KOLIHA–DRAZIN INVERTIBLES FORM A REGULARITY

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Contents

Summary

The axiomatic theory of Želazko defines a variety of general spectra where specified axioms are satisfied. However, there arise a number of spectra, usually defined for a single element of a Banach algebra, that are not covered by the axiomatic theory of Želazko. V. Kordula and V. Müller addressed this issue and created the theory of regularities. Their unique idea was to describe the underlying set of elements on which the spectrum is defined. The axioms of a regularity provide important consequences. We prove that the set of Koliha-Drazin invertible elements, which includes the Drazin invertible elements, forms a regularity. The properties of the spectrum corresponding to a regularity are also investigated.

Key terms:
Banach algebra; radical; spectrum; resolvent; quasinilpotent; nilpotent; spectral idempotent; isolated spectral point; accumulation point; regularity; Koliha-Drazin invertible; Drazin invertible; quasipolar; KD-spectrum; D-spectrum; Laurent expansion; poles of the resolvent.

Introduction

In the recent years the theory of regularities has been a research interest of various authors. In 1996, V. Kordula and V. Müller created the notion of regularities [12]. A non-empty subset R of a unital Banach algebra is called a regularity if it satisfies the following two conditions:

1. if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \Leftrightarrow a^n \in R$;
2. if a, b, c, d are mutually commuting elements of $\mathcal{A}$ satisfying $ac + bd = \mathbf{1}$, then $ab \in R \Leftrightarrow a, b \in R$.

The above axioms of regularities are weak enough so that there are many natural classes of elements in Banach algebras and operator theory satisfying them. These classes include the set of all invertible elements, as well as the sets of left and of right invertible elements. On the other hand the axioms are strong enough to provide important consequences, for example, the spectrum, $\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin R\}$, corresponding to a regularity R satisfies the Spectral Mapping Theorem, so that $\sigma_R(f(a)) = f(\sigma_R(a))$ for every $a \in \mathcal{A}$ and every function f analytic on a neighbourhood of $\sigma(a)$ which is non-constant on each component of its domain of definition.

In 2007 R.A. Lubansky [14] examined regularities in connection with Koliha-Drazin invertible elements (denoted by KD-invertibles). According to Koliha [11], an element $b \in \mathcal{A}$ is a Koliha-Drazin inverse, $b = a^{KD}$, of $a \in \mathcal{A}$ if

$$ab = ba, b = ab^2, a - a^2b \in QN(\mathcal{A}).$$

The Drazin index $i(a)$ of a is the nilpotency index of $a - a^2b$ if $a - a^2b \in N(\mathcal{A})$, and $i(a) = \infty$ otherwise. R.A. Lubansky proved that the set of KD-invertibles in a complex unital Banach algebra forms a regularity.

The focus of this dissertation is on the fact that the set of KD-invertibles forms a regularity. The work is organised into five chapters. Throughout this dissertation we used definitions and notation as set out in Aupetit [1], J.J. Koliha [11], E. Kreyszig [13] and V. Müller [15]. In the first chapter some basic definitions and results are dealt with, such as Banach algebras, various aspects of spectral theory, the Laurent expansion and the Bolzano–Weierstrass Theorem. In section 1.4, the Holomorphic Functional Calculus is introduced which is used in the study of Banach algebras. The basic properties of this Functional Calculus are given of which the Spectral Mapping Theorem, $\sigma(f(a)) = f(\sigma(a))$, is the most important. The spectral idempotent, as an essential tool, is used throughout this dissertation. Chapter 2 deals with a regularity and Chapter 3 with Koliha–Drazin and Drazin invertible elements where the definitions and results come from Koliha [11]. We include Harte's definition of quasipolar elements to elaborate on the more general definition of a Koliha–Drazin inverse. The main result of this dissertation is Theorem 4.1.6, where the concepts explained in Chapter 2 and Chapter 3 are united. It states that the set of KD-invertible elements forms a regularity as Lubansky pointed out. As a consequence of this (see Corollary 4.1.8) we see that the set of D–invertible elements forms a regularity, which is a well known result. In Chapter 5 we conclude our discussion with the D-spectrum and the KD-spectrum where we prove some interesting and relevant results in the context of spectral theory.

Chapter 1

Preliminaries

1.1 Banach Algebras

Definition 1.1.1

An *algebra* $\mathcal{A}$ over a field K is a vector space over K such that for all $a,\ b,\ c \in \mathcal{A}$ and $\alpha \in K$:

1. There is a unique product $ab \in \mathcal{A}$
2. $(ab)c = a(bc)$ (associative under multiplication)
3. $a(b + c) = ab + ac$ (left distributive over addition)
4. $(a + b)c = ac + bc$ (right distributive over addition)
5. $\alpha(ab) = (\alpha a)b = a(\alpha b)$

   If in addition for all $a,\ b \in \mathcal{A}$:

6. $ab = ba$,

   $\mathcal{A}$ is said to be *Abelian* or *commutative*.

   If there exists an element $\mathbf{1}_{\mathcal{A}} \in \mathcal{A}$, only called $\mathbf{1}$ if the context is clear, such that for all $a \in \mathcal{A}$:

7. $\mathbf{1}a = a\mathbf{1} = a$,

   then $\mathcal{A}$ is said to be an *algebra with identity/unit* and **1** is the *identity/unit*.

Such an algebra $\mathcal{A}$ is called a unital algebra. If $\mathcal{A}$ has identity **1**, then it is unique, since if **1** and **1'** are both identities, $\mathbf{1}' = \mathbf{1}'\mathbf{1} = \mathbf{1}$.
An element $a \in \mathcal{A}$ is said to be *invertible* if there exists an element $a^{-1} \in \mathcal{A}$, called the *inverse* of a, such that $aa^{-1} = a^{-1}a = \mathbf{1}$.
If the inverse exists, then it is unique, since if a^{-1} and b^{-1} are both inverses of a, then: $a^{-1} = a^{-1}\mathbf{1} = a^{-1}(ab^{-1}) = (a^{-1}a)b^{-1} = \mathbf{1}b^{-1} = b^{-1}$.
We will denote the *subset of* $\mathcal{A}$ *of all elements that are invertible in* $\mathcal{A}$ by $\mathcal{A}^{-1}$. $\mathcal{A}^{-1}$ is associative, contains the identity **1** and is closed under multiplication. Hence $\mathcal{A}^{-1}$ is a group.
An element of $\mathcal{A}$ which is not invertible in $\mathcal{A}$ is said to be *singular* in $\mathcal{A}$. For $K = \mathbb{R}$ or $K = \mathbb{C}$, $\mathcal{A}$ is called a *real* or *complex algebra* respectively.

Definition 1.1.2

A *subalgebra* $\mathcal{B}$ of an algebra $\mathcal{A}$ is a subspace of $\mathcal{A}$ that is closed under multiplication, which means that for $a, b \in \mathcal{B}$, $ab \in \mathcal{B}$.
Note that the unit of $\mathcal{A}$ does not necessarily belong to the subalgebra $\mathcal{B}$ and that $\mathcal{B}$ might have a unit different from the one in $\mathcal{A}$.

Definition 1.1.3

Let $\mathcal{A}$ be an algebra. An *algebra seminorm* in $\mathcal{A}$ is a function $\|\cdot\| : \mathcal{A} \to [0, \infty), a \mapsto \|a\|$ such that for all $a, b \in \mathcal{A}, \alpha \in \mathbb{C}$:

1. $\|\alpha a\| = |\alpha| \cdot \|a\|$

2. $\|a + b\| \leq \|a\| + \|b\|$

3. $\|ab\| \leq \|a\| \cdot \|b\|$

4. $\|\mathbf{1}\| = 1$

   An algebra norm in $\mathcal{A}$ is an algebra seminorm satisfying the following:

5. if $\|a\| = 0$, then $a = 0$.

Definition 1.1.4

A *normed algebra* is a pair $(\mathcal{A}, \|\cdot\|)$, where $\mathcal{A}$ is an algebra and $\|\cdot\|$ is an algebra norm in $\mathcal{A}$. A *Banach algebra* is a normed algebra which is complete in the norm defined on it.

Unless otherwise specified, all Banach algebras considered in this dissertation are complex and unital.

Definition 1.1.5

If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, then a homeomorphism $f : \mathcal{A} \to \mathcal{B}$ is a continuous bijective mapping whose inverse is continuous. If $\mathcal{A}$ and $\mathcal{B}$ are algebras then a linear mapping $\phi : \mathcal{A} \to \mathcal{B}$ is called a *homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{A}$ and $\phi(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$. If $\mathcal{A}$ and $\mathcal{B}$ are normed algebras, then a homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is *continuous* if $\|\phi\| = \sup\{\|\phi(a)\| : a \in \mathcal{A}, \|a\| = 1\} < \infty$.
An *isomorphism* is a continuous homomorphism ϕ satisfying the following:

$$\inf\{\|\phi(a)\| : a \in \mathcal{A}, \|a\| = 1\} > 0.$$

A homomorphism ϕ is called *isometrical* if $\|\phi(a)\| = \|a\|$ for all $a \in \mathcal{A}$.

If $\rho : \mathcal{A} \to \mathbb{C}$ is a nonzero linear functional such that for all $a, b \in \mathcal{A}$, $\rho(ab) = \rho(a)\rho(b)$, then it is called a *multiplicative linear function/character*. Hence a character is a linear functional that is a homomorphism.

Definition 1.1.6

Let $\mathcal{A}$ be a Banach algebra. A set $J \subset \mathcal{A}$ is called a *left(right) ideal* in $\mathcal{A}$ if J is a subspace of $\mathcal{A}$ and $aj \in J (ja \in J)$ for all $j \in J$ and $a \in \mathcal{A}$. If J is both a left and right ideal in $\mathcal{A}$, then it is called a *two-sided ideal* in $\mathcal{A}$. The ideal $J = \{0\}$ is called the *trivial ideal*.
An ideal (left, right or two-sided) is *proper* if $J \neq \mathcal{A}$; so J is proper if and only if $\mathbf{1}_{\mathcal{A}} \notin J$.
A *maximal ideal* is a proper ideal that is not properly contained in any proper ideal.
Throughout this text we shall refer to a two-sided ideal as an ideal in $\mathcal{A}$.

Example 1.1.7

1. $\mathbb{R}$ and $\mathbb{C}$ are both commutative Banach algebras with unit $\mathbf{1}$.

2. Let J be an ideal of a Banach algebra $\mathcal{A}$. $\mathcal{A}/J$ is an algebra with element $a + J$ where $a \in \mathcal{A}$; $\mathcal{A}/J$ is called the *quotient algebra* of $\mathcal{A}$ modulo J. The zero element is J and the unit is $\mathbf{1} + J$.
   Let $a + J, b + J$ be elements of $\mathcal{A}/J$ where $a, b \in \mathcal{A}$ and let $\alpha \in \mathbb{C}$.
   We define the following:
   Addition: $(a + J) + (b + J) = (a + b) + J$
   Multiplication: $(a + J)(b + J) = ab + J$
   Scalar multiplication: $\alpha(a + J) = \alpha a + J$.

The mapping $\phi : \mathcal{A} \rightarrow \mathcal{A}/J$ given by $\phi(a) = a + J$ for $a \in \mathcal{A}$, is called the canonical (quotient) mapping.
ϕ is a homomorphism, since $\phi(ab) = ab + J = \phi(a) \cdot \phi(b)$ and $\phi(\mathbf{1}) = \mathbf{1} + J$.
We write $[a] = a + J$ for the equivalence class and $a,\ b$ equivalent modula J means $a - b \in J$.
When J is a closed ideal, $\mathcal{A}/J$ is a Banach algebra with quotient norm
$\|[a]\| = \inf\limits_{u \in [a]} \|u\| = \inf\limits_{j \in J} \|a + j\| \leq \|a\|$. Lastly $\mathcal{A}/J$ is commutative whenever $\mathcal{A}$ is commutative.

From Definition 1.1.6 we can say that a left(right) ideal $J \subset \mathcal{A}$ is a maximal left(right) ideal if J is proper and if the only proper left(right) ideal containing J is J itself. One can easily prove that in a Banach algebra $\mathcal{A}$ every proper ideal is contained in a maximal ideal of the same kind by using Zorn's Lemma. It is also true that a maximal ideal of a Banach algebra $\mathcal{A}$ is closed in $\mathcal{A}$. *([15], Theorem 40, p. 16).*

Theorem 1.1.8 *([15], Theorem 41, p. 16).*

Let $\mathcal{A}$ be a Banach algebra.
The following sets are identical:

1. the intersection of all maximal left ideals in $\mathcal{A}$;
2. the intersection of all maximal right ideals in $\mathcal{A}$;
3. the set of all $c \in \mathcal{A}$ such that $\mathbf{1} - ac$ is invertible for every $a \in \mathcal{A}$;
4. the set of all $c \in \mathcal{A}$ such that $\mathbf{1} - ca$ is invertible for every $a \in \mathcal{A}$.

Definition 1.1.9

We denote the *radical* of $\mathcal{A}$ by rad$\mathcal{A}$ and it is the set of all $c \in \mathcal{A}$ with properties $(1) - (4)$ of Theorem 1.1.8. Note that rad$\mathcal{A}$ is a closed two–sided ideal of $\mathcal{A}$, since maximal ideals are closed sets. *([1], Corollary 3.2.2)*. We call $\mathcal{A}$ *semisimple* if rad$\mathcal{A} = \{0\}$.

1.2 Spectral Theory

Definition 1.2.1

If $\mathcal{A}$ is a Banach algebra with identity $\mathbf{1}$, then the *spectrum* of an element $a \in \mathcal{A}$, denoted by $\sigma^{\mathcal{A}}(a)$, or $\sigma(a)$ if the algebra is clear form the context, is defined as
$\sigma(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{A}^{-1}\}$.

Definition 1.2.2

If $\mathcal{A}$ is a Banach algebra with identity $\mathbf{1}$, then the *resolvent set* of an element $a \in \mathcal{A}$, denoted by $\rho(a)$: is defined as $\rho(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \in \mathcal{A}^{-1}\}$. The *resolvent* of an element $a \in \mathcal{A}$ is the function $\lambda \longmapsto (\lambda\mathbf{1} - a)^{-1}$ defined in the open set $\mathbb{C}\backslash\sigma(a)$ and denoted by $R(\cdot, a)$. From the above definitions we see that the spectrum of an element $a \in \mathcal{A}$ is the complement of its resolvent set, and that $\sigma(a) \cup \rho(a) = \mathbb{C}$.

Theorem 1.2.3 *([13], Theorem 7.7.1)*

Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. If $\|a\| < 1$ then $\mathbf{1} - a \in \mathcal{A}^{-1}$ and $(\mathbf{1} - a)^{-1} = \sum\limits_{j=0}^{\infty} a^j$.

Proof.

Assume $\|a\| < 1$. Then $\|a^j\| \leq \|a\|^j$ for $j = 0, 1, 2, \ldots$, since $\|ab\| \leq \|a\| \|b\|$ for all $a, b \in \mathcal{A}$. Thus $\sum_{j=0}^{\infty} \|a^j\| \leq \sum_{j=0}^{\infty} \|a\|^j$. Since $\|a\| < 1$, it follows that $\sum_{j=0}^{\infty} \|a\|^j$ converges, therefore by the Comparison Test for Series $\sum_{j=0}^{\infty} \|a^j\|$ converges. $\mathcal{A}$ is complete since it is a Banach space. The absolute convergence in the Banach space $\mathcal{A}$ implies that $\sum_{j=0}^{\infty} a^j$ converges. Let S denote its sum. Let $S_n = \mathbf{1} + a + a^2 + \cdots + a^n$. Then $(\mathbf{1} - a) S_n = S_n (\mathbf{1} - a) = \mathbf{1} - a^{n+1}$. We now let $n \to \infty$. Then $a^{n+1} \to 0$ since $\|a\| < 1$. Hence if $n \to \infty$, $S_n \to S$ and $a^{n+1} \to 0$. Since $(\mathbf{1} - a) S_n = \mathbf{1} - a^{n+1}$ and since multiplication is continuous in Banach algebras, we have $(\mathbf{1} - a) S = \mathbf{1}$. Similarly $S (\mathbf{1} - a) = \mathbf{1}$. So $(\mathbf{1} - a)$ is invertible with inverse S. Hence $(\mathbf{1} - a)^{-1} = S = \sum_{j=0}^{\infty} a^j$. ■

We already know that $\mathcal{A}^{-1}$, the set of all invertible elements of $\mathcal{A}$, is a group.

Theorem 1.2.4 *([13], Theorem 7.7.2)*

Let $\mathcal{A}$ be a complex Banach algebra with identity. Then $\mathcal{A}^{-1}$ is an open subset of $\mathcal{A}$.

Proof.

Let $a_0 \in \mathcal{A}^{-1}$. We need to show that every $a \in \mathcal{A}$ sufficiently close to a_0 belongs to $\mathcal{A}^{-1}$. Let $\|a - a_0\| < \frac{1}{\|a_o^{-1}\|}$. Let $c = a_0^{-1} a$ and $d = \mathbf{1} - c$. Then:

$$\begin{aligned}
\|d\| = \|-d\| &= \|c - \mathbf{1}\| \\
&= \left\|a_0^{-1} a - a_0^{-1} a_0\right\| \quad \text{(since } a_0 \in \mathcal{A}^{-1}\text{)} \\
&= \left\|a_0^{-1} (a - a_0)\right\| \\
&\leq \left\|a_0^{-1}\right\| \cdot \|a - a_0\| \quad \text{(by Definition 1.1.3)} \\
&< 1
\end{aligned}$$

Thus, $\|d\| < 1$ and by Theorem 1.2.3, $\mathbf{1} - d$ is invertible. Therefore $\mathbf{1} - d = c \in \mathcal{A}^{-1}$. Since $\mathcal{A}^{-1}$ is a group and $a_0,\ c \in \mathcal{A}^{-1},\ a = \left(a_0 a_0^{-1}\right) a = a_0 \left(a_0^{-1} a\right) = a_0 c \in \mathcal{A}^{-1}$.
Since $a_0 \in \mathcal{A}^{-1}$ was arbitrary, we have that $\mathcal{A}^{-1}$ is open. ■

Theorem 1.2.5 *([15], Theorem 16, p. 6)*

Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}$. Then the resolvent $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ is analytic in $\mathbb{C} \setminus \sigma(a)$.

Proof.
For $\lambda, \mu \notin \sigma(a)$ we have

$$\begin{aligned} & (\mu\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - a)^{-1} \\ = \ & (\mu\mathbf{1} - a)^{-1} \left((\lambda\mathbf{1} - a) - (\mu\mathbf{1} - a)\right)(\lambda\mathbf{1} - a)^{-1} \\ = \ & (\lambda - \mu)(\mu\mathbf{1} - a)^{-1}(\lambda\mathbf{1} - a)^{-1}, \end{aligned}$$

and so

$$\begin{aligned} & \lim_{\lambda \to \mu} \frac{(\mu\mathbf{1} - a)^{-1} - (\lambda\mathbf{1} - a)^{-1}}{(\lambda - \mu)} \\ = \ & \lim_{\lambda \to \mu} \frac{(\lambda - \mu)(\mu\mathbf{1} - a)^{-1}(\lambda\mathbf{1} - a)^{-1}}{(\lambda - \mu)} \\ = \ & (\lambda\mathbf{1} - a)^{-2} \end{aligned}$$

Thus the function $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ is analytic in $\mathbb{C} \setminus \sigma(a)$. ■

Theorem 1.2.6 *([1], Theorem 3.2.3)*

Let $\mathcal{A}$ be a Banach algebra, $a \in \mathcal{A}$ and $c \in \mathcal{A}^{-1}$. If $\|a-c\| < \frac{1}{\|c^{-1}\|}$, then $a \in \mathcal{A}^{-1}$. Moreover the mapping $a \mapsto a^{-1}$ is a homeomorphism from $\mathcal{A}^{-1}$ onto $\mathcal{A}^{-1}$.

Proof.
We have $a = c + a - c = c\left(\mathbf{1} + c^{-1}(a-c)\right)$.
Now $\left\|c^{-1}(a-c)\right\| \leq \|a-c\| \cdot \left\|c^{-1}\right\| < 1$ by using Definition 1.1.3 and the assumption $\|a-c\| < \frac{1}{\|c^{-1}\|}$. By Theorem 1.2.3 $\mathbf{1} + c^{-1}(a-c)$ is invertible.
Consequently, since $\mathcal{A}^{-1}$ is a group under multiplication, $a \in \mathcal{A}^{-1}$.
Hence

$$\begin{aligned}
a^{-1} &= \left[\mathbf{1} + c^{-1}(a-c)\right]^{-1} c^{-1} \\
&= \sum_{k=0}^{\infty} \left[c^{-1}(a-c)\right]^k c^{-1}, \text{ and so} \\
\left\|a^{-1} - c^{-1}\right\| &= \left\|\sum_{k=0}^{\infty} \left[c^{-1}(a-c)\right]^k c^{-1} - c^{-1}\right\| \\
&= \left\|\left(\sum_{k=0}^{\infty} \left[c^{-1}(a-c)\right]^k - \mathbf{1}\right)\left(c^{-1}\right)\right\| \\
&= \left\|\left(\sum_{k=1}^{\infty} \left[c^{-1}(a-c)\right]^k\right)\left(c^{-1}\right)\right\| \\
&= \left\|\left(c^{-1}\right)(a-c)\left(\sum_{k=0}^{\infty} \left[c^{-1}(a-c)\right]^k\right)\left(c^{-1}\right)\right\| \\
&\leq \left\|c^{-1}\right\|^2 \|a-c\| \sum_{k=0}^{\infty} \left(\left\|c^{-1}\right\| \cdot \|a-c\|\right)^k \\
&= \frac{\left\|c^{-1}\right\|^2 \|a-c\|}{\mathbf{1} - \|c^{-1}\| \, \|a-c\|} \\
&= 0 \quad \text{since} \quad \|a-c\| \to 0.
\end{aligned}$$

So $a \longmapsto a^{-1}$ is continuous, and since it is its own inverse, it is a homeomorphism. ∎

Note 1.2.7

For a fixed a, the mapping $\lambda \mapsto \lambda\mathbf{1} - a$ is continuous, for if $(\lambda_n) \subseteq \mathbb{C}$ and $\lambda_n \to \lambda$ as $n \to \infty$, then $\|(\lambda_n\mathbf{1} - a) - (\lambda\mathbf{1} - a)\| = |\lambda_n - \lambda| \, \|\mathbf{1}\| = |\lambda_n - \lambda| \to 0 \quad \text{as} \quad n \to \infty$.

Definition 1.2.8

Let a be an element of a Banach algebra $\mathcal{A}$. The *spectral radius* of a denoted $r(a)$, is defined to be the number $r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|$.

Note 1.2.9

The following is a well known expression for the spectral radius in terms of the norm:

$$r(a) = \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = \inf_n \|a^n\|^{\frac{1}{n}}.$$

It is called the Beurling–Gelfand formula. For the proof of the spectral radius formula see *([15], Theorem 22, p. 8)*.

Definition 1.2.10

If $a \in \mathcal{A}$ satisfies $r(a) = 0$, that is $\sigma(a) = \{0\}$, then a is said to be *quasinilpotent.* This means that an element a of a Banach algebra $\mathcal{A}$ is quasinilpotent if

$$\lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = 0.$$

The *set of quasinilpotent elements* of $\mathcal{A}$ is denoted by $QN(\mathcal{A})$. We thus have that
$a \in QN(\mathcal{A})$ if $\lambda\mathbf{1} - a \in \mathcal{A}^{-1}$ for all complex $\lambda \neq 0$. *([9], Definition 2)*

Definition 1.2.11

The *set of nilpotent elements* of $\mathcal{A}$ is $N(\mathcal{A}) = \{a \in \mathcal{A} : a^n = 0 \text{ for some } n \in \mathbb{N}\}$

Note that $N(\mathcal{A}) \subset QN(\mathcal{A})$. The converse holds if $\mathcal{A}$ is a finite dimensional Banach algebra.

We show that rad$\mathcal{A} \subseteq QN(\mathcal{A})$: From Theorem 1.1.8 we have that rad$\mathcal{A} = \{c \in \mathcal{A} : \mathbf{1} - ca \in \mathcal{A}^{-1} \text{ for all } a \in \mathcal{A}\}$. So for any complex $\lambda \neq 0$ and $c \in$ rad$\mathcal{A}$ we have that $\mathbf{1} - \frac{c}{\lambda} \in \mathcal{A}^{-1}$. Consequently, since $\mathcal{A}^{-1}$ is a group under multiplication, $\lambda(\mathbf{1} - \frac{c}{\lambda}) = \lambda\mathbf{1} - c$ is invertible for all $\lambda \neq 0$. Thus $\sigma(c) = \{0\}$ and therefore $c \in QN(\mathcal{A})$.

If $\mathcal{A}$ is a commutative Banach algebra, then rad$\mathcal{A} = QN(\mathcal{A})$.

Theorem 1.2.12 *([13], Theorem 7.7.3)*

If $\mathcal{A}$ is a complex Banach algebra with identity, then for every $a \in \mathcal{A}$, $r(a) \leq \|a\|$ and $\sigma(a)$ is compact.

Proof.

Let $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. First we show that $r(a) \leq \|a\|$. Suppose $|\lambda| > \|a\|$. Then $\left\|\lambda^{-1}a\right\| < 1$ and by Theorem 1.2.3 we have that $\mathbf{1} - \lambda^{-1}a$ is invertible. Hence $\lambda\left(\mathbf{1} - \lambda^{-1}a\right) = \lambda\mathbf{1} - a$ is also invertible, and so $\lambda \in \rho(a)$.

Therefore $|\lambda| \leq \|a\|$ for all $\lambda \in \sigma(a)$. Thus $r(a) \leq \|a\|$ (Definition 1.2.8). Since $r(a) \leq \|a\|$ we have that $\sigma(a)$ is bounded. We only need to show that $\sigma(a)$ is closed. To do so we prove that $\rho(a) = \mathbb{C}\backslash\sigma(a)$ is open. Let $\lambda_0 \in \rho(a)$, which means that $\lambda_0\mathbf{1} - a$ is invertible. From Theorem 1.2.4 we have that $\mathcal{A}^{-1}$ is open, so there exists an open neighbourhood $N \subset \mathcal{A}$ of $\lambda_0\mathbf{1} - a$ consisting only of invertible elements. From Theorem 1.2.6 and Note 1.2.7 we have $\lambda \mapsto (\lambda\mathbf{1} - a)^{-1}$ is continuous. Consequently there exists an open neighbourhood of λ_0, say

M, such that for each $\lambda \in M$, $\lambda\mathbf{1} - a$ is invertible, since continuity implies that the inverse of every open set is open. Thus $M \subseteq \rho(a)$. Now since $\lambda_0 \in \rho(a)$ was arbitrary, we see that there exists an open neighbourhood about every point of $\rho(a)$, contained in $\rho(a)$. So $\rho(a)$ is open and $\sigma(a)$ is closed. Therefore $\sigma(a)$ is compact. ■

Note 1.2.13 *([18], Theorem 3.3, p. 278)*

Suppose $\mathcal{A}$ is a Banach algebra, $a \in \mathcal{A}$ and $|\lambda| > r(a)$. Then $\lambda\mathbf{1} - a$ is invertible and $(\lambda\mathbf{1} - a)^{-1} = \sum_{n=1}^{\infty} \lambda^{-n} a^{n-1}$.

Note 1.2.14

- Theorem 1.2.12 shows that $\rho(a) \neq \emptyset$, since $\sigma(a) = \mathbb{C} \backslash \rho(a)$ is bounded.
- $\sigma(a) \neq \emptyset$ and compact *([1], Theorem 3.2.8)*
- If $\mathcal{A}$ is real, then the spectrum of $a \in \mathcal{A}$ may be empty.

Definition 1.2.15

An *accumulation point* λ of $\sigma(a)$ is a point $\lambda \in \mathbb{C}$ such that each neighbourhood of λ contains some point of $\sigma(a)$ other than λ. In other words $\lambda \in \operatorname{acc}(\sigma(a))$ if for any ε–neighbourhood of λ there exists $\delta \in \sigma(a)$ such that $0 < |\lambda - \delta| < \varepsilon$. We denote the *set of all accumulation points* of $\sigma(a)$ by $\operatorname{acc}(\sigma(a))$.

Definition 1.2.16

An *isolated spectral point* λ of $\sigma(a)$ is a point $\lambda \in \mathbb{C}$ such that there exists a neighbourhood of λ containing no other point of the spectrum. In other words $\lambda \in \text{iso}(\sigma(a))$ if there exists an ε–neighbourhood of λ, such that if $0 < |\lambda - \delta| < \varepsilon$, then $\delta \notin \sigma(a)$. We denote the *set of all isolated points* of $\sigma(a)$ by $\text{iso}(\sigma(a))$.

Note 1.2.17 *([2], Remark 10 and [18], p. 330)*

In the discussion to follow we define a pole of the resolvent. First observe that
$\sigma(a) = \text{acc}(\sigma(a)) \cup \text{iso}(\sigma(a))$. If $\mathcal{A}$ is a unital Banach algebra and $a \in \mathcal{A}$, then the resolvent function of a, $R(\cdot, a) : \rho(a) \to \mathcal{A}$ is holomorphic and $\text{iso}(\sigma(a))$ coincides with the set of isolated singularities of $R(\cdot, a)$.
For $\lambda_0 \in \text{iso}(\sigma(a))$, we have a Laurent expansion of $R(\cdot, a)$ in terms of $(\lambda - \lambda_0)$ such that $R(\lambda, a) = (\lambda\mathbf{1} - a)^{-1} = \sum_{n \geq 0} a_n (\lambda - \lambda_0)^n + \sum_{n \geq 1} b_n (\lambda - \lambda_0)^{-n}$, where a_n and b_n belong to $\mathcal{A}$, using the functional calculus. This representation is valid when
$0 < |\lambda - \lambda_0| < \delta$ for any δ such that $\sigma(a) \setminus \{\lambda_0\}$ lies on or outside the circle $|\lambda - \lambda_0| = \delta$.
Then λ_0 will be called a *pole of order k of* $R(\cdot, a)$ if and only if there exists $k \geq 1$ such that $b_k \neq 0$ and $b_m = 0$, for all $m \geq k+1$. Hence λ_0 is a pole of order k if and only if $b_k \neq 0$ and $b_{k+1} = 0$.
The *set of poles of* $R(\cdot, a)$ will be denoted by $\prod(a)$. If λ_0 is an isolated point of $\sigma(a)$ but not a pole of $R(\cdot, a)$, then we call λ_0 an *isolated essential singularity* of $R(\cdot, a)$. We denote the *set of isolated essential singularities* by $\text{IES}(a)$. So $\text{iso}(\sigma(a)) \setminus \prod(a) = \text{IES}(a)$.
If $\lambda_0 \in \text{iso}(\sigma(a)) \setminus \prod(a)$ then $\lambda_0\mathbf{1} - a \notin \mathcal{A}^{-1}$ and in the Laurent expansion
$(\lambda\mathbf{1} - a)^{-1} = \sum\limits_{n \geq 0} a_n (\lambda - \lambda_0)^n + \sum\limits_{n \geq 1} b_n (\lambda - \lambda_0)^{-n}$ an infinite number of b_n's are nonzero.

Definition 1.2.18 *([9], Definition 1 and [15], Definition 23, p. 9)*

If M is a subset of a Banach algebra $\mathcal{A}$, then the *commutant of* M is defined by $\text{comm}(M) = \{b \in \mathcal{A} : bm = mb,\ m \in M\}$ and the *double commutant* of M is defined by $\text{comm}^2(M) = \{b \in \mathcal{A} : bm' = m'b,\ \text{for all } m' \in \text{comm}(M)\}$.
In particular we have the following:
If $a \in \mathcal{A}$ then its commutant is the set

$$\text{comm}(a) = \{b \in \mathcal{A} : ab = ba\}$$

and its double commutant is the set

$$\text{comm}^2(a) = \{b \in \mathcal{A} : bc = cb \text{ for all } c \in \text{comm}(a)\}.$$

Lemma 1.2.19 *([15], Lemma 24, p. 9)*

Let $M,\ N$ be subsets of Banach algebra $\mathcal{A}$. Then:

1. $\text{comm}(M)$ is a closed subalgebra of $\mathcal{A}$;
2. $M \subset \text{comm}^2(M)$;
3. if $M \subset N$ then $\text{comm}(N) \subset \text{comm}(M)$ and $\text{comm}^2(M) \subset \text{comm}^2(N)$;
4. if M consists of mutually commuting elements, then $M \subset \text{comm}^2(M) \subset \text{comm}(M)$ and $\text{comm}^2(M)$ is a commutative Banach algebra.

The double commutant is a commutative Banach algebra containing the identity of $\mathcal{A}$.

Lemma 1.2.20

Let $\mathcal{A}$ be a Banach algebra. Take $a \in \mathcal{A}$. Then $\text{acc}(\sigma(\lambda\mathbf{1} - a)) = \lambda -$ $\text{acc}(\sigma(a))$.

Proof.

Let $\alpha \in \text{acc}(\sigma(\lambda\mathbf{1} - a))$. By the Spectral Mapping Theorem $\text{acc}(\sigma(\lambda\mathbf{1} - a)) = \text{acc}(\lambda - \sigma(a))$.
Take $\alpha = \lambda - \beta,\ \beta \in \sigma(a)$. Now for all $\varepsilon > 0$ there exists an $\alpha_n \in \sigma(\lambda\mathbf{1} - a)$,
$\alpha_n = \lambda - \beta_n,\ \beta_n \in \sigma(a)$, such that $\alpha_n \to \alpha$.
Hence $\lambda - \beta_n \to \lambda - \beta$, and therefore $\beta_n \to \beta$.
This means that $\beta \in \text{acc}(\sigma(a))$.
Consequently $\alpha = \lambda - \beta \in \lambda -$ $\text{acc}(\sigma(a))$.
Conversely, suppose $\alpha = \lambda - \beta \in \lambda -$ $\text{acc}(\sigma(a))$. So we assume $\beta \in \text{acc}(\sigma(a))$. Then by the definition of an accumulation point we have

$$|\beta_n - \beta| < \varepsilon \Rightarrow \beta_n \in \sigma(a) \text{ for all } n \in \mathbb{N}.$$

This means $\beta_n \to \beta$ for all $n \in \mathbb{N}$. So $\lambda - \beta_n \to \lambda - \beta$ for $\beta_n \in \sigma(a)$, since addition is continuous in a Banach algebra. Since $\beta_n \in \sigma(a)$, $\lambda - \beta_n \in \lambda - \sigma(a)$ for all β_n. So $\lambda - \beta$ must be an element of $\text{acc}(\lambda - \sigma(a)) = \text{acc}(\sigma(\lambda\mathbf{1} - a))$. ■

Theorem 1.2.21 *([15], Theorem 26, p. 9)*

Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras and $a \in \mathcal{A}$. Let $\psi : \mathcal{A} \to \mathcal{B}$ be a homomorphism. Then $\sigma^{\mathcal{B}}(\psi(a)) \subset \sigma^{\mathcal{A}}(a)$.

Theorem 1.2.22 *([15], Theorem 27, p. 9)*

Let $\mathcal{A}$ be a closed subalgebra of a Banach algebra $\mathcal{B}$ containing the same unit and let $a \in \mathcal{A}$. Then $\sigma^{\mathcal{B}}(a) \subset \sigma^{\mathcal{A}}(a)$.

Theorem 1.2.23 *([15], Theorem 11, p. 20)*

Let $\mathcal{A}$ be a commutative Banach algebra such that $a,\ b \in \mathcal{A}$ and $ab = ba$. Then

1. $\sigma(ab) \subset \sigma(a) \cdot \sigma(b)$ and $\sigma(a+b) \subset \sigma(a) + \sigma(b)$;

2. $r(ab) \leq r(a) \cdot r(b)$ and $r(a+b) \leq r(a) + r(b)$.

We now use Theorem 1.2.21 to prove very important properties of the radical.

Theorem 1.2.24 *([15], Theorem 43, p. 17)*

Let $\mathcal{A}$ be a Banach algebra. Then:

1. $\mathcal{A}/\text{rad}\mathcal{A}$ is semisimple;

2. an element $c \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $c+\text{rad}\mathcal{A}$ is invertible in $\mathcal{A}/\text{rad}\mathcal{A}$.

Proof.

1. Denote by $q : \mathcal{A} \rightarrow \mathcal{A}/\text{rad}\mathcal{A}$ the canonical projection, which means that for $c \in \mathcal{A}$, $q : c \mapsto c+\text{rad}\mathcal{A}$. If $c \in \mathcal{A}\backslash\text{rad}\mathcal{A}$, then $c \in \mathcal{A}$ and $c \notin\text{rad}\mathcal{A}$; then there exists a maximal left ideal J in $\mathcal{A}$ with $c \notin J$ since rad$\mathcal{A}$ is the intersection of all maximal left ideals in $\mathcal{A}$. Since rad$\mathcal{A} \subset J$, we now show that $J+\text{rad}\mathcal{A} = q(J)$ is a maximal left ideal in $\mathcal{A}/\text{rad}\mathcal{A}$. Take $j+\text{rad}\mathcal{A} \in J+\text{rad}\mathcal{A}$, $j \in J$ and $a+\text{rad}\mathcal{A} \in \mathcal{A}/\text{rad}\mathcal{A},\ a \in \mathcal{A}$.

   - To show $J+\text{rad}\mathcal{A}$ is a left ideal in $\mathcal{A}/\text{rad}\mathcal{A}$, let $r_1,\ r_2 \in \text{rad}\mathcal{A}$,
     $(a + r_1) \in \mathcal{A}+\text{rad}\mathcal{A}$ and $j + r_2 \in J+\text{rad}\mathcal{A}$.
     Then $(a + r_1)(j + r_2) = (a + r_1) j + (a + r_1) r_2 \in J+\text{rad}\mathcal{A}$.
   - The mapping $J \rightarrow J+\text{rad}\mathcal{A}$ is onto so since J is maximal, $J+\text{rad}\mathcal{A}$ is maximal. Also $q(c) = c+\text{rad}\mathcal{A} \notin q(J)$; since $c \notin J,\ c+\text{rad}\mathcal{A}$ is not a maximal left ideal in $\mathcal{A}/\text{rad}\mathcal{A}$. Thus $c + \text{rad}\mathcal{A} \notin \text{rad}(\mathcal{A}/\text{rad}\mathcal{A})$. Since c was an arbitrary element in $\mathcal{A}\backslash\text{rad}\mathcal{A}$, the algebra $\mathcal{A}/\text{rad}\mathcal{A}$ is semisimple.

2. If $c \in \mathcal{A}^{-1}$, then $q(c) \in (\mathcal{A}/\text{rad}\mathcal{A})^{-1}$, by Theorem 1.2.21.
   Conversely, if $q(c) \in (\mathcal{A}/\text{rad}\mathcal{A})^{-1}$, then there exists $d \in \mathcal{A}$ such that $cd \in \mathbf{1} + \text{rad}\mathcal{A}$, which means $cd - \mathbf{1} \in rad\mathcal{A}$. Similarly $dc - \mathbf{1} \in \text{rad}\mathcal{A}$. By Theorem 1.1.8 (3) and (4), the elements $\mathbf{1} + \mathbf{1}(cd - \mathbf{1}) = cd$ and $\mathbf{1} + \mathbf{1}(dc - \mathbf{1}) = dc$ are invertible.
   Hence $c \in \mathcal{A}^{-1}$. ■

The following Lemma give some conditions for a Banach algebra to be finite dimensional:

Lemma 1.2.25 *([10], Lemma 7 and [19], Lemma p. 4)*

Let $\mathcal{A}$ be a semisimple Banach algebra in which each element has a finite spectrum. Then $\mathcal{A}$ is finite dimensional.

1.3 Some complex analysis

Cauchy's integral formula will be given in the following theorem. It shows that if a function f is analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

Theorem 1.3.1 *([4], Theorem, Section 39)*

Let function f be analytic everywhere within and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz.$$

The theory of Laurent series is very helpful to understand the behaviour of functions around isolated points. If a function fails to be analytic at a point z_0, we use the Laurent series representation for $f(z)$ involving both positive and negative powers of $z - z_0$, as given in the following theorem.

Theorem 1.3.2 *([4], Theorem, Section 46)*

Let function f be analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, and let C denote any positively simple closed contour around z_0 and lying in that domain. Then, at each point z in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad (R_1 < |z - z_0| < R_2)$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \quad (n = 0,\ 1,\ 2,\ \ldots)$$

and

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}} \quad (n = 1,\ 2,\ \ldots).$$

This is called a Laurent series.

We mention another important result:

Theorem 1.3.3 *([4], Exercise 14, Section 57) (Bolzano–Weierstrass Theorem)*

An infinite set of points lying in a closed bounded region R has at least one accumulation point in R.

Another form of the Bolzano–Weierstrass Theorem states that a bounded sequence (a_n) has at least one limit point. By definition the sequence has infinitely many terms. A number a is a limit point of (a_n) if for every given $\varepsilon > 0$ we have $|a_n - a| < \varepsilon$ for infinitely many n.
(See [13], A.7, p. 620.)

1.4 Holomorphic Functional Calculus

The Holomorphic Functional Calculus deals with a broad class of functions in Banach algebras and is one of the main tools used throughout this dissertation. Let $H(a)$ denote the set of all complex valued functions f, each defined and analytic in an open neighbourhood of $\sigma(a)$. If $f(\lambda) = \alpha_0 + \alpha_1\lambda + \cdots + \alpha_n\lambda^n$ is a polynomial with coefficients $\alpha_i \in \mathbb{C}$, then for a Banach algebra $\mathcal{A}$ and $a \in \mathcal{A}$, the Banach algebra version of this polynomial is given by

$$f(a) = \alpha_0\mathbf{1} + \alpha_1 a + \cdots + \alpha_n a^n$$

which is obviously well-defined in $\mathcal{A}$. Note that $f(a)$ exists as an element of $\mathcal{A}$ whenever f is holomorphic on an open set containing $\sigma(a)$. This means that although $f(a)$ belongs to $\mathcal{A}$, some of its properties may depend on the behaviour of its complex counterpart $f(\lambda)$ on an open set containing $\sigma(a)$.

In the following definition we combine the defintions and notations as found in *([1]) and ([17])*.

Definition 1.4.1

Suppose K is compact in $\mathbb{C}$ and μ is a Borel measure on K and that $f : K \to \mathcal{A}$ is a continuous function from K into the Banach algebra $\mathcal{A}$ such that the scalar functions $\phi(f(\lambda))$ are integrable with respect to μ, for every bounded linear functional $\phi \in \mathcal{A}'$ and $\lambda \in K$.

The element $c = \int_k f(\lambda)\, d\mu$ is the unique element of $\mathcal{A}$ which satisfies
$\phi(c) = \phi\left(\int_k f(\lambda)\right) d\mu = \int_k \phi(f(\lambda))\, d\mu$ for every bounded linear functional $\phi \in \mathcal{A}'$.

For the existence and uniqueness of this integral, see *([17], Theorem 3.27)*.

The Holomorphic Functional Calculus, defined hereafter, may be viewed as a generalization of Cauchy's Theorem for complex functions.

Theorem 1.4.2 *([1], Theorem 3.3.3)*

Let $\mathcal{A}$ be a Banach algebra and let $a \in \mathcal{A}$. Suppose Ω is an open set containing $\sigma(a)$ and that Γ is an arbitrary smooth contour in Ω, surrounding $\sigma(a)$. Then for an analytic function f on Ω the element

$$f(a) = \frac{1}{2\pi i} \int_\Gamma f(\lambda)(\lambda \mathbf{1} - a)^{-1}\, d\lambda$$

is a well–defined element of $\mathcal{A}$, because $\lambda \mapsto (\lambda \mathbf{1} - a)^{-1}$ is defined and continuous on Γ. The mapping $f \mapsto f(a)$ from $H(\Omega)$, the algebra of holomorphic functions on Ω, into $\mathcal{A}$ has the following properties:

1. $(f_1 + f_2)(a) = f_1(a) + f_2(a)$, $f_1, f_2 \in H(\Omega)$;
2. $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a) = f_2(a) \cdot f_1(a)$, $\quad f_1, f_2 \in H(\Omega)$;
3. $1(a) = \mathbf{1}$ where $1(\lambda) = 1$ and

   $I(a) = a$ where $I(\lambda) = \lambda$;
4. If (f_n) converges to f uniformly on compact subsets of Ω, then $f(a) = \lim_{n\to\infty} f_n(a)$;
5. $\sigma(f(a)) = f(\sigma(a))$.

Property (5) is known as the Spectral Mapping Theorem. Furthermore properties (1) and (2) express the fact that the mapping $f \mapsto f(a)$ is a homomorphism from the algebra of holomorphic functions on Ω into $\mathcal{A}$. For an alternative formulation of the Holomorphic Functional Calculus in terms of power series see *([6], 4.7, p. 206)*.

1.5 Spectral Idempotents in Banach algebras

Definition 1.5.1

If $\mathcal{A}$ is a Banach algebra, then an element $p \in \mathcal{A}$ is called an *idempotent* or a projection if $p = p^2$. We denote the *set of all idempotents* of $\mathcal{A}$ by $\mathcal{A}^\bullet$. The zero and identity elements of $\mathcal{A}$ are called the *trivial idempotents* of $\mathcal{A}$. Unless explicitely stated, we assume all idempotents to be non–trivial.

Lemma 1.5.2 *([1], Remark, p. 40)*

If $\mathcal{A}$ is a Banach algebra and p is an idempotent in $\mathcal{A}$ then $\sigma(p) = \{0, 1\}$.

Proof.

Since p is an idempotent, $p - p^2 = 0$. This implies $\sigma\left(p - p^2\right) = \{0\}$, but, by the Spectral Mapping Theorem, $\sigma\left(p - p^2\right) = \left\{\lambda - \lambda^2 : \lambda \in \sigma(p)\right\}$ which implies that $\lambda - \lambda^2 = 0$ for all $\lambda \in \sigma(p)$. Hence $\lambda \in \{0, 1\}$. What is left to show, is that $\{0, 1\} \subseteq \sigma(p)$ when $p \neq 0$ and $p \neq \mathbf{1}$. Since $p^2 = p$ we have $p(\mathbf{1} - p) = (\mathbf{1} - p)p = 0$. This gives $\mathbf{1} - p \notin \mathcal{A}^{-1}$ and $p \notin \mathcal{A}^{-1}$ since, if $1 - p$ is invertible, then $p = 0$ and we have a contradiction and similarly, if p is invertible then $p = \mathbf{1}$, which is also a contradiction. Hence $\{0, 1\} \subseteq \sigma(p)$. ■

Definition 1.5.3

Let $a \in \mathcal{A}$ and let α be an isolated spectral point. The *spectral idempotent corresponding to* a *and* α is defined by $p_{(\alpha,\, \sigma(a))} = \frac{1}{2\pi i}\int_{\Gamma_\alpha} (\lambda\mathbf{1} - a)^{-1}\, d\lambda$ where Γ_α is a small circle with centre at α and separating α from the remaining spectrum. We write $p_\alpha = \frac{1}{2\pi i}\int_{\Gamma_\alpha} (\lambda\mathbf{1} - a)^{-1}\, d\lambda$ for the spectral idempotent corresponding to a and α when there is no ambiguity in the context. We write only p for the spectral idempotent corresponding to a and 0.

We now apply the above mentioned definition to the isolated point 0.

Definition 1.5.4

Let $0 \in \text{iso}(\sigma(a))$. Let U_0 be an open ball with centre 0 and U_1 an open set containing $\sigma(a) \setminus \{0\}$ such that U_0 and U_1 are separated in $\mathbb{C}$. Let Γ_0 be a circle in U_0 surrounding 0 and Γ_1 be a smooth contour in U_1 surrounding $\sigma(a) \setminus \{0\}$. Define $f(\lambda) = \begin{cases} 1, & \lambda \in U_0 \\ 0, & \lambda \in U_1 \end{cases}$, which is an element of $H(a)$. Then by Holomorphic Functional Calculus we see that

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} f(\lambda)(\lambda\mathbf{1} - a)^{-1}\, d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1}\, d\lambda \\ &= p_{(0,\sigma(a))} \\ &= p. \end{aligned}$$

Lemma 1.5.5

Let $0 \in \text{iso}(\sigma(a))$. Suppose $U_0,\ U_1,\ \Gamma_0,\ \Gamma_1$ and f are as above. Then p is indeed an idempotent.

Proof.

Since $f^2(\lambda) = f(\lambda)$ for all $\lambda \in U_0 \cup U_1$, we have by the Holomorphic Functional Calculus that:

$$\begin{aligned}
p^2 &= \left(\frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda\right)\left(\frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda\right)\\
&= \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f^2(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1}-a)^{-1}\,d\lambda\\
&= p
\end{aligned}$$

■

Lemma 1.5.6

Let $0 \in \text{iso}(\sigma(a))$. Suppose $U_0,\ U_1,\ \Gamma_0,\ \Gamma_1$ and f are as before. Then $ap = pa$ for all $a \in \mathcal{A}$.

Proof.

Let $a \in \mathcal{A}$. Then:

$$\begin{aligned}
ap &= a\left[\frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} f(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda\right]\\
&= \frac{1}{2\pi i}\int_{\Gamma_0} a(\lambda\mathbf{1}-a)^{-1}\,d\lambda\\
&= \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1}-a)^{-1}\,a d\lambda\\
&= \left[\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1}-a)^{-1}\,d\lambda\right]a\\
&= pa.
\end{aligned}$$

■

Lemma 1.5.7

Let $0 \in \mathrm{iso}(\sigma(a))$. Suppose $U_0,\ U_1,\ \Gamma_0,\ \Gamma_1$ and f are as before. Then $p \in \mathrm{comm}^2(a)$.

Proof.

If $za = az,\ a,\ z \in \mathcal{A}$, then

$$z(\lambda\mathbf{1} - a) = (\lambda\mathbf{1} - a)z \Rightarrow (\lambda\mathbf{1} - a)^{-1} z = z(\lambda\mathbf{1} - a)^{-1} \text{ for all } \lambda \notin \sigma(a)$$

and so

$$\begin{aligned} zp &= \frac{1}{2\pi i}\int_{\Gamma_0} z(\lambda\mathbf{1} - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1} z d\lambda \\ &= \left[\frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1} d\lambda\right] z \\ &= pz. \end{aligned}$$

This means that p commutes with every element which commutes with a; therefore p is an element of the bicommutant of a. This lemma is valid for any spectral idempotent p_α. ■

We are now going to define a few specific elements of the Banach algebra using the above criteria. Let $0 \in \mathrm{iso}(\sigma(a))$ and take $U_0,\ U_1,\ \Gamma_0,\ \Gamma_1$ and f as above (See Definition 1.5.4).

Definition 1.5.8

Consider the function

$$g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0 \end{cases}.$$

We then obtain from the Holomorphic Functional Calculus an element

$$\begin{aligned} b = g(a) &= \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} g(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}\,d\lambda. \end{aligned}$$

We also see that we may write

$$g(\lambda) = (\lambda + f(\lambda))^{-1}(1-f(\lambda)) = \begin{cases} \frac{1}{\lambda}, & \lambda\in U_1 \\ 0, & \lambda\in U_0 \end{cases}.$$

So

$$\begin{aligned} b = g(a) &= (a+f(a))^{-1}(1-f(a)) = (a+p)^{-1}(1-p) \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}\,d\lambda. \end{aligned}$$

We shall later see that b is indeed the KD-inverse of a.

Definition 1.5.9

Consider the function

$$\begin{aligned} h(\lambda) &= \lambda f(\lambda)\in H(a) \quad \text{with} \\ h(\lambda) &= \begin{cases} \lambda, & \lambda\in U_0 \\ 0, & \lambda\in U_1 \end{cases}. \end{aligned}$$

From the Holomorphic Functional Calculus we have

$$\begin{aligned} h(a) &= af(a) = ap \\ \text{and} \quad h(a) &= \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} \lambda f(\lambda)(\lambda\mathbf{1}-a)^{-1}\,d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} \lambda(\lambda\mathbf{1}-a)^{-1}\,d\lambda. \end{aligned}$$

Definition 1.5.10

Consider the function $i(\lambda) = \lambda,\ \lambda \in U_1 \cup U_0$, then i is obviously an analytic function on any neighbourhood of $\sigma(a)$ and we have

$$\begin{aligned} i(a) = a &= \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} i(\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} \lambda(\lambda\mathbf{1} - a)^{-1} d\lambda. \end{aligned}$$

Definition 1.5.11

Consider the function $j(\lambda) = f(\lambda) + \lambda;\ j(\lambda)$ is obviously an element of $H(a)$ and

$$j(\lambda) = \begin{cases} 1 + \lambda, & \lambda \in U_0 \\ \lambda, & \lambda \in U_1 \end{cases}.$$

Then $j(a) = f(a) + a = p + a$, since

$$\begin{aligned} j(a) &= \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} [f(\lambda) + \lambda](\lambda\mathbf{1} - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} f(\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda + \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} (\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1} d\lambda + \frac{1}{2\pi i}\int_{\Gamma_0 \cup \Gamma_1} \lambda(\lambda\mathbf{1} - a)^{-1} d\lambda \\ &= p + a \end{aligned}$$

Lemma 1.5.12

Let $\mathcal{A}$ be a Banach algebra and suppose $a \in \mathcal{A}$ such that $\sigma(a) \subset \{0, 1\}$. We then have that $p_{(0,\ \sigma(a))} + p_{(1,\ \sigma(a))} = \mathbf{1}$, or equivalently that $p_{(0,\ \sigma(a))} + p_{(0,\ \sigma(\mathbf{1}-a))} = \mathbf{1}$.

Proof.

We have three cases:

Case 1. Suppose $\sigma(a) = \{0\}$, so we only need to consider $p_{(0,\sigma(a))}$. Since $\sigma(a) = \{0\}$, $r(a) = 0$ in which case we have the Laurent expansion for any $|\lambda| > r(a)$, namely $(\lambda\mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} a^n \lambda^{-n-1}$. (See Note 1.2.13). So if Γ_0 is the circle around 0 with radius T, we must have that

$$\begin{aligned} p_{(0,\sigma(a))} &= \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=T} \sum_{n=0}^{\infty} a^n \lambda^{-n-1} d\lambda = \mathbf{1} \end{aligned}$$

Case 2. Suppose $\sigma(a) = \{1\}$. Then we only consider $p_{(1,\,\sigma(a))}$. By the Spectral Mapping Theorem we have $\sigma(\mathbf{1} - a) = \{0\}$. Again with Γ_0 and T as in case 1 we have, for $|\lambda| > r(\mathbf{1} - a)$, that

$$\begin{aligned} p_{(1,\,\sigma(a))} &= p_{(0,\,\sigma(\mathbf{1}-a))} \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda\mathbf{1} - (1 - a))^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int_{|\lambda|=R} \sum_{n=0}^{\infty} (\mathbf{1} - a)^n \lambda^{-n-1} d\lambda \\ &= \mathbf{1} \end{aligned}$$

Case 3. Suppose $\sigma(a) = \{0, 1\}$. Let U_0 be an open ball with centre 0 and U_1 an open ball with centre 1, such that U_0 and U_1 are separated in $\mathbb{C}$. Let Γ_0 be a circle in U_0 surrounding 0 and Γ_1 be a circle in U_1 surrounding 1. Then

$$\begin{aligned} p_{(0,\,\sigma(a))} &= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} f(\lambda) (\lambda\mathbf{1} - a)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_0} (\lambda\mathbf{1} - a)^{-1} \, d\lambda \end{aligned}$$

where $f(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \in U_0 \\ 0 & \text{if} \quad \lambda \in U_1 \end{cases}$.

Also

$$\begin{aligned} p_{(1,\, \sigma(a))} &= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} \ell(\lambda) (\lambda \mathbf{1} - a)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda \mathbf{1} - a)^{-1} \, d\lambda \end{aligned}$$

where $\ell(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda \in U_1 \\ 0 & \text{if} \quad \lambda \in U_0 \end{cases}$.

Since $f(\lambda) + \ell(\lambda) = 1$ for all $\lambda \in U_0 \cup U_1$ where $U_0 \cup U_1$ is a neighbourhood of $\sigma(a)$, we have that

$$\begin{aligned} p_{(0,\, \sigma(a))} + p_{(1,\, \sigma(a))} &= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} (f(\lambda) + \ell(\lambda)) (\lambda \mathbf{1} - a)^{-1} \, d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} (\lambda \mathbf{1} - a)^{-1} \, d\lambda \\ &= \mathbf{1}. \end{aligned}$$

■

Chapter 2

Regularities

2.1 Basic properties of regularities

Definition 2.1.1

A *regularity* R, in a complex unital Banach algebra $\mathcal{A}$, is defined as a non–empty subset R of $\mathcal{A}$ satisfying the following conditions:

1. if $a \in \mathcal{A}$ and $n \in \mathbb{N}$, then $a \in R \Leftrightarrow a^n \in R$;
2. if $a,\ b$ are *relatively prime elements* of $\mathcal{A}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$. (a, b are relatively prime if there exists $c,\ d \in \mathcal{A}$ such that $\{a,\ b,\ c,\ d\}$ is a commuting set and $ac + bd = \mathbf{1}$.)

Definition 2.1.2

A regularity $R \subset \mathcal{A}$ assigns to each $a \in \mathcal{A}$ a subset of $\mathbb{C}$. This mapping is called the spectrum of a corresponding to R, defined by $\sigma_R(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin R\}$.

In general $\sigma_R(a)$ is neither compact nor non–empty. Furthermore $\sigma_R(a) \subset \sigma(a)$ and $\sigma_R(\lambda\mathbf{1} - a) = \lambda - \sigma_R(a)$ for all $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

Example 2.1.3

Let $\mathcal{A}$ be a Banach algebra. The following sets are regularities:

2.1.3.1 $R_1 = \mathcal{A}$; the spectrum corresponding to R_1 is $\sigma_{R_1}(a) = \emptyset$.

2.1.3.2 $R_2 = \mathcal{A}^{-1}$; the corresponding spectrum is $\sigma_{R_2}(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{A}^{-1}\} = \sigma(a)$, the ordinary spectrum.

2.1.3.3 $R_3 = \mathcal{A}_\ell^{-1}$, the set of all left invertible elements of $\mathcal{A}$, the spectrum corresponding to R_3 is $\sigma_{R_3}(a) = \{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{A}_\ell^{-1}\} = \{\lambda \in \mathbb{C} : \mathbf{1} \notin \mathcal{A}(\lambda - a)\}$ where $\sigma_{R_3}(a)$ is the left spectrum. Similarly for $R_4 = \mathcal{A}_r^{-1}$.

Proposition 2.1.4 *([15], Proposition 2, p. 51)*

Let R be a regularity in a Banach algebra $\mathcal{A}$. Then:

1. $\mathbf{1} \in R$
2. $\mathcal{A}^{-1} \subset R$
3. If $a,\ b \in \mathcal{A},\ ab = ba$ and $a \in \mathcal{A}^{-1}$, then $ab \in R \Leftrightarrow b \in R$.
   In particular, if $a \in R$ and $\lambda \in \mathbb{C},\ \lambda \neq 0$, then $\lambda a \in R$.

Proof.

1. Choose $b \in R$. We have $\mathbf{1}.\mathbf{1} + b.0 = \mathbf{1}$ and $\mathbf{1} \cdot b \in R$. Thus $\mathbf{1} \in R$.
2. Let $c \in \mathcal{A}^{-1}$. By (1) $c \cdot c^{-1} = \mathbf{1} \in R$; also $c \cdot c^{-1} + c^{-1} \cdot 0 = \mathbf{1}$. From Definition 2.1.1 we have that $c \in R$.

   Therefore $\mathcal{A}^{-1} \subset R$.

3. $a \in \mathcal{A}^{-1}$, thus $aa^{-1} = \mathbf{1}$.

   We have that $aa^{-1} + b \cdot 0 = \mathbf{1}$.

   So by Definition 2.1.1 we have that $ab \in R \Leftrightarrow b \in R$. ■

In order to verify the axioms of a regularity one can use the following property $(P1)$:

Theorem 2.1.5 *([15], Theorem 4, p. 52)*

Let R be a non-empty subset of Banach algebra $\mathcal{A}$ satisfying the following:

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R \qquad (P1)$$

for all commuting elements $a,\ b \in \mathcal{A}$. Then R is a regularity.

Every spectrum corresponding to a regularity R satisfies the Spectral Mapping Theorem:

Theorem 2.1.6 *([15], Theorem 7, p. 53)*

Let R be a regularity in a Banach algebra $\mathcal{A}$ and let σ_R be the corresponding spectrum. Then $\sigma_R(f(a)) = f(\sigma_R(a))$ for every $a \in \mathcal{A}$ and every function f analytic on a neighbourhood of $\sigma(a)$ which is non–constant on each component of its domain of definition.

Proof.
Take $\mu \in \mathbb{C}$. It is only necessary to show that

$$\mu \notin \sigma_R(f(a)) \Leftrightarrow \mu \notin f(\sigma_R(a)). \qquad (1)$$

We know that $(f - \mu)$ has only a finite number of zeros $\lambda_1, \ldots, \lambda_n$ in $\sigma(a)$. So it can be written as

$$f(z) - \mu = (z - \lambda_1)^{k_1} \ldots (z - \lambda_n)^{k_n} \cdot g(z),$$

where g is a function analytic on a neighbourhood of $\sigma(a)$ and $g(z) \neq 0$ for $z \in \sigma(a)$. By the Holomorphic Functional Calculus we have that $f(a) - \mu\mathbf{1} = (a - \lambda_1\mathbf{1})^{k_1} \ldots (a - \lambda_n\mathbf{1})^{k_n} \,.\, g(a)$, where $g(a)$ is invertible by the Spectral Mapping Theorem for the ordinary spectrum. (See Theorem 1.4.2 (5).) So (1) is equivalent to

$$f(a) - \mu\mathbf{1} \in R \Leftrightarrow a - \lambda_i\mathbf{1} \in R \quad (i = 1, \ \ldots, \ n) \tag{2}$$

Since $g(a)$ is invertible and by applying Proposition 2.1.4 (3) and Definition 2.1.1 (2) we now see that it is sufficient to show that

$$(a - \lambda_1\mathbf{1})^{k_1} \ldots (a - \lambda_n\mathbf{1})^{k_n} \in R \Leftrightarrow (a - \lambda_i\mathbf{1})^{k_i} \in R \, (i = 1, \ldots, n) \tag{3}$$

We know that for all relatively prime polynomials p, q there exist polynomials $p_1, \ q_1$ such that $pp_1 + qq_1 = 1$. Thus by Holomorphic Functional Calculus

$$p(a)\, p_1(a) + q(a)\, q_1(a) = \mathbf{1}.$$

Now we may apply Definition 2.1.1 (2) inductively to get (3). This completes the proof. ■

2.2 Continuity of the spectrum corresponding to a regularity

We are now going to give some properties and results of a regularity R and the spectrum corresponding to R, namely $\sigma_R(a)$.

Properties 2.2.1 *([15], (P2), (P3), (P4), p. 55)*

Let R be a regularity in a Banach algebra $\mathcal{A}$ and $\sigma_R(a)$ the spectrum corresponding to R. We consider the following properties of R (or $\sigma_R(a)$):
(P2): Upper semicontinuity of $\sigma_R(a)$: if $a_n,\ a \in \mathcal{A},\ a_n \to a,\ \lambda_n \in \sigma_R(a_n)$ and if $\lambda_n \to \lambda$, then $\lambda \in \sigma_R(a)$.
(P3): Upper semicontinuity on commuting elements: if $a_n,\ a \in \mathcal{A},\ a_n \to a,\ a_n a = a a_n$ for every $n,\ \lambda_n \in \sigma_R(a_n)$ and $\lambda_n \to \lambda$, then $\lambda \in \sigma_R(a)$.
(P4): Continuity on commuting elements: if $a_n,\ a \in \mathcal{A},\ a_n \to a$ and a_n and a commute for every n, then $\lambda \in \sigma_R(a)$ if and only if there exists a sequence $\lambda_n \in \sigma_R(a_n)$ such that $\lambda_n \to \lambda$.

We observe that either (P2) or (P4) implies (P3).

If σ_R has property (P3) and we consider a constant sequence $a_n = a$, then the spectrum $\sigma_R(a)$ is closed for every $a \in \mathcal{A}$.

Proposition 2.2.2 *([15] Proposition 9, p. 55)*

Let $\mathcal{A}$ be a Banach algebra and R a regularity in $\mathcal{A}$. Let $\sigma_R(a)$ be the spectrum corresponding to the regularity R. Then the following conditions are equivalent:

1. $\sigma_R(a)$ is upper semicontinuous;

2. $\sigma_R(a)$ is closed for every $a \in \mathcal{A}$ and the mapping $a \mapsto \sigma_R(a)$ is upper semicontinuous;

3. R is an open subset of $\mathcal{A}$.

Chapter 3

The Koliha-Drazin inverse and the Drazin inverse

3.1 The representation of the Koliha-Drazin inverse and the Drazin inverse

Following notation in [11] and [15] we have the following definitions:

Definition 3.1.1

Let $\mathcal{A}$ be a complex Banach algebra. An element $a \in \mathcal{A}$ is called *regular (or relatively regular)* if there is a *generalized inverse* $b \in \mathcal{A}$ such that $aba = a$ and $bab = b$.

A reflexivity property exists between a relatively regular element and its generalized inverse in the sense that if b is a generalized inverse of a, then a is a generalized inverse of b. This set of regular elements includes the set of invertible elements, $\mathcal{A}^{-1} = \left\{a \in \mathcal{A} : aa^{-1} = a^{-1}a = \mathbf{1}\right\}$, as well as the set of idempotents, $\mathcal{A}^{\bullet} = \left\{a \in \mathcal{A} : \ a = a^2\right\}$.

Definition 3.1.2

We call an element $a \in \mathcal{A}$ *group invertible* if there exists a *group inverse* $b \in \mathcal{A}$ such that $a = aba,\ b = bab$ and $ab = ba.$ We denote the *set of group invertible elements by* $\mathcal{G}(\mathcal{A})$.

A reflexivity property exists between a group inverse in the sense that if b is a group inverse of a, then a is a group inverse of b. The term group indicates that $\{a, b\}$ generates an Abelian group with identity ab.

We are now going to introduce a generalized inverse which does not have the reflexivity property, but commutes with the element:

Definition 3.1.3

An element $a \in \mathcal{A}$ is called *Drazin invertible* (or D–invertible) if there is an element $b \in \mathcal{A}$ such that:

1. $ab = ba$;
2. $bab = b$;
3. $a^k ba = a^k$ for some $k \in \mathbb{Z}^+$.

Since $\mathbf{1} - ab \in \mathcal{A}^\bullet$ we see that the condition $a^k ba = a^k$ of this definition is equivalent to $[a(\mathbf{1} - ab)]^k = a^k (\mathbf{1} - ab) = 0$ which means that $a(\mathbf{1} - ab)$ is nilpotent of order k. If $a \in \mathcal{A}$ is Drazin invertible, then the least nonnegative integer k for which there exists $b \in \mathcal{A}$ satisfying these equations is called the *Drazin index* $i(a)$ of a. The element $b = a^D$ is called the *Drazin inverse* (or D-inverse) of a. We denote the *set of Drazin invertible elements* by $\mathcal{A}^D$ and the *subset of* $\mathcal{A}^D$ *consisting of elements with index* k by $\mathcal{D}^k(\mathcal{A})$. It is clear that the sets $\mathcal{D}^k(\mathcal{A})$ are mutually disjoint.

With the convention that $a^0 = \mathbf{1}$ we have $\mathcal{D}^0(\mathcal{A}) = \mathcal{A}^{-1}$ and then also by definition $\mathcal{D}^1(\mathcal{A}) = \mathcal{G}(\mathcal{A}) \setminus \mathcal{A}^{-1}$. We can extend the definition of Drazin invertibility to the case where $a(\mathbf{1} - ab) \in QN(\mathcal{A})$, called the Koliha-Drazin invertible element:

Definition 3.1.4

An element $a \in \mathcal{A}$ is called *Koliha–Drazin invertible* (or KD–invertible) if there is an element $b \in \mathcal{A}$ such that:

1. $ab = ba$;
2. $bab = b$;
3. $a(\mathbf{1} - ab) \in QN(\mathcal{A})$

The element $b = a^{KD}$ is called the *Koliha–Drazin inverse* (or KD-inverse) of a. We denote the *set of Koliha–Drazin invertible elements* by $\mathcal{A}^{KD}$. We have the following inclusion

$$\mathcal{A}^{-1} \subseteq \mathcal{G}(\mathcal{A}) \subseteq \mathcal{A}^{D} \subseteq \mathcal{A}^{KD}.$$

We also mention that the Drazin inverse is generally speaking not continuous except of course on the subset $\mathcal{D}^0(\mathcal{A}) = \mathcal{A}^{-1}$.

The following lemma gives another characterisation of KD–invertibility:

Lemma 3.1.5 *([11], Lemma 2.4)*

Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has a KD–inverse, a^{KD}, if and only if there exists an idempotent $p \in \mathcal{A}^{\bullet} \cap$ comm(a) such that $ap \in QN(\mathcal{A})$ and $a + p \in \mathcal{A}^{-1}$. The KD-inverse, a^{KD}, is unique and given by $a^{KD} = (a + p)^{-1}(\mathbf{1} - p)$.

Proof.

Suppose $p \in \mathcal{A}^{\bullet} \cap$ comm(a) such that $ap \in QN(\mathcal{A})$ and $a + p \in \mathcal{A}^{-1}$.

Set $b = (a + p)^{-1}(\mathbf{1} - p)$. We have $ab = ba$ and also that

$ab = a(a + p)^{-1}(\mathbf{1} - p) = (a + p)(a + p)^{-1}(\mathbf{1} - p) = \mathbf{1} - p$,

so that $ab^2 = (\mathbf{1} - p)b = b(\mathbf{1} - p) = b$ (since $bp = 0$).

Finally, $a - a^2 b = a(\mathbf{1} - ab) = ap \in QN(\mathcal{A})$.

Conversely, suppose a has a KD–inverse b satisfying $ab = ba,\ b = bab$ and $a(\mathbf{1} - ab) \in QN(\mathcal{A})$. Set $p = \mathbf{1} - ab$. Since $(ab)^2 = a\left(ab^2\right) = ab$, i.e. $ab \in A^{\bullet}$ we have that

$$p^2 = (\mathbf{1} - ab)^2 = \mathbf{1} - 2ab + (ab)^2 = \mathbf{1} - ab = p,$$

that is $p \in \mathcal{A}^{\bullet}$. Obviously $p \in$ comm(a) and $ap \in QN(\mathcal{A})$. Furthermore, since $ap \in QN(\mathcal{A}), \sigma(ap) = 0$, so that $\mathbf{1} + ap \in \mathcal{A}^{-1}$. Thus

$$\begin{aligned}(a + p)(b + p) &= ab + ap + pb + p \\ &= ab + ap + p(b + \mathbf{1}) \\ &= ab + ap + (\mathbf{1} - ab)(b + \mathbf{1}) \\ &= \mathbf{1} + ap \in \mathcal{A}^{-1}\end{aligned}$$

Thus $a + p \in \mathcal{A}^{-1}$, since $\mathcal{A}^{-1}$ is a group under multiplication.

From $(a + p)b = \mathbf{1} - p$ it follows that $b = (a + p)^{-1}(\mathbf{1} - p)$, which proves the uniqueness of $b = a^{KD}$. ■

The characterization of isolated spectral points of an element of $\mathcal{A}$ will provide the main tool for the development of another property of the inverse.
In the following theorem we use Definition 1.5.4 and all the notation specified there.

Theorem 3.1.6 *([11], Theorem 3.1)*

Let $\mathcal{A}$ be a complex unital Banach algebra. Let $a \in \mathcal{A}$.
Then $0 \notin \mathrm{acc}(\sigma(a))$ if and only if there is an idempotent $p \in \mathcal{A}^{\bullet} \cap \mathrm{comm}(a)$ such that $ap \in QN(\mathcal{A})$ and $p + a \in \mathcal{A}^{-1}$. (3.1.6.1)
Moreover, $0 \in \mathrm{iso}(\sigma(a))$ if and only if $p \neq 0$ in which case p is the spectral idempotent corresponding to a and 0.

Proof.
Clearly $a \in \mathcal{A}$ is invertible if and only if 3.1.6.1 holds with $p = 0$.
Suppose now $0 \in \mathrm{iso}(\sigma(a))$. We define the spectral idempotent associated with a and 0 as in Definition 1.5.4. Then, using the notation of Lemma 1.5.6, we have that $ap = pa$ and we also see from Definition 1.5.9 that $ap = h(a)$ where $h(\lambda) = \lambda f(\lambda) = \begin{cases} \lambda, & \lambda \in U_0 \\ 0, & \lambda \in U_1 \end{cases}$. By the Spectral Mapping Theorem we have that $\sigma(ap) = \sigma(h(a)) = h(\sigma(a)) = \{0\}$. Thus $\lambda\mathbf{1} - ap \in \mathcal{A}^{-1}$ for all $\lambda \neq 0$ and therefore $ap \in QN(\mathcal{A})$. By Definition 1.5.11, the function $j(\lambda) = f(\lambda) + \lambda$ is in $H(a)$, so that $j(a) = p + a$. Since $j(\lambda) \neq 0$ for all λ in a neighbourhood of $\sigma(a)$ and so on $\sigma(a)$, we have that $j(a) = p + a \in \mathcal{A}^{-1}$.
Conversely, suppose there exists a $p \in \mathcal{A}^{\bullet} \cap \mathrm{comm}(a)$ with $ap \in QN(\mathcal{A})$ and $a + p \in \mathcal{A}^{-1}$. For any λ, $\lambda\mathbf{1} - a = (\lambda\mathbf{1} - ap)\, p + (\lambda\mathbf{1} - (p + a))(\mathbf{1} - p)$. However, from Theorem 1.2.6, there is an $r > 0$, say $r = \frac{1}{\|(p+a)^{-1}\|}$, such that if $|\lambda| < r$, then $(\lambda\mathbf{1} - (a + p)) \in \mathcal{A}^{-1}$. Also since $ap \in QN(\mathcal{A})$ we have $\lambda\mathbf{1} - ap \in \mathcal{A}^{-1}$ for all $\lambda \neq 0$. So if $0 < |\lambda| < r$ we have that $(\lambda\mathbf{1} - a)^{-1}$ exists and

$(\lambda\mathbf{1}-a)^{-1}=(\lambda\mathbf{1}-ap)^{-1}p+(\lambda\mathbf{1}-(p+a))^{-1}(\mathbf{1}-p)$.

Since $p\neq 0,\ 0\in \text{iso}(\sigma(a))$. We now show that this p is indeed the spectral idempotent of a corresponding to 0. We take a function $f\in H(a)$ with $f=1$ in a neighbourhood of 0 and $f=0$ in a neighbourhood of $\sigma(a)\setminus\{0\}$. If we take the circle Γ_0 in the neighbourhood of 0 small enough such that $0<|\lambda|<r$, then

$$\begin{aligned} f(a) &= \frac{1}{2\pi i}\int_{\Gamma_0}(\lambda\mathbf{1}-a)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_0}(\lambda\mathbf{1}-ap)^{-1}pd\lambda+\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda\mathbf{1}-(p+a))^{-1}(\mathbf{1}-p)\,d\lambda \end{aligned}$$

Concerning the first part of this equation we know that if $|\lambda|>r(ap)=0$, (see Note 1.2.13), the Laurent expansion is given by $(\lambda\mathbf{1}-ap)^{-1}=\sum\limits_{n=0}^{\infty}\frac{(ap)^n}{\lambda^{n+1}}=\sum\limits_{n=0}^{\infty}\frac{a^np}{\lambda^{n+1}}$ and then the coefficient of λ^{-1} ensures that $\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda\mathbf{1}-ap)^{-1}pd\lambda=\frac{1}{2\pi i}\int_{\Gamma_0}\sum\limits_{n=0}^{\infty}\frac{a^np}{\lambda^{n+1}}d\lambda=p.$

For the second part of the equation we observe that $r=\frac{1}{\|(a+p)^{-1}\|}\leq\| a+p\|$ (since $1\leq\| a+p\|\|(a+p)^{-1}\|$) and then

$$\begin{aligned} & [\lambda\mathbf{1}-(p+a)]^{-1} \\ =\ & \left[(p+a)\left(\lambda(p+a)^{-1}-\mathbf{1}\right)\right]^{-1} \\ =\ & -(p+a)^{-1}\left[\mathbf{1}-\lambda(p+a)^{-1}\right]^{-1} \\ =\ & -(p+a)^{-1}\sum_{n=0}^{\infty}\lambda^n(p+a)^{-n}. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{1}{2\pi i}\int_{\Gamma_0}(\lambda\mathbf{1}-(p+a))^{-1}(\mathbf{1}-p)\,d\lambda \\ =\ & \frac{\mathbf{1}-p}{2\pi i}\int_{\Gamma_0}-(p+a)^{-1}\sum_{n=0}^{\infty}\lambda^n(p+a)^{-n}\,d\lambda \\ =\ & 0 \end{aligned}$$

So $f(a)=\frac{1}{2\pi i}\int_{\Gamma_0}(\lambda-a)^{-1}d\lambda=p.$ ■

Note 3.1.7 *([11], Note 3.4)*

In view of the Laurent expansion for the resolvent $(\lambda\mathbf{1}-a)^{-1}$ (see Note 1.2.17), we can also say that 0 is a pole of $(\lambda\mathbf{1}-a)^{-1}$ if and only if there is an idempotent $p\neq 0$ commuting with a such that $ap\in N(\mathcal{A})\,,\ a+p\in\mathcal{A}^{-1}$.

Definition 3.1.8 *([8], p. 257)*

An element $a\in\mathcal{A}$ is called *quasipolar* if there exists $b\in\mathcal{A}$ such that $ab=ba=p=p^2$ with $\|a^n(\mathbf{1}-p)\|^{\frac{1}{n}}\to 0$. The *set of quasipolar elements* is denoted by $QP(\mathcal{A})$. One may easily verify that $QP(\mathcal{A})=\{a\in\mathcal{A}:0\notin \text{acc}(\sigma(a))\}$ ([11], Theorem 3.2). According to Harte ([8], p. 257) an element a of a Banach algebra $\mathcal{A}$ is *quasipolar* if there is an idempotent $q\in\mathcal{A}$ commuting with a such that $a(\mathbf{1}-q)\in QN(\mathcal{A})\,,\ q\in(\mathcal{A}a)\cap(a\mathcal{A})$.

From Lemma 3.1.5, Theorem 3.1.6 and Definition 3.1.8 we thus have the following equivalence:

Theorem 3.1.9 *([11], Theorem 4.2)*

The following conditions on an element $a\in\mathcal{A}$ are equivalent

1. $a\in\mathcal{A}^{KD}$;
2. $0\notin \text{acc}(\sigma(a))$, thus $a\in QP(\mathcal{A})$;
3. there exists $p\in\mathcal{A}^{\bullet}\cap$ comm(a) such that $ap\in QN(\mathcal{A})$ and $a+p\in\mathcal{A}^{-1}$. In this case the KD-inverse is unique and given by $a^{KD}=(a+p)^{-1}(\mathbf{1}-p)$, where p is the spectral idempotent of a corresponding to 0. From Definition 1.5.8 we see that the integral representation of a^{KD} is $a^{KD}=\dfrac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}d\lambda$, where U_1 an open set containing $\sigma(a)\setminus\{0\}$ and Γ_1 is the smooth contour in U_1 surrounding $\sigma(a)\setminus\{0\}$.

3.2 The resolvent expansion and some characterizations of the inverses

We now give a version of the Laurent series for the resolvent of $a \in \mathcal{A}$ in a neighbourhood of an isolated spectral point 0.

Theorem 3.2.1 *([11], Theorem 5.1)*

Let $0 \in \text{iso}(\sigma(a))$ and let b be the Koliha-Drazin inverse of a. Then, on some punctured disc $0 < |\lambda| < r$,

$$(\lambda \mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} \frac{a^n p}{\lambda^{n+1}} - \sum_{n=0}^{\infty} \lambda^n b^{n+1}. \tag{3.2.1.1}$$

Proof.

Let $0 \in \text{iso}(\sigma(a))$. Let $p = \mathbf{1} - ab$ be the spectral idempotent of a corresponding to $\lambda = 0$; then by Theorem 3.1.9 the KD-inverse of a is $b = (a+p)^{-1}(\mathbf{1}-p)$. In some disc $|\lambda| < r$, with $r = \dfrac{1}{\left\| (p+a)^{-1} \right\|} > 0$, $\lambda \mathbf{1} - (a+p)$ is invertible (by Theorem 1.2.6). Also ap is quasinilpotent (by Lemma 3.1.5), which means that $\lambda \mathbf{1} - ap \in \mathcal{A}^{-1}$ for $\lambda \neq 0$. Therefore

$$\begin{aligned}
& (\lambda \mathbf{1} - a)^{-1} \\
= \; & (\lambda \mathbf{1} - ap)^{-1} p + (\lambda \mathbf{1} - (a+p))^{-1} (\mathbf{1} - p) \\
= \; & (\lambda)^{-1} \left(\mathbf{1} - \frac{ap}{\lambda} \right)^{-1} p + \left[-(a+p) \left(\mathbf{1} - \lambda (a+p)^{-1} \right) \right]^{-1} (\mathbf{1} - p) \\
= \; & (\lambda)^{-1} \sum_{n=0}^{\infty} \frac{(ap)^n}{\lambda^n} p - (a+p)^{-1} \left[\mathbf{1} - \lambda (a-p)^{-1} \right]^{-1} (\mathbf{1} - p) \\
= \; & \sum_{n=0}^{\infty} \frac{a^n p}{\lambda^{n+1}} - (a+p)^{-1} \sum_{n=0}^{\infty} \lambda^n (a+p)^{-n} (\mathbf{1} - p) \\
= \; & \sum_{n=0}^{\infty} \frac{a^n p}{\lambda^{n+1}} - \sum_{n=0}^{\infty} \lambda^n b^{n+1}
\end{aligned}$$

■

Remark.

In the case when $\mu \in \text{iso}(\sigma(a))$, Theorem 3.2.1 can be generalized as follows:

Let $\mu \in \text{iso}(\sigma(a))$. Then on some punctured disc $0 < |\lambda - \mu| < r$,

$$(\lambda \mathbf{1} - a)^{-1} = \sum_{n=0}^{\infty} \frac{(a - \mu \mathbf{1})^n p_\mu}{(\lambda - \mu)^{n+1}} - \sum_{n=0}^{\infty} (\lambda - \mu)^n (g)^{n+1}. \tag{3.2.1.2}$$

where $g = (a - \mu \mathbf{1} + p_\mu)^{-1} (\mathbf{1} - p_\mu)$ is the KD-inverse of $(a - \mu \mathbf{1})$.

We show that the KD-inverse belongs to the second commutant of a.

Theorem 3.2.2

Let $\mathcal{A}$ be a unital Banach algebra. The KD–inverse of $a \in \mathcal{A}$ belongs to the second commutant of a.

Proof.

Suppose $za = az, \ a, \ z \in \mathcal{A}$. Then $z(\lambda \mathbf{1} - a) = (\lambda \mathbf{1} - a) z$ implies $(\lambda \mathbf{1} - a)^{-1} z = z (\lambda \mathbf{1} - a)^{-1}$ for all $\lambda \notin \sigma(a)$. By the Holomorphic Functional Calculus, (see 3.1.9) the KD-inverse of a is given by $a^{KD} = \frac{1}{2\pi i} \int_{\Gamma_0 \cup \Gamma_1} g(\lambda) (\lambda \mathbf{1} - a)^{-1} d\lambda$, where

$$g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0 \end{cases}$$

and U_0 is an open ball about 0, U_1 is an open set containing $\sigma(a) \setminus \{0\}$, U_0 and U_1 are separated in $\mathbb{C}$, Γ_0 is a circle in U_0 surrounding 0 an Γ_1 is a smooth contour in U_1 surrounding $\sigma(a) \setminus \{0\}$. Then

$$\begin{aligned} za^{KD} &= \frac{1}{2\pi i}\int_{\Gamma_0\cap\Gamma_1} zg(\lambda)(\lambda\mathbf{1}-a)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} z\frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}d\lambda \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}zd\lambda \\ &= \left[\frac{1}{2\pi i}\int_{\Gamma_1} \frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}d\lambda\right]z \\ &= \left[\frac{1}{2\pi i}\int_{\Gamma_0\cap\Gamma_1} g(\lambda)(\lambda\mathbf{1}-a)^{-1}d\lambda\right]z \\ &= a^{KD}z. \end{aligned}$$ ■

We now state that the Drazin inverse is unique and belongs to the second commutant of a.

Theorem 3.2.3 *([7], Theorem 1)*

An element $a\in\mathcal{A}$ has at most one Drazin inverse. If it exists, the Drazin inverse belongs to the second commutant of a.

Proof.

Suppose b_1 and b_2 are Drazin inverses of a with corresponding integers k_1 and k_2 as in Definition 3.1.3. So $a^{k_1+1}b_1=a^{k_1}$ and $a^{k_2+1}b_2=a^{k_2},\ k_1,\ k_2\in\mathbb{Z}^+$.
Let $k=\max(k_1,k_2)$. Then obviously $b_1a^{k+1}=a^k=a^{k+1}b_2$ and $b_1=b_1^2=a$, $b_2=ab_2^2$. By induction we prove that for $k=1,2,...$ we have $b_1=b_1^{m+1}a^m$ and $b_2=a^mb_2^{m+1}$. In particular $b_1=b_1^{k+1}a^k$ and $b_2=a^kb_2^{k+1}$. Hence $b_1=b_1^{k+1}a^k=b_1^{k+1}b_2=b_1ab_2$ and similarly $b_2=b_2ab_1$ so that $b_1=b_2$.
We now show that $a^D=b\in comm^2(a)$. Suppose $ac=ca,\ a,\ c\in\mathcal{A}$.
Then if b denotes the unique Drazin inverse, we have

$$ba^mc=bca^m=bca^{m+1}b=ba^{m+1}cb=a^mcb.$$

Hence $b^{m+1}a^m c = a^m cb^{m+1}$. But $b = b^{m+1}a^m$ so that
$bc = b^{m+1}a^m c = a^m cb^{m+1} = cb^{m+1}a^m = cb.$ ■

Theorem 3.2.4

If $a,\ b \in \mathcal{A}^{KD}$ with $ab = ba$, then $a,\ b,\ a^{KD},\ b^{KD}$ all commute.

Proof.

From the integral definition of the KD-inverse given in Theorem 3.1.9, $a^{KD} = \dfrac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}\,d\lambda$ and $b^{KD} = \frac{1}{2\pi i}\int_{\Gamma_1}\frac{1}{\lambda}(\lambda\mathbf{1}-b)^{-1}\,d\lambda$, we see that since $(\lambda\mathbf{1}-a)\,b = b\,(\lambda\mathbf{1}-a)$ we must have that $(\lambda\mathbf{1}-a)^{-1}\,b = b\,(\lambda\mathbf{1}-a)^{-1}$ and so $ba^{KD} = \dfrac{1}{2\pi i}b\int_{\Gamma_1}\dfrac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}\,d\lambda = \dfrac{1}{2\pi i}\left[\int_{\Gamma_1}\dfrac{1}{\lambda}(\lambda\mathbf{1}-a)^{-1}\,d\lambda\right]b = a^{KD}b$. The other elements commute similarly. ■

Theorem 3.2.5 *([11], Theorem 5.5)*

If $a,\ b \in \mathcal{A}^{KD}$ with $ab = ba$, then $(ab)^{KD} = a^{KD}b^{KD}$.

Proof.

By Theorem 3.2.4 $a,\ b,\ a^{KD}$ and b^{KD} all commute, therefore $ab\left(a^{KD}b^{KD}\right) = a^{KD}b^{KD}ab$.
Also $ab\left(a^{KD}b^{KD}\right)^2 = a\left(a^{KD}\right)^2 b\left(b^{KD}\right)^2 = a^{KD}b^{KD}$.
Furthermore,
$ab-(ab)^2\,a^{KD}b^{KD} = \left(a-a^2a^{KD}\right)\left(b-b^2b^{KD}\right)+a^2a^{KD}\left(b-b^2b^{KD}\right)+b^2b^{KD}\left(a-a^2a^{KD}\right)$
and since the spectral radius is subadditive and submultiplicative in commutative subalgebras of $\mathcal{A}$ (Theorem 1.2.23) we must have, since $r\left(a-a^2a^{KD}\right) = 0$ and $r\left(b-b^2b^{KD}\right) = 0$, that $r\left(ab-(ab)^2\,a^{KD}b^{KD}\right) = 0$, and so $ab-(ab)^2\,a^{KD}b^{KD} \in QN(\mathcal{A})$. We have proven that $a^{KD}b^{KD}$ is the Koliha–Drazin inverse of ab, so $(ab)^{KD} = a^{KD}b^{KD}$. ■

Theorem 3.2.6 *([11], Theorem 5.7)*

If $a,\ b \in \mathcal{A}^{KD}$ with $ab = ba = 0$ then also $(a+b)^{KD}$ exists and $(a+b)^{KD} = a^{KD} + b^{KD}$.

Proof.
We know that $a,\ b,\ a^{KD},\ b^{KD}$ all commute, and therefore $ab^{KD} = ab\left(b^{KD}\right)^2 = 0$ and $a^{KD}b = ab\left(a^{KD}\right)^2 = 0$.
Hence $(a+b)\left(a^{KD}+b^{KD}\right) = \left(a^{KD}+b^{KD}\right)(a+b)$
and $(a+b)\left(a^{KD}+b^{KD}\right)^2 = a\left(a^{KD}\right)^2 + b\left(b^{KD}\right)^2 = a^{KD}+b^{KD}$.
Also $(a+b)-(a+b)^2\left(a^{KD}+b^{KD}\right) = \left(a-a^2a^{KD}\right)+\left(b-b^2b^{KD}\right) \in QN(\mathcal{A})$. This shows that $\left(a^{KD}+b^{KD}\right)$ is the KD–inverse of $a+b$, so $(a+b)^{KD} = a^{KD}+b^{KD}$. ■

Theorem 3.2.7 *([11], Theorem 5.4)*

Suppose that $a \in \mathcal{A}$ has the Koliha–Drazin inverse a^{KD} and that p is the spectral idempotent of a corresponding to 0. Then $(a^n)^{KD} = \left(a^{KD}\right)^n$ for all $n = 1,\ 2,\ \ldots$.
We get a similar result for $a \in \mathcal{A}^D$.

Proof.
It is given that a^{KD} is the Koliha–Drazin inverse of element $a \in \mathcal{A}$, which means that $aa^{KD} = a^{KD}a,\ a^{KD} = a\left(a^{KD}\right)^2 = a^{KD}aa^{KD}$ and $a - a^2a^{KD} \in QN(\mathcal{A})$. We also have $p = \mathbf{1} - aa^{KD}$ where $ap \in QN(\mathcal{A})$ and $a+p \in \mathcal{A}^{-1}$. Then:

- $$\begin{aligned} a^n\left(a^{KD}\right)^n &= a^n\left((a+p)^{-1}(\mathbf{1}-p)\right)^n \\ &= \left(a(a+p)^{-1}\right)^n(\mathbf{1}-p) \\ &= \left((a+p)^{-1}\right)^n a^n(\mathbf{1}-p) \\ &= \left((a+p)^{-1}(\mathbf{1}-p)\right)^n a^n \\ &= \left(a^{KD}\right)^n a^n \end{aligned}$$

- $(a^n)\left(\left(a^{KD}\right)^n\right)^2 = \left(a^{KD}\right)^n (a^n)\left(a^{KD}\right)^n = \left(a^{KD}aa^{KD}\right)^n = \left(a^{KD}\right)^n.$

- $$\begin{aligned} & a^n - (a^n)^2\left(a^{KD}\right)^n \\ = \;& a^n\left(\mathbf{1} - \left(a^n\left(a^{KD}\right)^n\right)\right) \\ = \;& a^n\left(\mathbf{1} - \left(a^{KD}a\right)^n\right) \\ = \;& a^n\left(\mathbf{1} - (\mathbf{1} - p)^n\right) \\ = \;& (ap)^n \in Q\mathcal{N}(\mathcal{A}) \text{ since } ap \in QN(\mathcal{A}) \end{aligned}$$

Thus $(a^n)^{KD} = \left(a^{KD}\right)^n.$ ■

We close this chapter with an interesting characterization of Drazin invertibility.
First we need the following Lemma.

Lemma 3.2.8

If $\sigma(a) = \{0\}$ and $a \in \mathcal{A}^D$ with index k, then $a^k = 0$.

Proof.
Since $a \in \mathcal{A}^D$ with index k, there exists an a^D such that $aa^D = a^D a,\ a^D = a^D a a^D$ and $a^k = a^{k+1}a^D$. Now $a^k - a^{k+1}a^D = 0$, so $a^k\left(\mathbf{1} - aa^D\right) = 0$. Since $\sigma(a) = \{0\}$ we have that $1 - aa^D = 1$ (See Theorem 1.5.12 Case 1) Therefore $a^k = 0$. ■

Theorem 3.2.9 *([3], Lemma 2)*

If $\mathcal{A}$ is a Banach algebra and $a \in \mathcal{A}$, with $a \notin \mathcal{A}^{-1}$ then $a \in \mathcal{D}^k(\mathcal{A})$ if and only if $0 \in \text{iso}(\sigma(a))$ and k is the least integer such that $\frac{1}{2\pi i}\int\limits_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1} d\lambda = 0$ where Γ_0 is a small circle surrounding 0 and separating 0 from $\sigma(a) \setminus \{0\}$.

Proof.

Let $a \notin \mathcal{A}^{-1}$ and $a \in \mathcal{D}^k(\mathcal{A})$. Then it follows from Theorem 3.1.9 and from the fact that $\mathcal{D}^k(\mathcal{A}) \subset \mathcal{A}^{KD}$, that $0 \notin \text{acc}(\sigma(a))$ and therefore $0 \in \text{iso}(\sigma(a))$.

Case 1: If $\sigma(a) = \{0\}$, then since $a \in \mathcal{D}^k(\mathcal{A})$ we have k is the least positive integer such that $a^k = 0$ (by Lemma 3.2.8). By the Holomorphic Functional Calculus, $a^k = \frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1} d\lambda$, where Γ_0 is a circle surrounding 0, (Theorem 1.4.2). Therefore $\frac{1}{2\pi i}\int\limits_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1} d\lambda = 0$.

Case 2: Let $\sigma(a) \neq 0$. From Definition 1.5.8 and Theorem 3.1.9 the Drazin inverse of a is given by

$$a^D = \frac{1}{2\pi i}\int\limits_{\Gamma_o \cup \Gamma_1} g(\lambda)(\lambda\mathbf{1} - a)^{-1} d\lambda,$$

where

$$g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0 \end{cases},$$

U_0 is an open ball about 0, U_1 is an open set containing $\sigma(a) \setminus \{0\}$, U_0 and U_1 are separated in $\mathbb{C}$, Γ_0 is a circle in U_0 surrounding 0 and Γ_1 is a smooth contour in U_1 surrounding $\sigma(a) \setminus \{0\}$. k is also the least positive integer such that $a^{k+1}a^D = a^k$. If we consider the equation $a^{k+1}a^D = a^k$, we have:

$$\begin{aligned} a^{k+1}a^D &= \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^k \lambda \left(\frac{1}{\lambda}\right) (\lambda\mathbf{1} - a)^{-1}\, d\lambda, \\ &= \frac{1}{2\pi i}\int_{\Gamma_1} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda \quad \text{and} \\ a^k &= \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda; \end{aligned}$$

so k is the least positive integer satisfying
$\dfrac{1}{2\pi i}\int_{\Gamma_1} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda = a^{k+1}a^D = a^k = \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda$. This means k is the least positive integer such that
$\frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda = 0$.
Conversely, assume $0 \in \text{iso}(\sigma(a))$ and assume k is the least positive integer such that
$\frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda = 0$. We now want to show that $a \in \mathcal{D}^k(\mathcal{A})$.

Case 1: If $\sigma(a) = \{0\}$. Then k is the least positive integer such that $a^k = 0$.
Hence $a \in \mathcal{D}^k(\mathcal{A})$ and

$$a^D = \frac{1}{2\pi i}\int_{\Gamma_0} g(\lambda)(\lambda\mathbf{1} - a)^{-1}\, d\lambda = 0.$$

Case 2: If $\sigma(a) \neq \{0\}$ then, since $0 \in \text{iso}(\sigma(a))$ we may find $U_0,\ U_1,\ \Gamma_0,\ \Gamma_1$ as in the first part of the proof. If we define

$$a^D = \frac{1}{2\pi i}\int_{\Gamma_0\cup\Gamma_1} g(\lambda)(\lambda\mathbf{1} - a)^{-1}\, d\lambda$$

with $g(\lambda) = \begin{cases} \frac{1}{\lambda}, & \lambda \in U_1 \\ 0, & \lambda \in U_0 \end{cases}$, then it follows by the assumption (k is the least integer such that $\frac{1}{2\pi i}\int_{\Gamma_0} \lambda^k (\lambda\mathbf{1} - a)^{-1}\, d\lambda = 0$), that $a \in \mathcal{D}^k(\mathcal{A})$ with Drazin inverse a^D. ■

Chapter 4

Koliha-Drazin invertibles and Drazin invertibles form regularities

4.1 $\mathcal{A}^{KD}$ and $\mathcal{A}^{D}$ form regularities

In this chapter it is sometimes necessary to write the spectral idempotent corresponding to a and 0, $p_{(0,\ \sigma(a))}$, only as p_a.

Lemma 4.1.1

Let $\mathcal{A}$ be a complex unital Banach algebra and let $a,\ b \in \mathcal{A}$ be relatively prime. Let $x,\ y \in \mathcal{A}$, $ab \in \mathcal{A}^{KD}$ and $u = b\,(ab)^{KD}$. Then $a, b, x, y\ \ (ab)^{KD}$, p_{ab} and u all commute.

Proof.

Since a, b are relatively prime, there exists $x, y \in \mathcal{A}$ such that a, b, x, y all commute. From Theorem 3.2.2 we have that $(ab)^{KD} \in \text{comm}^2\,(ab)$. From Lemma 1.5.7 we have that $p_{ab} \in \text{comm}^2\,(ab)$. Since $(ab)\,a = a\,(ba) = a\,(ab)$ we thus have $a(ab)^{KD} = (ab)^{KD}a$ and $ap_{ab} = p_{ab}a$. Furthermore $au = ab\,(ab)^{KD} = ba\,(ab)^{KD} = b\,(ab)^{KD}\,a = ua$. Also by

Theorem 3.1.9 $(ab)^{KD} p_{ab} = p_{ab} (ab)^{KD}$. To show that $(ab)^{KD}$ and p_{ab} commute with u, we note that $b(ab) = (ba)b = (ab)b$ and since $(ab)^{KD} \in \text{comm}^2(ab)$ that $b(ab)^{KD} = (ab)^{KD} b$. So

$$\begin{aligned} (ab)^{KD} u &= (ab)^{KD} \left(b (ab)^{KD}\right) \\ &= \left((ab)^{KD} b\right) (ab)^{KD} \\ &= b (ab)^{KD} (ab)^{KD} \\ &= u (ab)^{KD} . \end{aligned}$$

Trivially we can also show $p_{ab}u = up_{ab}$. ■

Theorem 4.1.2 *([14], Lemma 1.1)*

Let $\mathcal{A}$ be a complex unital Banach algebra and let $a, b \in \mathcal{A}$ be relatively prime. If $ab \in \mathcal{A}^{KD}$, then $a \in \mathcal{A}^{KD}$.

Proof.

We know the following:

4.1.2.1 $a, b \in \mathcal{A}$ are relatively prime, so there exist $x, y \in \mathcal{A}$ such that a, b, x, y all commute and $ax + by = \mathbf{1}$, so $by = \mathbf{1} - ax$.

4.1.2.2 We also have that $ab \in \mathcal{A}^{KD}$, which means that $(ab)(ab)^{KD} = (ab)^{KD}(ab)$, $(ab)^{KD} = (ab)^{KD}(ab)(ab)^{KD}$ and $(ab) - (ab)^2 (ab)^{KD} \in QN(\mathcal{A})$. So the spectral idempotent p_{ab} of ab exists and $p_{ab} = \mathbf{1} - (ab)(ab)^{KD}$, $(ab) p_{ab} \in QN(\mathcal{A})$ and $(ab) + p_{ab} \in \mathcal{A}^{-1}$.

We first show that $u = b\,(ab)^{KD}$ is the KD inverse of $a\,(\mathbf{1} - p_{ab})$ and that $v = xp_{ab}\,(axp_{ab})^{KD}$ is the KD–inverse of ap_{ab}. Then since $a\,(\mathbf{1} - p_{ab})\,ap_{ab} = 0 = ap_{ab}a\,(\mathbf{1} - p_{ab})$ we apply Theorem 3.2.6 to $a = a(\mathbf{1} - p_{ab}) + (ap_{ab})$ and obtain

$$\begin{aligned} a^{KD} &= (a\,(\mathbf{1} - p_{ab}) + (ap_{ab}))^{KD} \\ &= (a\,(\mathbf{1} - p_{ab}))^{KD} + (ap_{ab})^{KD}. \end{aligned}$$

I) We show that $u = b\,(ab)^{KD}$ is the KD–inverse of $a\,(\mathbf{1} - p_{ab})$, that is

$$(a\,(\mathbf{1} - p_{ab}))^{KD} = b\,(ab)^{KD}.$$

- $$\begin{aligned} ua\,(\mathbf{1} - p_{ab}) &= ua\left(\mathbf{1} - \left(\mathbf{1} - (ab)\,(ab)^{KD}\right)\right) \\ &= b\,(ab)^{KD}\,aab\,(ab)^{KD} \\ &= aab\,(ab)^{KD}\,b\,(ab)^{KD} \\ &= a\,(\mathbf{1} - p_{ab})\,u. \end{aligned}$$
- $u\,[a\,(\mathbf{1} - p_{ab})]\,u = u$:

$$\begin{aligned} (ab)^{KD}\,p_{ab} &= (ab)^{KD}\left(\mathbf{1} - ab\,(ab)^{D}\right) \\ &= (ab)^{KD} - (ab)^{KD}\,ab\,(ab)^{KD} = 0, \end{aligned}$$

so we have

$$\begin{aligned} & u\,(\mathbf{1} - p_{ab}) \\ =\; & b\,(ab)^{KD}\,(\mathbf{1} - p_{ab}) \\ =\; & b\,(ab)^{KD} - b\,(ab)^{KD}\,p_{ab} \\ =\; & b\,(ab)^{KD} \\ =\; & u \end{aligned}$$

Further

$$\begin{aligned} ua = au &= ab\,(ab)^{KD} \\ &= \mathbf{1} - \left(\mathbf{1} - ab\,(ab)^{KD}\right) \\ &= \mathbf{1} - p_{ab}. \end{aligned}$$

Thus $ua\,(\mathbf{1} - p_{ab})\,u = (\mathbf{1} - p_{ab})\,(\mathbf{1} - p_{ab})\,u = (\mathbf{1} - p_{ab})\,u = u.$

- $$a\,(\mathbf{1} - p_{ab}) - [a\,(\mathbf{1} - p_{ab})]^2\,u \in QN\,(\mathcal{A}) \ :$$

$$\begin{aligned} & a\,(\mathbf{1} - p_{ab}) - a\,(\mathbf{1} - p_{ab})\,ua\,(\mathbf{1} - p_{ab}) \\ = \ & a\,(\mathbf{1} - p_{ab}) - a\,(\mathbf{1} - p_{ab})\,(\mathbf{1} - p_{ab})\,(\mathbf{1} - p_{ab}) \\ = \ & a\,(\mathbf{1} - p_{ab}) - a\,(\mathbf{1} - p_{ab}) \\ = \ & 0 \in QN\,(\mathcal{A}) \end{aligned}$$

Therefore $(a\,(\mathbf{1} - p_{ab}))^{KD} = u.$

II) Now we show that $v = xp_{ab}\,(axp_{ab})^{KD}$ is the KD inverse of ap_{ab}. To do so we need to show that $axp_{ab} \in \mathcal{A}^{KD}$. Since $abp_{ab} \in QN\,(\mathcal{A})$, we have

$$axp_{ab} - (axp_{ab})^2 = (ax)\,(\mathbf{1} - ax)\,p_{ab} = axbyp_{ab} = (abp_{ab})\,xy \in QN\,(\mathcal{A}) \quad (4.1.2.3)$$

(abp_{ab} and xy commute, so we can apply Theorem1.2.23(2)). By the Spectral Mapping Theorem for the ordinary spectrum (Theorem 1.4.2(5)) with $f\,(\lambda) = \lambda - \lambda^2$ we have that $f\,(\sigma\,(axp_{ab})) = \sigma\,(f\,(axp_{ab})) = \sigma\left(axp_{ab} - (axp_{ab})^2\right) = \{0\}$
and so $\sigma\,(axp_{ab}) \subset \{0, 1\}$;
so $axp_{ab},\ 1 - axp_{ab} \in \mathcal{A}^{KD}$.

From Lemma 1.5.12 we thus have that

$$\begin{aligned} & p_{(0,\sigma(axp_{ab}))} + p_{(1,\sigma(axp_{ab}))} \\ = \ & p_{axp_{ab}} + p_{(1-axp_{ab})} \\ = \ & \mathbf{1} \end{aligned}$$

We now continue to show that $v = xp_{ab}(axp_{ab})^{KD}$ is the KD–inverse of ap_{ab}.

- v commutes with ap_{ab} :
  $axp_{ab} \in \mathcal{A}^{KD}$, so $(axp_{ab})(axp_{ab})^{KD} = (axp_{ab})^{KD}(axp_{ab})$,
  and then $(axp_{ab})^2(axp_{ab})^{KD} = (axp_{ab})^{KD}(axp_{ab})^2$,
  thus $ap_{ab}xp_{ab}(axp_{ab})^{KD} = xp_{ab}(axp_{ab})^{KD}ap_{ab}$.

- $$\begin{aligned} v(ap_{ab})v &= xp_{ab}(axp_{ab})^{KD}ap_{ab}xp_{ab}(axp_{ab})^{KD} \\ &= xp_{ab}(axp_{ab})^{KD}axp_{ab}(axp_{ab})^{KD} \\ &= xp_{ab}(axp_{ab})^{KD} \\ &= v \end{aligned}$$

- $ap_{ab} - (ap_{ab})^2 v \in QN(\mathcal{A})$:
  Since $p_{\mathbf{1}-axp_{ab}} = \mathbf{1} - (\mathbf{1} - axp_{ab})(\mathbf{1} - axp_{ab})^{KD}$ and $p_{axp_{ab}} = \mathbf{1} - axp_{ab}(axp_{ab})^{KD}$,

$$\begin{aligned} & ap_{ab} - ap_{ab}vap_{ab} \\ =\ & ap_{ab}\left(\mathbf{1} - xp_{ab}(axp_{ab})^{KD}ap_{ab}\right) \\ =\ & ap_{ab}\left(\mathbf{1} - axp_{ab}(axp_{ab})^{KD}\right) \\ =\ & ap_{ab}p_{axp_{ab}} \\ =\ & ap_{ab}\left[\mathbf{1} - p_{\mathbf{1}-axp_{ab}}\right] \\ =\ & ap_{ab}(\mathbf{1} - axp_{ab})(\mathbf{1} - axp_{ab})^{KD} \\ =\ & ap_{ab}(\mathbf{1} - ax)(\mathbf{1} - axp_{ab})^{KD} \\ =\ & ap_{ab}by(\mathbf{1} - axp_{ab})^{KD} \end{aligned}$$

We know that $ap_{ab}b = abp_{ab}$ is quasinilpotent and trivially $ap_{ab}b$ commutes with $y\left(1 - axp_{ab}\right)^{KD}$. Hence $ap_{ab}by\left(\mathbf{1} - axp_{ab}\right)^{KD} = ap_{ab} - \left(ap_{ab}\right)^2 v \in QN\left(\mathcal{A}\right)$.

Therefore $\left(ap_{ab}\right)^{KD} = v$.
We have shown that $a = a(1 - p_{ab}) + (ap_{ab}) \in \mathcal{A}^{KD}$. ■

Remark 4.1.3 *([14], Lemma 1.1)*

In fact, we have $a\left(1 - p_{ab}\right)$ is Drazin invertible as

$$\begin{aligned} p_{a(1-p_{ab})} &= 1 - ua\left(1 - p_{ab}\right) \\ &= 1 - \left(1 - p_{ab}\right)\left(1 - p_{ab}\right) \\ &= p_{ab}, \end{aligned}$$

and so

$$\begin{aligned} & a\left(1 - p_{ab}\right)\left(p_{a(1-p_{ab})}\right) \\ = & \ a\left(1 - p_{ab}\right)p_{ab} \\ = & \ a\left(p_{ab} - \left(p_{ab}\right)^2\right) \\ = & \ 0. \end{aligned}$$

Therefore $a(1 - p_{ab})(p_{a(1-p_{ab})})$ is nilpotent.

Remark 4.1.4 *([14], p. 139)*

It is interesting to note that from Theorem 4.1.2 we get $a^{KD} = b\left(ab\right)^{KD} + xp_{ab}\left(axp_{ab}\right)^{KD}$ and then $aa^{KD} = ab\left(ab\right)^{KD} + axp_{ab}\left(axp_{ab}\right)^{KD}$.

We need the following Lemma to prove that $\mathcal{A}^{KD}$ forms a regularity.

Lemma 4.1.5

Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}^{KD}$. We have $0 \notin \text{acc}(\sigma(a))$ if and only if $0 \notin \text{acc}(\sigma(a^n))$.

Proof.
If $0 \notin \text{acc}(\sigma(a))$, then either $0 \in \rho(a)$ or $0 \in \text{iso}(\sigma(a))$.
Now $0 \in \rho(a)$ if and only if $0 \in \rho(a^n)$ (since $\mathcal{A}^{-1}$ is a group under multiplication). Also from the Spectral Mapping Theorem we have $\sigma(a^n) = \{\lambda^n | \lambda \in \sigma(a)\}$ and therefore $0 \in \text{iso}(\sigma(a))$ if and only if $0 \in \text{iso}(\sigma(a^n))$. ■

Theorem 4.1.6 *([14], Theorem 1.2)*

The set $\mathcal{A}^{KD}$ of all KD–invertible elements in a complex unital Banach algebra $\mathcal{A}$ forms a regularity.

Proof.
We refer to Definition 2.1.1(1) and (2) of a regularity.

1. Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Suppose $a \in \mathcal{A}^{KD}$, then by Theorem 3.1.9 $0 \notin \text{acc}(\sigma(a))$. By Lemma 4.1.5 $0 \notin \text{acc}(\sigma(a^n))$ and so $a^n \in \mathcal{A}^{KD}$.
   Conversely, let $a^n \in \mathcal{A}^{KD}$. This holds for $n = 1, 2, \ldots$, so also for $n = 1$, thus $a \in \mathcal{A}^{KD}$.
   Hence $a \in \mathcal{A}^{KD} \Leftrightarrow a^n \in \mathcal{A}^{KD}$.

2. Let a, b be relatively prime. Then a, b commute. Suppose $a, b \in \mathcal{A}^{KD}$, then by Theorem 3.2.5 ab is KD–invertible.

   Conversely, let a, b be relatively prime and suppose $ab \in \mathcal{A}^{KD}$ then by Theorem 4.1.2 $a \in \mathcal{A}^{KD}$ and $b \in \mathcal{A}^{KD}$. This completes the proof. ■

According to Definition 3.1.3 an element $a \in \mathcal{A}$ is Drazin invertible if there is a $b \in \mathcal{A}$ such that $ab = ba,\ bab = b,\ a^k = a^{k+1}b$, for some nonnegative integer k. The following Lemma gives an equivalent definition:

Lemma 4.1.7 *([11], Lemma 2.1.)*

In a complex Banach algebra $\mathcal{A}$ with unit $\mathbf{1}$, Definition 3.1.3 is equivalent to $ab = ba,\ bab = b,$ $a - a^2b \in N(\mathcal{A})$.

Proof.

Suppose $ab = ba$ and $bab = b$. Then $p = \mathbf{1} - ab$ is an idempotent as $(ab)^2 = a\left(ab^2\right) = ab$. So $a^k - a^{k+1}b = a^k p = (ap)^k = (a(\mathbf{1} - ab))^k = \left(a - a^2b\right)^k = 0$ for any $k \geq 1$. So $a - a^2b \in N(\mathcal{A})$. This completes the proof. ■

Corollary 4.1.8 *([14], Corollary 1.3)*

The set $\mathcal{A}^D$ of all Drazin invertible elements of a complex unital Banach algebra $\mathcal{A}$ forms a regularity.

Proof.

We refer to Definition 2.1.1(1) and (2) of a regularity.

1. Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Suppose $a \in \mathcal{A}^D$. Then by Lemma 4.1.7 $aa^D = a^Da$, $a^Daa^D = a^D$ and $a - a^2a^D \in N(\mathcal{A})$. From Theorem 3.2.7, we know that

$(a^n)^D = (a^D)^n$ for all $n = 1,\ 2,\ \ldots$. So it is easy to see that $(a^n)(a^n)^D = (a^n)^D a^n$ and $(a^n)^D a^n (a^n)^D = (a^n)^D$, for all n. Since $w = a - a^2 a^D \in N(\mathcal{A})$, it can be shown using induction that $a^n - (a^n)^2 (a^D)^n = a^{n-1} w$.

Therefore $a^n - (a^n)^2 (a^D)^n \in N(\mathcal{A})$.

Thus $a^n \in \mathcal{A}^D$.

Conversely, assume $a^n \in \mathcal{A}^D$, $n = 1,\ 2,\ \ldots$. This holds for $n = 1$, therefore $a \in \mathcal{A}^D$.

2. Assume $a, b \in \mathcal{A}^D$ where a, b are relatively prime.

   By Theorem 3.2.5 we have that if $a, b \in \mathcal{A}$ are commuting KD–invertible elements, then ab is KD-invertible with $(ab)^{KD} = a^{KD} b^{KD}$. To prove that if $a, b \in \mathcal{A}$ are Drazin invertible with $ab = ba$, then ab is Drazin invertible, we still have to show that if $v^t = \left(a - a^2 a^D\right)^t = 0$ and $w^p = \left(b - b^2 a^D\right)^p = 0$, then $\left[ab - (ab)^2 (ab)^D\right]^q = 0$, for some $p, q, t \in \mathbb{N}$.

   So assume $v^t = 0$ and $w^p = 0$ and take $q = p + t$. Then

$$\begin{aligned}
& \left[(ab) - (ab)^2 (ab)^D\right]^q \\
= & \left[(a)\left(b - b^2 b^D\right) + b^2 b^D \left(a - a^2 a^D\right)\right]^{p+t} \\
= & \sum_{i=0}^{p+t} \binom{p+t}{i} \left((a)\left(b - b^2 b^D\right)\right)^{p+t-i} \left(b^2 b^D \left(a - a^2 a^D\right)\right)^i \\
= & \ 0
\end{aligned}$$

   since we have $p + t - i \geq p$ for $i = 0,\ 1,\ \ldots,\ t$ so that $\left(b - b^2 b^D\right)^{p+t-i} = 0$ for these i and $i > t$ for $i = t + 1,\ \ldots,\ t + p$ so that $\left(a - a^2 a^D\right)^i = 0$ for these i. So $ab \in \mathcal{A}^D$.

   Conversely, assume $ab \in \mathcal{A}^D$. Suppose that $ax + by = 1$ with $a,\ b,\ x,\ y$ commuting. From Theorem 4.1.2, $a, b \in \mathcal{A}^{KD}$ and $a^{KD} = b\,(ab)^{KD} + xp_{ab}\,(axp_{ab})^{KD}$ (see Remark 4.1.4).

   It remains to show $a, b \in \mathcal{A}^D$.

- First assume that $(ab)\,p_{ab} = 0$. From (4.1.2.3) we have that $(axp_{ab})^2 = axp_{ab} - abp_{ab}xy$, and hence that $(axp_{ab})^D = axp_{ab}$.

  Then

$$
\begin{aligned}
p_a &= 1 - aa^D \\
&= 1 - ab\,(ab)^D - (axp_{ab})\,(axp_{ab})^D \quad \text{(see Remark 4.1.4)} \\
&= 1 - ab\,(ab)^D - (axp_{ab})\,(axp_{ab}) \\
&= p_{ab} - a^2x^2p_{ab} \\
&= \left(\mathbf{1} - a^2x^2\right) p_{ab} \\
&= (\mathbf{1} + ax)\,byp_{ab} \quad \text{(since } 1 - ax = by\text{)} \\
\text{and} \quad ap_a &= (\mathbf{1} + ax)\,abp_{ab}y = 0 \quad \text{(from our assumption that } (ab)p_{ab} = 0\text{).}
\end{aligned}
$$

  Thus $a \in \mathcal{A}^D$. By symmetry $bp_b = 0$ and $b \in \mathcal{A}^D$.

- For the general case, assume that $(abp_{ab})^n = a^nb^np_{ab} = 0$ for $n \in \mathbb{N}$. There exist elements $x_n,\ y_n \in \mathcal{A}$ such that $a^nx_n + b^ny_n = 1$. (This is seen from the expansion of $(ax + by)^{2n-1}$.)

  Applying the first part of the proof to a^n and b^n we get that $a^np_a = a^np_a^n = (ap_a)^n = 0$. Thus $a \in \mathcal{A}^D$. By symmetry, $(bp_b)^n = 0$. Thus $b \in \mathcal{A}^D$. ■

Chapter 5

Properties of the KD-spectrum and the D-spectrum

5.1 The KD-spectrum and the D-spectrum

Definition 5.1.1

We define the KD*–spectrum* and the D*–spectrum* of an element $a \in \mathcal{A}$ as follows:
$\sigma_{KD}(a) = \left\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{A}^{KD}\right\}$;
$\sigma_{D}(a) = \left\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{A}^{D}\right\}$;
we may also define $\sigma_{\mathcal{D}^K(\mathcal{A})}(a) = \left\{\lambda \in \mathbb{C} : \lambda\mathbf{1} - a \notin \mathcal{D}^K(\mathcal{A})\right\}$.

From Theorem 4.1.6 and Corollary 4.1.8 we know that $\mathcal{A}^{KD}$ and $\mathcal{A}^{D}$ form regularities. Then by Theorem 2.1.6 the Spectral Mapping Theorem holds for $\sigma_{KD}(a)$ and for $\sigma_{D}(a)$:

Theorem 5.1.2 *([14], Theorem 1.4)*

Let $a \in \mathcal{A}$, where $\mathcal{A}$ is a complex unital Banach algebra. If f is any function holomorphic in an open neighbourhood of the ordinary spectrum, $\sigma(a)$, of a and non–constant on any component of $\sigma(a)$, then $f(\sigma_{KD}(a)) = \sigma_{KD}(f(a))$ and $f(\sigma_D(a)) = \sigma_D(f(a))$.

In Theorem 5.1.4 and Theorem 5.1.5 we are going to prove some of the properties of the KD–spectrum and the D–spectrum.

To do so we give a short discussion on the elements of the spectrum.

By acc$(\sigma(a))$ and iso$(\sigma(a))$ we denote the set of all accumulation points and isolated points of $\sigma(a)$ respectively. We know $\sigma(a) = \text{acc}(\sigma(a)) \cup \text{iso}(\sigma(a))$. Recall that $\prod(a)$ denotes the set of all poles of the resolvent $(\lambda\mathbf{1} - a)^{-1}$ and IES(a) denotes the set of essential singularities of $(\lambda\mathbf{1} - a)^{-1}$ (see p 13).

First we consider $\mu \in \text{acc}(\sigma(a))$:

$$\begin{aligned}
\mu \in \text{acc}(\sigma(a)) &\Leftrightarrow 0 \in \mu - \text{acc}(\sigma(a)) \\
&\Leftrightarrow 0 \in \text{acc}(\sigma(\mu\mathbf{1} - a)) \quad \text{By Lemma 1.2.20} \\
&\Leftrightarrow \mu\mathbf{1} - a \notin \mathcal{A}^{KD} \quad \text{By Theorem 3.1.9} \\
&\Leftrightarrow \mu \in \sigma_{KD}(a) \quad \text{By Definition 5.1.1}
\end{aligned}$$

Next we take $\mu \in \text{iso}(\sigma(a))$:

In this case $0 \in \text{iso}(\sigma(\mu\mathbf{1} - a))$.

By Theorem 3.1.9 $\mu\mathbf{1} - a$ has a Koliha–Drazin inverse. It is now possible that $\mu\mathbf{1} - a \in \mathcal{D}^k(\mathcal{A}) \subset \mathcal{A}^D$. By Definition 3.1.3 $[(\mu\mathbf{1} - a)p]^k = 0$ which means that μ is a

pole of order k of $(\lambda\mathbf{1}-a)^{-1}$ where k is a non–negative integer (See Note 3.1.7). It is also possible that $\mu\mathbf{1}-a \in \mathcal{A}^{KD}\backslash\mathcal{A}^{D}$. This means that $(\mu\mathbf{1}-a)\,p \in QN(\mathcal{A})$ and so from the Laurent expansion (3.2.1.2) μ is an essential singularity of $(\lambda\mathbf{1}-a)^{-1}$.
So iso $(\sigma(a))$ therefore consists of the poles of order $k \geq 1$ and those that are essential singularities. Thus IES $(a) =$ iso$(\sigma(a))\backslash \prod(a)$.

Theorem 5.1.3 *([14], Proposition 1.5)*

Let $a \in \mathcal{A}$ where $\mathcal{A}$ is a complex unital Banach algebra. Then:

1. $\sigma_{KD}(a) =$ acc$(\sigma(a))$;
2. $\sigma_D(a) =$ acc$(\sigma(a)) \cup ($iso$(\sigma(a)) \setminus (\prod(a)))$;
3. $\sigma_{KD}(a) \subset \sigma_D(a) \subset \sigma(a)$.

Proof.

1. Clear from the preceding discussion.
2. $\mu \in$ acc$(\sigma(a))$

   $\Leftrightarrow \mu \in \sigma_{KD}(a)$ (By Property 5.1.3(1))

   $\Leftrightarrow \mu\mathbf{1}-a \notin \mathcal{A}^{KD}$ (By Definition 5.1.1)

   $\Rightarrow \mu\mathbf{1}-a \notin \mathcal{A}^{D}$ (Since $\mathcal{A}^{D} \subset \mathcal{A}^{KD}$)

   $\Leftrightarrow \mu \in \sigma_D(a)$ (By Definition 5.1.1)

$\mu \in iso(\sigma(a))$:

There are two possibilities:

First possibility:

μ is a pole of order k of $(\lambda\mathbf{1} - a)^{-1}$

$\Leftrightarrow \mu\mathbf{1} - a \in \mathcal{D}^k(\mathcal{A}) \subset \mathcal{A}^{KD}$

$\Leftrightarrow \mu\mathbf{1} - a \in \mathcal{A}^D$ (since $\mathcal{A}^D \subset \mathcal{A}^{KD}$)

$\Leftrightarrow \mu \notin \sigma_D(a)$ (By Definition 5.1.1)

(So $\mu \in \prod(a)$ implies $\mu \notin \sigma_D(a)$).

Second possibility:

μ is an essential singularity

$\Leftrightarrow \mu\mathbf{1} - a \in \mathcal{A}^{KD} \backslash \mathcal{A}^D$

$\Leftrightarrow \mu \notin \sigma_{KD}(a)$ and $\mu \in \sigma_D(a)$

(So $\mu \in$ IES(a) implies $\mu \in \sigma_D(a)$).

3. The inclusion is clear. ■

Theorem 5.1.4 *([15], Proposition 1.5)*

Let $a \in \mathcal{A}$ where $\mathcal{A}$ is a complex Banach algebra with unit. Then:

1. $\sigma_{KD}(a)$ is closed;
2. $\sigma_D(a)$ is closed;

Proof.

1. $\sigma_{KD}(a)$ is closed since $\sigma_{KD}(a) = \text{acc}(\sigma(a))$ and the set $\text{acc}(\sigma(a))$ is closed.

2. Since $\sigma_D(a) = \text{acc}(\sigma(a)) \cup (\text{iso}(\sigma(a)) \setminus \prod(a))$, we have to consider the following conditions to see if it is closed:

2.1 Suppose $\sigma_D(a)$ is finite or $\emptyset$, then since both of them are closed we must have $\sigma_D(a)$ is closed.

2.2 Suppose there are finitely many elements in $\text{iso}(\sigma(a)) \setminus \prod(a)$, then since $\text{acc}(\sigma(a))$ is closed and the union of a finite number of closed sets is closed, $\sigma_D(a)$ must be closed.

2.3 Suppose $\sigma_D(a) = \text{acc}(\sigma(a)) \cup (\text{iso}(\sigma(a)) \setminus \prod(a))$ has infinitely many elements. Suppose (α_n) is a sequence in $\sigma_D(a)$ and let $\alpha_n \to \alpha$.

- If (α_n) has an infinite number of elements in $\text{acc}(\sigma(a))$, say $\alpha_{n_k} \subseteq \text{acc}(\sigma(a))$, then since $\text{acc}(\sigma(a))$ is closed we must have $\lim_{k\to\infty} \alpha_{n_k} = \alpha \in \text{acc}(\sigma(a)) \subseteq \sigma_D(a)$.
- If (α_n) has an infinite number of elements in $\text{iso}(\sigma(a)) \setminus \prod(a)$, say $\alpha_{n_k} \subseteq \text{iso}(\sigma(a)) \setminus \prod(a)$, then also $\alpha_{n_k} \subseteq \text{iso}(\sigma(a))$. So α_{n_k} is an infinite sequence in $\sigma(a)$ with $\lim_{k\to\infty} \alpha_{n_k} = \alpha$ (since the subsequences go to the same limit). Thus $\alpha \in \text{acc}(\sigma(a))$.

Therefore $\sigma_D(a)$ is closed, since it contains all its accumulation points. ■

Theorem 5.1.5 *([14], Proposition 1.5)*

Let $a \in \mathcal{A}$ where $\mathcal{A}$ is a complex Banach algebra with unit. Then:

1. $\sigma_{KD}(a) = \emptyset$ if and only if $\sigma(a)$ is a finite set;
2. $\sigma_D(a) = \emptyset$ if and only if $\sigma(a)$ consists of a finite number of points which are poles of the resolvent of a.

Proof.

1. Assume $\sigma(a)$ is finite. Then every point is isolated and $\text{acc}(\sigma(a)) = \sigma_{KD}(a) = \emptyset$.

Conversely, we assume $\sigma(a)$ is infinite and $\sigma_{KD}(a) = \text{acc}(\sigma(a)) = \emptyset$. So iso$(\sigma(a))$ is infinite and since $\sigma(a)$ is compact we have by Theorem 1.3.3 that $\sigma(a)$ has an accumulation point μ. Thus $\sigma_{KD}(a) \neq \emptyset$.

2. First we assume $\sigma_D(a) = \emptyset$. Then acc$(\sigma(a)) \cup (\text{iso}(\sigma(a)) \setminus \prod(a)) = \emptyset$, by Theorem 5.1.3(2). Consequently, by Theorem 5.1.3(1), $\sigma_{KD}(a) = \text{acc}(\sigma(a)) = \emptyset$ and also IES$(a) = \emptyset$. Since $\sigma(a) \neq \emptyset$, it must consist of isolated points which are poles of the resolvent. Again using Theorem 1.3.3, we have that this set of isolated points, consisting only of poles of the resolvent, must be finite.

   Conversely, assume $\sigma(a)$ consists of a finite number of points which are poles of the resolvent. Since, from Theorem 5.1.3(2),

$$\begin{aligned} \sigma_D(a) &= \text{acc}(\sigma(a)) \cup (\text{iso}(\sigma(a)) \setminus \textstyle\prod(a)) \\ &= \text{acc}(\sigma(a)) \cup \text{ IES}(a), \end{aligned}$$

   we must have that $\sigma_D(a) = \emptyset$. ■

5.2 Regularity related properties of the KD-spectrum and the D-spectrum

We mentioned property $(P1)$ in Theorem 2.1.5 which states that a regularity R is said to have property $(P1)$ if $ab \in R \Leftrightarrow a \in R$ and $b \in R$ whenever, $a, b \in \mathcal{A}$ commute. However, neither $\mathcal{A}^{KD}$ nor $\mathcal{A}^{D}$ possess property $(P1)$ as demonstrated by the following example.

Example 5.2.1 *([14], Example 1.6)*

Consider $\mathcal{A} = \ell^\infty$ with pointwise addition and multiplication.
If $a = \left(\frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{4},\ 0,\ \frac{1}{5},\ 0,\ \ldots\right)$ and $b = \left(0,\ \frac{1}{2},\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{4},\ \ldots\right)$,
then $ab = ba = 0 \in \mathcal{A}^D \subset \mathcal{A}^{KD}$. Now

$$\begin{aligned} \sigma(a) &= \left\{\lambda \mid \lambda - a \notin \mathcal{A}^{-1}\right\} \\ &= \left\{\lambda \mid \left(\lambda - \frac{1}{2},\ \lambda,\ \lambda - \frac{1}{3},\ \lambda,\ \ldots\right) \notin \mathcal{A}^{-1}\right\} \end{aligned}$$

Obviously $\lambda \in \{0, \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\}$ implies $\lambda \in \sigma(a)$. We thus have $0 \in \operatorname{acc}(\sigma(a))$. Similarly $0 \in \operatorname{acc}(\sigma(b))$. Thus neither a nor b is in $\mathcal{A}^{KD}$ or $\mathcal{A}^D$.

However, $\mathcal{A}^{KD}$ satisfies $(P1)$, mentioned in Theorem 2.1.5, only in a very special case.

Theorem 5.2.2 *([14], Theorem 1.7)*

Let $\mathcal{A}$ be a complex Banach algebra with unit Then the following conditions are equivalent:

1. $\mathcal{A}^{KD}$ has property $(P1)$;
2. $\mathcal{A}^{KD} = \mathcal{A}$;
3. $\sigma_{KD}(a) = \emptyset$ for all $a \in \mathcal{A}$;
4. each element of $\mathcal{A}$ has finite spectrum;
5. the quotient algebra $\mathcal{A}/\mathrm{rad}\mathcal{A}$ is finite dimensional.

Proof.

$1 \Rightarrow 2$: Suppose $\mathcal{A}^{KD}$ has property $(P1)$. As $0 \in \mathcal{A}^{KD}$ and 0 commutes with all elements $a \in \mathcal{A}$, we have that for any $a \in \mathcal{A}$, $0.a \in \mathcal{A}^{KD}$. By $(P1)$, $a \in \mathcal{A}^{KD}$ for all $a \in \mathcal{A}$. So $\mathcal{A}^{KD} = \mathcal{A}$.

$2 \Rightarrow 3$: Since $\mathcal{A}^{KD} = \mathcal{A}$, $\sigma_{KD}(a) = \{\lambda : \lambda\mathbf{1} - a \notin \mathcal{A}\}$. But $\lambda\mathbf{1} - a \in \mathcal{A}$ for all $a \in \mathcal{A}$. Thus $\sigma_{KD}(a) = \emptyset$.

$3 \Rightarrow 4$: Follows from Theorem 5.1.5(1).

$4 \Rightarrow 1$: If each element of $\mathcal{A}$ has finite spectrum, then all elements of the spectrum are isolated and each element of the algebra is KD–invertible. Then trivially, $\mathcal{A}^{KD}$ satisfies $(P1)$.

$4 \Leftrightarrow 5$: From Theorem 1.2.24(2) we have that $\sigma(a)$ is finite if and only if $\sigma(a + \text{rad}\mathcal{A})$ is finite. So if $\sigma(a)$ is finite, it means $\sigma(a + \text{rad}\mathcal{A})$ is also finite, then since $\mathcal{A}/\text{rad}\mathcal{A}$ is semisimple (see Theorem 1.2.24(1)) we have from Theorem 1.2.25 that $\mathcal{A}/\text{rad}\mathcal{A}$ is finite dimensional.

Obviously, $\mathcal{A}/\text{rad}\mathcal{A}$ finite dimensional, implies $\sigma(a + \text{rad}\mathcal{A})$ and thus $\sigma(a)$ is finite. ■

Finally, the spectral continuity properties (P2), (P3) and (P4) defined in Properties 2.2.1, hold for $\sigma_{KD}(a)$ and $\sigma_D(a)$ under a specific condition.

Theorem 5.2.3 *([14], Theorem 1.8)*

For a complex unital Banach algebra $\mathcal{A}$, the set of KD–invertible elements satisfies properties (P2), (P3) and (P4) if and only if $\mathcal{A}^{KD} = \mathcal{A}$.

Proof.
If $\mathcal{A}^{KD} = \mathcal{A}$ then it is trivial to show that these properties are satisfied.
Conversely, if $\mathcal{A}^{KD} \neq \mathcal{A}$, then there exists an element $a \in \mathcal{A} \backslash \mathcal{A}^{KD}$, that is, $0 \in \operatorname{acc}(\sigma(a))$, (see Theorem 3.1.9). Let $a_n = \frac{a}{n}$. Then $0 \in \frac{1}{n}\operatorname{acc}(\sigma(a))$ so that $0 \in \operatorname{acc}(\sigma(a_n))$ and $a_n \to 0$ as $n \to \infty$. However $0 \notin \operatorname{acc}(\sigma(0))$. This means that property (P3) does not hold. Hence, neither do properties (P2) or (P4). ■

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