model can supported also type-II integrable defects, which are related to another kind of Bäcklund transformation. It was also shown in [10] the conservation of the modified momentum and its corresponding transmission matrix was provided in [18]. The integrability properties are ensured by the existence of its defect matrix [14].

*However, its quantum aspects are not going to be considered in the whole of this thesis.

In this chapter, we firstly present the explicit construction of an infinite set of conserved quantities in the bulk theory using the inverse scattering approach. Then, we briefly review how to introduce type-I defects and compute the corresponding modified energy and momentum. Finally, we will investigate the existence of type-II defects in the classical sine-Gordon model and compute the respective defect contributions to the energy and momentum.

3.1 The bulk theory and the associated linear problem

Let us start by considering the Lagrangian density describing the sG theory, namely

$$\mathcal{L}_{sG} = \frac{1}{2} (\partial_t \varphi)^2 - \frac{1}{2} (\partial_x \varphi)^2 + \frac{m^2}{\beta^2} \cos(\beta \varphi), \quad (3.1)$$

where φ is a real scalar field, m is the mass parameter and β coupling constant which at classical level is not relevant and will be absorbed from now just taking $\varphi \rightarrow \varphi/\beta$. The equation of motion is given by

$$\partial_t^2 \varphi - \partial_x^2 \varphi = -m^2 \sin \varphi. \quad (3.2)$$

Note that the cosine term in the potential introduces the interactions into the theory. This Lagrangian density is invariant under the Z_2 transformation, $\varphi \rightarrow -\varphi$, and the potential possesses multiple vacuum solutions, namely $\varphi = \pi n$, $n \in \mathbb{Z}$, due to its periodic nature. It is important also to point out that under analytic continuation $\varphi \rightarrow i\varphi$, the sG model becomes the sinh-Gordon (shG) model which has a non-periodic (hyperbolic) potential and possesses a unique vacuum solution. In spite of these two models seem almost the same, have very different properties which can be found in the wide literature (see for instance [29]).

As we have mentioned before the sG bulk theory is classically integrable which means there are infinitely many conserved quantities that can be constructed. Let us start by considering the energy and momentum. From the Lagrangian, the total energy and momentum functionals are directly given by,

$$E = \int_{-\infty}^{\infty} dx J^0 = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_t \varphi)^2 + \frac{1}{2} (\partial_x \varphi)^2 - m^2 \cos \varphi \right], \quad (3.3)$$

$$P = \int_{-\infty}^{\infty} dx J^1 = \int_{-\infty}^{\infty} dx (\partial_t \varphi) (\partial_x \varphi). \quad (3.4)$$

where the respective currents J^0 and J^1 satisfy the conservation law $\partial_t J^0 = \partial_x J^1$, and then trivially we get

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} dx (\partial_t J^0) = \int_{-\infty}^{\infty} dx (\partial_x J^1) = [J^1]_{-\infty}^{\infty} = 0, \quad (3.5)$$

In general, following [26] the same can be done to achieve higher-order conserved quantities, which transform in higher representations of the $(1+1)$ -dimensional Lorentz group as $J_s^{(i)} \rightarrow e^{\alpha s} J_s^{(i)}$, where the integer s is called the (Lorentz) spin since $J_s^{(i)}$ rotates s times under a boost of $2\pi i$. Now, just considering the currents $J_s^0 = T_{s+1} + \Theta_{s-1}$ and $J_s^1 = T_{s+1} - \Theta_{s-1}$ such that

$$\partial_- T_{s+1} = \partial_+ \Theta_{s-1}, \quad (3.6)$$

we get

$$I_s = \int_{-\infty}^{\infty} dx J_s^0 = \int_{-\infty}^{\infty} (T_{s+1} + \Theta_{s-1}) dx, \quad (3.7)$$

an infinite set of conserved quantities. Now, to construct explicitly the conserved quantities we will use the auxiliary linear problem for the sine-Gordon model. As we already have mentioned, the equation of motion is derived as a compatibility condition for the associated linear problem,

$$\partial_x \Psi(x, t; \lambda) = U(x, t; \lambda) \Psi(x, t; \lambda), \quad (3.8)$$

$$\partial_t \Psi(x, t; \lambda) = V(x, t; \lambda) \Psi(x, t; \lambda), \quad (3.9)$$

where the Lax pair is given here by

$$U = \frac{1}{4i} (\partial_t \varphi) H + q(\lambda) E_+ + r(\lambda) E_-, \quad (3.10)$$

$$V = \frac{1}{4i} (\partial_x \varphi) H + A(\lambda) E_+ + B(\lambda) E_-, \quad (3.11)$$

where $\{H, E_{\pm}\}$ are the generators of the $\mathfrak{sl}(2)$ finite Lie algebra (see Appendix A.1) and the following fields have been defined,

$$q(\lambda) = -\frac{m}{4} (\lambda e^{\frac{i\varphi}{2}} - \lambda^{-1} e^{-\frac{i\varphi}{2}}), \quad r(\lambda) = \frac{m}{4} (\lambda e^{-\frac{i\varphi}{2}} - \lambda^{-1} e^{\frac{i\varphi}{2}}), \quad (3.12)$$

$$A(\lambda) = -\frac{m}{4} (\lambda e^{\frac{i\varphi}{2}} + \lambda^{-1} e^{-\frac{i\varphi}{2}}), \quad B(\lambda) = \frac{m}{4} (\lambda e^{-\frac{i\varphi}{2}} + \lambda^{-1} e^{\frac{i\varphi}{2}}). \quad (3.13)$$

As it was described in section (2.2), we can easily derived two conservation equations from the linear system, which can be written as

$$\partial_t \left[q\Gamma_{21} - \frac{i}{4} (\partial_t \varphi) \right] = \partial_x \left[A\Gamma_{21} - \frac{i}{4} (\partial_x \varphi) \right], \quad (3.14)$$

$$\partial_t \left[r\Gamma_{12} + \frac{i}{4} (\partial_t \varphi) \right] = \partial_x \left[B\Gamma_{12} + \frac{i}{4} (\partial_x \varphi) \right], \quad (3.15)$$

where the auxiliary functions $\Gamma_{21} = \Psi_2 \Psi_1^{-1}$ and $\Gamma_{12} = \Psi_1 \Psi_2^{-1}$ has been introduced. These functions satisfy the set of Ricatti equation,

$$\partial_x \Gamma_{21} = r + \frac{i}{2} (\partial_t \varphi) \Gamma_{21} - q (\Gamma_{21})^2, \quad (3.16)$$

$$\partial_x \Gamma_{12} = q - \frac{i}{2} (\partial_t \varphi) \Gamma_{12} - r (\Gamma_{12})^2, \quad (3.17)$$

Firstly, let us consider the equation (3.16) to solve Γ_{21} . Hence, expanding Γ_{21} as $\lambda \rightarrow \infty$

$$\Gamma_{21} = \sum_{n=0}^{\infty} \frac{\Gamma_{21}^{(n)}}{\lambda^n}, \quad (3.18)$$

we get,

$$\Gamma_{21}^{(0)} = i e^{-\frac{i\varphi}{2}}, \quad (3.19)$$

$$\Gamma_{21}^{(1)} = -\frac{i}{m} [\partial_t \varphi + \partial_x \varphi] e^{-\frac{i\varphi}{2}}, \quad (3.20)$$

$$\Gamma_{21}^{(2)} = e^{-\frac{i\varphi}{2}} \left[-\frac{2}{m^2} \partial_x (\partial_t \varphi + \partial_x \varphi) + \frac{i}{2m^2} (\partial_t \varphi + \partial_x \varphi)^2 + \sin \varphi \right], \quad (3.21)$$

$$\begin{aligned} \Gamma_{21}^{(3)} = & \frac{2i}{m} e^{-\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x^2 (\partial_t \varphi + \partial_x \varphi) - \frac{i}{m^2} (\partial_t \varphi + \partial_x \varphi) (\partial_x \partial_t \varphi + \partial_x^2 \varphi) - \cos \varphi (\partial_x \varphi) \right. \\ & \left. - \frac{1}{2} (\partial_t \varphi + \partial_x \varphi) e^{-i\varphi} \right]. \end{aligned} \quad (3.22)$$

Thus, we have a first infinite set of conserved quantities generated from

$$\mathbf{I}_1 = \int_{-\infty}^{\infty} dx \left[q \Gamma_{21} - \frac{i}{4} (\partial_t \varphi) \right]. \quad (3.23)$$

From the coefficients of the expansion of Γ_{21} , it is very easy to see that the charge $\mathbf{I}_1^{(+1)}$ trivially vanishes and $\mathbf{I}_1^{(0)}$ is basically a topological term. So the first non-vanishing charge is given by

$$\mathbf{I}_1^{(-1)} = \frac{1}{4mi} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_t \varphi + \partial_x \varphi)^2 - m^2 \cos \varphi \right]. \quad (3.24)$$

Now, if we consider the expansion of Γ_{21} as $\lambda \rightarrow 0$,

$$\Gamma_{21} = \sum_{n=0}^{\infty} \widehat{\Gamma}_{21}^{(n)} \lambda^n, \quad (3.25)$$

we obtain the following coefficients,

$$\widehat{\Gamma}_{21}^{(0)} = ie^{\frac{i\varphi}{2}}, \quad (3.26)$$

$$\widehat{\Gamma}_{21}^{(1)} = \frac{i}{m} [\partial_t \varphi - \partial_x \varphi] e^{\frac{i\varphi}{2}}, \quad (3.27)$$

$$\widehat{\Gamma}_{21}^{(2)} = e^{\frac{i\varphi}{2}} \left[-\frac{2}{m^2} \partial_x (\partial_t \varphi - \partial_x \varphi) + \frac{i}{2m^2} (\partial_t \varphi - \partial_x \varphi)^2 - \sin \varphi \right], \quad (3.28)$$

$$\begin{aligned} \widehat{\Gamma}_{21}^{(3)} &= \frac{2i}{m} e^{\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x^2 (\partial_t \varphi - \partial_x \varphi) - \frac{i}{m^2} (\partial_t \varphi - \partial_x \varphi) (\partial_x \partial_t \varphi - \partial_x^2 \varphi) + \cos \varphi (\partial_x \varphi) \right. \\ &\quad \left. - \frac{1}{2} (\partial_t \varphi - \partial_x \varphi) e^{-i\varphi} \right]. \end{aligned} \quad (3.29)$$

Then, we obtain again one trivial charge, one topological charge, and the first non-trivial charge which is given by,

$$\widehat{\Gamma}_1^{(+1)} = -\frac{1}{4mi} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_t \varphi - \partial_x \varphi)^2 - m^2 \cos \varphi \right]. \quad (3.30)$$

As we can see, this charge is not real and therefore it remains to add the hermitian conjugate of them for obtaining a real conserved quantity. In fact, we will see that these contributions naturally rise from the other conservation equation that we have derived in (3.17).

Let us consider then the conservation equation (3.17) to solve Γ_{12} recursively. Clearly, using the same scheme we can obtain the first coefficients for the auxiliary function. The results are listed down,

$$\Gamma_{12}^{(0)} = ie^{\frac{i\varphi}{2}}, \quad (3.31)$$

$$\Gamma_{12}^{(1)} = -\frac{i}{m} [\partial_t \varphi + \partial_x \varphi] e^{\frac{i\varphi}{2}}, \quad (3.32)$$

$$\Gamma_{12}^{(2)} = e^{\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x (\partial_t \varphi + \partial_x \varphi) + \frac{i}{2m^2} (\partial_t \varphi + \partial_x \varphi)^2 - \sin \varphi \right], \quad (3.33)$$

$$\begin{aligned} \Gamma_{12}^{(3)} &= \frac{2i}{m} e^{\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x^2 (\partial_t \varphi + \partial_x \varphi) + \frac{i}{m^2} (\partial_t \varphi + \partial_x \varphi) (\partial_x \partial_t \varphi + \partial_x^2 \varphi) - \cos \varphi (\partial_x \varphi) \right. \\ &\quad \left. - \frac{1}{2} (\partial_t \varphi + \partial_x \varphi) e^{i\varphi} \right], \end{aligned} \quad (3.34)$$

and correspondingly,

$$\widehat{\Gamma}_{12}^{(0)} = ie^{-\frac{i\varphi}{2}}, \quad (3.35)$$

$$\widehat{\Gamma}_{12}^{(1)} = \frac{i}{m} [\partial_t \varphi - \partial_x \varphi] e^{-\frac{i\varphi}{2}}, \quad (3.36)$$

$$\widehat{\Gamma}_{12}^{(2)} = e^{-\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x (\partial_t \varphi - \partial_x \varphi) + \frac{i}{2m^2} (\partial_t \varphi - \partial_x \varphi)^2 + \sin \varphi \right], \quad (3.37)$$

$$\begin{aligned} \widehat{\Gamma}_{12}^{(3)} &= \frac{2}{im} e^{-\frac{i\varphi}{2}} \left[\frac{2}{m^2} \partial_x^2 (\partial_t \varphi - \partial_x \varphi) + \frac{i}{m^2} (\partial_t \varphi - \partial_x \varphi) (\partial_x \partial_t \varphi - \partial_x^2 \varphi) + \cos \varphi (\partial_x \varphi) \right. \\ &\quad \left. - \frac{1}{2} (\partial_t \varphi - \partial_x \varphi) e^{-i\varphi} \right]. \end{aligned} \quad (3.38)$$

Therefore, from the generating function of the infinite conserved quantities, namely,

$$\mathbb{I}_2 = \int_{-\infty}^{\infty} dx \left[\frac{i}{4} (\partial_t \varphi) + r \Gamma_{12} \right], \quad (3.39)$$

we immediately obtain the following non-vanishing charges,

$$\mathbb{I}_2^{(-1)} = -\frac{1}{4mi} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_t \varphi + \partial_x \varphi)^2 - m^2 \cos \varphi \right], \quad (3.40)$$

$$\widehat{\mathbb{I}}_2^{(+1)} = \frac{1}{4mi} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} (\partial_t \varphi - \partial_x \varphi)^2 - m^2 \cos \varphi \right]. \quad (3.41)$$

Notice that $\mathbb{I}_2^{(-1)} = \mathbb{I}_1^{\dagger(-1)}$ and $\widehat{\mathbb{I}}_2^{(+1)} = \widehat{\mathbb{I}}_1^{\dagger(+1)}$, which allow us to define two real conserved quantities, namely,

$$\mathbb{I}^{(-1)} = i(\mathbb{I}_1^{(-1)} - \mathbb{I}_2^{(-1)}), \quad \widehat{\mathbb{I}}^{(+1)} = i(\widehat{\mathbb{I}}_1^{(+1)} - \widehat{\mathbb{I}}_2^{(+1)}). \quad (3.42)$$

Therefore, we finally can recover the usual expressions for the energy and momentum of the bulk sine-Gordon model by adding and subtracting the above results as follows,

$$E = m \left(\mathbb{I}^{(-1)} - \widehat{\mathbb{I}}^{(+1)} \right) = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \left\{ (\partial_t \varphi)^2 + (\partial_x \varphi)^2 \right\} - m^2 \cos \varphi \right], \quad (3.43)$$

$$P = m \left(\mathbb{I}^{(-1)} + \widehat{\mathbb{I}}^{(+1)} \right) = \int_{-\infty}^{\infty} dx (\partial_t \varphi) (\partial_x \varphi). \quad (3.44)$$

Then, we have recursively constructed some few first conserved quantities in the bulk theory of the sine-Gordon model through the inverse scattering method techniques. One of the advantages of this approach is that once the auxiliary functions Γ_{ij} are computed we can use them directly to derive the corresponding defect contributions. In the following section, we will use the results obtained to introduce the defect at a fixed point and consequently to compute the respective defects contributions to each integral of motion.

3.2 Review of type-I defect sine-Gordon theory

Just for the sake of completeness, let us review briefly the type-I defect in the sine-Gordon model following the pioneering work [7]. The starting point is the addition of a local term to the Lagrangian density, i.e

$$\begin{aligned} \mathcal{L} = & \theta(-x) \left(\frac{1}{2} (\partial_t \tilde{\varphi})^2 - \frac{1}{2} (\partial_x \tilde{\varphi})^2 + m^2 \cos \tilde{\varphi} \right) + \theta(x) \left(\frac{1}{2} (\partial_t \varphi)^2 - \frac{1}{2} (\partial_x \varphi)^2 + m^2 \cos \varphi \right) \\ & + \delta(x) \left(\frac{1}{2} (\tilde{\varphi} \partial_t \varphi - \varphi \partial_t \tilde{\varphi}) - 2m \left[\sigma \cos \left(\frac{\tilde{\varphi} + \varphi}{2} \right) + \frac{1}{\sigma} \cos \left(\frac{\tilde{\varphi} - \varphi}{2} \right) \right] \right), \end{aligned} \quad (3.45)$$

where the defect potential depends on the values of both fields $\tilde{\varphi}, \varphi$ at $x = 0$. The equations of motion for each region, $x < 0$ and $x > 0$, are the sine-Gordon equation for each field $\tilde{\varphi}$ and φ respectively, and the equations associated with defect conditions at $x = 0$ are given by,

$$\partial_t \varphi - \partial_x \tilde{\varphi} = m \left[\sigma \sin \left(\frac{\tilde{\varphi} + \varphi}{2} \right) + \frac{1}{\sigma} \sin \left(\frac{\tilde{\varphi} - \varphi}{2} \right) \right], \quad (3.46)$$

$$\partial_x \varphi - \partial_t \tilde{\varphi} = m \left[\sigma \sin \left(\frac{\tilde{\varphi} + \varphi}{2} \right) - \frac{1}{\sigma} \sin \left(\frac{\tilde{\varphi} - \varphi}{2} \right) \right], \quad (3.47)$$

where σ is a free parameter. These equations turn out to be “frozen” auto-Bäcklund transformation for the sine-Gordon model. So, we find that if the field $\tilde{\varphi}$ satisfies the sine-Gordon equation then the field φ also does. These transformations are not unique, and in fact in next sections we will present an alternative Bäcklund transformation for the sine-Gordon.

Using this particular Lagrangian framework, it was also shown in [7] that the energy and momentum corresponding to this type-I defect are conserved containing bulk and defect contributions. To see that using our approach, which allows us to compute defect contributions at any order, it is necessary to introduce the corresponding defect matrix.

3.2.1 Modified integrals of motion from the defect matrix

The simplest way to compute the defect matrix connecting the two auxiliary problems, namely $\tilde{\Psi} = K\Psi$, is obtained by choosing the following ansatz,

$$K = K_0 + \lambda^{-1} K_1, \quad (3.48)$$

which is solution of the differential equations

$$\partial_x K = \tilde{U}K - KU, \quad \partial_t K = \tilde{V}K - KV. \quad (3.49)$$

Solving grade by grade the above equation, we find that the defect matrix for the type-I case takes the following simple form,

$$K = \begin{bmatrix} e^{-\frac{i}{4}(\tilde{\varphi}-\varphi)} & \lambda^{-1} \sigma e^{-\frac{i}{4}(\tilde{\varphi}+\varphi)} \\ -\lambda^{-1} \sigma e^{\frac{i}{4}(\tilde{\varphi}+\varphi)} & e^{\frac{i}{4}(\tilde{\varphi}-\varphi)} \end{bmatrix}. \quad (3.50)$$

Now, by using the defect contribution equation (2.57) to the generating function of infinite integrals of motion (3.23), modified conserved quantities can be properly computed. In this case, the equation reads

$$D_1 = -\ln \left[K_{11} + K_{12} \Gamma_{21} \right] \Big|_{x=0}. \quad (3.51)$$

Hence, taking into account the both expansion in negative and positive powers of λ and the explicit form of the defect matrix (3.50), we found that

$$D_1^{(-1)} = -i\sigma e^{-i(\tilde{\varphi}+\varphi)/2}, \quad \widehat{D}_1^{(+1)} = \frac{i}{\sigma} e^{-i(\tilde{\varphi}-\varphi)/2} - \frac{1}{m} (\partial_t \varphi - \partial_x \varphi). \quad (3.52)$$

Now, repeating same procedure for the second generating function (3.39), one gets

$$D_2 = -\ln \left[K_{21} \Gamma_{12} + K_{22} \right] \Big|_{x=0}, \quad (3.53)$$

from which we obtain the following contributions,

$$D_2^{(-1)} = i\sigma e^{i(\tilde{\varphi}+\varphi)/2}, \quad \widehat{D}_2^{(+1)} = -\frac{i}{\sigma} e^{i(\tilde{\varphi}-\varphi)/2} - \frac{1}{m} (\partial_t \varphi - \partial_x \varphi). \quad (3.54)$$

As was expected, $D_2^{(-1)} = D_1^{\dagger(-1)}$ and $\widehat{D}_2^{(+1)} = \widehat{D}_1^{\dagger(+1)}$, which allow us to define two real defect contributions as follows,

$$\mathbb{D}^{(-1)} = i[D_1^{(-1)} - D_2^{(-1)}], \quad \widehat{\mathbb{D}}^{(+1)} = i[\widehat{D}_1^{(+1)} - \widehat{D}_2^{(+1)}]. \quad (3.55)$$

Then, the corresponding defect energy and momentum for the sine-Gordon model is recovered by adding and subtracting all the results obtained as follows,

$$E_D = m \left[\mathbb{D}^{(-1)} - \widehat{\mathbb{D}}^{(+1)} \right] = 2m \left[\sigma \cos \left(\frac{\tilde{\varphi} + \varphi}{2} \right) + \frac{1}{\sigma} \cos \left(\frac{\tilde{\varphi} - \varphi}{2} \right) \right], \quad (3.56)$$

$$P_D = m \left[\mathbb{D}^{(-1)} + \widehat{\mathbb{D}}^{(+1)} \right] = 2m \left[\sigma \cos \left(\frac{\tilde{\varphi} + \varphi}{2} \right) - \frac{1}{\sigma} \cos \left(\frac{\tilde{\varphi} - \varphi}{2} \right) \right], \quad (3.57)$$

which are in complete agreement with the results previously obtained from the Lagrangian point of view [7] (see also [21] to compare with the r -matrix approach).

3.3 Type-II defect sine-Gordon theory

The starting point to introduce a type-II defect is the Lagrangian density,

$$\mathcal{L} = \theta(-x)\mathcal{L}_{\tilde{\varphi}} + \theta(x)\mathcal{L}_{\varphi} + \delta(x)\mathcal{L}_D, \quad (3.58)$$

where

$$\mathcal{L}_D = \left[\frac{1}{2}(\tilde{\varphi}\partial_t\varphi - \varphi\partial_t\tilde{\varphi}) + \Lambda\partial_t(\tilde{\varphi} - \varphi) - \partial_t\Lambda(\tilde{\varphi} - \varphi) - B_0(\tilde{\varphi}, \varphi, \Lambda) \right]. \quad (3.59)$$

Here $\Lambda = \Lambda(t)$ is the additional degree of freedom associated to the defect. Besides the bulk equations for the two fields in their respective regions, the equations associated with the defect conditions are obtained and can be written as,

$$\partial_x\tilde{\varphi} - \partial_t\varphi + 2\partial_t\Lambda = -\frac{\partial B_0}{\partial\tilde{\varphi}}, \quad (3.60)$$

$$\partial_x\varphi - \partial_t\tilde{\varphi} + 2\partial_t\Lambda = \frac{\partial B_0}{\partial\varphi}, \quad (3.61)$$

$$\partial_t(\tilde{\varphi} - \varphi) = \frac{1}{2}\frac{\partial B_0}{\partial\Lambda}. \quad (3.62)$$

By introducing new variables $\varphi_+ = (\tilde{\varphi} + \varphi)/2$ and $\varphi_- = (\tilde{\varphi} - \varphi)/2$, we can rewrite them as,

$$\partial_t\varphi_+ - \partial_x\varphi_+ - 2\partial_t\Lambda = \frac{1}{2}\frac{\partial B_0}{\partial\varphi_-}, \quad (3.63)$$

$$\partial_t\varphi_- - \partial_x\varphi_- = \frac{1}{2}\left(\frac{\partial B_0}{\partial\varphi_+} + \frac{\partial B_0}{\partial\Lambda}\right), \quad (3.64)$$

$$\partial_x\varphi_- + \partial_t\varphi_- = -\frac{1}{2}\frac{\partial B_0}{\partial\varphi_+}, \quad (3.65)$$

These equations were originally derived in [10], where the conservation of energy and momentum was also shown and the defect potential was assumed to take the following form[†],

$$B_0(\varphi_+, \varphi_-, \Lambda) = -\frac{m}{2\sigma} [e^{-i(\varphi_+-\Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) + 4e^{i(\varphi_+-\Lambda)}] - \frac{m\sigma}{2} [e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) + 4e^{-i\Lambda}], \quad (3.66)$$

where η, σ are two free parameters. Then, we arrive at the following equations for the defect conditions,

$$i\partial_t\varphi_+ - i\partial_x\varphi_+ - 2i(\partial_t\Lambda) = \frac{m}{4\sigma} e^{-i(\varphi_+-\Lambda)} (e^{i\varphi_-} - e^{-i\varphi_-}) + \frac{m\sigma}{4} e^{i\Lambda} (e^{i\varphi_-} - e^{-i\varphi_-}), \quad (3.67)$$

$$i\partial_t\varphi_- - i\partial_x\varphi_- = \frac{m\sigma}{4} [e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) - 4e^{-i\Lambda}], \quad (3.68)$$

$$i\partial_t\varphi_- + i\partial_x\varphi_- = \frac{m}{4\sigma} [e^{-i(\varphi_+-\Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) - 4e^{i(\varphi_+-\Lambda)}]. \quad (3.69)$$

[†]To fit the notations is necessary to identify $i\varphi_+ \rightarrow p, i\varphi_- \rightarrow q$, and $i\Lambda \rightarrow \lambda$.

Notice that, the vacuum solution in the bulk $\tilde{\varphi} = \varphi = 0$ satisfies the above defect conditions, if the auxiliary field Λ takes the constant value,

$$\Lambda = -\frac{i}{2} \ln \left(\frac{4}{\eta} \right). \quad (3.70)$$

In the next section we will derive two possible forms for the defect matrix in order to compute the different defect contribution to the infinite set of conserved quantities.

3.3.1 Defect matrices

By considering a simple generalization of the ansatz (3.48) for the defect matrix, we will be able to derive the type-II Bäcklund transformation for the sine-Gordon which includes the defect degree of freedom, namely,

$$K_{ij} = \alpha_{ij} + \lambda^{-1} \beta_{ij} + \lambda^{-2} \gamma_{ij}. \quad (3.71)$$

In order to determine the explicit form of the defect matrix, we will work with the light-cone coordinates $x_{\pm} = (t \pm x)/2$, where the differential equations take the following form,

$$\partial_{\pm} K = K A_{\pm} - \tilde{A}_{\pm} K, \quad (3.72)$$

and it has been defined $A_{\pm} = -(V \pm U)$, i.e

$$A_{+} = \begin{bmatrix} \frac{i}{4} \partial_{+} \varphi & \frac{m}{2} \lambda e^{\frac{i\varphi}{2}} \\ -\frac{m}{2} \lambda e^{-\frac{i\varphi}{2}} & -\frac{i}{4} \partial_{+} \varphi \end{bmatrix}, \quad A_{-} = \begin{bmatrix} -\frac{i}{4} \partial_{-} \varphi & \frac{m}{2} \lambda^{-1} e^{-\frac{i\varphi}{2}} \\ -\frac{m}{2} \lambda^{-1} e^{\frac{i\varphi}{2}} & \frac{i}{4} \partial_{-} \varphi \end{bmatrix}. \quad (3.73)$$

In terms of the variables φ_{\pm} , we find from (3.71) and (3.72) that equations for the matrices α_{ij} , β_{ij} and γ_{ij} can be grouped into two decoupled subsets and consequently we find two different forms of the defect matrix:

(a) The first one is spanned by $\{\alpha_{11}, \alpha_{22}, \beta_{12}, \beta_{21}, \gamma_{11}, \gamma_{22}\}$ and leads to

$$\alpha_{11} = a_{11} e^{-\frac{i}{2} \varphi^{-}}, \quad \alpha_{22} = a_{11} e^{\frac{i}{2} \varphi^{-}}, \quad \gamma_{11} = c_{11} e^{\frac{i}{2} \varphi^{-}}, \quad \gamma_{22} = c_{11} e^{-\frac{i}{2} \varphi^{-}}. \quad (3.74)$$

From the form of the respective equations we note that in general it is possible to parametrize β_{21} by including an auxiliary field Λ ,

$$\beta_{21} = b_{21} e^{-i(\Lambda - \frac{\varphi_{+}}{2})}, \quad (3.75)$$

where b_{21} as well as a_{11} and c_{11} are arbitrary constants. We find then the equations involving $\partial_+\alpha_{11}, \partial_-\gamma_{11}, \partial_+\beta_{21}$ and $\partial_-\beta_{21}$ read,

$$\begin{aligned} i\partial_+\varphi_- &= \frac{m}{2a_{11}} \left(b_{21}e^{-i(\Lambda-\varphi_+)} + \beta_{12}e^{-\frac{i}{2}\varphi_+} \right), \\ i\partial_-\varphi_- &= -\frac{m}{2c_{11}} \left(\beta_{12}e^{\frac{i}{2}\varphi_+} + b_{21}e^{-i\Lambda} \right), \\ i\partial_+\Lambda &= -\frac{mc_{11}}{2b_{21}} e^{i(\Lambda-\varphi_+)} (e^{i\varphi_-} - e^{-i\varphi_-}), \\ i\partial_-(\Lambda - \varphi_+) &= \frac{ma_{11}}{2b_{21}} e^{i\Lambda} (e^{i\varphi_-} - e^{-i\varphi_-}), \end{aligned} \quad (3.76)$$

together with

$$\begin{aligned} \partial_+\beta_{12} &= -\frac{i}{2}(\partial_+\varphi_+) \beta_{12} + \frac{mc_{11}}{2} e^{\frac{i}{2}\varphi_+} (e^{i\varphi_-} - e^{-i\varphi_-}), \\ \partial_-\beta_{12} &= \frac{i}{2}(\partial_-\varphi_+) \beta_{12} - \frac{ma_{11}}{2} e^{-\frac{i}{2}\varphi_+} (e^{i\varphi_-} - e^{-i\varphi_-}). \end{aligned} \quad (3.77)$$

A solution for (3.77) compatible with (3.76) is found to be

$$\beta_{12} = -\frac{b_{21}}{4} e^{i(\Lambda-\frac{\varphi_+}{2})} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta), \quad (3.78)$$

where η is also an arbitrary constant. Therefore

$$K = \begin{bmatrix} e^{-\frac{i\varphi_-}{2}} - \frac{1}{\lambda^2} c_{11} e^{\frac{i\varphi_-}{2}} & -\frac{1}{4\lambda} b_{21} e^{i(\Lambda-\frac{\varphi_+}{2})} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \\ \frac{1}{\lambda} b_{21} e^{-i\Lambda} e^{\frac{i\varphi_+}{2}} & e^{\frac{i\varphi_-}{2}} - \frac{1}{\lambda^2} c_{11} e^{-\frac{i\varphi_-}{2}} \end{bmatrix}, \quad (3.79)$$

where we have chosen $a_{11} = 1$. Here, we pointed out that the type-I Bäcklund transformation can be rederived by taking $\text{Im}\Lambda \rightarrow \infty$, $\eta \rightarrow 4e^{2\text{Im}\Lambda}$ and $b_{21} \rightarrow 0$ while holding $b_{21}e^{-i\Lambda}$ constant, say ω . Then, we obtain

$$K \longrightarrow \begin{bmatrix} e^{-\frac{i\varphi_-}{2}} & -\frac{\omega}{\lambda} e^{-\frac{i\varphi_+}{2}} \\ \frac{\omega}{\lambda} e^{\frac{i\varphi_+}{2}} & e^{\frac{i\varphi_-}{2}} \end{bmatrix}. \quad (3.80)$$

As expected in this limit (3.76) reduce to

$$\partial_-\varphi_+ = -\frac{m}{\omega} \sin \varphi_-, \quad \partial_+\varphi_- = m\omega \sin \varphi_+. \quad (3.81)$$

Now, by introducing a new parameter $\sigma = -\frac{2}{b_{21}} = \frac{b_{21}}{2c_{11}}$, the equation (3.76) becomes

(with $a_{11} = 1$),

$$\begin{aligned}
i\partial_-(\varphi_+ - \Lambda) &= \frac{m\sigma}{4} e^{i\Lambda} (e^{i\varphi_-} - e^{-i\varphi_-}), \\
i\partial_+\Lambda &= -\frac{m}{4\sigma} e^{-i(\varphi_+ - \Lambda)} (e^{i\varphi_-} - e^{-i\varphi_-}), \\
i\partial_-\varphi_- &= \frac{m\sigma}{4} [e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) - 4e^{-i\Lambda}], \\
i\partial_+\varphi_- &= \frac{m}{4\sigma} [e^{-i(\varphi_+ - \Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) - 4e^{i(\varphi_+ - \Lambda)}], \tag{3.82}
\end{aligned}$$

and the expression for K in (3.79) takes the form

$$K = \begin{bmatrix} e^{-\frac{i\varphi_-}{2}} - \frac{1}{(\sigma\lambda)^2} e^{\frac{i\varphi_-}{2}} & \frac{1}{2(\sigma\lambda)} e^{i(\Lambda - \frac{\varphi_+}{2})} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \\ -\frac{2}{(\sigma\lambda)} e^{-i\Lambda} e^{\frac{i\varphi_+}{2}} & e^{\frac{i\varphi_-}{2}} - \frac{1}{(\sigma\lambda)^2} e^{-\frac{i\varphi_-}{2}} \end{bmatrix}. \tag{3.83}$$

By cross-differentiating the last two equations in (3.82), we find that if the field φ satisfies the sine-Gordon equation then $\tilde{\varphi}$ also does. Additionally by differentiating the second equation in (3.84) with respect to x_- , we obtain

$$\partial_- \partial_+ \Lambda = \frac{im^2}{16} e^{-i\varphi_+} [4e^{2i\Lambda} - (e^{i\varphi_-} + e^{-i\varphi_-})(4 - \eta e^{2i\Lambda})]. \tag{3.84}$$

This equation of motion for the field Λ depends on both fields $\tilde{\varphi}$ and φ . Finally, it is interesting to rewrite the defect matrix in a different and suggestive form, as follows,

$$K = e^{\frac{i\varphi}{4} \cdot H} \bar{K} e^{-\frac{i\tilde{\varphi}}{4} \cdot H}, \tag{3.85}$$

with

$$\begin{aligned}
\bar{K} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{(\sigma\lambda)} \begin{bmatrix} 0 & \frac{1}{2} e^{i\Lambda} (e^{-i\varphi} + e^{-i\tilde{\varphi}} + \eta e^{-i(\tilde{\varphi}+\varphi)/2}) \\ 2e^{-i\Lambda} e^{i(\tilde{\varphi}+\varphi)/2} & 0 \end{bmatrix} \\
&- \frac{1}{(\sigma\lambda)^2} \begin{bmatrix} e^{i(\tilde{\varphi}-\varphi)/2} & 0 \\ 0 & e^{-i(\tilde{\varphi}-\varphi)/2} \end{bmatrix}, \tag{3.86}
\end{aligned}$$

where H is the generator in the Cartan subalgebra of $\mathfrak{sl}(2)$.

(b) The second set of equations is specified by $\{\alpha_{12}, \alpha_{21}, \beta_{11}, \beta_{22}, \gamma_{12}, \gamma_{21}\}$, which satisfy

$$\begin{aligned}
\gamma_{12} &= c_{12}e^{-\frac{i}{2}\varphi_+}, & \gamma_{21} &= -c_{12}e^{\frac{i}{2}\varphi_+}, \\
\alpha_{21} &= -a_{12}e^{-\frac{i}{2}\varphi_+}, & \alpha_{12} &= a_{12}e^{\frac{i}{2}\varphi_+}, \\
\partial_+\alpha_{12} &= -\frac{i}{2}(\partial_+\varphi_+)\alpha_{12} + \frac{m}{2}e^{\frac{i}{2}(\varphi_++\varphi_-)}\beta_{11} - \frac{m}{2}e^{\frac{i}{2}(\varphi_+-\varphi_-)}\beta_{22}, \\
\partial_+\alpha_{21} &= \frac{i}{2}(\partial_+\varphi_+)\alpha_{21} - \frac{m}{2}e^{-\frac{i}{2}(\varphi_++\varphi_-)}\beta_{22} + \frac{m}{2}e^{-\frac{i}{2}(\varphi_+-\varphi_-)}\beta_{11}, \\
\partial_+\beta_{11} &= \frac{i}{2}(\partial_+\varphi_-)\beta_{11} - \frac{m}{2}e^{-\frac{i}{2}(\varphi_++\varphi_-)}\gamma_{12} - \frac{m}{2}e^{\frac{i}{2}(\varphi_+-\varphi_-)}\gamma_{21}, \\
\partial_+\beta_{22} &= -\frac{i}{2}(\partial_+\varphi_-)\beta_{22} + \frac{m}{2}e^{\frac{i}{2}(\varphi_++\varphi_-)}\gamma_{21} + \frac{m}{2}e^{-\frac{i}{2}(\varphi_+-\varphi_-)}\gamma_{12}, \tag{3.87}
\end{aligned}$$

and

$$\begin{aligned}
\partial_-\beta_{11} &= -\frac{i}{2}(\partial_-\varphi_-)\beta_{11} - \frac{m}{2}e^{\frac{i}{2}(\varphi_++\varphi_-)}\alpha_{12} - \frac{m}{2}e^{-\frac{i}{2}(\varphi_+-\varphi_-)}\alpha_{21}, \\
\partial_-\gamma_{12} &= \frac{i}{2}(\partial_-\varphi_+)\gamma_{12} + \frac{m}{2}e^{-\frac{i}{2}(\varphi_++\varphi_-)}\beta_{11} - \frac{m}{2}e^{-\frac{i}{2}(\varphi_+-\varphi_-)}\beta_{22}, \\
\partial_-\gamma_{21} &= -\frac{i}{2}(\partial_-\varphi_+)\gamma_{21} - \frac{m}{2}e^{\frac{i}{2}(\varphi_++\varphi_-)}\beta_{22} + \frac{m}{2}e^{\frac{i}{2}(\varphi_+-\varphi_-)}\beta_{11}, \\
\partial_-\beta_{22} &= \frac{i}{2}(\partial_-\varphi_-)\beta_{22} + \frac{m}{2}e^{-\frac{i}{2}(\varphi_++\varphi_-)}\alpha_{21} + \frac{m}{2}e^{\frac{i}{2}(\varphi_+-\varphi_-)}\alpha_{12}. \tag{3.88}
\end{aligned}$$

As before, we propose a suitable reparametrization by introducing the auxiliary field $\bar{\Lambda}$,

$$\beta_{22} = b_{22}e^{-i(\bar{\Lambda} + \frac{\varphi_-}{2})}, \tag{3.89}$$

where b_{22} as well as a_{12} and c_{12} are arbitrary constants. The equations involving $\partial_-\gamma_{12}, \partial_+\beta_{22}, \partial_-\beta_{22}$ and $\partial_+\alpha_{12}$ yields respectively,

$$i\partial_+\varphi_+ = -\frac{m}{2a_{12}} \left[b_{22}e^{-i(\bar{\Lambda} + \varphi_-)} - \beta_{11}e^{\frac{i}{2}\varphi_-} \right], \tag{3.90}$$

$$i\partial_-(\bar{\Lambda} + \varphi_-) = -\frac{m}{2} \frac{a_{12}}{b_{22}} e^{i\bar{\Lambda}} [e^{i\varphi_+} - e^{-i\varphi_+}], \tag{3.91}$$

$$i\partial_+\bar{\Lambda} = \frac{m}{2} \frac{c_{12}}{b_{22}} e^{i(\bar{\Lambda} + \varphi_-)} [e^{i\varphi_+} - e^{-i\varphi_+}], \tag{3.92}$$

$$i\partial_-\varphi_+ = \frac{m}{2c_{12}} \left[b_{22}e^{-i\bar{\Lambda}} - \beta_{11}e^{-\frac{i}{2}\varphi_-} \right]. \tag{3.93}$$

A solution of β_{11} compatible with (3.87) and (3.88) can be written as

$$\beta_{11} = -\frac{a_{12}c_{12}}{b_{22}} e^{i(\bar{\Lambda} + \frac{1}{2}\varphi_-)} (e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}), \tag{3.94}$$

where $\bar{\eta}$ is an arbitrary constant. K is then given in the following form

$$K = \begin{bmatrix} -\frac{1}{\lambda} \frac{a_{12}c_{12}}{b_{22}} e^{i(\bar{\Lambda} + \frac{1}{2}\varphi_-)} (e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) & a_{12}e^{\frac{i}{2}\varphi_+} + \frac{1}{\lambda^2} c_{12}e^{-\frac{i}{2}\varphi_+} \\ -a_{12}e^{-\frac{i}{2}\varphi_+} - \frac{c_{12}}{\lambda^2} e^{\frac{i}{2}\varphi_+} & \frac{b_{22}}{\lambda} e^{-i(\bar{\Lambda} + \frac{1}{2}\varphi_-)} \end{bmatrix} \quad (3.95)$$

Here we can also perform the type-I Bäcklund limit by fixing $a_{12} = 1$ and taking $\text{Im}\bar{\Lambda} \rightarrow \infty$, $\bar{\eta} \rightarrow \infty$ and $c_{12} \rightarrow 0$, while satisfying

$$b_{22}e^{-i\bar{\Lambda}} = \bar{\omega}, \quad -\frac{a_{12}\bar{c}_{12}}{b_{22}}e^{i\bar{\Lambda}} \rightarrow 0, \quad -\frac{a_{12}\bar{c}_{12}}{b_{22}}e^{i\bar{\Lambda}}\bar{\eta} = \bar{\omega}, \quad (3.96)$$

where $\bar{\omega}$ is a constant, we recover the structure of type I Bäcklund transformations,

$$\partial_+\varphi_+ = \frac{m\bar{\omega}}{a_{12}} \sin\varphi_-, \quad \partial_-\varphi_- = -\frac{ma_{12}}{\bar{\omega}} \sin\varphi_+. \quad (3.97)$$

Notice that eq. (3.97) and eq. (3.82) are related by the so-called charge conjugation symmetry $\mathbf{C} : (\text{soliton}) \rightarrow (\text{antisoliton})$, or equivalently the Z_2 symmetry by exchanging $\varphi \rightarrow -\varphi$.

3.3.2 Modified integrals of motion

Now, we will concerns with the defect contributions to the integrals of motion of the type-II Bäcklund transformations for the sine-Gordon model. Firstly, taking into account the type-II defect matrix K in (3.83) and using (3.51) and (3.53), we obtain the following results:

$$D_1^{(-1)} = -\frac{i}{2\sigma} e^{-i(\varphi_+ - \Lambda)} (e^{iq} + e^{-i\varphi_-} + \eta), \quad \widehat{D}_1^{(+1)} = \frac{i\sigma}{2} e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta), \quad (3.98)$$

$$D_2^{(-1)} = \frac{2i}{\sigma} e^{i(\varphi_+ - \Lambda)}, \quad \widehat{D}_2^{(+1)} = -2i\sigma e^{-i\Lambda}. \quad (3.99)$$

As before, the corresponding type II defect energy and momentum for the sine-Gordon model can be expressed by the introduction of the following real quantities,

$$\mathbb{D}^{(-1)} = i \left(D_1^{(-1)} - D_2^{(-1)} \right), \quad \widehat{\mathbb{D}}^{(+1)} = i \left(\widehat{D}_1^{(+1)} - \widehat{D}_2^{(+1)} \right), \quad (3.100)$$

leading to

$$\begin{aligned} E_D &= m \left(\mathbb{D}^{(-1)} - \widehat{\mathbb{D}}^{(+1)} \right) \\ &= \frac{m}{2\sigma} \left[4e^{i(\varphi_+ - \Lambda)} + e^{-i(\varphi_+ - \Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \right] + \frac{m\sigma}{2} \left[4e^{-i\Lambda} + e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \right], \end{aligned}$$

$$\begin{aligned} P_D &= m \left(\mathbb{D}^{(-1)} + \widehat{\mathbb{D}}^{(+1)} \right) \\ &= \frac{m}{2\sigma} \left[4e^{i(\varphi_+ - \Lambda)} + e^{-i(\varphi_+ - \Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \right] - \frac{m\sigma}{2} \left[4e^{-i\Lambda} + e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) \right]. \end{aligned}$$

The above results are in perfect agreement with the ones obtained from the Lagrangian approach in [10], simply by identifying $f = -m\mathbb{D}^{(-1)}$ and $g = m\widehat{\mathbb{D}}^{(+1)}$.

On the other hand, by considering the type-II defect matrix K in (3.95), we get

$$D_1^{(-1)} = -i\frac{\bar{c}_{12}}{b_{22}}e^{i(\varphi_- + \bar{\Lambda})}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) + \frac{1}{m}(\partial_t + \partial_x)(\varphi_+ + \varphi_-), \quad (3.101)$$

$$\widehat{D}_1^{(+1)} = -i\frac{a_{12}}{b_{22}}e^{i\bar{\Lambda}}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) - \frac{1}{m}(\partial_t - \partial_x)(\varphi_+ + \varphi_-), \quad (3.102)$$

$$D_2^{(-1)} = -i\frac{b_{22}}{a_{12}}e^{-i(\varphi_- + \bar{\Lambda})} + \frac{1}{m}(\partial_t + \partial_x)(\varphi_+ + \varphi_-), \quad (3.103)$$

$$\widehat{D}_2^{(+1)} = -i\frac{b_{22}}{\bar{c}_{12}}e^{-i\bar{\Lambda}} - \frac{1}{m}(\partial_t - \partial_x)(\varphi_+ + \varphi_-), \quad (3.104)$$

yielding in this case

$$E_D = m \left[\frac{\bar{c}_{12}}{b_{22}}e^{i(\bar{\Lambda} + \varphi_-)}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) - \frac{b_{22}}{a_{12}}e^{-i(\bar{\Lambda} + \varphi_-)} + \frac{b_{22}}{\bar{c}_{12}}e^{-i\bar{\Lambda}} - \frac{a_{12}}{b_{22}}e^{i\bar{\Lambda}}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) \right],$$

$$P_D = m \left[\frac{\bar{c}_{12}}{b_{22}}e^{i(\bar{\Lambda} + \varphi_-)}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) - \frac{b_{22}}{a_{12}}e^{-i(\bar{\Lambda} + \varphi_-)} - \frac{b_{22}}{\bar{c}_{12}}e^{-i\bar{\Lambda}} + \frac{a_{12}}{b_{22}}e^{i\bar{\Lambda}}(e^{i\varphi_+} + e^{-i\varphi_+} + \bar{\eta}) \right].$$

Notice that the above results reflect again the charge conjugation symmetry, $\varphi \rightarrow -\varphi$, already pointed out with respect to (3.97) and (3.81).

The Tzitzéica-Bullough-Dodd model

It was noticed some years ago that type-I defects can be also supported within the affine Toda model based on the root data of $a_n^{(1)}$ [8] from a Lagrangian point of view, where the simplest example is the sine/sinh-Gordon model based on $a_1^{(1)}$. Its defect matrix was investigated and a general form for the defect potential was derived. In [30] some aspects of the quantum description of the type-I defects within these Toda models were also provided. Additionally, the classical Lagrangian description of the type-II defects within the $a_n^{(1)}$ Toda models was proposed in [10], and its corresponding transmission matrix was also provided from a representation point of view.

Apart from the $a_n^{(1)}$ series, the type-II defects are also supported within the affine Toda model based on the twisted algebra $a_2^{(2)}$. This integrable model was firstly introduced by Tzitzéica in the study of hyperbolic surfaces, and also known as the Bullough-Dodd or Zhiber-Mikhailov-Shabat equation. This model describes a quite interesting system because it is the only relativistic two-dimensional theory which involves a single scalar field and is integrable both at the classical and at the quantum level apart from the sine-Gordon model.

In this chapter, we will present the basic aspects of the bulk theory firstly from the Lagrangian point of view and then we derive the conservation laws. Then, we discuss the integrability properties of the model with type-II defects computing explicitly its defect matrix and consequently establish its integrability by computing the corresponding modified conserved energy and momentum.

4.1 Bulk theory and associated linear problem

The Tzitzéica-Bullough-Dodd (TBD) model is described by the Lagrangian density^{*},

$$\mathcal{L} = \frac{1}{2} (\partial_t \phi)^2 - \frac{1}{2} (\partial_x \phi)^2 - \frac{1}{2} (2e^\phi + e^{-2\phi}), \quad (4.1)$$

where ϕ is a single scalar field satisfying the following equation of motion,

$$\partial_t^2 \phi - \partial_x^2 \phi = -e^\phi + e^{-2\phi}. \quad (4.2)$$

It is clear that this model does not possess Z_2 symmetry. This integrable field theory possesses soliton solutions which are complex [31, 32, 33]. The integrability properties of this model can be seen from its zero curvature representation, where its Lax pair is described by the twisted affine algebra $a_2^{(2)}$ and can be written as follows,

$$U = -\left(\frac{\partial_- v}{2v}\right) T_3 + \frac{\lambda}{2} \left(\frac{v}{\sqrt{2}} T_+ + \frac{1}{v^2} L_{-2}\right) + \frac{1}{2\lambda} \left(\frac{\sqrt{2}}{2} T_- + L_2\right), \quad (4.3)$$

$$V = \left(\frac{\partial_- v}{2v}\right) T_3 + \frac{\lambda}{2} \left(\frac{v}{\sqrt{2}} T_+ + \frac{1}{v^2} L_{-2}\right) - \frac{1}{2\lambda} \left(\frac{\sqrt{2}}{2} T_- + L_2\right), \quad (4.4)$$

where we have defined $v = e^\phi$ for simplicity and the light-cone notations $\partial_\pm = \partial_t \pm \partial_x$ has been used. The operators T_3, T_\pm and $L_{\pm 2}$ belong to the generators of the twisted affine algebra $a_2^{(2)}$ and its matricial representation are given in appendix A.2. In terms of these, we can write down the set of differential equations of the auxiliary linear problem as,

$$\partial_t \Psi_1 = \left(\frac{\partial_- v}{2v}\right) \Psi_1 + \left(\frac{i\lambda v}{2}\right) \Psi_2 + \left(\frac{1}{2\lambda}\right) \Psi_3, \quad (4.5)$$

$$\partial_t \Psi_2 = \left(\frac{i}{2\lambda}\right) \Psi_1 + \left(\frac{i\lambda v}{2}\right) \Psi_3, \quad (4.6)$$

$$\partial_t \Psi_3 = -\left(\frac{\lambda}{2v^2}\right) \Psi_1 + \left(\frac{i}{2\lambda}\right) \Psi_2 - \left(\frac{\partial_- v}{2v}\right) \Psi_3, \quad (4.7)$$

and

$$\partial_x \Psi_1 = -\left(\frac{\partial_- v}{2v}\right) \Psi_1 + \left(\frac{i\lambda v}{2}\right) \Psi_2 - \left(\frac{1}{2\lambda}\right) \Psi_3, \quad (4.8)$$

$$\partial_x \Psi_2 = -\left(\frac{i}{2\lambda}\right) \Psi_1 + \left(\frac{i\lambda v}{2}\right) \Psi_3, \quad (4.9)$$

$$\partial_x \Psi_3 = -\left(\frac{\lambda}{2v^2}\right) \Psi_1 - \left(\frac{i}{2\lambda}\right) \Psi_2 + \left(\frac{\partial_- v}{2v}\right) \Psi_3. \quad (4.10)$$

^{*}Where just for convenience we had dropped out the mass parameter m and the coupling constant β by reparametrizing $\phi \rightarrow \beta\phi$ and $x^\mu \rightarrow mx^\mu$.

Now, by defining the auxiliary functions $\Gamma_{21} = \Psi_2 \Psi_1^{-1}$ and $\Gamma_{31} = \Psi_3 \Psi_1^{-1}$, we can construct an infinite set of conservation laws from the equations (4.5) and (4.8) as follows,

$$\partial_t \left[-\frac{(\partial_- v)}{2v} + \left(\frac{i\lambda v}{2} \right) \Gamma_{21} - \frac{1}{2\lambda} \Gamma_{31} \right] = \partial_x \left[\frac{(\partial_- v)}{2v} + \left(\frac{i\lambda v}{2} \right) \Gamma_{21} + \frac{1}{2\lambda} \Gamma_{31} \right], \quad (4.11)$$

where the auxiliary functions satisfy the following coupled Riccati equations for the x -part,

$$\partial_x \Gamma_{21} = \left(\frac{(\partial_- v)}{2v} \right) \Gamma_{21} + \left(\frac{i\lambda v}{2} \right) (\Gamma_{31} - (\Gamma_{21})^2) - \frac{1}{2\lambda} (i - \Gamma_{21} \Gamma_{31}), \quad (4.12)$$

$$\partial_x \Gamma_{31} = \left(\frac{(\partial_- v)}{v} \right) \Gamma_{31} - \frac{\lambda}{2} \left(\frac{1}{v^2} + iv \Gamma_{21} \Gamma_{31} \right) - \frac{1}{2\lambda} (i\Gamma_{21} - (\Gamma_{31})^2), \quad (4.13)$$

and for the t -part,

$$\partial_t \Gamma_{21} = -\frac{(\partial_- v)}{2v} \Gamma_{21} + \frac{i\lambda v}{2} (\Gamma_{31} - (\Gamma_{21})^2) + \frac{1}{2\lambda} (i - \Gamma_{21} \Gamma_{31}), \quad (4.14)$$

$$\partial_t \Gamma_{31} = -\frac{(\partial_- v)}{v} \Gamma_{31} - \frac{\lambda}{2} \left(\frac{1}{v^2} + iv \Gamma_{21} \Gamma_{31} \right) + \frac{1}{2\lambda} (i\Gamma_{21} - (\Gamma_{31})^2). \quad (4.15)$$

Now, as usual these differential equations can be recursively solved by considering an expansion in non-positive powers of λ ,

$$\Gamma_{21} = \sum_{n=0}^{\infty} \frac{\Gamma_{21}^{(n)}}{\lambda^n}, \quad \Gamma_{31} = \sum_{n=0}^{\infty} \frac{\Gamma_{31}^{(n)}}{\lambda^n}. \quad (4.16)$$

The lowest coefficients are simply given by,

$$\Gamma_{21}^{(0)} = i(\mu v)^{-1}, \quad \Gamma_{31}^{(0)} = \mu v^{-2}, \quad (4.17)$$

$$\Gamma_{21}^{(1)} = -i(\partial_+ v)v^{-2}, \quad \Gamma_{31}^{(1)} = \mu^{-1}(\partial_+ v)v^{-3}, \quad (4.18)$$

$$\Gamma_{21}^{(2)} = -\frac{4i\mu}{3} \partial_x ((\partial_+ v)v^{-1}) v^{-1} + \frac{i\mu}{3} ((\partial_+ v)v^{-1})^2 v^{-1} + \frac{2i\mu}{3} (1 - v^{-3}), \quad (4.19)$$

$$\Gamma_{31}^{(2)} = \frac{2}{3} \partial_x ((\partial_+ v)v^{-1}) v^{-2} - \frac{2}{3} ((\partial_+ v)v^{-1})^2 v^{-2} - \frac{1}{3} (v^{-1} - v^{-4}), \quad (4.20)$$

where μ is an arbitrary constant satisfying $\mu^3 = -1$. Assuming sufficiently smooth decaying fields as $|x| \rightarrow \pm\infty$, the corresponding conserved quantities reads

$$I_1 = \int_{-\infty}^{\infty} dx \left[-\frac{(\partial_- v)}{2v} + \frac{i\lambda v}{2} \Gamma_{21} - \frac{1}{2\lambda} \Gamma_{31} \right]. \quad (4.21)$$

By substituting the expansion of the auxiliary functions into above definition, we get an infinite number of conserved charges $I_1^{(k)}$. It is very easy to check that the conserved quantities

corresponding to $k = 1$ is trivial, and for $k = 0$ we obtain a topological term. The first non-vanishing conserved charge is explicitly given by,

$$\mathbb{I}_1^{(-1)} = \frac{1}{3} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_+ v) v^{-1})^2 + \left(v + \frac{1}{2} v^{-2} \right) \right], \quad (4.22)$$

where without loss of generality, we have chosen $\mu = -1$. Then, repeating this procedure we can construct another set of conserved quantities corresponding to the expansion of the auxiliary functions in non-negative powers of λ , namely,

$$\Gamma_{21} = \sum_{n=0}^{\infty} \widehat{\Gamma}_{21}^{(n)} \lambda^n, \quad \Gamma_{31} = \sum_{n=0}^{\infty} \widehat{\Gamma}_{31}^{(n)} \lambda^n. \quad (4.23)$$

From the Ricatti equations we get,

$$\widehat{\Gamma}_{21}^{(0)} = i\mu, \quad \widehat{\Gamma}_{31}^{(0)} = \mu^{-1}, \quad (4.24)$$

$$\widehat{\Gamma}_{21}^{(1)} = 0, \quad \widehat{\Gamma}_{31}^{(1)} = -(\partial_- v) v^{-1}, \quad (4.25)$$

$$\widehat{\Gamma}_{21}^{(2)} = -\frac{2i}{3} \partial_x ((\partial_- v) v^{-1}) + \frac{i}{3} ((\partial_- v) v^{-1})^2 - \frac{i}{3} (v - v^{-2}), \quad (4.26)$$

$$\widehat{\Gamma}_{31}^{(2)} = -\frac{2\mu}{3} \partial_x ((\partial_- v) v^{-1}) + \frac{\mu}{3} ((\partial_- v) v^{-1})^2 - \frac{\mu}{3} (v - v^{-2}). \quad (4.27)$$

From these results and choosing $\mu = -1$, the first non-vanishing conserved charge is given by

$$\widehat{\mathbb{I}}_1^{(+1)} = \frac{1}{3} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_- v) v^{-1})^2 + \left(v + \frac{1}{2} v^{-2} \right) \right]. \quad (4.28)$$

Then, we clearly can combine $\mathbb{I}_1^{(-1)}$ and $\widehat{\mathbb{I}}_1^{(+1)}$ in order to obtain the usual energy and momentum quantities. However, it is not enough because we are not considering all the information coming from the Lax pair. So, it is also possible to construct another infinite sets of conserved quantities by considering two more conservation equations that can be derived from the equations (4.6), (4.7), (4.9) and (4.10), namely,

$$\partial_t \left[-\left(\frac{i}{2\lambda} \right) \Gamma_{12} + \left(\frac{i\lambda v}{2} \right) \Gamma_{32} \right] = \partial_x \left[\left(\frac{i}{2\lambda} \right) \Gamma_{12} + \left(\frac{i\lambda v}{2} \right) \Gamma_{32} \right], \quad (4.29)$$

$$\partial_t \left[\left(\frac{\partial_- v}{2v} \right) - \left(\frac{\lambda}{2v^2} \right) \Gamma_{13} - \left(\frac{i}{2\lambda} \right) \Gamma_{23} \right] = \partial_x \left[-\left(\frac{\partial_- v}{2v} \right) - \left(\frac{\lambda}{2v^2} \right) \Gamma_{13} + \left(\frac{i}{2\lambda} \right) \Gamma_{23} \right] \quad (4.30)$$

where we have introduced some other auxiliary functions $\Gamma_{12} = \Phi_1 \Phi_2^{-1}$, $\Gamma_{32} = \Phi_3 \Phi_2^{-1}$, $\Gamma_{13} = \Phi_1 \Phi_3^{-1}$, and $\Gamma_{23} = \Phi_2 \Phi_3^{-1}$. It is quite straightforward that these functions satisfy a set of

$\Gamma_{12}^{(0)} = iv$	$\Gamma_{32}^{(0)} = -iv^{-1}$	$\widehat{\Gamma}_{12}^{(0)} = i$	$\widehat{\Gamma}_{32}^{(0)} = -i$
$\Gamma_{13}^{(0)} = -v^2$	$\Gamma_{23}^{(0)} = iv$	$\widehat{\Gamma}_{13}^{(0)} = -1$	$\widehat{\Gamma}_{23}^{(0)} = i$
$\Gamma_{12}^{(1)} = -i(\partial_+ v)$	$\Gamma_{32}^{(1)} = 0$	$\widehat{\Gamma}_{12}^{(1)} = 0$	$\widehat{\Gamma}_{32}^{(1)} = -i(\partial_- v)v^{-1}$
$\Gamma_{13}^{(1)} = (\partial_+ v)v$	$\Gamma_{23}^{(1)} = 0$	$\widehat{\Gamma}_{13}^{(1)} = (\partial_- v)v^{-1}$	$\widehat{\Gamma}_{23}^{(1)} = -i(\partial_- v)v^{-1}$

Table 4.1: The zero-th and first order coefficients.

Ricatti equations that can be written for the x -part as follows,

$$\partial_x \Gamma_{12} = \left(\frac{i\lambda v}{2} \right) - \left(\frac{\partial_- v}{2v} \right) \Gamma_{12} - \left(\frac{1}{2\lambda} \right) \Gamma_{32} - \left(\frac{i\lambda v}{2} \right) (\Gamma_{12} \Gamma_{32}) + \left(\frac{i}{2\lambda} \right) (\Gamma_{12})^2, \quad (4.31)$$

$$\partial_x \Gamma_{32} = - \left(\frac{i}{2\lambda} \right) - \left(\frac{\lambda}{2v^2} \right) \Gamma_{12} + \left(\frac{\partial_- v}{2v} \right) \Gamma_{32} + \left(\frac{i}{2\lambda} \right) (\Gamma_{12} \Gamma_{32}) - \left(\frac{i\lambda v}{2} \right) (\Gamma_{32})^2, \quad (4.32)$$

$$\partial_x \Gamma_{13} = - \left(\frac{1}{2\lambda} \right) - \left(\frac{\partial_- v}{2v} \right) \Gamma_{13} + \left(\frac{i\lambda v}{2} \right) \Gamma_{23} + \left(\frac{i}{2\lambda} \right) (\Gamma_{13} \Gamma_{23}) + \left(\frac{\lambda}{2v^2} \right) (\Gamma_{13})^2, \quad (4.33)$$

$$\partial_x \Gamma_{23} = \left(\frac{i\lambda v}{2} \right) - \left(\frac{i}{2\lambda} \right) \Gamma_{13} - \left(\frac{\partial_- v}{2v} \right) \Gamma_{23} + \left(\frac{\lambda}{2v^2} \right) (\Gamma_{13} \Gamma_{23}) + \left(\frac{i}{2\lambda} \right) (\Gamma_{23})^2, \quad (4.34)$$

and for the t -part,

$$\partial_t \Gamma_{12} = \left(\frac{i\lambda v}{2} \right) + \left(\frac{\partial_- v}{2v} \right) \Gamma_{12} + \left(\frac{1}{2\lambda} \right) \Gamma_{32} - \left(\frac{i\lambda v}{2} \right) (\Gamma_{12} \Gamma_{32}) - \left(\frac{i}{2\lambda} \right) (\Gamma_{12})^2, \quad (4.35)$$

$$\partial_t \Gamma_{32} = \left(\frac{i}{2\lambda} \right) - \left(\frac{\lambda}{2v^2} \right) \Gamma_{12} - \left(\frac{\partial_- v}{2v} \right) \Gamma_{32} - \left(\frac{i}{2\lambda} \right) (\Gamma_{12} \Gamma_{32}) - \left(\frac{i\lambda v}{2} \right) (\Gamma_{32})^2, \quad (4.36)$$

$$\partial_t \Gamma_{13} = \left(\frac{1}{2\lambda} \right) + \left(\frac{\partial_- v}{2v} \right) \Gamma_{13} + \left(\frac{i\lambda v}{2} \right) \Gamma_{23} - \left(\frac{i}{2\lambda} \right) (\Gamma_{13} \Gamma_{23}) + \left(\frac{\lambda}{2v^2} \right) (\Gamma_{13})^2, \quad (4.37)$$

$$\partial_t \Gamma_{23} = \left(\frac{i\lambda v}{2} \right) + \left(\frac{i}{2\lambda} \right) \Gamma_{13} + \left(\frac{\partial_- v}{2v} \right) \Gamma_{23} + \left(\frac{\lambda}{2v^2} \right) (\Gamma_{13} \Gamma_{23}) - \left(\frac{i}{2\lambda} \right) (\Gamma_{23})^2. \quad (4.38)$$

As was already shown, these equations can be recursively solved by introducing an expansion of the respective auxiliary functions in positive and/or negative powers of the spectral parameter λ . Doing so, after a lengthy calculation the first few coefficients for these auxiliary functions can be determined, and the results are shown in tables 4.1 and 4.2.

Now, from equations (4.29) and (4.30) we obtain directly the following two generating

$\Gamma_{12}^{(2)} = -\frac{4}{3} \left[\partial_x(\partial_+ v) + \frac{1}{2v} (\partial_- v) (\partial_+ v) + \frac{1}{2} (v^{-1} - v^2) \right]$
$\Gamma_{32}^{(2)} = \frac{2}{3i} \left[\partial_x(\partial_+ v)v^{-2} + \frac{1}{2v^3} (\partial_- v) (\partial_+ v) + \frac{1}{2} (v^{-3} - 1) \right]$
$\widehat{\Gamma}_{12}^{(2)} = \frac{2}{3i} \left[\partial_x(\partial_- v)v^{-1} - \frac{1}{2v^2} (\partial_- v) (\partial_+ v) - \frac{1}{2} (v^{-2} - v) \right]$
$\widehat{\Gamma}_{32}^{(2)} = \frac{4}{3i} \left[-\partial_x(\partial_- v)v^{-1} + \frac{1}{2v^2} (\partial_- v) (\partial_+ v) + \frac{1}{2} (v^{-2} - v) \right]$
$\Gamma_{13}^{(2)} = -\frac{2}{3} \left[\partial_x(\partial_+ v)v + \frac{1}{2} (\partial_- v) (\partial_+ v) + \frac{1}{2} (1 - v^3) \right]$
$\Gamma_{23}^{(2)} = \frac{2}{3iv} \left[\partial_x(\partial_+ v)v + \frac{1}{2} (\partial_- v) (\partial_+ v) + \frac{1}{2} (1 - v^3) \right]$

Table 4.2: Second-order coefficients.

functions of the conserved quantities,

$$\mathbb{I}_2 = \int_{-\infty}^{\infty} dx \left[-\left(\frac{i}{2\lambda}\right) \Gamma_{12} + \left(\frac{iv\lambda}{2}\right) \Gamma_{32} \right], \quad (4.39)$$

$$\mathbb{I}_3 = \int_{-\infty}^{\infty} dx \left[\left(\frac{\partial_- v}{2v}\right) - \left(\frac{\lambda}{2v^2}\right) \Gamma_{13} - \left(\frac{i}{2\lambda}\right) \Gamma_{23} \right]. \quad (4.40)$$

Then, by substituting the respective expansion of each auxiliary function and using the coefficients showed in tables 4.1 and 4.2, we immediately get the first few non-vanishing conserved quantities, which are explicitly given by,

$$\mathbb{I}_2^{(-1)} = \mathbb{I}_3^{(-1)} = \frac{1}{3} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_+ v)v^{-1})^2 + \left(v + \frac{1}{2}v^{-2}\right) \right], \quad (4.41)$$

$$\widehat{\mathbb{I}}_2^{(+1)} = \widehat{\mathbb{I}}_3^{(+1)} = \frac{1}{3} \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_- v)v^{-1})^2 + \left(v + \frac{1}{2}v^{-2}\right) \right]. \quad (4.42)$$

From the above results, we can notice that there is a simple combination of all these contributions giving us the usual energy and momentum conserved quantities. In fact, if we define the following conserved quantities,

$$\mathbb{I}^{(-1)} = \mathbb{I}_1^{(-1)} + \mathbb{I}_2^{(-1)} + \mathbb{I}_3^{(-1)} = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_+ v)v^{-1})^2 + \left(v + \frac{1}{2}v^{-2}\right) \right], \quad (4.43)$$

$$\widehat{\mathbb{I}}^{(+1)} = \widehat{\mathbb{I}}_1^{(+1)} + \widehat{\mathbb{I}}_2^{(+1)} + \widehat{\mathbb{I}}_3^{(+1)} = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} ((\partial_- v)v^{-1})^2 + \left(v + \frac{1}{2}v^{-2}\right) \right], \quad (4.44)$$

the conserved energy and momentum can be written as

$$E = \frac{(\mathbb{I}^{(-1)} + \widehat{\mathbb{I}}^{(+1)})}{2} = \int_{-\infty}^{\infty} dx \left\{ \frac{1}{2} [(\partial_x \phi)^2 + (\partial_t \phi)^2] + \left(e^\phi + \frac{1}{2} e^{-2\phi} \right) \right\}, \quad (4.45)$$

$$P = \frac{(\mathbb{I}^{(-1)} - \widehat{\mathbb{I}}^{(+1)})}{2} = \int_{-\infty}^{\infty} dx \frac{(\partial_x v)(\partial_t v)}{v^2} = \int_{-\infty}^{\infty} dx (\partial_x \phi)(\partial_t \phi). \quad (4.46)$$

4.2 Type-II defect theory

The starting point to discuss the type-II defect in the TBD model is the following Lagrangian density,

$$\mathcal{L} = \theta(-x)\mathcal{L}_{\tilde{\phi}} - \theta(x)\mathcal{L}_{\phi} + \delta(x) \left[\frac{1}{2}(\phi\partial_t\tilde{\phi} - \tilde{\phi}\partial_t\phi) - \Lambda\partial_t(\tilde{\phi} - \phi) + \partial_t\Lambda(\tilde{\phi} - \phi) - B_0(\tilde{\phi}, \phi, \Lambda) \right]. \quad (4.47)$$

Here, the defect conditions can be written in terms of the field coordinates $\phi_+ = (\tilde{\phi} + \phi)/2$ and $\phi_- = (\tilde{\phi} - \phi)/2$ as follows,

$$\partial_t\phi_+ + \partial_x\phi_+ - 2\partial_t\Lambda = -\frac{1}{2} \frac{\partial B_0}{\partial \phi_-}, \quad (4.48)$$

$$\partial_t\phi_- - \partial_x\phi_- = \frac{1}{2} \frac{\partial B_0}{\partial \phi_-}, \quad (4.49)$$

$$\partial_t\phi_- + \partial_x\phi_- = -\frac{1}{2} \left(\frac{\partial B_0}{\partial \phi_-} + \frac{\partial B_0}{\partial \Lambda} \right), \quad (4.50)$$

where the defect potential is given in this case by the following form,

$$B_0(\phi_+, \phi_-, \Lambda) = -\xi \left[2ie^{\phi_+ - \Lambda} (e^{\phi_-} + e^{-\phi_-}) + e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 \right] - \frac{1}{\xi} \left[\frac{1}{4} e^{-2(\phi_+ - \Lambda)} - ie^\Lambda \right] \quad (4.51)$$

where ξ is a free parameter. From these follows the equations of motion,

$$\partial_t\phi_+ + \partial_x\phi_+ - 2\partial_t\Lambda = \xi e^{-2\Lambda} (e^{2\phi_-} - e^{-2\phi_-}) + i\xi e^{-\Lambda + \phi_+} (e^{\phi_-} - e^{-\phi_-}), \quad (4.52)$$

$$\partial_t\phi_- - \partial_x\phi_- = -\frac{1}{2} \left(2i\xi e^{\phi_+ - \Lambda} (e^{\phi_-} + e^{-\phi_-}) - \frac{e^{-2(\phi_+ - \Lambda)}}{2\xi} \right), \quad (4.53)$$

$$\partial_t\phi_- + \partial_x\phi_- = -\frac{1}{2} \left(\frac{i}{\xi} e^\Lambda + 2\xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 \right). \quad (4.54)$$

The above defect conditions represent a type-II Bäcklund transformation for the TBD model frozen at $x = 0$, firstly reported in [10][†]. It is worth noting that this Bäcklund differs from others found in the literature [34, 35, 36]. Using this Lagrangian setting, it was shown also in [10] that this Bäcklund conserves the momentum. In addition, the Lax representation was used in [14] to prove that in fact this defect is integrable, and that will be presented in the next section.

[†]By simply associating $\Lambda \rightarrow -\lambda$, $\partial_\pm \rightarrow \partial_\mp$, and $\phi \rightarrow -\phi$

4.2.1 Defect matrix and modified integrals of motion

Let us now consider the following form for the matrix K [37],

$$K_{ij} = \alpha_{ij} + \frac{1}{\lambda}\beta_{ij} + \frac{1}{\lambda^2}\delta_{ij} + \frac{1}{\lambda^3}\gamma_{ij}. \quad (4.55)$$

is a solution for the differential equations,

$$\partial_{\pm}K = KA_{\pm} - \tilde{A}_{\pm}K, \quad (4.56)$$

where it has been used the light-cone coordinates and A_{\pm} is given by,

$$A_+ = \begin{pmatrix} 0 & -i\lambda e^{\phi} & 0 \\ 0 & 0 & -i\lambda e^{\phi} \\ \lambda e^{-2\phi} & 0 & 0 \end{pmatrix}, \quad A_- = \begin{pmatrix} -\partial_- \phi & 0 & -\frac{1}{\lambda} \\ -\frac{i}{\lambda} & 0 & 0 \\ 0 & -\frac{i}{\lambda} & \partial_- \phi \end{pmatrix}, \quad (4.57)$$

Then, equation (4.56) decomposes into three independent systems of equations. We will consider the set involving the variables $\{\alpha_{11}, \alpha_{22}, \alpha_{33}, \beta_{13}, \beta_{21}, \beta_{32}, \delta_{12}, \delta_{23}, \delta_{31}, \gamma_{11}, \gamma_{22}, \gamma_{33}\}$ such that

$$K = \begin{bmatrix} \alpha_{11} + \frac{1}{\lambda^3}\gamma_{11} & \frac{1}{\lambda^2}\delta_{12} & \frac{1}{\lambda}\beta_{13} \\ \frac{1}{\lambda}\beta_{21} & \alpha_{22} + \frac{1}{\lambda^3}\gamma_{22} & \frac{1}{\lambda^2}\delta_{23} \\ \frac{1}{\lambda^2}\delta_{31} & \frac{1}{\lambda}\beta_{32} & \alpha_{33} + \frac{1}{\lambda^3}\gamma_{33} \end{bmatrix}. \quad (4.58)$$

Equations (4.56) with K given above are satisfied for

$$\gamma_{11} = \gamma_{22} = \gamma_{33} = \nu, \quad (4.59)$$

$$\alpha_{11} = \xi \left(\frac{\nu}{\tilde{v}} \right) = \alpha, \quad \alpha_{22} = \xi, \quad \alpha_{33} = \xi \left(\frac{\tilde{v}}{\nu} \right) = \frac{\xi^2}{\alpha}, \quad (4.60)$$

where we have defined again $\nu = e^{\phi}$ and ν and ξ are in principle two arbitrary constants. Introducing the following reparametrization,

$$\beta_{21} = \frac{Y}{\tilde{v}} = \frac{\alpha Y}{\xi \nu}, \quad (4.61)$$

the matrix K given in (4.58) takes the form,

$$K = \begin{bmatrix} \alpha + \nu \lambda^{-3} & \frac{2\xi \nu \nu}{\alpha Y} (\alpha + \xi) \lambda^{-2} & \frac{2\xi^2 \nu^2 \nu}{\alpha^2 Y^2} (\alpha + \xi)^2 \lambda^{-1} \\ \frac{\alpha Y}{\xi \nu} \lambda^{-1} & \xi + \nu \lambda^{-3} & \frac{2\xi \nu \nu}{\alpha Y} (\alpha + \xi) \lambda^{-2} \\ \frac{\alpha Y^2}{2\nu^2 \xi^2} \lambda^{-2} & \frac{Y}{\nu} \lambda^{-1} & \xi^2 \frac{1}{\alpha} + \nu \lambda^{-3} \end{bmatrix}. \quad (4.62)$$

The fields α and Y satisfy the following partial differential equations,

$$\partial_+ \alpha - i \frac{\alpha Y}{\xi} - \frac{2\nu}{Y^2} (\alpha + \xi)^2 = 0, \quad (4.63)$$

$$\frac{1}{v} \partial_+ Y - \frac{Y}{v^2} \partial_+ v + \frac{2\nu \xi}{\alpha v Y} (\alpha + \xi) + i \frac{Y^2}{2v\xi} = 0, \quad (4.64)$$

$$\partial_- \alpha - i \frac{2\nu v \xi}{Y} (\alpha + \xi) + \frac{\alpha^2 Y^2}{2v^2 \xi^2} = 0, \quad (4.65)$$

$$\frac{\partial_- Y}{v} + i \xi \left(\frac{\xi}{\alpha} - 1 \right) = 0. \quad (4.66)$$

We now define the functions

$$\phi_+ = \frac{\tilde{\phi} + \phi}{2}, \quad \phi_- = \frac{\tilde{\phi} - \phi}{2}, \quad \alpha = \xi e^{-2\phi_-}. \quad (4.67)$$

Then, after a few manipulations the equations (4.63) to (4.66) become

$$\partial_+ \phi_- = -\frac{1}{2} \left(\frac{i}{\xi} e^\Lambda + 2\xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 \right), \quad (4.68)$$

$$\partial_- \phi_- = -\frac{1}{2} \left(2i\xi e^{\phi_+ - \Lambda} (e^{\phi_-} + e^{-\phi_-}) - \frac{e^{-2(\phi_+ - \Lambda)}}{2\xi} \right), \quad (4.69)$$

$$\partial_+ (\phi_+ - \Lambda) = \xi e^{-2\Lambda} (e^{2\phi_-} - e^{-2\phi_-}), \quad (4.70)$$

$$\partial_- \Lambda = i\xi e^{\phi_+ - \Lambda} (e^{-\phi_-} - e^{\phi_-}), \quad (4.71)$$

where $Y \equiv e^\Lambda$ and ν is taken to be equal to 1. Note that if the above equations are considered to be frozen at $x = 0$, the sum of the equations (4.70) and (4.71) together with (4.68) and (4.69) we obtain the type-II defect conditions for the TBD model derived previously by variational principle in (4.52)-(4.54). From these equations it is clear from (4.70) and (4.71) that the compatibility condition $\partial_+ \partial_- \phi = \partial_- \partial_+ \phi$ is satisfied. In addition, cross-differentiating (4.68) and (4.69) we find the field equation satisfied by Λ , namely

$$\partial_+ \partial_- \Lambda = \frac{1}{2} \left(e^{\tilde{\phi}} + e^\phi \right) (1 + 8i\xi^2 e^{-3\Lambda}), \quad (4.72)$$

which depends on the fields $\tilde{\phi}$ and ϕ as it was expected. Of course, when we just consider the defect conditions, i.e the frozen Bäcklund transformation, Λ only depends on t and is defined by the three equations (4.52)-(4.54). Finally, it is worth noting that the vacuum solution in the bulk $\tilde{\phi} = \phi = 0$ satisfies the defect conditions, if the auxiliary field is one of three roots that satisfies the equation,

$$(e^\Lambda)^3 = 8i\xi^2. \quad (4.73)$$

Let us now to compute the modified conserved charges coming from the defect contributions for the Tzitzéica-Bullough-Dodd model. By considering first the set of infinite charges given by (4.21), we have that the defect contributions are explicitly given by

$$D_1 = -\ln \left[K_{11} + K_{12}\Gamma_{21} + K_{13}\Gamma_{31} \right] \Big|_{x=0}. \quad (4.74)$$

As it was already emphasized, from the above formula we can derive the two sets of defect contribution by considering the corresponding expansion of the auxiliary functions in positive and negative powers of λ . In particular for the $\lambda^{\pm 1}$ -terms, we obtain

$$D_1^{(-1)} = 2\xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2, \quad \widehat{D}_1^{(+1)} = 2i\xi e^{(\phi_+ - \Lambda)} (e^{\phi_-} + e^{-\phi_-}). \quad (4.75)$$

As we now know, to obtain the exact form of the corresponding defect contributions to the energy and momentum, we need to consider the others conservation equations and consequently the other charges given in (4.39) and (4.40). Applying the same steps to derive the defect contributions, one gets

$$D_2 = -\ln \left[K_{21}\Gamma_{12} + K_{22} + K_{23}\Gamma_{32} \right] \Big|_{x=0}, \quad (4.76)$$

$$D_3 = -\ln \left[K_{31}\Gamma_{13} + K_{32}\Gamma_{23} + K_{33} \right] \Big|_{x=0}, \quad (4.77)$$

leading to

$$D_2^{(-1)} = -\frac{i}{\xi} e^\Lambda, \quad \widehat{D}_2^{(+1)} = 2i\xi e^{(\phi_+ - \Lambda)} (e^{\phi_-} + e^{-\phi_-}), \quad (4.78)$$

$$D_3^{(-1)} = -\frac{i}{\xi} e^\Lambda, \quad \widehat{D}_3^{(+1)} = \frac{1}{2} e^{2(\Lambda - \phi_+)}, \quad (4.79)$$

Now, defining by analogy the following two conserved quantities,

$$\mathbb{D}^{(-1)} = D_1^{(-1)} + D_2^{(-1)} + D_3^{(-1)} = 2\xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 - \frac{2i}{\xi} e^\Lambda, \quad (4.80)$$

$$\widehat{\mathbb{D}}^{(+1)} = \widehat{D}_1^{(+1)} + \widehat{D}_2^{(+1)} + \widehat{D}_3^{(+1)} = 4i\xi e^{(\phi_+ - \Lambda)} (e^{\phi_-} + e^{-\phi_-}) + \frac{1}{2} e^{2(\Lambda - \phi_+)}, \quad (4.81)$$

we can write down the defect energy and momentum as follows,

$$E_D = \frac{(\mathbb{D}^{(-1)} + \widehat{\mathbb{D}}^{(+1)})}{2} = \xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 - \frac{i}{\xi} e^\Lambda + 2i\xi e^{(\phi_+ - \Lambda)} (e^{\phi_-} + e^{-\phi_-}) + \frac{1}{4} e^{2(\Lambda - \phi_+)},$$

$$P_D = \frac{(\mathbb{D}^{(-1)} - \widehat{\mathbb{D}}^{(+1)})}{2} = \xi e^{-2\Lambda} (e^{\phi_-} + e^{-\phi_-})^2 - \frac{i}{\xi} e^\Lambda - 2i\xi e^{(\phi_+ - \Lambda)} (e^{\phi_-} + e^{-\phi_-}) - \frac{1}{4} e^{2(\Lambda - \phi_+)}.$$

These are exactly the defect energy and momentum which are obtained using the Lagrangian formalism. In particular, we can note that as it was expected $E_D = -B_0$ in (4.51).

The Grassmannian massive Thirring model

The massive Thirring model is a two-dimensional exactly solvable quantum field theory which describes Dirac fermions with local (self-)interaction, namely a current-current interaction [38]. This model has been widely studied not only because of its purely theoretical relevance but also because of its interesting connections with the physics of strongly correlated systems in one spatial dimension [39] and for its quantum equivalence with the sine-Gordon model [40, 41]. In this thesis we are not concerned with its importance as a solvable quantum field theory model but rather restrict our attention to its integrability properties from a classical point of view. The classical version of the model is described by Grassmannian fields and its integrability properties has already been established in the bulk theory [42, 43].

Some years ago, it was shown that the massive Thirring model can be restricted to the half-line $(-\infty, 0)$, without losing integrability by imposing suitable boundary conditions and the relation to the boundary sine-Gordon was provided [44, 45]. These ideas motivated to consider the natural connection between the defect sine-Gordon model and the possible existence of integrable defects within the Thirring model related to its Bäcklund transformation.

In this chapter, we will discuss the integrability properties of the Grassmannian massive Thirring (GMT) model in the presence of the type-II defects, which are naturally related with its Bäcklund transformation [46]. Firstly, we present some basic aspects of the bulk theory and construct the conservation laws using our method. Then, we provide the Lagrangian formalism and derive the defect conditions preserving the modified momentum. Finally, we construct the defect matrix to ensure that the defects do not spoil the integrability.

5.1 The bulk theory and associated linear problem

The massive Thirring model [38] is a theory of self-coupled Dirac fields in two-dimensional space-time defined by the following Lagrangian density

$$\mathcal{L} = \bar{\psi} \left(i\gamma^\mu \overleftrightarrow{\partial}_\mu - m \right) \psi - \frac{g}{2} j_\mu j^\mu, \quad (5.1)$$

where g is the coupling constant and $j^\mu = \bar{\psi}\gamma^\mu\psi$ is the fermionic current. In two-dimensional space-time the γ -matrices are given in terms of the Pauli matrices, namely $\gamma^0 = \sigma_1, \gamma^1 = -i\sigma_2$ and $\gamma_5 = \sigma_3$. The metric tensor is simply defined by $g^{00} = -g^{11} = 1$ and $g^{01} = 0$. This massive model possesses $U(1)$ -vectorial invariance and the massless case $U(1)$ axial invariance, i.e

$$\psi(x, t) \longrightarrow \psi'(x, t) = e^{i\alpha v} \psi(x, t), \quad (5.2)$$

$$\psi(x, t) \longrightarrow \psi'(x, t) = e^{i\alpha_A \gamma_5} \psi(x, t), \quad \text{for } m = 0. \quad (5.3)$$

By using the fields components, $\psi = (\psi_1(x, t), \psi_2(x, t))$, it is possible to rewrite the Lagrangian density in the following way

$$\begin{aligned} \mathcal{L} &= \frac{i}{2} \psi_1 (\partial_t - \partial_x) \psi_1^\dagger + \frac{i}{2} \psi_1^\dagger (\partial_t - \partial_x) \psi_1 + \frac{i}{2} \psi_2 (\partial_t + \partial_x) \psi_2^\dagger + \frac{i}{2} \psi_2^\dagger (\partial_t + \partial_x) \psi_2 \\ &- m (\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) - g (\psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2), \end{aligned} \quad (5.4)$$

Then, the equations of motion which are obtained from this Lagrangian have the form,

$$i(\partial_t - \partial_x) \psi_1 = m \psi_2 + g \psi_2^\dagger \psi_2 \psi_1, \quad (5.5)$$

$$i(\partial_t + \partial_x) \psi_2 = m \psi_1 + g \psi_1^\dagger \psi_1 \psi_2, \quad (5.6)$$

$$i(\partial_t - \partial_x) \psi_1^\dagger = -m \psi_2^\dagger - g \psi_2^\dagger \psi_2 \psi_1^\dagger, \quad (5.7)$$

$$i(\partial_t + \partial_x) \psi_2^\dagger = -m \psi_1^\dagger - g \psi_1^\dagger \psi_1 \psi_2^\dagger. \quad (5.8)$$

These equations can be described as a compatibility condition of the following associated linear problem,

$$\partial_x \Psi(x, t; \lambda) = U(x, t; \lambda) \Psi(x, t; \lambda), \quad (5.9)$$

$$\partial_t \Psi(x, t; \lambda) = V(x, t; \lambda) \Psi(x, t; \lambda). \quad (5.10)$$

where the three-wave function Ψ has the form $(\Psi_1, \Psi_2, \Psi_3)^T$ and the Lax pair U, V are 3×3 matrices belong to the $\mathfrak{sl}(2, 1)$ Lie algebra (see Appendix A.3), and can be explicitly written as follows,

$$U = \frac{ig\rho_-}{2} h_1 + \frac{im}{2} (\lambda^2 - \lambda^{-2}) (h_1 + 2h_2) + r_1 E_{-(\alpha_1 + \alpha_2)} + r_2 E_{-\alpha_2} + q_1 E_{\alpha_1 + \alpha_2} + q_2 E_{\alpha_2} \quad (5.11)$$

$$V = -\frac{ig\rho_+}{2} h_1 + \frac{im}{2} (\lambda^{-2} + \lambda^2) (h_1 + 2h_2) - q_2 E_{-(\alpha_1 + \alpha_2)} - q_1 E_{-\alpha_2} - r_2 E_{\alpha_1 + \alpha_2} - r_1 E_{\alpha_2} \quad (5.12)$$

where just for simplicity, we have also defined $\rho_{\pm} = (\psi_2^\dagger \psi_2 \pm \psi_1^\dagger \psi_1)$ and the functions

$$q_1 = -i\sqrt{\frac{mg}{2}} (\lambda\psi_1 + \lambda^{-1}\psi_2), \quad q_2 = i\sqrt{\frac{mg}{2}} (\lambda\psi_1^\dagger - \lambda^{-1}\psi_2^\dagger), \quad (5.13)$$

$$r_1 = -i\sqrt{\frac{mg}{2}} (\lambda\psi_1^\dagger + \lambda^{-1}\psi_2^\dagger), \quad r_2 = i\sqrt{\frac{mg}{2}} (\lambda\psi_1 - \lambda^{-1}\psi_2). \quad (5.14)$$

Now, the set of differential equations (5.9) and (5.10) read in components,

$$\partial_x \Psi_1 = \left[\frac{ig}{2} \rho_- + \frac{im}{2} (\lambda^2 - \lambda^{-2}) \right] \Psi_1 + q_1 \Psi_3, \quad (5.15)$$

$$\partial_x \Psi_2 = - \left[\frac{ig}{2} \rho_- - \frac{im}{2} (\lambda^2 - \lambda^{-2}) \right] \Psi_2 + q_2 \Psi_3, \quad (5.16)$$

$$\partial_x \Psi_3 = r_1 \Psi_1 + r_2 \Psi_2 + im (\lambda^2 - \lambda^{-2}) \Psi_3, \quad (5.17)$$

and

$$\partial_t \Psi_1 = - \left[\frac{ig}{2} \rho_+ - \frac{im}{2} (\lambda^2 + \lambda^{-2}) \right] \Psi_1 - r_2 \Psi_3, \quad (5.18)$$

$$\partial_t \Psi_2 = \left[\frac{ig}{2} \rho_+ + \frac{im}{2} (\lambda^2 + \lambda^{-2}) \right] \Psi_2 - r_1 \Psi_3, \quad (5.19)$$

$$\partial_t \Psi_3 = -q_2 \Psi_1 - q_1 \Psi_2 + im (\lambda^2 + \lambda^{-2}) \Psi_3. \quad (5.20)$$

Now, by defining the auxiliary functions $\Gamma_{21} = \Psi_2 \Psi_1^{-1}$ and $\Gamma_{31} = \Psi_3 \Psi_1^{-1}$, we obtain a first conservation law from (5.15) and (5.18), namely,

$$\partial_t \left[q_1 \Gamma_{31} + \frac{ig}{2} \rho_- \right] + \partial_x \left[r_2 \Gamma_{31} + \frac{ig}{2} \rho_+ \right] = 0, \quad (5.21)$$

where Γ_{21} and Γ_{31} satisfy the following coupled Riccati equations for the x -part,

$$\partial_x \Gamma_{21} = -(ig\rho_-) \Gamma_{21} + q_2 \Gamma_{31} - q_1 \Gamma_{21} \Gamma_{31}, \quad (5.22)$$

$$\partial_x \Gamma_{31} = r_1 + r_2 \Gamma_{21} - \frac{i}{2} [g\rho_- - m (\lambda^2 - \lambda^{-2})] \Gamma_{31}, \quad (5.23)$$

and for the t -part,

$$\partial_t \Gamma_{21} = (ig\rho_+) \Gamma_{21} - r_1 \Gamma_{31} + r_2 \Gamma_{21} \Gamma_{31}, \quad (5.24)$$

$$\partial_t \Gamma_{31} = -q_2 - q_1 \Gamma_{21} + \frac{i}{2} [g\rho_+ + m (\lambda^2 + \lambda^{-2})] \Gamma_{31}. \quad (5.25)$$

Now, by firstly considering an expansion in inverse powers of λ for the auxiliary functions as,

$$\Gamma_{ij}(x, t; \lambda) = \sum_{k=1}^{\infty} \frac{\Gamma_{ij}^{(k)}(x, t)}{\lambda^k}, \quad (5.26)$$

and inserting this in the Riccati equations (5.22) and (5.23) we find that the first coefficients of the expansion are given by

$$\Gamma_{31}^{(1)} = \sqrt{\frac{2g}{m}} \psi_1^\dagger, \quad \Gamma_{31}^{(2)} = - \left(\sqrt{\frac{2g}{m}} \psi_1 \right) \Gamma_{21}^{(1)}, \quad (5.27)$$

$$\Gamma_{31}^{(3)} = -\frac{2i}{m} \left[\sqrt{\frac{2g}{m}} (\partial_x \psi_1^\dagger) + i \sqrt{\frac{mg}{2}} (\psi_2^\dagger - \psi_1 \Gamma_{21}^{(2)}) + \frac{ig}{2} \sqrt{\frac{2g}{m}} (\psi_2^\dagger \psi_2) \psi_1^\dagger \right], \quad (5.28)$$

where $\Gamma_{21}^{(1)}$ and $\Gamma_{21}^{(2)}$ satisfy the following differential equations,

$$\partial_x \Gamma_{21}^{(1)} = -ig (\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1) \Gamma_{21}^{(1)}, \quad (5.29)$$

$$\partial_x \Gamma_{21}^{(2)} = -ig (\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1) \Gamma_{21}^{(2)} + \frac{2g}{m} (\psi_1^\dagger \partial_x \psi_1^\dagger) + 2ig \psi_1^\dagger \psi_2^\dagger. \quad (5.30)$$

Then, we find out in this case that the associated generating function of the conserved quantities are given by,

$$\mathbb{I}_1 = \int_{-\infty}^{\infty} dx \left[q_1 \Gamma_{31} + \frac{ig}{2} \rho_- \right], \quad (5.31)$$

and substituting the respective coefficients for the auxiliary function Γ_{31} in the expansion in λ , we found that the lowest conserved quantities are given,

$$\mathbb{I}_1^{(0)} = \frac{ig}{2} \int_{-\infty}^{\infty} dx \left[\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right], \quad (5.32)$$

$$\mathbb{I}_1^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} (\psi_1 \partial_x \psi_1^\dagger) + ig (\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2) + \frac{ig^2}{m} (\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1) \right]. \quad (5.33)$$

Now, to compute a second infinite set of conserved quantities we have to expand the auxiliary functions around $\lambda = 0$, i.e, in positive powers of λ , as follows

$$\Gamma_{ij}(x, t; \lambda) = \sum_{k=1}^{\infty} \widehat{\Gamma}_{ij}^{(k)}(x, t) \lambda^k. \quad (5.34)$$

By following the same procedure, we obtain that the respective first few coefficients for each expansion are given by,

$$\widehat{\Gamma}_{31}^{(1)} = -\sqrt{\frac{2g}{m}} \psi_2^\dagger, \quad \widehat{\Gamma}_{31}^{(2)} = - \left(\sqrt{\frac{2g}{m}} \psi_2 \right) \widehat{\Gamma}_{21}^{(1)}, \quad (5.35)$$

$$\widehat{\Gamma}_{31}^{(3)} = -\frac{2i}{m} \left[\sqrt{\frac{2g}{m}} (\partial_x \psi_2^\dagger) - i \sqrt{\frac{mg}{2}} (\psi_1^\dagger + \psi_2 \widehat{\Gamma}_{21}^{(2)}) - \frac{ig}{2} \sqrt{\frac{2g}{m}} (\psi_1^\dagger \psi_1) \psi_2^\dagger \right], \quad (5.36)$$

with

$$\partial_x \widehat{\Gamma}_{21}^{(1)} = ig \left(\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right) \widehat{\Gamma}_{21}^{(1)}, \quad (5.37)$$

$$\partial_x \widehat{\Gamma}_{21}^{(2)} = ig \left(\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right) \widehat{\Gamma}_{21}^{(2)} - \frac{2g}{m} \left(\psi_2^\dagger \partial_x \psi_2^\dagger \right) - 2ig \psi_1^\dagger \psi_2^\dagger. \quad (5.38)$$

Then, we have that the corresponding first charges associated to this expansion of Γ_{31} are given as follows,

$$\widehat{\Gamma}_1^{(0)} = -\frac{ig}{2} \int_{-\infty}^{\infty} dx \left[\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right], \quad (5.39)$$

$$\widehat{\Gamma}_1^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} \left(\psi_2^\dagger \partial_x \psi_2^\dagger \right) - ig \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) - \frac{ig^2}{m} \left(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right) \right]. \quad (5.40)$$

Clearly, these charges are not totally real and therefore it is necessary to add the hermitian conjugate terms. To do that, we need to consider other contributions coming from two more conservation equations that can be derived using (5.16), (5.17), (5.19) and (5.20), namely

$$\partial_t \left[q_2 \Gamma_{32} - \frac{ig}{2} \rho_- \right] + \partial_x \left[r_1 \Gamma_{32} - \frac{ig}{2} \rho_+ \right] = 0, \quad (5.41)$$

$$\partial_t \left[r_1 \Gamma_{13} + r_2 \Gamma_{23} \right] + \partial_x \left[q_2 \Gamma_{13} + q_1 \Gamma_{23} \right] = 0, \quad (5.42)$$

where we have introduced some other auxiliary functions $\Gamma_{12} = \Psi_1 \Psi_2^{-1}$, $\Gamma_{32} = \Psi_3 \Psi_2^{-1}$, $\Gamma_{13} = \Psi_1 \Psi_3^{-1}$, and $\Gamma_{23} = \Psi_2 \Psi_3^{-1}$. It is very easy to check that the set of Riccati equations satisfied by these auxiliary functions can be written as,

$$\partial_x \Gamma_{12} = (ig \rho_-) \Gamma_{12} + q_1 \Gamma_{32} - q_2 \Gamma_{12} \Gamma_{32}, \quad (5.43)$$

$$\partial_x \Gamma_{32} = r_2 + r_1 \Gamma_{12} + \frac{i}{2} (g \rho_- + m(\lambda^2 - \lambda^{-2})) \Gamma_{32}, \quad (5.44)$$

$$\partial_x \Gamma_{13} = q_1 + \frac{i}{2} [g \rho_- - m(\lambda^2 - \lambda^{-2})] \Gamma_{13} + r_2 \Gamma_{13} \Gamma_{23}, \quad (5.45)$$

$$\partial_x \Gamma_{23} = q_2 - \frac{i}{2} [g \rho_- + m(\lambda^2 - \lambda^{-2})] \Gamma_{23} - r_1 \Gamma_{13} \Gamma_{23}, \quad (5.46)$$

and

$$\partial_t \Gamma_{12} = -(ig \rho_+) \Gamma_{12} - r_2 \Gamma_{32} - r_1 \Gamma_{12} \Gamma_{32}, \quad (5.47)$$

$$\partial_t \Gamma_{32} = -q_1 - q_2 \Gamma_{12} - \frac{i}{2} [g \rho_+ - m(\lambda^2 + \lambda^{-2})] \Gamma_{32} \quad (5.48)$$

$$\partial_t \Gamma_{13} = -r_2 - \frac{i}{2} [g \rho_+ + m(\lambda^2 + \lambda^{-2})] \Gamma_{13} - q_1 \Gamma_{13} \Gamma_{23} \quad (5.49)$$

$$\partial_t \Gamma_{23} = -r_1 + \frac{i}{2} [g \rho_+ - m(\lambda^2 + \lambda^{-2})] \Gamma_{23} + q_2 \Gamma_{13} \Gamma_{23}. \quad (5.50)$$

Now, these equations are solved by expanding each of the auxiliary functions in positive and negative powers of the spectral parameter λ . Performing similar computations, the first few coefficients for these auxiliary functions can be determined, and the results read

$$\Gamma_{23}^{(1)} = \sqrt{\frac{2g}{m}} \psi_1^\dagger, \quad \Gamma_{23}^{(1)} = \Gamma_{32}^{(1)} = \sqrt{\frac{2g}{m}} \psi_1, \quad \widehat{\Gamma}_{13}^{(1)} = -\widehat{\Gamma}_{32}^{(1)} = \sqrt{\frac{2g}{m}} \psi_2, \quad (5.51)$$

$$\widehat{\Gamma}_{23}^{(1)} = \sqrt{\frac{2g}{m}} \psi_2^\dagger, \quad \Gamma_{32}^{(2)} = \sqrt{\frac{2g}{m}} \psi_1^\dagger, \quad \widehat{\Gamma}_{32}^{(2)} = -\sqrt{\frac{2g}{m}} \psi_2^\dagger, \quad (5.52)$$

$$\Gamma_{13}^{(2)} = \Gamma_{23}^{(2)} = \widehat{\Gamma}_{13}^{(2)} = \widehat{\Gamma}_{23}^{(2)} = 0, \quad (5.53)$$

and

$$\Gamma_{32}^{(3)} = \frac{2}{m} \left[i\sqrt{\frac{2g}{m}} (\partial_x \psi_1) + \sqrt{\frac{mg}{2}} (\psi_2 + \psi_1^\dagger \Gamma_{12}^{(2)}) + \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_2^\dagger \psi_2) \psi_1 \right], \quad (5.54)$$

$$\widehat{\Gamma}_{32}^{(3)} = \frac{2}{m} \left[-i\sqrt{\frac{2g}{m}} (\partial_x \psi_2) + \sqrt{\frac{mg}{2}} (\psi_1 - \psi_2^\dagger \widehat{\Gamma}_{12}^{(2)}) + \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_1^\dagger \psi_1) \psi_2 \right], \quad (5.55)$$

$$\Gamma_{13}^{(3)} = \frac{2}{m} \left[-i\sqrt{\frac{2g}{m}} (\partial_x \psi_1) - \sqrt{\frac{mg}{2}} \psi_2 - \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_2^\dagger \psi_2) \psi_1 \right], \quad (5.56)$$

$$\widehat{\Gamma}_{13}^{(3)} = \frac{2}{m} \left[-i\sqrt{\frac{2g}{m}} (\partial_x \psi_2) + \sqrt{\frac{mg}{2}} \psi_1 + \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_1^\dagger \psi_1) \psi_2 \right], \quad (5.57)$$

$$\Gamma_{23}^{(3)} = \frac{2}{m} \left[i\sqrt{\frac{2g}{m}} (\partial_x \psi_1^\dagger) - \sqrt{\frac{mg}{2}} \psi_2^\dagger - \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_2^\dagger \psi_2) \psi_1^\dagger \right], \quad (5.58)$$

$$\widehat{\Gamma}_{23}^{(3)} = \frac{2}{m} \left[-i\sqrt{\frac{2g}{m}} (\partial_x \psi_2^\dagger) - \sqrt{\frac{mg}{2}} \psi_1^\dagger - \frac{g}{2} \sqrt{\frac{2g}{m}} (\psi_1^\dagger \psi_1) \psi_2^\dagger \right], \quad (5.59)$$

together with the following relations,

$$\partial_x \Gamma_{12}^{(1)} = ig (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \Gamma_{12}^{(1)}, \quad (5.60)$$

$$\partial_x \widehat{\Gamma}_{12}^{(1)} = -ig (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \widehat{\Gamma}_{12}^{(1)}, \quad (5.61)$$

$$\partial_x \Gamma_{12}^{(2)} = ig (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \Gamma_{12}^{(2)} + \frac{2g}{m} (\psi_1 \partial_x \psi_1) - 2ig (\psi_1 \psi_2), \quad (5.62)$$

$$\partial_x \widehat{\Gamma}_{12}^{(2)} = -ig (\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \widehat{\Gamma}_{12}^{(2)} - \frac{2g}{m} (\psi_2 \partial_x \psi_2) + 2ig (\psi_1 \psi_2). \quad (5.63)$$

Now, we will compute the corresponding conserved quantities from the conservation equations (5.41) and (5.42), namely

$$\mathbb{I}_2 = \int_{-\infty}^{\infty} dx \left[q_2 \Gamma_{32} - \frac{ig}{2} \rho_- \right], \quad (5.64)$$

$$\mathbb{I}_3 = \int_{-\infty}^{\infty} dx \left[r_1 \Gamma_{13} + r_2 \Gamma_{23} \right]. \quad (5.65)$$

Therefore, by a straightforward substitution of the each expansion coefficient, we easily get the following results,

$$\mathbb{I}_2^{(0)} = -\widehat{\mathbb{I}}_2^{(0)} = -\frac{ig}{2} \int_{-\infty}^{\infty} dx \left[\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right], \quad \mathbb{I}_3^{(0)} = \widehat{\mathbb{I}}_3^{(0)} = 0, \quad (5.66)$$

and

$$\mathbb{I}_2^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} \left(\psi_1^\dagger \partial_x \psi_1 \right) + ig \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) + \frac{ig^2}{m} \left(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right) \right], \quad (5.67)$$

$$\widehat{\mathbb{I}}_2^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} \left(\psi_2^\dagger \partial_x \psi_2 \right) - ig \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) - \frac{ig^2}{m} \left(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right) \right], \quad (5.68)$$

$$\mathbb{I}_3^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} \left(\psi_1^\dagger \partial_x \psi_1 + \psi_1 \partial_x \psi_1^\dagger \right) + 2ig \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) + \frac{2ig^2}{m} \left(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right) \right], \quad (5.69)$$

$$\widehat{\mathbb{I}}_3^{(2)} = \int_{-\infty}^{\infty} dx \left[-\frac{2g}{m} \left(\psi_2^\dagger \partial_x \psi_2 + \psi_2 \partial_x \psi_2^\dagger \right) - 2ig \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) - \frac{2ig^2}{m} \left(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right) \right]. \quad (5.70)$$

Then, from all these conserved quantities together with the ones derived in (5.32), (5.33), (5.39), and (5.40), we can notice that

$$\mathbb{I}_1^{(n)} + \mathbb{I}_2^{(n)} = \mathbb{I}_3^{(n)}, \quad \widehat{\mathbb{I}}_1^{(n)} + \widehat{\mathbb{I}}_2^{(n)} = \widehat{\mathbb{I}}_3^{(n)}. \quad (5.71)$$

Therefore, it is convenient to define the following quantities,

$$\mathbb{I}^{(0)} = (\mathbb{I}_1^{(0)} - \mathbb{I}_2^{(0)} - \mathbb{I}_3^{(0)}), \quad \widehat{\mathbb{I}}^{(0)} = (\widehat{\mathbb{I}}_1^{(0)} - \widehat{\mathbb{I}}_2^{(0)} - \widehat{\mathbb{I}}_3^{(0)}), \quad (5.72)$$

$$\mathbb{I}^{(2)} = (\mathbb{I}_1^{(2)} + \mathbb{I}_2^{(2)} + \mathbb{I}_3^{(2)}), \quad \widehat{\mathbb{I}}^{(2)} = (\widehat{\mathbb{I}}_1^{(2)} + \widehat{\mathbb{I}}_2^{(2)} + \widehat{\mathbb{I}}_3^{(2)}), \quad (5.73)$$

in order to get the usual conserved number of occupation, energy and momentum for the GT model by performing a simple combination, namely

$$N = \frac{1}{2ig} \left[\mathbb{I}^{(0)} - \widehat{\mathbb{I}}^{(0)} \right] = \int_{-\infty}^{\infty} dx \left[\psi_2^\dagger \psi_2 + \psi_1^\dagger \psi_1 \right], \quad (5.74)$$

$$E = \frac{m}{8ig} \left[\mathbb{I}^{(2)} - \widehat{\mathbb{I}}^{(2)} \right] = \int_{-\infty}^{\infty} dx \left[\frac{i}{2} \left(\psi_1 \partial_x \psi_1^\dagger + \psi_1^\dagger \partial_x \psi_1 - \psi_2 \partial_x \psi_2^\dagger - \psi_2^\dagger \partial_x \psi_2 \right) \right. \\ \left. + m \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) + g \psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 \right], \quad (5.75)$$

$$P = \frac{m}{8ig} \left[\mathbb{I}^{(2)} + \widehat{\mathbb{I}}^{(2)} \right] = \int_{-\infty}^{\infty} dx \left[\frac{i}{2} \left(\psi_1 \partial_x \psi_1^\dagger + \psi_1^\dagger \partial_x \psi_1 + \psi_2 \partial_x \psi_2^\dagger + \psi_2^\dagger \partial_x \psi_2 \right) \right]. \quad (5.76)$$

5.2 The defect theory

In this section we introduce a defect placed at the origin $x = 0$, and then derive the corresponding defect contributions to the modified conserved quantities firstly by implementing defect conditions on the field through the addition of the respective boundary term to the bulk action, and secondly by performing a Bäcklund transformation between the two auxiliary wave-functions on both sides of the defect.

5.2.1 Lagrangian description

The Lagrangian density for GMT model with jump defect can be written as follows,

$$\mathcal{L} = \theta(-x) \mathcal{L}_-(\tilde{\psi}_1, \tilde{\psi}_2) + \theta(x) \mathcal{L}_+(\psi_1, \psi_2) + \delta(x) \mathcal{L}_D(\tilde{\psi}_1, \tilde{\psi}_2, \psi_1, \psi_2, X), \quad (5.77)$$

where \mathcal{L}_\mp represents the bulk Lagrangian density (5.4) describing the massive Dirac fields at $x < 0$ and $x > 0$ respectively, and the defect conditions can be derived from the local Lagrangian density,

$$\begin{aligned} \mathcal{L}_D &= \frac{ia}{2m} [X^\dagger(\partial_t X) - (\partial_t X^\dagger)X] + \frac{i}{2} [\tilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \tilde{\psi}_1 + \tilde{\psi}_2^\dagger \psi_2 - \psi_2^\dagger \tilde{\psi}_2] \\ &+ \frac{1}{2} [i(\psi_1 - \tilde{\psi}_1) + a(\psi_2 + \tilde{\psi}_2)] X^\dagger + \frac{1}{2} [i(\psi_1^\dagger - \tilde{\psi}_1^\dagger) - a(\psi_2^\dagger + \tilde{\psi}_2^\dagger)] X \\ &- \frac{ga}{4m} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 + \tilde{\psi}_2^\dagger \tilde{\psi}_2 + \psi_2^\dagger \psi_2] X^\dagger X, \end{aligned} \quad (5.78)$$

where X and X^\dagger are auxiliary fields and a is the Bäcklund parameter. For $x = 0$, the equations corresponding to defect conditions are

$$X = (\tilde{\psi}_1 + \psi_1) + \frac{ia}{2m} \tilde{\psi}_1 X^\dagger X = ia^{-1}(\psi_2 - \tilde{\psi}_2) - \frac{g}{2m} X^\dagger X \psi_2, \quad (5.79)$$

$$X^\dagger = (\tilde{\psi}_1^\dagger + \psi_1^\dagger) - \frac{ia}{2m} \tilde{\psi}_1^\dagger X^\dagger X = -ia^{-1}(\psi_2^\dagger - \tilde{\psi}_2^\dagger) - \frac{g}{2m} X^\dagger X \psi_2^\dagger, \quad (5.80)$$

together with their respective time derivatives,

$$\partial_t X = \frac{m}{2a}(\psi_1 - \tilde{\psi}_1) - \frac{im}{2}(\psi_2 + \tilde{\psi}_2) - \frac{ig}{4} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 + \tilde{\psi}_2^\dagger \tilde{\psi}_2 + \psi_2^\dagger \psi_2] X, \quad (5.81)$$

$$\partial_t X^\dagger = \frac{m}{2a}(\psi_1^\dagger - \tilde{\psi}_1^\dagger) + \frac{im}{2}(\psi_2^\dagger + \tilde{\psi}_2^\dagger) + \frac{ig}{4} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 + \tilde{\psi}_2^\dagger \tilde{\psi}_2 + \psi_2^\dagger \psi_2] X^\dagger. \quad (5.82)$$

These equations are the respective “frozen” Bäcklund transformations, and if we also consider the x -derivatives

$$\partial_x X = \frac{m}{2a}(\psi_1 - \tilde{\psi}_1) + \frac{im}{2}(\psi_2 + \tilde{\psi}_2) - \frac{ig}{4} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 - \tilde{\psi}_2^\dagger \tilde{\psi}_2 - \psi_2^\dagger \psi_2] X, \quad (5.83)$$

$$\partial_x X^\dagger = \frac{m}{2a}(\psi_1^\dagger - \tilde{\psi}_1^\dagger) - \frac{im}{2}(\psi_2^\dagger + \tilde{\psi}_2^\dagger) + \frac{ig}{4} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 - \tilde{\psi}_2^\dagger \tilde{\psi}_2 - \psi_2^\dagger \psi_2] X^\dagger, \quad (5.84)$$

they become exactly the Bäcklund transformations for the classical anticommuting Thirring model [46]. We notice that although the local Lagrangian density used in [47] appears to differ from (5.78), they are actually equivalent Lagrangians. Indeed, that the two representations describe the same problem can be seen by eliminating the quadratic term $X^\dagger X$ from (5.78) using equations (5.79) and (5.80).

As has already been mentioned, this model possesses an infinite family of conserved charges and now we will derive defect contribution through the Lagrangian formalism. The lowest integral of motion considered is the number of occupation N given by ,

$$N = \int_{-\infty}^0 dx \left(\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \tilde{\psi}_2^\dagger \tilde{\psi}_2 \right) + \int_0^\infty dx \left(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \right). \quad (5.85)$$

which, by using the Bäcklund transformations it can be easily shown that the modified number of occupation is given by [48],

$$N = N + N_D = N + \frac{a}{m} X^\dagger X. \quad (5.86)$$

Since $N_D^2 = 0$, we can also notice that (5.79) can be rewritten as,

$$X = \tilde{\psi}_1 e^{\frac{igN_D}{4}} + \psi_1 e^{-\frac{igN_D}{4}} = ia^{-1} \left(\psi_2 e^{\frac{igN_D}{4}} - \tilde{\psi}_2 e^{-\frac{igN_D}{4}} \right). \quad (5.87)$$

Now, the energy can be derived using the Noether theorem, and is given by

$$E = \int_{-\infty}^0 dx \left[\frac{i}{2} \left(\tilde{\psi}_1 \partial_x \tilde{\psi}_1^\dagger + \tilde{\psi}_1^\dagger \partial_x \tilde{\psi}_1 - \tilde{\psi}_2 \partial_x \tilde{\psi}_2^\dagger - \tilde{\psi}_2^\dagger \partial_x \tilde{\psi}_2 \right) + m \left(\tilde{\psi}_2^\dagger \tilde{\psi}_1 + \tilde{\psi}_1^\dagger \tilde{\psi}_2 \right) + g \tilde{\psi}_1^\dagger \tilde{\psi}_1 \tilde{\psi}_2^\dagger \tilde{\psi}_2 \right] \\ + \int_0^\infty dx \left[\frac{i}{2} \left(\psi_1 \partial_x \psi_1^\dagger + \psi_1^\dagger \partial_x \psi_1 - \psi_2 \partial_x \psi_2^\dagger - \psi_2^\dagger \partial_x \psi_2 \right) + m \left(\psi_2^\dagger \psi_1 + \psi_1^\dagger \psi_2 \right) + g \psi_1^\dagger \psi_1 \psi_2^\dagger \psi_2 \right]. \quad (5.88)$$

Using just the field equations we find that the time-derivative can be written as,

$$\frac{dE}{dt} = \frac{i}{2} \left[\tilde{\psi}_1 \partial_t \tilde{\psi}_1^\dagger + \tilde{\psi}_1^\dagger \partial_t \tilde{\psi}_1 - \tilde{\psi}_2 \partial_t \tilde{\psi}_2^\dagger - \tilde{\psi}_2^\dagger \partial_t \tilde{\psi}_2 - \psi_1 \partial_t \psi_1^\dagger - \psi_1^\dagger \partial_t \psi_1 + \psi_2 \partial_t \psi_2^\dagger + \psi_2^\dagger \partial_t \psi_2 \right] \Big|_{x=0}, \quad (5.89)$$

then, the modified conserved energy is $\mathcal{E} = E + E_D$ [47], where

$$E_D = \frac{ia}{m} (X \partial_t X^\dagger - (\partial_t X) X^\dagger) - \frac{i}{2} (\tilde{\psi}_1 \psi_1^\dagger + \tilde{\psi}_1^\dagger \psi_1 + \tilde{\psi}_2 \psi_2^\dagger + \tilde{\psi}_2^\dagger \psi_2) \\ - \frac{ga}{2m} \tilde{\psi}_1^\dagger \tilde{\psi}_1 \psi_1^\dagger \psi_1 - \frac{g}{2ma} \tilde{\psi}_2^\dagger \tilde{\psi}_2 \psi_2^\dagger \psi_2. \quad (5.90)$$

We may eliminate the X field by noting the following relation,

$$\frac{ia}{m} (X^\dagger \partial_t X - \partial_t X^\dagger X) = i \left[\tilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \tilde{\psi}_1 + \tilde{\psi}_2^\dagger \psi_2 - \psi_2^\dagger \tilde{\psi}_2 \right], \quad (5.91)$$

then (5.90) becomes,

$$E_D = \frac{i}{2} \left[\tilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \tilde{\psi}_1 + \tilde{\psi}_2^\dagger \psi_2 - \psi_2^\dagger \tilde{\psi}_2 \right] - \frac{ag}{2m} (\tilde{\psi}_1^\dagger \tilde{\psi}_1 \psi_1^\dagger \psi_1) - \frac{g}{2ma} (\tilde{\psi}_2^\dagger \tilde{\psi}_2 \psi_2^\dagger \psi_2). \quad (5.92)$$

Finally, let us consider the canonical momentum,

$$P = \int_{-\infty}^0 dx \left[\frac{i}{2} (\tilde{\psi}_1 \partial_x \tilde{\psi}_1^\dagger + \tilde{\psi}_1^\dagger \partial_x \tilde{\psi}_1 + \tilde{\psi}_2 \partial_x \tilde{\psi}_2^\dagger + \tilde{\psi}_2^\dagger \partial_x \tilde{\psi}_2) \right] + \int_0^\infty dx \left[\frac{i}{2} (\psi_1 \partial_x \psi_1^\dagger + \psi_1^\dagger \partial_x \psi_1 + \psi_2 \partial_x \psi_2^\dagger + \psi_2^\dagger \partial_x \psi_2) \right]. \quad (5.93)$$

Using the field equations (5.5)–(5.8) we obtain

$$\frac{dP}{dt} = \left[m(\tilde{\psi}_1 \tilde{\psi}_2^\dagger + \tilde{\psi}_2 \tilde{\psi}_1^\dagger) - g\tilde{\psi}_1^\dagger \tilde{\psi}_2^\dagger \tilde{\psi}_2 \tilde{\psi}_1 + \frac{i}{2} (\tilde{\psi}_1^\dagger \partial_t \tilde{\psi}_1 + \tilde{\psi}_1 \partial_t \tilde{\psi}_1^\dagger + \tilde{\psi}_2^\dagger \partial_t \tilde{\psi}_2 + \tilde{\psi}_2 \partial_t \tilde{\psi}_2^\dagger) \right]_{x=0} - \left[m(\psi_1 \psi_2^\dagger + \psi_2 \psi_1^\dagger) - g\psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 + \frac{i}{2} (\psi_1^\dagger \partial_t \psi_1 + \psi_1 \partial_t \psi_1^\dagger + \psi_2^\dagger \partial_t \psi_2 + \psi_2 \partial_t \psi_2^\dagger) \right]_{x=0}, \quad (5.94)$$

Considering the boundary conditions (5.79) - (5.82) the right hand side becomes a total time derivative. Thus, we found [47]

$$P_D = \frac{ia}{m} (X \partial_t X^\dagger - (\partial_t X) X^\dagger) - \frac{i}{2} (\tilde{\psi}_1 \psi_1^\dagger + \tilde{\psi}_1^\dagger \psi_1 + 3\tilde{\psi}_2 \psi_2^\dagger + 3\tilde{\psi}_2^\dagger \psi_2) - \frac{ga}{2m} \tilde{\psi}_1^\dagger \tilde{\psi}_1 \psi_1^\dagger \psi_1 + \frac{g}{2ma} \tilde{\psi}_2^\dagger \tilde{\psi}_2 \psi_2^\dagger \psi_2, \quad (5.95)$$

so that $\mathcal{P} = P + P_D$ is conserved and by using (5.91) we also find that

$$P_D = \frac{i}{2} \left[\tilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \tilde{\psi}_1 - \tilde{\psi}_2^\dagger \psi_2 + \psi_2^\dagger \tilde{\psi}_2 \right] - \frac{ag}{2m} (\tilde{\psi}_1^\dagger \tilde{\psi}_1 \psi_1^\dagger \psi_1) + \frac{g}{2ma} (\tilde{\psi}_2^\dagger \tilde{\psi}_2 \psi_2^\dagger \psi_2). \quad (5.96)$$

Then, we have derive the defect contributions for the lowest conserved quantities in the Lagrangian framework. The integrability of the model involves also higher conservation laws encoded within the Lax pair formalism which will be presented in the following sections.

5.2.2 Modified integrals of motion from the defect matrix

The relation between two different solutions of the respective auxiliary linear problems, say $\tilde{\Psi}$ and Ψ , is given by

$$\tilde{\Psi}(x, t; \lambda) = K(x, t; \lambda) \Psi(x, t; \lambda), \quad (5.97)$$

where the defect matrix K satisfied the gauge transformations,

$$\partial_t K = \tilde{V} K - K V, \quad \partial_x K = \tilde{U} K - K U. \quad (5.98)$$

Here, for the GMT model we assume that the defect matrix can be written by the following λ -expansion [48],

$$K = K_0 + \lambda K_1 + \lambda^{-1} K_{-1}. \quad (5.99)$$

After some manipulations we found that a totally consistent solution for K is then given by,

$$K = \begin{bmatrix} K_- & 0 & \sqrt{\frac{2g}{m}} X \\ 0 & K_+ & -\sqrt{\frac{2g}{m}} X^\dagger \\ \sqrt{\frac{2g}{m}} X^\dagger & -\sqrt{\frac{2g}{m}} X & -(\lambda + i(\lambda a)^{-1}) \end{bmatrix} \quad (5.100)$$

where the elements K_\pm are given by,

$$\begin{aligned} K_\pm &= \lambda \exp \left[\pm \frac{iga}{2m} X^\dagger X \right] - i(\lambda a)^{-1} \exp \left[\mp \frac{iga}{2m} X^\dagger X \right] \\ &= [\lambda - i(\lambda a)^{-1}] \pm \frac{iga}{2m} [\lambda + i(\lambda a)^{-1}] X^\dagger X, \end{aligned} \quad (5.101)$$

Let us now implement a defect placed at the origin $x = 0$, the respective auxiliary wave-functions $\tilde{\Psi}$ in $x < 0$ and Ψ in $x > 0$, and then consider the defect contributions to the conserved quantities. As we have already discussed, the entries of the defect matrix determine the modified conserved quantities from (2.57). First of all, let us consider the first set of conserved quantities given by (5.31) in the presence of a defect,

$$\mathcal{J}_1 = \int_{-\infty}^0 dx \left[\tilde{q}_1 \tilde{\Gamma}_{31} + \frac{ig}{2} \tilde{\rho}_- \right] + \int_0^{\infty} dx \left[q_1 \Gamma_{31} + \frac{ig}{2} \rho_- \right]. \quad (5.102)$$

Then, taking the time-derivative and using the formula we found that $\mathcal{J}_1 + D_1$ is conserved, where the defect contribution D_1 to this first set of conserved quantities is explicitly given by

$$D_1 = -\ln \left[K_{11} + K_{12} \Gamma_{21} + K_{13} \Gamma_{31} \right] \Big|_{x=0}. \quad (5.103)$$

Hence, by taking the both expansions for negative and positive powers of λ and the explicit form of the defect matrix (5.100), we get

$$D_1^{(0)} = \left(\frac{iga}{2m} \right) X^\dagger X, \quad \hat{D}_1^{(0)} = - \left(\frac{iga}{2m} \right) X^\dagger X, \quad (5.104)$$

$$D_1^{(2)} = -\frac{g}{m} X^\dagger X - \frac{2g}{m} X \psi_1^\dagger, \quad \hat{D}_1^{(2)} = -\frac{ga^2}{m} X^\dagger X + \frac{2iag}{m} X \psi_2^\dagger. \quad (5.105)$$

In a similar way, repeating the computations for the other two generating functions (5.64) and (5.65), we find that the respective defect contributions are given by,

$$D_2 = -\ln [K_{21}\Gamma_{12} + K_{22} + K_{23}\Gamma_{32}] \Big|_{x=0}, \quad D_3 = -\ln [K_{31}\Gamma_{13} + K_{32}\Gamma_{23} + K_{33}] \Big|_{x=0}. \quad (5.106)$$

Using them, we obtain

$$D_2^{(0)} = -\left(\frac{iga}{2m}\right) X^\dagger X, \quad \widehat{D}_2^{(0)} = \left(\frac{iga}{2m}\right) X^\dagger X, \quad D_3^{(0)} = 0 = \widehat{D}_3^{(0)}, \quad (5.107)$$

$$D_2^{(2)} = \frac{g}{m} X^\dagger X - \frac{2g}{m} X^\dagger \psi_1, \quad \widehat{D}_2^{(2)} = \frac{ga^2}{m} X^\dagger X - \frac{2iag}{m} X^\dagger \psi_2, \quad (5.108)$$

$$D_3^{(2)} = -\frac{2g}{m} X^\dagger \psi_1 - \frac{2g}{m} X \psi_1^\dagger, \quad \widehat{D}_3^{(2)} = -\frac{2iag}{m} X^\dagger \psi_2 + \frac{2iag}{m} X \psi_2^\dagger. \quad (5.109)$$

As it was expected, we also have the relations $D_3^{(n)} = D_1^{(n)} + D_2^{(n)}$ and $\widehat{D}_3^{(n)} = \widehat{D}_1^{(n)} + \widehat{D}_2^{(n)}$. Then, defining by analogy the following defect quantities,

$$\mathbb{D}^{(0)} = D_1^{(0)} - D_2^{(0)} - D_3^{(0)} = \left(\frac{iga}{m}\right) X^\dagger X, \quad (5.110)$$

$$\widehat{\mathbb{D}}^{(0)} = \widehat{D}_1^{(0)} - \widehat{D}_2^{(0)} - \widehat{D}_3^{(0)} = -\left(\frac{iga}{m}\right) X^\dagger X, \quad (5.111)$$

$$\mathbb{D}^{(2)} = D_1^{(2)} + D_2^{(2)} + D_3^{(2)} = -\frac{4g}{m} X^\dagger \psi_1 - \frac{4g}{m} X \psi_1^\dagger, \quad (5.112)$$

$$\widehat{\mathbb{D}}^{(2)} = \widehat{D}_1^{(2)} + \widehat{D}_2^{(2)} + \widehat{D}_3^{(2)} = -\frac{4iag}{m} X^\dagger \psi_2 + \frac{4iag}{m} X \psi_2^\dagger, \quad (5.113)$$

the corresponding defect number of occupation, energy and momentum can be written in the following way,

$$N_D = \frac{1}{2ig} \left(\mathbb{D}^{(0)} - \widehat{\mathbb{D}}^{(0)} \right) = \frac{a}{m} X^\dagger X, \quad (5.114)$$

$$E_D = \frac{m}{8ig} \left(\mathbb{D}^{(2)} - \widehat{\mathbb{D}}^{(2)} \right) = \frac{i}{2} \left[(X^\dagger \psi_1 + X \psi_1^\dagger) - ia(X^\dagger \psi_2 - X \psi_2^\dagger) \right], \quad (5.115)$$

$$P_D = \frac{m}{8ig} \left(\mathbb{D}^{(2)} + \widehat{\mathbb{D}}^{(2)} \right) = \frac{i}{2} \left[(X^\dagger \psi_1 + X \psi_1^\dagger) + ia(X^\dagger \psi_2 - X \psi_2^\dagger) \right]. \quad (5.116)$$

Notice that by eliminating the auxiliary fields X and X^\dagger , we get

$$E_D = \frac{i}{2} \left[\widetilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \widetilde{\psi}_1 + \widetilde{\psi}_2^\dagger \psi_2 - \psi_2^\dagger \widetilde{\psi}_2 \right] - \frac{ag}{2m} (\widetilde{\psi}_1^\dagger \widetilde{\psi}_1 \psi_1^\dagger \psi_1) - \frac{g}{2ma} (\widetilde{\psi}_2^\dagger \widetilde{\psi}_2 \psi_2^\dagger \psi_2), \quad (5.117)$$

$$P_D = \frac{i}{2} \left[\widetilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \widetilde{\psi}_1 - \widetilde{\psi}_2^\dagger \psi_2 + \psi_2^\dagger \widetilde{\psi}_2 \right] - \frac{ag}{2m} (\widetilde{\psi}_1^\dagger \widetilde{\psi}_1 \psi_1^\dagger \psi_1) + \frac{g}{2ma} (\widetilde{\psi}_2^\dagger \widetilde{\psi}_2 \psi_2^\dagger \psi_2). \quad (5.118)$$

Then, we have particularly derived in an alternative way the defect energy and momentum for the Grassmannian Thirring model in the presence of type-II defects. These results are in complete agreement with the ones obtained based on variational principles [47].

The Bosonic massive Thirring model

The classical version of the massive Thirring model is soluble via classical inverse scattering method in both cases anticommuting (Grassmann) and commuting (or Bosonic) variables [23, 42, 43]. In this chapter, motivated by the possibility of having Thirring solitons we shall focus our attention to explore the consequences of integrability of the Bosonic massive Thirring (BMT) model in the presence of defects. As it was pointed out in [49], there has been a great interest in the study of physical systems supporting Thirring-like soliton propagation, for instance optical Thirring solitons in quadratic media [50, 51], atomic vapors with electromagnetically induced transparency [52] and photorefractive media [53]. In addition, atomic Thirring solitons can be supported in Bose-Einstein condensates [54, 55, 56].

On the other hand, the BMT model is particularly related with the derivative nonlinear Schrödinger (DNLS) model, which is also an integrable model [57]. In [58] it was shown that the Lax operator for BMT model can be generated by “fusing” two Lax operators of DNLS model with different spectral parameters.

In this chapter, after doing a briefly review of the bulk theory, we will derive the Bäcklund transformations for the BMT model directly from the defect matrix. In addition, we compute the defect contributions to the modified conserved energy and momentum. The N-soliton solution for this model was derived in [43] by using the inverse scattering method. Here, we will construct the one-soliton and two-soliton solutions for the model using the well-known dressing method [59, 60]. Finally, we verify the Bäcklund transformations by examining the behaviour of a single soliton passing through the defect.

6.1 The bulk BMT model and the linear problem

The BMT model is described by the Lagrangian density

$$\begin{aligned} \mathcal{L} &= \frac{i}{2}\phi_1(\partial_t - \partial_x)\phi_1^\dagger - \frac{i}{2}\phi_1^\dagger(\partial_t - \partial_x)\phi_1 + \frac{i}{2}\phi_2(\partial_t + \partial_x)\phi_2^\dagger - \frac{i}{2}\phi_2^\dagger(\partial_t + \partial_x)\phi_2 \\ &+ m(\phi_2^\dagger\phi_1 + \phi_1^\dagger\phi_2) + g(\phi_1^\dagger\phi_1\phi_2^\dagger\phi_2), \end{aligned} \quad (6.1)$$

where m is the mass parameter, g is the coupling constant and $\phi = (\phi_1(x, t), \phi_2(x, t))$ is a two-component massive bosonic field. This Lagrangian density is also invariant under $U(1)$ transformations as well. The field equations are given by,

$$i(\partial_t - \partial_x)\phi_1 = m\phi_2 + g\phi_2^\dagger\phi_2\phi_1, \quad (6.2)$$

$$i(\partial_t + \partial_x)\phi_2 = m\phi_1 + g\phi_1^\dagger\phi_1\phi_2, \quad (6.3)$$

$$i(\partial_t - \partial_x)\phi_1^\dagger = -m\phi_2^\dagger - g\phi_2^\dagger\phi_2\phi_1^\dagger, \quad (6.4)$$

$$i(\partial_t + \partial_x)\phi_2^\dagger = -m\phi_1^\dagger - g\phi_1^\dagger\phi_1\phi_2^\dagger, \quad (6.5)$$

As it is well known the BMT model in the bulk is integrable [43] and the associated linear problem can be formulated by using the two-dimensional representation of the $\mathfrak{sl}(2)$ algebra as follows,

$$\partial_t\Psi(x, t; \lambda) = V(x, t; \lambda)\Psi(x, t; \lambda), \quad (6.6)$$

$$\partial_x\Psi(x, t; \lambda) = U(x, t; \lambda)\Psi(x, t; \lambda), \quad (6.7)$$

where the auxiliary field $\Psi = (\Psi_1, \Psi_2)^T$ is a two-vector and the Lax pair can be written in a compact form as,

$$U = \frac{i}{4}[g\rho_- - m(\lambda^2 - \lambda^{-2})]H + q(\lambda)E_+ + r(\lambda)E_-, \quad (6.8)$$

$$V = -\frac{i}{4}[g\rho_+ + m(\lambda^2 + \lambda^{-2})]H + B(\lambda)E_+ + C(\lambda)E_-, \quad (6.9)$$

where for convenience we have defined $\rho_\pm = (\phi_2^\dagger\phi_2 \pm \phi_1^\dagger\phi_1)$, and the following functions,

$$B(\lambda) = \frac{i\sqrt{mg}}{2}(\lambda\phi_1 - \lambda^{-1}\phi_2), \quad q(\lambda) = \frac{i\sqrt{mg}}{2}(\lambda\phi_1 + \lambda^{-1}\phi_2), \quad (6.10)$$

$$C(\lambda) = -\frac{i\sqrt{mg}}{2}(\lambda\phi_1^\dagger - \lambda^{-1}\phi_2^\dagger), \quad r(\lambda) = -\frac{i\sqrt{mg}}{2}(\lambda\phi_1^\dagger + \lambda^{-1}\phi_2^\dagger). \quad (6.11)$$

Now, we define the auxiliary function $\Gamma_{21} = \Psi_2\Psi_1^{-1}$. Then, by using the system of linear equations we have that the conservation equation can be written in the following form,

$$\partial_t \left[q\Gamma_{21} + \frac{ig}{4}\rho_- \right] = \partial_x \left[B\Gamma_{21} - \frac{ig}{4}\rho_+ \right]. \quad (6.12)$$

The auxiliary function Γ_{21} satisfies the following Riccati equations

$$\partial_x \Gamma_{21} = r - \frac{i}{2} [g\rho_- - m(\lambda^2 - \lambda^{-2})] \Gamma_{21} - q\Gamma_{21}^2, \quad (6.13)$$

$$\partial_t \Gamma_{21} = C + \frac{i}{2} [g\rho_+ + m(\lambda^2 + \lambda^{-2})] \Gamma_{21} - B\Gamma_{21}^2. \quad (6.14)$$

Now, we expand Γ_{21} in inverse powers of λ around ∞ ,

$$\Gamma_{21}(x, t; \lambda) = \sum_{k=0}^{\infty} \frac{\Gamma_{21}^{(k)}(x, t)}{\lambda^k}. \quad (6.15)$$

Using the Riccati equation, each expansion coefficient $\Gamma_{21}^{(k)}(x, t)$ can be obtained easily in a recursive way. The first coefficients are given by

$$\Gamma_{21}^{(1)} = \sqrt{\frac{g}{m}} \phi_1^\dagger, \quad \Gamma_{21}^{(2)} = 0, \quad \Gamma_{21}^{(3)} = \sqrt{\frac{g}{m}} \left[-\frac{2i}{m} (\partial_x \phi_1^\dagger) + \phi_2^\dagger + \frac{g}{m} (\phi_2^\dagger \phi_2) \phi_1^\dagger \right]. \quad (6.16)$$

Considering, as usual, the bosonic fields $\phi_i(x, t)$ vanish at $|x| \rightarrow \infty$, the corresponding generating function for the conserved quantities reads

$$\mathbf{I}_1 = \int_{-\infty}^{\infty} dx \left[q\Gamma_{21} + \frac{ig}{4} \rho_- \right], \quad (6.17)$$

and substituting (6.15) in the expression for \mathbf{I}_1 , we get an infinite number of conserved quantities given by the expansion

$$\mathbf{I}_1 = \sum_{k=0}^{\infty} \frac{\mathbf{I}_1^{(k)}}{\lambda^{2k}}. \quad (6.18)$$

Then, the first two conserved quantities are explicitly given by

$$\mathbf{I}_1^{(0)} = \frac{ig}{4} \int_{-\infty}^{\infty} dx \left[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \right], \quad (6.19)$$

$$\mathbf{I}_1^{(2)} = -\frac{ig}{m} \int_{-\infty}^{\infty} dx \left[i\phi_1 (\partial_x \phi_1^\dagger) - \frac{m}{2} (\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) - \frac{g}{2} (\phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2) \right]. \quad (6.20)$$

In addition, there is another set of conserved quantities that can be computed taking an expansion of $\Gamma_{21}(x, t; \lambda)$ in positive powers of λ ,

$$\Gamma_{21}(x, t; \lambda) = \sum_{k=0}^{\infty} \widehat{\Gamma}_{21}^{(k)}(x, t) \lambda^k. \quad (6.21)$$

In a very similar way, the first coefficients are,

$$\widehat{\Gamma}_{21}^{(1)} = -\sqrt{\frac{g}{m}}\phi_2^\dagger, \quad \widehat{\Gamma}_{21}^{(2)} = 0, \quad \widehat{\Gamma}_{21}^{(3)} = \sqrt{\frac{g}{m}} \left[-\frac{2i}{m}(\partial_x \phi_2^\dagger) - \phi_1^\dagger - \frac{g}{m}(\phi_1^\dagger \phi_1) \phi_2^\dagger \right]. \quad (6.22)$$

Substituting in (6.17), we will now obtain that the conserved quantities read

$$\mathbb{I}_1 = \sum_{k=0}^{\infty} \widehat{\Gamma}_1^{(k)} \lambda^{2k}, \quad (6.23)$$

where, the first two of them have been computed schematically, and the result is the following

$$\widehat{\Gamma}_1^{(0)} = -\frac{ig}{4} \int_{-\infty}^{\infty} dx \left[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \right], \quad (6.24)$$

$$\widehat{\Gamma}_1^{(2)} = -\frac{ig}{m} \int_{-\infty}^{\infty} dx \left[i\phi_2(\partial_x \phi_2^\dagger) + \frac{m}{2} (\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) + \frac{g}{2} (\phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2) \right]. \quad (6.25)$$

Then, we have found two infinite set of independent conserved quantities as consequence of the two possible choices for the λ -expansion of the auxiliary function $\Gamma_{21}(x, t; \lambda)$, i.e, around $\lambda = 0$ and $\lambda = \infty$. Now, by considering the second conservation law from the linear system (6.6) and (6.7), we get

$$\partial_t \left[r\Gamma_{12} - \frac{ig}{4}\rho_- \right] = \partial_x \left[C\Gamma_{12} + \frac{ig}{4}\rho_+ \right], \quad (6.26)$$

where we have introduced a new auxiliary function $\Gamma_{12} = \Psi_1 \Psi_2^{-1}$, which satisfies

$$\partial_x \Gamma_{12} = q + \frac{i}{2} [g\rho_- - m(\lambda^2 - \lambda^{-2})] \Gamma_{12} - r\Gamma_{12}^2, \quad (6.27)$$

$$\partial_t \Gamma_{12} = B - \frac{i}{2} [g\rho_+ + m(\lambda^2 + \lambda^{-2})] \Gamma_{12} - C\Gamma_{12}^2. \quad (6.28)$$

Then, using the same scheme we can obtain recursively the first few coefficients for the auxiliary function $\Gamma_{12}(x, t; \lambda)$ by considering the corresponding expansion in negative and positive powers of λ . Doing that, the results obtained are:

$$\Gamma_{12}^{(1)} = \sqrt{\frac{g}{m}}\phi_1, \quad \Gamma_{12}^{(2)} = 0, \quad \Gamma_{12}^{(3)} = \sqrt{\frac{g}{m}} \left[\frac{2i}{m}(\partial_x \phi_1) + \phi_2 + \frac{g}{m}(\phi_2^\dagger \phi_2) \phi_1 \right], \quad (6.29)$$

$$\widehat{\Gamma}_{12}^{(1)} = -\sqrt{\frac{g}{m}}\phi_2, \quad \widehat{\Gamma}_{12}^{(2)} = 0, \quad \widehat{\Gamma}_{12}^{(3)} = \sqrt{\frac{g}{m}} \left[\frac{2i}{m}(\partial_x \phi_2) - \phi_1 - \frac{g}{m}(\phi_1^\dagger \phi_1) \phi_2 \right]. \quad (6.30)$$

From the conservation equation (6.26), the second generating function of the conserved quantities can be written as follows,

$$\mathbb{I}_2 = \int_{-\infty}^{\infty} dx \left[r\Gamma_{12} - \frac{ig}{4}\rho_- \right]. \quad (6.31)$$

Substituting the corresponding coefficients of the auxiliary functions for each expansion in λ , we obtained the following conserved quantities,

$$\mathbb{I}_2^{(0)} = \widehat{\mathbb{I}}_2^{(0)} = -\frac{ig}{4} \int_{-\infty}^{\infty} dx \left[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \right], \quad (6.32)$$

$$\mathbb{I}_2^{(2)} = -\frac{ig}{m} \int_{-\infty}^{\infty} dx \left[i\phi_1^\dagger (\partial_x \phi_1) + \frac{m}{2} (\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) + \frac{g}{2} (\phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2) \right], \quad (6.33)$$

$$\widehat{\mathbb{I}}_2^{(2)} = -\frac{ig}{m} \int_{-\infty}^{\infty} dx \left[i\phi_2^\dagger (\partial_x \phi_2) - \frac{m}{2} (\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) - \frac{g}{2} (\phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2) \right]. \quad (6.34)$$

We note that the usual number of occupation, energy and momentum for the BMT model are then given by,

$$N = \frac{1}{ig} \left[(\mathbb{I}_1^{(0)} - \mathbb{I}_2^{(0)}) - (\widehat{\mathbb{I}}_1^{(0)} - \widehat{\mathbb{I}}_2^{(0)}) \right] = \int_{-\infty}^{\infty} dx \left[\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 \right], \quad (6.35)$$

$$E = \frac{im}{2g} \left[(\mathbb{I}_1^{(2)} - \mathbb{I}_2^{(2)}) - (\widehat{\mathbb{I}}_1^{(2)} - \widehat{\mathbb{I}}_2^{(2)}) \right] \quad (6.36)$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{i}{2} (\phi_1 \partial_x \phi_1^\dagger - \phi_1^\dagger \partial_x \phi_1 - \phi_2 \partial_x \phi_2^\dagger + \phi_2^\dagger \partial_x \phi_2) - m(\phi_2^\dagger \phi_1 + \phi_1^\dagger \phi_2) - g \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2 \right],$$

$$P = \frac{im}{2g} \left[(\mathbb{I}_1^{(2)} - \mathbb{I}_2^{(2)}) + (\widehat{\mathbb{I}}_1^{(2)} - \widehat{\mathbb{I}}_2^{(2)}) \right] \quad (6.37)$$

$$= \int_{-\infty}^{\infty} dx \left[\frac{i}{2} (\phi_1 \partial_x \phi_1^\dagger - \phi_1^\dagger \partial_x \phi_1 + \phi_2 \partial_x \phi_2^\dagger - \phi_2^\dagger \partial_x \phi_2) \right].$$

6.2 Modified integrals of motion

As it was already shown, in order to compute the defect contributions to each bulk integral of motion, it is necessary to know the explicit form of the elements of the defect matrix. Using the following ansatz for λ expansion of K ,

$$K = K_0 + \lambda K_1 + \lambda^{-1} K_{-1}, \quad (6.38)$$

a totally consistent defect matrix can be written in the following form [48]

$$K = \begin{bmatrix} -\sqrt{\frac{m}{g}} \left[\lambda e^{-i\alpha} - i(\lambda a)^{-1} e^{i\alpha} \right] & X \\ -X^\dagger & \sqrt{\frac{m}{g}} \left[\lambda e^{i\alpha} + i(\lambda a)^{-1} e^{-i\alpha} \right] \end{bmatrix}, \quad (6.39)$$

where

$$2\alpha = \arcsin \left[\frac{ga}{2m} X^\dagger X \right], \quad (6.40)$$

which it turns to be related to the modified number of occupation by $4\alpha = gN_D$. Here, the boundary bosonic fields X and X^\dagger satisfy the following algebraic relations,

$$X = \tilde{\phi}_1 e^{i\alpha} + \phi_1 e^{-i\alpha} = \frac{i}{a} \left[\phi_2 e^{i\alpha} - \tilde{\phi}_2 e^{-i\alpha} \right], \quad (6.41)$$

$$X^\dagger = \tilde{\phi}_1^\dagger e^{-i\alpha} + \phi_1^\dagger e^{i\alpha} = \frac{1}{ia} \left[\phi_2^\dagger e^{-i\alpha} - \tilde{\phi}_2^\dagger e^{i\alpha} \right], \quad (6.42)$$

and the respective time-derivatives,

$$\begin{aligned} \partial_t X &= \frac{m}{2a} \left(\phi_1 e^{i\alpha} - \tilde{\phi}_1 e^{-i\alpha} \right) - \frac{im}{2} \left(\tilde{\phi}_2 e^{i\alpha} + \phi_2 e^{-i\alpha} \right) \\ &\quad - \frac{ig}{4} \left[\tilde{\phi}_1^\dagger \tilde{\phi}_1 + \phi_1^\dagger \phi_1 + \tilde{\phi}_2^\dagger \tilde{\phi}_2 + \phi_2^\dagger \phi_2 \right] X, \end{aligned} \quad (6.43)$$

$$\begin{aligned} \partial_t X^\dagger &= \frac{m}{2a} \left(\phi_1^\dagger e^{-i\alpha} - \tilde{\phi}_1^\dagger e^{i\alpha} \right) + \frac{im}{2} \left(\tilde{\phi}_2^\dagger e^{-i\alpha} + \phi_2^\dagger e^{i\alpha} \right) \\ &\quad + \frac{ig}{4} \left[\tilde{\phi}_1^\dagger \tilde{\phi}_1 + \phi_1^\dagger \phi_1 + \tilde{\phi}_2^\dagger \tilde{\phi}_2 + \phi_2^\dagger \phi_2 \right] X^\dagger, \end{aligned} \quad (6.44)$$

together with,

$$\begin{aligned} \partial_x X &= \frac{m}{2a} \left(\phi_1 e^{i\alpha} - \tilde{\phi}_1 e^{-i\alpha} \right) + \frac{im}{2} \left(\tilde{\phi}_2 e^{i\alpha} + \phi_2 e^{-i\alpha} \right) \\ &\quad - \frac{ig}{4} \left[\tilde{\phi}_1^\dagger \tilde{\phi}_1 + \phi_1^\dagger \phi_1 - \tilde{\phi}_2^\dagger \tilde{\phi}_2 - \phi_2^\dagger \phi_2 \right] X, \end{aligned} \quad (6.45)$$

$$\begin{aligned} \partial_x X^\dagger &= \frac{m}{2a} \left(\phi_1^\dagger e^{-i\alpha} - \tilde{\phi}_1^\dagger e^{i\alpha} \right) - \frac{im}{2} \left(\tilde{\phi}_2^\dagger e^{-i\alpha} + \phi_2^\dagger e^{i\alpha} \right) \\ &\quad + \frac{ig}{4} \left[\tilde{\phi}_1^\dagger \tilde{\phi}_1 + \phi_1^\dagger \phi_1 - \tilde{\phi}_2^\dagger \tilde{\phi}_2 - \phi_2^\dagger \phi_2 \right] X^\dagger, \end{aligned} \quad (6.46)$$

where a is a real parameter, $\tilde{\phi}$ is the field defined on $x < 0$ and ϕ the field defined on $x > 0$. The compatibility condition $\partial_t \partial_x X = \partial_x \partial_t X$ can be easily checked to be satisfied. The expressions (6.41)–(6.46) are identified with the auto-Bäcklund transformations for the BMT model, which have been reported for the first time in [48].

Once a defect matrix is given by (6.39), the defect contribution to the modified conserved quantities can be calculated using (2.57). Firstly, let us consider the generating function of conserved quantities (6.17), then the corresponding defect contributions are given by,

$$D_1 = -\ln [K_{11} + K_{12}\Gamma_{21}] \Big|_{x=0}. \quad (6.47)$$

Now, by considering the corresponding expansions in powers of λ , we find

$$D_1^{(2)} = \frac{i}{a} e^{2i\alpha} + \frac{g}{m} X \phi_1^\dagger e^{i\alpha}, \quad (6.48)$$

$$\hat{D}_1^{(2)} = -ia e^{-2i\alpha} - \frac{ia g}{m} X \phi_2^\dagger e^{-i\alpha}. \quad (6.49)$$

Performing the same procedure for the generating function (6.31),

$$D_2 = -\ln [K_{22} + K_{21}\Gamma_{12}] \Big|_{x=0}, \quad (6.50)$$

we obtain the following coefficients,

$$D_2^{(2)} = -\frac{i}{a} e^{-2i\alpha} + \frac{g}{m} X^\dagger \phi_1 e^{-i\alpha}, \quad (6.51)$$

$$\widehat{D}_2^{(2)} = ia e^{2i\alpha} + \frac{ia g}{m} X^\dagger \phi_2 e^{i\alpha}. \quad (6.52)$$

Finally, we find that the defect energy and momentum for the BMT model can be written in the following way,

$$\begin{aligned} E_D &= \frac{im}{2g} \left[(D_1^{(2)} - D_2^{(2)}) - (\widehat{D}_1^{(2)} - \widehat{D}_2^{(2)}) \right] = -\frac{m}{2g} \left(a + \frac{1}{a} \right) (e^{2i\alpha} + e^{-2i\alpha}) \\ &\quad + \frac{i}{2} \left(X \phi_1^\dagger e^{i\alpha} - X^\dagger \phi_1 e^{-i\alpha} \right) - \frac{a}{2} \left(X \phi_2^\dagger e^{-i\alpha} + X^\dagger \phi_2 e^{i\alpha} \right), \end{aligned} \quad (6.53)$$

$$\begin{aligned} P_D &= \frac{im}{2g} \left[(D_1^{(2)} - D_2^{(2)}) + (\widehat{D}_1^{(2)} - \widehat{D}_2^{(2)}) \right] = \frac{m}{2g} \left(a - \frac{1}{a} \right) (e^{2i\alpha} + e^{-2i\alpha}) \\ &\quad + \frac{i}{2} \left(X \phi_1^\dagger e^{i\alpha} - X^\dagger \phi_1 e^{-i\alpha} \right) + \frac{a}{2} \left(X \phi_2^\dagger e^{-i\alpha} + X^\dagger \phi_2 e^{i\alpha} \right). \end{aligned} \quad (6.54)$$

We can note that it is possible to rewrite these results in an alternative form by using the Bäcklund transformations (6.41)–(6.46), as follows

$$E_D = \frac{i}{2} \left[\left(\phi_1^\dagger \widetilde{\phi}_1 - \widetilde{\phi}_2^\dagger \phi_2 \right) e^{2i\alpha} - \left(\widetilde{\phi}_1^\dagger \phi_1 - \phi_2^\dagger \widetilde{\phi}_2 \right) e^{-2i\alpha} \right] - \frac{m}{g} \left(a + \frac{1}{a} \right) \cos(2\alpha), \quad (6.55)$$

$$P_D = \frac{i}{2} \left[\left(\phi_1^\dagger \widetilde{\phi}_1 + \widetilde{\phi}_2^\dagger \phi_2 \right) e^{2i\alpha} - \left(\widetilde{\phi}_1^\dagger \phi_1 + \phi_2^\dagger \widetilde{\phi}_2 \right) e^{-2i\alpha} \right] + \frac{m}{g} \left(a - \frac{1}{a} \right) \cos(2\alpha). \quad (6.56)$$

These expressions for the defect energy and momentum seem not to have been reported elsewhere in the literature before [15], and constitute a very important result in order to address in future works the question of the Lagrangian formalism as well as quantum aspects like the transmission matrix.

Since the integrable defect conditions for the BMT model have already been determined by giving the corresponding auto-Bäcklund transformations, the integrability of the model in the presence of defects, following the integrability criteria adopted in this thesis, is provided by the existence of the defect matrix and the explicit computations of the modified conserved quantities. However, as we have mentioned it is necessary a Lagrangian description of this type of defects, probably by considering a generalization of the framework presented in section 2.1, in order to determine the constraints over the defect potential which it is expected to be related to expression (6.55) in some way.

6.3 Dressing solutions

In this section, we show how the soliton solutions of the BMT model can be constructed by using the dressing method. In this formulation we start with the zero curvature representation using the following Lax pair in the light-cone coordinates*

$$L_+ = i\partial_+ + A_+, \quad L_- = i\partial_- + A_-, \quad (6.57)$$

where

$$A_+ = \sqrt{mg} \phi_1 E_+^{(0)} - \sqrt{mg} \phi_1^\dagger E_-^{(+1)} - \frac{g}{2} (\phi_1^\dagger \phi_1) H^{(0)} - \frac{m}{2} H^{(+1)} + \alpha_+ C \quad (6.58)$$

$$A_- = -\sqrt{mg} \phi_2 E_+^{(-1)} + \sqrt{mg} \phi_2^\dagger E_-^{(0)} - \frac{g}{2} (\phi_2^\dagger \phi_2) H^{(0)} - \frac{m}{2} H^{(-1)} + \alpha_- C \quad (6.59)$$

are the Lax connections taking values in the $\mathfrak{sl}(2)$ affine Kac-Moody algebra, and α_\pm are two new fields that do not have any influence in the dynamics of the field ϕ_i but are necessary in the whole construction. From the zero curvature condition $[L_+, L_-] = 0$, we get the field equations, namely

$$i\partial_- \phi_1 = m\phi_2 + g(\phi_2^\dagger \phi_2) \phi_1, \quad (6.60)$$

$$i\partial_+ \phi_2 = m\phi_1 + g(\phi_1^\dagger \phi_1) \phi_2, \quad (6.61)$$

$$i\partial_- \phi_1^\dagger = -m\phi_2^\dagger - g(\phi_2^\dagger \phi_2) \phi_1^\dagger, \quad (6.62)$$

$$i\partial_+ \phi_2^\dagger = -m\phi_1^\dagger - g(\phi_1^\dagger \phi_1) \phi_2^\dagger, \quad (6.63)$$

together with the following equations

$$i\partial_+ (\phi_2^\dagger \phi_2) - i\partial_- (\phi_1^\dagger \phi_1) - 2m(\phi_1^\dagger \phi_2 - \phi_2^\dagger \phi_1) = 0, \quad (6.64)$$

$$i\partial_+ \alpha_- - i\partial_- \alpha_+ + mg(\phi_1^\dagger \phi_2) + \frac{m^2}{2} = 0. \quad (6.65)$$

Equation (6.64) is a straightforward consequence of the field equations (6.60)-(6.63), and (6.65) determines the dependence of the fields α_\pm in terms of the massive fields ϕ_i .

The key ingredient of the dressing procedure is the existence of two gauge transformations $\Theta_+ = \exp(\mathcal{G}_\geq)$ and $\Theta_- = \exp(\mathcal{G}_\leq)$ mapping the vacuum in a non-trivial configuration, i.e.

$$A_\mu^{vac} \longrightarrow A_\mu \equiv \Theta_\pm^{-1} i\partial_\mu \Theta_\pm + \Theta_\pm^{-1} A_\mu^{vac} \Theta_\pm, \quad \mu = \{\pm\}. \quad (6.66)$$

As consequence of the graded structure, the form of the Lax connection is preserved by these transformations. Since A_μ and A_μ^{vac} satisfy the zero curvature condition, they are of the form

$$A_\mu = iT\partial_\mu T^{-1}, \quad A_\mu^{vac} = iT_0\partial_\mu T_0^{-1} \quad (6.67)$$

*The light-cone coordinates are defined as $x_\pm = \frac{1}{2}(t \pm x)$.

where T and T_0 are group elements. From the equivalence of the two dressing transformations (6.66) we find that

$$\Theta_- \Theta_+^{-1} = T_0 \rho T_0^{-1}, \quad (6.68)$$

where ρ is a constant group element. In order to construct systematically soliton solutions we now define the vacuum configuration,

$$\phi_1^{(0)} = \phi_2^{(0)} = \phi_1^{\dagger(0)} = \phi_2^{\dagger(0)} = 0, \quad \alpha_{\pm}^{(0)} = \mp \frac{im^2 x_{\mp}}{4}. \quad (6.69)$$

and Lax connections (6.58) and (6.59) become

$$A_+^{vac} = -\frac{m}{2} H^{(+1)} - \frac{im^2 x_-}{4} C, \quad A_-^{vac} = -\frac{m}{2} H^{(-1)} + \frac{im^2 x_+}{4} C. \quad (6.70)$$

They are associated to the following linear problem

$$i\partial_{\pm} T_0 = -A_{\pm}^{vac} T_0, \quad (6.71)$$

which is solved as follows,

$$T_0 = e^{-ix_- h^- - ix_+ h^+}, \quad \text{with} \quad h^{\pm} \equiv \frac{m}{2} H^{(\pm 1)}. \quad (6.72)$$

The dressing matrices Θ_{\pm} are now determined by the gauge transformation (6.66) with

$$\Theta_+ = e^{m^{(0)}} e^{m^{(1)}} e^{m^{(2)}} \dots \quad \Theta_- = e^{l^{(0)}} e^{l^{(-1)}} e^{l^{(-2)}} \dots \quad (6.73)$$

where Θ_+ is constructed from elements $m(k)$ of a subalgebra containing grade $k \geq 0$, while Θ_- is constructed from elements $l(k)$ of a subalgebra containing grade $k \leq 0$. From (6.66) we get the following results for the first few elements $m(k)$ and $l(k)$,

$$m(0) = \chi_+ \sigma_3^{(0)} + \nu_+ C, \quad l(0) = (i\pi - \chi_+) \sigma_3^{(0)} + \nu_- C, \quad (6.74)$$

$$m(1) = \sqrt{\frac{g}{m}} \left[\phi_2 E_+^{(0)} + \phi_2^{\dagger} E_-^{(1)} \right], \quad l(-1) = -\sqrt{\frac{g}{m}} \left[\phi_1^{\dagger} E_-^{(0)} + \phi_1 E_+^{(-1)} \right], \quad (6.75)$$

$$m(2) = a_+ H^{(1)}, \quad l(-2) = a_- H^{(-1)}, \quad (6.76)$$

where the fields ϕ_i satisfy the equations of motion (6.60)-(6.63), and the fields $\chi_+, \nu_{\pm}, a_{\pm}$ satisfy the following equations,

$$i\partial_+ \chi_+ = -\frac{g}{2} (\phi_1^{\dagger} \phi_1), \quad i\partial_- \chi_+ = \frac{g}{2} (\phi_2^{\dagger} \phi_2), \quad (6.77)$$

$$i\partial_+ \nu_+ = \alpha_+ - \alpha_+^{(0)}, \quad i\partial_- \nu_+ = \alpha_- - \alpha_-^{(0)} - ma_+ - \frac{g}{2} (\phi_2^{\dagger} \phi_2), \quad (6.78)$$

$$i\partial_- \nu_- = \alpha_- - \alpha_-^{(0)}, \quad i\partial_+ \nu_- = \alpha_+ - \alpha_+^{(0)} + ma_- - \frac{g}{2} (\phi_1^{\dagger} \phi_1). \quad (6.79)$$

The (x_+, x_-) -dependence of the fields is given explicitly by the right-hand-side of eq. (6.68). In fact, the solutions can be calculated by taking the expectation value between states of a given representation of $\widehat{\mathfrak{G}}$. As usual, we consider the highest weight representation of the $\mathfrak{sl}(2)$ affine Kac-Moody algebra. Firstly, let $|\lambda_0\rangle$ and $|\lambda_1\rangle$ be the corresponding highest weight states, satisfying

$$\begin{aligned} H^{(0)}|\lambda_i\rangle &= \delta_{i,1}|\lambda_i\rangle, & C|\lambda_i\rangle &= |\lambda_i\rangle, \\ H^{(n)}|\lambda_i\rangle &= E_+^{(0)}|\lambda_i\rangle = E_{\pm}^{(n)}|\lambda_i\rangle = 0, & \text{for } i = 0, 1, \quad n > 0, \end{aligned} \quad (6.80)$$

and then we define the τ -functions as follows

$$\tau_i = \langle \lambda_i | \Theta_- \Theta_+^{-1} | \lambda_i \rangle = \langle \lambda_i | T_0 \rho T_0^{-1} | \lambda_i \rangle, \quad i = 1, 2. \quad (6.81)$$

The soliton solutions are obtained by choosing the constant element $\rho = e^V$, as the exponential of an eigenvector V of the elements of algebra h^{\pm} . This eigenvector can be constructed in the following way

$$V_{\pm}(\gamma) = \sum_{n \in \mathbb{Z}} \gamma^{-n} E_{\pm}^{(n)}, \quad (6.82)$$

satisfying the following commutation relations

$$[h^+, V_{\pm}(\gamma)] = \pm m \gamma V_{\pm}(\gamma), \quad (6.83)$$

$$[h^-, V_{\pm}(\gamma)] = \pm \frac{m}{\gamma} V_{\pm}(\gamma). \quad (6.84)$$

It is clear from (6.83) and (6.84) that $V_+(\gamma)$ and $V_-(-\gamma)$ have the same eigenvalue. From (6.68) we obtain

$$T_0 e^{\mu_{\pm} V_{\pm}(\gamma)} T_0^{-1} = \exp [e^{\mp \Gamma} \mu_{\pm} V_{\pm}(\gamma)] \cong 1 + \mu_{\pm} e^{\mp \Gamma} V_{\pm}(\gamma), \quad (6.85)$$

where $\Gamma = im(x_+ \gamma + \gamma^{-1} x_-)$ and the vertex operator V_{\pm} satisfy,

$$V_+(\gamma_1) V_+(\gamma_2) \rightarrow 0, \quad V_-(\gamma_1) V_-(\gamma_2) \rightarrow 0, \quad \text{as } \gamma_1 \rightarrow \gamma_2. \quad (6.86)$$

In general, the N-soliton solution is obtained taking $\rho = e^{\mu_1 V(\gamma_1)} e^{\mu_2 V(\gamma_2)} \dots e^{\mu_N V(\gamma_N)}$, being μ_k some arbitrary parameters and the vertex functions satisfy the commutation relation

$$[h^{(n)}, V(\gamma_k)] = f(n, \gamma_k) V(\gamma_k). \quad (6.87)$$

6.3.1 The one-soliton solution

Using the highest weight representation of $\mathfrak{sl}(2)$ we obtain the one-soliton solution from the vacuum configuration, as follows

$$e^{(\nu_- - \nu_+)} = \tau_0, \quad e^{-2\chi_+} = -\frac{\tau_1}{\tau_0}, \quad (6.88)$$

$$\phi_1 = \sqrt{\frac{m}{g}} \frac{\tau_2}{\tau_1}, \quad \phi_2 = \sqrt{\frac{m}{g}} \frac{\tau_4}{\tau_0}, \quad (6.89)$$

$$\phi_1^\dagger = \sqrt{\frac{m}{g}} \frac{\tau_3}{\tau_0}, \quad \phi_2^\dagger = \sqrt{\frac{m}{g}} \frac{\tau_5}{\tau_1}, \quad (6.90)$$

where we have introduced the tau-functions

$$\begin{aligned} \tau_0 &= \langle \lambda_0 | G | \lambda_0 \rangle, & \tau_2 &= \langle \lambda_0 | E_-^{(+1)} G | \lambda_0 \rangle, & \tau_4 &= \langle \lambda_1 | G E_-^{(0)} | \lambda_1 \rangle, \\ \tau_1 &= \langle \lambda_1 | G | \lambda_1 \rangle, & \tau_3 &= \langle \lambda_1 | E_+^{(0)} G | \lambda_1 \rangle, & \tau_5 &= \langle \lambda_0 | G E_+^{(-1)} | \lambda_0 \rangle, \end{aligned} \quad (6.91)$$

with $G = T_0 \rho T_0^{-1}$ and $\rho = e^V$. Firstly, we can notice that there are two possible solutions corresponding to the choice of $V = \mu_1 V_+(\gamma_1)$, given by

$$\nu_+ = \nu_-, \quad \chi_+ = \frac{i\pi}{2}, \quad \phi_1 = \sqrt{\frac{m}{g}} \mu_1 \gamma e^{-\Gamma_1}, \quad \phi_2 = \sqrt{\frac{m}{g}} \mu_1 e^{-\Gamma_1}, \quad \phi_1^\dagger = \phi_2^\dagger = 0, \quad (6.92)$$

and by choosing $V = \mu_2 V_-(\gamma_2)$, we obtain

$$\nu_+ = \nu_-, \quad \chi_+ = \frac{i\pi}{2}, \quad \phi_1 = \phi_2 = 0, \quad \phi_1^\dagger = \sqrt{\frac{m}{g}} \mu_2 e^{\Gamma_2}, \quad \phi_2^\dagger = \sqrt{\frac{m}{g}} \frac{\mu_2}{\gamma} e^{\Gamma_2}. \quad (6.93)$$

These solutions are not really interesting because of the inconsistency with the interpretation of the dagger fields ϕ_i^\dagger as the corresponding complex conjugate of the fields ϕ_i . Then, we construct the one-soliton solution of the system using the fact that $V_+(\gamma)$ and $V_-(\gamma)$ have the same eigenvalue. In fact, by choosing $\rho = e^{\mu_1 V_+(\gamma_1)} e^{\mu_2 V_-(\gamma_2)}$ and computing the matrix elements we get the following solution,

$$\begin{aligned} \tau_0 &= 1 + \mu_1 \mu_2 \left[\frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2} \right] e^{-\Gamma_1 + \Gamma_2}, & \tau_1 &= 1 + \mu_1 \mu_2 \left[\frac{\gamma_1}{\gamma_1 - \gamma_2} \right]^2 e^{-\Gamma_1 + \Gamma_2}, \\ \tau_2 &= \mu_1 \gamma_1 e^{-\Gamma_1}, & \tau_3 &= \mu_2 e^{\Gamma_2}, & \tau_4 &= \mu_1 e^{-\Gamma_1}, & \tau_5 &= \frac{\mu_2}{\gamma_2} e^{\Gamma_2} \end{aligned} \quad (6.94)$$

where $\Gamma_k = im(x_+ \gamma_k + \gamma_k^{-1} x_-)$. Then, we are interested in the case where ϕ_k^\dagger corresponds to the complex conjugate of ϕ_k , i.e. in the limit $\gamma_2 \rightarrow -\gamma_1$, which provides a suitable one-soliton solution for the BMT model.

The result is

$$e^{(\nu_- - \nu_+)} = 1 - \Omega e^{-2\Gamma_1}, \quad e^{-2\chi_+} = - \left[\frac{1 + \Omega e^{-2\Gamma_1}}{1 - \Omega e^{-2\Gamma_1}} \right], \quad (6.95)$$

$$\phi_1 = \sqrt{\frac{m}{g}} \left[\frac{\mu_1 \gamma_1 e^{-\Gamma_1}}{1 + \Omega e^{-2\Gamma_1}} \right], \quad \phi_2 = \sqrt{\frac{m}{g}} \left[\frac{\mu_1 e^{-\Gamma_1}}{1 - \Omega e^{-2\Gamma_1}} \right], \quad (6.96)$$

$$\phi_1^\dagger = \sqrt{\frac{m}{g}} \left[\frac{\mu_2 e^{-\Gamma_1}}{1 - \Omega e^{-2\Gamma_1}} \right], \quad \phi_2^\dagger = -\sqrt{\frac{m}{g}} \frac{\mu_2}{\gamma_1} \left[\frac{e^{-\Gamma_1}}{1 + \Omega e^{-2\Gamma_1}} \right], \quad (6.97)$$

where we have introduced the parameter $\Omega = \frac{\mu_1 \mu_2}{4}$. Considering m and g to be real, and γ_1 purely imaginary, from (6.95)-(6.97) one gets that the parameters μ_1 and μ_2 must satisfy the following relation,

$$\mu_2 = -\gamma_1 \mu_1^*, \quad (6.98)$$

We can also notice that for an appropriated choice of the parameters, it is possible to show the equivalence with the one-soliton solution found by Orfanidis [61].

6.3.2 The two-soliton solution

Now let us show that the two-soliton solution can be also calculated from the vacuum solution (6.69) using the dressing transformation. We will do it using only the algebraic properties of the $\mathfrak{sl}(2)$ affine Lie algebra. According to the approach above, there is an element ρ in the group satisfying (6.81). Consider the constant group element as

$$\rho = e^{\mu_1 V_+(\gamma_1)} e^{\mu_2 V_-(\gamma_2)} e^{\mu_3 V_+(\gamma_3)} e^{\mu_4 V_-(\gamma_4)}. \quad (6.99)$$

The explicit form for the solution is calculated by computing the corresponding matrix elements from the group element G ,

$$\begin{aligned} G &= 1 + \mu_1 e^{-\Gamma_1} V_+(\gamma_1) + \mu_2 e^{\Gamma_2} V_+(\gamma_2) + \mu_3 e^{-\Gamma_3} V_+(\gamma_3) + \mu_4 e^{\Gamma_4} V_+(\gamma_4) \\ &+ \mu_1 \mu_2 e^{-\Gamma_1 + \Gamma_2} V_+(\gamma_1) V_-(\gamma_2) + \mu_1 \mu_3 e^{-\Gamma_1 - \Gamma_3} V_+(\gamma_1) V_+(\gamma_3) + \mu_1 \mu_4 e^{-\Gamma_1 + \Gamma_4} V_+(\gamma_1) V_-(\gamma_4) \\ &+ \mu_2 \mu_3 e^{\Gamma_2 - \Gamma_3} V_-(\gamma_2) V_+(\gamma_3) + \mu_2 \mu_4 e^{\Gamma_2 + \Gamma_4} V_-(\gamma_2) V_-(\gamma_4) + \mu_3 \mu_4 e^{-\Gamma_3 + \Gamma_4} V_+(\gamma_3) V_-(\gamma_4) \\ &+ \mu_1 \mu_2 \mu_3 e^{-\Gamma_1 + \Gamma_2 - \Gamma_3} V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) + \mu_1 \mu_2 \mu_4 e^{-\Gamma_1 + \Gamma_2 + \Gamma_4} V_+(\gamma_1) V_-(\gamma_2) V_-(\gamma_4) \\ &+ \mu_1 \mu_3 \mu_4 e^{-\Gamma_1 - \Gamma_3 + \Gamma_4} V_+(\gamma_1) V_+(\gamma_3) V_-(\gamma_4) + \mu_2 \mu_3 \mu_4 e^{\Gamma_2 - \Gamma_3 + \Gamma_4} V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4) \\ &+ \mu_1 \mu_2 \mu_3 \mu_4 e^{-\Gamma_1 + \Gamma_2 - \Gamma_3 + \Gamma_4} V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4). \end{aligned} \quad (6.100)$$

The explicit form of the matrix elements is given in Appendix A.4. The solution one obtains is given as follows,

$$\begin{aligned}\tau_0 &= 1 - \frac{\mu_1\mu_2}{4}e^{-2\Gamma_1} - \frac{\mu_3\mu_4}{4}e^{-2\Gamma_3} - (\mu_1\mu_4 + \mu_2\mu_3)e^{-(\Gamma_1+\Gamma_3)} \left[\frac{\gamma_1\gamma_3}{(\gamma_1 + \gamma_3)^2} \right] \\ &+ \frac{1}{16}(\mu_1\mu_2\mu_3\mu_4)e^{-2(\Gamma_1+\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^4\end{aligned}\quad (6.101)$$

$$\begin{aligned}\tau_1 &= 1 + \frac{\mu_1\mu_2}{4}e^{-2\Gamma_1} + \frac{\mu_3\mu_4}{4}e^{-2\Gamma_3} + (\gamma_1^2\mu_1\mu_4 + \gamma_3^2\mu_2\mu_3)e^{-(\Gamma_1+\Gamma_3)} \left[\frac{1}{(\gamma_1 + \gamma_3)^2} \right] \\ &+ \frac{1}{16}(\mu_1\mu_2\mu_3\mu_4)e^{-2(\Gamma_1+\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^4\end{aligned}\quad (6.102)$$

$$\begin{aligned}\tau_2 &= \mu_1\gamma_1e^{-\Gamma_1} + \mu_3\gamma_3e^{-\Gamma_3} - \frac{1}{4}\mu_1\mu_2\mu_3e^{-(2\Gamma_1+\Gamma_3)} \left[\frac{\gamma_3(\gamma_1 - \gamma_3)^2}{(\gamma_1 + \gamma_3)^2} \right] \\ &- \frac{1}{4}\mu_1\mu_3\mu_4e^{-(\Gamma_1+2\Gamma_3)} \left[\frac{\gamma_1(\gamma_1 - \gamma_3)^2}{(\gamma_1 + \gamma_3)^2} \right],\end{aligned}\quad (6.103)$$

$$\begin{aligned}\tau_3 &= \mu_2e^{-\Gamma_1} + \mu_4e^{-\Gamma_3} + \frac{1}{4}\mu_2\mu_3\mu_4e^{-(\Gamma_1+2\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^2 \\ &+ \frac{1}{4}\mu_1\mu_2\mu_4e^{-(2\Gamma_1+\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^2,\end{aligned}\quad (6.104)$$

$$\begin{aligned}\tau_4 &= \mu_1e^{-\Gamma_1} + \mu_3e^{-\Gamma_3} + \frac{1}{4}\mu_1\mu_2\mu_3e^{-(2\Gamma_1+\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^2 \\ &+ \frac{1}{4}\mu_1\mu_3\mu_4e^{-(\Gamma_1+2\Gamma_3)} \left[\frac{\gamma_1 - \gamma_3}{\gamma_1 + \gamma_3} \right]^2,\end{aligned}\quad (6.105)$$

$$\begin{aligned}\tau_5 &= -\frac{\mu_2}{\gamma_1}e^{-\Gamma_1} - \frac{\mu_4}{\gamma_3}e^{-\Gamma_3} + \frac{1}{4}\mu_2\mu_3\mu_4e^{-(\Gamma_1+2\Gamma_3)} \left[\frac{(\gamma_1 - \gamma_3)^2}{\gamma_1(\gamma_1 + \gamma_3)^2} \right] \\ &+ \frac{1}{4}\mu_1\mu_2\mu_4e^{-(2\Gamma_1+\Gamma_3)} \left[\frac{(\gamma_1 - \gamma_3)^2}{\gamma_3(\gamma_1 + \gamma_3)^2} \right].\end{aligned}\quad (6.106)$$

Notice that, if m and g are considered to be real constants, there are two possibilities in order to ϕ_k^\dagger corresponds to the complex conjugate of ϕ_k : in the case of γ_1 and γ_3 to be purely imaginary numbers, and the parameters μ_k satisfying the following conditions,

$$\mu_2 = -\gamma_1\mu_1^*, \quad \mu_4 = -\gamma_3\mu_3^*. \quad (6.107)$$

The second possibility corresponds to the situation when $\gamma_3^* = -\gamma_1$, and as consequence $\Gamma_3^* = \Gamma_1$. In this case, we need that the parameters μ_k satisfy the following conditions,

$$\mu_4^* = \gamma_1\mu_1, \quad \mu_2^* = \gamma_3\mu_3. \quad (6.108)$$

6.3.3 The Breather solution

Let us choose the convenient parameters in order to obtain a breather solution of the BMT model. We know that the space-time dependence of the solutions are given by the exponentials $\exp \Gamma_k$, where $\Gamma_k = im(x_+ \gamma_k + \gamma_k^{-1} x_-)$. Then, to obtain the localized solutions periodic in time we made the following choice of parameters,

$$\gamma_1 = -ie^{i\theta}, \quad \gamma_3^* = -\gamma_1, \quad \mu_1 = \frac{2i}{\tan \theta}, \quad \mu_2 = \frac{2}{\tan \theta}. \quad (6.109)$$

Then we have that $\Gamma_3^* = \Gamma_1$ and the parameter μ_3 and μ_4 are given by the conditions (6.108). In addition, one gets from the form of Γ_1 that,

$$\Gamma_1 = kx + i\omega t, \quad \Gamma_3 = kx - i\omega t, \quad (\Gamma_1 + \Gamma_3) = 2kx, \quad (6.110)$$

with

$$k = m \cos \theta, \quad \text{and} \quad \omega = m \sin \theta. \quad (6.111)$$

Then, if one makes these choices of the parameters, we obtain

$$\chi_+ = \pm i \left(\frac{\pi}{2} \pm \arctan \left[\left(\frac{\sqrt{m^2 - \omega^2}}{\omega^2} \right) \frac{m^2 + \sqrt{m^2 - \omega^2} \cos(2\omega t)}{\sinh(2\sqrt{m^2 - \omega^2} x)} \right] \right). \quad (6.112)$$

6.4 Bäcklund solutions

In this section, we want to discuss the type of solutions derived from the auto-Bäcklund transformations (6.41)-(6.46). Particularly, we are interested to show that these solutions are in totally consistency with the ones given by the dressing method. Firstly, we noted that there is a closed relation between the field χ_+ appearing in the dressing procedure and the defect contribution to the number of particle conserved quantity N_D . In fact, from (6.77) it follows that

$$\frac{\partial \chi_+}{\partial t} = \frac{ig}{4} (\phi_1^\dagger \phi_1 - \phi_2^\dagger \phi_2). \quad (6.113)$$

Then, we get the relation

$$N_D = \frac{4i}{g} (\tilde{\chi}_+ - \chi_+) \Big|_{x=0}, \quad (6.114)$$

where χ_+ is given in general by (6.88). So, this expression gives us a closed relation between the dressing solutions and the Bäcklund solutions.

6.4.1 The one-soliton solution

Now, we construct the one-soliton solution for the BMT model by performing the Bäcklund transformation starting from the vacuum solution,

$$\tilde{\phi}_1 = \tilde{\phi}_2 = \tilde{\phi}_1^\dagger = \tilde{\phi}_2^\dagger = 0, \quad \tilde{\chi}_+ = \frac{i\pi}{2}. \quad (6.115)$$

From the explicit form of χ_+ for one-soliton solution given by (6.95) and using the Bäcklund transformations, we obtained

$$X = \sqrt{\frac{m}{g}} \left[\frac{-i\mu_1 a^{-1} e^{-\Gamma}}{[1 + |\Omega|^2 e^{-4\Gamma}]^{\frac{1}{2}}} \right], \quad X^\dagger = \sqrt{\frac{m}{g}} \left[\frac{i\mu_1^* a^{-1} e^{-\Gamma}}{[1 + |\Omega|^2 e^{-4\Gamma}]^{\frac{1}{2}}} \right], \quad (6.116)$$

and

$$\phi_1 = \sqrt{\frac{m}{g}} \left[\frac{i\mu_1 a^{-1} e^{-\Gamma}}{1 + \Omega e^{-2\Gamma}} \right], \quad \phi_2 = \sqrt{\frac{m}{g}} \left[\frac{\mu_1 e^{-\Gamma}}{1 - \Omega e^{-2\Gamma}} \right], \quad (6.117)$$

$$\phi_1^\dagger = -\sqrt{\frac{m}{g}} \left[\frac{i\mu_1^* a^{-1} e^{-\Gamma}}{1 - \Omega e^{-2\Gamma}} \right], \quad \phi_2^\dagger = \sqrt{\frac{m}{g}} \left[\frac{\mu_1^* e^{-\Gamma}}{1 + \Omega e^{-2\Gamma}} \right]. \quad (6.118)$$

where μ_1 is an arbitrary constant. Thus, we have found exactly the one-soliton solution for the BMT model firstly obtained by the dressing method, with $\gamma = ia^{-1}$. Then, it shows that our Bäcklund transformation are compatible not only with the integrability of the model in the presence of the defect, but also with the soliton solutions obtained by dressing method.

6.4.2 One-soliton/defect interaction

We are now interested in investigating the behaviour of single soliton solution passing through a defect. The defect condition (6.41), which determines how the soliton scatters with the defect, can be written in a more convenient way, namely

$$[\tilde{\tau}_2 \tau_0 + \tau_2 \tilde{\tau}_0] - \sigma [\tau_4 \tilde{\tau}_1 - \tilde{\tau}_4 \tau_1] = 0, \quad (6.119)$$

in terms of the tau-functions,

$$\tau_0 = 1 - \Omega e^{-2\Gamma}, \quad \tau_1 = 1 + \Omega e^{-2\Gamma}, \quad \tau_2 = \mu_1 \gamma e^{-\Gamma}, \quad (6.120)$$

$$\tau_3 = \mu_2 e^{-\Gamma}, \quad \tau_4 = \mu_1 e^{-\Gamma}, \quad \tau_5 = \frac{\mu_2}{\gamma} e^{-\Gamma}. \quad (6.121)$$

where $\sigma = ia^{-1}$ is the Bäcklund parameter. Then, we get

$$\begin{aligned} & \tilde{\mu}_1 \tilde{\gamma} e^{-\tilde{\Gamma}} + \mu_1 \gamma e^{-\Gamma} - \tilde{\mu}_1 \tilde{\gamma} \Omega e^{-\tilde{\Gamma}-2\Gamma} - \mu_1 \gamma \tilde{\Omega} e^{-\Gamma-2\tilde{\Gamma}} = \\ & -\sigma \tilde{\mu}_1 e^{-\tilde{\Gamma}} + \sigma \mu_1 e^{-\Gamma} - \sigma \tilde{\mu}_1 \Omega e^{-\tilde{\Gamma}-2\Gamma} + \sigma \mu_1 \tilde{\Omega} e^{-\Gamma-2\tilde{\Gamma}}, \end{aligned} \quad (6.122)$$

From this relation it is clear that following conditions over the parameters must be satisfied,

$$\tilde{\gamma} = \gamma, \quad \tilde{\mu}_1 = \left[\frac{\sigma - \gamma}{\sigma + \gamma} \right] \mu_1. \quad (6.123)$$

In a similar way, from the defect condition (6.42), we obtain

$$\tilde{\gamma} = \gamma, \quad \tilde{\mu}_2 = \left[\frac{\sigma - \gamma}{\sigma + \gamma} \right] \mu_2. \quad (6.124)$$

Then, it is worth noting that the jump-defect preserves the soliton velocity and the only effect of the interaction soliton-defect is a phase shift. In addition, we also have that the limiting cases when $\tilde{\mu}_1 = 0$ or $\mu_1 = 0$ do exist, and correspond to the situation where $\gamma = |\sigma|$ with $a > 0$, and $\gamma = |\sigma|$ with $a < 0$ respectively. Clearly, these cases indicate creation and absorption of the soliton. As $a \rightarrow \infty$, the parameter $\tilde{\mu}_1 \rightarrow -\mu_1$, which means that if the defect parameter is large the soliton will invert its shape, changing its character from soliton to anti-soliton or vice-versa. As $a \rightarrow 0$, we obtain $\tilde{\mu}_1 = \mu_1$, indicating that there is not defect and the soliton shape is preserved as expected. In addition, when $\gamma > \sigma$ the incoming soliton is delayed.

6.4.3 Two soliton–two soliton solution

For this time we consider the two soliton solution given explicitly by the equations (6.101)–(6.106) for both sides of the defect. Then, the two Bäcklund relations are satisfied providing the following relations between the parameters hold,

$$\tilde{\mu}_1 = \left[\frac{\sigma - \gamma_1}{\sigma + \gamma_1} \right] \mu_1, \quad \tilde{\mu}_3 = \left[\frac{\sigma - \gamma_3}{\sigma + \gamma_3} \right] \mu_3, \quad (6.125)$$

$$\tilde{\mu}_2 = \left[\frac{\sigma - \gamma_1}{\sigma + \gamma_1} \right] \mu_2, \quad \tilde{\mu}_4 = \left[\frac{\sigma - \gamma_3}{\sigma + \gamma_3} \right] \mu_4, \quad (6.126)$$

defining the corresponding phase shifts. These results clearly show that each soliton approaching a defect interact with it independently of one another, each being delayed. Notice also that, given a particular value of the defect (Bäcklund) parameter σ then the defect can absorb at most one soliton or antisoliton but not both because in general $\gamma_1 \neq \gamma_3$. These features are similar to the ones pointed out for the soliton solutions of the sine-Gordon model [62], and based on that we suggest that these soliton solutions of the BMT model might also be used to model logic gates. A more complete description of transitions of soliton solutions through a jump-defect was better described in [48], where indirect evidence of the well-known permutability theorem was outlined as well.

Conclusions and future directions

In this thesis we have studied the classical description of integrable field theories in the presence of type-II defects. We have shown that the original integrability properties of the bulk theory is preserved even after the introduction of suitable internal boundary conditions (defects), using essentially the inverse scattering formalism. As it was mentioned before, our definition of integrability concerns with the existence of a constructive way of finding solutions for the equations of motion of the models as well as a sufficient (infinity) number of integrals of motion. We have focused our attention mainly on the derivation of such conserved quantities which was achieved principally with the help of a general formula (2.57) that allowed us to compute the defect contributions to the modified conserved quantities. To do that, we have solved systematically a set of coupled Riccati equations for each model. The contribution of the defect to all orders was explicitly identified in terms of a defect matrix. In addition, we have derived all the defect matrices for each model studied in this thesis, from which we have recognized the corresponding integrable defect condition which corresponds to frozen Bäcklund transformation of the integrable equation. In particular, we have computed explicitly the defect contribution for the energy and momentum for each model, recovering previous results derived from their Lagrangian descriptions, except for the BMT model which surprisingly seems not to be described within any of the Lagrangian frameworks proposed until now. Then, we might expect that a generalisation of the existing schemes so far will encompass the BMT model which is interestingly related with the derivative nonlinear Schrödinger model (DNLS).

It should be emphasized that the approach adopted in this thesis provides a sufficient condition to address the question of integrable defects in classical field theories. It is remarkable that the defect matrices we have derived in this work are essentially “on-shell”, which means that its components have non-vanishing Poisson brackets with the fields in the bulk. As it was already pointed out in section 2.4, the natural way to address the question of involutivity of the modified conserved charges could be to work within the Hamiltonian formulation of the classical inverse scattering method using essentially the classical r -matrix of the bulk theory and a modified transition matrix to include the defect contributions. A recent proposal to achieve this goal was made in [24, 63, 64] for the NLS, sG and sigma models. In spite of, the results obtained for the integrable defects conditions seem not to be same of the ones obtained from the Lagrangian and the “on-shell” Lax approaches, in these works was shown the involutivity of the corresponding modified conserved charges in a formal way. The analysis following the line of these ideas for the massive Thirring models using this approach is in progress.

On the other hand, it should be interesting to explore the quantum aspects of integrable type-II in purely fermionic and supersymmetric field theories. In fact, as it was already noticed before some results on transmission matrices for type-I and type-II sine-Gordon, Tzitzéica-Bullough-Dodd and the $a_n^{(1)}$ affine Toda model, have been obtained [27, 18]. These transmission matrices are typically infinite dimensional, and there exist two methods at least to compute them: the first one is by solving quadratic relations appearing in the defect Yang-Baxter equation (DYBE); and the second one is by solving a linear intertwining condition between a suitable infinite dimensional representation given in terms of a pair of generalized creation and annihilation, and a finite dimensional representation of the underlying Borel subalgebra [28, 18]. In particular, for the massive Thirring model this transmission matrix has not been studied and then we hope that some of the results discussed in this thesis can help as guide in that direction.

Finally, it is worth noting that there is not any approach to integrable type-II defects in non-relativistic models until now. It should be interesting to investigate the consequences of integrability of this kind of defects for instance in the NLS model. In particular, a study of the interactions between soliton solutions with type-II defects deserves special attention in order to provide an interpretation and consequently to identify a possible application of the extra degree of freedom presents in the type-II defect theory. Some of these questions are expected to be developed in the near future.

A.1 The $\mathfrak{sl}(2)$ affine Lie algebra

The generators $\{H^{(n)}, E_{\pm}^{(n)}, C, D\}$ of the $\mathfrak{sl}(2)$ affine Lie algebra satisfy the commutation relations,

$$\begin{aligned}
 [H^{(m)}, H^{(n)}] &= 2mC\delta_{m+n,0}, \\
 [H^{(m)}, E_{\pm}^{(n)}] &= \pm 2E_{\pm}^{(m+n)}, \\
 [E_{+}^{(m)}, E_{-}^{(n)}] &= H^{(m+n)} + mC\delta_{m+n,0}, \\
 [D, T^{(n)}] &= nT^{(n)},
 \end{aligned} \tag{A.1}$$

where $T^{(n)} \equiv \{H^{(m)}, E_{\pm}^{(n)}\}$. The principal grading for the $\mathfrak{sl}(2)$ affine Lie algebra is generated by the operator

$$Q = 2D + \frac{1}{2}H^{(0)}, \quad \text{with} \quad D \equiv \lambda \frac{d}{d\lambda}. \tag{A.2}$$

Then, the above grading operator decomposes the algebra $\widehat{\mathfrak{G}}$ into subalgebras generated by elements of positive, negative and zero grades respectively,

$$\widehat{\mathfrak{G}} = \widehat{\mathfrak{G}}_{+} \oplus \widehat{\mathfrak{G}}_{0} \oplus \widehat{\mathfrak{G}}_{-}. \tag{A.3}$$

In addition, we can notice that $\{H = H^{(0)}, E_{\pm} = E_{\pm}^{(0)}\}$ are the corresponding generators for the $\mathfrak{sl}(2)$ finite Lie algebra. For further details see [65].

A.2 The $a_2^{(2)}$ twisted Lie algebra

The generators of the algebra $a_2^{(2)}$ are given by T_3^m , T_\pm^m , and L_j^r , with $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$ and $j, k = 0, \pm 1, \pm 2$, and satisfy the following commutation relations,

$$\begin{aligned}
\left[T_3^{(m)}, T_3^{(n)} \right] &= 2m \delta_{m+n,0} C, & \left[T_+^{(m)}, T_-^{(n)} \right] &= 2T_3^{(m+n)} + 4m \delta_{m+n,0} C, \\
\left[T_3^{(m)}, T_\pm^{(n)} \right] &= \pm T_\pm^{(m+n)}, \\
\left[T_3^{(m)}, L_k^{(r)} \right] &= k L_k^{(m+r)}, & \left[T_\pm^{(m)}, L_k^{(r)} \right] &= \sqrt{6-k(k \pm 1)} L_{k \pm 1}^{(m+r)}, \\
\left[L_k^{(r)}, L_{-k}^{(s)} \right] &= (-1)^k \left(\frac{k}{2} T_3^{(r+s)} + r \delta_{r+s,0} C \right) \\
\left[L_0^{(r)}, L_{\pm 1}^{(s)} \right] &= -\frac{\sqrt{6}}{4} T_\pm^{(r+s)}, & \left[L_0^{(r)}, L_{\pm 2}^{(s)} \right] &= 0 \\
\left[L_1^{(r)}, L_{-2}^{(s)} \right] &= \frac{1}{2} T_-^{(r+s)}, & \left[L_{-1}^{(r)}, L_2^{(s)} \right] &= \frac{1}{2} T_+^{(r+s)}, \\
\left[L_1^{(r)}, L_2^{(s)} \right] &= 0, & \left[L_{-1}^{(r)}, L_{-2}^{(s)} \right] &= 0.
\end{aligned} \tag{A.4}$$

The principal grading is generated by the operator $Q = T_3^{(0)} + 6D$, decomposing the affine algebra into elements of positive, negative and zero grades. For a more complete presentation of this twisted affine Kac-Moody algebra see [33].

We notice that the Lax pair for the TBD model (4.3) and (4.4) is given in terms of the generators $T_3^{(0)} \equiv T_3$ of grade zero, $T_+^{(0)} \equiv T_+$ and $L_{-2}^{(1/2)} \equiv L_{-2}$ of grade +1, $T_-^{(0)} \equiv T_-$ and $L_{+2}^{(-1/2)} \equiv L_2$ with grade -1. A suitable finite matrix representation is given by,

$$\begin{aligned}
T_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, & T_+ &= \sqrt{2}i \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & T_- &= -\sqrt{2}i \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
L_0 &= \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & L_1 &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & L_{-1} &= \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\
L_2 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, & L_{-2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{A.5}$$

A.3 The $\mathfrak{sl}(2, 1)$ affine Lie algebra

Consider the $\hat{\mathfrak{sl}}(2, 1)$ super Lie algebra with its generators given by

$$\begin{aligned}
h_1 &= \alpha_1 \cdot H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & h_2 &= \alpha_2 \cdot H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\
E_{\alpha_1} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E_{-\alpha_1} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E_{\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, & (A.6) \\
E_{-\alpha_2} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, & E_{\alpha_1+\alpha_2} &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & E_{-(\alpha_1+\alpha_2)} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\end{aligned}$$

where α_1 is a bosonic root and $\alpha_2, \alpha_1 + \alpha_2$ are the fermionic roots. The $\mathfrak{sl}(2, 1)$ affine algebra is decomposed according to the *grading operator* $Q = 2d + \frac{1}{2}h_1$, where d is the *derivation operator* satisfying $[d, T_a^{(n)}] = nT_a^{(n)}$. Here $T_a^{(n)}$ denotes both $H_i^{(n)}$ and $E_\alpha^{(n)}$. The hierarchy is further specified by the constant grade one element $E = E^{(1)}$, as follows

$$E^{(2n+1)} = h_1^{(n+1/2)} + 2h_2^{(n+1/2)} = K_2^{(2n+1)}, \quad (A.7)$$

The grading operator Q together with the judicious choice of E decomposes the affine superalgebra $\hat{\mathfrak{G}} = \hat{\mathfrak{sl}}(2, 1)$ into $\hat{\mathfrak{G}} = \mathcal{K} \oplus \mathcal{M}$, where the Kernel $\mathcal{K} = \{x \in \hat{\mathfrak{G}} | [x, E] = 0\}$ of E , and its complement \mathcal{M} are given by

$$\mathcal{K} = \{K_1^{(2n+1)}, K_2^{(2n+1)}, M_1^{(2n+1)}, M_2^{(2n)}\}, \quad (A.8)$$

$$\mathcal{M} = \{F_1^{(2n+3/2)}, F_2^{(2n+1/2)}, G_1^{(2n+1/2)}, G_2^{(2n+3/2)}\}, \quad (A.9)$$

where the bosonic generators are

$$M_1^{(2n+1)} = -(E_{\alpha_1}^{(n)} - E_{-\alpha_1}^{(n+1)}), \quad M_2^{(2n)} = h_1^{(n)}, \quad (A.10)$$

$$K_1^{(2n+1)} = -(E_{\alpha_1}^{(n)} + E_{-\alpha_1}^{(n+1)}), \quad K_2^{(2n+1)} = h_1^{(n+1/2)} + 2h_2^{(n+1/2)}, \quad (A.11)$$

and the fermionic generators are

$$F_1^{(2n+3/2)} = (E_{\alpha_1+\alpha_2}^{(n+1/2)} - E_{\alpha_2}^{(n+1)}) + (E_{-(\alpha_1+\alpha_2)}^{(n+1)} - E_{-\alpha_2}^{(n+1/2)}), \quad (A.12)$$

$$F_2^{(2n+1/2)} = -(E_{\alpha_1+\alpha_2}^{(n)} - E_{\alpha_2}^{(n+1/2)}) + (E_{-(\alpha_1+\alpha_2)}^{(n+1/2)} - E_{-\alpha_2}^{(n)}), \quad (A.13)$$

$$G_1^{(2n+1/2)} = (E_{\alpha_1+\alpha_2}^{(n)} + E_{\alpha_2}^{(n+1/2)}) + (E_{-(\alpha_1+\alpha_2)}^{(n+1/2)} + E_{-\alpha_2}^{(n)}), \quad (A.14)$$

$$G_2^{(2n+3/2)} = -(E_{\alpha_1+\alpha_2}^{(n+1/2)} + E_{\alpha_2}^{(n+1)}) + (E_{-(\alpha_1+\alpha_2)}^{(n+1)} + E_{-\alpha_2}^{(n+1/2)}) \quad (A.15)$$

These generators satisfy the following (anti)-commutation relations (see for instance [66]),

$$\left[K_1^{(n)}, K_2^{(m)} \right] = 0, \quad \left\{ F_1^{(n+\frac{1}{2})}, F_2^{(m+\frac{1}{2})} \right\} = 0, \quad (\text{A.16})$$

$$\left[F_1^{(n+\frac{1}{2})}, K_1^{(m)} \right] = F_2^{(n+m+\frac{1}{2})}, \quad \left[F_1^{(n+\frac{1}{2})}, K_2^{(m)} \right] = -F_2^{(n+m+\frac{1}{2})}, \quad (\text{A.17})$$

$$\left[F_2^{(n+\frac{1}{2})}, K_1^{(m)} \right] = F_1^{(n+m+\frac{1}{2})}, \quad \left[F_2^{(n+\frac{1}{2})}, K_2^{(m)} \right] = -F_1^{(n+m+\frac{1}{2})}, \quad (\text{A.18})$$

$$\left\{ F_1^{(n+\frac{1}{2})}, F_1^{(m+\frac{1}{2})} \right\} = -\left\{ F_2^{(n+\frac{1}{2})}, F_2^{(m+\frac{1}{2})} \right\} = 2E^{(n+m+1)} \quad (\text{A.19})$$

$$\left\{ F_2^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})} \right\} = -\left\{ F_1^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})} \right\} = 2M_1^{(n+m+1)} \quad (\text{A.20})$$

$$\left\{ F_1^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})} \right\} = -\left\{ F_2^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})} \right\} = 2M_2^{(n+m+1)} \quad (\text{A.21})$$

$$\left[M_1^{(n)}, F_1^{(m+\frac{1}{2})} \right] = G_1^{(n+m+\frac{1}{2})}, \quad \left[M_1^{(n)}, F_2^{(m+\frac{1}{2})} \right] = G_2^{(n+m+\frac{1}{2})} \quad (\text{A.22})$$

$$\left[M_2^{(n)}, F_1^{(m+\frac{1}{2})} \right] = -G_2^{(n+m+\frac{1}{2})}, \quad \left[M_2^{(n)}, F_2^{(m+\frac{1}{2})} \right] = -G_1^{(n+m+\frac{1}{2})} \quad (\text{A.23})$$

$$\left[M_1^{(n)}, K_1^{(m)} \right] = 2M_2^{(n+m)}, \quad \left[M_1^{(n)}, K_2^{(m)} \right] = 0 \quad (\text{A.24})$$

$$\left[M_2^{(n)}, K_1^{(m)} \right] = 2M_1^{(n+m)}, \quad \left[M_2^{(n)}, K_2^{(m)} \right] = 0 \quad (\text{A.25})$$

$$\left[G_1^{(n+\frac{1}{2})}, K_1^{(m)} \right] = -G_2^{(n+m+\frac{1}{2})}, \quad \left[G_1^{(n+\frac{1}{2})}, K_2^{(m)} \right] = -G_2^{(n+m+\frac{1}{2})}, \quad (\text{A.26})$$

$$\left[G_2^{(n+\frac{1}{2})}, K_1^{(m)} \right] = -G_1^{(n+m+\frac{1}{2})}, \quad \left[G_2^{(n+\frac{1}{2})}, K_2^{(m)} \right] = -G_1^{(n+m+\frac{1}{2})}, \quad (\text{A.27})$$

$$\left\{ G_1^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})} \right\} = 0 \quad (\text{A.28})$$

$$\left\{ G_1^{(n+\frac{1}{2})}, G_1^{(m+\frac{1}{2})} \right\} = -\left\{ G_2^{(n+\frac{1}{2})}, G_2^{(m+\frac{1}{2})} \right\} = 2(K_2^{(n+m+1)} - K_1^{(n+m+1)}) \quad (\text{A.29})$$

$$\left[M_1^{(n)}, G_1^{(m+\frac{1}{2})} \right] = -F_1^{(n+m+\frac{1}{2})}, \quad \left[M_1^{(n)}, G_2^{(m+\frac{1}{2})} \right] = -F_2^{(n+m+\frac{1}{2})} \quad (\text{A.30})$$

$$\left[M_2^{(n)}, G_1^{(m+\frac{1}{2})} \right] = -F_2^{(n+m+\frac{1}{2})}, \quad \left[M_2^{(n)}, G_2^{(m+\frac{1}{2})} \right] = -F_1^{(n+m+\frac{1}{2})} \quad (\text{A.31})$$

$$\left[M_1^{(n)}, M_2^{(m)} \right] = -2K_1^{(n+m)}, \quad (\text{A.32})$$

A.4 Matrix elements

Let us consider the constant group element as

$$\rho = e^{\mu_1 V_+(\gamma_1)} e^{\mu_2 V_-(\gamma_2)} e^{\mu_3 V_+(\gamma_3)} e^{\mu_4 V_-(\gamma_4)}. \quad (\text{A.33})$$

The two-soliton solution of the BMT model was obtained from the following matrix elements,

$$\langle \lambda_0 | E_-^{(+1)} V_+(\gamma_1) | \lambda_0 \rangle = \gamma_1, \quad \langle \lambda_1 | E_+^{(0)} V_-(\gamma_1) | \lambda_1 \rangle = 1, \quad (\text{A.34})$$

$$\langle \lambda_0 | V_-(\gamma_2) E_+^{(-1)} | \lambda_0 \rangle = \frac{1}{\gamma_2}, \quad \langle \lambda_1 | V_+(\gamma_1) E_-^{(0)} | \lambda_0 \rangle = 1, \quad (\text{A.35})$$

$$\langle \lambda_0 | V_+(\gamma_1) V_-(\gamma_2) | \lambda_0 \rangle = \langle \lambda_0 | V_-(\gamma_2) V_+(\gamma_1) | \lambda_0 \rangle = \frac{\gamma_1 \gamma_2}{(\gamma_1 - \gamma_2)^2}, \quad (\text{A.36})$$

$$\langle \lambda_1 | V_+(\gamma_1) V_-(\gamma_2) | \lambda_1 \rangle = \langle \lambda_1 | V_-(\gamma_2) V_+(\gamma_1) | \lambda_1 \rangle = \frac{\gamma_1^2}{(\gamma_1 - \gamma_2)^2}, \quad (\text{A.37})$$

and

$$\langle \lambda_0 | V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4) | \lambda_0 \rangle = \left[\frac{\gamma_1 \gamma_2 \gamma_3 \gamma_4 (\gamma_1 - \gamma_3)^2 (\gamma_2 - \gamma_4)^2}{(\gamma_1 - \gamma_2)^2 (\gamma_3 - \gamma_4)^2 (\gamma_1 - \gamma_4)^2 (\gamma_2 - \gamma_3)^2} \right], \quad (\text{A.38})$$

$$\langle \lambda_1 | V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4) | \lambda_1 \rangle = \left[\frac{\gamma_1^2 \gamma_3^2 (\gamma_1 - \gamma_3)^2 (\gamma_2 - \gamma_4)^2}{(\gamma_1 - \gamma_2)^2 (\gamma_3 - \gamma_4)^2 (\gamma_2 - \gamma_3)^2 (\gamma_1 - \gamma_4)^2} \right], \quad (\text{A.39})$$

$$\langle \lambda_0 | E_-^{(+1)} V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) | \lambda_0 \rangle = \left[\frac{\gamma_1 \gamma_2 \gamma_3 (\gamma_1 - \gamma_3)^2}{(\gamma_1 - \gamma_2)^2 (\gamma_2 - \gamma_3)^2} \right], \quad (\text{A.40})$$

$$\langle \lambda_0 | E_-^{(+1)} V_+(\gamma_1) V_+(\gamma_3) V_-(\gamma_4) | \lambda_0 \rangle = \left[\frac{\gamma_1 \gamma_3 \gamma_4 (\gamma_1 - \gamma_3)^2}{(\gamma_3 - \gamma_4)^2 (\gamma_1 - \gamma_4)^2} \right], \quad (\text{A.41})$$

$$\langle \lambda_1 | E_+^{(0)} V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4) | \lambda_1 \rangle = \left[\frac{\gamma_3^2 (\gamma_2 - \gamma_4)^2}{(\gamma_2 - \gamma_3)^2 (\gamma_3 - \gamma_4)^2} \right], \quad (\text{A.42})$$

$$\langle \lambda_1 | E_+^{(0)} V_+(\gamma_1) V_-(\gamma_2) V_-(\gamma_4) | \lambda_0 \rangle = \left[\frac{\gamma_1^2 (\gamma_2 - \gamma_4)^2}{(\gamma_1 - \gamma_2)^2 (\gamma_1 - \gamma_4)^2} \right], \quad (\text{A.43})$$

$$\langle \lambda_1 | V_+(\gamma_1) V_+(\gamma_3) V_-(\gamma_4) E_-^{(0)} | \lambda_1 \rangle = \frac{\gamma_4^2 (\gamma_1 - \gamma_3)^2}{(\gamma_1 - \gamma_4)^2 (\gamma_3 - \gamma_4)^2}, \quad (\text{A.44})$$

$$\langle \lambda_1 | V_+(\gamma_1) V_-(\gamma_2) V_+(\gamma_3) E_-^{(0)} | \lambda_1 \rangle = \frac{\gamma_2^2 (\gamma_1 - \gamma_3)^2}{(\gamma_1 - \gamma_2)^2 (\gamma_2 - \gamma_3)^2}, \quad (\text{A.45})$$

$$\langle \lambda_0 | V_-(\gamma_2) V_+(\gamma_3) V_-(\gamma_4) E_+^{(-1)} | \lambda_0 \rangle = \frac{\gamma_3^3 (\gamma_2 - \gamma_4)^2}{\gamma_2 \gamma_4 (\gamma_2 - \gamma_3)^2 (\gamma_3 - \gamma_4)^2}, \quad (\text{A.46})$$

$$\langle \lambda_0 | V_+(\gamma_1) V_-(\gamma_2) V_-(\gamma_4) E_+^{(-1)} | \lambda_0 \rangle = \frac{\gamma_1^3 (\gamma_2 - \gamma_4)^2}{\gamma_2 \gamma_4 (\gamma_1 - \gamma_2)^2 (\gamma_1 - \gamma_4)^2}, \quad (\text{A.47})$$

So, the tau-functions are explicitly given by,

$$\begin{aligned} \tau_0 &= 1 + \mu_1\mu_2e^{-\Gamma_1+\Gamma_2} \left[\frac{\gamma_1\gamma_2}{(\gamma_1-\gamma_2)^2} \right] + \mu_1\mu_4e^{-\Gamma_1+\Gamma_4} \left[\frac{\gamma_1\gamma_4}{(\gamma_1-\gamma_4)^2} \right] \\ &+ \mu_2\mu_3e^{\Gamma_2-\Gamma_3} \left[\frac{\gamma_2\gamma_3}{(\gamma_2-\gamma_3)^2} \right] + \mu_3\mu_4e^{-\Gamma_3+\Gamma_4} \left[\frac{\gamma_3\gamma_4}{(\gamma_3-\gamma_4)^2} \right] \\ &+ \mu_1\mu_2\mu_3\mu_4e^{-\Gamma_1+\Gamma_2-\Gamma_3+\Gamma_4} \left[\frac{\gamma_1\gamma_2\gamma_3\gamma_4(\gamma_1-\gamma_3)^2(\gamma_2-\gamma_4)^2}{(\gamma_1-\gamma_2)^2(\gamma_3-\gamma_4)^2(\gamma_1-\gamma_4)^2(\gamma_2-\gamma_3)^2} \right], \end{aligned} \quad (\text{A.48})$$

$$\begin{aligned} \tau_1 &= 1 + \mu_1\mu_2e^{-\Gamma_1+\Gamma_2} \left[\frac{\gamma_1^2}{(\gamma_1-\gamma_2)^2} \right] + \mu_1\mu_4e^{-\Gamma_1+\Gamma_4} \left[\frac{\gamma_1^2}{(\gamma_1-\gamma_4)^2} \right] \\ &+ \mu_2\mu_3e^{\Gamma_2-\Gamma_3} \left[\frac{\gamma_3^2}{(\gamma_2-\gamma_3)^2} \right] + \mu_3\mu_4e^{-\Gamma_3+\Gamma_4} \left[\frac{\gamma_3^2}{(\gamma_3-\gamma_4)^2} \right] \\ &+ \mu_1\mu_2\mu_3\mu_4e^{-\Gamma_1+\Gamma_2-\Gamma_3+\Gamma_4} \left[\frac{\gamma_1^2\gamma_3^2(\gamma_1-\gamma_3)^2(\gamma_2-\gamma_4)^2}{(\gamma_1-\gamma_2)^2(\gamma_3-\gamma_4)^2(\gamma_2-\gamma_3)^2(\gamma_1-\gamma_4)^2} \right], \end{aligned} \quad (\text{A.49})$$

$$\begin{aligned} \tau_2 &= \mu_1\gamma_1e^{-\Gamma_1} + \mu_3\gamma_3e^{-\Gamma_3} + \mu_1\mu_2\mu_3e^{-\Gamma_1+\Gamma_2-\Gamma_3} \left[\frac{\gamma_1\gamma_2\gamma_3(\gamma_1-\gamma_3)^2}{(\gamma_1-\gamma_2)^2(\gamma_2-\gamma_3)^2} \right] \\ &+ \mu_1\mu_3\mu_4e^{-\Gamma_1-\Gamma_3+\Gamma_4} \left[\frac{\gamma_1\gamma_3\gamma_4(\gamma_1-\gamma_3)^2}{(\gamma_3-\gamma_4)^2(\gamma_1-\gamma_4)^2} \right], \end{aligned} \quad (\text{A.50})$$

$$\begin{aligned} \tau_3 &= \mu_2e^{\Gamma_2} + \mu_4e^{\Gamma_4} + \mu_2\mu_3\mu_4e^{\Gamma_2-\Gamma_3+\Gamma_4} \left[\frac{\gamma_3^2(\gamma_2-\gamma_4)^2}{(\gamma_2-\gamma_3)^2(\gamma_3-\gamma_4)^2} \right] \\ &+ \mu_1\mu_2\mu_4e^{-\Gamma_1+\Gamma_2+\Gamma_4} \left[\frac{\gamma_1^2(\gamma_2-\gamma_4)^2}{(\gamma_1-\gamma_2)^2(\gamma_1-\gamma_4)^2} \right], \end{aligned} \quad (\text{A.51})$$

$$\begin{aligned} \tau_4 &= \mu_1e^{-\Gamma_1} + \mu_3e^{-\Gamma_3} + \mu_1\mu_2\mu_3e^{-\Gamma_1+\Gamma_2-\Gamma_3} \left[\frac{\gamma_2^2(\gamma_1-\gamma_3)^2}{(\gamma_1-\gamma_2)^2(\gamma_2-\gamma_3)^2} \right] \\ &+ \mu_1\mu_3\mu_4e^{-\Gamma_1-\Gamma_3+\Gamma_4} \left[\frac{\gamma_4^2(\gamma_1-\gamma_3)^2}{(\gamma_1-\gamma_4)^2(\gamma_3-\gamma_4)^2} \right], \end{aligned} \quad (\text{A.52})$$

$$\begin{aligned} \tau_5 &= \frac{\mu_2}{\gamma_2}e^{\Gamma_2} + \frac{\mu_4}{\gamma_4}e^{\Gamma_4} + \mu_2\mu_3\mu_4e^{\Gamma_2-\Gamma_3+\Gamma_4} \left[\frac{\gamma_3^3(\gamma_2-\gamma_4)^2}{\gamma_2\gamma_4(\gamma_2-\gamma_3)^2(\gamma_3-\gamma_4)^2} \right] \\ &+ \mu_1\mu_2\mu_4e^{-\Gamma_1+\Gamma_2+\Gamma_4} \left[\frac{\gamma_1^3(\gamma_2-\gamma_4)^2}{\gamma_2\gamma_4(\gamma_1-\gamma_2)^2(\gamma_1-\gamma_4)^2} \right]. \end{aligned} \quad (\text{A.53})$$

We can check that these tau-functions satisfy the equations (6.60)-(6.63) and (6.77) for any values of the parameters μ_k and γ_k , with $k = 1, \dots, 4$. The two-soliton solution for the BMT model is obtained by taking the limits $\gamma_2 \rightarrow -\gamma_1$ and $\gamma_4 \rightarrow -\gamma_3$.

B.1 Type-I and type-II boundary sine-Gordon theory

Firstly, we consider the type-I defect potential for the sine-Gordon model B_0 given in (3.45),

$$B_0 = 2m \left[\sigma \cos \left(\frac{\varphi + \tilde{\varphi}}{2} \right) + \frac{1}{\sigma} \cos \left(\frac{\varphi - \tilde{\varphi}}{2} \right) \right]. \quad (\text{B.1})$$

In order to define the sG theory restricted to the half-line ($x > 0$), we perform a suitable limit on the left field within the action, by taking $\tilde{\varphi}$ to be a constant k . It is convenient to redefine the Bäcklund parameter as $\sigma = e^{-\xi/2}$. Then, we obtain

$$\tilde{B}_0 = 2M \cos \left(\frac{\varphi - \varphi_0}{2} \right), \quad (\text{B.2})$$

for some constants M and φ_0 , which satisfy the following relations,

$$M \sin \left(\frac{\varphi_0}{2} \right) = 2m \sinh \left(\frac{\xi}{2} \right) \sin \left(\frac{k}{2} \right), \quad (\text{B.3})$$

$$M \cos \left(\frac{\varphi_0}{2} \right) = 2m \cosh \left(\frac{\xi}{2} \right) \cos \left(\frac{k}{2} \right). \quad (\text{B.4})$$

The boundary potential (B.2) was first found by Ghoshal and Zamolodchikov in [2], and it was shown that an infinite subset of conserved charges survived after introducing the boundary by using the inverse scattering method [67].

From the form of the boundary potential (B.2) we find that the corresponding boundary condition is given by,

$$\partial_x \varphi|_{x=0} = M \sin\left(\frac{\varphi - \varphi_0}{2}\right), \quad (\text{B.5})$$

where the UV parameters M and φ_0 are determined in terms of the Bäcklund parameter σ and the constant k by the inverse relations of the (B.3) and (B.4), as follows

$$M^2 = m^2 \left(\sigma^2 + \frac{1}{\sigma^2} + 2 \cos k \right), \quad (\text{B.6})$$

$$\tan\left(\frac{\varphi_0}{2}\right) = \left(\frac{1 - \sigma^2}{1 + \sigma^2}\right) \tan\left(\frac{k}{2}\right). \quad (\text{B.7})$$

As it was pointed out in [68], the boundary condition (B.5) is compatible with the Bäcklund transformation. We can also notice that if we exploit the discrete symmetry of the sG action under $\varphi \rightarrow \pm\varphi$ when $x \rightarrow -x$, we could take $\tilde{\varphi}(x) = \pm\varphi(-x)$, and in this case we obtain the following boundary potential,

$$B_0 = 2m [\sigma^{\mp 1} + \sigma^{\pm 1} \cos \varphi] |_{x=0}, \quad (\text{B.8})$$

with the simple boundary condition,

$$\partial_x \varphi|_{x=0} = 2m\sigma^{\pm 1} \sin \varphi|_{x=0}, \quad (\text{B.9})$$

which corresponds to the trivial Dirichlet problem when $\varphi_0 = 0$ or equivalently $k = 0$. This boundary condition was proved to be integrable [69, 70, 17]. The behaviour of the soliton solutions for the sine-Gordon equation with this boundary condition were studied [71].

Now, we consider the type-II defect potential for the sine-Gordon model (3.66),

$$B_0 = -\frac{m}{2\sigma} [e^{-i(\varphi_+ - \Lambda)} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) + 4e^{i(\varphi_+ - \Lambda)}] - \frac{m\sigma}{2} [e^{i\Lambda} (e^{i\varphi_-} + e^{-i\varphi_-} + \eta) + 4e^{-i\Lambda}], \quad (\text{B.10})$$

and performing the half-line limit by taking $\tilde{\varphi}$ to be a constant κ , we obtain

$$\mathcal{L}_B = 2\varphi(\partial_t \Lambda) - [f_+(\varphi)e^{i\Lambda} + f_-(\varphi)e^{-i\Lambda}], \quad (\text{B.11})$$

with

$$f_+(\varphi) = -\frac{m}{2\sigma\omega} [\omega e^{-i\varphi} + (\eta + (\sigma\omega)^2) e^{-i\varphi/2} + \sigma e^{i\varphi/2} + (1 + \eta\omega\sigma^2)], \quad (\text{B.12})$$

$$f_-(\varphi) = -\frac{2m}{\sigma} [\omega e^{i\varphi} + \sigma^2], \quad (\text{B.13})$$

where we have three free constant parameters σ, η , and $\omega \equiv e^{i\kappa/2}$. Notice that, redefining $\Lambda \rightarrow \Lambda + \widehat{\Lambda}(\varphi)$ changes the Lagrangian density (B.11) by a total time derivative. A similar form for this type-II boundary potential has been recently suggested in [72]

B.2 The boundary Thirring model

We now are interested in deriving the boundary potential for Thirring model from the type-II defect potential (5.78), namely

$$\begin{aligned} \mathcal{L}_D &= \frac{ia}{2m} [X^\dagger(\partial_t X) - (\partial_t X^\dagger)X] + \frac{i}{2} [\tilde{\psi}_1^\dagger \psi_1 - \psi_1^\dagger \tilde{\psi}_1 + \tilde{\psi}_2^\dagger \psi_2 - \psi_2^\dagger \tilde{\psi}_2] \\ &+ \frac{1}{2} [i(\psi_1 - \tilde{\psi}_1) + a(\psi_2 + \tilde{\psi}_2)] X^\dagger + \frac{1}{2} [i(\psi_1^\dagger - \tilde{\psi}_1^\dagger) - a(\psi_2^\dagger + \tilde{\psi}_2^\dagger)] X \\ &- \frac{ga}{4m} [\tilde{\psi}_1^\dagger \tilde{\psi}_1 + \psi_1^\dagger \psi_1 + \tilde{\psi}_2^\dagger \tilde{\psi}_2 + \psi_2^\dagger \psi_2] X^\dagger X. \end{aligned} \quad (\text{B.14})$$

To define the Thirring model restricted to the half-line ($x > 0$), we use the $U(1)$ invariance of the Thirring action to perform the following transformation on the left fields, namely,

$$\tilde{\psi}_1(0^-) \longrightarrow e^{i\mu} \psi_2^\dagger(0^+), \quad \tilde{\psi}_2(0^-) \longrightarrow -e^{i\mu} \psi_1^\dagger(0^+), \quad (\text{B.15})$$

$$\tilde{\psi}_1^\dagger(0^-) \longrightarrow e^{-i\mu} \psi_2(0^+), \quad \tilde{\psi}_2^\dagger(0^-) \longrightarrow -e^{-i\mu} \psi_1(0^+), \quad (\text{B.16})$$

where μ is just a phase commuting with the components of ψ . By replacing, we obtain

$$\mathcal{L}_D \longrightarrow \mathcal{L}_{\text{free}} + \mathcal{L}_B, \quad (\text{B.17})$$

where

$$\mathcal{L}_{\text{free}} = \frac{i}{2} \left[e^{i\mu} \left(\psi_2^\dagger(x) \psi_1^\dagger(0^+) - \psi_1^\dagger(x) \psi_2^\dagger(0^+) \right) - e^{-i\mu} \left(\psi_1(x) \psi_2(0^+) - \psi_2(x) \psi_1(0^+) \right) \right], \quad (\text{B.18})$$

is the Lagrangian density from which the free boundary conditions raise, namely

$$\psi_1^\dagger(0) = -e^{-i\mu} \psi_2(0), \quad \psi_2^\dagger(0) = -e^{-i\mu} \psi_1(0), \quad (\text{B.19})$$

which are equivalent to have the fermionic current $j^0|_{x=0} = \psi_1^\dagger(0)\psi_1(0) + \psi_2^\dagger(0)\psi_2(0) = 0$. On the other hand, the boundary Lagrangian density \mathcal{L}_B is given by,

$$\begin{aligned} \mathcal{L}_B &= \frac{ia}{2m} (X^\dagger \partial_t X - (\partial_t X^\dagger)X) + \frac{1}{2} \left[i \left(\psi_1(x) - e^{i\mu} \psi_2^\dagger(0^+) \right) + a \left(\psi_2(x) - e^{i\mu} \psi_1^\dagger(0^+) \right) \right] X^\dagger \\ &+ \frac{1}{2} \left[i \left(\psi_1^\dagger(x) - e^{-i\mu} \psi_2(0^+) \right) - a \left(\psi_2^\dagger(x) - e^{-i\mu} \psi_1(0^+) \right) \right] X. \end{aligned} \quad (\text{B.20})$$

By varying the total boundary action we obtain the following boundary conditions at $x = 0$,

$$X = \psi_1 + e^{i\mu} \psi_2^\dagger = ia^{-1} (\psi_2 + e^{i\mu} \psi_1^\dagger), \quad (\text{B.21})$$

$$X^\dagger = \psi_1^\dagger + e^{-i\mu} \psi_2 = -ia^{-1} (\psi_2^\dagger + e^{i\mu} \psi_1^\dagger). \quad (\text{B.22})$$

and its corresponding time-derivatives,

$$\partial_t X = \frac{m}{2a} \left[(\psi_1 - e^{i\mu} \psi_2^\dagger) - ia(\psi_2 - e^{i\mu} \psi_1^\dagger) \right]_{x=0}, \quad (\text{B.23})$$

$$\partial_t X^\dagger = \frac{m}{2a} \left[(\psi_1^\dagger - e^{-i\mu} \psi_2) + ia(\psi_2^\dagger - e^{-i\mu} \psi_1) \right]_{x=0}. \quad (\text{B.24})$$

The above boundary Lagrangian for the GMT model seems to have the same structure of the boundary derived in [44, 45]. The auxiliary fermionic field X and X^\dagger are expected to be related with the boundary field operators used in these works.

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