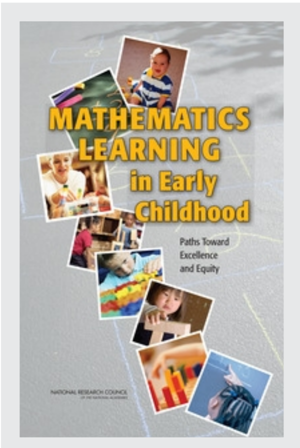




# Mathematics Learning in Early Childhood: Paths Toward Excellence and Equity

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**AUTHORS**

Committee on Early Childhood Mathematics, National Research Council

# MATHEMATICS LEARNING in Early Childhood

## Paths Toward Excellence and Equity

Committee on Early Childhood Mathematics

Christopher T. Cross, Taniesha A. Woods,
and Heidi Schweingruber, *Editors*

Center for Education
Division of Behavioral and Social Sciences and Education

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## COMMITTEE ON EARLY CHILDHOOD MATHEMATICS

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**MARY ANN KASPER,** *Senior Program Assistant*

# Preface

Early childhood education has risen to the top of the national policy agenda with recognition that ensuring educational success and attainment must begin in the earliest years of schooling. There is now a substantial body of research to guide efforts to support young children's learning. Over the past 15 years, great strides have been made in supporting young children's literacy. This report summarizes the now substantial literature on learning and teaching mathematics for young children in hopes of catalyzing a similar effort in mathematics.

The need for this study was recognized and championed by the National Research Council's (NRC's) Mathematical Sciences Education Board following the publication in 2001 of *Adding It Up: Helping Children Learn Mathematics*. The tireless efforts of board member Sharon Griffin and then board director David Mandel led the design of this project, which is a comprehensive examination of the evidence base that can guide mathematics education (teaching and learning) for children ages 2 through 6. It represents the further extension of a portfolio of NRC reports focused on mathematics learning and teaching that includes *Adding It Up: Helping Children Learn Mathematics* (2001); *Eager to Learn: Educating Our Preschoolers* (2001); *How Students Learn: Mathematics in the Classroom* (2005); and *On Evaluating Curricular Effectiveness: Judging the Quality of K-12 Mathematics Evaluations* (2004).

The majority of support for this study was provided by the U.S. Department of Health and Human Services, Administration for Children and Families, Office of Head Start. In particular, we thank Frank Fuentes, deputy director of the Office of Head Start, Administration for Children and Families; Amanda Bryans, director of the Educational Development and Partnership Division, Office of Head Start, Administration for Children and Families; and Michele Plutro, education specialist, Office of Head Start, Administration for Children and Families. Additional funding was provided by the Office of Planning, Research, and Evaluation in the Administration for Children and Families, under the leadership of Mary Bruce Webb.

Other sponsors who contributed to the project include the Ewing Marion Kauffman Foundation, under the guidance of Margo Quiriconi and Karen Norwood, and the National Institute of Child Health and Human Development, under the leadership of Daniel Berch and James Griffin. In addition, the National Academies President's Fund provided partial support for the study.

Our work was also advanced by the contributions of able consultants and staff and the input of outside experts. Throughout the study process, the committee benefited from presentations or written input by individuals with a range of perspectives: W. Steven Barnett, National Institute for Early Education Research, Rutgers, The State University of New Jersey; Linda Bevilacqua, Core Knowledge Foundation; Toni Bickart, Creative Curriculum, Teaching Strategies; Bruce D. McCandliss, Sackler Institute for Developmental Psychobiology, Weill Medical College of Cornell University; Holly Rhodes, consultant; Elisa Rosman, consultant for the Georgetown University Center for Child and Human Development; Lawrence Schweinhart, High/Scope Educational Research Foundation; Catherine Snow, Harvard Graduate School of Education; and Prentice Starkey, Graduate School of Education, University of California, Berkeley.

The committee also thanks those who wrote papers that were invaluable to our discussions: Sarah Archibald, Consortium for Policy Research in Education, University of Wisconsin-Madison; Kathryn Bouchard Chval, College of Education, University of Missouri; Jason Downer, Center for the Advanced Study of Teaching and Learning, University of Virginia; Shalom Fisch, MediaKidz Research and Consulting; Michael Goetz, University of Wisconsin, Madison; Bridget K. Hamre, Curry School of Education, University of Virginia; Marilou Hyson, National Association for the Education of Young Children and George Mason University; Carolyn R. Kilday, Graduate Student, Curry School of Education, University of Virginia; Pat McGuire, Graduate Student Curry Leadership Foundations and Policy, School of Education, University of Virginia; Barbara Reys, Department of Learning, Teaching, and Curriculum, University of Missouri; Catherine Scott-Little, Human Development and Family Studies Department, University of North Carolina, Greensboro; and John Switzer, Department of Learning, Teaching, and Curriculum, University of Missouri.

This report has been reviewed in draft form by individuals chosen for their diverse perspectives and technical expertise, in accordance with procedures approved by the Report Review Committee of the NRC. The purpose of this independent review is to provide candid and critical comments that will assist the institution in making its published report as sound as possible and to ensure that the report meets institutional standards for objectivity, evidence, and responsiveness to the study charge. The review comments and draft manuscript remain confidential to protect the integrity of the deliberative process.

We thank the following individuals for their review of this report: Arthur Baroody, Curriculum and Instruction, University of Illinois, Urbana-Champaign; Elena Bodrova, Mid-continent Research for Education and Learning, Lakewood, CO; Karen S. Cook, Department of Sociology, Institute for Research in the Social Sciences, Stanford University; Sharon A. Griffin, Department of Education, Clark University; Jacqueline A. Jones, Division of Early Childhood Education, New Jersey Department of Education; Constance Kamii, Curriculum and Instruction, University of Alabama; Michèle M. M. Mazzocco, Psychiatry and Behavioral Sciences, Johns Hopkins School of Medicine and Math Skills Development Project, Kennedy Krieger West Campus, Baltimore, MD; Sally Moomaw, College of Education, Criminal Justice, and Human Services, University of Cincinnati; Donald G. Saari, Institute for Mathematical Behavioral Sciences, University of California, Irvine; Maria Shea Terrell, Department of Mathematics, Cornell University; and Karen L. Worth, Center for Science Education, Education Development Center, Inc., Newton, MA.

Although the reviewers listed above have provided many constructive comments and suggestions, they were not asked to endorse the conclusions or recommendations nor did they see the final draft of the report before its release. The review of this report was overseen by Jeremy Kilpatrick, Department of Mathematics and Science Education, University of Georgia, Athens, and Charles (Randy) Gallistel, Rutgers University, Rutgers Center for Cognitive Science, The State University of New Jersey. Appointed by the NRC, they were responsible for making certain that an independent examination of this report was carried out in accordance with institutional procedures and that all review comments were carefully considered. Responsibility for the final content of this report rests entirely with the authoring committee and the institution.

We are also grateful to the work of others at the NRC, including Christine McShane, senior editor, Division of Behavioral and Social Sciences and Education (DBASSE), whose work greatly improved the text of the report; Kirsten Sampson Snyder, DBASSE reports officer, who worked with us through several revisions of the report; and Yvonne Wise, DBASSE production editor, who managed the report through final publication. As well, we are thankful to those who assisted committee members with literature searches or background research, including Patricia Harvey, Julie Shuck, and Matthew Von Hendy, at the National Academies.

The committee appreciates the support provided by the Center for

Education, under the leadership of Patricia Morison. Taniesha Woods, the study director, provided invaluable support and guidance to the committee throughout the study. We could not have asked for a better colleague. Senior program assistant Mary Ann Kasper masterfully handled all the logistical aspects of this project, including our four committee meetings. We are also grateful for the leadership and support of Heidi Schweingruber, deputy director of the Board on Science Education, who provided much thoughtful counsel throughout this process and contributed substantially to editing the report in the final stages.

Christopher T. Cross, *Chair*
Committee on Early Childhood Mathematics

# Contents

## APPENDIXES

# Summary

Mathematics education has risen to the top of the national policy agenda as part of the need to improve the technical and scientific literacy of the American public. The new demands of international competition in the 21st century require a workforce that is competent in and comfortable with mathematics. There is particular concern about the chronically low mathematics and science performance of economically disadvantaged students and the lack of diversity in the science and technical workforce. Particularly alarming is that such disparities exist in the earliest years of schooling and even before school entry.

Recognizing the increasing importance of mathematics and encouraged by a decade of success in improving early literacy, the Mathematical Sciences Education Board of the Center for Education at the National Research Council established the Committee on Early Childhood Mathematics. The committee was charged with examining existing research in order to develop appropriate mathematics learning objectives for preschool children; providing evidence-based insights related to curriculum, instruction, and teacher education for achieving these learning objectives; and determining the implications of these findings for policy, practice, and future research.

The committee found that, although virtually all young children have the capability to learn and become competent in mathematics, for most the potential to learn mathematics in the early years of school is not currently realized. This stems from a lack of opportunities to learn mathematics either in early childhood settings or through everyday experiences in homes and in communities. This is particularly the case for economically disadvantaged children, who start out behind in mathematics and will remain so without extensive, high-quality early mathematics instruction.

In fact, well before first grade, children can learn the ideas and skills that support later, more complex mathematics understanding. There is expert consensus that two areas of mathematics are particularly important for young children to learn: (1) number, which includes whole number, operations, and relations; and (2) geometry, spatial thinking, and measurement. A rich body of research provides insight into how children's proficiency develops in both areas and the instruction needed to support it. The committee used this evidence to develop research-based *teaching-learning paths* to guide policy and practice in early childhood education.

Examination of current standards, curricula, and instruction in early childhood education revealed that many early childhood settings do not provide adequate learning experiences in mathematics. The relative lack of high-quality mathematics instruction, especially in comparison to literacy, reflects a lack of attention to mathematics throughout the childhood education system, including standards, curriculum, instruction, and the preparation and training of the teaching workforce.

For example, many widely used early childhood curricula do not provide sufficient guidance on mathematics pedagogy or content. When early childhood classrooms do have mathematics activities, they are often presented as part of an integrated or embedded curriculum, in which the teaching of mathematics is secondary to other learning goals. Emerging research indicates, however, that learning experiences in which mathematics is a supplementary activity rather than the primary focus are less effective in promoting children's mathematics learning than experiences in which mathematics is the primary goal. Finally, education and training for most teachers typically places heavy emphasis on children's social-emotional development and literacy, with much less attention to mathematics. In fact, academic activities such as mathematics can be a context in which social-emotional development and the foundations of language and literacy flourish.

As noted, opportunities to experience high-quality mathematics instruction are especially important for low-income children. These children, on average, demonstrate lower levels of competence with mathematics prior to school entry, and the gaps persist or even widen over the course of schooling. Providing young children with extensive, high-quality early mathematics instruction can serve as a sound foundation for later learning in mathematics and contribute to addressing long-term systematic inequities in educational outcomes.

The committee found that although the research to date about how young children develop and learn key concepts in mathematics has clear implications for practice, the findings are neither widely known nor implemented by early childhood educators or those who teach them. To ensure that all children enter elementary school with the mathematical foundation they need for success requires that individuals throughout the early childhood education system—including the teaching workforce, curriculum developers, program directors, and policy makers—transform their approach to mathematics education in early childhood by supporting, developing, and implementing research-based practices and curricula.

## RECOMMENDATIONS

**Recommendation 1: A coordinated national early childhood mathematics initiative should be put in place to improve mathematics teaching and learning for all children ages 3 to 6.**

A number of specific recommendations for action follow from this overarching recommendation. The specific steps and the individuals or organization that must be involved in enacting them are outlined below. We provide further guidance about how to enact these steps in Chapter 9.

**Recommendation 2: Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to other topics. The mathematical process goals should be integrated in these content areas. Children should understand the concepts and learn the skills exemplified in the teaching-learning paths described in this report.**

**Recommendation 3: All early childhood programs should provide high-quality mathematics curricula and instruction as described in this report.**

**Recommendation 4: States should develop or revise their early childhood learning standards or guidelines to reflect the teaching-learning paths described in this report.**

**Recommendation 5: Curriculum developers and publishers should base their materials on the principles and teaching-learning paths described in this report.**

**Recommendation 6: An essential component of a coordinated national early childhood mathematics initiative is the provision of professional development to early childhood in-service teachers that helps them (a)**

**to understand the necessary mathematics, the crucial teaching-learning paths, and the principles of intentional teaching and curriculum and (b) to learn how to implement a curriculum.**

**Recommendation 7: Coursework and practicum requirements for early childhood educators should be changed to reflect an increased emphasis on children's mathematics as described in the report. These changes should also be made and enforced by early childhood organizations that oversee credentialing, accreditation, and recognition of teacher professional development programs.**

**Recommendation 8: Early childhood education partnerships should be formed between family and community programs so that they are equipped to work together in promoting children's mathematics.**

**Recommendation 9: There is a need for increased informal programming, curricular resources, software, and other media that can be used to support young children's mathematics learning in such settings as homes, community centers, libraries, and museums.**

# Part I

# Introduction and Research on Learning

# 1

# Introduction

> For centuries, many students have learned mathematical knowledge—whether the rudiments of arithmetic computation or the complexities of geometric theorems—without much understanding. . . . Of course, many students tried to make whatever sense they could of procedures such as adding common fractions or multiplying decimals. No doubt many students noticed underlying regularities in the computations they were asked to perform. Teachers who themselves were skilled in mathematics might have tried to explain those regularities. But mathematics learning has often been more a matter of memorizing than of understanding.
>
> Today it is vital that young people understand the mathematics they are learning. Whether using computer graphics on the job or spreadsheets at home, people need to move fluently back and forth between graphs, tables of data, and formulas. To make good choices in the marketplace, they must know how to spot flaws in deductive and probabilistic reasoning as well as how to estimate the results of computations. . . . Public policy issues of critical importance hinge on mathematical analyses. (pp. 15-16)

These words are from an earlier National Research Council (NRC) report called *Adding It Up: Helping Children Learn Mathematics* (National Research Council, 2001a). It focused on examining the evidence about school mathematics and outlining what it means to be mathematically proficient from prekindergarten to eighth grade. The report offers much to guide current policy and practice in elementary and middle schools across the nation. Yet the report also draws attention to the importance of what happens before children enter formal schooling: “Young children show a remarkable ability to formulate, represent, and solve simple mathematical problems and to reason and explain their mathematical activities. They are positively disposed to do and to understand mathematics when they first encounter it" (p. 6).

However, not much attention has been paid historically to teaching mathematics to young children before they enter the period of formal schooling. This stems, at least in part, from generally negative attitudes about mathematics on the part of the American public as well as to beliefs that early childhood education should consist of a nurturing environment that promotes social-emotional development, with academic content primarily focusing on language and literacy development. In fact, a majority of parents report that a positive approach to learning and language development is more important for young children than mathematics (Cannon and Ginsburg, 2008). When asked which subject was more important for her child to learn and why, one mother said (p. 249):

> Language. Definitely. I mean obviously they're both [math and language] very important. But you can find people, even adults, who never learn math. I think that you could survive much better [without mathematics] than if you never learn language. I think communication is so important. If you could learn to be expressive, you could hire someone to do your math for you.

Families are agents of cultural transmission, which includes conveying attitudes about mathematics. Often, mathematics is not viewed as important to young children's cognitive development and later academic success. Evidence shows, however, that learning mathematics is vital for children's early years and for later success in mathematics as well as better overall academic outcomes in such areas as literacy, science, and technology (e.g., Duncan et al., 2007; National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002).

In addition, early childhood teachers are often uncomfortable teaching mathematics (Clements and Sarama, 2007; Copley, 2004; Ginsburg et al., 2006; Lee and Ginsburg, 2007a). Many teachers avoid teaching mathematics because of their own negative early experiences with mathematics. The quote below, by a pre-service teacher attending a top-ranked university, is illustrative:

> Overall, my personal experiences with math have not been good. . . . Throughout [my] elementary [schooling] it was either you were right or wrong. . . . As a result, I found math very boring and confusing. I am not a natural math learner. . . . I do not like the idea of teaching math to others, because I feel like I am not competent enough to teach math. I remember how hard it was when I was teaching adding and subtracting to first graders, especially when some of them did not understand it. I panicked

> when I made some mistakes myself in adding and subtracting. (Personal communication, comments by student of H. Ginsburg, Teachers College, Columbia University, September 2007.)

In recent years, however, interest in mathematics as a key aspect of early childhood education has increased across both the policy and the practice communities. In 2000, the National Council of Teachers of Mathematics (NCTM), in their revision of the 1989 standards for elementary and secondary school mathematics, included prekindergarten for the first time. Also in 2000, a conference of early childhood and mathematics educators was held to focus more explicitly on standards for preschool and kindergarten children (Clements, Sarama, and DiBiase, 2004). In 2002, Good Start, Grow Smart, an early childhood-focused White House initiative, resulted in the linking of federal funding to the requirement that all states develop voluntary early learning guidelines in language, literacy, and mathematics. The now-suspended National Reporting System for assessing learning outcomes for children participating in Head Start programs, begun in 2002, originally specified four areas of focus for assessment, one of which was early mathematical skills (the other three were language-related: comprehension of spoken English, vocabulary, and letter naming) (National Research Council, 2008). Also in 2002, the National Association for the Education of Young Children and the NCTM approved a joint position statement, "Early Childhood Math: Promoting Good Beginnings," which included recommendations to guide both policy and practice.

In 2006, following on its efforts to improve language and literacy outcomes for the children it serves, the Office of Head Start turned its attention to early mathematics. It convened a mathematics working group composed of parents, local staff, researchers, and other experts in early mathematics learning and has since moved forward on developing strategies for helping Head Start and Early Head Start programs support the early mathematics learning of infants, toddlers, and preschoolers.

## LEARNING FROM THE RESEARCH

Clearly there is growing interest in including mathematics among the learning goals for young children and in improving the teaching of mathematics in developmentally appropriate ways. Over the past several decades, significant investments have been made in research on early development and learning, much of which is ripe for examination and synthesis as it applies to early mathematics.

In the past decade, the NRC has uncovered and synthesized key aspects of the knowledge about learning and development in early childhood. In the reports *From Neurons to Neighborhoods: The Science of Early Child-*

*hood Development* (National Research Council and Institute of Medicine, 2000) and *Eager to Learn: Educating Our Preschoolers* (National Research Council, 2001b) the NRC directed its attention to early childhood institutions, their financing or lack of same, considerations of health and nutrition, and the social, emotional, and cultural components of this territory as they also focused special attention on early literacy. The report *Early Childhood Assessment: Why, What, and How* (National Research Council, 2008) identifies important outcomes for children from birth to age 5 and outlines the quality and purposes of developmental assessments. Although mathematics received attention to some degree in these studies, it was not a central focus of this work.

The NRC study that resulted in the report *How People Learn* (National Research Council, 1999) drew on a large body of research in cognition to offer a set of powerful findings about teaching and learning at all levels and all subjects that, since its publication, have rippled across the research community. The most recent follow-on publication, *How Students Learn: History, Mathematics, and Science in the Classroom* (National Research Council, 2005a), provides several concrete examples of how this research on student learning can translate into improved practice, including one example in early childhood mathematics. Some additional examples of research in this territory also surfaced in *Mathematical and Scientific Development in Early Childhood* (National Research Council, 2005b), which captures the discussion at an NRC workshop. The previously mentioned report, *Adding It Up* (National Research Council, 2001a), synthesized the research on mathematics learning in prekindergarten through eighth grade and provided advice to educators, researchers, publishers, policy makers, and parents. Taken together, these prior initiatives have helped set the stage for an in-depth examination of early learning in mathematics.

## THE COMMITTEE'S CHARGE

In order to synthesize and distill the key lessons from the relevant research, the NRC established the Committee on Early Childhood Mathematics in 2007. The majority of support for the study was provided by the Office of Head Start, under the auspices of the U.S. Department of Health and Human Services; supplementary funding was also provided by the National Institute of Child Health and Human Development, the Ewing Marion Kauffman Foundation, and the NRC. In recognition of the interdisciplinary nature of this work, the committee consists of experts in mathematics, psychology, neuroscience, early childhood education, and teacher education, as well as early childhood practitioners and policy makers. The committee worked on the study over an 18-month period.

The committee charge is as follows:

> To synthesize and analyze the past research on early childhood mathematics from a number of disciplinary fields, draw out the implications for policy and practice affecting young children as they move through the preschool years and begin formal schooling, and provide research-based guidance to increase the number of young children, especially vulnerable children, prepared to get off to a strong start in learning mathematics during their first years of schooling. It is designed to capitalize on the research literature in the field and consider its various implications for policy makers, practitioners and parents.
>
> The committee will assemble the pertinent research literature from the multiple disciplines that have focused attention on the teaching and learning of mathematics by young children. They will analyze this literature in order to develop (1) appropriate mathematics learning objectives for preschool students; and (2) critical evidence-based insights related to curriculum, instruction, and teacher education for achieving these learning objectives. Finally, they will determine the implications of these findings for policy, practice, parent-child relations, future data collection and further research.

See Box 1-1 for questions that the committee might address as part of its charge.

**BOX 1-1**
**Questions the Committee Might Address**

- What does existing research tell us about what preschool children can know about mathematics, and how they develop this knowledge?
- Learning of which mathematical knowledge, skills, and concepts in the preschool years increases the likelihood of successful mathematics learning in school and beyond?
- What do international comparisons with respect to both preschoolers and primary grades students tell us about the nature of early mathematics learning and prospects for its improvement in the United States? What approaches in other countries with respect to interventions and ongoing support could usefully be applied here?
- What policies and practices best lay the foundation for successful mathematics learning?
- What can parents, preschool teachers, and other adults who interact with young children do to promote their mathematical development?
- How can we support the mathematical development of preschool teachers so that they will be able to promote young children's mathematical development?
- How can further research in cognitive development and preschool education be focused to address issues that will lead to improvement in children's mathematical proficiency?

The committee cast its net widely to examine as much of the relevant research as possible. For some issues, the evidence base was limited: Throughout the report, we attempt to recognize and acknowledge the limitations of the evidence base and, at the end of the report, suggest some areas in which the scope and quality of research can be strengthened. The committee was not able to pursue in depth the entire array of possible issues related to mathematics education during early childhood; for example, we lacked the time, resources, and expertise to do a comprehensive international comparative analysis of early childhood education in mathematics. We do discuss the literature on the role of language as a shared cultural experience that shapes children's mathematical learning. In addition, neither program evaluation nor accountability, both of which are important to children's early mathematics education programs, is discussed at length in the report.

In addressing the charge, although the committee did examine research related to the development of number and space concepts for the very early years (i.e., infancy through age 3), our focus was on children ages 3 through 6 and early mathematics education—which includes learning, teaching, teacher education, and curriculum. The committee paid special attention to the learning and teaching practices that underscore mathematical development in children from age 3 through the end of kindergarten. This age range was chosen as the focus because it provides children with key cognitive and social development opportunities associated with successful entry into formal schooling. Evidence demonstrates that preschool-age children are excited about learning and enjoy activities that develop their mathematics competencies (Gelman, 1980; Ginsburg et al., 2006; National Research Council, 2001b; Saxe et al., 1987); this period is thus critical for maintaining and enhancing motivation to learn, especially for children from disadvantaged backgrounds, because enriching early learning experiences can enable them to begin kindergarten on a more level footing with their more advantaged peers.

The committee has put particular emphasis on the need to translate research on early childhood mathematics into practice for *all* children. Still, young children from disadvantaged backgrounds show lower levels of mathematics achievement than children from middle-class and higher status backgrounds (Clements and Sarama, 2007; Ginsburg and Russell, 1981; Hughes, 1986; Jordan, Huttenlocher, and Levine, 1994; Saxe et al., 1987; Starkey and Klein, 2000; Starkey, Klein, and Wakeley, 2004). The committee paid particular attention to issues of equity in early mathematics education throughout the report because of evidence indicating that, whereas all young children can benefit from intentional mathematics instruction, children who are at risk because of particular life circumstances (e.g., low socioeconomic status) will fall further behind their more affluent peers over the course of their schooling if they do not receive more intensive mathematics teaching (Starkey and Klein, 2000).

The committee held four meetings, which provided opportunities for discussions with practitioners, researchers, and other experts in the field of early childhood education. These discussions helped committee members develop a better understanding of the history and positions in the various stakeholder communities as well as the reasoning behind their positions. Our analyses draw on a variety of sources. The committee examined relevant summary data produced by government agencies and professional organizations. We reviewed a wide body of interdisciplinary research and commissioned a number of research synthesis papers by experts. Often, practitioners and policy makers state that the research community is too far removed from what is actually happening in the classroom, causing researchers to make recommendations that cannot be realistically implemented. The committee is keenly aware of this concern, and thus we attempt to put forth here policy recommendations that are grounded in research as well as the action steps necessary to implement them.

## THE EARLY CHILDHOOD EDUCATION AND CARE DELIVERY SYSTEM

One important issue that influenced the committee's thinking about recommendations for policy and practice is the multifaceted and complex nature of the early childhood education "system." Before the beginning of formal schooling, children spend their days in a wide variety of settings. If they are not cared for at home by their parents or relatives, children typically receive care through the country's early education and child care system, which consists of a loosely sewn-together patchwork of different kinds of programs and providers that vary widely in their educational mission and whether they are explicitly designed to provide education services. Data from the nationally representative Early Childhood Longitudinal Study, Birth (ECLS-B) cohort show that about 60 percent of preschool-age children are in center-based care (including Head Start settings), about 21 percent of children are in home-based care arrangements, and about 20 percent have no formal child care arrangements (see Table 1-1) (Jacobson Chernoff, McPhee, and Park, 2007).

In addition, about 43 percent of children younger than age 6 live in low-income families (Chau and Douglas-Hall, 2007). The high cost of high-quality early education and care is unaffordable for many low- and middle-income families (Zigler, Gilliam, and Jones, 2006). For example, the average annual cost for full-day center-based care for preschool-age

**TABLE 1-1** Children Participating in Regular Nonparental Education and Early Care, 2005-2006 (percentage)

| | Home-Based | | Center-Based | | | |
|---|---|---|---|---|---|---|
| Characteristic | Relative Care | Nonrelative Care | Non-Head Start | Head Start | Multiple Arrangements | No Regular Nonparental Arrangement |
| Total | 13 | 8 | 45 | 13 | 2 | 20 |
| **Child Race/Ethnicity** | | | | | | |
| White, non-Hispanic | 11 | 9 | 53 | 7 | 2 | 18 |
| Black, non-Hispanic | 14 | 4 | 37 | 25 | 3 | 16 |
| Hispanic | 16 | 6 | 31 | 19 | 1 | 27 |
| Asian, non-Hispanic | 16 | 3 | 55 | 6 | 2[a] | 18 |
| American Indian and Alaska Native, non-Hispanic | 14 | 5 | 29 | 31 | 1[a] | 20 |
| Other, non-Hispanic | 19 | 9 | 40 | 12 | 2[a] | 18 |
| **Socioeconomic Status[b]** | | | | | | |
| Lowest 20 percent | 15 | 5 | 22 | 25 | 2 | 31 |
| Middle 60 percent | 15 | 7 | 44 | 13 | 2 | 20 |
| Highest 20 percent | 6 | 11 | 71 | 1 | 2 | 10 |

NOTE: Percentages do not sum to 100 because of rounding error.

[a]Standard error is more than one third as large as estimate.

[b]Socioeconomic status (SES) is a measure of social standing. This SES variable reflects the socioeconomic status of the household at the time of the preschool parent interview in 2005. The components used to create the measure of SES were as follows: father/male guardian's education; mother/female guardian's education; father/male guardian's occupation; mother/female guardian's occupation; and household income. SES was collapsed first into quintiles, then into a 20/60/20 percent distribution by collapsing the middle three quintiles.

SOURCE: Jacobson Chernoff, McPhee, and Park (2007).

children ranges from $3,794 in Mississippi to $10,920 in the District of Columbia (National Association of Child Care Resource and Referral Agencies, 2007).

An increase in women's participation in the workforce has also contributed to the demand for high-quality preschool and child care (National Research Council, 2001b). Over the past four and a half decades, women's participation in the workforce has grown from 38 percent in 1960 to 60 percent in 2002 (U.S. Census Bureau, 2003), with 59 percent of mothers of 4-year-olds working outside the home (Jacobson Chernoff, McPhee, and Park, 2007).

Head Start is a large, federally funded early childhood program that promotes school readiness for economically disadvantaged children and families; the program provides comprehensive child development services (education, health, nutritional, social, and other services). In fiscal year 2007, the program served 908,412 children, most of whom were 3- and 4-year-olds (87 percent). The reach of the program is large—in 2007 there were over 49,000 Head Start classrooms located in over 18,000 centers—which makes its policies and practices influential in early childhood education.

Of the 60 percent of children in the United States who attend center-based care, approximately 22 percent are enrolled in state-funded preschool, which is the largest source of public prekindergarten (Barnett et al., 2007). Increasingly, states are moving toward state-funded preschool education to provide early education and care for children, particularly for those whose families would otherwise not be able to afford it. Georgia and Oklahoma, for example, have public preschool programs that enroll (if parents choose) 4-year-olds across the state (Barnett et al., 2007). Voluntary universal preschool is one policy option that has been suggested as a way to provide opportunities for all children, regardless of family income, to receive high-quality early education and care (Zigler et al., 2006).

However, some have argued against voluntary universal preschool in favor of programs that target low-income children (e.g., Ceci and Papierno, 2005; Fuller, 2007). Ceci and Papierno (2005), for example, suggest that targeted programs are more effective in terms of financial and educational benefits because they use (often limited) early education funds to help the most disadvantaged children.

Revisions to legislation and new policy initiatives have also shaped early childhood education policy in recent years. For example, beginning with the National Education Goals of 1990, the No Child Left Behind (NCLB) Act of 2001, and continuing through the 2007 reauthorization of the Head Start Act, interest in young children's preparation for school has increased. Central aims of these pieces of legislation are to support young children's development and learning so that they make a successful transition into kindergarten and to provide equitable educational opportunities for all students.

With the implementation of NCLB, many school districts began to place a major emphasis on the academic success of students in the early elementary grades. NCLB testing requirements do not begin until children reach third grade, but implications from the law exist for lower grades and preschool programs. The emphasis that has been placed on accountability for early childhood learning has caused concern among researchers, parents, and early education stakeholders because of the strong focus on academic development rather than the combination of academic and social-emotional development. This tension is not new; the early childhood education community has grappled with the notion that preschool programs should be more focused on academics, in contrast to the idea that they should focus instead on children's social-emotional development. The consequences of accountability systems have brought an increased emphasis and disagreements about what should be the focus of early education and care.

In addition to NCLB, Good Start, Grow Smart, President Bush's plan to strengthen early learning (White House, n.d.), promoted accountability for preschool children's learning outcomes in literacy and mathematics and also called for program improvements in language and literacy development. A major premise of this initiative was to close the achievement gap between socioeconomic and racial/ethnic groups. Until recently, the focus of these efforts was targeted at improving literacy and language development (e.g., Reading First and Early Reading First). However, with recent research clearly demonstrating the importance of early childhood mathematics to later success in reading and mathematics, policy makers are beginning to see the value of investing in early childhood mathematics. As discussed more fully in Chapter 8, policies aimed at changing or improving the education and learning of 3- to 6-year-olds still need to consider the diverse range of settings and characteristics of those who will do the teaching in these settings.

## ORGANIZATION OF THE REPORT

The report is organized into four parts. Part I focuses on the research on learning and summarizes the nearly 30 years of research demonstrating that young children are able to learn foundational mathematics. As these chapters show, preschool-age children possess a well-developed understanding of informal mathematics (Ginsburg, Klein, and Starkey, 1998), and they are able to learn complex mathematics before school entry (Clements and Sarama, 2007; Ginsburg et al., 2006).

Chapter 2 provides an overview of the important mathematical thinking processes and mathematical ideas for the early childhood period, summarizing the areas in which children need foundational learning opportunities. Chapter 3 reviews the evidence about how young children's everyday mathematics learning begins in infancy with the proximal environments in which they develop. More specifically, it focuses on cognitive development and includes a discussion of the research on infancy. Chapter 4 examines individual variation in children's mathematics learning and performance, with particular attention to mathematics learning disabilities. The chapter also considers sources of individual variation, such as familial practices, and group variation, such as socioeconomic status and race/ethnicity.

Part II focuses on a sequence of milestones for children in the core areas of number (including whole number, relations, and operations) and geometry and measurement. Chapter 5 focuses on number and operations, and Chapter 6 on geometry and measurement.

In Part III the committee turns to topics of implementation of mathematics learning and teaching in the classroom context. Chapter 7 covers the research concerning standards, curriculum, teaching, and formative assessment. Chapter 8 focuses on the early childhood workforce and examines issues of teacher education and professional development.

Part IV contains the committee's synthesis of its major conclusions and outlines the recommendations that flow from these conclusions, focusing particularly on what changes are needed to improve the quality of mathematics learning for young children. The committee also lays out an agenda for future research.

Appendix A is a glossary that defines terminology used throughout the report and Appendix B supplements Chapter 6. Appendix C presents biographical sketches of committee members and staff.

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# 2

# Foundational Mathematics Content

Mathematics provides a powerful means for understanding and analyzing the world. Mathematical ways of describing and representing quantities, shapes, space, and patterns help to organize people's insights and ideas about the world in systematic ways. Some of these mathematical systems have become such a fundamental part of people's everyday lives—for example, counting systems and methods of measurement—that they may not recognize the complexity of the ideas underpinning them. In fact, the mathematical ideas that are suitable for preschool and the early grades reveal a surprising intricacy and complexity when they are examined in depth. At the deepest levels, they form the foundations of mathematics that have been studied extensively by mathematicians over centuries (e.g., see Grattan-Guinness, 2000) and remain a current research topic in mathematics.

In this chapter, we provide an overview of the mathematical ideas that are appropriate for preschool and the early grades and discuss some of the more complex mathematical ideas that build on them. These foundational ideas are taken for granted by many adults and are not typically examined in high school or college mathematics classes. Thus, many people with an interest in early childhood education may not have had adequate opportunities in their preparation to examine these ideas. Chapters 5 and 6 examine these ideas again in some detail, from the perspective of how children come to understand them and the conceptual connections they make in doing so.

This chapter has four sections. The first two describe mathematics for young children in two core areas: (1) number and (2) geometry and measurement. These ideas, which are important preparation for school and for life, are also genuinely mathematical, with importance from a mathematician's perspective. Moreover, they are interesting to children, who enjoy engaging with these ideas and exploring them.

The third section describes mathematical process goals, both general and specific. The general process goals are used throughout mathematics, in all areas and at every level, including in the mathematics for very young children. The specific process goals are common to many topics in mathematics. These process goals must be kept in mind when considering the teaching and learning of mathematics with young children.

The fourth section describes connections across the content described in the first two sections as well as to important mathematics that children study later in elementary school. These connections help to demonstrate the foundational nature of the mathematics described in the first two sections.

## NUMBER CONTENT

Number is a fundamental way of describing the world. Numbers are abstractions that apply to a broad range of real and imagined situations—five children, five on a die, five pieces of candy, five fingers, five years, five inches, five ideas. Because they are abstract, numbers are incredibly versatile ways of explaining the world. "Yet, in order to communicate about numbers, people need representations—something physical, spoken, or written" (National Research Council, 2001, p. 72). Understanding number and related concepts includes understanding concepts of quantity and relative quantity, facility with counting, and the ability to carry out simple operations. We group these major concepts into three core areas: number, relations, and operations. Box 2-1 summarizes the major ideas in each core area. Developing an understanding of number, operations, and how to represent them is one of the major mathematical tasks for children during the early childhood years.

### The Number Core

The number core concerns the list of counting numbers 1, 2, 3, 4, 5, . . . and its use in describing how many things are in collections. There are two distinctly different ways of thinking about the counting numbers: on one hand, they form an ordered list, and, on the other hand, they describe cardinality, that is, how many things are in a set. The notion of 1-to-1 correspondence bridges these two views of the counting numbers and is also central to the notion of cardinality itself. Another subtle and important aspect of numbers is the way one writes (and says) them using the base 10

**BOX 2-1**
**Overview of Number, Relations, and Operations Core**

**The Number Core: Perceive, Say, Describe/Discuss, and Construct Numbers**

**Cardinality:** giving a number word for the numerosity of a set obtained by perceptual subitizing (immediate recognition of 1 through 3) or conceptual subitizing (using a number composition/decomposition for larger numerosities), counting, or matching.

**Number word list:** knowing how to say the sequence of number words.

**1-to-1 counting correspondences:** counting objects by making the 1-to-1 time and spatial correspondences that connect a number word said in time to an object located in space.

**Written number symbols:** reading, writing, and understanding written number symbols (1, 2, 3, etc.).

**Coordinations across the above,** such as using the number word list in counting and counting to find the cardinality of a set.

**The Relations Core: Perceive, Say, Describe/Discuss, and Construct the Relations More Than, Less Than, and Equal To on Two Sets by**

Using general perceptual, length, density strategies to find which set is more than, less than, or equal to another set, and then later.

Using the unitizing count and match strategies to find which set is more than, less than, or equal to another set, and then later.

Seeing the difference between the two sets, so the relational situation becomes the additive comparison situation listed below.

**The Operations Core: Perceive, Say, Describe/Discuss, and Construct the Different Addition and Subtraction Operations (Compositions/Decompositions of Numbers)**

**Change situations:** addition change plus situations (start + change gives the result) and subtraction change minus situations (start – change gives the result).

**Put together/take apart situations:** put together two sets to make a total; take apart a number to make two addends.

**Compose/decompose numbers:** Move back and forth between the total and its composing addends: "I see 3. I see 2 and 1 make 3."

**Embedded number triads:** Experience a total and addends hiding inside it as a related triad in which the addends are embedded within the total.

**Additive comparison situations:** Comparing two quantities to find out how much more or how much less one is than the other (the Relations Core precedes this situation).

system. The top section of Box 2-1 provides an overview of the number core from the perspective of children's learning; this is discussed in more detail in Chapter 5. Here we discuss the number core from a mathematical perspective, as a foundation for the discussion of children's learning.

### *Numbers Quantify: They Describe Cardinality*

Numbers tell "how many" or "how much." In other words, numbers communicate how many things there are or how much of something there is. One can use numbers to give specific, detailed information about collections of things and about quantities of stuff. Initially, some toy bears in a basket may just look like "some bears," but if one knows there are seven bears in the basket, one has more detailed, precise information about the collection of bears.

Numbers themselves are an abstraction of the notion of quantity because any given number quantifies an endless variety of situations. We use the number 3 to describe the quantity of three ducks, three toy dinosaurs, three people, three beats of a drum, and so on. We can think of the number 3 as an abstract, common aspect that all these limitless examples of sets of three things share.

How can one grasp this common aspect that all sets of three things share? At the heart of this commonality is the notion of 1-to-1 correspondence. Any two collections of three things can be put into 1-to-1 correspondence with each other. This means that the members of the first collection can be paired with the members of the second collection in such a way that each member of the first collection is paired with exactly one member of the second collection, and each member of the second collection is paired with exactly one member of the first collection. For example, each duck in a set of three ducks can be paired with a single egg from a set of three eggs so that no two ducks are paired with the same egg, no two eggs are paired with the same duck, and no ducks or eggs remain unpaired.

### *The Number List*

The counting numbers can be viewed as an infinitely long and ordered list of distinct numbers. The list of counting numbers starts with 1, and every number in the list has a unique successor. This creates a specific order to the counting numbers, namely 1, 2, 3, 4, 5, 6, . . . . It would not be correct to leave a number out of the list, nor would it be correct to switch the order in which the list occurs. Also, every number in the list of counting numbers appears only once, so it would be wrong to repeat any of the numbers in the list.

The number list is useful because it can be used as part of 1-to-1 object counting to tell how many objects are in a collection. Although the number of objects in small collections (up to 3 or 4) can be recognized immediately—this is called *subitizing*—in general, one uses the number list to determine the number of objects in a set by counting. Counting allows one to quantify exactly collections that are larger than can be immediately recognized. To count means to list the counting numbers in order, usually starting at 1, but sometimes starting at another number, as in 5, 6, 7, . . . . (Other forms of counting include "skip counting," in which one counts every second, or third, or fourth, etc., number, such as 2, 4, 6, . . . , and counting backward, as in 10, 9, 8, 7, . . . .)

Although adults take it for granted because it is so familiar, the connection between the list of counting numbers and the number of items in a set is deep and subtle. It is a key connection that children must make. There are also subtleties and deep ideas involved in saying and writing the number list, which adults also take for granted because their use is so common. Because of the depth and subtlety of ideas involved in the number list and its connection to cardinality, and because these ideas are central to all of mathematics, it is essential that children become fluent with the number list (see Box 2-2).

***Connecting the number list with cardinality.*** In essence, counting is a way to make a 1-to-1 correspondence between each object (in which the

**BOX 2-2**
**The Importance of Fluency with the Number List**

All of the work on the relations/operation core in kindergarten serves a double purpose. It helps children solve larger problems and become more fluent in their Level 1 solution methods. It also helps them reach fluency with the number word list in addition and subtraction situations, so that the number word list can become a representational tool for use in the Level 2 counting of solution methods. To get some sense of this process, try to add or subtract using the alphabet list instead of the number word sequence. For counting on, you must start counting with the first addend and then keep track of how many words are counted on. Many adults cannot start counting within the alphabet from D or from J because they are not fluent with this list. Nor do they know their fingers as letters (How many fingers make F?), so they cannot solve D + F by saying D and then raising a finger for each letter said after D until they have raised F fingers. It is these prerequisites for counting on that kindergarten children are learning as they count, add, and subtract many, many times. Of course as they do this, they will also begin to remember certain sums and differences as composed/decomposed triads (as *number facts*).

objects can be any discrete thing, from a doll, to a drumbeat, to the idea of a unicorn) and a prototypical set, namely a set of number words. For example, when a child counts a set of seven bears, the child makes a 1-to-1 correspondence between the list 1, 2, 3, 4, 5, 6, 7 and the collection of bears. To count the bears, the child says the number word list 1, 2, 3, 4, 5, 6, 7 while pointing to one new bear for each number. As a result, each bear is paired with one number, each number is paired with one bear, and there are no unpaired numbers or bears once counting is completed. The pairing could be carried out in many different ways (starting with any one of the bears and proceeding to any other bear next, and so on), but any single way of making such a 1-to-1 correspondence by counting establishes that there are seven bears in the set.

A key characteristic of object counting is that the last number word has a special status, as it specifies the total number of items in a collection. For example, when a child counts a set of seven bears, the child counts 1, 2, 3, 4, 5, 6, 7, pointing to one bear for each number. The last number that is said, 7, is not just the last number in the list, but also indicates that there are seven bears in the set (i.e., cardinality of the set). Thus when counting the 7 bears, the counter shifts from a counting reference (to 7 as the last bear when counting) to a cardinal reference when referring to 7 as the number of bears in all. Counting therefore provides another way to grasp the abstract idea that all sets of a fixed number of things share a common characteristic—that when one counts two sets that have the same number of objects, the last counting word said will be the same for both.

Another key observation about counting is that, for any given number in the list of counting numbers, the next number in the list tells how many objects are in a set that has one more object than do sets of the given number of objects. For example, if there are five stickers in a box and one more sticker is put into the box, then one knows even without counting them all again that there will now be six stickers in the box, because 6 is the next number in the counting list. Generally each successive counting number describes a quantity that is one more than the quantity that the previous number describes.

In a sense, then, counting is adding: Each counting number adds one more to the previous collection (see Figure 2-1). Of course, if one counts backward, then one is subtracting. These observations are essential for children's early methods of solving addition and subtraction problems. Also, each step in the counting process can be thought of as describing the total number of objects that have been counted so far.

***The number word list and written number symbols in the base 10 place-value system.*** Each number in the number list has a unique spoken name and can be represented by a unique written symbol. The names and symbols for the initial numbers in the list have been passed along by tradition, but

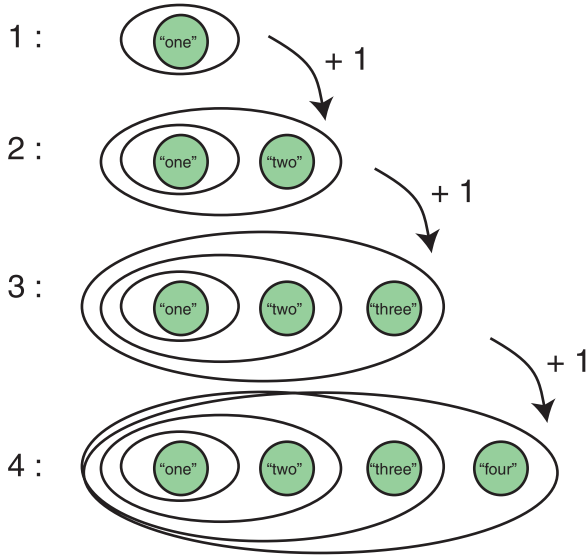


**FIGURE 2-1** Each counting number describes a quantity that is one more than the previous number describes.

the English names of the first 10 (or so) counting numbers and the symbols of the first 9 counting numbers are arbitrary and could have been different. For example, instead of the English word "three," one could be using "bik" or "Russell" or any other word, such as the words for "three" in other languages. Instead of the symbol 3, one could use a symbol that looks completely different.

The list of counting numbers needs to go on and on in order to count ever larger sets. So the problem is how to give a unique name to each number. Different cultures have adopted many different solutions to this problem (e.g., Menninger, 1958/1969; see Chapter 4 of this volume for a discussion of counting words in different languages). The present very efficient solution to this problem was not obvious and was in fact a significant achievement in the history of human thought (Menninger, 1958/1969). Even though the first nine counting numbers, 1, 2, 3, 4, 5, 6, 7, 8, 9, are represented by distinct, unrelated symbols, some mechanism for continuing to list numbers without resorting to creating new symbols indefinitely is desirable.

The decimal system (or base 10 system) is the ingenious system used today to write (and say) counting numbers. The decimal system allows one to use only the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 to write any counting number as a string of digits (such a written representation of a number is often called a *numeral*).

The system is called a base 10 system because it uses 10 distinct digits and is based on repeated groupings by 10. The use of only 10 digits to write any counting number, no matter how large, is achieved by using *place value*. That is, the meaning of a digit in a written number depends (in a very specific way) on its placement. The details about using the decimal system

**BOX 2-3**
**Using the Decimal System to Write the List of Counting Numbers**

Each of the first nine counting numbers (or number words) "one, two, . . . , nine," requires only one digit to write, 1, 2, . . . , 9. Each digit stands for that many things—in other words, that many "ones," as indicated at the top of Figure 2-2. Each of these digits is viewed as being in the "ones place."

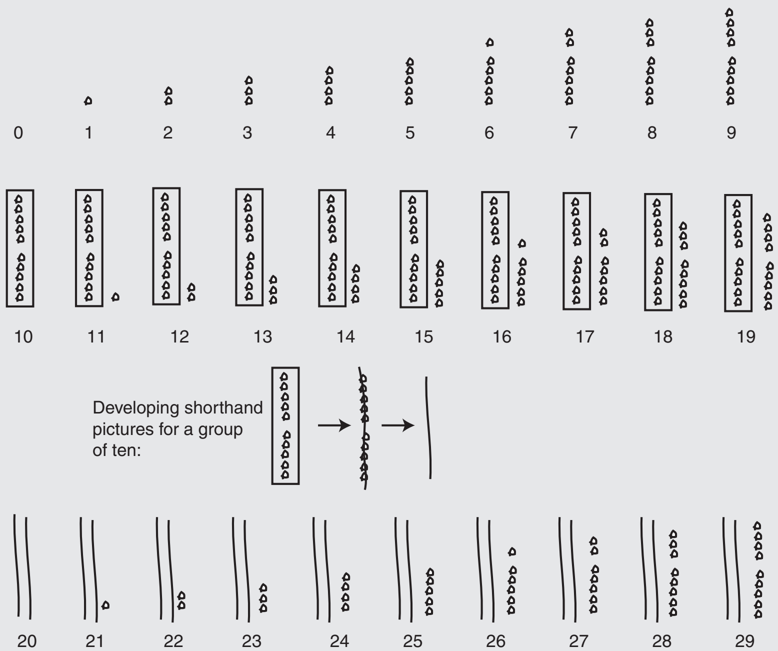


**FIGURE 2-2** Decimal system 1.

The next counting number, ten, requires two digits to write. The 1 stands for 1 ten and the 0 stands for 0 ones, and 10 stands for the combined amount in 1 ten and 0 ones. This way of describing and writing the number ten requires thinking of it as a single group of ten—in other words, as a new entity in its own right, which is created by joining 10 separate things into a new coherent whole, as indicated in the figure by the way 10 dots are shown grouped to form a single unit of 10.

to write the list of counting numbers are given in Box 2-3: A key idea is to create larger and larger units, which are the values of places farther and farther to the left, by taking the value of each place to be 10 times the value of the previous place to its right. One can think of doing this by bundling together 10 of the previous place's value. The greater and greater values

In each of the next two-digit counting numbers, 11, 12, 13, 14, 15, . . . , 20, 21, 22, . . . , 30, 31, . . . , 97, 98, 99, the digit on the right stands for that many ones, so one says this digit is in the "ones place," and the digit on the left stands for that many tens, so one says it is in the "tens place"; the number stands for the combined amount in those tens and ones. For example, in 37, the 3 stands for 3 tens, the 7 stands for 7 ones, and 37 stands for the combined amount in 3 tens and 7 ones. Notice that from 20 on, the way one says number words follows a regular pattern that fits with the way these numbers are written. But the way one says 11 through 19 does not fit this pattern. In fact, 13 through 19 are said backward, because the ones digit is said before the tens digit is indicated.

The number 99 is the last two-digit counting number, and it stands for the combined amount in 9 tens and 9 ones (see Figure 2-3). The next counting number will be the number of dots there are when one more dot is added to the dots on the left of the figure. This additional dot "fills up" a group of ten, as indicated in the middle of the figure. Now there are 10 tens, but there isn't a digit that can show this many tens in the tens place. So the 10 tens are bundled together to make a new coherent whole, as indicated on the right in Figure 2-3, which is called a hundred. From 0 to 9 hundreds can be recorded in the place to the left of the tens place, which is called the hundreds place. So the next counting number after 99 is written as 100, in which the 1 stands for 1 hundred, and the 0s stand for 0 tens and 0 ones.

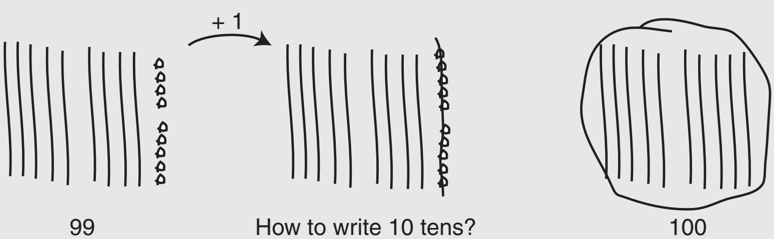


**FIGURE 2-3** Decimal system 2.

The decimal system has a systematic way to make new larger units by bundling 10 previously made units and recording the new unit one place to the left of the given unit's place. Just as 10 ones make a new unit of 10, which is recorded to the left of the ones place, 10 tens make a new unit of a hundred, which is recorded to the left of the tens place, and 10 hundreds make a new unit of a thousand, which is recorded to the left of the hundreds place. This pattern continues on and on to new places on the left.

of the places allow any number, no matter how large, to be expressed as a combination of between 0 and 9 of each place's value. In this way, every counting number can be expressed in a unique way as a numeral made of a string of digits. (See Howe, 2008, for a further discussion of the decimal system and place value.)

Even though most countries around the world now use this system of written numerals, they still use their own list of counting words that relate closely, or not so closely, to the written system of numerals. English and other European lists of counting words have various aspects that do not fit the decimal system so well and that create difficulties in learning the system. These, and ways to compensate for these difficulties, are discussed in Chapter 4.

## The Relations/Operations Core

Numbers do not exist in isolation. They make up a coherent system in which numbers can be compared, added, subtracted, multiplied, and divided. Just as numbers are abstractions of the notion of quantity, the relations "less than," "greater than," and "equal to" and the operations of addition, subtraction, multiplication, and division are abstractions of comparing, combining, and separating quantities. These relations and operations apply to a wide variety of problems. The middle and bottom sections of Box 2-1 are an overview of the relations core and the operations core for young children (which concerns only addition and subtraction, not multiplication or division).

### *Comparing*

In some cases it is visually evident that there are more things in one collection than in another, such as in the case of the two sets of beads shown at the top of Figure 2-4. But in other cases it is not immediately clear which collection (if either) has more items in it.

A basic way to compare two collections of objects is by direct matching (as in the middle of Figure 2-4). If a child has a collection of black beads and another collection of white beads, and if these collections are placed near each other, the child can place each black bead with one and only one white bead. If there is at least one extra white, then there are more whites; if at least one extra black, then more blacks. And if none is left over, then the two groups have the same number (although one may not know and does not need to know exactly what number it is).

When direct matching is not possible, a child can count the number of beads in two collections to determine which collection (if either) has more beads or if they both have the same number of beads. A key observation

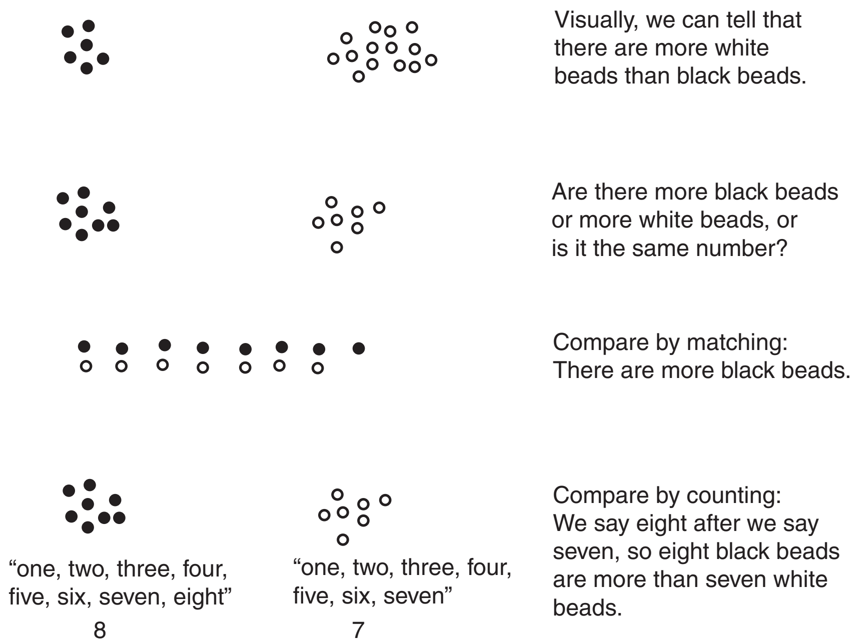


**FIGURE 2-4** Comparing.

about using counting to compare is that a number that is said later in the counting word list corresponds to a collection that has a greater number of objects than does a collection corresponding to a number earlier in the sequence. For example, one knows that there are more beads in a collection of eight black beads than there are in a collection of seven white beads because 8 occurs later in the counting list than 7 (see the bottom of Figure 2-4). Counting thus provides a more advanced way to compare sets of things than direct matching because it relies on knowledge about how numbers compare. Counting is also a more powerful way to compare sets of things than direct matching because it allows sets that are not in close proximity to be compared.

A key point about comparing collections of objects is that counting can be used to do so, and it relies on the link between the number list and cardinality: Numbers later in the list describe greater cardinalities than do numbers earlier in the list. Finding out which collection is more than another collection is easier than determining exactly how many more that collection has than the other, which can be formulated as an addition or subtraction problem. This more specific version of comparison is discussed in the next section.

*Addition and Subtraction Story Problems and Situations*

Addition and subtraction are used to relate amounts before and after combining or taking away, to relate amounts in parts and totals, or to say precisely how two amounts compare. Story problems and situations that can be formulated with addition or subtraction occur in a wider variety than just the simplest and most common "add to" and "take away" story problems. Methods that young children can use to solve addition and subtraction story problems, again, rely on a fluent link between the number list and cardinality. Later methods (in first grade or so) also rely on decomposing numbers and on an initial understanding of the base 10 system, namely that the numbers 11 through 19 can be viewed as a ten and some ones.

Box 2-4 describes the different types of story problems or situations that can be formulated with addition or subtraction. Viewed from a more

**BOX 2-4**
**Types of Addition/Subtraction Situations**

**Change Plus and Change Minus Situations**

Change situations have three quantitative steps over time: start, change, result. Most children before first grade solve only problems in which the result is the unknown quantity. In first grade, any quantity can be the unknown number. Unknown start problems are more difficult than unknown change problems, which are more difficult than unknown result problems.

Change plus: Start quantity + change quantity = result quantity: "Two bunnies sat on the grass. One more bunny hopped there. How many bunnies are on the grass now?"

Change minus: Start quantity – change quantity = result quantity: "Four apples were on the table. I ate two apples. How many apples are on the table now?"

**Put Together/Take Apart Situations**

In these situations, the action is often conceptual instead of physical and may involve a collective term like "animal": "Jimmy has one horse and two dogs. How many animals does he have?"

In put together situations, two quantities are put together to make a third quantity: "Two red apples and one green apple were on the table. How many apples are on the table?"

In take apart situations, a total quantity is taken apart to make two quantities: "Grandma has three flowers. How many can she put in her red vase and how many in her blue vase?"

These situations are decomposing/composing number situations in which children shift from thinking of the total to thinking of the addends. Working with differ-advanced perspective, most of these situations can be formulated in a natural way with an equation of the form

$$A + B = C \quad \text{or} \quad A - B = C$$

in which two of the three numbers in the equation are known and the problem is to determine the other number that makes the equation true. The types of situations that are naturally formulated with these equations are *change plus* and *change minus* situations, *put together* situations, and *comparison* situations.

In change plus and change minus situations, there is a starting quantity (A), an amount by which this quantity changes (B), and the resulting quantity (C). Problems in which A and B are the known amounts and C is to be determined are the classic, most readily recognized addition and

ent numbers helps them learn number triads related by this total-addend-addend relationship, which they can use when adding and subtracting. Eventually with much experience, children move to thinking of embedded number situations in which one considers the total and the two addends (partners) that are "hiding inside" the total simultaneously instead of needing to shift back and forth.

Equations with the total alone on the left describe take apart situations: $3 = 2 + 1$. Such equations help children understand that the = sign does not always mean *makes* or *results in* but can also mean *is the same number as*. This helps with algebra later.

**Comparison Situations**

Children first learn the comparing relations equal to, more than, and less than for two groups of things or two numbers. They find out which one is bigger and which one is smaller (or if they are equal) by matching and by counting.

Eventually first grade children come to see the third quantity involved in a more than/less than situation: the amount more or less (the difference). Children then can solve additive comparison problems in which a larger quantity is compared to a smaller quantity to find the difference. Children may write different equations to show such comparisons and may also still solve by matching or counting. As with the other addition and subtraction situations, any of the three quantities can be unknown. The language involved in such situations is complex because the comparing sentence gives two kinds of information. "Julie has six more than Lucy" says both that "Julie has more than Lucy" and that the amount more is six. This is a difficult linguistic structure for children to understand and to say.

NOTE: Researchers use different names for these types of addition and subtraction situations, and some finer distinctions can be made within the categories. However, there is widespread agreement about the basic types of problem situations despite the use of different terminology.

subtraction problems. Reversing the action in change minus or change plus situations shows the connection between subtraction and addition. For example, if Whitney had 9 dinosaurs and gave away 3 dinosaurs, how many dinosaurs did Whitney have left? This problem can be formulated with the subtraction equation, 9 – 3 = ? Starting with the dinosaurs Whitney has left, if she gets the 3 dinosaurs back, she will have her original 9 dinosaurs, which can be expressed with the addition equation ? + 3 = 9. Subtraction problems can thus be reformulated in terms of addition, which connects subtraction to addition.

In put together situations, there are two parts, A and B, which together make a whole amount, C. These situations are formulated in a natural way with an addition equation, A + B = C.

Change plus, change minus, and put together problems in which either A or B (the start quantity, the change quantity, or one of the two parts) is unknown involve an interesting reversal between the operation that formulates the problem and the operation that can be used to solve the problem from a more advanced perspective. For example, consider this "change unknown" problem: "Matt had 5 cards. After he got some more cards, he had 8. How many cards did Matt get?" This problem can be formulated with the addition equation 5 + ? = 8. Although young children will solve this problem by adding on to 5 until they reach 8 (perhaps with actual cards or other objects), older children and adults may solve the problem by subtracting, 8 – 5 = 3, which uses the opposite operation than the addition equation that was used to formulate the problem.

Comparison situations concern precise comparisons between two different quantities, A and C. Instead of simply saying that A is greater than, less than, or equal to C, the situation concerns the exact amount by which the two quantities differ. If C is B more than A, then the situation can be formulated with the equation A + B = C. If C is B less than A, then the situation can be formulated with the equation A – B = C. To consider this precise difference, B, requires one to conceptually create a collection that is not physically present separately in the situation. This difference is either that part of the larger collection that does not match the smaller collection, or it is those objects that must be added to the smaller collection to match the larger collection. Of course, these matches can be done by counting and with specific numbers rather than just by matching. Note that these situations are called additive comparison situations even when formulated with subtraction (A – B = C when C is B less than A) to distinguish them from multiplicative comparison situations, which can be formulated in terms of multiplication or division. Students solve multiplicative comparison problems in the middle and later elementary grades.

In take apart situations, a total amount, C, is known and the problem is to find the ways to break the amount into two parts (which do not have to be equal). Take apart situations are most naturally formulated with an equation of the form

$$C = A + B$$

in which C is known and all the possible combinations of A and B that make the equation true are to be found. There are usually many different As and Bs that make the equation true.

## GEOMETRY/MEASUREMENT CONTENT

Geometry and measurement provide additional, powerful systems for describing, representing, and understanding the world. Both support many human endeavors, including science, engineering, art, and architecture. Geometry is the study of shapes and space, including two-dimensional (2-D) and three-dimensional (3-D) space. Measurement is about determining the size of shapes, objects, regions, quantities of stuff, or quantifying other attributes. Through their study of geometry and measurement, children can begin to develop ways to mentally structure the spaces and objects around them. In addition, these provide a context for children to further develop their ability to reason mathematically.

Every 3-D object or 2-D shape, even very simple ones, has multiple aspects that can be attended to: the overall shape, the particular parts and features of the object or shape, and the relationships among these parts and with the whole object or shape. In determining the size of a shape or object, one must first decide on which particular aspect or measurable attribute to focus.

Space (both 3-D and 2-D) could be viewed initially as an empty, unstructured whole, but objects that are placed or moved within the space begin to structure it. The beginnings of the Cartesian structure of space, a central idea in mathematics, are seen when square tiles are placed in neat arrays to form larger rectangles and when cubical blocks are stacked and layered to make larger box-shaped structures. These are also examples of composing and decomposing shapes and objects more generally. Composing and decomposing shapes and objects are part of a foundation for later reasoning about fractions and about area and volume.

Viewing or imagining an object from different perspectives in space and moving or imagining how to move an object through space to fit in a particular spot links spatial relations with the parts and features of objects and shapes.

Just as numbers are an abstraction of quantity, the ideal, theoretical shapes (2-D and 3-D) of geometry are an abstraction of their approximate physical versions. The angles in a rectangular piece of paper aren't exactly right angles, the edges aren't perfectly straight line segments, and the paper, no matter how thin, has a thickness to it that makes it a solid 3-D shape rather than only 2-D. Measurements of actual physical objects are never exact, either. Even so, valid reasoning about ideal geometric shapes and ideal theoretical measurements can be aided with approximate physical shapes and measurements.

## Measurement

In its most basic form, measurement is the process of determining the size of an object. But the size of an object can be described in different ways, depending on the attribute one chooses. For example, the size of a tower made of cube-shaped blocks might be described by the height of the tower (a length) or in terms of the number of blocks in the tower (a volume). The size of the floor of a room that is covered in square tiles can be described in terms of the number of tiles on the floor (an area). The most important measurable attributes in mathematics are length, area, and volume.

To measure a quantity (with respect to a given measurable attribute, such as length, area, or volume), a unit must be chosen. Once a unit is chosen, the size of an object (with respect to the given measurable attribute) is the number of those units it takes to make (the chosen attribute of) the object.

For length, a stick, for example, 1 foot long, could be chosen to be a unit. With respect to that unit of length, the length of a toy train is the number of those sticks (all identical) needed to lay end to end alongside the train from the front to the end.

For area, a square tile, such as a tile that is 1 inch by 1 inch, could be chosen to be a unit. With respect to that unit of area, the area of a rectangular tray is the number of those tiles (all identical) it takes to cover the tray without gaps or overlaps. Although squares need not be used for units of area, they make especially useful units because they line up in neat rows and columns and fill rectangular regions completely without gaps or overlaps.

For volume, a cube-shaped block, such as a block that is 1 inch by 1 inch by 1 inch, could be chosen to be a unit. With respect to that unit of volume, the volume of a box is the number of those cubes (all identical) it takes to fill the box without any gaps. Although cubes need not be used for units of volume, they make especially useful units because they line up in neat rows and columns and stack in neat layers to fill box shapes completely without gaps.

Once a unit has been chosen, a measurement is a number of those units (e.g., 3 inches, 6 square inches, 12 cubic inches). So measurement is a generalization of cardinality, which describes how many things are in a collection. For young children, measurements will generally be restricted to whole numbers, but measurement is a natural context in which fractions arise. To fill a bucket with sand, a child might pour in 4 full cups of sand and another cup that is only half full of sand, so that the volume of the bucket is approximately 4½ cups.

An important but subtle idea about units, which children learn gradually, is that when measuring a given object, the larger the unit used to measure, the smaller the total number of units. For example, suppose there are two sizes of sticks to use as units of length: short sticks and longer sticks. More short sticks than long ones are needed to measure the same length. In other words, there is an inverse relation between the size of a measuring unit and the number of units needed to measure some characteristic.

Young children may also not grasp the importance of using standard units, which allow one to compare objects that are widely separated in space or time (see Chapter 3 for further discussion).

## 2-D Shapes

Shapes found in nature, such as flowers, leaves, tree trunks, and rocks, are complex, intricate, and 3-D rather than 2-D. In contrast, the familiar 2-D shapes studied in geometry, such as triangles, rectangles, and circles, are relatively simple. Compared with most shapes in the natural world, these shapes are relatively easy to draw or create and also to describe and analyze. Many manufactured objects, such as tabletops and appliances, have parts that are approximate triangles, rectangles, or circles. Many shapes in the natural world are approximate combinations of parts of these simpler geometric shapes. For example, a birch leaf might look like a triangle joined to a half-circle.

Although geometric shapes can be described and discussed informally and children can simply be told the names of some prototypical examples of these shapes (for ease of reference and discussion), these shapes also have mathematical definitions, which teachers should know.

### *Parts and Features of 2-D Shapes*

Geometric shapes have parts and features that can be observed and analyzed. The shapes all have an "inside region" and an "outer boundary." Distinguishing the inside region of a 2-D shape from its outer boundary is an especially important foundation for understanding the distinction between the perimeter and area of a shape in later grades. Except for circles, the outer boundary of the common 2-D geometric shapes consists of straight sides, and the nature of these sides and their relationships to each other are important characteristics of a shape. One can attend to the number of sides and the relative length of the sides: Are all the sides of the same length, or are some longer than others? Where two sides meet, there is a corner point or vertex (plural: vertices). One interesting observation is that the number of vertices is the same as the number of sides. One can attend to how "pointy" a shape is at its vertex. In this case, one is attending to the angle formed by the sides that meet at the vertex. In some shapes, all the angles are the same, such as rectangles. In some shapes, some angles are the same and others are different, such as a rhombus that is not a square. The study of geometry is not only about seeing shapes as wholes; it's about finding and analyzing their properties and features.

#### *Additional Characteristics of 2-D Shapes Beyond Their Defining Characteristics*

In studying shapes, young children's attention will be drawn to the many different characteristics and features of a given shape. But from a more advanced standpoint, mathematicians have made definitions of shapes precise and spare by selecting only some of the characteristics of a shape as defining characteristics. For example, the definition of a triangle is a 2-D shape with three straight sides. A triangle also has three vertices and three angles, but these are not mentioned in the definition of triangle. Similarly, the opposite sides in a rectangle are the same length, but this is not mentioned in the definition of rectangle. Young children, however, can observe and describe these additional properties of shapes. For example, when one folds a rectangle out of paper by folding right angles, one can see that the opposite sides of the rectangle are the same length. The rectangle wasn't constructed with the explicit intent of making opposite sides the same length, yet it turns out that way. Similarly, if one joins four sticks end to end to make a quadrilateral and if the sticks were chosen so that the opposite sides are the same length, one can see that the opposite angles are also the same. Although the shape wasn't constructed with the explicit intent of making opposite angles the same, it nevertheless turns out that way.

### 3-D Shapes

The common simple geometric 3-D shapes are cubes, prisms, cylinders, pyramids, cones, and spheres. Many common objects are approximate versions of these ideal, theoretical shapes. For example, a building block is a rectangular prism, and a party hat can be in the shape of a cone. As with 2-D shapes, the study of 3-D shapes is not only about seeing these shapes as wholes and learning their names, but also about finding and analyzing their properties and features.

The 3-D geometric shapes have parts and features that can be observed. The shapes all have an "inside" and an "outer surface." The outer surface may consist of several parts. For example, the outer surface of a prism can consist of rectangles. If the outer surface of a 3-D shape consists of flat surfaces, these are often called faces. For example, a long wooden building block has two faces at each end that are small rectangles and four faces around the middle that are long rectangles. Faces are joined along straight edges, and edges meet at points called vertices. Children might observe that some shapes (like that building block) have pairs of faces on opposite sides that are the same (congruent). Children might also observe that some shapes, like cylinders (like a pole or a can), cones (like a party hat), and spheres (like a ball), have outer surfaces that are not flat.

Although the outer surface of a 3-D shape is usually visible, unless one cuts the shape open, or the shape is made of clear plastic, or the shape is hollow and a face can be removed to look inside, one must usually imagine and visualize the inside. One exception is rooms, which are often (roughly) in the shape of a rectangular prism, and which one experiences from the inside. Distinguishing the inside of a 3-D shape from its outer surface is an especially important foundation for understanding the distinction between the surface area and volume of a shape in later grades.

### Composing and Decomposing Shapes

Just as 10 ones can be composed to make a single unit of 10, shapes can also be composed to make new, larger shapes. And just as a 10 can be decomposed into 10 ones, so too shapes can be decomposed to make new, smaller shapes. Figure 2-5 presents a few examples of relationships among shapes obtained by composing and decomposing shapes based on equilateral triangles. Figure 2-6 shows relationships among shapes obtained by composing and decomposing rectangles.

Composing and decomposing 2-D shapes is an important foundation for understanding area in later grades. In particular, viewing rectangles as

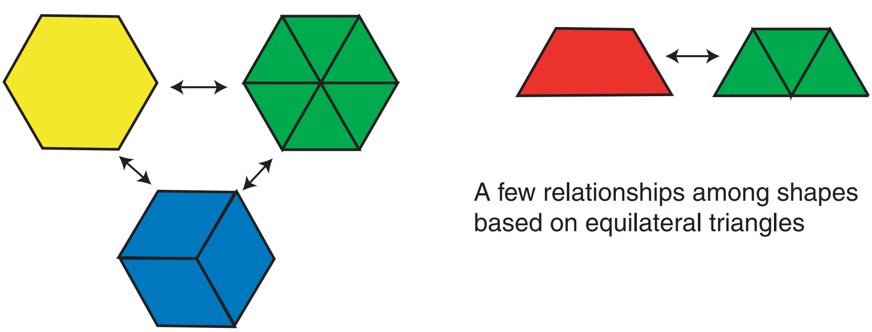


**FIGURE 2-5** Relationships among shapes based on equilateral triangles.

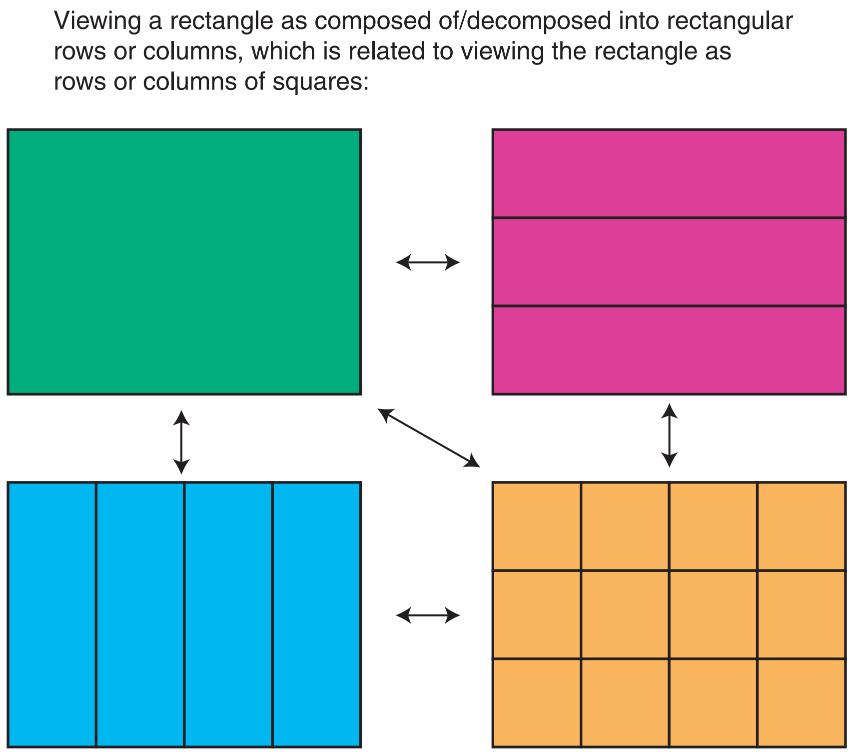


**FIGURE 2-6** Relationships among rectangles.

composed of rows and columns of squares, as illustrated in Figure 2-6, is key to understanding areas of rectangles.

Likewise, composing and decomposing 3-D shapes is an important foundation for understanding volume in later grades. In particular, viewing rectangular prisms as composed of layers of rows and columns of cubes is key to understanding volumes of rectangular prisms (see Figure 2-7). Also, reasoning about fractions often takes place in a context of reasoning about decomposing shapes into pieces.

Composition and decomposition is discussed in greater detail in the section on mathematical connections across content areas and to later mathematics.

### Motion, Relative Location, and Spatial Structuring

Part of the study of geometry is the analysis of both 2-D and 3-D space. A flat tabletop or piece of paper (imagined to extend infinitely in all

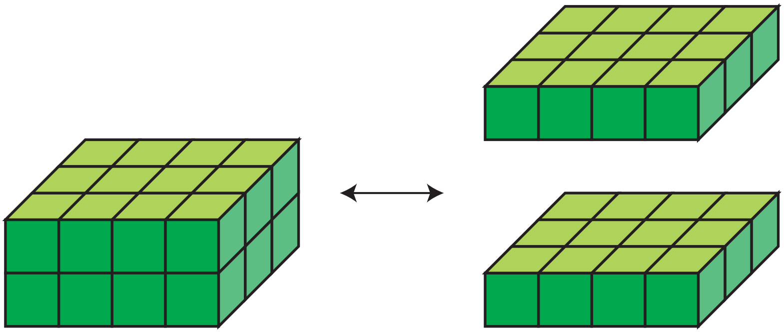


**FIGURE 2-7** Viewing a rectangular box as composed of layers of rows and columns.

directions) is a model for 2-D space. The space around one is a model for 3-D space. For young children, the study of space begins with movement through space and with describing relative location in space.

Space is oriented by relative location. Think of one object as at a fixed location in 3-D space. Another object may be above or below the fixed object, which indicates relative location along a vertical axis (line). Another object may be in front of or behind a fixed object, or it may be to the left or right of a fixed object. These two descriptions indicate relative location along two distinct (and perpendicular) horizontal axes (lines). A related way to begin to structure space is to join squares into neat arrays of rows and columns for 2-D space and to stack cubes in layers of rows and columns for 3-D space.

Although objects can be moved through space in many different ways, in 2-D space (think of a 2-D shape on a tabletop) there are some special motions that are of particular interest in advanced geometry that are also accessible to young children. Using elementary school terminology, these motions are called slides, flips, and turns (and in more advanced settings they are called translations, reflections, and rotations).

A slide moves a shape in a single direction for a specified distance without turning the shape. A flip reflects the shape across a line (so that the top and bottom of the shape become reversed). A turn rotates the shape around a fixed point with a specified amount of turning (e.g., a half turn or a quarter turn). (Technically, the center point of rotation need not be the center of the shape or even within the shape, although for young children it will be chosen that way.)

## MATHEMATICAL PROCESS GOALS

In addition to coming to understand the specific mathematical concepts discussed so far, children need to develop proficiency in the reasoning processes used in mathematics. In this section we describe two categories of mathematical processes: (1) general mathematical reasoning processes, which are central in every content area and at every level of mathematics, and (2) specific mathematical reasoning processes, which weave through many different content areas. Note that many of the specific reasoning processes were already touched on in the discussions of number, geometry, and measurement. In fact, these specific processes represent powerful, cross-cutting ideas that connect multiple concepts, procedures, or problems and can help children begin to see coherence across topics in mathematics. One major goal of early education should be to stimulate and foster mathematical reasoning.

### General Mathematical Reasoning Processes

The National Council of Teachers of Mathematics (NCTM) identified five process standards essential for meaningful and substantive mathematics learning and teaching (National Council of Teachers of Mathematics, 2000): (1) representing (including analyzing representations mathematically and visualizing internally), (2) problem solving, (3) reasoning, (4) connecting, and (5) communicating. These processes are vehicles for children to deepen, extend, elaborate, and refine their thinking and to explore ideas and lines of reasoning. According to NCTM, these processes are to be continually interwoven throughout the teaching and learning of mathematics content—even at the preschool level (see Chapters 5 and 6 for further discussion).

Representing is central in mathematics. Mathematics at every level uses simplified pictures or diagrams to represent a situation and subject it to mathematical analysis. For example, a child hears the story of The Three Bears. She forms a mental image of the three, with the papa bear largest in size, the mama bear next, and then the baby bear. She draws a crude picture of the three, or perhaps uses stick figures, or even lines. All of these are representations—the mental image, the picture, the stick figures, and the lines. The child can use the representations to reason about the objects and to explore ideas about size. Is the mama bear smaller than the papa bear? Is she also bigger than the baby bear? How can she be both bigger and smaller at the same time? Much later, the student can represent this situation as

A > B and B > C and reason that, if this is the case, then A > C. Here the representations are mathematical in the conventional sense. But when used to understand a situation quantitatively or geometrically, images and simple drawings are no less mathematical than are such representations as written numbers or equations, which are universally recognized as mathematical.

According to mathematical educators, "problem solving and reasoning are the heart of mathematics" (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002, p. 6). In fact, solving problems is both a goal of mathematics learning and a mechanism for doing so. Young children will need support to formulate, struggle with, and solve problems and to reflect on the reasoning they use in doing so. By developing their ability to reason mathematically, children will begin to note patterns or regularities in the world and across the mathematical ideas they are introduced to. They will become increasingly sophisticated in their ability to recognize and analyze the mathematics inherent in the world around them.

Connecting and communicating are particularly important in the preschool years. Children must learn to describe their thinking (reasoning) and the patterns they see, and they must learn to use the language of mathematical objects, situations, and notation. Children's informal mathematical experiences, problem solving, explorations, and language provide bases for understanding and using this formal mathematical language and notation. The informal and formal representations and experiences need to be continually connected in a nurturing "math talk" learning community, which provides opportunities for all children to talk about their mathematical thinking and produce and improve their use of mathematical and ordinary language. Children also need to connect ideas across different domains of mathematics (e.g., geometry and number) and across mathematics and other subjects (e.g., literacy) and aspects of everyday life.

### *Applying the Process Standards: Mathematizing*

Together, the general mathematical processes of reasoning, representing, problem solving, connecting, and communicating are mechanisms by which children can go back and forth between abstract mathematics and real situations in the world around them. In other words, they are a means both for making sense of abstract mathematics and for formulating real situations in mathematical terms—that is, for *mathematizing* the situations they encounter.

The power of mathematics lies in its ability to unify a wide variety of situations and thereby to apply a common problem-solving strategy in seemingly disparate examples. For example, the number 3 applies not only to concrete situations, such as three pencils or three apples, but also to any collection of three things, real or imagined. Thus, the addition problem 3 + 2 = ? provides an abstract formulation for a vast number of actual situations in the world around one. The abstract nature of mathematics is part of its power: Because it is abstract, it can apply to a virtually limitless number of situations. But for children to use this mathematical power requires that they take situations and problems from the world around them and formulate them in mathematical terms. In other words, it requires children to *mathematize* situations.

Mathematizing happens when children can create a model of the situation by using mathematical objects (such as numbers or shapes), mathematical actions (such as counting or transforming shapes), and their structural relationships to solve problems about the situation. For example, children can use blocks to build a model of a castle tower, positioning the blocks to fit with a description of relationships among features of the tower, such as a front door on the first floor, a large room on the second floor, and a lookout tower on top of the roof. Mathematizing often involves representing relationships in a situation so that the relationships can be quantified.

For example, if there are three green toy dinosaurs in one box and five yellow toy dinosaurs in another box, children might pair up green and yellow dinosaurs and then determine that there are two more yellow dinosaurs than green ones because there are two yellow dinosaurs that do not have a green partner. With experience and guidance, children create increasingly abstract representations of the mathematical aspects of the situation. For example, drawing five circles instead of five yellow ducks or drawing a rectangle to represent the side of a box of tissues and, later, writing an equation to model a situation. Children become able to visualize these mathematical attributes mentally, which helps in various kinds of problem solving. Children also need eventually to learn to read and to write formal mathematical notation, such as numerals (1, 2, 6, 10) and other symbols (=, +, −) and to use these symbols in mathematizing situations. Thus, mathematizing involves reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing what is first understood on an intuitive and informal level in the context of everyday activity (Clements and Sarama, 2007).

### Specific Mathematical Reasoning Processes

Mathematics learning in early childhood requires children to use several specific mathematical reasoning processes, also known as "big ideas," across domains. These big ideas are overarching concepts that connect multiple concepts, procedures, or problems within or across domains or topics and are a particularly important aspect of the process of forming connections. Big ideas "invite students to look beyond surface features of procedures and concepts and see diverse aspects of knowledge as having the same underlying structure" (Baroody, Feil, and Johnson, 2007, p. 26).

### *Unitizing*

Unitizing—finding or creating a mathematical unit—occurs in numerical, geometric, and spatial contexts. When children count, they must create mental units of what they are going to count: single cats, the paws on several cats, or groups of two cats. To measure length, children must select a unit of length measure (for example, they will lay along a length and then count new crayons, feet stepped heel-to-toe along some distance, or inch lengths). To create repeating patterns, children must select and repeat a unit. For example, they might make a bead necklace by repeatedly stringing two cubes then a sphere (their unit). In designing a block building, they might repeatedly place a square, then a triangular block, repeating that unit around the top of their building. When making designs or pictures with pattern blocks, children might join several shapes to make a unit that they repeat throughout the design. To begin to understand the base 10 place-value system, children must be able to view ten ones as forming a single unit of ten. Research suggests a link between being able to view a collection of shapes as a higher order unit and being able to view two-digit numbers as groups of tens and some ones (Clements et al., 1997; Reynolds and Wheatley, 1996). Because the concept of unit underlies core ideas in number and in geometry and measurement, it has been recommended as a central focus for early childhood mathematics education (Sophian, 2007).

### *Decomposing and Composing*

Decomposing and composing are used throughout mathematics at every level and in all topics. In the realm of numbers and operations (addition, subtraction, multiplication, and division), composing and decomposing are used in recognizing the number of objects in a collection, in the meaning of the operations themselves, and in the place-value system. Children can sometimes quickly determine the number of objects in a small collection by viewing the collection as composed of two immediately recognizable collections, such as seeing four counters as composed of a set of three counters and another counter. Composing and decomposing are the basis for the operations of addition and subtraction and later for the operations of multiplication and division. Some key steps toward developing proficiency with arithmetic involve decomposing and composing. Children must be able to decompose numbers from 1 to 10 into all possible pairs and to recognize numbers from 11 to 19 as composed of a ten and some ones. The base 10 place-value system relies on repeated bundling in groups of ten. Proficiency with multidigit addition and subtraction requires being able to compose ten ones as one ten and to decompose one ten as ten ones.

In geometry, shapes can be viewed as composed of other shapes, such as viewing a trapezoid as made from three triangles, or viewing a house shape as made from a triangle placed above a square. Children can compose rows of squares to make rectangles (see Figure 2-6). Many 3-D shapes seen in everyday life can be viewed as composed of shapes that are found in sets of building blocks (or at least approximately so). A juice box might look like a rectangular prism with a (sideways) triangular prism on top. Children can compose layers of cubes to make larger cubes and rectangular prisms.

In measurement, units are composed to make larger units and decomposed to make smaller units. Measurement itself requires viewing the attribute to be measured as composed of units. In effect, using a unit of measure to partition a continuous quantity, such as a length or area, into discrete and equal size pieces transforms it into a countable quantity.

### *Relating and Ordering*

Relating and ordering allow one to decide which is more and which is less in various domains: number, length, area. Having children see and discuss relating and ordering across domains can deepen mathematical understanding. By broadening the ways in which things can be compared, children are led to the idea of different measurable attributes. For example, two stacks of blocks might be made from the same number of blocks, but one stack might be taller than the other. Relating is a first step toward measurement, because measurement is a quantified form of relating. A measurement specifies how many of one thing (the unit) it takes to make the other thing (the attribute that is measured). When relating and number are joined via measurement, both realms are extended. On one hand, relating becomes more precise when it becomes measurement, and, on the other hand, numbers extend into fractions and decimals in the context of measurement. For example, a bucket of sand might be filled with 2½ smaller pails of sand.

### *Looking for Patterns and Structures and Organizing Information*

Looking for patterns and structures and organizing information (including classifying) are crucial mathematical processes used frequently in mathematical thinking and problem solving. They also have been viewed as distinct content areas in early childhood mathematics learning. Such pattern "content" usually focuses on repeated patterns, such as *abab* or *abcabc*, that are done with colors, sounds, body movements, and so forth (such as the bead and block patterning examples discussed in the section on unitizing). Such activities are appropriate in early childhood and can help to introduce children to seeing and describing patterns more broadly in mathematics. The patterns *abab*, *abcabc*, and *aabbaabb* can be learned by many young children, and many children in kindergarten can do more complex patterns (Clements and Sarama, 2007). Learning to see the unit in one direction (from left to right or from top to bottom or bottom to top) (*ab* in *abab*, *abc* in *abcabc*) and then repeating it consistently is the core of such repeated pattern learning. Learning to extend a given pattern to other modalities (for example, from color to shape, sounds, and body movements) is an index of abstracting and generalizing the pattern.

Counting involves some especially important patterns that go beyond simple repeating patterns. For example, the pattern of counting is a critical idea in number. The list of counting numbers has an especially important and intricate pattern, which involves a coordinated cycling of the digits 0 through 9 in the ones, tens, hundreds, etc., places (see Box 2-2). Although this intricate pattern will not be fully understood by children until later in elementary school, the foundation for this understanding is laid in early childhood as they identify and use the repeating patterns in the number words to 100.

Organizing information, including classifying, has also been seen as early childhood mathematics content, as children use attribute blocks and other collections of entities in which attributes are systematically varied so that they can sort them in multiple ways. Attribute blocks usually vary in color, shape, size, and sometimes thickness, so that children can sort on any of these dimensions and also describe a given block using multiple terms. For example, in small groups, a teacher may first ask children to sort the blocks on one or two dimensions: "Find all the big blue blocks." As children become more proficient, the teacher adds challenge, such as "Find the small blue thin rectangle." Later on, in preschool and in kindergarten, the teacher may ask children to generate their own descriptions of how groups of blocks are similar and different.

Recognizing patterns and organizing information are part of recognizing structure. At all levels in mathematics, one looks for structure. Some experiences in recognizing structure can be part of a foundation for later algebraic thinking. For example, recognizing that if there were 3 birds and then 2 more birds flew in versus if there were 2 birds at first and then 3 more birds flew in results in the same total number of birds either way is a step toward recognizing the commutative property of addition, that $a + b = b + a$ for all numbers a and b.

Although these content examples of looking for patterns and structures and organizing information are appropriate activities, they form a small part of the mathematics content for early childhood. Similarly, the specific skills in these examples are but a small part of the role that these processes play in mathematics.

## MATHEMATICAL CONNECTIONS

In this section we discuss some of the main connections across content areas of early childhood mathematics and into later mathematics. Mathematics as a whole is a web of interconnected ideas, and the mathematics of early childhood is no exception. Mathematics is also deep, in that every mathematical idea, including those of early childhood, is embedded in long chains of related ideas. As this section shows, the foundational and achievable mathematical ideas discussed in the previous sections are tightly interwoven with each other and with other important ideas that are studied later in mathematics.

### Connections in Structuring Numbers, Shapes, and Space

Throughout mathematics, structure is found and analyzed by composing and decomposing. A group of objects can be joined to form a new composite object. An object can be decomposed to reveal its finer structure. Some of the most important connections in elementary mathematics concern structuring of numbers and space via composition and decomposition. We now discuss several of these connections.

#### *Making Units by Grouping*

Numbers are structured by composition because the decimal place-value system relies on grouping by tens. In the realm of number, 10 individual counters are viewed as forming a single composite unit of 10. A geometric version of this grouping idea occurs when several shapes are put together to form another larger shape, which is then viewed as a unified shape in its own right, such as if the unified shape is seen as a possible substitute for another shape or as able to fill a space in a puzzle.

When children (or adults) make a repeating pattern, they might focus mainly on maintaining a certain order. But repeating patterns can also be viewed as made from a single composite unit that is copied over and over. This is not unlike viewing the counting numbers as a sequence that is structured in groups of 10 (see Figure 2-8).

Repeating patterns and, more generally, making groups of equal size are the basis for multiplication and division. Later in elementary school, when children skip count by fives, by counting 5, 10, 15, 20, . . . to list the multiples of 5, this pattern can be viewed as a growing pattern, but it can also be viewed as counting every fifth entry in a repeating pattern of 5. When children study division with remainders (in around fourth or fifth grade), they may observe a repeating pattern in the remainders. For example, when dividing successive counting numbers by 5, say, the remainders cycle through 0, 1, 2, 3, and 4.

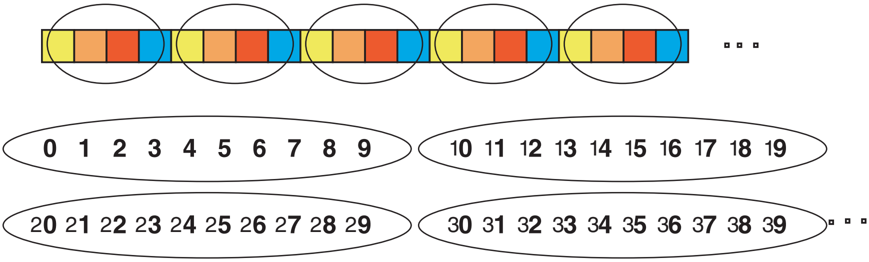


**FIGURE 2-8** A repeating pattern is formed by repeating a unit. In counting, the ones digits form a repeating pattern.

*Groups of Groups: Numbers, Shapes, and 2-D Space*

The compositional structure of the decimal system is more complex than just making groups of 10 from 10 ones, since every 10 groups of 10 are composed into a unit of 100. A geometric version of this group's idea occurs when shapes are put together to form a new, composite shape, and composite shapes are then put together to make another composite shape—a composite of the composite shapes.

An especially important case of geometric structuring as composites of composites occurs when analyzing rectangles and their areas. When considering the area of a rectangle, one views the rectangle as composed of identical square tiles that cover the rectangle without gaps or overlaps. Each square tile has area one square unit. The area of the rectangle (in square units) is the number of squares that cover the rectangle. Although these squares can be counted one by one, to develop and understand the *length* × *width* formula for the area of a rectangle, the squares must be seen as grouped, either into rows or into columns (see Figure 2-6). Each row has the same number of squares in it, and the number of rows in the rectangle is equal to the number of squares in a column (likewise, each column has the same number of squares in it, and the number of columns is the number of squares in a row). Because of this grouping structure, the area of the rectangle is # *rows* × # *in each row* or *length* × *width* (square units). Similarly, the decimal system has a multiplicative structure because 100 is formed (by definition) by making 10 groups of 10, and so 100 = 10 × 10.

The idea of structuring rectangles as arrays of squares can be extended to structuring an entire infinite plane (in the imagination) as an infinite array of squares. This idea of a plane structured by an infinite array is essentially the idea of the Cartesian coordinate plane, in which each point in the plane is described by a pair of numbers that indicate its location relative to two coordinate lines (axes) (see Figure 2-9).

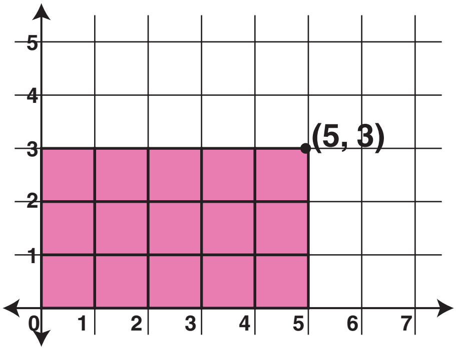


**FIGURE 2-9** The coordinate plane.

*Groups of Groups of Groups: Numbers, Shapes, and 3-D Space*

The compositional structure of the decimal system consists not only of making groups of 10 from 10 ones and groups of 100 from 10 groups of 10, but also groups of 1,000 from 10 groups of 100, so that 1,000 = 10 × 10 × 10. The grouping structure of the decimal system continues in such a way that all successive groupings are obtained by repeatedly grouping by 10. The geometric counterpart of this grouping structure of the decimal system takes one into 3-D space and then higher dimensional space. Just as 2-D rectangles can be structured as 2-D arrays of squares, so, too, 3-D rectangular prisms (box shapes) can be structured as 3-D arrays of cubes. As in the case of rectangles, the multiplicative structure of a 3-D array of cubes explains why one multiplies the three dimensions of length, width, and height of a box to find its volume. Box shapes can be built as layers of identical cubes, as in Figure 2-12, and each layer can be viewed as groups of rows, so a box built from cubes can be viewed as a group of a group of cubes in the same way that 1,000 is 10 groups of 10 groups of 10.

When one extends the array structure of rectangular prisms to all of 3-D space, one gets essentially the idea of coordinate space, in which the location of each point in space is described by a triple of numbers that indicate its location relative to three coordinate lines.

*Motion, Decomposing and Composing, Symmetry, and Properties of Arithmetic*

The properties (or laws) of arithmetic are the fundamental structural properties of addition and multiplication from which all of arithmetic is derived. These properties include the commutative properties of addition and of multiplication, the associative properties of addition and multiplication, and the distributive property of multiplication over addition. The commutative properties of addition and multiplication state that

$$A + B = B + A \text{ for all numbers A, B}$$

$$A \times B = B \times A \text{ for all numbers A, B.}$$

The associative properties of addition and multiplication state that

$$A + (B + C) = (A + B) + C \text{ for all numbers A, B, C}$$

$$A \times (B \times C) = (A \times B) \times C \text{ for all numbers A, B, C.}$$

The distributive property states that

$$A \times (B + C) = A \times B + A \times C \text{ for all numbers A, B, C.}$$

Each property can be illustrated by moving and reorganizing objects, sometimes also by decomposing and recomposing a grouping, and sometimes even in terms of symmetry.

The report *Adding It Up: Helping Children Learn Mathematics* has a good discussion and an illustration of the commutative and associative properties of addition, the commutative and associative properties of multiplication, and the distributive property (National Research Council, 2001, Chapter 3 and Box 3-1). The commutative property of addition is illustrated by switching the order in which two sets are shown. The commutative property is especially useful in conjunction with counting on strategies for solving addition problems (see Chapter 5 for further discussion of children's problem-solving strategies for addition and subtraction). For example, instead of counting on 6 from 2 to calculate 2 + 6, a child can switch the problem to 6 + 2 and count on 2 from 6. The associative property involves starting with three separate sets, two of which are close together, separating the two that are close together, and moving one of those sets to reassociate with the other set. The associative property of addition is used in make-a-ten methods, when one number is decomposed so that one of the pieces can be recomposed with another number to make a group of 10.

Early experiences with properties of addition then extend to multiplication in third and fourth grade. The commutative and associative properties of multiplication and the distributive property are essential to understanding relationships among basic multiplication facts and to understanding multidigit multiplication and division. For example, knowing that $3 \times 5 = 5 \times 3$ and that $7 \times 8$ can be obtained by adding $5 \times 8$ and $2 \times 8$ lightens the load in learning the multiplication tables. The commutative property of multiplication is illustrated by decomposing a rectangular array in two different ways: by the rows or by the columns (as shown in Figure 2-6) or in terms of a rotation (see National Research Council, 2001, Box 3-1). The associative property of multiplication can be illustrated by decomposing a 3-D array (or box shape built of blocks) in different ways (one way is shown in Figure 2-7). The distributive property is illustrated by viewing objects as grouped in two different ways (see National Research Council, 2001, Box 3-1).

The properties of multiplication can be illustrated with arrays and rectangles, and they are also visible in the multiplication tables, which contain many relationships and have important structure. One structural aspect of the multiplication tables is their diagonal symmetry. This diagonal symmetry corresponds with the commutative property of multiplication, namely that $a \times b = b \times a$ for all numbers a and b. Recognizing this symmetry allows children to learn multiplication facts more efficiently. In other words, once they know the upper right-hand triangular portion of the multiplication tables in around third grade, they can fill in the rest of the table by reflecting across the diagonal (see Figure 2-10).

Patterns associated with horizontal or vertical shifts (slides) can also be seen in the multiplication tables. For example, the entries in two columns are related by the column that is associated with the amount of shift between the columns (see Figure 2-10). This structural relationship corresponds with the distributive property.

## Connections in Measurement and Number: Fractions

Once children encounter measurement situations, the possibility of fractions arises naturally. Fractions can be shown well in the context of

| × | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

The diagonal symmetry in the multiplication table is due to the commutative property.

shifting over by 2 adds the 2× column

| × | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| 3 | 3 | 6 | 9 | 12 | 15 | 18 | 21 | 24 | 27 | 30 |
| 4 | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 36 | 40 |
| 5 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
| 6 | 6 | 12 | 18 | 24 | 30 | 36 | 42 | 48 | 54 | 60 |
| 7 | 7 | 14 | 21 | 28 | 35 | 42 | 49 | 56 | 63 | 70 |
| 8 | 8 | 16 | 24 | 32 | 40 | 48 | 56 | 64 | 72 | 80 |
| 9 | 9 | 18 | 27 | 36 | 45 | 54 | 63 | 72 | 81 | 90 |
| 10 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 |

The relationship among the columns is due to the distributive property, e.g., $6 \times 7 = 6 \times 5 + 6 \times 2$.

**FIGURE 2-10** Symmetry and relationships in the multiplication table.

length and on number lines (in around second or third grade). A number line is much like an infinitely long ruler, so number lines can be viewed as unifying measurement and number in a one-dimensional space. A number on a number line can be thought of as representing the length from 0 to the number (see Figure 2-11).

Because of the close connection between number lines and length, number lines are difficult for children below about second grade. In contrast, the number paths on most number board games used for preschoolers are a count model, not a number line. There is a path of squares, circles, or rocks, each has a number on it, and players move along this path by counting the squares or other objects or saying the number on them as they move. These are appropriate for younger children because they can support their knowledge of counting, cardinality, comparing, and number symbols.

In measurement, there is an important relationship between the size of a unit and the number of units it takes to make a given, fixed quantity. For example, if the triangle in Figure 2-5 is designated to have 1 unit of area, then the hexagon has an area of 6 units. But if one picks a new unit of area, such as designating the area of the rhombus in Figure 2-5 to be 1 unit, which is twice the size of the triangle, then the hexagon has an area of only 3 units.

Later in elementary school (in around second grade), children see this inverse relationship between the size of a unit of measurement and the number of units it takes to make a given quantity reflected in the inverse relationship between the ordering of the counting numbers and the ordering of the unit fractions (see Figure 2-12).

### Connections in Data Analysis, Number, and Measurement

To use data to answer (or address) a question, one must analyze the data, which often involves classifying the data into different categories,

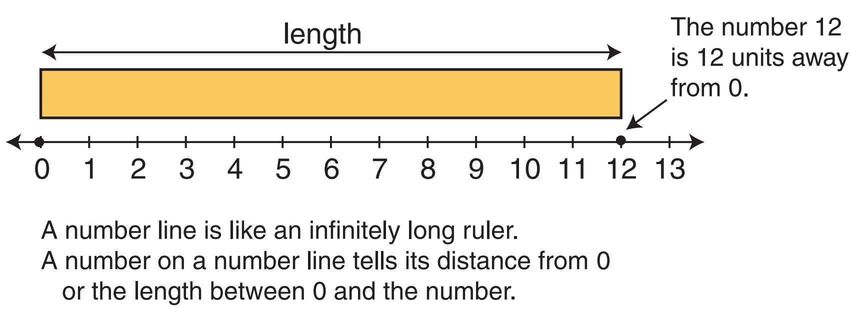


**FIGURE 2-11** Number lines relate numbers to lengths.

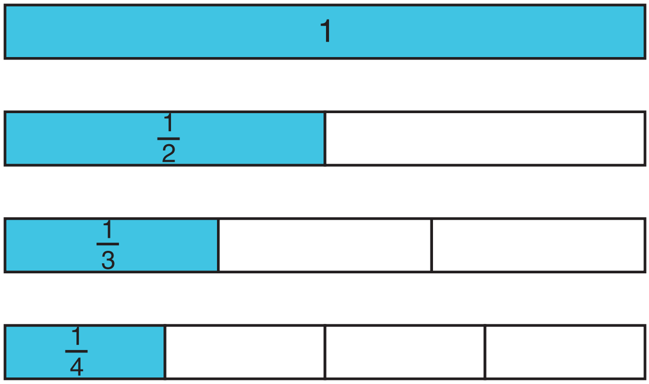


**FIGURE 2-12** 1 > ½ > ⅓ > ¼.

displaying the categorized data graphically, and describing or comparing the categories. Because the process of describing or comparing categories usually involves number or measurement, number and measurement are central to data analysis, and data analysis provides a context to which number and measurement can be applied.

The collection of data should ideally start with a question of interest to children. For example, children in a class might be interested in how everyone got to school in the morning and might wonder what way was most popular. To answer this question, children might divide themselves into different groups according to how they got to school in the morning (by bus, by car, by walking, or by bike). The children could then make "real graphs" (graphs made of real objects) either by lining up in their categories or by each placing a small toy or token to represent a bus, a car, a pair of shoes, or a bike into predrawn squares, as shown on the left in Figure 2-13 (the predrawn squares ensure that each object occupies the same amount of space in the graph). Instead of a real graph, children could display the data somewhat more abstractly in a pictograph by lining up sticky notes in categories, as on the right in the figure. Each child places a sticky note in the category for how the child got to school.

In general, pictographs use small, identical pictures to represent data. In this case, each sticky note stands for a single piece of data and functions as a small picture in a pictograph. Children can then use these real graphs or pictographs to answer such questions as "How many children rode a bus to get to school today?" or "Did more children ride in a car or walk to school today?" or even "If it were raining today, how do you think the graph might be different?" Data displays that are used in posing and answering such quantitative questions serve a purpose and help children mathematize their daily experiences. In contrast, data displays that are only

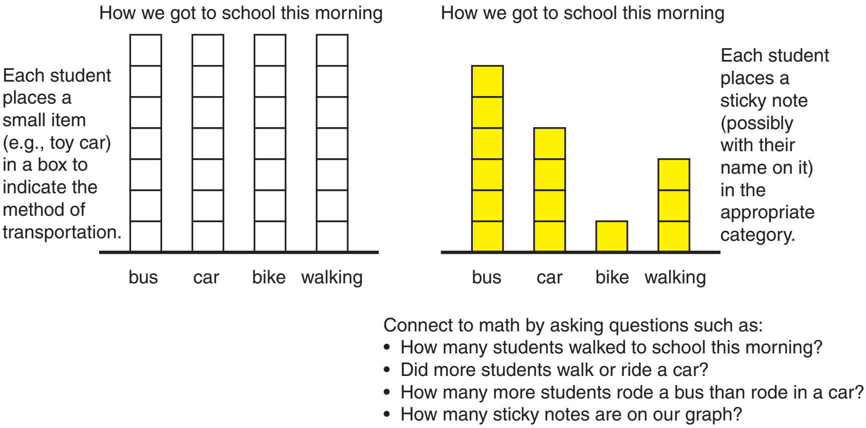


**FIGURE 2-13** A template for a "real graph" and a pictograph made with sticky notes.

made but not discussed are not likely to help children develop or extend their mathematical thinking.

In around second or third grade, once children have worked with linear measurement, they can begin to work with bar graphs. One can think of bar graphs as arising from pictographs by fusing the separated entries in a pictograph to make the bars in a bar graph. In this way, the discrete counting of separate entries in a pictograph gives way to the length measurement of a bar in a bar graph.

In third grade or so, once children have begun to skip count and to multiply, the entries in a pictograph can be used to represent more than one single piece of data. For example, each picture might represent 2 pieces of data or 10 pieces of data.

## SUMMARY

This chapter describes the foundational and achievable mathematics content for young children. The focus of this chapter is on the mathematical ideas themselves rather than on the teaching or learning of these ideas. These mathematical ideas are often taken for granted by adults, but they are surprisingly deep and complex. There are two fundamental areas of mathematics for young children: (1) number and (2) geometry and measurement as identified in NCTM's Curriculum Focal Points and outlined by this committee. There are also important mathematical reasoning processes that children must engage in. This chapter also describes some of the most important connections of the mathematics for young children to later mathematics.

In the area of number, a fundamental idea is the connection between the counting numbers as a list and for describing how many objects are in a set. We can represent arbitrarily large counting numbers in an efficient, systematic way by means of the remarkable decimal system (base 10). We can use numbers to compare quantities without matching the quantities directly. The operations of addition and subtraction allow us to describe how amounts are related before and after combining or taking away, how parts and totals are related, and to say precisely how two amounts compare.

In the area of geometry and measurement, a fundamental idea is that geometric shapes have different parts and aspects that can be described, and they can be composed and decomposed. To measure the size of something, one first selects a specific measurable attribute of the thing, and then views the thing as composed of some number of units. The shapes of geometry can be viewed as idealized and simplified approximations of objects in the world. Space has structure that derives from movement through space and from relative location within space. An important way to think about the structure of 2-D and 3-D space comes from viewing rectangles as composed of rows and columns of squares and viewing box shapes as composed of layers of rows and columns of cubes.

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# 3

# Cognitive Foundations for Early Mathematics Learning

Over the past two decades, a quiet revolution in developmental psychology and related fields has demonstrated that children have skills and concepts relevant to mathematics learning that are present early in life, and that most children enter school with a wealth of knowledge and cognitive skills that can provide a foundation for mathematics learning. At the same time, these foundational skills are not enough—children need rich mathematical interactions, both at home and at school in order to be well prepared for the challenges they will meet in elementary school and beyond. (Chapter 4 discusses supporting children's mathematics at home, and Chapters 5 and 6 discuss children's mathematical development and related instructional practices.) The knowledge and interest that children show about number and shape and other mathematics topics provide an important opportunity for parents and preschool teachers to help them develop their understanding of mathematics (e.g., Gelman, 1980; Saxe, Guberman, and Gearhart, 1987; Seo and Ginsburg, 2004).

In this chapter we review research on the mathematical development of infants and young children to characterize both the resources that most children bring to school and the limitations of preschoolers' understanding of mathematics. Because this literature is vast, it is not possible to do it justice in a single chapter. However, we attempt to provide an overview of key issues and research findings relevant to early childhood education settings. These include

- What is the nature of early universal starting points? These are generally thought to provide an important foundation for subsequent mathematical development (e.g., Barth et al., 2005; Butterworth, 2005; Dehaene, 1997; but see Holloway and Ansari, 2008, and Rips, Bloomfield, and Asmuth, 2008, for contrasting views). We examine two domains that are foundational to mathematics in early childhood: (1) number, including operations, and (2) spatial thinking, geometry, and measurement.

- What are some of the important developmental changes in mathematical understandings in these domains that occur during the preschool years?
- What is the relation of mathematical development to more general aspects of development needed for learning mathematics, such as the ability to regulate one's behavior and attention?

## EVIDENCE FOR EARLY UNDERSTANDING OF NUMBER

### Preverbal Number Knowledge

Delineating the starting points of knowledge in important domains is a major goal in developmental psychology. These starting points are of theoretical importance, as they constrain models of development. They are also of practical importance, as a basic tenet of instruction is that teaching that makes contact with the knowledge children have already acquired is likely to be most effective (e.g., Clements et al., 1999). Thus, it is not surprising that infant researchers have been actively mapping out the beginnings of preverbal number knowledge—knowledge that appears to be shared by humans from differing cultural backgrounds as well as with other species, and thus part of their evolutionary endowment (e.g., Boysen and Berntson, 1989; Brannon and Terrace, 1998, 2000; Brannon et al., 2001; Cantlon and Brannon, 2006; Dehaene, 1997; Dehaene, Dehaene-Lambertz, and Cohen, 1998, Meck and Church, 1983). A large body of research has examined a set of numerical skills, including infants' ability to discriminate between different set sizes, their ability to recognize numerical relationships, and their ability to understand addition and subtraction transformations. The study of numerical knowledge in infants represents a major departure from previously held views, which were heavily influenced by Piaget's (1941/1965) number conservation findings and stage theory. These older findings showed that children do not conserve number in the face of spatial transformations until school age, and they led many to believe that before this age children lack the ability to form concepts of number (see Mix, Huttenlocher, and Levine, 2002, for a review). Although Piaget recognized that children acquire some mathematically relevant skills at earlier ages, success on the conservation task was widely regarded as the sine qua non of numerical understanding.

Beginning in the 1960s and 1970s, researchers began to actively examine early numerical competencies, which led to a revised understanding of children's numerical competence. This research identified a great deal of competence in preschool children, including counting and matching strategies that children use on Piaget's conservation of number task (see the discussion in Chapter 5).

As we detail, infant and toddler studies have largely focused on the natural numbers (also called counting numbers). However, they have also examined representations of fractional amounts and proportional relations as well as geometric relationships, shape categories, and measurement. Moreover, although there is some disagreement in the field about the interpretation of the findings of infant and toddler studies as a whole, these findings are generally viewed as showing strong starting points for the learning of verbal and symbolic mathematical skills.

### *Infants' Sensitivity to Small Set Size*

Infant studies typically use habituation paradigms to examine whether infants can discriminate between small sets of objects, either static or moving (Antell and Keating, 1983; Starkey and Cooper, 1980; Strauss and Curtis, 1981; Van Loosbroek and Smitsman, 1990; Wynn, Bloom, and Chiang, 2002). In a typical habituation study, infants are repeatedly shown sets containing the same number of objects (e.g., 2) until they become bored and their looking time decreases to a specified criterion. The infant is then shown a different set size of objects or the same set size, and looking times are recorded. Longer looking times indicate that the infant recognizes that the new display is different from an earlier display. Results show that infants (ranging in age from 1 day old to several months old) can discriminate a set of two objects from a set of three objects, yet they are unable to discriminate four objects from six objects, even though the same 3:2 ratio is involved. These findings indicate that infants' ability to discriminate small set sizes is limited by number rather than by ratio. Huttenlocher, Jordan, and Levine (1994) suggest that infants' ability to discriminate small sets (2 versus 3) could be based on an approximate rather than on an exact sense of number.

Several studies suggest that the early quantitative sensitivity displayed by infants for small set sizes is actually based on their sensitivity to amount (surface area or contour length) which covaries with numerosity, rather than on number per se (Clearfield and Mix, 1999, 2001). That is, unless these variables are carefully controlled, the more items there are, the greater the amount of stuff there is. In studies that independently vary number and amount, Clearfield and Mix (1999, 2001) found that infants ages 6 to 8 months detected a change in amount (contour length or area) but not a change in number. Thus, if they were habituated to a set of two items, they did not dishabituate to a set of three items if that set was equivalent to the original set in area or contour length.

However, recent findings indicate that infants are sensitive to both continuous quantity and to number (Cordes and Brannon, 2008, in press; Kwon et al., 2009). Furthermore, Cordes and Brannon (2008) report that, although 6-month-old infants are sensitive to a two-fold change in number, they are sensitive to a three-fold change only in cumulative area across elements, suggesting that early sensitivity to set size may be more finely tuned than early sensitivity to continuous quantity. Other studies that provide support for early number sensitivity include a study showing that 6-month-old infants can discriminate between small sets of visually presented events (puppet jumps) (e.g., Wynn, 1996). This result is not subject to the alternative explanation of discrimination based on amount rather than number, like the findings involving sets of objects. However, it is possible that even though the rate and duration of the events have been controlled in these studies, infants' discrimination is based on nonnumerical cues, such as rhythm (e.g., Demany, McKenzie, and Vurpillot, 1977; Mix et al., 2002). Indeed, in one study in which the rate of motion was not a reliable cue to numerosity, 6-month-olds did not discriminate old and new numerosities (Clearfield, 2004).

A set size limitation also is seen in the behavior of 10- to 14-month-olds on search tasks (Feigenson and Carey, 2003, 2005; Feigenson, Carey, and Hauser, 2002). For example, in one study 12-month-olds saw crackers placed inside two containers. The toddlers chose the larger hidden quantity for 1 versus 2 and 2 versus 3 crackers, but they failed to do so on 3 versus 4, 2 versus 4, and 3 versus 6 crackers (Feigenson, Carey, and Hauser, 2002). The authors suggest that this failure is due to the set size limitation of the object file system.[1] When cracker size was varied, the toddlers based their search on the total cracker amount rather than on number. Similarly, 12- to 14-month-olds searched longer in a box in which two balls had been hidden after they saw the experimenter remove one ball, than they did in a box in which one ball had been hidden and the experimenter removed one ball (in actuality there were no more balls in either box, as the experimenter surreptitiously removed the remaining ball). They also succeeded on 3 versus 2 balls but failed on 4 versus 2 balls. That is, they did not search longer in a box in which four balls were hidden and they saw two removed than in a box in which they had seen two hidden and two were removed. The failure

[1]The object file system refers to the representation of an object in a set that consists of small numbers, the objects are in a 1-to-1 correspondence with each mental symbol, and there is no summary representation of set size (e.g., three items are represented as "this," "this," "this" rather than "a set of three things") (Carey, 2004).

on 2 versus 4, which has the same ratio as the 2 versus 1 problem, suggests that they were using the object file system rather than the analog magnitude system, which is second system that represents large set sizes (4 or more) approximately. Furthermore, in this study, the toddlers based their search on the number of objects they saw hidden rather than on the total object volume. Thus, at least by 12 months of age, it appears that children can represent the number of objects in sets up to three (Feigenson and Carey, 2003). A subsequent study shows that this set size limit can be extended to four if spatiotemporal cues allow the toddlers to represent the sets as two sets of two (Feigenson and Halberda, 2004).

### *Infants' Sensitivity to Large Set Size*

Recent studies have shown that infants can approximate the number of items in large sets of visual objects (e.g., Brannon, 2002; Brannon, Abbott, and Lutz, 2004; Xu, 2003; Xu and Spelke, 2000; Xu, Spelke, and Goddard, 2005), events (puppet jumps) (Wood and Spelke, 2005), and auditory sets (Lipton and Spelke, 2003) that are well beyond the range of immediate apprehension of numerosity (*subitizing* range). Consistent with the accumulator model, which refers to a nonverbal counting mechanism that provides approximate numerical representations in the form of analog magnitudes, infants' discrimination of large sets is limited by the ratio of the two sets being compared rather than by set size. Thus, at 6 months of age, when infants are habituated to an array of dots, they dishabituate to a new set as long as the ratio between two sets is at least 2:1. By 10 months of age, infants are able to discriminate visual and auditory sets that differ by a 2:3 ratio but not by a 4:5 ratio (Lipton and Spelke, 2003, 2004; Xu and Arriaga, 2007). Importantly, these studies controlled for many continuous variables, suggesting that the discriminations were based on number rather than amount (e.g., Brannon, Abbott, and Lutz, 2004; Cordes and Brannon, 2008; Xu, 2003; Xu and Spelke, 2000).

### *Do Infants Have a Concept of Number?*

Infants may be able to discriminate between sets of different sizes but have no notion that all sets that have the same numerosity form a category or equivalence class (the mathematical term for such a category). This notion is referred to as the cardinality concept (e.g., the knowledge that three flowers, three jumps, three sounds, and three thoughts are equivalent in number). Number covers such matters as the list of counting numbers (e.g., 1, 2, 3, . . .) and its use in describing how many things are in collections. It also covers the ordinal position (e.g., first, second, third, . . .), the idea of cardinal value (e.g., how many are there?), and the various operations on number (e.g., addition and subtraction). The notion of 1-to-1 correspondences connects the counting numbers to the cardinal value of sets. Another important aspect of number is the way one writes and says them using the base 10 system (see Chapters 2 and 5 for further discussion). Knowledge of number is foundational to children's mathematical development and gradually develops over time, so not all aspects of the number are present during the earliest years.

Several studies (e.g., Starkey, Spelke, and Gelman, 1990; Strauss and Curtis, 1984) examined whether infants understand that small sets that share their numerosity but contain different kinds of entities form a category (e.g., two dogs, two chicks, two jumps, two drumbeats). Starkey and colleagues (1990) examined this question by habituating infants to sets of two or three aerial photographs of different household objects. At test, infants were shown novel photographs that alternated between sets of two and sets of three. Infants dishabituated to the novel set size, suggesting that they considered different sets of two (or three) as similar. Whereas these studies might be regarded as suggesting that infants form numerical equivalence classes over visual sets containing disparate objects, these studies may have tapped infants' sensitivity to continuous amount rather than number, as described above (Clearfield and Mix, 1999, 2001). That is, unless careful controls are put in place, sets with two elements will on average be smaller in amount than sets of three elements (e.g., Clearfield and Mix, 1999, 2001; Mix et al., 2002).

Findings showing that infants consider two objects and two sounds to form a category would not be subject to this criticism and thus could be considered as strong evidence for abstract number categories. In an important study, Starkey, Spelke, and Gelman (1983) tested whether infants have such categories. While the results seemed to indicate that 7-month-olds regarded sets of two (or three) objects and drumbeats as similar, several attempts to replicate these important findings have called them into question (Mix, Levine, and Huttenlocher, 1997; Moore et al., 1987). Thus, whether infants have an abstract concept of number that allows them to group diverse sets that share set size remains an open question. The findings, reviewed below, showing that 3-year-olds have difficulty matching visual and auditory sets on the basis of number, and that this skill is related to knowledge of conventional number words, suggest that the ability to form equivalence classes over sets that contain different kinds of elements may depend on the acquisition of conventional number skills. Kobayashi, Hiraki, Mugitani, and Hasegawa (2004) suggest that the methods used may be too abstract to tap this intermodal knowledge and that when the sounds made are connected to objects, for example, the sound of an object landing on a surface, evidence of abstract number categories may be revealed at younger ages, perhaps even in infants.

### *Infant Sensitivity to Changes in Set Size*

Several studies report that infants track the results of numerically relevant transformations—adding or taking away objects from a set. That is, when an object is added to a set, they expect to see more objects than were previously in the set and when an object is taken away, they expect to see fewer objects than were previously in the set. Wynn (1992a) found that after a set was transformed by the addition or subtraction of an object, 5-month-old infants looked longer at the "impossible" result (e.g., 1 + 1 = 1) than at the "correct" result. However, as for numerical discrimination, subsequent studies suggest that their performance may reflect sensitivity to continuous (cumulative size of objects) amount rather than to numerosity (Feigenson, Carey, and Spelke, 2002). For the problem 1 + 1, infants looked longer at 2, the expected number of objects, when the cumulative size of the two objects was changed than at three, the impossible number of objects, when the cumulative size of the objects was correct—that is, when the cumulative area of the three objects was equivalent to the area that would have resulted from the 1 + 1 addition.

Cohen and Marks (2002) suggested an alternative explanation for Wynn's results. In particular, they suggest that the findings could be attributable to a familiarity preference rather than to an ability to carry out numerical transformations. For the problem 1 + 1 = 2, they point out that infants more often see one object, as there was a single object in the first display of every trial and thus, based on familiarity, look more at 1 (the incorrect answer). A similar argument was made for looking more at 2 for the problem 2 – 1.

Although their findings support this hypothesis, a more recent study by Kobayashi et al. (2004) provides evidence that infants look longer at 1 + 1 = 3 and 1 + 2 = 3 than at 1 + 1 + 2 and 1 + 2 = 3 when the first addend is a visual object and the second addend consists of a tone(s). This paradigm cannot be explained by the familiarity preference because, for each problem, infants see only one element on the stage.

### *Order Relations*

A few studies have examined infants' sensitivity to numerical order relations (more than, less than). One habituation study showed that 10- and 12-month-olds discriminated equivalent sets (e.g., a set of two followed by another set of two) from nonequivalent sets (e.g., a set of two followed by a set of three) (Cooper, 1984). In another study, Cooper (1984) habituated 10-, 12-, 14-, and 16-month-old infants to sequences that were nonequivalent. In the "less than" condition, the first display in the pair was always less than the second (e.g., infants were shown two objects followed by three objects). The reverse order was shown for the "greater than" condition. At test, the 14- and 16-month-olds showed more interest in the opposite relation than the one that was shown, suggesting that they represented the less than and greater than relations, whereas 10- and 12-month-olds did not. However, Brannon (2002) presents evidence that infants are sensitive to numerical order relations by 11 months of age.

### *Summary*

The results of infant studies using small set sizes show that, very early in life, infants have a limited ability to discriminate sets of different sizes from each other (e.g., 2 versus 3 but not 4 versus 6). The set size limitation has been interpreted as reflecting one of two core systems for number—the object file system. They also expect the appropriate result from small number addition and subtraction transformations (e.g., $1 + 1 = 2$ and $2 - 1 = 1$), at least when amount covaries with number. Somewhat later, by 10 months of age, infants discriminate equivalent from nonequivalent sets, and by 14 months of age they discriminate greater than from less than relationships. Because many of these studies did not control for continuous variables that covary with number (i.e., contour length and surface area), the basis of infant discriminations is debated. However, recent studies indicate that infants are sensitive to both number of objects in small sets and to continuous variables, and they may be more sensitive to number than to cumulative surface area. Infant studies also have examined sensitivity to approximate number by using larger sets of items (e.g., 8 versus 16). These studies have found that infants can discriminate between sets with a 2:1 ratio by age 6 months and between those with 2:3 ratios by age 9 months as long as all set sizes involved are greater than or equal to 4, that is, 6-month-olds fail to discriminate 2 versus 4.

We also note that infants' early knowledge of number is largely implicit and has important limitations that are discussed below. There were no number words involved in any of the studies described above. This means that learning the number words and relating them to sets of objects is a major new kind of learning done by toddlers and preschoolers at home and in care and education centers. This learning powerfully extends numerical knowledge, and children who acquire this knowledge at earlier ages are provided with a distinct advantage.

## Mental Number Representations in Preschool Children

Just as much of the infant research has a focus on theorizing about and researching the nature of infant representations of number, so, too, does some research on toddlers and preschool children. The goal is to understand how and when young children represent small and larger numbers. To do this, special tasks are used that involve hidden objects, so that children must use mental representations to solve the task. Sometimes objects are shown initially and are then hidden, and sometimes objects are never shown and numbers are given in words. These tasks are quite different from situations in which young children ordinarily learn about numbers in the home or in care and educational centers, and they can do tasks in home and naturalistic settings considerably earlier than they can solve these laboratory tasks (e.g., Mix, 2002). In home and in care and educational settings, numbers are presented with objects (things, fingers), and children and adults may see, or count, or match, or move the objects. The objects do not disappear, and they are not hidden. Children's learning under these ordinary conditions is described in Chapter 5. Here we continue to focus on theoretical issues of representations of numbers.

*Small Set Sizes*

Like infants, 2- to 3-year-olds show more advanced knowledge of number than would be predicted by previous views. As noted previously, conservation of number was considered to be a hallmark of number development (Piaget, 1941/1965). However, Gelman's (1972) "magic experiment" showed that much younger children could conserve number if the spatial transformation was less salient and much smaller set sizes were used. In this study, 3- to 6-year-olds were told that either a set of two mice or a set of three mice was the "winner." The two sets were then covered and moved around. After children learned to choose the winner, the experimenter altered the winner set, either by changing the spatial arrangement of the mice or by adding or subtracting a mouse. Even the 3-year-olds were correct in recognizing that the rearrangement maintained the status of the winner, whereas the addition and subtraction transformations did not.

Huttenlocher, Jordan, and Levine (1994) examined the emergence of exact number representation in toddlers. They posited that mental models representing critical mathematical features—the number of items in the set and the nature of the transformation—were needed to exactly represent the results of a calculation. Similarly, Klein and Bisanz (2000) suggest that young children's success in solving nonverbal calculations depends on their ability to hold and manipulate quantitative representations in working memory as well as on their understanding of number transformations.

Huttenlocher, Jordan, and Levine (1994) gave children ages 2 to 4 a numerosity matching task and a calculation task with objects (called nonverbal; see Box 3-1). On the matching task, children were shown a set of disks that was subsequently hidden under a box. They were then asked to lay out the same number of disks. On the calculation task, children were

**BOX 3-1**
**Clarifying Experimental Misnomers**

Researchers have used tasks in which two conditions vary in two important ways, such as in Huttenlocher, Newcombe, and Sandberg (1994). In one condition, children are first shown objects, and then the objects are hidden. Number words are not used in this condition. In the other condition, children never see objects but must imagine or generate them (e.g., by raising a certain number of fingers). Here the numbers involved are conveyed by using number words, either as a story problem or just as words (e.g., "2 and 1 make what?"). In their reports, researchers call the first condition *nonverbal* and the second condition *verbal*. But these labels are a bit misleading, because they sound as if nonverbal and verbal are describing the children's solution methods. In this report we use language that mentions both aspects that were varied: with objects (called nonverbal) and without objects (called verbal).

shown a set of disks that was subsequently covered. Following this, items were either added or taken away from the original set. The child's task was to indicate the total number of disks that were hidden by laying out the same number of disks ("Make yours like mine").

On both the matching and transformation tasks, performance increased gradually with age. Children were first successful with problems involving low numerosities, such as 1 and 2, gradually extending their success to problems involving higher numerosities. Importantly, when children responded incorrectly, their responses were not random, but rather were approximately correct. Approximately correct responses were seen in children as young as age 2, the youngest age group included in the study. On the basis of these findings, Huttenlocher, Jordan, and Levine (1994) argue that representations of small set sizes begin as approximate representations and become more exact as children develop the ability to create a mental model. Exactness develops further and extends to larger set sizes when children map their nonverbal number representations onto number words.

Toddlers' performance on numerosity matching tasks indicates that, as they get older, they get better at representing quantity abstractly. This achievement appears to be related to the acquisition of number words (Mix, 2008). Mix showed that preschoolers' ability to discriminate numerosities is highly dependent on the similarity of the objects in the sets. Thus, 3-year-olds could match the numerosities of sets consisting of pictures of black dots to highly similar black disks. Between ages 3 and 5, children were able to match the numerosities of increasingly dissimilar sets (e.g., black dots to pasta shells and black dots to sequential black disks at age 3½; black dots to heterogeneous sets of objects at age 4).

The abstractness of preschoolers' numerical representations was also assessed in a study (Mix, Huttenlocher, and Levine, 1996) examining their ability to make numerical matches between auditory and visual sets, an ability that Starkey, Spelke, and Gelman (1990) had attributed to infants. The researchers presented 3- and 4-year-olds with a set of two or three claps and were asked to point to the visual array that corresponded to the number of claps. The 3-year-olds performed at chance on this task, but by age 4, the children performed significantly above chance. In contrast, both age groups performed above chance on a control task that involved matching sets of disks to pictures of dots. Another study assessed the effect of the heterogeneity of sets on the ability of 3- to 5-year-olds to make numerical matches and order judgments. The results replicate Mix's (1999b) finding that the heterogeneity of sets decreases children's ability to make equivalence matches. However, heterogeneity versus homogeneity of sets did not affect their ability to make order judgments (i.e., to judge which of two sets is smaller) (Cantlon et al., 2007).

Mix (2002) has also examined the emergence of numerical knowledge through a diary study of her son, Spencer. In this study, she found indications of earlier knowledge than the experiments described above might indicate. Spencer was able to go into another room and get exactly two dog biscuits for his two dogs at 21 months of age, long before children succeed on the homogeneous or heterogeneous set matching tasks described above. Indeed, Spencer himself had failed to perform above chance on these laboratory tasks. Thus, it appears that early knowledge of numerical equivalence may arise piecemeal, and first in highly contextualized situations. For Spencer, his earliest numerical equivalence matches occurred in social situations (e.g., biscuits for dogs, sticks for guests). Whether this is a general pattern or whether there are wide individual differences in such behaviors is an open question (also see Mix, Sandhofer, and Baroody, 2005, for a review).

Levine, Jordan, and Huttenlocher (1992) compared the ability of preschool children to carry out calculations involving numerosities of up to six with objects (called nonverbal) and without objects (called verbal) (the former calculations were similar to those described above in the Huttenlocher, Jordan, and Levine, 1994 study). The calculations without objects (called verbal) were given in the form of story problems ("Ellen has 2 marbles and her father gives her 1 more. How many marbles does she have altogether?") and in the form of number combinations (e.g., "How much is 2 and 1?"). Children ages 4 to 5½ performed significantly higher on the calculation task when they could see objects and transformations than on the calculation tasks when they could not see objects or transformations. This was true for both addition and subtraction calculations. This difference in performance between nonverbal and verbal calculations was particularly marked for children from low socioeconomic backgrounds. Children from low socioeconomic backgrounds appear to have more difficulty accessing the numerical meaning of the number words (Jordan et al., 2006), which may be related to their exposure to cultural learning tools (e.g., number symbols, number words) (see Chapter 4 for further discussion).

*Large Set Sizes*

To investigate how preschoolers carry out approximate calculations with large numbers, 5-year-olds were presented with comparison and addition problems shown on a computer screen (Barth et al., 2005, 2006). On comparison problems, they were shown a set of blue dots (set sizes ranged from 10 to 58) that were then covered up. Next, they were shown a set of red dots and were asked whether there are more blue dots or red dots. On addition problems, they were shown a set of blue dots that were then covered up. They were then shown another set of blue dots that moved behind the same occluder. Finally, they were shown a set of red dots and were asked whether there were more blue dots or red dots. Subsequent experiments showed that children performed as well when the red dots were presented as a sequence of auditory tones as when they were presented visually. In each condition, performance was above chance and equivalent on comparison and addition problems, decreasing as the ratio of the red to blue dots approached 1. The ratio dependence of performance indicates that children are using the analogue magnitude system. This system differs from the exact representations of larger numbers that are built up by working with objects arranged in groups of tens and ones (see Chapter 5).

*Summary*

Toddlers and preschoolers continue to build on the two representational systems identified for infants: the object file system, which is limited to sets of three or less and provides a representation for each element in a set but no summary representation of set size, and the analogue magnitude system, which provides an approximate summary representation of set size but no representation of the individual elements in a set and no way to differentiate between adjacent set sizes, such as 10 and 11 (Carey, 2004; Feigenson, Dehaene, and Spelke, 2004; Spelke and Kinzler, 2007). Existing research also shows that children's early numerical knowledge is highly context-dependent, often depending on the presence of objects or fingers to represent sets. Although their numerical abilities are limited, young children have considerably more numerical competence than was inferred from Piaget's research. They are even building early informal knowledge in many other mathematics areas besides representation of the counting numbers (see sections below). However, the path from informal to formal knowledge is not necessarily a smooth one.

Impressive growth of numerical competence from age 2 to age 6 is stimulated by children's learning of important cultural numerical tools: spoken number words, written number symbols, and cultural solution methods, like counting and matching. As shown by Wynn's (1990, 1992b) research, the acquisition of the understanding of the cardinal meanings of number words is a protracted process. In a longitudinal study, Wynn found that it takes about a year for a child to move from succeeding in giving a set of "one" when requested to do so to being able to give the appropriate number for all numbers in his or her count list. The acquisition of such symbolic knowledge is important in promoting the abstractness of number concepts, that is, the concept of cardinality (that all sets of a given numerosity form an equivalence class). It is also important in promoting the exactness of number representations and the understanding of numerical relations, as only children who have acquired this knowledge understand that adding one item to a set means moving to the very next number in the count list (Sarnecka and Carey, 2008). The research concerning these cultural learning achievements is summarized in Chapter 5 in identifying foundational and achievable goals for teaching and learning. It is discussed in Chapter 4 as a major source of socioeconomic differences, connected to differential exposure to talk about number at home and at preschool.

## DEVELOPMENT OF SPATIAL THINKING AND GEOMETRY

Spatial thinking, like numerical thinking, is a fundamental component of mathematics that has its roots in foundational skills that emerge early in life. Spatial thinking is critical to a variety of mathematical topics, including geometry, measurement, and part-whole relations (e.g., Ansari et al., 2003; Fennema and Sherman, 1977, 1978; Guay and McDaniel, 1977; Lean and Clements, 1981; Skolnick, Langbort, and Day, 1982; see Chapter 6, this volume). Spatial thinking has been found to be a significant predictor of achievement in mathematics and science, even controlling for overall verbal and mathematical skill (e.g., Clements and Sarama, 2007; Hedges and Chung, in preparation; Lean and Clements, 1981; Shea, Lubinski, and Benbow, 2001; Stewart, Leeson, and Wright, 1997; Wheatley, 1990). One reason that spatial thinking is predictive of mathematics and science achievement is because it provides a way to conceptualize relationships in a problem prior to solving it (Clements and Sarama, 2007).

The mental functions encompassed by spatial thinking include categorizing shapes and objects and encoding the categorical and metric relations among shapes and objects. Spatial thinking is also crucial in representing object transformations and the outcomes of these transformations (e.g., rotation, translation, magnification, and folding) as well as perspective changes that occur as one moves to new locations. Spatial thinking is involved in navigating in the environment to reach goal locations and to find one's way back to one's starting point. Use of spatial symbolic systems, including language, maps, graphs, and diagrams, and spatial tools, such as measuring devices, extend and refine the ability to think spatially.

As is the case for the development of number knowledge, recent research has shown strong starting points for spatial thinking. In contrast to Piaget's view, which is in opposition to the gradual unfolding of spatial skills over the course of development, recent evidence shows that infants are able to code spatial information about objects, shapes, distances, locations, and spatial relations. This early emergence of spatial skills is consistent with an evolutionary perspective that emphasizes the adaptive importance of navigation for all mobile species (e.g., Newcombe and Huttenlocher, 2000, 2006; Wang and Spelke, 2002). That said, humans are unique in that their spatial skills are extended through symbolic systems, such as spatial language, measurement units, maps, graphs, and diagrams. Thus, it is not surprising that the trajectory of children's spatial development depends heavily on their spatially relevant experiences, including those involving spatial language and spatial activities, such as block building, puzzle play, and experience with certain video games.

## Starting Points in Infancy

Even young infants are able to segment their complex visual environments into objects that have stable shapes, using such principles as cohesion, boundedness, and rigidity (e.g., see Spelke, 1990). Infants also perceive the similarities between three-dimensional objects and photographs of these objects (DeLoache, Strauss, and Maynard, 1979). In habituation studies, infants show sensitivity to shape similarities across exemplars (e.g., Bomba and Siqueland, 1983). In addition, they are able to recognize invariant aspects of a shape shown from different angles of view (e.g., Slater and Morison, 1985).

Infants are also capable of forming categories of spatial relations—a claim that is widely supported; however, different views exist regarding the developmental sequence for children's understanding of space categories (Quinn, 1994, 2004; van Hiele, 1986). As stated by Bruner, Goodnow, and Austin (1956), categorization entails treating instances that are discriminable as the same. Using this criterion, Quinn showed that 3-month-old infants are sensitive to the categories above versus below (e.g., Quinn, 1994) and left versus right (e.g., Quinn, 2004). Both of these categories involve the relationship of an object and a single referent object (e.g., a horizontal or vertical bar). However, infants are not able to code the relationship between an object and a diagonal bar, showing that certain kinds of spatial relationship are privileged over others. Somewhat later, at 6 to 7 months, they are sensitive to the category of *between* relationships (Quinn et al., 1999). This spatial category is more complex than above/below or left/right, as it involves the relation of an object to two referent objects (e.g., two bars). At around this same age, infants form other, rather subtle spatial concepts. For example, they are sensitive to the functional difference between a container and a cylindrical object that does not have a bottom, even though these objects are highly similar visually (Aguiar and Baillargeon, 1998; Baillargeon, 1995).

Infants and toddlers also have impressive abilities to locate objects in space using both landmarks and geometric cues. Infants as young as 5 months are also able to use enclosed spaces that define a shape (e.g., walls of a sandbox) to code the location of objects (Newcombe, Huttenlocher, and Learmonth, 1999). By 12 months, children code distance and direction and use this information to search for objects hidden in displays (Bushnell et al., 1995). By 16 to 17 months, they are able to use the rectangular shape of an enclosure as well as landmark cues (both adjacent to the hiding location and at a distance from it) to search for objects (Hermer and Spelke, 1994, 1996; Huttenlocher, Newcombe, and Sandberg, 1994; Learmonth, Newcombe, and Huttenlocher, 2001).

### Mental Transformation of Shapes

Mental rotation (the ability to visualize and manipulate the movement of two-dimensional and three-dimensional objects) and spatial visualization (holding a shape in mind and finding the shape in more complex figures, combining shapes, or matching orientations) are fundamental spatial skills essential for mathematics learning (Linn and Peterson, 1985). Several recent studies have shown that preschool children are able to mentally rotate shapes in the picture plane. In one study, Marmor (1975) showed that children as young as age 5 years are able to mentally rotate visual images in the picture plane to determine whether one image is the same as another. Similarly, Levine et al. (1999) showed that children as young as age 4½ are able to perform above chance on mental transformations involving rotation and translation.

In tasks requiring spatial visualization (e.g., holding an image, such as a block letter, in mind for later comparison to a standard block letter), children between ages 4 and 5 perform poorly unless the visualized image is in the same orientation as the comparison object, whereas children between ages 6 and 10 were not adversely affected by differences in orientation (Smothergill et al., 1975). Furthermore, spatial ability in manipulating orientation at age 7, but not at ages 3 to 5 (Rod-and-Frame Test, Preschool

Embedded Figures Test), predicted spatial visualization abilities much later, at age 18 (Ozer, 1987). This developmental shift in spatial visualization ability is most likely to reflect differences in mental rotation ability and perspective-taking. Thus, when children are better able to mentally manipulate images held in mind (e.g., imagining the letter "F" and mentally rotating it clockwise or counterclockwise), they will be more accurate at determining how these images will appear from various viewpoints.

Similarly, when a child is asked to imagine what an object would look like from another person's perspective, this task is more easily accomplished when the child can mentally imagine the scene and move either themselves or the objects in order to match another person's perspective of the scene. For example, a child is sitting at a desk that has a toy car to the left of a pencil on top of the desk. A teacher is sitting on the other side of the desk, opposite the child, and asks the child to arrange the toy car and the pencil so that they would match what the teacher sees. The task becomes easier if the child can imagine the desk with the two objects and mentally "walk" to the other side of the desk to figure out the answer (pencil on the left, toy car on the right) or can imagine the objects and mentally rotate them so that they are in the 180 degree position.

As mental rotation and perspective-taking ability increase over time, such factors as changes in orientation become less problematic in tasks in which one must match something displayed in a different orientation than the visualized object. The fact that early spatial visualization measures during preschool were not correlated with later spatial visualization may suggest that the foundations for spatial abilities, such as mental rotation and perspective-taking, are molded in these formative years and are highly susceptible to change, more so than during later elementary education. This has important implications for findings that display gender differences in spatial performance on such tasks as mental rotation by age 4½ (Levine et al., 1999) and socioeconomic differences by second grade (Levine et al., 2005). That is, these differences in spatial ability may largely be the result of experiential differences during early childhood, and the preschool period may be an especially important time to begin addressing these issues through educational programs that foster spatial learning.[2]

The early emergence of mental rotation ability may be related to preschoolers' success with map use. Given simple maps, 4-year-olds and a majority of 3-year-olds can locate a hidden object in a sandbox (Huttenlocher, Newcombe, and Vasilyeva, 1999), children ages 3 to 5½

[2]Recent evidence shows a sex difference in mental rotation for 4- and 5-month-old infants that is not attributable to experimental factors (see Moore and Johnson, 2008; Quinn and Liben, 2008). Implications from these studies suggest there may be an advantage in early spatial learning for boys.

can find a hidden toy in an open field (Stea et al., 2004) and children ages 5 and 6 can navigate the hallways of an unfamiliar school (Sandberg and Huttenlocher, 2001). In order to succeed on these tasks, children must recognize the correspondence between the map and the actual space of a similar shape, scale distance (which we discuss further in the section on measurement), and perform mental rotation of the map with respect to actual space. Successful use of maps among preschoolers has occurred when the maps were oriented with respect to the space and mental rotation was limited to the vertical plane (in order to match ground-based perception of the space). Increasing the complexity of mental rotations required to realign spaces causes maps to become increasingly difficult for preschool children and is most likely to explain some of the difficulty children show in interpreting maps even into the elementary school years (Liben and Downs, 1989; Liben and Yekel, 1996; Piaget and Inhelder, 1967; Uttal, 1996; Wallace and Veek, 1995). Although the level of sophistication in mental transformation matures dramatically throughout childhood, the initial ability to mentally transform objects in space at the preschool age allows for productive interactions with spatial representations, such as maps.

### Learning Spatial Terms: Relation to Spatial Mathematical Skills

As summarized above, infants form spatial categories from an early age (e.g., Quinn, 1994, 2004). These visual categories may lay the foundation for the later learning of the spatial terms that label these categories (e.g., Mandler, 1992). However, it is also possible that linguistic input guides the learning of spatial concepts, highlighting certain preverbal spatial concepts and not others, perhaps leading to the formation of new spatial concepts. An example of how language can shape a preexisting nonverbal concept is provided by recent evidence showing that English-speaking infants form categories for tight/loose fit, a relation that is labeled in Korean but not in English (e.g., Casasola and Cohen, 2002; Hespos and Spelke, 2004; McDonough, Choi, and Mandler, 2003). By 29 months of age, English-speaking infants still categorized tight-fit containment relations when these were contrasted with loose-fit containment, but they no longer categorized loose-fit containment. By adulthood, English speakers do not pay attention to fit, categorizing tight and loose fit as "in" (McDonough, Choi, and Mandler, 2003). Thus, in this case exposure to English seems to play a selective function, highlighting some preexisting categories (in versus on) while downplaying others (tight fit/loose fit).

Exposure to spatial language during spatial experiences also appears to be particularly useful in "the learning and retention [of spatial concepts by] . . . inviting children to store the information and its label" (Gentner, 2003, pp. 207-208). Gentner found that children who heard specific spatial labels during a laboratory experiment that involved hiding objects ("I'm putting this on/in/under the box") were better able to find the objects than children who heard a general reference to location ("I'm putting this here"). Moreover, this was true even two days later, without further exposure to the spatial language (Loewenstein and Gentner, 2005). Similarly, Szechter and Liben (2004) observed parents and children in the lab as they read a children's book with spatial-graphic content. These researchers found a relation between the frequency with which parents drew children's attention to spatial-graphic content during book reading (e.g., "the rooster is really tiny now") and children's performance on spatial-graphic comprehension tasks.

Cannon, Levine, and Huttenlocher (2007) have also examined the parents' use of spatial language during puzzle play in a longitudinal study in which parent-child dyads were observed during naturalistic interactions every four months from age 26 to 46 months. Their findings show that puzzle play is correlated with children's mental rotation skill at 54 months for boys and girls. However, for girls but not boys, amount of parent spatial language during puzzle play (controlling for overall language input) is also a significant predictor of mental rotation skill at 54 months. This finding may be related to gender differences in the way in which spatial information is coded (e.g., Kail, Carter, and Pellegrino, 1979; Lourenco, Huttenlocher, and Fabian, under review).

## Understanding of Geometric Shape and Shape Composition

Various proposals have influenced views on children's shape categories. Piaget and Inhelder (1967) proposed a developmental sequence in which children first discriminate objects on the basis of topological features (e.g., a closed shape, which has an internal space defined by the closed boundary, versus an open shape, which has no defined internal or external boundaries) and only later on the basis of Euclidean features, such as rectilinear versus curvilinear. Still later, according to this theory, children are able to discriminate among rectilinear shapes (e.g., squares and diamonds). However, this sequence has been called into question on the basis of evidence that young children are able to represent the projective (e.g., curvilinear or rectilinear) as well as the Euclidean aspects of shape (e.g., Clements and Battista, 1992; Ginsburg et al., 2006; Kato, 1986; Lovell, 1959).

A different stage framework, proposed by van Hiele (1986), posits that children first identify shapes at the visual level on the basis of their appearance, then represent shapes at the "descriptive" level on the basis of their properties, and finally progress to more formal kinds of geometric thinking that are based on logical reasoning abilities. Consistent with van Hiele's first stage, preschoolers' early shape categories are centered on prototypes and the similarity of perceptual surface qualities of a shape are used to determine category inclusion. For example, preschoolers do not accept an inverted triangle as a triangle or nonisosceles triangles as triangles (e.g., Clements et al., 1999). Moreover, they tend to regard squares as a distinct category and not as a special kind of rectangle with four sides that are equal in length (Clements et al., 1999). Preschoolers sometimes overextend shape labels to nonexemplars. For example, they sometimes extend the label "rectangle" to right trapezoids as well as to nonrectangular parallelograms that have two sides that are much longer than the other two (Hannibal and Clements, 2008).

By the elementary school years, children's shape categories incorporate deeper knowledge of rules and theories that are definitional (Burger and Shaughnessy, 1986; Satlow and Newcombe, 1998). The timing of the shift from relying on characteristic perceptual features to relying on defining features varies depending on the shape. For example, Satlow and Newcombe (1998) report that this shift occurs between ages 3 and 5 for circles and rectangles, prior to second grade for triangles, and during second grade for pentagons. During the preschool years, the main change in shape categories is an increasing tendency to accept atypical exemplars of shapes as members of the category—that is, to extend shape categories beyond prototypical examples (e.g., Burger and Shaughnessy, 1986; Usiskin, 1987). The ability to broaden shape categories to include nonprotoypical examples depends on exposure to a variety of exemplars rather than to just prototypical examples such as equilateral and isosceles triangles (e.g., Clements et al., 1999). Neither Piaget's nor van Heile's stage theories recognize preschoolers' ability to represent and categorize shapes.

Children's learning of specific spatial terms also helps highlight spatial categories. These spatial terms include shape words (e.g., circle, square, triangle, rectangle), as well as words describing spatial features (e.g., curved, straight, line, side, corner, angle), spatial dimensions (e.g., big, little, tall, short, wide, narrow), and spatial relationships (e.g., in front of, behind, next to, between, over, under). Between ages 2 and 4, children learn terms for novel shapes more readily than other features, such as novel color or texture words (Heibeck and Markman, 1987; O'Hanlon and Roberson, 2006). Fuson and Murray (1978) found that over 60 percent of 3-year-old children could name a circle, a square, and a triangle. By age 5, 85 percent of children could name a circle, 78 percent a square, and 80 percent a triangle. In addition, 44 percent could correctly name a rectangle (Klein, Starkey, and Wakesley, 1999). In a shape word comprehension study, results were similar.

Clements et al. (1999) report that over 90 percent of children, ranging in age from 3 years 5 months to 4 years 4 months, could correctly point out a circle, and by age 6 years, 99 percent of children could do so. Only a few children in the younger group incorrectly chose an ellipse or another curved shape. For a square these numbers were also high yet somewhat lower: 82 percent of children in the younger group responded correctly, and 91 percent of 6-year-olds did so. Some children in the younger group incorrectly identified nonsquare rhombi as squares. Accuracy for triangles and rectangles was significantly lower (60 and 50 percent, respectively, for children ranging in age from 4 to 6).

Children also learn spatial words for shape dimensions (e.g., big, small, tall, short, wide, narrow) and words for the relationships of shapes (e.g., in, on, under, in front of, behind, between). For example, Clark (1972) reports that for each pair of dimensional adjectives, children learned the unmarked term before the marked term, that is, they learned big before little. Note that asking how big something is does not presuppose its being big or little, whereas asking how little something is carries the presupposition that one is asking about little things. The same is true for other pairs such as tall/short. The learning of these terms, like other words, is highly related to their frequency of occurrence in child-directed speech (e.g., Levine et al., 2008).

Children who hear greater amounts of spatial language have been found to perform at higher levels on a variety of nonverbal spatial tasks, including the WPPSI-3 Block Design subtest and a mental rotation task (Levine et al., 2008). This correlation may rest on the association of spatial language and spatial activities. Furthermore, spatial language may serve to focus children's attention on spatial relationships and lead to deeper processing of this information (e.g., forming categories of shapes and spatial relations). However, parents' spatial language to 3- to 5-year-old children has been found to occur more frequently during such activities as block and puzzle play than during other activities, such as book reading (Levine et al., 2008; Shallcross et al., 2008). Furthermore, higher amounts of parent spatial language occur during guided block play in which there is a goal than during free play with blocks (Shallcross et al., 2008). Thus, it is possible that spatial activities, spatial language, or both promote the development of spatial skills, such as block building and mental rotation.

## Summary

As for number, there are strong starting points during infancy for learning about space, including shapes, locations, distances, and spatial relations. These early starting points, however, like those for number, undergo major developments during the preschool years and beyond. Moreover, developmental rates and the competencies achieved are highly dependent on access to spatial activities, spatial language, and learning opportunities at home and at school.

Children are equipped to comprehend and reason about shape at an earlier age and in more complex detail than originally thought. By preschool, they benefit from learning about a variety of shapes, both typical and atypical, and this knowledge is impacted by their acquisition of spatial language. Language input and spatial activities appear to be highly influential in the development of spatial categories and spatial skills during the preschool years.

## DEVELOPMENT OF MEASUREMENT

Measurement is a fundamental aspect of mathematics, which "bridges two main areas of school mathematics—geometry and number" through the attachment of number to spatial dimensions (National Council of Teachers of Mathematics, 2000). The development of measurement skills usually starts with directly comparing objects along one dimension. Thus, children generally succeed in measuring length prior to area and volume (Hart, 1984; but see Curry and Outhred, 2005, for early success in measuring volume when the task involves successive filling of a container).

Certain skills, such as sensitivity to variations in amount, can be thought of as precursors to mature measurement skills and have been observed in infants. The ability to directly compare the lengths of objects is an early emerging skill and initially appears to be perceptually based (Boulton-Lewis, 1987). Infants demonstrate awareness of variations in amount in one dimension (e.g., noticing height) as early as 4 months (Baillargeon, 1991) and can discriminate between two objects based on height at 6 months (Gao, Levine, and Huttenlocher, 2000). For example, 6-month-old infants and 2-year-old toddlers are able to discriminate the length of dowels when they appear in the presence of a constant, aligned standard but not when there is no standard available with which to compare them (Huttenlocher, Duffy, and Levine, 2002).

Subsequent studies show that infants and toddlers are responding to the relative size of the standard and the test objects (Duffy, Huttenlocher, and Levine, 2005a, 2005b). This result is in line with the theory (Bryant, 1974) that relative coding precedes absolute coding. The ability to discriminate lengths in a more precise manner (distinguishing two heights that are fairly close without a present, aligned standard) develops some time between ages 2 and 4. However, even by age 4, children's sensitivity to variations in size is often influenced by the relation between two objects.

This early reliance on a standard to assess size may seem to contrast with findings by Piaget and his colleagues showing that young children do not spontaneously use a standard to measure objects (Piaget, Inhelder, and Szeminska, 1960). Piaget and colleagues argue that before school age, children's ability to encode metric information is limited because they lack the ability to make transitive inferences that are involved in measurement—that is, if A = B (the measure) and B = C, then A and B are equivalent. However, unlike Piaget and colleagues' task, in which the child was required to spontaneously use a stick to compare the heights of two towers that were not aligned, the experiments showing much earlier skill involve a visually aligned standard.

So far we have been discussing the development of the ability to discriminate linear extents and not the understanding of equivalence/nonequivalence of these extents, or sensitivity to amount transformations (adding or subtracting amounts). Although, as reviewed above, researchers have examined these topics with respect to discrete sets, there is little work on these topics with respect to continuous amounts. However, some evidence indicates that the ability to order continuous amounts is present at least by the preschool years. For example, Brainerd (1973) found that kindergartners could arrange three balls of clay according to weight and could arrange three sticks according to length.

## Understanding Units and Conventional Measurement

Early sensitivity to linear extent in relation to a standard is far from the mature ability to measure length. It is not until age 8 that children typically succeed in discriminating between objects of different lengths when there is not a constant aligned standard present. This ability is much closer to conventional measurement than the skill displayed by children up to age 4 (Duffy et al., 2005a). These changes in sensitivity to variation in amount from age 4 to age 8 may be related to exposure to conventional measurement at school and the ability to form and maintain images with certain attributes. However, developing a sophisticated conceptual understanding of linear measurement has a surprisingly long developmental time course (e.g., Copeland, 1979; Hiebert, 1981, 1984; Miller, 1984, 1989).

Conventional measurement involves several basic operations. First, it is important to realize that the units must be equal in size and must be specified. Second, the chosen unit must be repeated if it is smaller than the object being measured. Finally, the chosen unit must be subdivided when a whole unit does not fully cover the object or the remaining part of an object (Nunes, Light, and Mason, 1993).

Young children have difficulty understanding the importance of using an equal size unit. Miller (1984) showed that preschoolers between ages 4 and 5 have difficulty appreciating that the size of pieces (or units) must remain constant in measurement situations. In a well-known example, Miller found that preschool children who are asked to divide candy evenly among children consider it fair to break the last piece in half if they run out of pieces. In other words, as long as everyone gets a piece, they are not concerned that the pieces are unequal in size. In a study in which 5- and 6-year-olds were asked to make rulers by writing in the numbers, Nunes and Bryant (1996) found that they failed to space the numbers even approximately equally. In another study reported by Nunes and Bryant, children ages 5 to 7 had no trouble answering whether a 7 cm or 6 cm ribbon is longer. However, when asked whether a 2 inch or 2 cm ribbon is longer, 5-year-olds performed at chance. Although 7-year-olds performed above chance, they still performed significantly worse when the units were unequal than when they were equal, even though all the children knew that an inch is longer than a centimeter. Even first through third graders have difficulty understanding the importance of equal size units on rulers. Pettito (1990) gave children in elementary school a choice of rulers with which they could measure a line. She found that the majority of first and second graders were content to use a ruler with units that varied in size—in fact, only about half the third graders chose the standard unit.

Preschool children also have difficulty understanding that changes in the units of measure change the numerical answer (1 foot = 12 inches), but they do not change the length of the object being measured. Preschool children commonly fail to grasp the fundamental property of a unit, that a whole object can be segmented into parts of various sizes without changing the overall amount of what is being measured. They often count discrete parts of objects as being examples of a whole rather than grouping objects and counting amounts based on meaningful units (e.g., the two halves of a plastic egg each count as eggs versus the combination of the two pieces is one egg) (Shipley and Shepperson, 1990; Sophian and Kailihiwa, 1998). Similarly, Galperin and Georgiev (1969) gave kindergarten children two equal cups of rice and had them empty the cups by putting spoonfuls of rice into piles on a table using either a tablespoon or a teaspoon. When asked which group of piles contained more rice (correct answer is neither), a majority of the children chose the one made with the teaspoon because it contained more piles rather than choosing the group made with the tablespoon, which had fewer but larger piles. Thus, they were influenced by their propensity to count the overall number of piles. In this sense, children's skill at counting can interfere with their understanding of measurement. These findings highlight that part-whole relationships are fundamental to understanding the relationship between units and wholes (see Sophian, 2002, for a review).

A mature understanding of units of measure also entails the realization that the smaller the size of the unit, the larger the number of units the object will encompass. Research by Sophian, Garyantes, and Chang (1997) showed that preschool children have difficulty understanding this inverse relation, but that with instruction they can learn it. Young children do demonstrate some understanding of measurement principles, such as the inverse relation between unit size and the number of units after training or when measurement activities are set in a familiar context (e.g., part of a normal everyday routine or using familiar objects). Sophian (2002) taught preschool children ages 3 and 4 to correctly judge whether more small objects or more large ones would fit in a designated space. In pretest trials, the children incorrectly chose the larger object, but after six demonstration trials of watching the experimenter place objects of the two sizes, one by one, into two identical containers, they performed significantly better on posttest trials. These results identify the difficulties very young children have with understanding units and suggest that preschoolers (ages 2 years, 9 months to 4 years, 7 months) benefit from instructional intervention highlighting the relation between unit size and number. Thus, young children show some understanding of fundamental mathematical concepts that are relevant to measurement if given the opportunity to explore these concepts in interactive, supportive contexts.

### Scaling and Proportion

Children demonstrate early use of fundamental skills related to measurement and proportional reasoning in their use of maps. A critical factor for success in map use is scaling, which is related to measurement and proportional thinking. Scaling refers to the ability to code distance and understand how distance on a map corresponds to distance in the real world (Huttenlocher et al., 1999). Newcombe and Huttenlocher (2000, 2005) review the hierarchical nature of spatial coding, suggesting that various systems of coding spatial location are available, and their use depends on a mix of factors (e.g., cue salience in the external environment, complexity of movements required for action by the viewer). Furthermore, the availability of these systems appears as early as 6 months for both externally referenced and viewer-centered systems, which is much earlier than is predominantly reported in the literature. In relation to map use, children not only need to code locations in space but also to accommodate changes in scale, which requires a form of measurement (e.g., comparing the distance between two locations on a map and the corresponding distance between two locations in the real world). Scaling has been assumed to involve proportional reasoning and therefore to occur much later in development, between ages 10 and 12 (Piaget and Inhelder, 1967). However, evidence of early success using maps by children ages 3 to 6 indicates that scaling, at least in these cases, may represent a precursor to more precise measurement and is accomplished using spatial coding (Huttenlocher et al., 1999; Sandberg and Huttenlocher, 2001; Stea et al., 2004).

## REGULATING BEHAVIOR AND ATTENTION

Infants' and young children's mathematical development also takes place in the context of cognitive and behavior regulation, which when stimulated and supported can promote mathematical learning. Research suggests that executive function is more strongly associated with successful transition to formal schooling than IQ or entry-level reading or mathematics skills (Diamond, 2008). Executive function is defined as having three core components (Diamond, 2008). The first is inhibitory control, which is the ability to stay on task and do what is most necessary, even in the face of an inclination or impulse to do something else. The second is working memory, which is the ability to keep information in mind while still manipulating it or changing it mentally; "working memory may be thought of as a short-term 'working space' that can temporarily hold information while a participant is involved in other tasks" (Passolunghi, Vercelloni, and Schadee, 2007, p. 166). In mathematics specifically, this allows for performing mental arithmetic, such as addition or subtraction. The third component is cognitive flexibility, which allows for shifting between different tasks, demands, priorities, or perspectives. As Diamond explains, executive function, particularly the inhibitory control component, is very similar to self-regulation but tends to focus more on cognitive tasks and less on social situations. Multiple executive function skills may be valuable in early math learning. These include the ability to stay on task and ignore distractions, the ability to follow the teacher's directions, the ability to keep two strategies in mind at the same time, the motivation to succeed, the ability to plan and reflect on one's actions, and the ability to cooperate (Leong, n.d.; McClelland et al., 2007).

The link between mathematics success and executive function may have different underlying causes. Blair and colleagues (2007) review neuroscience research indicating that, in adults, there may be a relationship between mathematical skills and executive functioning at the neural level. Reviewing changes in mathematics curriculum for children, they also found that, increasingly, automatized knowledge is emphasized less and tasks that require executive function skills (pattern-solving, relational reasoning, and geometry concepts) are emphasized more. This is an area that will continue to shed light on the relationship between executive function and mathematical development as more research is conducted.

Some studies have explicitly found a link between executive function and early math skills.[3] In a study of 170 Head Start children, Blair and Razza (2007) found that multiple aspects of self-regulation (including inhibitory control, effortful control, and false belief understanding, along with fluid intelligence) all made independent contributions to children's early math knowledge. Similarly, McClelland and colleagues (2007) administered the Heads-to-Toes task to more than 300 preschool-age children. The

[3]Different studies use different terms for concepts encompassed by executive function, such as self-regulation and behavioral regulation.

Heads-to-Toes task asks children to do the opposite of what the instructor tells them. So, for example, if the instructor asks the children to touch their head, they are to touch their toes. This task measures behavioral regulation (a component of self-regulation), in that it requires children to employ inhibitory control, attention, and working memory. The researchers found that behavioral regulation scores significantly predicted emergent math scores. The researchers conclude that "strengthening attention, working memory, and inhibitory control skills prior to kindergarten may be an effective way to ensure that children also have a foundation of early academic skills" (p. 956). Espy and colleagues (2004) specifically studied the roles of working memory and inhibitory control with almost 100 preschoolers. They found that both components of executive function contributed to the children's mathematical proficiency, with inhibitory control being the most prominent. Passolunghi and colleagues (2007) studied 170 6-year-olds in Italy. They examined the roles of working memory, phonological ability, numerical competence, and IQ in predicting math achievement. They found that working memory skills significantly predicted math learning at the beginning of elementary school (primary school in Italy).

## SUMMARY

This chapter underscores that young children have more mathematics knowledge, in terms of number and spatial thinking, than was previously believed. Very early in life, infants can distinguish between larger set sizes, for example 8 versus 16 items, but their ability to do so is only approximate and is limited by the ratio of the number of items in the sets. The set size limitation is thought to reflect one of the two core systems for number (Feigenson, Dehaene, and Spelke, 2004; Spelke and Kinzler, 2007). Furthermore, young infants' early knowledge of quantity is implicit, in that they do not use number words, which means that learning number words and relating them to objects is one of the major developmental tasks to occur during early childhood.

Toddlers and preschool children move from the implicit understanding of number seen during infancy to formal number knowledge. Spoken number words, written number symbols, and cultural solution methods are important tools that support this developmental progression.

Young children also learn about space, including shapes, locations, distances, and spatial relations, which also go through major development during the early childhood years. Children's acquisition of spatial language plays an important role in the development of spatial categories and skills. In addition to learning about number and shape, early childhood also includes development of measurement, which is a fundamental aspect of mathematics that connects geometry and number. Young children's understanding of measurement begins with length, which is perceptually based, and an important feature of their learning during this period is that they have difficulty understanding units of measure. Young children can become successful at this when given appropriate instruction.

It is also important to note that across early childhood, mathematical development that is situated in an environment that promotes regulation of cognitive activities and behavior can improve mathematical development. More specifically, when young children have an opportunity to practice staying on task, to keep information in mind while manipulating or changing it mentally, and to practice shifting between differing tasks, mathematics learning is improved and in turn improves these regulatory processes.

Although we discuss universal starting points for mathematics development in this chapter, there are, of course, differences in children's mathematical development. The next chapter explores variation in children's mathematical development and learning outcomes and the sources of this variation. We also discuss the role of the family and informal mathematics learning experiences in supporting children's mathematical development.

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# 4

# Developmental Variation, Sociocultural Influences, and Difficulties in Mathematics

There is evidence that most children bring foundational resources and knowledge about mathematics to school. However, this is not the whole story. Research findings reveal enormous discrepancies in young children's levels of mathematics competence, and these discrepancies appear to be larger in the United States than they are in some other countries (e.g., China) (Starkey and Klein, 2008). This chapter describes the kinds of differences that exist and reviews what is known about the nature and sources of developmental variations among children.

Most children bring core number sense or number competencies to school (National Research Council, 2001). Number sense refers to interconnected knowledge of numbers and operations. Although preverbal number sense begins in infancy and appears to be universal, preschool and kindergarten number sense involves understanding of number words and symbols, which is heavily influenced by experience and instruction. The number sense children bring to kindergarten is highly predictive of their later mathematics achievement. The term "number sense" means different things in different fields of research, and almost no two researchers define it in exactly the same way (Gersten, Jordan, and Flojo, 2005; Jordan et al., 2006). The term "number sense" is used in this chapter because much of the research summarized here uses it. When the discussion is more general, the term "number competencies" is used along with number sense to remind the reader that we are talking about knowledge and skills that can be taught and learned. The word "competencies" is used as a balanced term meaning both knowledge and skills. The competencies encompassed by the term "number sense" as used here are described more fully in Chapter 5.

Despite strong universal starting points, striking individual differences in number sense emerge early in life and are present by the time children enter preschool (e.g., Klibanoff et al., 2006). These differences are apparent both on standardized tests (e.g., Arnold et al., 2002; Starkey, Klein, and Wakeley, 2004) and on specific measures tapping early number competencies, such as determining set size, comparing sets, and carrying out calculations (e.g., Entwisle and Alexander, 1990; Ginsburg and Russell, 1981; Griffin, Case, and Siegler, 1994; Jordan, Huttenlocher, and Levine, 1992; Levine et al., in preparation; Saxe, Guberman and Gearheart, 1987). The level of number sense in kindergarten is highly of predictive future mathematics success in first through third grades (Fuchs et al., 2007; Jordan, Glutting, and Ramineni, in press; Locuniak and Jordan, in press; Mazzocco and Thompson, 2005) as well as into the later school years (Duncan et al., 2007).

In this chapter, we explore individual differences in children's mathematics competence. We begin by describing the differences associated with key social groups specifically defined by socioeconomic status, gender, race/ethnicity, and English language ability. We then discuss the contextual factors and early experiences that appear to be linked to these differences, giving particular attention to the role of the family and language. We then discuss learning disabilities. We end with a brief discussion of potential intervention.

## GROUP DIFFERENCES IN MATHEMATICS PERFORMANCE

Researchers have explored several key social factors that are linked to systematic, average differences in children's mathematical performance. Socioeconomic status (SES), which includes income level as well as level of parental education, is strongly linked to differences in mathematics competence. Evidence related to gender differences in mathematics competence is less clear, although some differences have been found.

### Socioeconomic Status

Mathematical skills of young children from low-income families lag behind those of their middle-income peers. Preschoolers who attend Head Start Programs perform significantly below children who attend preschools serving middle-income children on standardized tests of mathematical readiness (Ehrlich and Levine, 2007). The gulf between low- and middle-income children is wide and includes spatial/geometric and measurement as well as number competencies (Clements, Sarama, and Gerber, 2005; Klein and Starkey, 2004; Saxe et al., 1987).

Jordan and colleagues (Jordan et al., 2006, 2007) found that low-income children enter kindergarten far behind their middle-income peers on tasks assessing counting skills, knowledge of number relations (e.g., recognizing which number is smaller), and number operations. Moreover, longitudinal assessment over six data points revealed that low-income children were four times more likely than their middle-income peers to show flat growth in these areas throughout kindergarten and early first grade. Underlining the importance of early number sense to school success, the researchers found that level of performance on a battery assessing number sense in kindergarten as well as rate of growth between kindergarten and first grade accounted for 66 percent of the variance in mathematics learning at the end of first grade (Jordan et al., 2007). In other words, number sense in kindergarten in strongly related to competence in mathematics at the end of first grade and the rate of growth over the first grade year. Income status, gender, age, and reading ability did not account for additional variance in first grade mathematics outcomes over and above initial performance and growth in number sense. This suggests that SES differences found at the end of first grade are due to initial differences in number sense in kindergarten.

Several studies indicate that SES differences in preschoolers' number skills are more marked on tasks tapping number skills without objects (called verbal tasks) than on tasks tapping number skills with objects (called nonverbal tasks). When kindergarten and first grade children are presented with verbal calculation problems with no objects, either as number combination problems ("How much is 3 and 2?") or story problems ("Mike had 3 pennies. Jen gave him 2 more pennies. How many pennies does Mike have now?"), middle-income children perform much better than do low-income children (Jordan et al., 2006; Jordan, Huttenlocher, and Levine, 1992; Jordan, Levine, and Huttenlocher, 1994). Middle-income children also achieve at a faster rate on calculation problems without objects in kindergarten (Jordan et al., 2006, 2007). In contrast, SES differences are smaller if the same calculations are presented in a nonverbal format with objects (e.g., the child is shown 3 disks that are then hidden with a cover. The tester then slides 2 disks under the cover and the child indicates how many are now hidden).

Jeong and Levine (2005) have shown that knowing number words is associated with very early performance on numerosity matching tasks that do not require verbal responses (e.g., matching arrays of visual dots). Specifically, performance on these tasks is more exact for children who have acquired the meaning of a few number words. For instance, 2- to 3-year-olds were more exact in their ability to match small set sizes when they have better knowledge of the cardinal meanings of number words. Although low-income children performed worse than middle-income children on such numerosity matching tasks, this difference was eliminated if answers that were plus or minus 1 from the correct answer were counted as correct

(Ehrlich, Levine, and Goldin-Meadow, 2006). Thus, low-SES preschoolers appear to have approximate representations of set sizes and number words at a time when their higher SES peers have gained exact representations. Therefore, low-SES preschoolers need experiences to learn number words and to use them to help on these matching tasks.

The sources of these differences are difficult to pinpoint. Research on children's early experiences point to the amount of support for mathematics at home as well as other language and contextual factors. Some findings show that young children from low-income families receive less support for mathematics in their home environment than do their middle-income peers (Blevins-Knabe and Musun-Miller, 1996; Holloway et al., 1995; Saxe et al., 1987; Starkey et al., 1999). Compounding the situation, public preschool programs serving low-income families tend to provide fewer learning opportunities and supports for mathematical development than ones serving middle-income families (Clements and Sarama, 2008). These factors are discussed in greater detail in the section on the influence of context and experience.

## Gender

Results and opinions vary regarding gender differences in early mathematics. Some studies have no revealed gender differences in mathematics performance (e.g., Clements and Sarama, 2008; Lachance and Mazzocco, 2006; Levine, Jordan, and Huttenlocher, 1992; Sarama et al., 2008). Some have found differences favoring boys: Jordan et al. (2006) found small but statistically significant gender effects on calculation with objects and on numerical estimation. In particular, boys had an edge over girls even when income level, age, and reading ability were controlled for in the analyses, and there were more boys than girls in the highest performing group. However, Coley's (2002) analysis of the Early Childhood Longitudinal Study database indicated small advantages in kindergarten in different areas for each gender: Girls were somewhat better in recognizing numbers and shapes, and boys were somewhat better in numerical operations.

Some research with older children indicates that girls in the primary grades may tend to use less advanced strategies than do boys (Fennema et al., 1998), and other work suggests no gender differences in the mathematics performance of older students (Hyde et al., 2008). Recent research (e.g., Carr et al., 2007) suggests spatial skills may promote the use of more advanced computational strategies, and boys seem to have an advantage in the more general area of spatial cognition, even in preschool. There are differences in the mean level of performance of boys and of girls on mental rotation tasks by 4½ years of age, ranging from small but significant differences (Levine et al., 1999) to large differences with girls performing at chance levels (Rosser et al., 1984). Preschool boys also perform better than preschool girls on solving problems involving mazes (e.g., Fairweather and Butterworth, 1977; Wechsler, 1967; Wilson, 1975) and are faster at copying a three-dimensional Lego (plastic blocks) model (Guiness and Morley, 1991). However, it appears that at least some of these differences are created by lack of particular types of experiences (Ebbeck, 1984).

Spatial skill may reflect or at least interact with greater engagement of boys than girls in spatial activities, such as building with Legos (Baenninger and Newcombe, 1989). Young boys typically spend more time playing with Legos and putting puzzles together than do girls, suggesting that engagement in spatial activities promotes skill development (Levine et al., 2005). The amount of puzzle play for both boys and girls was related to the mental transformation performance (McGuinness and Morley, 1991). Parents' spatial language may be more important for girls than for boys; use of such language by parents related to mental transformation performance of girls but not of boys (Cannon, Levine, and Huttenlocher, 2007). Boys tend to be more interested in movement and action from the first year of life and girls more focused on social interactions (e.g., Lutchmaya and Baron-Cohen, 2002). Boys also may gesture more on spatial tasks (e.g., Ehrlich, Levine, and Goldin-Meadow, 2006), indicating that encouraging gesture, especially for girls, may be helpful in spatial learning.

Given the finding that boys seem to have an advantage in spatial cognition and that this seems to result partly from the number of experiences they have that support such learning, it seems particularly important for both numerical and spatial learning that girls be given opportunities for spatial learning. Importantly, intervention studies with preschoolers using a research-based mathematics curriculum did not find an interaction with gender, indicating that girls can learn as much as boys in both numerical and spatial tasks (Clements and Sarama, 2008; Sarama et al., 2008). Simple modifications to everyday preschool activities, such as block building (Kersh, Casey, and Young, in press) and the use of stories about spatial topics (Casey et al., 2008), have been shown to be effective in developing girls' spatial cognition. Teachers should ensure that girls play with blocks and provide them with challenges that ensure that they extend their block-building skills, such as building windows, bridges, and arches.

## Race and Ethnicity

Over the past several decades, research has found differences in children's mathematics learning outcomes as a function of their race/ethnicity (e.g., Ginsburg and Russell, 1981). This section discusses differences in mathematics learning outcomes, but readers should keep in mind that using a fixed trait based on a single dimension can lead to a cultural deficit model

(Lubienski, 2007). Racial/ethnic groups are heterogeneous, and children in particular racial/ethnic groups have mathematical knowledge and skills that range from low to high mastery levels.

Generally, African American, Hispanic, and American Indian/Alaska Native children achieve at lower levels than their white peers in mathematics (National Center for Education Statistics, 2007). Few data exist on early childhood mathematics teaching and learning in relation to race/ethnicity, but one can extrapolate from K-12 studies. Findings suggest that this achievement disparity is related to differences in mathematics learning before school entry and fewer meaningful pedagogical experiences once children of color enter school (Magnuson and Waldfogel, 2008). For example, the National Assessment of Educational Progress (NAEP) survey data show that fourth grade black and Hispanic students and those with low SES report that mathematics mainly consists of memorizing facts, a belief that is negatively correlated with achievement even after controlling for race/ethnicity and SES (Lubienski, 2006, 2007). Furthermore, teachers' reports indicate that black and Hispanic children were more likely to be routinely assessed with multiple-choice tests than white students (Lubienski, 2006). These practices do not represent the best pedagogy for high-quality mathematics education (National Council of Teachers of Mathematics, 2000).

Teachers who build on children's everyday mathematical experiences promote genuine mathematics learning (Civil, 1998; Ladson-Billings, 1995). For example, Ladson-Billings (1995) found that urban and suburban students' community experiences shaped the way they approached a mathematics problem-solving task and that students' differing approaches to learning could be used by teachers to inform their instruction. Instructional practices that extend children's out-of-school experiences are more likely to produce meaningful mathematics learning.

## English Language Learners

Surprisingly little research has examined the mathematics performance of English language learners. Findings for other subject areas show that children who have limited proficiency in English perform more poorly than their native English-speaking peers in other academic subjects (McKeon, 2005). A major issue for educating English language learners (ELL) is the language of instruction (Barnett et al., 2007; Genesee et al., 2006). In research conducted by Barnett and colleagues (2007) with 3- and 4-year-olds, they tested whether children in a two-way immersion (English and Spanish) or those in English-only programs made gains in English language measures of mathematics, vocabulary development, and literacy. They found that children in both types of programs made gains on all academic measures and the two-way immersion classrooms saw improvements in Spanish language development for both ELL and English-speaking children without losses to English language learning (Barnett et al., 2007). It is important to note that classrooms in both types of program employed a licensed teacher and an assistant with a child development associate credential. A review of the K-12 literature on the language of instruction provides evidence that conflicts with the findings of Barnett and colleagues; specifically, Lindholm-Leary and Borasato (2006) suggest that bilingual education may be related to more positive educational outcomes for older ELL students. Given these disparate findings, additional research in high-quality early childhood settings on this topic is warranted.

One of the few studies focused specifically on mathematics competence with this population of students suggests there may not be performance differences in mathematics. Secada (1991) found that first grade Hispanic students were not at a disadvantage to their native English-speaking peers in solving addition and subtraction word problems. However, with the growing number of ELL in the student population, it vital that more attention be paid to the relationship between language status and early mathematics learning so that early childhood education can effectively accommodate and support these children.

## INFLUENCE OF CONTEXT AND EXPERIENCE

As noted in the previous section, research has identified consistent, average differences in mathematics competence and performance depending on membership in a particular social group. Why group membership is linked to such differences is a complicated question. Research suggests that early experiences play an important role in shaping the observed differences. In this section we explore the contributions of context and early experience. We begin with a general discussion of the role of families in shaping early experience, including parents' knowledge and beliefs about mathematics, and the support they provide for mathematics through engagement in mathematics activities. We then look more specifically at how differences in experiences at home are linked to the observed SES differences in performance. Finally, we consider the role of language in mathematics learning.

### Role of Families

Families are one of the critical social settings in which children develop and learn (Bronfenbrenner, 2000; Iruka and Barbarin, 2008). Families influence children's development in many ways, including parenting practices, provision of resources, interactions with school, and involvement in the community (Weiss, Caspe, and Lopez, 2006; Woods and Kurtz-Costes, 2007). Parents have different attitudes, values, and beliefs in raising young children, which result in difference emphasis on educational activities in the home. Families support mathematics learning through their activities at home, conversations, attitudes, materials they provide to their children, expectations they have about their performance, the behaviors they model, and the games they play. Parents also build connections with their children's educational settings—all of which can shape children's early mathematics development.

*Parents' Knowledge and Beliefs About Early Childhood Mathematics*

Although there are only a few empirical studies about parental beliefs and behaviors related to early mathematics, those that exist suggest that parents place more importance on literacy development (Barbarin et al., 2008). Barbarin and colleagues examined the beliefs of parents whose children were enrolled in public prekindergarten regarding the skills children need to be prepared for school. Mathematical skills and such tasks as counting were rated less important than other social and cognitive tasks. Specifically, language/early literacy was mentioned 50 percent of the time, whereas numeracy was mentioned only 3.5 percent of the time (Barbarin et al., 2008). Similarly, Cannon and Ginsburg (2008) found that mothers thought it was more important that their children learn daily living skills and develop language skills in preschool than that their children learn mathematical skills. Most mothers in the study reported they themselves spent more time teaching their children language skills than mathematics skills at home.

*Engagement in Mathematics Activities*

Children's mathematical competence is supported and shaped by the math-related activities they engage in as part of their daily lives (Benigno and Ellis, 2008). Parenting practices in which parents engage children in conversations about number concepts, play with puzzles and shapes, encourage counting, and use number symbols to represent quantity in their interactions in the physical world can facilitate mathematics learning (see Box 4-1 for examples of how parents can engage children in mathematics activities). Acquiring mathematics knowledge involves more than learning numbers. It also includes learning shapes and patterns. It is facilitated by conversations about what children are doing when they compute, solve puzzles, and develop patterns and discussions of why they took a particular approach to a problem.

In fact, one study demonstrates how parents and their children can engage in mathematics-related activities. In a groundbreaking study of

**BOX 4-1**
**Supporting Children's Mathematics at Home**

Parents play an important role in supporting mathematics learning through the mathematics-related activities in which they engage their children. Incorporating mathematics-focused activities during play is one strategy for enhancing mathematics. Another is to capitalize on situations in which mathematics is a natural part of everyday tasks, such as grocery shopping or cooking. During daily activities, parents can:

- Observe their children carefully, seeing what they do and encouraging and extending their fledgling use of number symbols and processing.
- Say the number word list. For example, they can count small food items or the number of cups at the table.
- Ask children to tell them about their problem solving. For example, they can ask "What did you mean by that?" or "Why did you do it that way?"
- Engage in activities that involve playing with blocks, building things, and board games.

Given the prevalence of the Internet, television, and videogames in the lives of children, even young children (for a review, see Fisch, 2008), these means of communication provide interesting opportunities for impacting early mathematics skills. Fisch (2008) provides a review of existing media that include a mathematical component. These include television shows, such as *Sesame Street*; mathematics-based software games, such as *Building Blocks* and *Millie's Math House*; websites that include mathematics content, such as that of Sesame Street and Disney; and electronic, interactive toys.

The Internet can be a tool to help families devise mathematics-related activities for their young children. Such websites as *FAMILY MATH*, from the Lawrence Hall of Science at the University of California, Berkeley, can provide this kind of help. Although there are no effectiveness data available for this website, *FAMILY MATH* offers fun activities that maintain mathematical integrity and uses inexpensive materials that families may already have at home (see http://sv.berkeley.edu/showcase/pages/fm_act.html).

early childhood mathematics in family contexts, Saxe and colleagues (1987) found that many of the children in the 78 families they studied, both low and middle income, were spontaneously engaging in number-related activities (counting toys, using numbers in play, etc.), but the nature of their numerical knowledge and environment differed. Mothers in the study reported that both they and their children had a high level of interest in number play, but middle-income children performed better than low-income children on both the cardinality and arithmetic tasks.

There are numerous opportunities on a daily basis for children and families to explore mathematical terms and concepts. These include mealtimes, shopping, playtime, sports, television, and reading (Benigno and Ellis, 2008). In fact, Blevins-Knabe and Musun-Miller (1996) provide evidence to support the effects of parental modeling, reporting a relation between parental participation in number activities and children's involvement in similar activities. Moreover, they found that parental reports of children's number activities at home predicted their scores on a standardized test of early mathematical ability.

Several studies suggest that exposure to the language and symbol system of mathematics powerfully extends the universal starting points of children's quantitative knowledge and contributes to observed differences in mathematics competence. This is true in terms of exposure to the language of mathematics in preschool (Klibanoff et al., 2006) as well as at home between ages 14 and 30 months (Levine et al., in preparation). These studies show that the range of number words used in these settings is enormous. For example, in the home study, a longitudinal project in which families were visited every 4 months for five 90-minute sessions during which they were asked to go about their normal activities, the use of number words ranged from a low of 3 to a high of 175 instances. Similarly, in the classroom studies, the amount of number input provided by teachers during a 1-hour period that included circle time ranged from 1 to 104 coded instances.

While research suggests that families do incorporate mathematics into their everyday lives, they may also need reminders of the importance of mathematics. An observational study of 39 preschoolers and their families (Tudge and Doucet, 2004) found that the children engaged in a very low rate of explicit mathematics lessons over the course of a day and also demonstrated low levels of mathematics-related play. Of the mathematics lessons that were observed, the most common were lessons involving numbering, and the most common types of mathematical play involved toys that featured numbers (puzzles, computer programs, etc.). Furthermore, parents may overestimate their children's mathematical skills. Fluck and colleagues (2005) found that parents believed their children had a much better grasp of the concept of cardinality (beyond mere counting) than the children actually displayed.

### *Differences in Children's Experiences and Learning Opportunities as a Function of Socioeconomic Status*

Evidence suggests that SES differences in children's mathematics competence are linked to parallel differences in experiences provided in the home. For parents in some low-SES families, involvement in fostering the acquisition of mathematics skills in their children may be hampered by multiple factors. Poverty and uncertainty related to inadequate resources and residential instability can easily become all-consuming, leaving room for little else. Parents in low-SES families, though concerned about their children's education, may feel less ready to assist them due to limitations in their own education, the strains of inadequate financial resources, unmet mental health needs, and specific discomfort with their own mathematical skills and a lack of awareness of the importance of early mathematics development (for research on the effects of poverty on parenting see, e.g., Knitzer and Lefkowitz, 2006; McLoyd, 1990; see Clements and Sarama, 2007, for a specific discussion of low-income families and mathematics).

Research shows that low-income parents provide fewer mathematics activities than middle-class parents (Starkey et al., 1999). This includes free activities, such as those that are integrated into everyday experiences and made-up games, suggesting that, to some extent, lack of financial resources does not explain the difference. Starkey and Klein (2008) suggest that the difference may instead stem from educational background and exposure to mathematics courses. The difference may be resource-based as well. Ramani and Siegler (2008), in a study of board game activities, found that, although 80 percent of middle-class preschool-age children reported playing one or more board games outside preschool, only 47 percent of Head Start children did so. However, such board games could easily be made and used at home.

It is also vital to remember that, in many cases, children and families from low-SES backgrounds are involved with many more agencies and programs than their more well-off peers. "Exploring the contribution of these additional settings is important because interpreting SES effects as emanating exclusively from the family or the child means that policy and program interventions may focus too narrowly as they attempt to improve the educational outcomes of low-SES children" (Aikens and Barbarin, 2008, p. 236). Policy makers, researchers, and practitioners should not neglect the importance of the interactions and experiences of the multiple contexts and the nature of development in everyday life. Thus, at the level of a mother and child interacting in a larger social context unique to cultural environments, the entire dynamic may influence a child's learning and specifically reinforce or hinder the development of mathematical thinking and understanding.

The SES gap prior to preschool entry suggests that the home environment plays a major role, yet it is important to note that formal preschool programs do not appear to be ameliorating it. In fact, the gap widens during the preschool years. "In the United States, neither the home nor preschool learning environments of low-SES children provide sufficient enrichment to close or even maintain early SES-related differences in mathematical knowledge" (Starkey and Klein, 2008, p. 266). The issue of how to better support low-income children in mathematics and address the gap is taken up in detail in Chapter 7.

## Role of Language

Languages vary in the ways they represent mathematical concepts. This variation appears to be linked to variation in children's mathematics learning. For example, several recent studies have shown that characteristics of speakers' language influence the quantitative skills of children and adults. One set of studies provides evidence that variations in the structure of a morphological marker, which refers to a language element that identifies quantity in different languages, is associated with the age at which children learn the meaning of specific cardinal numbers. That is, children who speak a language that marks the singular-plural distinction through a morphological marker (e.g., the *s* on the end of dog*s*, which indicates that the word is plural, is the morphological marker) acquire the meanings of small cardinal numbers sooner than children whose language does not make such a distinction (e.g., LeCorre, Li, and Lee, 2004; Li et al., 2003; Sarnecka et al., 2007). Even more strikingly, recent evidence has shown that adults in cultural groups with few number words perform worse than adults from cultural groups with more elaborated number systems in matching set sizes, performing arithmetic operations, and other cognitive tasks requiring knowledge of exact numbers (Gordon, 2004; Pica et al., 2004). There is also a large body of evidence regarding the implications of number naming systems for mathematics learning.

### *Language Differences in Number Names*

Language differences in number names have received in-depth attention in the literature. Such differences appear to be linked to the ease with which children learn to count, an essential task during early childhood. Names and symbols for numbers can be (and have been) generated according to a bewildering variety of systems (see Ifrah, 1985; Menninger, 1958/1969). Because the base-ten system is so familiar and widespread and because humans have 10 fingers, it may appear that the development of a base-ten system is somehow natural and inevitable. Historically, base 4 and base 8 systems were also common (Menninger, 1958/1969). However, most modern languages now use systems that are organized around a base of 10, although languages vary in the consistency and transparency of that structure. For example, number words in English, Spanish, and Chinese differ in important ways. In all three languages, number names can be described to a first approximation as a base-ten system, but the languages differ in the clarity and consistency with which the base-ten structure is reflected in actual number names.

Representations for numbers from 1 to 9 consist of an unsystematically organized list. There is no way to predict that "5" or "five" or "wu" comes after "4," "four," and "si," in the Arabic numeral, English, or

Chinese systems, respectively. Names for numbers above 10 also diverge in interesting ways among the three languages. The Chinese number-naming system maps directly onto the Hindu-Arabic number system used to write numerals. For example, a word-for-word translation of "shi qi" (17) into English produces "ten-seven." English has unpredictable names for "11" and "12" that bear only a historical relation to "one" and "two" from the Old Saxon *ellevan* (one left over) and *twelif* (two left over) (Menninger, 1958/1969). Whether the boundary between 10 and 11 is marked in some way is very significant, because this is the first potential clue to the fact that number names are organized according to a base-ten system.

English names for teen numbers beyond twelve do have an internal structure, but this relation is obscured by phonetic modifications of many of the elements from those used for 1 through 10 (e.g., "ten" becomes "teen," "three" becomes "thir," and "five" becomes "fif"). Furthermore, the order of formation reverses place value compared with the Hindu-Arabic and Chinese systems (and with English names above 20), naming the smaller value before the larger value (e.g., say "fourteen" but write 14 with the 4 second). Spanish follows the same basic pattern for English to begin the teens, although there may be a clearer parallel between "uno, dos, tres" and "once, doce, trece" than between "one, two, three" and "eleven, twelve, thirteen." The biggest difference between Spanish and English is that, after 15, number names in Spanish abruptly take on a different structure. Thus, the name for 16 in Spanish "diez y seis" (literally "ten and six"), follows the same basic structure as do Arabic numerals and Chinese number names (starting with the tens value and then naming the ones place), rather than the structure used by teens names in English from 13 to 19 and by teens names in Spanish from 11 to 15 (starting with the ones place and then naming the tens value).

Above 20, all these number-naming systems converge on the Chinese structure of naming the larger value before the smaller one, consistent with the order of writing the values in numerals. Despite this convergence, the systems continue to differ in the clarity of the connection between decade names and the corresponding unit values. Chinese numbers are consistent in forming decade names by combining a unit value and the base (10). Decade names in English and Spanish generally can be derived from the name for the corresponding unit value, with varying degrees of phonetic modification (e.g., "five" becomes "fif" in English as in fifty rather that fivety, "cinco" becomes "cincuenta" in Spanish) and some notable exceptions, primarily the special name for twenty ("veinte") used in Spanish.

### *Consequences for Learning to Count*

Although languages differ in the length and complexity of the irregular portion of the system of names that must be learned, in general children must learn quite a few number names prior to coming across data supporting the induction that they are dealing with an ordered base-ten system of names. Looking at the extent to which differences in learning reflect differences in counting terms can assess effects of number-naming systems on children's early mathematics.

Research on children's acquisition of number names (Fuson, Richards, and Briars, 1982; Miller and Stigler, 1987; Siegler and Robinson, 1982) suggests that children in America learn to recite the list of number names through at least the teens in essentially a rote learning task. When first counting above twenty, U.S. preschoolers often produce idiosyncratic number names, indicating that they fail to understand the base-ten structure underlying larger number names, often counting "twenty-eight, twenty-nine, twenty-ten, twenty-eleven, twenty-twelve." This kind of mistake is extremely rare for Chinese children, indicating that the base-ten structure of number names is more accessible for learners of Chinese than it is for children learning to count in English.

The cognitive consequences of the relative complexity of English number names are not limited to obstacles placed in the way of early counting. Speakers of English and other European languages (Fuson, Fraivillig, and Burghardt, 1992; Séron et al., 1992) face a complex task in learning to write Arabic numerals, one more difficult than that faced by speakers of Chinese (compare the mapping between name and numeral for "twenty-four" with that for "fourteen" in the two languages). Work by Miura and her colleagues (Miura, 1987; Miura and Okamoto, 1989; Miura et al., 1988, 1993) suggests that the lack of transparency of base-ten markings in English has conceptual consequences as well. They have found that speakers of languages whose number names are patterned after Chinese (including Korean and Japanese) are better able than speakers of English and other European languages to represent numbers using base-ten blocks and to perform other place-value tasks. Because school arithmetic algorithms are largely structured around place value, this indication that the complexity of number names affects the ease with which children acquire this basic concept is a finding with real educational significance.

When learning to count, children must acquire a combination of conventional knowledge about number names (they must learn their own cultural number word list in order), a conceptual understanding of the mathematics principles that underlie counting, and an ability to apply this knowledge to mathematical problem solving. Language differences during preschool appear to be limited to the first aspect of learning to count. For example, Miller and colleagues (1995) found no differences between Chinese and U.S. preschoolers in the extent to which they violated counting principles when counting objects, or in their ability to use counting to produce sets of a given size in the course of a game. The effects of differences in number-name structure on early mathematical development appear to be very specific to those aspects of mathematics that require one to learn and use these symbol systems. These effects have implications for learning Arabic numerals and thus for acquiring the primary symbol system used in school-based mathematics.

The nature and timing of differences in early counting between Chinese-speaking and English-speaking preschoolers correspond to predictions based on the morphology of number names. Evidence from object counting indicates that these differences are also limited to aspects of counting that involve number naming. Miller and colleagues (1995) looked at children's object counting for sets that were small (3-6 items), medium (7-10 items), and large (14-17 items). They found that Chinese-speaking children were significantly more likely to report the correct number word for a set than English-speaking children, but this was entirely due to the greater likelihood of Chinese children to correctly recite the sequence of names. The task of completely coordinating saying number words and designating objects in counting is quite difficult for many young preschoolers, and equally so for U.S. and Chinese children: 37 percent of U.S. and 38 percent of Chinese preschoolers either pointed to an object and did not produce the number name or the reverse. Double counting or skipping objects was even more common, but again did not differ between the Chinese and U.S. preschoolers.

### *Consequences for Using the Base-ten Structure in Problem Solving*

The structure of number names is associated with a specific, limited difference in the course of counting acquisition between English-speaking and Chinese-speaking children. One area in which there may be conceptual consequences of these linguistic differences is in children's understanding of the base-ten principle that underlies the structure of Arabic numerals. This structure is a feature of a particular representational system rather than a fundamental mathematical fact, but it is a feature that is incorporated into many of the algorithms children learn for performing arithmetic and thus is a powerful concept in early mathematical development. Because English number names do not show a base-ten structure as consistently or as early in the count sequence as do Chinese number names, English-speaking children's conceptual understanding of this base-ten structure may be delayed compared with their Chinese-speaking peers.

Miura and her colleagues (Miura, 1987; Miura and Okamoto, 2003; Miura et al., 1993) have looked at the base-ten understanding of two groups of first grade children: speakers of East Asian languages, whose number-naming systems incorporate a clear base-ten structure, usually based on Chinese, and speakers of European languages, which generally do not show a clear base-ten structure in their number names. The primary task used is asking children to represent the cardinal value associated with a given number name using sets of blocks representing units and tens. Children whose native language is Chinese, Korean, or Japanese are consistently more likely to represent numbers as sets of tens and ones as either a first or second choice than are children whose native language is English, French, or Swedish.

Ho and Fuson (1998) compared the performance of Chinese-speaking preschool children in Hong Kong with English-speaking children in Britain and the United States. They found that half of the Chinese-speaking 5-year-olds (but none of the English-speaking children) who could count to at least 50 were able to take advantage of the base-ten structure of number names to quickly determine the answer to addition problems of the form "10 + n = ?," compared with other problems. Fuson and Kwon (1992) argued that the Chinese number-naming structure facilitates the use of a tens-complement strategy for early addition. In this approach, when adding numbers whose sum is greater than 10 (e.g., 8 + 7), the smaller addend is partitioned into the tens-complement of the first addend (2) and the remainder (5); the answer is 10 plus that remainder (10 + 5). In Chinese-structured number-naming systems, the answer corresponds to the result of the calculation ("shi wu" – "10 5"); in English, there is an additional step as the answer is converted into a different number name ("fifteen"). Fuson and Kwon reported that most Korean first graders they tested used this method before it was explicitly taught in school. Explicit instruction may be required for English-speaking children, but there is evidence that it can be quite successful, even with children from at-risk populations. Fuson and her colleagues (Fuson, Smith, and Lo Cicero, 1997) report success with explicitly teaching low-SES urban first graders about the base-ten structure of numbers, with the result that their end-of-year arithmetic performance approximated that reported for East Asian children.

## LEARNING DISABILITIES IN MATHEMATICS

Mathematics learning disabilities appear in 6 to 10 percent of the elementary school population (Barberisi et al., 2005). Many more children struggle in one or more mathematics content area at some point during their school careers (Geary, 2004). Although less research has been devoted to mathematical than to reading disabilities (Geary and Hoard, 2001; Ginsburg, 1997), considerable progress has been made over the past two decades with respect to understanding the nature of the mathematics difficulties and disabilities that children experience in school (Gersten, Jordan, and Flojo, 2005).

## Characteristics of Learning Difficulties

Poor computational fluency is a signature characteristic of mathematics learning disabilities in elementary school (e.g., Geary, 2004; Hasselbring, Goin, and Bransford, 1988; Jordan and Montani, 1997; Jordan, Hanich, and Kaplan, 2003a, 2003b; Ostad, 1998; Russell and Ginsburg, 1984). Computational fluency refers to accurate, efficient, and flexible computation with basic operations. Weak knowledge of facts reduces cognitive and attentional resources that are necessary for learning advanced mathematics (Goldman and Pellegrino, 1987). Computational fluency deficits can be reliably identified in the first few years of school and, if not addressed, are very persistent throughout elementary and middle school (Jordan, Hanich, and Kaplan, 2003b).

Children around the world move through a learning path of levels of solution methods for addition and subtraction problems. These levels become progressively more abstract, abbreviated, embedded, and complex. As they move through the levels, many children use a mix of strategies that vary according to number size and aspects of the problem situation (Geary and Burlinghman-Dubree, 1989; Siegler and Jenkins, 1989; Siegler and Robinson, 1982; Siegler and Shipley, 1995).

In contrast, young children with a mathematics learning disability rely on the most primitive Level 1 methods for extended periods in elementary school, do not use efficient counting procedures (e.g., counting on from the larger addend), and make frequent counting errors while learning to add and subtract (Geary, 1990). They also lag behind other children in the accuracy and linearity of their number line estimates (Geary et al., 2007). Researchers have differentiated children with a specific mathematics learning disability from those with a comorbid learning disability in both mathematics and reading. Jordan and colleagues (Hanich et al., 2001; Jordan, Hanich, and Kaplan, 2003a; Jordan, Kaplan, and Hanich, 2002) as well as other researchers (e.g., Geary, Hamson, and Hoard, 2000; Landerl, Bevan, and Butterworth, 2004) suggest that the nature of the mathematical deficits is similar for both groups, although children with the comorbid condition show lower performance overall. What differentiates children with a mathematics-only disability from those with combined mathematics and reading learning disabilities is that the former group performs better on word problems in mathematics, which depend on language comprehension as well as calculation facility. The potential for catching up in mathematics is much better for children with a mathematics-only disability, who can exploit their relative strength in general language to compensate for their deficiencies with numbers.

Some research shows that mathematics learning disabilities can be traced to early weaknesses in number, number relationships, and number operations as opposed to more general cognitive deficits (e.g., Gersten et al., 2005; Malofeeva et al., 2004). Weak number competency is reflected in poorly developed counting procedures, slow fact retrieval, and inaccurate computation, all characteristics of the disability (Geary et al., 2000; Jordan, Hanich, and Kaplan, 2003a). Skill with number combinations is tied to fundamental number knowledge (Baroody and Rosu, 2006; Locuniak and Jordan, in press). Accurate and efficient counting procedures can lead to strong connections between a problem and its solution (Siegler and Shrager, 1984). Developmental dyscalculia, a severe form of mathematics disability that has a known neurological basis, is explained more by domain-specific impairments in number knowledge than by domain-general deficits related to memory, spatial processing, or language (Butterworth and Reigosa, 2007). Although debate continues about the underpinnings of mathematics learning disabilities and diagnostic criteria (e.g., Geary et al., 2007), weakness in number sense appears to be a common theme in the literature. This finding has instructional implications for young children's mathematics education. Specifically, early interventions that focus on number sense have the potential to improve children's mathematics learning outcomes.

## Helping High-Risk Children

Early number competencies serve as a foundation for learning formal mathematics (Griffin et al., 1994; Miller, 1992). Deficits in these can prevent children from benefiting from formal mathematics instruction when they enter school, regardless of whether they are associated with environmental disadvantages or with genuine learning differences or disabilities (Baroody and Rosu, 2006; Griffin, 2007). In a recent study, Jordan and colleagues (in press) found that poor mathematics achievement is mediated by low number sense regardless of children's social class. That is, deficits in number sense are a better predictor of poor mathematics achievement than SES when all else is equal. Implications of this work suggest that children from low-income backgrounds and those with mathematics difficulties would benefit from a mathematics intervention during the early years (Jordan et al., in press).

Number competencies appear to have neurological origins, with their core components (e.g., subitization and approximate number representations) developing without much formal instruction (Berch, 2005; Dehaene, 1997; Feigenson, Dehaene, and Spelke, 2004). These early foundations provide support for learning more complex number skills involving number words, number comparisons, and counting. Children with mathematics difficulties seem to have problems with the symbolic system of number, rather than the universal analog magnitude system. Knowledge of the symbolic number system is heavily influenced by experience and instruction (Geary, 1995; Levine et al., 1992). Engaging young children in number activities (e.g., a mother or preschool teacher asking a child to give her 4 cookies) and simple games (e.g., board games that emphasize 1-to-1 correspondences, counting, and moving along number paths) are important for strengthening foundations and building conventional number knowledge (Gersten et al., 2005, Klibanoff et al., 2006; Levine et al., in preparation). Case and Griffin (1990) report that number sense learning is closely associated with children's home experiences with number concepts (e.g., reading number books with children). Moreover, efforts to teach number-related skills to high-risk kindergartners show promise for improving mathematics achievement (Griffin et al., 1994). In a recent study, Ramani and Siegler (2008) showed that playing a number board game that involved counting on squares on a number path improved the performance of 5-year-olds from low-income backgrounds on counting, numeral identification, numerical magnitude estimation, and number line estimation, and that the gains held after a follow-up several weeks later. Importantly, children playing this game said the number words written on the squares as they counted on one or two more, rather than saying "one" or "two" as they counted on. Playing games to help children master basic number, counting, and arithmetic concepts and skills has long been advocated by mathematics educators (e.g., Baroody, 1987; Ernest, 1986; Wynroth, 1986)—a proposition that is supported by research (for reviews, see, e.g., Baroody, 1999; Bright, Harvey, and Wheeler, 1985).

The effects of weaknesses in early mathematics, if not addressed, are likely to be felt throughout the school years and beyond. There is good reason to believe that early intensive instruction, both at home and at school, will give children the background they need to achieve at grade level in elementary school mathematics and help "shape the course of their mathematical journey" (Griffin, 2007, p. 392).

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# Part II

# Teaching-Learning Paths

In Part II, we lay out a sequence of milestones for children ages 2-7 in the core areas of number (which includes whole number, relations, and operations) and geometry and measurement. We call this sequence a *teaching-learning path*. A teaching-learning path consists of the significant steps in learning in a particular topic; each new step in the learning path builds on the earlier steps. These paths are based on research that shows that young children generally follow particular paths when learning number-relations-operations and geometry-measurement (Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b; Ginsburg, 1983). Of course, learning is a continuous process, but to overview the process, we have identified four related steps organized by age/grade. The four steps move from children 2 and 3 years old, to children age 4 or in prekindergarten, to children in kindergarten, to children in Grade 1. Grade 1 is included to indicate how the knowledge from the earlier step is used—and vital for doing well—in Grade 1.

For our purposes, we define the core mathematical ideas as those that are mathematically central and coherent, consistent with the thinking of children who have had adequate mathematical experiences, and generative of future learning. Thus, they are foundational mathematically and developmentally. They are achievable for children of these ages. That is, they are consistent with children's ways of thinking, developing, and learning when they have experience with mathematics ideas. In addition, they are interesting to children. The committee recommends that all children learn this mathematics by the end of kindergarten.

In Chapter 2, we discussed why these core ideas are important mathematically. Here we focus on how they develop in children who have opportunity to learn them as the ideas become increasingly sophisticated and interconnected over these years. Relationships among the ideas as well as some of children's common errors are also discussed. Vital ideas for Grade 1 are briefly overviewed to indicate how younger children's knowledge is developed and extended into Grade 1.

As noted in Chapter 2, we are building on earlier efforts to articulate appropriate mathematics content for young children. In 1989, the National Council of Teachers of Mathematics (NCTM) issued *Curriculum and Evaluation Standards for School Mathematics*. This document described 13 curriculum standards for the grade band K-4 (as well as for the grade bands 5-8 and 9-12). Although these standards have been influential, they do not describe the mathematics to be learned in detail and did not give guidance by grade level, nor for children younger than kindergarten.

In 2000, NCTM released *Principles and Standards for School Mathematics* (PSSM) after an extensive process of revision of the 1989 standards. Prekindergarten (pre-K) was included this time, in the grade band pre-K–2. PSSM described five content standards—number and operations, algebra, geometry, measurement, and data analysis and probability—and five process standards—problem solving, reasoning and proof, communication, connections, and representations—for each of four grade bands (pre-K–2, 3-5, 6-8, 9-12), covering all of school mathematics from pre-K through the end of high school. Although PSSM discussed the mathematics to be learned at the grade bands in greater detail than the 1989 standards did, it still did not specify what was to be learned at individual grade levels.

Recognizing the need for more in-depth attention to prekindergarten, early childhood educators and mathematics educators convened in 2000 and publish a conference report on the development of mathematics standards for young children. The resulting book, *Engaging Young Children in Mathematics: Findings of the 2000 National Conference on Standards for Preschool and Kindergarten Mathematics Education* (Clements, Sarama, and DiBiase, 2004), contains 17 recommendations for early childhood mathematics education. They concern equity, programs, teaching, teachers and their development, assessment, appropriate mathematics for young children, and broader efforts to inform stakeholders and encourage collaboration in early childhood education and addressing the need for age/grade level standards. That report grouped the mathematics content for early childhood into four topic areas: number and operations, geometry, measurement, and algebra, patterns, and data analysis.

In 2002, the National Association for the Education of Young Children (NAEYC) and NCTM approved a joint position statement, "Early Childhood Mathematics: Promoting Good Beginnings." The statement includes 10 research-based recommendations to guide practice and 4 policy recommendations. The statement includes sample charts of learning paths related to a number goal and a geometry goal with activity examples.

In 2006, NCTM released *Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence* (hereafter *Curriculum Focal Points*). These were developed in response to inconsistency in placement of topics by grade level in the United States and the lack of focus ("a mile wide and an inch deep") typical of U.S. mathematics curricula. Although much shorter than PSSM (and developed over a much shorter time), this report gives grade-level recommendations for each individual grade from pre-K to grade 8. These grade-level recommendations do not specify a full curriculum but rather describe the most significant mathematical concepts and skills at each grade level. There are three focal points at each grade level, each of which is a coherent cluster of skills and ideas, sometimes cutting across NCTM's five content strands. *Curriculum Focal Points* recommends that instruction at a grade level should devote the vast majority of attention to the content identified in the three focal points (p. 6). At pre-K and kindergarten, the three focal points concern number and operations, geometry, and measurement.

In addition to the three focal points at each grade level, *Curriculum Focal Points* describes connections, which consist of related content, including contexts and material to receive continuing development from previous grade levels. At pre-K, the connections concern data analysis, number and operations, and algebra. At kindergarten, the connections concern data analysis, geometry, and algebra. Collectively, these previous reports form the basis for the descriptions of foundational and achievable mathematics content in this report. The current report provides guidance on the two most critical mathematical areas during early childhood: number and operation and geometry and measurement, and as will be discussed later, number and operations is the area where young children need to spend the most time. Meaningful learning experiences in these content areas provide young children with the foundation that is necessary for them to be successful in later mathematics.

## SUPPORTING LEARNING IN MATHEMATICS

Our view of children is one of powerful and intrinsically motivated mathematics learners who, in a supportive physical and social environment, spontaneously learn some aspects of mathematics and make connections and extensions. However, children need adult guidance to help them learn the many culturally important aspects of mathematics, such as language and counting. In preschools and care centers, all children will bring to each mathematical topic area some initial competencies and knowledge on which to build. The major teaching challenge is to build a mathematical learning and teaching environment in which children will learn at least the basics of each topic area. This will enable them to practice and build on their own knowledge, with guidance from adults, peers, and family members, and be supported to move through learning paths to learn the foundational and achievable content identified in this report. These teaching and learning environments need to be consistent with the process goals outlined in Chapter 2, and they need to support children to be active in thinking about and discussing mathematical ideas.

Children require significant amounts of time to develop the foundational mathematical skills and understandings they have the desire and potential to learn and that they will need for success at school. Although some children have a sufficiently enriched home environment and enough mathematically focused interactions with family members so that they develop many of the necessary foundational mathematical understandings and skills at home, others do not. For the sake of equity, preschool programs should help children develop foundational mathematical understandings and skills; high-quality preschool programs that devote sufficient time to mathematics are able to do so (see Chapter 7). Even children who learn mathematical ideas at home will benefit from a consistent high-quality program experience in the preschool and kindergarten years. It is therefore critical that sufficient time is devoted to mathematics instruction in preschool programs so that children develop the foundational mathematical skills and understandings described here. Time must be allocated not only for the more formal parts of mathematics instruction and discussions that occur in the whole group or in small groups, but also for children to elaborate and extend their mathematical thinking by exploring, creating, and playing.

The time that is allotted for mathematics in early childhood programs must be allocated across various topics. The typical description of mathematics content is divided into the five strands of number and operations, algebra, geometry, measurement, and data analysis and probability. These are used to describe and categorize all of school mathematics, from pre-K through high school, and these strands are intended to receive different amounts of emphasis at different grade levels.

The committee is concerned that inclusion of all five strands for young children has led some programs and teachers to spread their mathematics time equally across these different content areas, thus spreading mathematical experiences too thinly and not going deeply enough into the core foundations that children need to establish firmly. It is important to concentrate on number and operations and on geometry and measurement in the early childhood period, with a greater portion of time spent on number and operations. Number is critically important to all of later mathematics. Geometry and measurement play an important supporting role in the development of number concepts and are themselves important to later mathematics. In addition, research on programs that result in positive learning gains for children indicate that children need sufficient time working with these ideas in order to achieve a level of proficiency that prepares them for continuing success in mathematics. Of course, many activities overlap these topic areas and could be counted in either if there is a balanced focus on both. The time spent on number and operations and on geometry and measurement can also include connections to data analysis and patterns, as listed in *Curriculum Focal Points* and discussed in the chapters of Part II.

The kind of learning involved in various number and operation components and in various aspects of geometry and measurement is different, as we describe. Major themes of these variations in the kinds of learning are the need for achieving fluency, the use of patterns, generalizing, and extending. All of these require many repeated experiences with the same numbers and related similar tasks. This is part of what makes learning mathematics require so much time focused on mathematical content.

Mathematics is a participant sport. Children must play it frequently to become good at it. They do need frequent modeling of correct performance, discussion about the concepts involved, and frequent feedback about their performance. Both modeling and feedback can come from other students as well as from adults, and feedback also sometimes comes from the situation. All children must have sustained and frequent times in which they themselves enact the core mathematical content and talk about what they are doing and why they are doing it. In mathematics learning, effort creates ability.

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# 5

# The Teaching-Learning Paths for Number, Relations, and Operations

In this chapter we describe the teaching-learning paths for number, relations, and operations at each of the four age/grade steps (2- and 3-year-olds, 4-year-olds [prekindergarten], kindergarten, and Grade 1). As noted, the four steps are convenient age groupings, although, in fact, children's development is continuous. There is considerable variability in the age at which children do particular numerical tasks (see the reviews of the literature in Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b; also see Chapter 4). However, a considerable amount of this variability comes from differences in the opportunities to learn these tasks and the opportunity to practice them with occasional feedback to correct errors and extend the learning. Once started along these numerical learning paths, children become interested in consolidating and extending their knowledge, practicing by themselves and seeking out additional information by asking questions and giving themselves new tasks. Home, child care, and preschool and school environments need to support children in this process of becoming a self-initiating and self-guiding learner and facilitate the carrying out of such learning. Targeted learning path time is also needed—time at home or in an early childhood learning center—that will support children in consolidating thinking at one step and moving along the learning path to the next step.

Although we consider the mathematics goals described in this and the next chapter foundational and achievable for all children in the designated age range for that step, we recognize that some children's learning will be advanced while others' functioning will be significantly behind. Children at particular ages/grades may be able to work correctly with larger numbers or more complex geometric ideas than those we specify in the various tables and text. Each subsequent step assumes that children have had sufficient experiences with the topics in the previous step to learn the earlier content well. (See Box 5-1 for a discussion of what it means to learn something well.) However, many children can still learn the content at a given step without having fully mastered the previous content if they have sufficient time to learn and practice the more challenging content. Of course, some children have difficulty in learning certain kinds of mathematical concepts, and a few have really significant difficulties. But most children are capable of learning the foundational and achievable mathematics content specified in the learning steps outlined here.

In both the number and operations and the geometry and measurement core areas, children learn about the basic numerical or geometric concepts and objects (numbers, shapes), and they also relate those objects and compose/decompose (operate on) them. Therefore, each core area begins by discussing the basic objects and then moves to the relations and operations on them. In all of these, it is important to consider how children perceive, say, describe/discuss, and construct these objects, relations, and operations.

The development of the elements of the number core across ages is described first, and then the development of the relations and operations core

**BOX 5-1**
**Learning Something Well**

In most aspects of the number and the relations/operation core, children need a great deal of practice doing a task, even after they can do it correctly. The reasons for this vary a bit across different aspects, and no single word adequately captures this need, because the possible words often have somewhat different meanings for different people.

*Overlearning* can capture this meaning, but it is not a common word and might be taken to mean something learned beyond what is necessary rather than something learned beyond the initial level of correctness. *Automaticity* is a word with technical meaning in some psychological literature as meaning a level of performance at which one can also do something else. But to some people it carries only a sense of rote performance. *Fluency* is the term used by several previous committees, and we have therefore chosen to continue this usage. *Fluency* also carries for some a connotation of flexibility because a person knows something well enough to use it adaptively. We find this meaning useful as well as the usual meaning of doing something rapidly and relatively effortlessly. Research on reading in early childhood has recently used *fluency* only in the latter sense as measured by performance on standardized tests of reading, such as the Dynamic Indicators of Basic Early Literacy Skills (DIBELS). We do not mean *fluency* to be restricted to this rote sense. By *fluent* we mean accurate and (fairly) rapid and (relatively) effortlessly with a basis of understanding that can support flexible performance when needed.

is summarized. These cores are quite related, and their relationships are discussed. Box 5-2 summarizes the steps along the teaching-learning paths in the core areas. As children move from age 2 through kindergarten, they learn to work with larger and more complicated numbers, make connections across the mathematical contents of the core areas, learn more complex strategies, and move from working only with objects to using mental representations. This journey is full of interesting discoveries and patterns that can be supported at home and at care and education centers.

## THE NUMBER CORE

The four mathematical aspects of the number core identified in Chapter 2 involve culturally specific ways that children learn to perceive, say, describe/discuss, and construct numbers. These involve

1. Cardinality: Children's knowledge of cardinality (how many are in a set) increases as they learn specific number words for sets of objects they see (*I want two crackers*).
2. Number word list: Children begin to learn the ordered list of number words as a sort of chant separate from any use of that list in counting objects.
3. 1-to-1 counting correspondences: When children do begin counting, they must use one-to-one counting correspondences so that each object is paired with exactly one number word.
4. Written number symbols: Children learn written number symbols through having such symbols around them named by their number word (*That is a two*).

Initially these four aspects are separate, and then children make vital connections. They first connect saying the number word list with 1-to-1 correspondences to begin counting objects. Initially this counting is just an activity without an understanding of the total amount (cardinality). If asked the question *How many are there?* after counting, children may count again (repeatedly) or give a number word different from the last counted word. Connecting counting and cardinality is a milestone in children's numerical learning path that coordinates the first three aspects of the number core.

As noted, we divide the teaching-learning path into four broad steps. In Step 1, for 2- and 3-year-olds, children learn about the separate aspects of number and then begin to coordinate them. In Step 2, for approximately 4-year-olds/prekindergartners, children extend their understanding to larger numbers. In Step 3, for approximately 5-year-olds/kindergartners, children integrate the aspects of number and begin to use a ten and some ones in teen numbers. In Step 4, approximately Grade 1, children see, count, write, and work with tens-units and ones-units from 1 to at least 100.

## BOX 5-2
## Overview of Steps in the Number, Relations, and Operations Core

### Steps in the Number Core

**Step 1 (ages 2 and 3):** Beginning 2- and 3-year-olds learn the number core components for very small numbers: cardinality, number word list, 1-1 counting correspondences, and written number symbols; later 2- and 3-year-olds coordinate these number core components to count **n** things and, later, say the number counted.

**Step 2 (age 4/prekindergarten):** Extend all four core components to larger numbers and also use conceptual subitizing if given learning opportunities to do so.

**Step 3 (age 5/kindergarten):** Integrate all core components, see a ten and some ones in teen numbers, and relate ten ones to one ten and extend the core components to larger numbers.

**Step 4 (Grade 1):** See, say, count, and write tens-units and ones-units from 1 to 100.

### Steps in the Relations (More Than/Less Than) Core

**Step 1 (ages 2 and 3):** Use perceptual, length, and density strategies to find which is more for two numbers ≤ 5.

**Step 2 (age 4/prekindergarten):** Use counting and matching strategies to find which is more (less) for two numbers ≤ 5.

**Step 3 (age 5/kindergarten):** Kindergartners show comparing situation with objects or in a drawing and match or count to find out which is more and which is less for two numbers ≤ 10.

**Step 4 (Grade 1):** Solve comparison word problems that ask, "How many more (less) is one group than another?" for two numbers ≤ 18.

### Steps in the Addition/Subtraction Operations Core

**Step 1 (ages 2 and 3):** Use subitized and counted cardinality to solve situation and oral number word problems with totals ≤ 5; these are much easier to solve if objects present the situation rather than the child needing to present the situation and the solution.

**Step 2 (age 4/prekindergarten):** Use conceptual subitizing and cardinal counting of objects or fingers to solve situation, word, and oral number word problems with totals ≤ 8.

**Step 3 (age 5/kindergarten):** Use cardinal counting to solve situation, word, oral number word, and written numeral problems with totals ≤ 10.

**Step 4 (Grade 1):** Use counting on solution procedures to solve all types of addition and subtraction word problems: Count on for problems with totals ≤ 18 and find subtraction as an unknown addend.

### Step 1 (Ages 2-3)

At this step, children first begin to learn the core components of number: cardinality, the number word list, 1-to-1 correspondences, and written number symbols (see Box 5-3).

**BOX 5-3**
**Step 1 in the Number Core**

Children at particular ages/grades may exceed the specified numbers and be able to work correctly with larger numbers. The numbers for each age/grade are the foundational and achievable content for children at this age/grade. The major types of new learning for each age/grade are given in italics. Each level assumes that children have had sufficient learning experiences at the lower level to learn that content; many children can still learn the content at a level without having fully mastered the content at the lower level if they have sufficient time to learn and practice.

**Beginning 2- and 3-Year-Olds Learn the Number Core Components**

**Cardinality:** *How many animals (crackers, fingers, circles, . . . )?* uses perceptual subitizing to give the number for 1, 2, or 3 things.

**Number word list:** *Count as high as you can (no objects to count)* says 1 to 6.

**1-to-1 counting correspondences:** *Count these animals (crackers, fingers, circles, . . . )* or *How many animals (crackers, fingers, circles, . . . )?* counts accurately 1 to 3 things with 1-1 correspondence in time and in space.

**Written number symbols:** *This (2, 4, 1, etc.) is a______?* knows some symbols; will vary.

**Later 2- and 3-Year-Olds Coordinate the Number Core Components**

**Cardinality:** *Continues to generalize perceptual subitizing to new configurations and extends to some instances of conceptual subitizing for 4 and 5:* can give number for 1 to 5 things.

**Number word list:** *Continues to extend and may be working on the irregular teen patterns and the early decade twenty to twenty-nine, etc., pattern:* says 1 to 10.

**1-to-1 counting correspondences:** *Continues to generalize to counting new things, including pictures, and to extend accurate correspondences to larger sets (accuracy will vary with effort):* counts accurately 1 to 6 things.

**Written number symbols:** *Continues to learn new symbols if given such learning opportunities.*

**Coordinates counting and cardinality into cardinal counting** in which the last counted word tells how many and (also or later) tells the cardinality (the number in the set).

*Cardinality*

The process of identifying the number of items in a small set (cardinality) has been called *subitizing*. We will call it *perceptual subitizing* to differentiate it from the more advanced form we discuss later for larger numbers called *conceptual subitizing* (see Clements, 1999). For humans, the process of such verbal labeling can begin even before age 2 (see Chapter 3). It first involves objects that are physically present and then extends to nonpresent objects visualized mentally (for finer distinctions in this process, see Benson and Baroody, 2002). This is an extremely important conceptual step for attaching a number word to the perceived cardinality of the set. In fact, there is growing evidence that the number words are critical to toddlers' construction of cardinal concepts of even small sets, like three and four and possibly one and two (Benson and Baroody, 2002; Spelke, 2003; also see Baroody, Lai, and Mix, 2006; and Mix, Sanhofer, and Baroody, 2005).

Children generally learn the first 10 number words by rote first and do not recognize their relation to quantity (Fuson, 1988; Ginsburg, 1977; Lipton and Spelke, 2006; Wynn, 1990). They do, however, begin to learn sets of fingers that show small amounts (cardinalities). This is an important process, because these finger numbers will become tools for adding and subtracting (see research literature summarized in Clements and Sarama, 2007; Fuson, 1992a, 1992b). Interestingly, the conventions for counting on fingers vary across cultures (see Box 5-4).

In order to fully understand cardinality, children need to be able to both generalize and extend the idea. That is, they need to generalize from a specific example of two things (two crackers), to grasp the "two-ness" in any set of two things. They also need to *extend* their knowledge to larger and larger groups—from one and two to three, four, and five, although these are more difficult to see and label (Baroody, Lai, and Mix, 2006; Ginsburg, 1989). Children's early notions of cardinality and how and when they learn to label small sets with number words are an active area of research at present. The timing of these insights seems to be related to the grammatical structure of the child's native language (e.g., see the research summarized in Sarnecka et al., 2007).

Later on, children can learn to quickly see the quantity in larger sets if these can be decomposed into smaller subitized numbers (e.g., *I see two and three, and I know that makes five*). Following Clements (1999), we call such a process *conceptual subitizing* because it is based on visually apprehending the pair of small numbers rather than on counting them. Conceptual subitizing requires relating the two smaller numbers as addends within the conceptually subitized total. With experience, the move from seeing the smaller sets to seeing and knowing their total becomes so rapid that one can experience this as seeing 5 (rather than as seeing 2 and 3). Children may also learn particular patterns, such as the 5 pattern on a die. Because these

**BOX 5-4**
**Using Fingers to Count: Cultural Differences**

Around the world, most children learn from their family one of the three major ways of raising (or in some cultures, lowering) fingers to show numbers. All of these methods can be seen in centers or schools with children coming from different parts of the world, as well as some less frequent methods (the Indian counting on cracks of fingers with the thumb, Japanese lowering and raising fingers). The most common way is to raise the thumb first and then the fingers in order across to the small finger. Another way is to raise the index finger, then the next fingers in order to the smallest finger, and then the thumb. The third way is to begin with the little finger and move across in order to the thumb. The first way is very frequent throughout Latin America, and the third way also is used by some children coming from Latin America. The second way is the most usual in the United States. It is the common way to show ages (for example, *I am two years old* by holding up the index and largest finger). This method allows children to hold down unused fingers with their thumb. But the other two methods show numbers in a regular pattern going across the fingers. Children in a center or school where children show numbers on fingers in different ways may come to use multiple methods. Because fingers are such an important tool for numerical problem solving, it is probably best not to force a child to change his or her method of showing numbers on fingers if it is well established. It is important for teachers to be aware and accepting of these differences.

kinds of patterns can also be considered in terms of addends that compose them, they are included in conceptual subitizing. Such patterns can help older children learn mathematically important groups, such as five and ten; these are discussed in the later levels and in the relations and operation core discussion of addition and subtraction composing/decomposing.

Children also learn to assign a number to sets of entities they hear but do not see, such as drum beats or ringing bells. There is relatively little research on auditory quantities, and they play a much smaller role in everyday life or in mathematics than do visual quantities. For these reasons, and because auditory quantities relate to music and rhythm and body movements, it seems sensible to have some activities in the classroom in which children repeat simple or complex sets they hear (clap clap or, later, clap clap clap pause clap clap), tell the number they hear (of bells, drumbeats, feet stamping, etc.), and produce sounds with body movements for particular quantities (*Let me hear three claps*).

In home and care/educational settings, it is important that early experiences with subitizing be provided with simple objects or pictures. Textbooks or worksheets often present sets that discourage subitizing and depict collections of objects that are difficult to count. Such complicating factors include embedded or overlapping pictures, complex noncompact things or pictures (e.g., detailed animals of different sizes rather than circles or squares), lack of symmetry, and irregular arrangements (Clements and Sarama, 2007).

The importance of facilitating subitizing is underscored by a series of studies, which first found that children's spontaneous tendency to focus on numerosity was related to counting and arithmetic skills, then showed that it is possible to enhance such spontaneous focusing, and then found that doing so led to better competence in cardinality tasks (Hannula, 2005). Increasing spontaneous focusing on numerosity is an example of helping children mathematize their environment (seek out and use the mathematical information in it). Such tendencies can stimulate children's self-initiated practice in numerical skills because they notice those features and are interested in them.

### *Number Word List*

A common activity in many families and early childhood settings is helping a child learn the list of number words. Children initially may say numbers in the number word list in any order, but rapidly the errors take on a typical form. Children typically say the first part of the list correctly, and then may omit some numbers in the next portion of the list, or they say a lot of numbers out of order, often repeating them (e.g., one, two, three, four, five, eight, nine, four, five, two, six) (Fuson, 1988; Fuson, Richards, and Briars, 1982; Miller and Stigler, 1987; Siegler and Robinson, 1982). Children need to continue to hear a correct number list to begin to include the missing numbers and to extend the list.

Children can learn and practice the number word list by hearing and saying it without doing anything else, or it can be heard or said in coordination with another activity. Saying it alone allows the child to concentrate on the words, and later on the patterns in the words. However, it is also helpful to practice in other ways to link the number words to other aspects of the number core. Saying the words with actions (e.g., jumping, pointing, shaking a finger) can add interest and facilitate the 1-to-1 correspondences in counting objects. Raising a finger with each new word can help in learning how many fingers make certain numbers, and flashing ten fingers at each decade word can help to emphasize these words as made from tens.

### *Counting: 1-to-1 Correspondences*

In order to count a group of objects the person counting must use some kind of action that matches each word to an object. This often involves moving, touching or pointing to each object as each word is said. This counting action requires two kinds of correct matches (1-to-1 correspondences): (1) the matching in a moment of *time* when the action occurs and a word is said, and (2) the matching in *space* where the counting action points to an object once and only once. Children initially make errors in both of these kinds of correspondences (e.g., Fuson, 1988; Miller et al., 1995). They may violate the matching in time by pointing and not saying a word or by pointing and saying two or more words. They may also violate the matching in space by pointing at the same object more than once or skipping an object; these errors are often more frequent than the errors in time.

Four factors strongly affect counting correspondence accuracy: (1) amount of counting experience (more experience leads to fewer errors), (2) size of set (children become accurate on small sets first), (3) arrangement of objects (objects in a line make it easier to keep track of what has been counted and what has not), and (4) effort (see research reviewed in Clements and Sarama, 2007, and in Fuson, 1988). Small sets (initially up to three and later also four and five) can be counted in any arrangement, but larger sets are easier to count when they are arranged in a line. Children ages 2 and 3 who have been given opportunities to learn to count objects accurately can count objects in any arrangement up to 5 and count objects in linear arrangements up to 10 or more (Clements and Sarama, 2007; Fuson, 1988).

In groundbreaking research, Gelman and Gallistel (1978) identified five counting principles that stimulated a great deal of research about aspects of counting. Her three how-to-count principles are the three mathematical aspects we have just discussed: (1) the *stable order principle* says that the number word list must be used in its usual order, (2) the *one-one principle* says that each item in a set must be tagged by a unique count word, and (3) the *cardinality principle* says that the last number word in the count list represents the number of objects in the set. Her two what-to-count principles are mathematical aspects we have also discussed: (1) the *abstraction principle* states that any combination of discrete entities can be counted (e.g., heterogeneous versus homogeneous sets, abstract entities, such as the number of days in a week) and (2) the *order irrelevance principle* states that a set can be counted in any order and yield the same cardinal number (e.g., counting from right to left versus left to right).

Gelman took a strong position that children understand these counting principles very early in counting and use them in guiding their counting activity. Others have argued that at least some of these principles are understood only after accurate counting is in place (e.g., Briars and Siegler, 1984). Still others, taking a middle ground between the "principles before" view and the "principles after" view, suggest that there is a mutual (e.g., iterative) relation between understanding the count principles and counting skill (e.g., Baroody, 1992; Baroody and Ginsburg, 1986; Fuson, 1988; Miller, 1992; Rittle-Johnson and Siegler, 1998).

Each of these aspects of counting is complex and does not necessarily exist as a single principle that is understood at all levels of complexity at once. Children may initially produce the first several number words and not even separate them into distinct words (Fuson, Richards, and Briars, 1982). They may think that they need to say the number word list in order as they count, but early on they cannot realize the implication that they need a unique last counted word, or they would not repeat words so frequently as they say the number word list.

The what-to-count principles also cover a range of different understandings. It takes some time for children to learn to count parts of a thing (Shipley and Shepperson, 1990; Sophian and Kailihiwa, 1998), a later use of the abstraction principle. And the order irrelevance principle (counting in any order will give the same result) seems to be subject to expectations about what is conventional "acceptable" counting (e.g., starting at one end of a row rather than in the middle) as well as involving, later on, a deeper understanding of what is really involved in 1-to-1 correspondence: Count-

**BOX 5-5**
**Common Counting Errors**

There are some common counting errors made by young children as they learn the various principles that underpin successful counting. Counting requires effort and continued attention, and it is normal for 4-year-olds to make some errors and for 5-year-olds to make occasional errors, especially on larger sets (of 15 or more for 4-year-olds and of 25 or more for 5-year-olds). Younger children may initially make quite a few errors. It is much more important for children to be enthusiastic counters who enjoy counting than for them to worry so much about errors that they are reluctant to count. If one looks at the proportion of objects that receive one word and one point, children's counting often is pretty accurate. Letting errors go sometimes or even somewhat frequently if children are trying hard and just making the top four kinds of errors is fine as long as children *understand* that correct counting requires one point and one word for each object and are trying to do that. As with many physical activities, counting will improve with practice and does not need to be perfect each time. Teachers do not have to monitor children's counting all of the time. It is much more important for all children to get frequent counting practice and watch and help each other, with occasional help and corrections from the teacher.

Very young children counting small rows with high effort make more errors in which their say-point actions do not correspond than errors in the matching of the points and objects. Thus, they may need more practice coordinating their actions of saying one word and pointing at an object. Energetic collective practice in which children rhythmically say the number word list and move down their hand with a finger pointed as each word is said can be helpful. To vary the practice, the words can sometimes be said loudly and sometimes softly, but always with emphasis (a regular beat). The points can involve a large motion of the whole arm or a smaller motion, but, again, in a regular beat with each word. Coordinating these actions ing is correct if and only if each object receives one number word (LeFevre et al., 2006). An aspect of the 1-to-1 principle that is difficult even for high school students or adults to execute is remembering exactly which objects they have already counted with a large fixed set of objects scattered irregularly around (such as in a picture) (Fuson, 1988).

The principles are useful in understanding children's learning to count, but they should not be taken as simplistic statements that describe knowledge that is all-or-nothing or that has a simple relationship to counting skill. It can be helpful for teachers or parents to make statements of various aspects of counting (e.g., *Remember that each object needs one point and one number word, You can't skip any, Remember where you started in the circle so you stop just before that.*). But children will continue to make counting errors even when they understand the task, because counting is a complex activity (see Box 5-5).

of saying and pointing is the goal for overcoming this type of error. For variety, these activities can involve other movements, such as marching around the room with rhythmic arm motions or stamping a foot saying a count word each time.

Counting an object twice or skipping over an object are errors made occasionally by 4-year-olds and even by 5-year-olds on larger sets. These seem to stem from momentary lack of attention rather than lack of coordination. Trying hard or counting slowly can reduce these errors. However, when two counts of the same set disagree, many children of this age think that their second count is correct, and they do not count again. Learning the strategy of counting a third time can increase the accuracy of their counts. If children are skipping over many objects, they need to be asked to *count carefully and don't skip any*.

Young children sometimes make multiple count errors on the last object. They either find it difficult to stop or think they need to say a certain number of words when counting and just keep on counting so they say that many. When they say the number word list, more words are better, so they need to learn that saying the number word list when counting objects is controlled by the number of objects. Reminding them that even the last object only gets one word and one point can help. They also may need the physical support of holding their hand as they reach to point to the last object so that the hand can be stopped from extra points and the last word is said loudly and stretched out (e.g., *fii-i-i-ve*) to inhibit saying the next word.

Regularity and rhythmicity are important aspects of counting. Activities that increase these aspects can be helpful to children making lots of correspondence errors. Children who are not discouraged about their counting competence generally enjoy counting all sorts of things and will do so if there are objects they can count at home or in a care or education center. Counting in pairs to check each other find and correct errors is often fun for the pairs. Counting in other activities, such as building towers with blocks, should also be encouraged.

*Written Number Symbols*

Learning to read written number symbols is quite variable and depends considerably on the written symbols in children's environment and how often these are pointed out and read with a number word so that they can learn the symbol-word pair (Clements and Sarama, 2007; Mix, Huttenlocher, and Levine, 2002). Unlike much of the number core discussed so far, learning these pairs is rote learning with hardly any possibility of finding and using sequential information. Component parts of particular numbers, or an overall impression (e.g., an 8 looks like a snowman) can be identified and discussed using perceptual learning principles (Baroody, 1987; Baroody and Coslick, 1998; Gibson, 1969; Gibson and Levin, 1975). Learning to recognize the numerals is not a hugely difficult task, and 2- and 3-year-olds can often read some numerals; 4-year-olds can learn to read many of the numerals to 10. Kindergarten children with such experiences can then concentrate on reading and understanding the numerals for the teens, and first graders can master the cardinal tens and ones connections in the numerals from 20 to 100 (see discussions at those levels).

Learning to write number symbols (numerals) is a much more difficult task than is reading them and often is not begun until kindergarten. Writing numerals requires children to have an accurate mental image of the symbol, which entails left-right orientation, and a motor plan to translate the mental image into the correct sequence of motor actions to form a numeral (e.g., see details in Baroody, 1987; Baroody and Coslick, 1998; Baroody and Kaufman, 1993). Some numerals are much easier than others. The loops in 6 and 9, the curve and straight line in the 2, and the crossovers in the 8 are difficult but can be mastered by kindergarten children with effort. The easier numerals 1, 3, 4, 5, and 7 can often be mastered earlier. Whenever children do learn to write numerals, learning to write correct and readable numerals is not enough. They must become fluent at writing numerals (i.e., writing numerals must become overlearned) so that writing them as part of a more complex task is not so slow or effortful as to be discouraging when solving several problems. It is common for children at this step and even later to reverse some numerals (such as 3) because the left-right orientation is difficult for them. This will become easier with age and experience.

*Coordinating the Components of the Number Core*

We discussed above how children coordinate their knowledge of the number word list and 1-to-1 correspondences in time and in space to count groups of objects in space. They also gradually generalize what they can count and extend their accurate counting to larger sets and to sets in various arrangements not in a row (circular, disorganized). However, accuracy for the latter comes quite late, except for small sets (Fuson, 1988). Generalization of counting involves taking as a unit each object they are counting so that each object can receive one count word. For example, when they are counting toy animals, each animal is a unit regardless of how big it is, what color it is, or what kind of animal it is. Later, 2- and 3-year-olds continue to generalize the range of objects they can count. Children with little experience with print may have more difficulty counting pictures of objects rather than objects themselves, and so they may especially need practice counting pictures of objects (Murphy and Wood, 1981).

The next crucial coordination of components is connecting counting and cardinality (Fuson, 1988; Gelman and Gallistel, 1978). When counting things (objects or pictures), the counting action matches each count word to one thing (see discussion above and in Chapter 2). But a cardinal number word refers to how many things there are in the whole set of things. So when anyone counts, they must at the end of the counting action make a mental shift from thinking of the last counted word as referring to *the last counted thing* to thinking of that word as referring to *all of the things* (the number of things in the whole set, i.e., the cardinality of the set). For example, when counting 7 toy animals 1, 2, 3, 4, 5, 6, 7, the 7 refers to the one last animal you count when you say 7. But then you must shift to thinking of all of the animals and think of the 7 as meaning all of them: There are 7 animals. This is a major conceptual milestone for young children.

When children discover this relationship, they tend to apply it to all counts no matter the size of the set of objects (Fuson, 1988). Therefore, this is a type of rule/principle of learning that children immediately generalize and apply fairly consistently. It is relatively easy to teach children that the last word said in counting tells how many there are (see Fuson, 1988). For example, a statement of this principle followed by three demonstrations followed by another statement of the principle was sufficient to move 20 of 22 children ages 2 years 8 months to 3 years 11 months who did not use the principle to using it (Fuson, 1988).

However, not all children really understand cardinality, even when they understand the importance of the last counted word (Fuson, 1988). Some children initially understand only that the last word answers the "How many?" question. They do not fully grasp the more abstract idea of cardinality. Thus, they give their last counted word when asked how many there are, but they do not point to all of the objects when asked the cardinality question *"Show me the seven animals."* Instead, they point at the last animal again. It is important to note that responding with the last word is progress. Earlier when asked *"How many are there?"* children may have recounted or given a number other than the last counted word. Children who recount are understanding the question *"How many are there?"* as a request to count, not as a cardinal request. Such children may recount several if the question is repeated and may protest *But I already did it* or *I already said it* because they don't understand the reason for the repeated requests (to them, each count is a correct response to the *How many are there?* question). Children making the other error (giving a number that differs from the last word) are understanding that the question *How many are there?* is a request for cardinal information about the whole set, but they do not yet understand that the cardinal information is given by counting, and, in particular, by the last word said in counting.

Verbal knowledge is also required for full competence in discriminating the use of individual number words for each thing counted versus the use of the final number word to refer to the whole set. Even children who gesture correctly to show their count meaning (gesture to one thing) or their cardinal meaning (gesture to the whole set) may struggle with correct verbal expressions (see Box 5-6). Mastering these is a later achievement that will be learned with modeling and practice.

**BOX 5-6**
**Learning the Correct Counting Language**

Learning the singular and plural forms that go with counting (single) and with cardinal (plural) references to objects takes some time. Here are typical examples of errors that children initially make while they are sorting out all of these conceptual and linguistic issues. After children counted a row of objects, they were asked a count-reference question and a cardinality-reference question (the order varied across children). The count-reference question was *Is this the soldier (chip) where you said* ***n****?* where **n** was the last word said by the child. The experimenter asked the question three times and pointed to the last item, the next-to-last item, and all the items in the row. The cardinal-reference question was *Are these the* ***n*** *soldiers (chips)?* The correct answer was always in the middle, because research indicated that young children have a strong bias toward choosing the last alternative. In the examples below, children spontaneously verbalized cardinality or counting references that disagreed with their gesture.

| | |
|---|---|
| Response to cardinality question: | *Those are five soldiers*, said as child points to the last soldier. |
| Response to cardinality question: | *This one's the five chips*, said as child points to the last chip. |
| Response to cardinality question: | *This is the six soldiers*, said as child points to each soldier (said six times). |
| Response to cardinality question: | *This is the four chips*, said as child points to the last chip. |
| Response to cardinality question: | *This is where I said chip four*, said as child's hands gesture to all of the chips. |
| Response to count question: | *All of these animals I said five.* |

SOURCE: Fuson (1988, p. 232).

### Step 2 (Age 4 or Prekindergarten)

As children become acquainted with the components of number, they extend cardinal counting and conceptual subitizing to larger numbers. The major advances for children at this step who have had opportunities at home or in a care center to learn the previous foundational and achievable number core content involve extending their competency to larger numbers. This means that teachers or caregivers who must support children at different levels, or support a mixture of children who have learned and those who have not had sufficient opportunity to learn the previous number core content, can frequently combine these groups by allowing children to choose set sizes with which they feel comfortable and can succeed (see Box 5-7).

**BOX 5-7**
**Step 2 in the Number Core**
**Age 4 or Prekindergarten**

**Extend Cardinal Counting and Conceptual Subitizing to Larger Numbers**

Children at particular ages/grades may exceed the specified numbers and be able to work correctly with larger numbers. The numbers for each age/grade are the foundational and achievable content for children at this age/grade. The major types of new learning for each age/grade are given in italics. Each level assumes that children have had sufficient learning experiences at the lower level to learn that content; many children can still learn the content at a level without having fully mastered the content at the lower level if they have sufficient time to learn and practice.

**Cardinality:** *Extends conceptual subitizing to 5-groups with 1, 2, 3, 4, 5 to see 6 through 10:* can see the numbers 6, 7, 8, 9, 10 as 5 + 1, 5 + 2, 5 + 3, 5 + 4, 5 + 5 and can relate these to the fingers (5 on one hand). May do other such numerical compose/decompose patterns also.

**Number word list:** *Continues to extend and learns the irregular teen patterns and extends the early decade twenty to twenty-nine, etc., pattern to higher decades:* says 1 to 39.

**1-to-1 counting correspondences:** *Continues to generalize to counting new things and to extend accurate correspondences to larger sets (accuracy will vary with effort):* counts accurately 1 to 15 things in a row.

**Written number symbols:** *Continues to learn new symbols if given such learning opportunities:* reads 1 to 10; writes some numerals.

**Reverses the cardinal counting principle (the count-to-cardinal shift) to count out n things (makes the cardinal-to-count shift):** *Must have fluent counting to have the attentional space to remember the number to which you're counting so you can stop there.*

### *Cardinality*

Children at this level continue to extend to larger numbers their conceptual subitizing of small groups to make a larger number, for example, *I see one thumb and four fingers make my five fingers* (this is part of the relation and operation core and is discussed more there). The 5-groups are particularly important and useful. These 5-groups provide a good way to understand the numbers 6, 7, 8, 9, 10 as 5 + 1, 5 + 2, 5 + 3, 5 + 4, 5 + 5 (see Figure 5-1). The convenient relationship to fingers (5 on one hand) provides a kinesthetic component as well as a visual aspect to this knowledge. Without focused experience with 5-groups, children's notions of the numbers 6 through 10 tend to be hazy beyond a general sense that the numbers are getting larger. Knowing the 5-groups is helpful at the next level, as children add and subtract numbers 6 through 10; the patterns are problem-solving tools that can be drawn or used mentally. Children in East Asia learn and use these 5-group patterns throughout their early numerical learning (Duncan, Lee, and Fuson, 2000). Children can continue to experience and begin remembering other addends that make totals (e.g., 3 and 3 make 6, 8 is 4 and 4).

### *Number Word List*

As noted, beyond the first ten words, which are arbitrary in most languages (e.g., see the extensive review in Menninger, 1958/1969), most languages begin to have patterns that make them easier to learn. English, however, has irregularities that are challenging for children. A major difficulty in understanding the meaning of the teens words is that English words do not explicitly say the *ten* that is in the teen number (*teen* does not mean *ten* even to many adults), so English-speaking children can benefit

5-groups that show 6 as 5 + 1, 7 as 5 + 2, 8 as 5 + 3,
9 as 5 + 4, and 10 as 5 + 5

| 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|
| ooooo | ooooo | ooooo | ooooo | ooooo |
| o | oo | ooo | oooo | ooooo |

**FIGURE 5-1** Five groups to understand the numbers 6, 7, 8, 9, and 10.

from visual representations that show the ten inside teen numbers in order to understand what quantities these words represent (see the discussion in the kindergarten level).

There are two patterns in the English number words from 20 to 100 that children need to understand if each word is to have its value as some number of tens and some number of ones, as in Chinese words (52 is said as *five ten two*). One is the irregular pattern in the decade words that name the tens multiples: *twenty (twin-tens), thirty (three-tens), forty, fifty (five tens), sixty, seventy, eighty, ninety*. As with the teens, the relationships of the decade words to the numbers below ten become really clear only for the last four words because only then are the *six*, *seven*, *eight*, *nine* said. The irregularities in *twenty* through *fifty* interfere with seeing the meaning of these words as two tens, three tens, four tens, five tens, etc., and thus with learning these in order by using the list below ten, as Chinese-speaking children can do (see Chapter 4). Also, as with the teen words, the *ten* is not said explicitly but is said as a different suffix, *–ty*. Therefore, as discussed later for Grade 1, children need to work explicitly with groups of tens and ones to understand these meanings for the number words from 20 to 100.

The second pattern is the pattern of a decade word followed by the decade word with the numbers one through nine: *twenty, twenty-one, twenty-two, twenty-three, . . . , twenty-nine*. Children can begin to learn this second pattern quite early. Because the transition to ten and the teens words is not clear in English, children often initially do not stop at *twenty-nine* but continue to count *twenty-nine, twenty-ten, twenty-eleven, twenty-twelve, twenty-thirteen* (Fuson, 1988). This error can be a mixture of not yet understanding that the pattern ends at nine and difficulty stopping the usual counting at nine in order to shift to another decade.

Children in the United States tend to learn the pattern of the decade word followed by a number (1-9) before learning the order of the decade words (e.g., Fuson, 1988; Fuson, Richards, and Briars, 1982; Miller and Stigler, 1987; Siegler and Robinson, 1982). Although some 2- and 3-year-olds begin learning and practicing the patterns for the teens and decade words, the teen pattern can be mastered by almost all 4-year-olds with support and practice, as can the early decades (two cycles of the pattern from twenty through thirty-nine). Many 4-year-olds learn more than this, but mastering the correct order of the decades and using this with the ***n**-ty through **n**-ty-nine* pattern is for many children a kindergarten achievement (e.g., Fuson, 1988; Fuson, Richards, and Briars, 1982; Miller et al., 1995). Structured learning experiences can decrease the time it takes to learn this pattern of decades to 100, but without such experiences this learning effort can continue even to age 6. Counting by tens to 100 to learn this decade sequence is a goal for kindergarten and is discussed in that section.

*Counting Correspondences*

At this step, children extend considerably the set size they are able to count accurately. They move from considerable inaccuracy with counting larger sets to only occasional errors, even with large sets of 15 and above, unless the sets are arranged in a disorganized way and children are not able to move objects to keep track of which have been counted (i.e., make a counted and an uncounted pile) (Fuson, 1988). As before, effort continues to be important. Children who are tired or discouraged may make many more errors than they make after a simple prompt to try hard or count slowly. Children at this step also continue to generalize what they can count.

Children at this step are working on counting linear arrangements correctly in the teens or above, and many make few errors, showing considerably more accuracy than children a year younger (Clements and Sarama, 2007; Fuson, 1988). Of course, accurate counting also depends on knowing an accurate number word list, so accuracy with these larger sets depends on three things:

1. Knowing the patterns discussed above in the number word list so that a correct number word list can be said.
2. Correctly assigning one number word to one object (1-to-1 correspondence).
3. Keeping track of which objects have already been counted so that they are not counted more than once.

Differentiating counted from uncounted entities is most easily done by moving objects into a counted set, but this is not possible with things that cannot be moved, such as pictures in a book. For pictures or objects that cannot be moved, counting objects arranged in a row is easiest because one can start at the end of a row and continue to the other end. However, if objects are arranged in a circle, children may initially count on and on around the circle. Strategies for keeping track of messy, large sets continue to develop for many years (Fuson, 1988), with even adults not being entirely accurate.

Children in kindergarten who have had adequate counting experiences earlier continue to extend their counting of objects as high as 100, often with correct correspondences (and perhaps occasional errors). There may or may not still be errors in the number word list.

*Written Number Symbols*

Children at this step continue to extend the number of written number symbols they can read, now often reading many of the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. However, the 10 at this level means ten ones, the counted number *ten* that comes after *nine*. Not until the next level does it come to mean what the 10 symbols actually say: 1 ten and 0 ones. Children at this level can begin to write some numerals, often beginning with the easier numerals 1, 3, 4, and 7.

### *Counting Out "n" Things*

Children at this level make one major conceptual advance. They move from knowing that the last number stated represents the amount in the group to knowing how to count out a given number of objects (Clements and Sarama, 2007; Fuson, 1988). Lots of counting of objects and saying the number word list enables their counting to become fluent enough that they can count out a specified number of things, for example, count out 6 things. Counting out **n** things requires a child to remember the number **n** while counting. This is more difficult for larger numbers because the child has to remember the number longer. So children may initially count past **n** because their counting is not fluent (overlearned) enough to count a long sequence of words, remember a number, and monitor with each count whether they have reached the number yet. Counting out a specified number is needed for solving addition and subtraction problems and for doing various real-life tasks, so this is an important milestone. Children can practice this conceptual task by counting out **n** things for various family and school purposes; such practice can also occur in game-like activities.

Counting out **n** things also requires a conceptual advance that is the reverse of learning that the last count word tells how many there are. To count out 6 things, a child is being told how many there are (a cardinal meaning) and must then shift to a count meaning of that 6 in order to monitor the count words as they are said (*Have I said 6 yet?*) so that they can stop when they say 6 as a counting word that corresponds to one object. They then have the set of 6 things they need.

## Step 3 (Kindergarten)

At this step children work to integrate all of the core components of number. They are able to see that teen numbers are made up of tens and some ones. They also can come to understand that ten ones make one group of ten (see Box 5-8).

Kindergarten children can begin the process with seeing and making tens in teen numbers, and first graders can continue the process for tens and ones in numbers 20 to 100. At both grades this process helps children integrate the number components into a related web of cardinal, counting, and written number symbol knowledge. The first conceptual step is for chil-

**BOX 5-8**
**Step 3 in the Number Core**
**Age 5 or Kindergarten**

**Integrate All Core Components, See a Ten and Some Ones in Teen Numbers, Relate Ten Ones to One Ten, and Extend the Core Components to Larger Numbers**

Children at particular ages/grades may exceed the specified numbers and be able to work correctly with larger numbers. The numbers for each age/grade are the foundational and achievable content for children at this age/grade. The major types of new learning for each age/grade are given in italics. Each level assumes that children have had sufficient learning experiences at the lower level to learn that content; many children can still learn the content at a level without having fully mastered the content at the lower level if they have sufficient time to learn and practice.

**Cardinality:** *Extends conceptual subitizing to a new visual group, a group of tens:* can see a ten in each teen number (18 = 10 + 8).

**Number word list:** *Extends to learn all of the decades in order as a new number word list counting by tens; uses this decade order with the decade pattern to count to 100 by ones:* says the tens list 10, 20, 30, . . . , 90, 100; says 1 to 100 by ones.

**1-to-1 counting correspondences:** *Continues to extend accurate correspondences to larger sets; accuracy will still vary with effort:* counts 25 things in a row with effort.

**Written number symbols:** *Coordinates knowledge of symbols 1 to 9 to write teen numbers:* reads and writes 1 to 19; reads 1 to 100 arranged in groups of ten when counting 1 to 100.

**Integrates all of the above for teen numbers** so that ten ones = 1 ten, relating the unitary cardinality relationship *ten ones + eight ones make eighteen ones* to the written symbols 18 as 10 with an 8 on top of the 0 ones in ten.

dren to understand each cardinal teen number as consisting of two groups: 1 group of ten things and a group of the ones (the extra over ten). So, for example, 11 is 1 group of ten and 1 one (11 = 10 + 1), 14 is 1 group of ten and 4 ones (14 = 10 + 4), and 18 is 1 group of ten and 8 ones (18 = 10 + 8). The second crucial understanding that builds on the above is that ten ones equal one ten. That is, the written teen number symbols such as 18 mean 1 group of ten (1 ten rather than ten ones) and 8 ones. Being able to see ten ones as one ten is a crucial step on the learning path.

It can be helpful for English-speaking children to have experiences seeing 18 things separated into ten and eight and relating these quantities to both the number words "*eighteen is ten and eight*" and to the written number symbols (18). It may also be helpful to use the written symbol version of this as 18 = 10 + 8. Repeated experiences with all of these relationships can help children overcome the second kind of typical error in writing teen numbers, in which children write first what they say first. They hear *eighteen* and know that teens have a 1 in them (they may not yet think of this as one ten) and so they write 81.

Kindergarten children can also experience and learn all of the decade words in order from 20 to 100. Doing so while looking at a list of these number symbols grouped in tens can help to reinforce the pattern of the groups of ten.

Many states require that kindergarten children understand some aspects of money, but sometimes they have goals that are not sensible for this age group, even children who have had strong earlier mathematical experiences. The mathematical aspects of money that are most appropriate are the groups of ten pennies in dimes and the groups of five pennies in nickels. Children have been working with these cardinal groups of tens in this level and with 5-groups in the 4-year-old/prekindergarten level, so it is easy to build this understanding by extending this knowledge to coins by using any visual support that relates a 5-group of pennies to one nickel and one 10-group of pennies to one dime. Such supports were used successfully for first graders to construct the relationships for understanding two-digit numbers described next for first graders (Fuson, Smith, and Lo Cicero, 1997; Hiebert et al., 1997).

Learning the values of a dime and a nickel are of course particularly complicated because their values are not in the order of the sizes of the coins. In size, a dime < a penny < a nickel, but in value a penny < a nickel < a dime. For this reason, it is too difficult to work with these coins alone rather than with visual supports that show the values of these coins in pennies, as discussed above. Counting mixed collections of dimes, nickels, and pennies requires shifting counts from counting by tens when counting dimes to counting by fives when counting nickels to counting by ones when counting pennies. Such shifts are too complex for many children at this level, especially if they are looking at the coins rather than looking at their values as pennies. Practice just on the names of the coins and on their visual features, rather than on their value as ones, fives, or tens, is also not appropriate. It is the quantitative values that are mathematically important.

### Step 4 (Grade 1)

At this step children see, say, count, and write tens and ones from 1 to 100 (see Box 5-9). To do this, they build on the integrations among cardinality, counting, and written number symbols that they have made in kindergarten. The major advance has two parts. First, children learn to count by two different units, units of ten and units of one. Second, they learn to shift from counting by units of ten to counting by units of one so that they can count cardinal sets up to 100. Children who have mastered

**BOX 5-9**
**Step 4 in the Number Core**
**Grade 1**

**See, Say, Count, and Write Tens-Units and Ones-Units from 1 to 100**

Children at particular ages/grades may exceed the specified numbers and be able to work correctly with larger numbers. The numbers for each age/grade are the foundational and achievable content for children at this age/grade. The major types of new learning for each age/grade are given in italics. Each level assumes that children have had sufficient learning experiences at the lower level to learn that content; many children can still learn the content at a level without having fully mastered the content at the lower level if they have sufficient time to learn and practice.

**Cardinality:** *Relates patterns in number word list to 100 to quantities of tens and of ones:* can see the tens and ones quantities in numbers from 10 to 99 (e.g., 68 = 60 + 8); sees the 60 both as 60 ones (*sixty*) and as *6 tens*; can make drawn quantities to show tens and ones.

**Number word list:** *May count groups of ten using a tens list (1 ten, 2 tens, etc.) as well as the decade list 10, 20, 30, . . . .*

**1-to-1 counting correspondences:** *Extends counting single units to counting a group of ten as a 10-unit and shifts from counting these units of ten to counting by ones when counting left-over ones units:* arranges things in groups of ten (or uses prearranged groups or drawings) and counts the groups by tens and then shifts to a count by ones for the leftover single things: 10, 20, 30, 40, 50, 60, 61, 62, 63, 64, 65, 66, 67, 68, or 1 ten, 2 tens, 3 tens, 4 tens, 5 tens, 6 tens, 6 tens and 1 one, 6 tens and 2 ones, 6 tens and 3 ones, 6 tens and 4 ones, 6 tens and 5 ones, 6 tens and 6 ones, 6 tens and 7 ones, 6 tens and 8 ones.

**Written number symbols:** *Extends reading and writing to all two-digit numbers 1 to 99 and understands that the tens digit refers to groups of tens and the ones digit refers to groups of ones;* also sees that the 0 from the tens number is hiding behind the ones number so can see 68 as 60 + 8.

**Integrates all of the above for numbers 1 to 100** so that ***n–ty*** = ***n tens*** (e.g., 60 is 6 tens); the counting by tens and by ones represents sets of tens and of ones; a 2-digit numeral like 68 = 60 + 8 and 68 also means 6 tens and 8 ones.

the kindergarten concept that *ten ones equal one ten* can learn to use visual representations of tens that show each ten as *one ten*.

Children at this step need to be able to make drawings of tens and of ones so that they can represent numbers to use when adding and subtracting. Making such drawings can also help with the consolidation of the two-digit numerals, for example, 68 = 60 + 8 as *sixty plus eight* and as *six tens plus eight*. Place value cards in which the ones card covers the 0 in the tens card can also help eliminate the typical errors of children hearing 68 as *sixty eight* and therefore writing what they hear: 608 instead of 68.

## THE RELATIONS AND OPERATIONS CORE

The main mathematical categories in the relations and operations core were discussed in Chapter 2, and the steps through which our four age groups move were summarized in Box 5-2. These steps are elaborated in Box 5-10.

In the relations core, children learn to perceive, say, discuss, and create the relations *more than*, *less than*, and *equal to* on two sets. Initially they use general perceptual, or length, or density strategies to decide whether one set is *more than*, *less than*, or *equal to* another set. Gradually these are replaced by more accurate strategies: They match the entities in the sets to find out which has leftover entities, or they count both sets and use understandings of *more than/less than* order relations on numbers (see research reviewed in Clements and Sarama, 2007; Fuson, 1988). Eventually, in Grade 1, children begin to see the third set potentially present in relational situations, the *difference* between the smaller and the larger set (see research reviewed in Fuson, 1992a, 1992b). In this way, relational situations become the third kind of addition/subtraction situations: comparison situations.

In the operations core, children learn to see addition and subtraction situations in the real world by focusing on the mathematical aspects of those situations and making a model of the situation (called *mathematizing* these situations, as explained in Chapter 2). Initially such mathematizing may involve only focusing on the number of objects involved rather than on their color or their use (*I see* ***two*** *red spoons and* ***one*** *blue spoon*) and using those same objects to find the answer by refocusing on the total or counting it (*I see* ***three*** *spoons in all*). The three types of addition/subtraction situations that children must learn to solve were discussed in Chapter 2 and summarized in Box 2-4. These types are change plus/change minus, put together/take apart (sometimes called combine), and comparisons.

Addition and subtraction situations, and the word problems that describe such situations, provide many wonderful opportunities for learning language. Word problems are short and fairly predictable texts, so children can vary words in them while keeping much of the text. This enables them to say word problems in their own words and help everyone's understanding. English language learners can repeat such texts and vary particular words as they wish, all with the support of visual objects or acted-out situations. Although children need to learn the special mathematics vocabulary involved in addition and subtraction, these problems also give them wonderful opportunities to integrate art (drawing pictures) and language practice and pretend play while also generalizing their growing mathematical knowledge.

**BOX 5-10**
**Steps in Addition/Subtraction Operations and Relations**

**Step 1 (ages 2 and 3)**

- Use subitized and counted cardinality to solve situation and oral number word problems with totals ≤ 5.
- Act out numerical situations with objects and say them in words; see answer at the end.
- Determine that something is bigger or has more using perceptual, length, and density strategies.

Examples of problems they can solve:
- Change plus: Two blocks and two blocks make four blocks.
- Change minus: Four apples take away one apple is three apples.
- Put together/take apart: I see three apples. I see two and one make three.

**Step 2 (age 4/prekindergarten)**

- Use conceptual subitizing and cardinal counting to solve situation, word, and oral number word problems with totals ≤ 8.
- Solve numerical situations and word problems by modeling actions with objects, fingers, or mentally (or just know the answer); or see or count the answer.
- Solve number word problems by modeling actions with objects, fingers, or mentally (or just know the answer); or see or count the answer.
- Learn the partners for 3, 4, 5 (e.g., 5 = 4 + 1, 5 = 3 + 2).
- For relations, understand and say *this is/has less/fewer than that.*
- For more than/less than relations with totals ≤ 5, act out or show situation, and count or match to solve.

Examples of problems they can solve:
- Change plus: Two and two make ?

## Levels in Children's Numerical Solution Methods

There is a large research base from around the world describing three levels through which children's numerical solution methods for addition and subtraction situations move (e.g., see the research summarized in Baroody, 1987, 2004; Baroody, Lai, and Mix, 2006; Clements and Sarama, 2007, 2008; Fuson, 1988, 1992a, 1992b; Ginsburg, 1983; Saxe, 1982; Sophian, 1984). These levels are summarized in Box 5-11. At all levels, the solution methods require mathematizing the real-world situation (or later the word problem or the problem represented with numbers) to focus on only the

- Change minus: Four take away one is ?
- Put together/take apart: Three has ? and ?

**Step 3 (Kindergarten)**

- Use conceptual subitizing and cardinal counting to solve situation, word, oral number word, and written numeral problems with totals ≤ 10.
- For word problems, model action with objects or fingers or a math drawing and count or see to solve; write an expression or equation.
- For oral or written numeral problems, use fingers, objects, or a math drawing to solve.
- Engage in learning the partners for 6, 7, 8, 9, 10.
- For relations, act out or show with objects or a drawing, then count or match to solve.
- Use =, ≠ symbols.

**Step 4 (Grade 1)**

- Use Level 2 or Level 3 solution procedures: count on or use a derived fact method for problems with totals ≤ 18 and find subtraction as an unknown addend.
- Solve change plus problems by counting on to find the total 6 + 3 = ?
- Solve change minus problems by counting on to find the unknown addend 9 – 6 = ? is 6 + ? = 9.
- Solve put together/take apart problems by counting on to find the unknown addend 6 + ? = 9.
- Advanced first graders use Level 3 solution procedures: (a) doubles and doubles ± 1. (b) they experience make-a-ten methods: 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14; 14 – 8 is 8 + ? = 14, so 8 + 2 + 4 = 14, so ? = 6 (not all children master these in Grade 1).
- Solve comparison situations or determine how much/many more/less by counting or matching for totals ≤ 10, then for totals ≤ 18.

mathematical aspects—the numbers of things and the additive or subtractive operation in the situation. As we discuss each level, we also describe ways in which children can be helped to learn methods appropriate for that level and the prerequisite knowledge. Children need opportunities to relate strategies to actual objects or pictures of objects and to discuss and explain their thinking.

The solution methods at Level 1 use direct modeling of every object. In direct modeling children must carry out the actions in the situation using actual objects or fingers. Until around age 6, children primarily use such direct modeling to solve situations presented in objects, word problems

**BOX 5-11**
**Levels in Children's Numerical Solution Methods**

**Level 1: Direct modeling** of all quantities in a situation; used at the first three number/operation levels:

**Counting all:** Count out things or fingers for one addend, count out things or fingers for the other addend, and then count all of the things or fingers.
**Take away:** Count out things or fingers for the total, take away the known addend number of things or fingers, and then count the things or fingers that are left.

**Level 2: Count on** can be done in first grade (some children can do so earlier): They use embedded number understanding to see the first addend within the total and so see that they do not need to count all of the total, but instead could make a cardinal-to-count shift and count on from the first addend.

**Count on to find the total:** On fingers or with objects or with conceptual subitizing, children keep track of how many words to count on so that they stop when they have counted on the second addend number of words and the last word they say is the total:
$6 + 3 = ?$ would be "six, seven, eight, nine, so the total is nine. I counted on 3 more from 6 to make 9."
After learning counting on from the first addend, children learn to count on from the larger addend.
**Count on to find the unknown addend:** Children stop counting when they say the total, and the fingers (or other keeping track method) tell the answer (the unknown addend number of words they counted on past the first addend).
$6 + ? = 9$ would be "six, seven, eight, nine, so I added on 3 to 6 to make 9. I counted on 3 more from 6 to make 9. Three is my unknown addend."

(situations expressed in words, perhaps with an accompanying picture), oral numerical problems such as *three plus two*, and written numerical problems such as $3 + 2$. Chapter 4 summarized research reporting that more children from low-income families had trouble with the last three kinds of problems than with the first kind and than did their middle-income peers. Therefore, such children especially need help and practice in generating models using objects or fingers for such situations.

At Grade 1, children who have not yet moved to the Level 2 general counting on methods (see Box 5-8 and Box 5-11 for more details) can do so with help. In these methods, children shift from the cardinal meaning of the first addend to the counting meaning as they count on from it: For $5 + 2$, they think *five*, shift to the counting word *five* in the number word list, and count on two more words—*five, six, seven*. This ability to count on can be

**Level 3: Derived fact methods** in which known facts are used to find related facts (mastery by some/many at first grade).

Doubles are totals of two of the same addend: 1 + 1, 2 + 2, 3 + 3, etc., up to 9 + 9. These are learned by many children in the United States because of the easy pattern in their totals (2, 4, 6, 8, etc.). **Doubles ± 1** is a Level 3 more advanced strategy that uses a related double to find the total of two addends in which one addend is one more or less than the other addend (6 + 7 = 6 + 6 + 1 = 12 + 1 = 13).

**Make-a-ten methods** are general methods for adding or subtracting to find a teen total by changing a problem into an easier problem involving 10. Children first make a 10 from the first addend and then learn to make a 10 from the larger addend.

**Make a ten to find a total:** 8 + 6 becomes 10 + 4 by separating the 6 into the amount that makes 10 with the 8. Then solving 6 = 2 + ? gives the leftover 4 within the 6 to become the ones number in the teen total: 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14.

**Make a ten to find an unknown addend:** 14 – 8 = ? is 8 + ? = 14, so 8 + 2 is 10 plus the 4 in 14 makes 14. So 8 + 6 = 14. In this method subtraction requires adding, which is easier than making a ten to find a total. The first step can also be thought of as subtracting the 8 from 10.

**Three prerequisites** for fluency with make-a-ten methods can be built up before first grade:

1. knowing the number that makes 10 (the partner to 10) for each number 3 to 9;
2. knowing each teen number as a 10 and some ones (e.g., knowing that 14 = 10 + 4 and that 10 + 4 = 14 without counting); and
3. knowing all the partners of numbers 3 to 9 so that the second number can be broken into a partner to make 10 and the leftover partner that will make the teen number.

facilitated by children's earlier work with embedded number experiences of finding partners of a total (e.g., *Inside seven, I see five and two*) and by fluency with the count word sequence, so they can begin counting from any number (most 2-, 3-, and 4-year-olds need to start at 1 when counting and cannot start from just any number). With larger second addends, children also need a method of keeping track of how many they have counted on. These counting on methods are sufficient for all further quantitative work, especially if children are helped to see subtraction as finding an unknown addend, so that they can use counting on to find that addend. Counting down to subtract is difficult, and children make many errors at it (Baroody, 1984; Fuson, 1984). Just counting backward is difficult, and children make various count-cardinal errors in counting down. Counting forward to find an unknown addend for subtraction (e.g., solving 9 – 5 = ? as 5 + ? = 9) is much easier and can make subtraction as easy as addition (e.g., Fuson, 1986b; Fuson and Willis, 1988). It also emphasizes addition and subtraction as inverse operations.

The derived fact methods (Level 3) are mastered by some children at Grade 1, depending on how many of the prerequisites shown in Box 5-8 have been made accessible for 4- and 5-year-olds and then have been practiced so that they become fluent. These methods require recomposing the given numbers into a new, easier problem (e.g., 9 + 4 becomes 10 + 3). The make-a-ten methods are taught in East Asian countries and are very useful in multidigit computation (see the discussion in Chapter 2). The prerequisites are discussed later in the summaries of the 4- and 5-year-olds because children can begin building these prerequisites then. Enabling 4- and 5-year-olds to learn the prerequisites for the counting on and derived facts methods can help low-income children to learn more advanced strategies, which fewer of them do now. This can also help children with learning difficulties in mathematics because they often continue to use the Level 1 modeling methods for too many years unless they are helped to learn more advanced strategies. The general counting on methods for addition and subtraction can be learned meaningfully and done accurately and rapidly by most children in Grade 1 (Fuson, 2004).

Throughout the process of learning and using more advanced approaches to solving addition and subtraction problems, children also become fluent with individual sums and differences. Small numbers, such as plus 1 and minus one, and doubles (2 + 2, 3 + 3) become fluent early. Others become fluent over time.

## Step 1 (Ages 2 and 3)

Children at this step use subitized and counted cardinality to solve situation and oral number word problems. They also use perceptual, length, and density strategies to find which is more with totals ≤ 5 (see Box 5-10).

### *Relations: More Than, Equal To, Less Than*

Children ages 2 and 3 begin to learn the language involved in relations (Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b; Ginsburg, 1977). *More* is a word learned by many children before they are 2. Initially it is an action directive that means: *Give me more of this*. But gradually children become able to use perceptual subitizing and length or density strategies to judge which of two sets has more things: *She has more than I have*. Such comparisons may not be correct at this age level if the sets are larger than three because children focus on length or on density and cannot yet coordinate these dimensions or use the strategies of matching or counting effectively (see research reviewed in Fuson, 1988, 1992a, 1992b, and in Clements and Sarama, 2007, 2008).

*Operations: Addition and Subtraction*

The 2- and 3-year-old children can solve change plus/change minus situations and put together/take apart situations with small numbers (totals ≤ 5) if the situation is presented with objects or if they are helped to use objects to model these situations (Clements and Sarama, 2007; Fuson, 1988). Children can have experience in learning how to do such adding and subtracting from family members, in child care centers, and from media such as television and CDs. Children may subitize groups of one and two or count these or somewhat larger numbers. To find the total, they may count or put together the subitized quantity into a pattern that is also just seen and not really counted (e.g., *two and two make four*).

### Step 2 (Age 4 or Prekindergarten)

At this step, children learn to use conceptual subitizing and cardinal counting to solve situation, word, and oral number word problems with totals ≤ 8 and begin to count and to match to find out which set has more or less (see Box 5-10).

Cardinal counters at this age level can extend their understanding of relations and of all of the addition/subtraction situations and generalize them to a wider range of settings because their real-world knowledge is more extensive than it was at the previous level. Children can now also count out a specified number of objects, so they can carry out the count all and take away solution methods (Level 1 in Box 5-11) for numbers in their counting accuracy range. They also begin to use counting and matching as well as the earlier perceptual strategies to find which of two sets is more and begin to learn the meaning of the word *less*.

*Relations*

Children at this level continue to use the perceptual strategies they used earlier (general perceptual, length, density) but they can also begin to use matching and counting to find *which is less* and *which is more* (see research summarized in Clements and Sarama, 2007, 2008; Fuson, 1988, 1992a, 1992b; Sophian, 1988). However, they can also be easily misled by perceptual cues. For example, the classic tasks used by Piaget (1941/1965) involved two rows of objects in which the objects in one row were moved apart so one row was longer (or occasionally, moved together so one row was shorter). Many children ages 4 and 5 would say that the longer row has more. These children focused either on length or on density, but they could not notice and coordinate both. However, when asked to count in such situations, many 4-year-olds can count both rows accurately, remember both count words, and change them to cardinal numbers and find the order relation on the cardinal numbers (Fuson, 1988). Thus, many 4-year-olds need encouragement to count in *more than/less than/equal to* situations, especially when the perceptual information is misleading.

To use matching successfully to find more than/less than, children may need to learn how to match by drawing lines visually to connect pairs or draw such matching lines if the compared sets are drawn on paper. Then they need to know that the number with any extra objects is more than the other set. It is also helpful to match using actual objects.

To use counting successfully, children need to be able to count both sets accurately and remember the first count result while counting the second set. Here is another example of the need for fluency in counting (see Box 5-1). Without such fluency, some children forget their first count result by the time they have counted the second set. They need more counting practice in such situations. Children also need to know order relations on cardinal numbers. They need to learn the general pattern that most children do derive from the order of the counting words: *The number that tells more is farther along (said later) in the number word list than the smaller number* (e.g., Fuson, Richards, and Briars, 1982). Activities in which children make sets for both numbers, match them in rows and count them, and discuss the results can help them establish this general pattern.

There was an early period in which the counting and matching research had not been done and many researchers and educators suggested that teachers had to wait until children conserved number (said that rows in the classic Piagetian task were equal even in the face of misleading perceptual transformations) to do any real number activities, such as adding and subtracting. However, newer research shows that there is a crucial stage for 4- and 5-year-olds in which using counting and matching are important to learn and can lead to correct relational judgments (see the research summarized in Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b). It is true that children typically do not understand that the rows are equal out of a logical necessity until age 6 or 7 (sometimes not until age 8). These older children (ages 6-7) judge the rows to be equal based on mental transformations that they apply to the situation. They do not see the need to count or match after one row is made shorter or longer by moving objects in it together or apart to see that they are equal. They are certain that simply moving the objects in the set does not change the numerosity. This is what Piaget meant by conservation of number. But children can work effectively with situations involving more and less long before they demonstrate this meaning of conservation of number.

For progress in relations, it is important that children hear, and try to use, the less common comparative terms such as *less, shorter, smaller* instead of only hearing or using *more, taller, bigger*. Initially some children think that *less* means *more* because almost all of their experience has been focused on selecting the set with *more* (e.g., Fuson, Carroll, and Landis, 1996). So children need to hear many examples of *fewer* and *less*, although it is not vital that they differentiate these from each other because that is difficult (*fewer* is used with things you can count, *less* is used with measured quantities and with numbers). Teachers can also use the comparative terms (for example, *bigger* and *smaller* rather than just *big* and *small*) so that children gain experience with them, although all children may not become fluent in their use at this level.

### *Operations*

Problems expressed in words (word problems) can now be solved, although many children may need to act out some word problems in order to understand the meanings of the situation or of some of the words (see research summarized in Fuson 1992a, 1992b). Through such experiences relating actions and words, children gradually extend their vocabulary of words that mean to add—*in all, put together, altogether, total*—and of words that mean to subtract—*are left, take away, eat, break*. Discussing and sharing solutions to word problems and acting out addition/subtraction situations can provide extended experiences for language learning. Children can begin posing such word problems as well as solving them, although many will need help with asking the questions, the most difficult aspect of posing word problems. As with all language learning, it is very important for children to talk and to use the language themselves, so having them retell a word problem in their own words is a powerful general teaching strategy to extend their knowledge and give them practice speaking in English.

Drawing the solution actions using circles or other simple shapes instead of pictures of real objects can be helpful. The two addends can be separated just by space or encircled separately or separated by a vertical line segment. Some children can also begin to make mathematical drawings to show their solutions. Teacher and child drawings leave a visual record of the full solution that facilitates children's reflecting on the solution, as well as discussing and explaining it. For children, making math drawings is also a creative activity in which they are somehow showing in space actions that occur over time. Children do this in various interesting ways that can lead to productive discussions.

Children also become able to use their fingers to add or to subtract using the direct modeling solution methods counting all or taking away (see Box 5-11, Level 1). When counting all, they will count out and raise fingers for the first addend, then for the second addend, and then count all of the raised fingers. (See Box 5-4 for a discussion of different conventions for counting on fingers.)

Some children learn at home or in a care center to put the addends on separate hands, while others continue on to the next fingers for the second addend. The former method makes it easier to see the addends, and the latter method makes it easier to see the total. Both methods can be modeled by the teacher. As children become more and more familiar with which group of fingers makes 4 or 5 or 7 fingers, they may not even have to count out the total because they can feel or see the total fingers. Similarly, children using the method of putting fingers on separate hands eventually can just raise the fingers for the addends without counting out the fingers. But they do need initially to count the total. Children who put addends on separate hands may have difficulty with problems with addends over 5 (e.g., 6 + 3) because one cannot put both such numbers on a separate hand. They can, however, continue raising fingers from 6 fingers. Because these problems involve adding 1 or 2, such continuations of 1 and 2 are relatively easy.

By now children who have had experience with adding and subtracting situations when they were younger can generalize to solve decontextualized problems that are posed numerically, as in *Two and two make how many?* (Clements and Sarama, 2007; Fuson, 1988). For some small numbers, children may have solved such a problem so many times that they know the answer as a verbal statement: *Two and two make four.* If such knowledge is fluent, children may be able to use it to solve a more complex unknown addend problem. For example, *Two and how many make four? Two.*

For larger numbers, children will need to use objects or fingers to carry out a counting all or taking away solution procedure (Box 5-11) (see research summarized in Fuson 1992a, 1992b). Children will learn new composed/decomposed numerical triads as they have such experiences. The doubles that involve the same addends (2 is 1 and 1, 4 is 2 and 2, 6 is 3 and 3, 8 is 4 and 4) are particularly easy for children to learn because the perceptual and verbal task is simplified by have the same addends (e.g., see research summarized in Fuson, 1992a, 1992b). The visual 5-groups (e.g., 8 is made from 5 and 3) discussed for the number core are also useful. Research about powerful patterns for conceptual subitizing for very small numbers would be helpful, including the extent to which flexibility is important beyond a single powerful visual core that will work for all numbers.

The put together/take apart situations, and especially the take apart situation, can be used to provide varied numerical experiences with given numbers that help children see all of the addends (*partners*) hiding inside a given number. For example, children can take apart five to see that it can be made from a three and two and also from four and one. Later on these decomposed/composed triads can be symbolized by equations, such as 5 =

3 + 2 and 5 = 4 + 1, giving children experiences with the meaning of the = symbol as *is the same number as* and with algebraic equations with one number on the left. Initially children shift from seeing the total and then seeing the partners (addends), but with experience and fluency, they can simultaneously see the addend within the total. This is called *embedded numbers*: The two addends are embedded within the total. Such embedded numbers, along with the number word sequence skill of starting counting at any number, allow children to move to the second level of addition/subtraction solution procedures, *counting on*. Initially composed/decomposed number triads and even embedded number triads are constructed with small numbers using conceptual subitizing, but eventually counting is used with larger numbers to construct larger triads.

Many children from low-income backgrounds cannot initially solve such oral numerical problems, even with very small numbers (see Chapter 4). They need opportunities to learn and practice the Level 1 solution methods with objects and with fingers and experience composing/decomposing numbers to be able to see the addends (partners) hiding inside the small numbers 3, 4, 5. Such alternating focusing on the total and then on the partners (addends) will enable them to answer such oral numerical problems and also begin the learning path toward embedded numbers that is vital for the Level 2 addition/subtraction solution methods.

**Step 3 (Kindergarten)**

At this step, children extend cardinal counting and use math drawings as well as objects to solve situation, word, oral number word, written numeral, and which-is-more/less problems with totals ≤ 10 (see Box 5-10). Written work, including worksheets, is appropriate in kindergarten if it follows up on activities with objects or presents supportive visualizations. Children at these ages need practice that builds fluency after related experiences with objects to build mathematical understanding, and they need experience relating symbols for quantities to actual or drawn quantities.

Kindergarten children can extend their addition and subtraction problem solving to all problems with totals ≤ 10. Close to half of these problems have one addend of six or more. For these problems, knowing the 5-patterns using fingers for 6 through 10 can be helpful (5 + 1 = 6, 5 + 2 = 7, etc., to 5 + 5 = 10). All children can begin to make math drawings themselves, even for these larger numbers. This allows them to reflect on and discuss their solution methods. Math drawings involving circles or other simple shapes also enable more advanced children to explore problems with totals greater than ten. It is difficult to solve such problems with fingers until one advances to the general counting on solution methods (see Box 5-11, Level 2), which typically does not occur until Grade 1. Children can discuss general patterns they see in addition and subtraction, such as +1 is just the next counting number or −1 is the number just before. Children can discuss adding and subtracting 0 and the pattern it gives: adding or subtracting 0 does not change the original number, so the result (the answer) is the same as the original number. Many children can now informally use the commutative property (A + B = B + A) especially when one number is small (e.g., Baroody and Gannon, 1985; Carpenter et al., 1993; DeCorte and Verschaffel, 1985; for a review of the literature, see Baroody, Wilkins, and Tiilikainen, 2003). Experience with put together addition situations in which the addends do not have different roles provides better support for learning the commutative property than does experience with the change situation (see research described in Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b) because these addends have such different roles in the action. To the child, it actually feels different to have 1 and then get 8 more than to have 8 and get 1 more. It feels better to gain 8 instead of gaining 1, even though you end up with the same amount. In contrast, the numerical work on put together/take apart partners facilitates understanding that the order in which one adds does not matter. Looking at composed/decomposed triads with the same addends also enables children to see and understand commutativity in these examples (for example, see that 9 = 1 + 8 and 9 = 8 + 1 and that the addends are just switched in order but still total the same).

All of the work on the relations/operation core in kindergarten serves a double purpose. It helps children solve larger problems and become more fluent in their Level 1 direct modeling solution methods. It also helps them reach fluency with the number word list in addition and subtraction situations, so that the number word list can become a representational tool for use in the counting on solution methods.

Different children learn and remember some sums and differences at each level, and it is very useful to know these for small numbers, for example for totals ≤ 8. But the more important step at the kindergarten level is that children are learning general numerical solution methods that they can extend to larger numbers. Simultaneously they are becoming fluent with these processes and with the number word list, so that they can advance to the Level 2 counting on methods that are needed to solve single-digit sums and differences with totals over ten. Children later in the year can begin to practice the number word list prerequisite for counting on by starting to count at a given number instead of always at one.

Kindergarten children are also working on all of the prerequisites for the Level 3 derived fact methods, such as make-a-ten (see Box 5-11). One prerequisite, seeing the tens in teen numbers, was discussed in the number core. The other two prerequisites involve knowing partners of numbers (decomposed/composed numerical triads) to permit flexible breaking apart and combining of numbers to turn them into teen addition or subtraction problems. For example, all of the following addition problems—9 + 2, 9 + 3, 9 + 4, . . . , 9 + 9—require the same first step: 9 needs 1 more to make ten, so separate the second number into 1 + ?. This triad then becomes 9 + 1 + ? = 10 + ?, which is an easier problem to solve if you know the tens in teen numbers. However, each problem requires a different second step: decomposing the second number to identify the rest of the second addend that will be added to ten (prerequisite 3 for derived facts methods in Box 5-11). For example, 9 + 4 = 9 + 1 + **3** = 10 + 3 = 13, but 9 + 6 = 9 + 1 + 5 = 10 + 5 = 15. So kindergarten children need experiences with finding and learning the partners of various numbers under 10.

Children's counting and matching knowledge is now sufficient to extend to relations on sets up through 10 and to more abstract ways of presenting such relational situations as two rows of drawings that can be matched by drawing lines connecting them. As discussed above for Step 2, children will be more accurate when these objects are already matched instead of being visually misleading (for example, the longer row has less). They therefore can start with the simpler nonmisleading situations and extend to the visually misleading situations when they have mastered such matched situations. Again, differentiating length and number meanings of *more* will be helpful (which *looks like more* and which *really is more*). Children who have not had sufficient experiences matching objects at Step 2 will need such experiences to support the more advanced activities in which matching is done by drawing lines.

Working with the terms *more* and *less* can also be an opportunity to discuss and emphasize that length units used in measuring a length must touch each other and cover the whole length from beginning to end to get an accurate length measurement. But things children are counting can be spread apart or moved around and they will still have the same number of things. Comparing objects spaced evenly in two rows can also be related to picture graphs, which record numbers of different kinds of data as a row of the same pictures (see the Chapter 2 discussion in the Mathematical Connections section). Activities in which children compare two rows of drawings by counting or matching them can be considered as using picture graphs if each drawing in one row is the same. What is important about such activities is that children talk about them using comparison language (*There are more suns than clouds* or *There are fewer clouds than suns*) and describe how they found their answer.

Children at this level can also prepare for the comparison problems at Grade 1 by beginning to equalize two related sets. For example, for a row of 5 above a row of 7, they can be asked to add more to the row of 5 to make it equal to the row of 7 and write their addition 5 + 2. This 2 is the difference between 5 and 7, it is the amount extra 7 has, so such exercises help children begin to see this third quantity in the comparison situation.

### *Writing Equations*

There is not sufficient evidence to indicate the best time for teachers to start writing addition and subtraction problems in equations or for students to do so. The equation form can be confusing to some students even in Grade 1, and students may confuse the symbols + = and –. This confusion and limited meanings for the = sign often continue for many years and are of concern for the later learning of algebra. Because the fundamental aspect of an equation is that the sides are equal to each other, it is important for children to learn to conceptually chunk each side. Thus, some children may need extensive experience just with expressions, such as 3 + 2 or 7 – 5, before these are used in equations. These forms might be introduced before the full equation is introduced, perhaps even with 4-year-olds. It may also help for the teacher to circle or underline these expressions to indicate that this group of symbols is a chunk that represents a single number. Future research directed at such issues of when and how to write such pre-equation forms would be helpful.

The other issue with equations is the form of the equation to write. As mentioned earlier, it is important for later algebraic understanding of acceptable forms of equations for children to see equations with only one number on the left, such as 6 = 4 + 2 to show that 6 breaks apart to make 4 and 2. This equation form can be written for take apart situations in which the total is being separated into two parts, for example, *Grammy has 6 flowers. She put four flowers in one vase and two flowers in the other vase*. Children can show this situation with objects or fingers (*Count out 6 objects and then separate them into 4 and 2*) or make a math drawing of it while the teacher records the situation in an equation. This form can also be used in practice activities with objects in which children find all of the partners (addends) of a given number. For example, children can make 5 using two different colors of objects, and each color can show the partners. The teacher can record all of the partners that children find: 5 = 1 + 4, 5 = 2 + 3, 5 = 3 + 2, 5 = 4 + 1. This can be in a situation (*Let's find all of the ways that Grammy can put her 5 flowers in her 2 vases*) or just an activity with numbers (*Let's find all of the partners of 5*).

Change plus and change minus situations can be recorded by equations with only one number on the right because that is the action in these situations (see Box 2-4), for example, 3 + 1 = 4 or 5 – 2 = 3. In these equations the = sign is really more like an arrow, meaning *gives* or *results in*. As discussed, this is often the only meaning of = that students in the United States know, and this interferes with their use of algebra. So it is really important that they also see and use forms like 5 = 3 + 2 to show the numbers hiding inside a number, the partners (addends) that make that number.

### Step 4 (Grade 1)

At this step, children build on their earlier number and relations/operation knowledge and skills to advance to Level 2 counting on solution methods. They also come to understand that addition is related to subtraction and can think of subtraction as finding an unknown addend (see Box 5-10).

Grade 1 addition and subtraction is the culmination of all of the number core and relations/operation core experiences and expertise that have been building since birth, for those who have been given sufficient opportunities to build such competence. Foundational and achievable relations and operations content for Grade 1 children is summarized in Box 5-9.

For all of the earlier experiences to come together into the Level 2 counting on solution methods, some children may still need some targeted practice in beginning counting at any number instead of always starting at one (one of the prerequisites for counting on). It is also helpful to begin counting on in some kind of structured visual setting, so that children can conceptualize the relationships between the counting and cardinal meanings of number words.

Counting on is not a rote method. It requires a shift in word meaning for the first addend from its cardinal meaning of the number in that first addend to a counting meaning, as children count on from that first addend to the total. Children then must shift from that last counted word to its cardinal meaning of how many objects there are in total. For example, seeing circles for both addends in a row with the problem printed above enables children to count both addends and then count all to find the total (their usual Level 1 direct modeling solution method). But after several times of counting all, they can be asked what number they say when they count the last circle in the group of 6 and whether they need to count all of the objects or could they just start at 6. Going back and forth between this counting on and the usual counting all enables children to see that counting on is just an abbreviation of counting all, in which the initial counts are omitted (e.g., Fuson, 1982; Fuson and Secada, 1986; Secada, Fuson, and Hall, 1983).

| | | |
|---|---|---|
| | 6 + 3 | *Six* is a cardinal number. |
| o o o o o o | o o o | |
| 1 2 3 4 5 6 | 7 8 9 | *Six* here is a count number when counting all. |

siiiixxx 7 8 9 To count on like this, a child must shift from the cardinal meaning above to the count meaning of *six* and then keep counting 7, 8, 9.

Trying this with different problems enables many children to see this general pattern and begin counting on. Transition strategies, such as counting 1, 2, 3, 4, 5, 6 very quickly or very softly or holding the 6 (*siiiiiiixxxxxx*), have been observed in students who are learning counting on by themselves; these can be very useful in facilitating this transition to counting on (e.g., Fuson, 1982; Fuson and Secada, 1986; Secada, Fuson, and Hall, 1983). Some weaker students may need explicit encouragement to *trust the six* and to let go of the initial counting of the first addend, and they may need to use these transitional methods for a while.

Counting on has two parts, one for each addend. The truncation of the final counting all by starting with the cardinal number of the first addend was discussed above. Counting on also requires keeping track of the second addend—of how many you count on so that you count on from the first addend exactly the number of the second addend. When the number is small, such as for 6 + 3, most children use perceptual subitizing to keep track of the 3 counted on. This keeping track might be visual and involve actual objects, fingers, or drawn circles. But it can also use a mental visual image (some children say they see 3 things in their head and count them). Some children use auditory subitizing (they say they hear 7, 8, 9 as three words). For larger second addends, children use objects, fingers, or conceptual subitizing to keep track as they count on. For 8 + 6, they might think of 6 as 3 and 3 and count with groups of three: 8 9 10 11 12 13 14 with a pause after the 9 10 11 to mark the first three words counted on. Other children might use a visual (*I saw 3 circles and another 3 circles*) or an auditory rhythm to keep track of how many words they counted on. So here we see how the *perceptual subitizing* and the *conceptual subitizing*, which begin very early, come to be used in a more complex and advanced mathematical process. This is how numerical ideas build, integrating the levels of thinking visually/holistically and thinking about parts into a complex new conceptual structure that relates the parts and the whole. Children can discuss the various methods of keeping track, and they can be helped to use one that will work for them. Almost all children can learn to use fingers successfully to keep track of the second addend.

Many experiences with composing/decomposing (finding partners hiding inside a number) can give children the understanding that a total is any number that has partners (addends) that compose it. When subtracting, they have been seeing that they take away one of those addends, leaving the other one. These combine into the understanding that subtracting means finding the unknown addend. Therefore, children can always solve subtraction problems by a forward method that finds the unknown addend, thus avoiding the difficult and error-prone counting down methods (e.g., Baroody, 1984; Fuson, 1984, 1986b). So 14 – 8 = ? can be solved as 8 + ? = 14, and students can just count on from 8 up to 14 to find that 8 plus 6 more is 14.

Some first graders will also move on to Level 3 derived fact solution methods (see Box 5-11) such as doubles plus or minus one and the general method that works for all teen totals: the make-a-ten methods taught in East Asia (see Chapter 4 and, e.g., Geary et al., 1993; Murata, 2004). These make-a-ten methods are particularly useful in multidigit addition and subtraction, in which one decomposes a teen number into a ten to give to the next column while the leftover ones remain in their column. More children will be able to learn make-a-ten methods if they have learned the prerequisites for them in kindergarten or even in Grade 1.

The comparison situations compare a large quantity to a smaller quantity to find the difference. These are complex situations that are usually not solvable until Grade 1. The third quantity, the difference, is not physically present in the situation, and children must come to see the differences as the extra leftovers in the bigger quantity or the amount the smaller quantity needs to gain in order to be the same as the bigger quantity. The language involved in comparison situations is challenging, because English gives two kinds of information in the same sentence. Consider, for example, the sentence *Emily has five more than Tommy*. This says both that Emily has more than Tommy and that she has five more. Many children do not initially hear the five. They will need help and practice identifying and using the two kinds of information in this kind of sentence (see the research reviewed in Clements and Sarama, 2007, 2008; Fuson, 1992a, 1992b; Fuson, Carroll, and Landis, 1996).

Learning to mathematize and model addition and subtraction situations with objects, fingers, and drawings is the foundation for algebraic problem solving. More difficult versions of the problem situations can be given from Grade 1 on. For example, the start or change number can be the unknown in change plus problems: *Joey drew 5 houses and then he drew some more. Now he has 9 houses. How many more houses did he draw?* Children naturally model the situation and then reflect on their model (with objects, fingers, or a drawing) to solve it (see research summarized in Clements and Sarama, 2007; Fuson, 1992a, 1992b). From Grade 2 on they can also learn to represent the situation with a situation equation (e.g., 5 + ? = 9 as in the example above, or ? + 4 for an unknown start number) and then reflect on that to solve it. This process of mathematizing (including representing the situation) and then solving the situation representation is algebraic problem solving.

## Issues in Learning Relations and Operations

### *The Extensive Learning Path for Addition and Subtraction*

The teaching-learning path we describe shows that even the most advanced solution strategies for adding and subtracting single-digit numbers have their roots before age 2 and may not culminate until Grade 1 or even Grade 2. The paths also illustrate how children coordinate several different complex kinds of understandings and skills beginning with *perceptual subitizing* through *conceptual subitizing* and then counting and matching to employ more sophisticated problem-solving strategies. This makes it clear that one cannot characterize the learning of single-digit addition and subtraction as simply "memorizing the facts" or "recalling the facts," as if children had been looking at an addition table of numbers and memorizing these. Children do remember particular additions and subtractions as early as age 2, but each of these has some history as perceptually or conceptually subitized situations, counted situations over many examples, or additions/subtractions derived from other known additions/subtractions. It is therefore much more appropriate to set learning goals that use the terminology *fluency with single-digit additions and their related subtractions* rather than the terms *recalled* or *memorized facts*. The latter terms imply simplistic rote teaching/learning methods that are far from what is needed for deep and flexible learning.

### *The Mental Number Word List as a Representational Tool*

We have demonstrated how children come to use the number word list (the number word sequence) as a mental tool for solving addition and subtraction problems. They are able to use increasingly abbreviated and abstract solution methods, such as counting on and the make-a-ten methods. The number words themselves have become unitized mental objects to be added, subtracted, and ordered as their originally separate sequence, counting, and cardinal meanings become related and finally integrated over several years into a truly numerical mental number word sequence. Each number can be seen as embedded within each successive number and as seriated: related to the numbers before and after it by a linear ordering created by the order relation *less than* applied to each pair of numbers (see Box 5-12). This is what Piaget (1941/1965) called *truly operational cardinal number*: Any number in the sequence displays both class inclusion (the embeddedness) and seriation (see also Kamii, 1985). But this fully Piagetian integrated sequence will not be finished for most children until Grade 1 or Grade 2, when they can do at least some of the Step 3 derived fact solution methods, which depend on the whole teaching-learning path we have discussed.

**BOX 5-12**
**Ordering and Ordinal Numbers**

There is frequent confusion in the research literature in the use of the terms *ordered* or *ordering*, *ordinal number*, and *order relation*. Some of this confusion stems from the fact that adults can flexibly and fluently use the counting, cardinal, and ordinal meaning of number words without needing to consciously think about the different meanings. As a result, they may not be able to differentiate the meanings very clearly. But young children learn the meanings separately and need to connect them.

When counting to find the total number in a set, the order for connecting each number word to objects is arbitrary and could be done in any order. As noted previously, the last number takes on a cardinal meaning and refers to the total numbers of items counted. Thus, the cardinal meaning of a number refers to a set with that many objects. Cardinal numbers can be used to create an order relation. That is the idea that one set has more members than another set. An order relation (one number or set is less than or more than another number or set) tells how two quantities are related. This order relation produces a linear ordering on these numbers or sets. An ordinal number tells where in the ordering a particular number or set falls. A child can subitize for the small ordinal numbers (see whether an object in an ordered set is first, second, or third), but needs to count for larger ordinal numbers and shift from a count meaning to an ordinal meaning (e.g., count *one, two, three, four, five, six, seven* [count meaning]. *That person is seventh* [ordinal meaning and ordinal work] *in the line to buy tickets.*).

We have not emphasized ordinal words in this chapter because they are so much more difficult than are cardinal words, and children learn them much later (e.g., Fuson, 1988). Although 4- and 5-year-olds could learn to use the ordinal words *first*, *second*, and *last*, it is not crucial that they do so. The ordinal words *first* through *tenth* could wait until Grade 1.

Many researchers have noted how the number word list turns into a mental representational tool for adding and subtracting. A few researchers have called this a *mental number line*. However, for young children this is a misnomer, because children in kindergarten and Grade 1 are using the number word list (sequence) as a count model: Each number word is taken as a unit to be counted, matched, added, or subtracted. In contrast, a number line is a length model, like a ruler or a bar graph, in which numbers are represented by the length from zero along a line segmented into equal lengths. Young children have difficulties with the number line representation because they have difficulty seeing the units—they need to see things, so they focus on the numbers instead of on the lengths. So they may count the starting point 0 and then be off by one, or they focus on the spaces and are confused by the location of the numbers at the end of the spaces. The report *Adding It Up: Helping Children Learn Mathematics* (National

Research Council, 2001a) recognized the difficulties of the number line representation for young children and recommended that its use begin at Grade 2 and not earlier.

The number line is particularly important when one wants to show parts of one whole, such as one-half. In early childhood materials, the term *number line* or *mental number line* often really means a number path, such as in the common early childhood games in which numbers are put on squares and children move along a numbered path. Such number paths are count models—each square is an object that can be counted—so these are appropriate for children from age 2 through Grade 1. Some research summarized in Chapter 3 did focus on children's and adult's use of the analog magnitude system to estimate large quantities or to say where specified larger numbers fell along a number line. Again, it is not clear, especially for children, whether they are using a mental number list or a number line; the crucial research issue is the change in the spacing of the numbers with age, and this could come either from children's use of a mental number list or a number line. The use of number lines, such as in a ruler or a bar graph scale, is an important part of measurement and is discussed in Chapter 6. But for numbers, relations, and operations, physical and mental number word lists are the appropriate model.

### *Variability in Children's Solution Methods*

The focus of this chapter is on how children follow a learning path from age 2 to Grade 1 in learning important aspects of numbers, relations, and operations. We continually emphasize that there is variability within each age group in the numbers and concepts with which a given child can work. As summarized in Chapter 3, much of this variability stems from differences in opportunities to learn and to practice these competencies, and we stress how important it is to provide such opportunities to learn for all children. We close with a reminder that there is also variability within a given individual at a given time in the strategies the child will use for a given kind of task. Researchers through the years have shown that children's strategy use is marked by variability both within and across children (e.g., Siegler, 1988; Siegler and Jenkins, 1989; Siegler and Shrager, 1984). Even on the same problem, a child might use one strategy at one point in the session, and another strategy at another point. As children gain proficiency, they gradually move to more mature and efficient strategies, rather than doing so all at once. The variability itself is thought to be an important engine of cognitive change. Similarly, as discussed above, accuracy can vary with effort, particularly with counting. The variability in the use of strategies within or across children can provide important opportunities to discuss different methods and extend understandings of all participants. The variability in results with different levels of effort can lead to discussions about how learning mathematics depends on effort and practice and that everyone can get better at it if they practice and try hard. Effort creates competencies that are the building blocks for the next steps in the learning path for numbers, relations, and operations.

## SUMMARY

The teaching-learning path described in this chapter shows how young children learn, integrate, and extend their knowledge about cardinality, the number word list, 1-to-1 counting correspondences, and written number symbols in successive steps from age 2 to 7. Much of this knowledge requires specific cultural knowledge—for example, the number word list in English, counting, matching, vocabulary about relations and operations. Children require extensive, repeated experiences with small numbers and then similar experiences with larger and larger numbers. Counting must become very fluent, so that it can become a mental representational tool for problem solving. As we have shown, even young children can have experiences in the teaching-learning path that support later algebraic learning. To move through the steps in the teaching-learning path, children require teaching and interaction in the context of explicit, real-world problems with feedback and opportunities for reflection provided. They also require accessible situations in which they can practice (consolidate), deepen, and extend their learning and their own.

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# 6

# The Teaching-Learning Paths for Geometry, Spatial Thinking, and Measurement

Geometry, spatial thinking, and measurement make up the second area of mathematics we emphasize for young children. In this chapter we provide an overview of children's development in these domains, lay out the teaching-learning paths for children ages 2 through kindergarten in each broad area, and discuss instruction to support their progress through these teaching-learning paths. As in Chapter 5, the discussion of instruction is closely tied to the specific mathematical concepts covered in the chapter. Chapter 7 provides a more general overview of effective instruction.

## GEOMETRY AND SPATIAL THINKING

The Dutch mathematician Hans Freudenthal stated that geometry and spatial thinking are important because "Geometry is grasping space. And since it is about the education of children, it is grasping that space in which the child lives, breathes, and moves. The space that the child must learn to know, explore, and conquer, in order to live, breath and move better in it. Are we so accustomed to this space that we cannot imagine how important it is for us and for those we are educating?" (Freudenthal, 1973, p. 403). This section describes the two major ways children understand that space, starting with smaller scale perspectives on geometric shape, including composition and transformation of shapes, and then turning to larger spaces in which they live. Although the research on these topics is far less developed than in number, it does provide guidelines for developing young children's learning of both geometric and spatial abilities.

## Shape

Shape is a fundamental idea in mathematics and in development. Beyond mathematics, shape is the basic way children learn names of objects, and attending to the objects' shapes facilitates that learning (Jones and Smith, 2002).

### *Steps in Thinking About Shape*

Children tend to move through different levels in thinking as they learn about geometric shapes (Clements and Battista, 1992; van Hiele, 1986). They have an innate, implicit ability to recognize and match shapes. But at the earliest, prerecognition level, they are not explicitly able to reliably distinguish circles, triangles, and squares from other shapes. Children at this level are just starting to form unconscious visual schemes for the shapes, drawing on some basic competencies. An example is pattern matching through some type of feature analysis (Anderson, 2000; Gibson et al., 1962) that is conducted after the visual image of the shape is analyzed by the visual system (Palmer, 1989).

At the next level, children think visually or holistically about shapes (i.e., syncretic thought, a fusion of differing systems; see Clements, Battista, and Sarama, 2001; Clements and Sarama, 2007b) and have formed schemes, or mental patterns, for shape categories. When first built, such schemes are holistic, unanalyzed, and visual. At this visual/holistic step, children can recognize shapes as wholes but may have difficulty forming separate mental images that are not supported by perceptual input. A given figure is a rectangle, for example, because "it looks like a door." They do not think about shapes in terms of their attributes, or properties. Children at this level of geometric thinking can construct shapes from parts, but they have difficulty integrating those parts into a coherent whole.

Next, children learn to describe, then analyze, geometric figures. The culmination of learning at this descriptive/analytic level is the ability to recognize and characterize shapes by their properties. Initially, they learn about the parts of shapes—for example, the boundaries of two-dimensional (2-D) and three-dimensional (3-D) shapes—and how to combine them to create geometric shapes (initially imprecisely). For example, they may explicitly understand that a closed shape with three straight sides is a triangle. In the teaching-learning path articulated in Table 6-1, this is called the "thinking about parts" level.

Children then increasingly see relationships between parts of shapes, which are properties of the shapes. For instance, a student might think of a parallelogram as a figure that has two pairs of parallel sides and two pairs of equal angles (angle measure is itself a relation between two sides, and

**TABLE 6-1** Space and Shapes in Two Dimensions

| | Goals | | |
|---|---|---|---|
| Steps/Ages (Level of Thinking) | A. Perceive, Say, Describe/Discuss, and Construct Objects in 2-D Space | B. Perceive, Say, Describe/Discuss, and Construct Spatial Relations in 2-D Space | C. Perceive, Say, Describe/Discuss, and Construct Compositions and Decompositions in 2-D Space |
| **Step 1 (Ages 2 and 3)** | | | |
| Thinking visually/holistically | Recognition and informal description (including at least circles, squares, then triangles, rectangles). | Recognize shapes in many different orientations and sizes. Trial-and-error geometric movements (informal, not quantified).<br>• Use relational language, including vertical directionality terms as "up" and "down," referring to a 2-D environment.<br>• Informally recognizes area as filling 2-D space (e.g., "I need more papers to cover this table"). | Solve simple puzzles involving things in the world.<br>Create pictures by representing single objects, each with a different shape. |
| Thinking about parts | Shapes by number of sides (starting with restricted cases, e.g., prototypical equilateral triangle, square). | | |
| **Step 2 (Age 4)** | | | |
| Thinking visually/holistically | Recognition and informal description at multiple orientations, sizes, and shapes (includes circles and half/quarter circles, squares and rectangles, triangles, and others [the pattern block rhombus, trapezoids, hexagons regular]). | Recognize shapes (to the left) in many different orientations, sizes, and shapes (e.g., "long" and "skinny" rectangles and triangles).<br>• Match shapes by using geometric motions to superimpose them.<br>• Use relational words of proximity, such as "beside," "next to," and "between," referring to a 2-D environment. | |

*continued*

**TABLE 6-1** Continued

| | Goals | | |
|---|---|---|---|
| Steps/Ages (Level of Thinking) | A. Perceive, Say, Describe/Discuss, and Construct Objects in 2-D Space | B. Perceive, Say, Describe/ Discuss, and Construct Spatial Relations in 2-D Space | C. Perceive, Say, Describe/Discuss, and Construct Compositions and Decompositions in 2-D Space |
| Thinking about parts | Describe and name shapes by number of sides (up to the number they can count).<br>Describe and name shapes by number of corners (vertices). | Move shapes using slides, flips, and turns.<br>• Use relational language involving frames of reference, such as "to this side of," "above."<br>• Compare areas by superimposition.<br>For rectangular spaces<br>• Tile a rectangular space with physical tiles (squares, right triangles, and rectangles with unit lengths) and guidance. | Move shapes using slides, flips, and turns to combine shapes to build pictures.<br>For rectangular spaces<br>• Copy a design shown on a grid, placing squares onto squared-grid paper. |
| Relating parts and wholes | Sides of same/ different length.<br>• Right vs. nonright angles. | Predict effects of rigid geometric motions. | Combine shapes with intentionality, recognizing them as new shapes.<br>• In an "equilateral triangle world," create pattern block blue rhombus, trapezoid, and hexagons from triangles. |
| **Step 3 (Age 5)** | | | |
| Thinking visually/ holistically | Recognition and informal description, varying orientation, sizes, shapes (includes all above, as well as octagons, parallelograms, convex/concave figures). | | |

**TABLE 6-1** Continued

| | Goals | | |
|---|---|---|---|
| Steps/Ages (Level of Thinking) | A. Perceive, Say, Describe/Discuss, and Construct Objects in 2-D Space | B. Perceive, Say, Describe/Discuss, and Construct Spatial Relations in 2-D Space | C. Perceive, Say, Describe/Discuss, and Construct Compositions and Decompositions in 2-D Space |
| Thinking about parts | Shape by number of sides and corners (including new shapes). | Create and record original compositions made using squares, right triangles, and rectangles on grid paper. Extend to equilateral grids and pattern blocks (those with multiples of 60° and 120° angles).<br>• Begin to use relational language of "right" and "left."<br>• Draw a complete covering of a rectangle area. Count squares in rectangular arrays correctly and (increasingly) systematically. | |
| Relating parts and wholes | Measure of sides (simple units), gross comparison of angle sizes. | Compare area using superimposition.<br>• For rectangular regions, draw and count by rows (initially may only count some rows as rows).<br>• Identify and create symmetric figures using motions (e.g., paper folding; also mirrors as reflections). | Composition on grids and in puzzles with systematicity and anticipation, using a variety of shape sets (e.g., pattern blocks; rectangular grids with squares, right triangles, and rectangles; tangrams). |

NOTE: Most of the time should be spent on 2-D, about 85 percent (there are many beneficial overlapping activities).

equality of angles another relation). Owing usually to a lack of good experiences, many students do not reach this level until late in their schooling. However, with appropriate learning experiences, even preschoolers can begin to develop this level of thinking. In Table 6-1 this is called the "relating parts and wholes" level.

*Development of Shape Concepts*

What ideas do preschool children form about common shapes? Decades ago, Fuson and Murray (1978) reported that, by 3 years of age, over

60 percent of children could name a circle, a square, and a triangle. More recently, Klein, Starkey, and Wakeley (1999) reported the shape-naming accuracy of 5-year-olds as circle, 85 percent; square, 78 percent; triangle, 80 percent; rectangle, 44 percent. In one study (Clements et al., 1999), children identified circles quite accurately (92, 96, and 99 percent for 4-year-olds, 5-year-olds, and 6-year-olds, respectively), and squares fairly well (82, 86, and 91 percent). Young children were less accurate at recognizing triangles and rectangles, although their averages (e.g., 60 percent for triangles for all ages 4-6) were not remarkably smaller than those of elementary students (64-81 percent). Their visual prototype for a triangle seems to be of an isosceles triangle. Their average for rectangles was a bit lower (just above 50 percent for all ages). Children's prototypical image of a rectangle seems to be a four-sided figure with two long parallel sides and "close to" square corners. Thus, young children tended to accept long parallelograms or right trapezoids as rectangles.

In a second study (Hannibal and Clements, 2008), children ages 3 to 6 sorted a variety of manipulable forms. Certain mathematically irrelevant characteristics affected children's categorizations: skewness, aspect ratio, and, for certain situations, orientation. With these manipulatives, orientation had the least effect. Most children accepted triangles even if their base was not horizontal, although a few protested. Skewness, or lack of symmetry, was more important. Many rejected triangles because "the point on top is not in the middle." For rectangles, many children accepted nonright parallelograms and right trapezoids. Also important was aspect ratio, the ratio of height to base. Children preferred an aspect ratio near one for triangles; that is, about the same height as width. Children rejected both triangles and rectangles that were "too skinny" or "not wide enough."

## Spatial Structure and Spatial Thinking

Spatial thinking includes two main abilities: spatial orientation and spatial visualization and imagery. Other important competencies include knowing how to represent spatial ideas and how and when to apply such abilities in solving problems.

### *Spatial Orientation*

Spatial orientation involves knowing where one is and how to get around in the world. As shown in Chapter 3, spatial orientation is, like number, a core cognitive domain, for which competencies, including the ability to actively and selectively seek out information, are present from birth (Gelman and Williams, 1997). Children have cognitive systems that are based on their own position and their movements through space, and external references. They can learn to represent spatial relations and movement through space using both of these systems, eventually mathematizing their knowledge.

Children as young as age 2 can implicitly use knowledge of multiple landmarks and distances between them to determine or remember locations. By about age 5, they can explicitly represent that information, even interpreting or creating simple models of spaces, such as their classroom. Similarly, they can implicitly use distance and direction when they move at age 1-2. They do so more reliably when they move themselves, another justification for providing children of all ages with opportunities to explore large spaces in which they can navigate safely. By age 4, children explicitly use distance and direction and reason about their locations. For example, they can point to one location from another, even though they never walked a path that connected the two (Uttal and Wellman, 1989).

Language for spatial relationships is acquired in a consistent order, even across different languages (Bowerman, 1996). The first terms acquired are *in*, *on*, and *under*, along with such vertical directionality terms as *up* and *down*. These initially refer to transformations (e.g., "on" not as a smaller object on top of another, but only as making an object become physically attached to another; Gopnik and Meltzoff, 1986). Children then learn words of proximity, such as "beside" and "between." Later, they learn words referring to frames of reference, such as "in front of," "behind." The words "left" and "right" are learned much later, and are the source of confusion for several years.

In these early years, children also can learn to analyze what others need to hear in order to follow a route through a space. Such learning is dependent on relevant experiences, including language. Learning and using spatial terminology can affect spatial competence (Wang and Spelke, 2002). For example, teaching preschoolers the spatial terms "left" and "right" helped them reorient themselves more successfully (Shusterman and Spelke, 2004). However, language provides better support for simpler representations, and more complex spatial relationships are difficult to capture verbally. In such cases, children benefit from learning to interpret and use external representations, such as models or drawings.

Young children can begin to build mental representations of their spatial environments and can model spatial relationships of these environments. When very young children tutor others in guided environments, they build geometrical concepts (Filippaki and Papamichael, 1997). Such environments might include interesting layouts inside and outside classrooms, incidental and planned experiences with landmarks and routes, and frequent discussion about spatial relations on all scales, including distinguishing parts of their bodies (Leushina, 1974/1991), describing spatial movements (forward, back), finding a missing object ("under the table that's next to the door"), putting objects away, and finding the way back home from an excursion. As for many areas of mathematics, verbal interaction is important. For example, parental scaffolding of spatial communication helped both 3- and 4-year-olds perform direction-giving tasks, in which they had to clarify the directions (disambiguate) by using a second landmark ("it's in the bag on the table"), which children are more likely to do the older they are. Both age groups benefited from directive prompts, but 4-year-olds benefited more quickly than younger children from nondirective prompts (Plumert and Nichols-Whitehead, 1996). Children who received no prompts never disambiguated, showing that interaction and feedback from others is critical to certain spatial communication tasks.

Children as young as 3½ to 5 years of age can build simple but meaningful models of spatial relationships with toys, such as houses, cars, and trees (Blaut and Stea, 1974), although this ability is limited until about age 6 (Blades et al., 2004). Thus, younger children create relational, geometric correspondences between elements, which may still vary in scale and perspective (Newcombe and Huttenlocher, 2000).

As an example, children might use cutout shapes of a tree, a swing set, and a sandbox in the playground and lay them out on a felt board as a simple map. These are good beginnings, but models and maps should eventually move beyond overly simple iconic picture maps and challenge children to use geometric correspondences. Four questions arise: direction (which way?), distance (how far?), location (where?), and identification (what objects?). To answer these questions, children need to develop a variety of skills. They must learn to deal with mapping processes of abstraction, generalization, and symbolization. Some map symbols are icons, such as an airplane for an airport, but others are more abstract, such as circles for cities. Children might first build with objects, such as model buildings, then draw pictures of the objects' arrangements, then use maps that are miniaturizations and those that use abstract symbols. Teachers need to consistently help children connect the real-world objects to the representational meanings of map symbols.

As noted in Chapter 4, equity in the education of spatial thinking is an important issue. Preschool teachers spend more time with boys than girls and usually interact with boys in the block, construction, sand play, and climbing areas and with girls in the dramatic play area (Ebbeck, 1984). Boys engage in spatial activities more than girls at home, both alone and with caretakers (Newcombe and Sanderson, 1993). Such differences may interact with biology to account for early spatial skill advantages for boys (note that some studies find no gender differences (e.g., Brosnan, 1998, Chapter 15; Ehrlich, Levine, and Goldin-Meadow, 2006; Jordan et al., 2006; Levine et al., 1999; Rosser et al., 1984).

### *Spatial Visualization and Imagery*

Spatial images are internally experienced, holistic representations of objects that are to a degree isomorphic to their referents (Kosslyn, 1983). Spatial visualization is understanding and performing imagined movements of 2-D and 3-D objects. To do this, you need to be able to create a mental image and manipulate it, showing the close relationship between these two cognitive abilities.

An image is not a "picture in the head." It is more abstract, more malleable, and less crisp than a picture. It is often segmented into parts. Some images can cause difficulties, especially if they are too inflexible, vague, or filled with irrelevant details. People's first images are static. They can be mentally recreated, and even examined, but not transformed. For example, one might attempt to think of a group of people around a table. In contrast, dynamic images can be transformed. For example, you might mentally "move" the image of one shape (such as a book) to another place (such as a bookcase, to see if it will fit). In mathematics, you might mentally move (slide) and rotate an image of one shape to compare that shape to another one. Piaget argued that most children cannot perform full dynamic motions of images until the primary grades (Piaget and Inhelder, 1967, 1971). However, preschool children show initial transformational abilities (Clements et al., 1997a; Del Grande, 1986; Ehrlich et al., 2005; Levine et al., 1999). With guidance, 4-year-olds and some younger children can generate strategies for verifying congruence for some tasks, moving from more primitive strategies, such as edge matching (Beilin, 1984; Beilin, Klein, and Whitehurst, 1982) to the use of geometric transformations and superposition. Interventions can improve the spatial skills of young children, especially when embedded in a story context (Casey, 2005). Computers are especially helpful, as the screen tools make motions more accessible to reflection and thus bring them to an explicit level of awareness for children (Clements and Sarama, 2003; Sarama et al., 1996).

Similarly, other types of imagery can be developed. Manipulative work with shapes, such as tangrams (a puzzle consisting of seven flat shapes, called tans, which are put together in different ways to form distinct geometric shapes), pattern blocks, and other shape sets, provides a valuable foundation (Bishop, 1980). After such explorations, it is useful to engage children in puzzles in which they see only the outline of several pieces and have them find ways to fill in that outline with their own set of tangrams. Similarly, children can begin to develop a foundation for spatial structuring by forming arrays with square tiles and cubes (this is discussed in more detail in the section on measurement).

Also challenging to spatial visualization and imagery are "snapshot" activities (Clements, 1999b; Yackel and Wheatley, 1990). Children briefly see a simple arrangement of pattern blocks, then try to reproduce it. The configuration is shown again for a couple of seconds as many times as necessary. Older children can be shown a line drawing and try to draw it themselves (Yackel and Wheatley, 1990). This often creates interesting discussions revolving around "what I saw."

Spatial visualization and imagery have been positively affected by interventions that emphasize building and composing with 3-D shapes (Casey et al., in press). Another series of activities described above that develops imagery is the sequence of tactile-kinesthetic exploration of shapes.

## Achievable and Foundational Geometry and Spatial Thinking

Although longitudinal research is needed, extant research provides guidance about which geometric and spatial experiences are appropriate for and achievable by young children and will contribute to their mathematical development. First, of the mathematics children engage in spontaneously in child-centered school activities, the most frequent deals with shape and pattern. Second, each of the recently developed, research-based preschool mathematics curricula includes geometric and spatial activities (Casey, Paugh, and Ballard, 2002; Clements and Sarama, 2004; Ginsburg, Greenes, and Balfanz, 2003; Klein, Starkey, and Ramirez, 2002), with some of these featuring such a focus in 40 percent or more of the activities. Third, pilot-testing has shown that these activities were achievable and motivating to young children (Casey, Kersh, and Young, 2004; Clements and Sarama, 2004; Greenes, Ginsburg, and Balfanz, 2004; Starkey, Klein, and Wakeley, 2004), and formal evaluations have revealed that they contributed to children's development of both numerical and spatial/geometric concepts (Casey and Erkut, 2005, in press; Casey et al., in press; Clements and Sarama, 2007c, in press; Starkey et al., 2004, 2006).

Fourth, previous work has shown that well-designed activities can effectively build geometric and spatial skills and general reasoning abilities (e.g., Kamii, Miyakawa, and Kato, 2004). Fifth, results with curricula in Israel that involved only spatial and geometric activities (Eylon and Rosenfeld, 1990) are remarkably positive. Children gained in geometric and spatial skills and showed pronounced benefits in the areas of arithmetic and writing readiness (Razel and Eylon, 1990). Similar results have been found in the United States (Swaminathan, Clements, and Schrier, 1995). Children are better prepared for all school tasks when they gain the thinking tools and representational competence of geometric and spatial sense.

In this section, we describe teaching-learning paths for spatial and geometric thinking in 2-D and 3-D contexts. For each area outlined below, children should be engaged in activities that cover a range of difficulty, including perceive, say, describe/discuss, and construct (measurement in one, two, and three dimensions is described in the following section). Tables 6-1 and 6-2 summarize development of spatial and geometric thinking, as well as measurement, in two and three dimensions. Ages are grouped in the same was as in the previous chapter in order to illustrate how children's engagement with mathematics should build and develop over the prekindergarten years.

In the tables, children's competence within each band is described on the basis of the level of sophistication in their thinking. These levels are called *thinking visually/holistically*, *thinking about parts*, and *relating parts and wholes*.

## Step 1 (Ages 2 and 3)

### *2-D and 3-D Objects*

Very young children match shapes implicitly in their play. Working at the visual/holistic level (see Table 6-1), they can describe pictures of objects of all sorts, using the shape implicitly in their recognition. By age 2 to 3, they also learn to name shapes, with 2-D shapes being more familiar in most cultures, beginning with the familiar and symmetric circle and square and extending to at least prototypical triangles. Although they may name 3-D shapes by the name of one of its faces (calling a cube a square), their ability to match 2-D to corresponding 2-D (and similar for 3-D) indicates their intuitive differentiation of 2-D and 3-D shapes.

Children also learn to recognize and name additional shapes, such as triangles and rectangles—at least in their prototypical forms—and can begin to describe them in their own words. With appropriate knowledge of number, they can begin to describe these shapes by the number of sides they have, just starting to learn the concepts and terminology of the thinking about parts level of geometric thinking.

### *Spatial Relations*

From the first year of life, children develop an implicit ability to move objects. They also learn relationship language, such as "up" and "down" and similar vocabulary. They learn to apply that vocabulary in both 3-D contexts and in 2-D situations, such as the "bottom" of a picture that they are drawing on a horizontal surface.

### *Compositions and Decompositions*

At the visual/holistic level, children can solve simple puzzles involving things in the world (e.g., wooden puzzles with insets for each separate ob-

**TABLE 6-2** Space and Shapes in Three Dimensions

| Steps/Ages (Levels of Thinking) | Goals: A. Perceive, Say, Describe/Discuss, and Construct Objects in 3-D Space | B. Perceive, Say, Describe/Discuss, and Construct Spatial Relations in 3-D Space | C. Perceive, Say, Describe/ Discuss, and Construct Compositions and Decompositions in 3-D Space |
|---|---|---|---|
| **Step 1 (Ages 2 and 3)** | | | |
| Thinking visually/ holistically | See and describe pictures of objects of all sorts (3-D to 2-D).* | Understand and use relational language, including "in," "out," "on," "off," and "under," along with such vertical directionality terms as "up" and "down. | Represent real-world objects with blocks that have a similar shape.<br>• Combine unit blocks by stacking. |
| Thinking about parts | Discriminate between 2-D and 3-D shapes intuitively, marked by accurate matching or naming. | | |
| **Step 2 (Age 4)** | | | |
| Thinking visually/ holistically | Describe the difference between 2-D and 3-D shapes, and names common 3-D shapes informally and with mathematical names ("ball"/sphere; "box" or rectangular prism, "rectangular block," or "triangular block"; "can"/cylinder). | Match 3-D shapes.<br>• Uses relational words of proximity, such as "beside," "next to" and "between," "above," "below," "over," and "under." | |
| Thinking about parts | Identify faces of 3-D objects as 2-D shapes and name those shapes.<br>• Use relational language involving frames of reference such as "in front of," "in back of," "behind," "before." | Identify (matches) the faces of 3-D shapes to (congruent) 2-D shapes, and match faces of congruent 2-D shapes, naming the 2-D shapes.<br>• Represent 2-D and 3-D relationships with objects. | Combine building blocks, using multiple spatial relations. |

**TABLE 6-2** Continued

| | Goals | | |
|---|---|---|---|
| Steps/Ages (Levels of Thinking) | A. Perceive, Say, Describe/Discuss, and Construct Objects in 3-D Space | B. Perceive, Say, Describe/Discuss, and Construct Spatial Relations in 3-D Space | C. Perceive, Say, Describe/ Discuss, and Construct Compositions and Decompositions in 3-D Space |
| Relating parts and wholes | Informally describe why some blocks "stack well" and others do not. | | Compose building blocks to produce composite shapes. Produce arches, enclosures, corners, and crosses systematically. |
| **Step 3 (Age 5)** | | | |
| Thinking visually/ holistically | Name common 3-D shapes with mathematical terms (spheres, cylinder, rectangle, prism, pyramid). | | |
| Thinking about parts | Begin to use relational language of "right" and "left." | Fill rectangular containers with cubes, filling one layer at a time. | |
| Relating parts and wholes | Describe congruent faces and, in context (e.g., block building), parallel faces of blocks. | Understand and can replicate the perspective of a different viewer. | Substitution of shapes. Build complex structures. • Build structures from pictured models. |

NOTE: Less time on 3-D than on 2-D, about 10 percent of the time on 3-D.

*Research indicates that very young children mainly use shape for object identification. Research says children with lower socioeconomic status have difficulty with describing objects and need to learn the vocabulary to do so.

ject pictured). They create pictures with geometric shapes (circles, circle sections, polygons), often representing single objects with different shapes, but eventually combining shapes to make, for example, the body of a vehicle or an animal. That is, initially children manipulate shapes individually, but they are unable to combine them to compose a larger shape. For example, they might use a single shape for a sun, a separate shape for a tree, and another separate shape for a person. Initially, they cannot accurately match shapes to even simple frames.

Later, children learn to place 2-D shapes contiguously to form pictures. In free-form "make a picture" tasks, for example, each shape used represents a unique role or function in the picture (e.g., one shape for one leg). Children can fill simple frame-based shapes puzzles using trial and error, but they may have limited ability to use turns or flips to do so; they cannot use motions to see shapes from different perspectives. Thus, children view shapes only as wholes and see few geometric relationships between shapes or between parts of shapes (i.e., a property of the shape).

Composition with 3-D shapes usually begins with stacking blocks. Children then learn to stack congruent blocks and make horizontal "lines." Next they build a vertical and horizontal structure, such as a floor or a simple wall. Later, some 3-year-olds begin to extend their buildings in multiple directions, possibly creating arches, enclosures, corners, and crosses, but often using unsystematic trial and error and simple addition of pieces.

## Step 2 (Age 4)

### *2-D and 3-D Objects*

Beginning at the visual/holistic level, preschoolers learn to recognize a wide variety of shapes, including shapes that are different sizes and are presented at different orientations. They also begin to recognize that geometric figures can belong to the same shape class, but have different measures and proportions. Similarly, preschoolers learn to describe the differences between 2-D and 3-D shapes informally. They also learn to name common 3-D shapes informally and with mathematical names (ball/sphere, box/rectangular prism, rectangular block, triangular block, can/cylinder). They name and describe these shapes, first using their own descriptions and increasingly adopting mathematical language. For example, "diamond" gives way to "rhombus" and "corners" become "angles" (or vertices). Eventually, they adopt the terminology of the thinking about parts level, such as identifying shapes as triangles because they have three sides. Faces of 3-D shapes are identified as specific 2-D shapes.

Such descriptions build geometric concepts, as well as reasoning skills and language. They encourage children to view shapes analytically. Children begin to describe some shapes in terms of their properties, such as saying that squares have four sides of equal length, and thus make initial forays into thinking at the relating parts and wholes step. They informally describe the properties of blocks in functional contexts, such as that some blocks roll and others do not.

### *Spatial Relations*

Also beginning at the visual/holistic level, preschool children learn to extend their vocabulary of spatial relations with such terms as "beside," "next to," and "between," which they can apply in 3-D and 2-D spaces. Later, they extend this to terms that involve frames of reference, such as "to the side of," "above," and "below."

Later, at the thinking about parts level, preschoolers recognize "matching" shapes at different orientations. They can learn to check if pairs of 2-D shapes are congruent by using geometry motions intuitively, moving from less accurate strategies, such as side-matching, or using lengths, to the use of superimposition (placing one shape on top of the other). They begin to use the geometric motions of slides, flips, and turns explicitly and intentionally, in discussing their solutions to puzzles or in applying such motions in computer environments to manipulate shapes. They learn to predict the effects of geometric motions, thus laying the foundation for thinking at the relating parts and wholes level.

Children also begin to be able to cover a rectangular space with physical tiles and represent their tilings with simple drawings, although they may initially leave gaps in each and may not align all the squares. This is mainly a competence of spatial structuring but it has close connections to the ability to construct compositions in 2-D space.

Preschoolers also learn about the parts of 3-D shapes, using motions to match the faces of 3-D shapes to 2-D shapes and representing 2-D and 3-D relationships with objects. For example, they may make a simple model of the classroom, using a rectangular block for the teacher's desk, small cubes for chairs, and so forth.

### *Compositions and Decompositions*

At the thinking about parts level, preschoolers can place shapes contiguously to form pictures in which several shapes play a single role (e.g., a leg might be created from three contiguous squares), but they use trial and error and do not anticipate creation of new geometric shapes. When filling in a frame or picture outline, children use gestalt configuration or one component, such as side length (Sarama et al., 1996). For example, if several sides of the existing arrangement form a partial boundary of a shape (instantiating a schema for it), children can find and place that shape. If such cues are not present, they match by a side length. Children may attempt to match corners but do not understand angle as a quantitative entity, so they try to match shapes into corners of existing arrangements in which their angles do not fit. Rotating and flipping are used, usually by trial and error, to try different arrangements (a "picking and discarding" strategy). Thus, they can complete a frame that suggests placement of the individual shapes but in which several shapes together may play a single semantic role in the picture.

Later, preschoolers begin to develop relating parts and wholes thinking.

For example, they might combine pattern block shapes (angles that are multiples of 30°) to make composites that they recognize as new shapes and to fill puzzles with growing intentionality and anticipation ("I know what will fit"). Shapes are chosen using angles as well as side lengths. The equilateral triangle world of pattern blocks provides a microworld, in which matching by sides (all of which are equal in length or double the unit length), fitting angles (multiples of 30°), and composing (two equilateral triangles can "make" the blue rhombus, a rhombus and a triangle make a trapezoid, etc.) are facilitating at this beginning step. Eventually, children consider several alternative shapes with angles equal to the existing arrangement. Rotation and flipping are used intentionally (and mentally, i.e., with anticipation) to select and place shapes (Sarama et al., 1996). Children can fill complex frames (Sales, 1994) or cover regions (Mansfield and Scott, 1990).

Related to their ability to tile the rectangular section of a plane, children can copy designs made from squares (and, for some, also isosceles right triangles) and place these shapes onto squared-grid paper. This square-based microworld is simple and not only facilitates composition, but also develops the foundations of much of mathematics (spatial structuring, multiplication, area, volume, coordinates, etc.).

Using 3-D shapes, preschoolers combine building blocks using multiple spatial relations, extending in multiple directions and with multiple points of contact among components, showing flexibility in integrating parts of the structure. Thus, they can reliably produce arches, enclosures, corners, and crosses, including enclosures that are several blocks in height. Later, they can learn to compose building blocks with anticipation, understanding what 3-D shape will be produced with a composition of 2 or more other (simple, familiar) 3-D shapes.

## Step 3 (Age 5)

### *2-D and 3-D Objects*

Kindergartners learn to recognize additional shapes, such as parallelograms, and, more importantly, learn to describe why a certain figure is classified into a given class of shapes (at the relating parts and wholes level). They may therefore discuss that parallelograms have two pairs of sides that are equal in length and two pairs of angles of equal size. This remains just the beginning of this type of thinking, as concepts of parallelism, perpendicularity, and angle measure develop over many years thereafter.

Kindergartners also learn the names of more 3-D shapes, such as spheres, cylinders, prisms, and pyramids. They describe congruent faces of such shapes and begin to understand and discuss such properties as parallel faces in some contexts (e.g., building with blocks).

### *Spatial Relations*

Kindergartners begin to use relational terms "right" and "left" in both 3-D and 2-D contexts, using scaffolds and other guidance as needed. They can also continue to develop the ability to tile a plane with square tiles without gaps and begin to represent such a tiling by drawing. They can learn to count the squares in their tiling, using more systematic strategies for keeping track, such as counting one row at a time. Finally, kindergartners can understand and can replicate the perspectives of different viewers. These competencies reflect an initial development of thinking at the relating parts and wholes level.

### *Compositions and Decompositions*

Kindergartners continue to develop the ability to intentionally and systematically combine shapes to make new shapes and complete puzzles. They do so with increasing anticipation, based on the shapes' attributes, indicating development of mental images of the component shapes. A significant advance is that they can combine shapes with different properties, extending the pattern block (30°) shapes common at early steps to such shapes as tangrams (angles multiples of 45°), and with sets of various shapes that include angles that are multiples of 15° as well as sections of circles.

Using 3-D shapes, kindergartners can substitute a composite shape for a congruent whole shape. They learn to build complex structures, such as bridges with multiple arches, with ramps and stairs at the ends. They can build structures with cubes or building blocks from 2-D pictures of these structures. Children of this age also can learn to move squares and right triangles on grids to create original designs. They can also record these designs on squared-grid paper.

## Instruction to Support the Teaching-Learning Paths

### *Learning and Teaching About Shape*

Without good experiences—"educative" rather than "mis-educative" (Dewey, 1933)—students often rely on impoverished visual prototypes that they develop based on limited examples and limited experiences with language. In contrast, good experiences include providing a variety of examples—for example, with triangles, not all equilateral or isosceles, and not all with a horizontal base, as well as discussions about triangles and their attributes that go beyond simple memorized definitions. Most children in the United States do not have these good experiences. Teachers and curriculum writers assume that children in early childhood classrooms have little or no knowledge of geometric figures. And teachers have had few experiences with geometry in their own education or in their professional development. Thus, it is unsurprising that most classrooms exhibit limited geometry instruction. One early study found that kindergarten children had a great deal of knowledge about shapes and matching shapes before instruction began. Their teacher tended to elicit and verify this prior knowledge but did not add content or develop new knowledge. That is, about two-thirds of the interactions had children repeat what they already knew (Thomas, 1982). Furthermore, many of their attempts to add content were mathematically inaccurate ("every time you cut a square, you get two triangles").

Such neglect is reflected in student achievement. U.S. students are not prepared for learning more sophisticated geometry, especially when compared with students of other nations (Carpenter et al., 1980; Fey et al., 1984; Kouba et al., 1988; Starkey et al., 1999; Stevenson, Lee, and Stigler, 1986; Stigler, Lee, and Stevenson, 1990). In some international studies, they score at or near the bottom in every geometry task (Beaton et al., 1997; Lappan, 1999).

The research reviewed to this point suggests that development of geometric knowledge is fueled by experience and education, not just maturation. If the shape categories children experience are limited, so will be their concepts of shapes. If the examples and nonexamples children experience are rigid, so will be their mental prototypes. Many children learn to accept only isosceles triangles, for example. Others learn richer concepts, even at a young age. Such children are likely to have had good experiences with shapes, including rich, varied examples and nonexamples and discussions about shapes and their characteristics.

Good experiences should begin early. Children need to experience varied examples and nonexamples and understand the attributes of shapes that are mathematically relevant as well as those (orientation, size) that are not. So, examples of triangles and rectangles should include a wide variety of shapes, including "long," "skinny," and "fat" examples. Direct empirical support for this finding is strongest for 4-year-olds, who are motivated to explore shape (Seo and Ginsburg, 2004) and have achieved substantial gains in geometric knowledge through curricular interventions, often surpassing the concepts of much older students in business-as-usual curricula (Casey and Erkut, in press; Casey et al., in press; Clements and Sarama, 2007c, in press; Starkey et al., 2006; Starkey, Klein, and Wakeley, 2004).

Beyond perceiving and naming shapes, children can and should discuss the parts and attributes of shapes. Again, there are several reasons for this recommendation. First, such descriptive activity encourages children to move beyond visual prototypes to the use of mathematical criteria. Second, discussions redirect attention and build strong concepts, mutually affecting and benefiting mental images (Clements and Sarama, 2007b). Third, these types of discussions are interesting to, and beneficial for, children as young as ages 3 and 4 (as the evaluations of the research-based curricula show; see also Spitler, Sarama, and Clements, 2003). Instructional activities that promote such reflection and discussion include building shapes from components. For example, children might build squares and other polygons with toothpicks and marshmallows. They might also form shapes with their bodies, either singly or with their friends.

Another sequence of activities involves tactile-kinesthetic exploration of shapes (feeling shapes hidden in a box). Such nonvisual exploration of shapes does not allow simple matching to prototypes. Instead, they force children to carefully put the parts of the shape into relationship with each other. First, teachers place a small number of shapes on the table and hide a shape congruent to one of these in the box (Clements and Sarama, 2007a). Children feel the shape and point to the matching shape, then pull out the hidden shape to check. Later, children do not have the shapes on the table. Instead, they have to name the shape they are feeling. Even later, they have to describe the shape without using its name, so that their friends could name the shape. In this way, children learn the properties of the shape, moving from intuitive to explicit knowledge.

The sequence in Table 6-1 indicates that 3-year-olds may begin to associate certain shapes with a known small number, even if only at an intuitive level. In comparison, 4-year-olds can explicitly adopt terminology of the thinking about parts step, illustrated by a preschooler stating that an obtuse triangle "must be a triangle, because it has three sides." As 4-year-olds start to see that some shapes have four sides that are the same length, they begin a long journey into the relating parts and wholes level of geometric thinking. Kindergartners can explicitly discuss why they call a certain shape a rectangle. Teachers might start by having children gather rectangles and have them describe why their shapes are rectangles in their own words. They could also show children a variety of shapes and have them decide whether they were or were not rectangles and why. Another useful instructional task is to challenge children to use sticks or straws of varying lengths to make triangles. Older children could draw a series of rectangles, increasing in size. Some children increase the lengths in both dimensions (e.g., length and width), some in only one dimension, leading to rich discussions.

Early childhood curricula traditionally introduce shapes in four basic categories: circle, square, triangle, and rectangle. The separation of the square and the rectangle sets up a misconception that violates the mathematical relationship between these shapes: A square is a rectangle; it is a special kind of rectangle in which all sides are the same length. The idea that a square is not a rectangle, however, is rooted by age 5 (Clements et al., 1999; Hannibal and Clements, 2008). It is time to change the presentation of squares as an isolated set. Instead, recent approaches present many examples of squares and rectangles, varying orientation, size, and so forth, including squares as examples of rectangles. If children say "that's a square," teachers respond that it is a square that is a special type of rectangle, using double-naming ("it's a square-rectangle"). This approach has been shown to be successful with preschoolers and kindergartners (Clements and Sarama, 2007c, in press; Clements, Sarama, and Wilson, 2001; Sarama and Clements, 2002). Kindergarten and first graders can discuss general categories, such as quadrilaterals and triangles, counting the sides of various figures to choose their category. They can then build hierarchical relationships of subsets of these general categories (Kay, 1987).

Children should also learn about composing and decomposing shapes from other shapes. This competence is significant in that the concepts and actions of creating and then iterating units and higher order units in the context of constructing patterns, measuring, and computing are established bases for mathematical understanding and analysis (Clements et al., 1997b; Reynolds and Wheatley, 1996; Steffe and Cobb, 1988). In addition, there is empirical support that this type of composition corresponds with, and supports, children's ability to compose and decompose numbers (Clements et al., 1996).

The sequence in Table 6-1 is based on a series of developmental studies describing children's capabilities (Clements, Sarama, and Wilson, 2001; Mansfield and Scott, 1990; Sales, 1994; Sarama, Clements, and Vukelic, 1996). These studies were synthesized into an empirically verified developmental progression that identified skills that are achievable for children at different ages, especially if provided opportunities to learn (Clements, Wilson, and Sarama, 2004). Starting with a lack of competence in composing geometric shapes, they gain abilities to combine shapes into pictures, and finally synthesize combinations of shapes into new shapes (composite shapes). As further evidence, interventions at the preschool level have shown notable gains in this ability for 2-D shapes (Casey and Erkut, in press). Intentional interventions with 3-D shape construction (i.e., building with unit blocks) have also resulted in statistically significant gains (Casey et al., in press).

Many activities develop these abilities. With a variety of groups of shapes, such as pattern blocks, tangrams, or groups with a greater variety of shapes, children can be encouraged to combine shapes creatively to create pictures and designs. Noting children's developmental level, teachers can make suggestions and pose challenges that will facilitate their learning of more sophisticated thinking.

Outline puzzles that can be filled with those same groups of shapes are also motivating and particularly useful because they can be designed to promote a particular level of thinking. Teachers can then view children's active problem solving and provide them with puzzles that will be attainable but challenging—that is, that will promote their development to the next level of thinking for 2-D geometric composition.

Similar teaching strategies can develop composition of 3-D shapes. Discussions about children's own creative constructions may make explicit ideas about length and symmetry, among others. Also, problems can be designed to encourage spatial and mathematical thinking and sequenced to match developmental progressions (Casey et al., in press; Kersh, Casey, and Young, 2008) early problem for children might be to build an enclosure with walls that are at least two blocks high and include an arch. This introduces the problem of bridging, which involves balance, measurement, and estimation. A second problem might be to build more complex bridges, such as ones with multiple arches and ramps or stairs at the end. This introduces planning and seriation. The third problem might be to build a complex tower with at least two floors, or stories. Children could be provided with cardboard ceilings, so they to make the walls fit the constraints of the cardboard's dimensions.

The recommended approaches and activities in this section have been performed successfully with 3- and 4-year-olds in classrooms serving low- and middle-income children, with strong positive results on child outcomes (Clements and Sarama, 2007c, in press; Starkey et al., 2006; Starkey, Klein, and Wakeley, 2004).

### *Use of Manipulatives, Pictures, and Computers*

Research suggests that the use of manipulatives can help young children develop geometric and spatial thinking (Clements and McMillen, 1996). Using a greater variety of manipulatives is beneficial (Greabell, 1978). Such tactile-kinesthetic experiences as body movement and manipulating geometric solids help young children learn geometric concepts (Gerhardt, 1973; Prigge, 1978). Children also fare better with solid cutouts than with printed forms, the former encouraging the use of more senses (Stevenson and McBee, 1958). However, such benefits are not straightforward or certain (Clements, 1999a; National Mathematics Advisory Panel, 2008). These materials must be used in the context of a complete mathematics program to intentionally develop specific skills and concepts. Also, from the beginning, manipulatives should be used to help children—even young children—develop mental representations that are increasingly abstract.

Pictures can also support learning. Children as young as 5 or 6 (but not most younger children) can use information in pictures to build a pyramid, for example (Murphy and Wood, 1981). Thus, pictures can give students an immediate, intuitive grasp of certain geometric ideas. Instructionally, pictures need to be sufficiently varied so the ideas that students form are not too limited. With experience, children can become sophisticated in interpreting geometric relationships in pictures. Diagrams are also useful tools for visualizing numerical and arithmetic problems, and the more experience children have with the geometric and measurement attributes of pictures and shapes, the more competence they will have in constructing and interpreting such diagrams. However, research indicates that it is rare for pictures to be superior to manipulatives. In fact, in some cases, pictures may not differ in effectiveness from instruction with symbols (Sowell, 1989). The reason may lie not so much in the nonconcrete nature of the pictures as in their nonmanipulability—that is, that children cannot act on them as flexibly and extensively. This is one reason that manipulatives on computers—even though 2-D—can benefit learning and teaching.

In fact, computers may have some specific advantages (Clements and McMillen, 1996). For example, some computer manipulatives offer more flexibility than their noncomputer counterparts. Computer-based pattern blocks, for example, can be composed and decomposed in more ways than physical pattern blocks. As another example, children and teachers can save and later retrieve any arrangement of computer manipulatives. Similarly, computers allow storage and replay of sequences of actions on manipulatives. Computers can also be used to carry out mathematical processes that are difficult or impossible to perform with physical manipulatives. For example, a computer environment might automatically draw shapes symmetrical to anything the child constructs or draws.

As a final illustration, computers can help children become aware of, and mathematize, their actions. For example, very young children can move puzzle pieces into place, but they do not think about their actions. Using the computer, however, helps children become aware of and describe these motions (Clements and Battista, 1991; Johnson-Gentile, Clements, and Battista, 1994).

Manipulatives—physical or computer—are one tool that can assist children in constructing mathematical meaning. They do not always do that, however, and the point of using them lies not in their use in promoting manipulations or random play, but to develop abstract ideas. In this view, manipulatives are successful to the extent that they become unnecessary because children have built mental images and concepts that they use for mathematical thinking (Clements, 1999a).

## MEASUREMENT

Geometric measurement connects and enriches the two critical domains of geometry and number. Children's understanding of measurement has its roots in infancy and the preschool years, and it grows over many years, as the research described in Chapter 3 shows. Even preschoolers can be guided to learn important concepts if provided appropriate measurement experiences. They naturally encounter and discuss quantities (Seo and Ginsburg, 2004). They initially learn to use words that represent quantity or magnitude of a certain attribute. Then they compare two objects directly and recognize equality or inequality (Boulton-Lewis, Wilss, and Mutch, 1996). At age 4-5, most children can learn to overcome perceptual cues and make progress in reasoning about and measuring quantities. They are ready to learn to measure, connecting number to the quantity, yet the average child in the United States, with limited measurement experience, exhibits limited understanding of measurement until the end of the primary grades. We examine this development in more detail for the attribute of length.

## Length Measurement

Length is a characteristic of an object found by quantifying how far it is between the end points of the object. Distance is often used similarly to quantify how far it is between any two points in space. Measuring length or distance consists of two aspects: (1) identifying a unit of measure and *subdividing* (mentally and physically) the object by that unit; (2) placing that unit end to end (*iterating*) alongside the object. Subdividing and unit iteration are complex mental accomplishments that are too often ignored in traditional measurement curriculum materials and instruction. Many researchers therefore go beyond the physical act of measuring to investigate children's understandings of measuring as covering space and quantifying that covering. Appendix B describes concepts that are basic to understanding length measurement.

Before kindergarten, many children lack understanding of measurement ideas and procedures, such as lining up end points when comparing the lengths of two objects. Even 5- to 6-year-olds, given a demarcated ruler, write in numerals haphazardly, with little regard to the size of the spaces. Few use zero as a starting point, showing a lack of understanding of the origin concept. At age 4-5, however, many children can, with opportunities to learn, become less dependent on perceptual cues and thus make progress in reasoning about or measuring quantities. From kindergarten to Grade 2, children can significantly improve in measurement knowledge (Ellis, 1995). They learn to represent length with a third object, using transitivity to compare the length of two objects that are not compared directly in a wider variety of contexts (Hiebert, 1981). They can also use given units to find the length of objects and associate higher counts with longer objects (Hiebert, 1981, 1984). Some 5-year-olds and most 7-year-olds can use the concept of unit to make inferences about the relative size of objects; for example, if the numbers of units are the same, but the units are different, the total size is different (Nunes and Bryant, 1996).

Children as young as kindergartners may be proficient with a conventional ruler and understand quantification in limited measurement contexts. However, their skill decreases when features of the ruler deviate from the convention. Thus, measurement is supported by characteristics of measurement tools, but children still need to develop understanding of key measurement concepts. For example, they may initially iterate a unit leaving gaps between subsequent units or overlapping adjacent units (Horvath and Lehrer, 2000; Lehrer, 2003). These children may think of measuring as the physical activity of placing units along a path in some manner, rather than the activity of covering the space/length of the object with no gaps. Furthermore, children often begin counting at the numeral 1 on a ruler (Lehrer, 2003) or, when counting paces heel-to-toe, start their count with the movement of the first foot, missing the first foot and counting the second foot as one (Lehrer, 2003; Stephan et al., 2003). Again, children may not be thinking about measuring as covering space. Rather, the numerals on a ruler (or the placement of a foot) signify when to start counting, not an amount of space that has already been covered (i.e., 1 is the space from the beginning of the ruler to the hash mark, not the hash mark itself). Many children initially find it necessary to iterate the unit until it "fills up" the length of the object and will not extend the unit past the end point of the object they are measuring (Stephan et al., 2003). Finally, many children do not understand that units must be of equal size. They will even measure with tools subdivided into different size units and conclude that quantities with more units are larger (Ellis et al., 2000). This may be a deleterious side effect of counting, in which children learn that the size of objects does not affect the result of counting (Mix, Huttenlocher, and Levine, 2002). However, the researchers base this interpretation on the assumption that units are always "given" in counting contexts. In fact, there are counting contexts in which this is not the case, such as counting whole toy people constructed in two parts, top and bottom, when some are fastened and some are separated (Sophian and Kailihiwa, 1998).

Thus, significant development occurs in the early childhood years. However, the foundational ideas about length are usually not integrated, even by the primary grades. For example, children may still not understand the importance of, or be able to create, equal size units (Clements et al., 1997a; Lehrer, Jenkins, and Osana, 1998; Miller, 1984). This indicates that children have not necessarily differentiated fully between counting discrete objects and measuring. Even if they show competence with rulers and are given identical units, children may not spontaneously iterate those they have if they do not have a sufficient number to measure an object (Lehrer, Jenkins, and Osana, 1998)—even when the units are rulers themselves (Clements, 1999c). Some children can or do not mentally partition the object to be measured.

Many recent curricula or other instructional guides advise a sequence of instruction in which children compare lengths, measure with nonstandard units, incorporate the use of manipulative standard units, and measure with a ruler (Clements, 1999c; Kamii and Clark, 1997). The basis for this sequence is, explicitly or implicitly, the theory of measurement of Piaget et al. (1960). The argument is that this approach motivates children to see the need for a standard measuring unit.

Although such an approach has been shown to be effective, it may not be necessary to follow a nonstandard-to-standard units approach. For example, Boulton-Lewis et al. (1996) found that children used nonstandard units unsuccessfully but were successful at an earlier age with standard units and measuring instruments. The researchers concluded that nonstandard units are not a good way to initially help children understand the need for standardized conventional units in the length measuring process. Just as interesting were children's strategy preferences. Children of every age preferred to use standard rulers, even though their teachers were encouraging them to use nonstandard units.

Furthermore, children measured correctly with a ruler before they could devise a measurement strategy using nonstandard units. To realize that arbitrary units are not reliable, a child must reconcile the varying lengths and numbers of arbitrary units. Emphasizing nonstandard units too early may defeat the purpose it is intended to achieve. That is, early emphasis on various nonstandard units may interfere with children's development of the basic measurement concepts required to understand the need for standard units. In contrast, using manipulative standard units, or even standard rulers, is less demanding and appears to be a more interesting and meaningful real-world activity for young children (Boulton-Lewis et al., 1996). These findings have been supported by additional research (Boulton-Lewis, 1987; Clements and Battista, 2001; Clements et al., 1997b; Héraud, 1989).

Thus, early experience measuring with different units may be exactly the wrong thing to do. Another study (Nunes, Light, and Mason, 1993) suggests that children can meaningfully use rulers before they reinvent such ideas as units and iteration. In it, children ages 6 to 8 communicated about lengths using string, centimeter rulers, or one ruler and one broken ruler starting at 4 cm. The traditional ruler supported the children's reasoning more effectively than the string; their accurate performance almost doubled. Their strategies and language (it is as long as the "little line just after three") indicated that children gave "correct responses based on rigorous procedures, clearly profiting from the numerical representation available through the ruler" (p. 46). They did even better with the *broken* ruler than the string, showing that they were not just reading numbers off the ruler. The unusual context confused children only 20 percent of the time. The researchers concluded that conventional units already chosen and built into the ruler do not make measurement more difficult. Indeed, children benefited from the numerical representation provided by even the broken ruler.

Such research has led several authors to argue that early rule use should be encouraged, not avoided or delayed (Clements, 1999c; Nührenbörger, 2001; Nunes et al., 1993). Rulers allow children to connect instruction to their previous measurement experiences with conventional tools. In contrast, dealing with informal, 3-D units deemphasizes the one-dimensional (1-D) nature of length and focuses on the counting of discrete objects. In this way, it deemphasizes both the zero point and the iteration of line segment lengths as units (Bragg and Outhred, 2001).

The Piagetian-based argument, that children must conserve length before they can make sense of ready-made systems, such as rulers (or computer tools, such as those discussed in the following section), may be an overstatement. Findings of these studies support a Vygotskian perspective (Ellis et al., 2000; Miller, 1989), in which rulers are viewed as cultural instruments children can appropriate. That is, children can use rulers, appropriate them, and so build new mental tools. Not only do children prefer using rulers, but also they can use them meaningfully and in combination with manipulable units to develop understanding of length measurement. In general, measurement procedures can serve as cognitive tools (Miller, 1989) developed to solve certain practical problems and organize the way children think about amount. Measurement concepts may originally be organized in terms of the contexts and procedures used to judge, compare, or measure specific attributes (Miller, 1989). If so, transformations that do not change length but do change number, such as cutting, may be particularly difficult for children, more so than traditional conservation questions. Children need to learn to distinguish the different attributes (e.g. length, number) and learn which transformations affect which attributes.

Another Piagetian idea, from the field of social cognition, is that conflict is the genesis of cognitive growth. One series of studies, however, indicates that this is not always so. If two strategies, measurement and direct comparison, were in conflict, children learned little and benefited little from verbal instruction. However, if children saw that the results of measurement and direct comparison agreed, then they were more likely to use measurement later than were children who observed both procedures but did not have the opportunity to compare their results (Bryant, 1982). This is a case in which presenting children with conflicting information (between strategies or between results of measuring with different units) too soon is unhelpful or deleterious.

Whatever the specific instructional approach taken, research demonstrates several implications. Measurement should not be taught as a simple skill. It is a complex combination of concepts and skills that develops over years. Teachers who understand the foundational concepts of measurement will be better able to interpret children's understanding and ask questions that will lead them to construct these ideas. Both research with children and interviews with teachers support the claims that (a) the principles of measurement are difficult for children, (b) they require more attention in school than is usually given, (c) time needs to first be spent in informal measurement, in which the use of measurement principles is evident, and (d) transition from informal to formal measurement needs much more time and care, with instruction in formal measure always returning to basic principles (see Irwin, Vistro-Yu, and Ell, 2004).

The sequence in Table 6-3 summarizes achievable goals in linear measurement that have been employed in pilot-testing of research-based curricula (Casey et al., 2004; Clements and Sarama, 2004; Greenes et al., 2004; Starkey et al., 2004). Again, evaluations confirm the appropriateness of the sequencing (Clements and Sarama, 2007c, in press; Starkey et al., 2004, 2006).

## Area Measurement and Spatial Structuring

Area is an amount of 2-D surface that is contained within a boundary. Area measurement assumes that a suitable 2-D region is chosen as a unit, congruent regions have equal areas, regions do not overlap, and the area of the union of two regions that do not overlap (disjoint union) is the sum of their areas (Reynolds and Wheatley, 1996). Thus, finding the area of a region can be thought of as tiling (or equal partitioning) a region with a 2-D unit of measure. Such understandings are complex, and children develop them over time. These area understandings do not develop well in traditional U.S. instruction (Carpenter et al., 1975), not only for young children, but also for preservice teachers (Enochs and Gabel, 1984). A study of children from Grades 1, 2, and 3 revealed little understanding of area measurement (Lehrer, Jenkins, and Osana, 1998). Asked how much space a square (and a triangle) cover, 41 percent of children used a ruler to measure length. Although area measurement is typically emphasized in the intermediate grades, the literature suggests that some less formal aspects of area measurement can be introduced in earlier years. Concepts that are essential to understanding and learning area measurement are described in Appendix B. One especially important one, spatial structuring, is discussed next.

Nascent awareness of area is often noticed in informal observations, such as when a child asks for pieces of colored paper to cover their table. A way to more formally assess children's understanding of area is through comparison tasks. Some researchers report that preschoolers use only one dimension or one salient aspect of the stimulus to compare the area of two surfaces (Bausano and Jeffrey, 1975; Maratsos, 1973; Mullet and Paques, 1991; Piaget et al., 1960; Raven and Gelman, 1984; Russell, 1975; Sena

**TABLE 6-3** Linear Measurement (Space in One Dimension)

| | Goals | | |
|---|---|---|---|
| Steps/ Ages (Levels of Thinking) | A. Perceive, Say, Describe/Discuss, and Construct Objects in 1-D Space | B. Perceive, Say, Describe/Discuss, and Construct Spatial Relations in 1-D Space | C. Perceive, Say, Describe/Discuss, and Construct Compositions and Decompositions in 1-D Space |
| **Step 1 (Ages 2 and 3)** | | | |
| Thinking visually/ holistically | Informally recognize length as extent of 1-D space. Compare 2 objects directly, noting equality or inequality. | | Informally combine objects in linear extent. |
| **Step 2 (Age 4)** | | | |
| Thinking about parts | Compare the length of two objects by representing them with a third object.<br>• Initial measurement by laying units end to end, often with units that are notably square or cubical (to facilitate physical concatenation). | | Understand that lengths can be concatenated. |
| Relating parts and wholes | Seriate up to six objects by length (e.g., connecting cube towers). | | |
| **Step 3 (Age 5)** | | | |
| Thinking about parts | Measure by repeated use of a unit, moving from units that are notably square or cubical to those that more closely embody one dimension (e.g., sticks or stirrers). | | |
| Relating parts and wholes | Seriate any number of objects by length, even if differences between consecutive lengths are not palpable perceptually.<br>• Initial measurement with simple unit rulers, including sticks with unit lengths marked off and other unit rulers.<br>• Explore the relationship between the size and number of units.<br>Interpret bar graphs to answer questions such as "more," "less," as well as simple trends, using length of the bars. | | Add two lengths to obtain the length of a whole. |

NOTE: Less time on 1-D than on 2-D; about 5 percent of the time on 1-D.

and Smith, 1990). For example, 4- and 5-year-olds may match only one side of figures when attempting to compare their areas (Silverman, York, and Zuidema, 1984). Others claim that children can integrate more than one feature of a region but judge areas with additive combination, for example, making implicit area judgments based on the longest single dimension (Mullet and Paques, 1991) or height + width rules (Cuneo, 1980; Rulence-Paques and Mullet, 1998). Children ages 6 to 8 use a linear extent rule, such as the diagonal of a rectangle. Only after this age do most children move to explicit use of spatial structuring of multiplicative rules to solve those studies' tasks. Note that this does not imply formal use of multiplication, but only that their estimates are best approximated by the area formula.

In most of these studies, children did not interact with the materials. Doing so often changes their strategies and improves their estimates. Children as young as age 3 are more likely to make estimates consistent with multiplicative rules when using manipulatives than when just asked to make a perceptual estimation. For example, they are more accurate when they are asked to count out the right number of square tiles to cover a floor and put them in a cup (Miller, 1984). Similarly, children ages 5 to 6 were more likely to use strategies consistent with multiplicative rules after playing with the stimulus materials (Wolf, 1995).

A more accurate strategy for comparing areas than visual estimation is superimposition. Children as young as age 3 have a rudimentary concept of area based on placing regions on top of one another, but it is not until age 5 or 6 that their strategy is accurate and efficient. As an illustration, when asked to manipulate regions, preschoolers in one study used superimposition instead of the less precise strategies of laying objects side-by-side or comparing single sides, both of which use one dimension at best in estimating the area (Yuzawa, Bart, and Yuzawa, 2000). Again, the facilitative effect of manipulation is shown. Children were given target squares or rectangles and asked to choose one that was equal to two standard rectangles in area. They performed better when they placed the standard figures on the targets than when they made perceptual judgments. They also performed better when one target could be overlapped completely with the standard figures (even in the perceptual condition, which suggests that they performed a mental superposition).

Higher steps in thinking about area may have their roots in the internalization of such procedures as placing figures on one another, which may be aided by cultural tools (manipulatives) or scaffolding by adults (see Vygotsky, 1934/1986). For example, kindergartners who were given practice with origami (paper folding) increased the spontaneous use of the procedure of placing one figure on another for comparing sizes (Yuzawa et al., 1999). Because origami practice includes the repeated procedure of folding one sheet into two halves, origami practice might facilitate the development of an area concept, which is related to the spontaneous use of the procedure.

To measure, a unit must be established. Teachers often assume that the product of two lengths structures a region into an area of 2-D units for students. However, the construction of a 2-D array from linear units is nontrivial. Young children often cannot partition and conserve area and instead use counting as a basis for comparing. For example, when it was determined that one share of pieces of paper cookie was too little, preschoolers cut one of that share's pieces into two and handed them both back, apparently believing that the share was now "more" (Miller, 1984).

As with length measurement, children often cover space, but they do not initially do so without gaps or overlapping (i.e., they do not tile the region with units). They also initially do not extend units over the boundaries when a subdivision of that unit is needed to fill the surface (Stephan et al., 2003). Even more limiting, children often choose units that physically resemble the region they are covering; for example, choosing bricks to cover a rectangular region and beans to cover an outline of their hands (Lehrer, 2003; Lehrer, Jenkins, and Osana, 1998; Nunes et al., 1993). They also mix different shapes (and areas), such as rectangular and triangular, to cover the same region and accept a measure of "7" even if the seven covering shapes are of different sizes (84 percent of primary grade children; Lehrer, Jenkins, and Osana, 1998). These concepts have to be developed before children can use iteration of equal units to measure area with understanding. Once these problems have been solved, children need to structure 2-D space into an organized array of units to achieve multiplicative thinking in determining volume, a concept to which we now turn.

### Volume Measurement

Volume introduces even more complexity, not only in adding a third dimension and thus presenting a significant challenge to students' spatial structuring, but also in the very nature of the materials that are measured using volume. This leads to two ways to measure volume, illustrated by "packing" a space, such as a 3-D array with cubic units, and "filling" with iterations of a fluid unit that takes the shape of the container. For the latter, the unit structure may be psychologically 1-D for some children (i.e., simple iterative counting that is not processed as geometric 3-D), especially, for example, in filling a cylindrical jar in which the (linear) height corresponds to the volume (Curry and Outhred, 2005). Given the possible complexities, is either of these more or less appropriate for young children, beyond, say, informal experiences?

For children in Grades 1-4, competence in filling volume (e.g., estimating and measuring the number of cups of rice that filled a container) was about equivalent to their competence in corresponding length tasks (Curry and Outhred, 2005). The relationship is consistent with the notion that the structure of the task is 1-D, exemplified by some students' treating the height of the rice in the container as if it were a unit length and iterating it, either mentally or using their fingers, up the side of the container. Some students performed better on length, others on filling volume, giving no evidence of a relationship between the two. The task contained some extra demands, such as creating equal measurements; even many first graders made sure that the cup was not over- or underfilled for each iteration. In another study, 3- and 4-year-olds understood that unit size affects the measurement of the object's volume (Sophian, 2002). Thus, simple experience with filling volume may be appropriate for young children.

On the other hand, packing volume is more difficult than length and area (Curry and Outhred, 2005). Most children had little idea of how to estimate or measure on packing tasks. There were substantial increases from Grades 2 to 4, but even the older students' scores were below the corresponding scores for the area task. Furthermore, there was a suggestion that understanding of area is a prerequisite to understanding packing volume. Therefore, children should have many experiences building with blocks and filling boxes with cubes. A developmental progression is provided in Table 6-2. A full conceptual understanding of 3-D space will develop only over several years for most children.

## Achievable and Foundational Measurement in One, Two, and Three Dimensions

In this section, we describe children's development of measurement in one, two, and three dimensions. We do not consider measurement of nongeometric attributes, such as weight/mass, capacity, time, and color, because these are more appropriately considered in science and social studies curricula. Again, for each area outlined below, children should be engaged in activities that cover a range of difficulty, including perceive, say, describe/discuss, and construct. Table 6-3 outlines the path for measurement of length.

### Step 1 (Ages 2 and 3)

*Objects and Spatial Relations*

Young children naturally encounter and discuss quantities in their play (Ginsburg, Inoue, and Seo, 1999). They first learn to use words that represent quantity or magnitude of a certain attribute. Facilitating this language is important not only to develop communication abilities, but for the development of mathematical concepts. Simply using labels such as

"Daddy/Mommy/Baby" and "big/little/tiny" helped children as young as 3 years to represent and apply higher order seriation abilities, even in the face of distracting visual factors, an improvement equivalent to a 2-year gain.

At the visual/holistic level (see Table 6-3), children begin by informally recognizing length as extent of 1-D space. For example, they may remark of a road made with building blocks, "This is long." They can then compare two objects directly and recognize and describe their equality (e.g., "You are just as tall as I am!") or inequality (e.g., "My pencil is longer than yours") in length.

*Compositions and Decompositions*

At the visual/holistic level, children compose lengths intuitively. For example, they may lay building blocks along a path to "make a long road."

**Step 2 (Age 4)**

*Objects and Spatial Relations*

At the thinking about parts level, preschool children learn to compare the length of two objects by representing them with a third object and using transitive reasoning (i.e., indirect comparison) (Boulton-Lewis et al., 1996). Again, language, such as the differences between counting-based terms (e.g., a toy, two trucks) and mass terms (e.g., some sand), can help children form relationships between counting and continuous measurement (Huntley-Fenner, 2001).

Preschoolers also begin actual measurement by laying physical units end to end and counting them to measure a length. However, they may not recognize the need for equal-length units and initially may make errors, such as leaving gaps between units. One way to engage in discussions of such concepts is to apply the resulting measures to comparison situations. These concepts and skills develop in parallel with competencies in seriating lengths, which emerge last and mark the first level of thinking about relating parts and wholes.

Preschoolers also begin to be able to cover a rectangular space with physical tiles and represent their tilings with simple drawings, although they may leave gaps in each and may not align all the squares.

*Compositions and Decompositions*

At the thinking about parts level, preschoolers understand that lengths can be concatenated in this way. This understanding, initially implicit, is revealed as children operate on objects.

## Step 3 (Age 5)

### *Objects and Spatial Relations*

Kindergartners move to more sophisticated understanding at the thinking about parts level by measuring via the repeated use of a unit. However, they initially may not be precise in such iterations. Beginning to develop aspects of thinking at the level of relating parts and wholes, they can explore the concept of the inverse relationship between the size of the unit of length and the number of units required to cover a specific length or distance, recognizing it at least at an intuitive level. However, they may not appreciate the need for identical units. Work with manipulative units of standard measure (e.g., 1 inch or 1 cm), along with related use of rulers and consistent discussion, will help children learn both the concepts and procedures of linear measurement.

Kindergartners also can learn to fill containers with cubes, filling one layer at a time, intentionally, all of which involves relationships at the thinking about parts level of thinking. In a similar vein, they can learn to accurately count the number of squares in a rectangular array, using increasingly systematic strategies, including counting in rows or columns. They represent a complete covering of a rectangle's area (although initially there may be some inaccuracies, such as in the alignment of drawn shapes).

### *Compositions and Decompositions*

Kindergartners understand length composition explicitly. For example, they can add to lengths to obtain the length of the whole. They can use a simple ruler (or put a length of connecting cubes together) to measure one plastic snake and measure the length of another snake to find the total of their lengths. Or, more practically, they can measure all sides of a table with unmarked (foot) rulers to measure how much ribbon they would need to decorate the perimeter of the table. Their use of rows or columns in covering a rectangular area also implies at least an implicit composition of units into a composite unit.

## Instruction to Support the Teaching-Learning Path

### *Length*

To move children through the teaching-learning path, teachers of the youngest children should observe children in their play, because they encounter and discuss measurable quantities frequently (Ginsburg, Inoue, and Seo, 1999). Using such words as "bigger/larger/smaller," and, as soon as possible, "longer/shorter" and "taller/shorter" directs children's attention to these attributes and also helps them apply seriation abilities. Teachers should listen carefully to see how they are interpreting and using language (e.g., length as the distance between end points or as "one end sticking out").

Children should be given a variety of experiences comparing the size of objects. Once they can do so by direct comparison, they should compare several items to a single item, such as finding all the objects in the classroom longer than their forearm. Ideas of transitivity can then be explicitly discussed. Next, children should engage in experiences that allow them to connect number to length. Teachers should provide children with both conventional rulers and manipulative units using standard units of length, such as centimeter cubes (specifically labeled "length-units"; from Dougherty and Slovin, 2004). As they explore with these tools, the ideas of length-unit iteration (e.g., not leaving space between successive length-units), correct alignment (with a ruler), and the zero-point concept can be developed. Having older (or more advanced) children draw, cut out, and use their own rulers can be used to discuss these aspects explicitly.

In all activities, teachers should focus on the meaning that the numerals on the ruler have for children, such as enumerating lengths rather than discrete numbers. In other words, classroom discussions should focus on "What are you counting?" with the answer in length-units. Given that counting discrete items often correctly teaches children that the length-unit size does not matter, teachers should plan experiences and reflections on the nature of properties of the length-unit in various discrete counting and measurement contexts. Comparing results of measuring the same object with manipulatives and with rulers and using manipulative length-units to make their own rulers help children connect their experiences and ideas.

In second or third grade, teachers might introduce the need for standard length-units and the relation between the size and number of length-units. The relationship between the size and number of length-units, the need for standardization of length-units, and additional measuring devices can be explored at this time. The early use of multiple nonstandard length-units would not be used until this point (see Carpenter and Lewis, 1976). Instruction focusing on children's interpretations of their measuring activity can enable them to use flexible starting points on a ruler to indicate measures successfully (Lubinski and Thiessen, 1996). Without such attention, children are just reading off whatever ruler number aligns with the end of the object into the intermediate grades (Lehrer, Jenkins, and Osana, 1998).

By kindergarten, length is used in other areas, such as understanding addition and graphing. For example, bar graphs use length to represent counts or measures. Kindergartners can answer such questions as "more" and "less," as well as simple trends, using length of the bars.

Emphasis on children's solving real measurement problems and, in so doing, building and iterating units, as well as units of units, helps them develop strong concepts and skills. Teachers should help children closely connect the use of manipulative units and rulers. When conducted in this way, measurement tools and procedures become tools for mathematics and tools for thinking about mathematics (Clements, 1999c; Miller, 1984, 1989). Well before first grade, children have begun the journey toward that end.

### *Area*

Children need to structure an array to understand area as truly 2-D (see Appendix B). Play with structured materials, such as unit blocks, pattern blocks, and tiles, can lay the groundwork for children's spatial structuring, although achieving the conceptual benchmark will not be achieved until after the primary grades for most children, even with high-quality instruction. In brief, the too-frequent practice of simple counting of units to find area (achievable by preschoolers) leading directly to teaching formulas may not build the requisite foundational concepts (Lehrer, 2003). Instead, educators should build on young children's initial spatial intuitions and appreciate their need to construct the idea of measurement units—including development of a measurement sense for standard units, for example, finding common objects in the environment that have a unit measure. Children need to have many experiences covering quantities with appropriate measurement units, counting those units, and spatially structuring the object they are to measure, in order to build a firm foundation for eventual use for formulas. For example, children might build rectangular arrays with square tiles and learn to count the number of manipulatives used in each. Eventually, they need to link counting by groups to reflect the structure of rectangular arrays, for example, counting the squares in an array by skip-counting the number in each row.

This long developmental process usually only begins in the years before first grade. However, we should also appreciate the importance of these early conceptualizations. For example, 3- and 4-year-olds' use of a linear rating scale to judge area, even if using an additive rule, indicates an impressive level of quantitative ability and, according to some, nascent mental structures for algebra at an early age (Cuneo, 1980).

Competencies in the major realms of geometry/spatial thinking and number are connected throughout development. The earliest competencies may share common perceptual and representational origins (Mix, Huttenlocher, and Levine, 2002). Infants are sensitive to both the amount of liquid in a container (Gao, Levine, and Huttenlocher, 2000) and the distance away a toy is hidden in a long sandbox (Newcombe, Huttenlocher, and Learmonth, 1999). Visual-spatial deficits in early childhood are detrimental to children's development of numerical competencies (Semrud-Clikeman and Hynd, 1990; Spiers, 1987). Other evidence shows specific spatially related learning disabilities in arithmetic, possibly more so for boys than girls (Share, Moffitt, and Silva, 1988). Primary school children's thinking about units and units of units was found to be consistent in both spatial and numerical problems (Clements et al., 1997a). In this and other ways, specific spatial abilities appear to be related to other mathematical competencies (Brown and Wheatley, 1989; Clements and Battista, 1992; Fennema and Carpenter, 1981; Wheatley, Brown, and Solano, 1994). Geometric measurement connects the spatial and numeric realms explicitly.

## SUMMARY

This chapter describes geometry and spatial thinking and measurement, which comprise the second essential domain for young children's mathematical development. The research in this domain is less developed than for number, but it does provide guidance for educators regarding what young children can and should do to develop competence in these areas. The teaching-learning path for geometry and spatial relations demonstrates how young children move through levels of thinking as they learn about 2-D and 3-D objects. The use of manipulatives, pictures, and computers play an important role in facilitating children's progress along this path. Early childhood teachers should help children extend their thinking by building on simple conventional models (e.g., child represents classroom with cut out pictures) and challenge them by asking them to use geometric correspondences (e.g., direction—which way?, identification—which object?) to solve problems.

Measurement, the second major area covered in this chapter, connects and enriches the two crucial domains of geometry and number. The teaching-learning path for measurement describes children's developing competence in linear measurement and initial steps toward understanding areas and volume. The teaching-learning path outlined for length emphasizes the need to provide experiences that allow children to compare the size of objects and to connect number to length. Children also need opportunities to solve real measurement problems which can help build their understanding of units, length-unit iteration, correct alignment and the zero-point concept. Children's early competency in measurement is facilitated by play with structured materials, such as unit blocks, pattern blocks, and tiles and strengthened through opportunities to reflect on and discuss their experiences.

It is important to note that the potential of young children's learning in geometry and measurement if a conscientious, sequenced development of spatial thinking and geometry were provided to them throughout their earliest years is not yet known. Research on the learning of shapes and certain aspects of visual literacy suggests the inclusion of these topics in the early years can be powerful. Specific spatial abilities appear to be related to other mathematical competencies and geometric measurement connects the spatial and numeric realms explicitly (Brown and Wheatley, 1989; Clements and Battista, 1992; Fennema and Carpenter, 1981; Wheatley et al., 1994). However, there is insufficient evidence on the effects (efficacy and efficiency) of including such topics as congruence, similarity, transformations, and angles in curricula and teaching at specific age levels (Clements and Sarama, 2007b; National Mathematics Advisory Panel, 2008). Such research, as well as longitudinal research on many such topics, is needed.

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# Part III

# Contexts for Teaching and Learning

# 7

# Standards, Curriculum, Instruction, and Assessment

In this chapter, we address the topic of effective mathematics curriculum and teaching—what is known about how teachers can effectively support children's learning of important foundational mathematics content. We begin the chapter with a description and analysis of current state standards for early learning. Standards are intended to influence the development of curriculum and assessment tools, and therefore they have the potential to serve as a bridge between what research says about children's learning and the kinds of teaching and learning that actually occur.

Next, the chapter provides an overview about the state of mathematics teaching and learning experiences in early childhood settings and reviews the literature on effective practices for teaching young children mathematics. Following this is a discussion of formative assessment, an essential and often overlooked element of effective instruction. The chapter concludes with a discussion of research on effective curricula.

## DEFINITIONS

To enhance understanding of the content of this chapter, we first define some of the most frequently used early childhood education terminology.

### Teacher-Initiated and Child-Initiated Experiences

Early childhood practices are often described as either teacher-initiated or child-initiated. *Teacher-initiated* or *teacher-guided* means that teachers plan and implement experiences in which they provide explicit information, model or demonstrate skills, and use other teaching strategies in which they take the lead. Teacher-initiated learning experiences are determined by the teacher's goals and direction, but they should also reflect children's active engagement (Epstein, 2007). Ideally, teacher-initiated instruction actively involves children. Indeed, when appropriately supportive and focused, teacher-initiated instruction can lead to significant learning gains (French and Song, 1998; Howes et al., 2008). In practice, however, most teacher-initiated instruction is associated with the passive engagement of children (Pianta et al., 2005).

By contrast, *child-initiated* or *child-guided* means that children acquire knowledge and skills through their own exploration and through interactions with objects and with peers (Epstein, 2007, p. 2). Child-initiated experience emanates primarily from children's interests and actions with support from teachers. For child-initiated learning to occur, teachers organize the environment and materials and provide the learning opportunities from which children make choices (Epstein, 2007). Teachers thoughtfully observe children during child-initiated activity, gauging their interactions and the provision of new materials, as well as reorganization of the environment, to support their continued learning and development.

During optimal child-initiated experience, teachers are not passive, nor are children entirely in control—although this ideal is not always realized in practice. For example, classroom observational research reveals that teachers tend to spend little time with children during free play (Seo and Ginsburg, 2004), or they focus their interactions on behavior management rather than on helping children learn (Dickinson and Tabors, 2001; Kontos, 1999).

## Instruction and Intentional Teaching

In early childhood education, the term *instruction* is most often used to mean "direct instruction," implying that teachers are entirely in control and children are passive recipients of information. The term is also used pejoratively to refer to drill and practice on isolated skills. *Direct instruction* is more accurately defined as situations in which teachers give information or present mathematics content directly to children. The National Mathematics Advisory Panel (2008) uses the term *explicit instruction* to refer to the many ways that teachers can intentionally structure children's experiences so that they support learning in mathematics.

Throughout the day and across various contexts—whole group, small group, centers, play, and routines—teachers need to be active and draw on a repertoire of effective teaching strategies. This skill in adapting teaching to the content, type of learning experience, and individual child with a clear learning target as a goal is called *intentional teaching* (Epstein, 2007;

National Association for the Education of Young Children, 1997). To be effective, intentional teaching requires that teachers use formative assessment to determine where children are in relation to the learning goal and to provide the right kind and amount of support for them to continue to make progress. Intentional teaching is useful to get beyond the dichotomies that arise when teaching is characterized as either teacher-directed or child-initiated.

### Integrated and Focused Curriculum

Early childhood curriculum is often integrated across content domains or subject matter disciplines. *Integration* is the blending together of two or more content areas in one activity or learning experience (Schickedanz, 2008). The purpose of an integrated curriculum is to make content meaningful and accessible to young children. Integration also enables more content to be covered during the limited school day.

Integration typically occurs in two ways. One approach is to add a mathematics content goal to a storybook reading. In this situation, language and literacy goals related to storybook reading are primary, and mathematics learning is secondary. Another way of integrating curriculum is to use a broad topic of study, a theme (such as animals or plants), or a project of interest to children through which mathematics content goals are addressed. Projects are extended investigations into a topic that intellectually engages and interests children, such as how to create a garden or build a house (Katz and Chard, 1989). In both of these approaches to integration, mathematics learning is a secondary objective, rather than the primary focus of attention. In this report, we use both *integrated learning experience* and *secondary focus* on mathematics (which some studies have referred to as *embedded mathematics*) to reflect the teaching/exposure to mathematics content as an ancillary activity.

By contrast, *focused* curriculum or primary focus on mathematics refers to experiences in which mathematics is the major learning goal. A focused mathematics curriculum should also be meaningful and connect to children's interests and prior knowledge. In this report, we use the terms, "primary focus on mathematics" and "focused mathematics time" to refer to dedicated time for a learning experience with mathematics as the primary goal.

## STANDARDS FOR CHILDREN'S MATHEMATICS LEARNING

State standards for students' learning have had an increasingly important role in education over at least the past decade, particularly in K-12 education. More recently, standards have begun to play a role in early childhood education as well. Standards have great potential for shaping instruction, curricula, and assessment; however, the impact of standards on learning depends heavily on the content and specific learning goals laid out in them.

The number of states with published early learning standards has grown over the past eight years from 27 in 2002 to 49 as of 2008. To inform their early learning standards in mathematics, states have used a variety of National Council of Teachers of Mathematics (NCTM) resources, including *Principles and Standards for School Mathematics* (2000) (14 states) and *Early Childhood Mathematics: Promoting Good Beginnings*, issued by NCTM and National Association for the Education of Young Children (NAEYC). *Engaging Young Children in Mathematics* (2004) is also a widely recognized guide for state early learning standards.

*Curriculum Focal Points* (National Council of Teachers of Mathematics, 2006), the most recent set of guidelines provided by NCTM, was developed after most states had already established their standards. The *Curriculum Focal Points* provides guidance about the most significant mathematical concepts and skills (i.e., number and operations, geometry and measurement) that should be addressed during children's early education. *Curriculum Focal Points* also has a clear emphasis on the PSSM process standards, which are essential for meaningful and substantive mathematics learning. The process strands of communication, reasoning, representation, connections, and particularly problem solving allow children to understand their mathematics learning as a coherent and connected body of knowledge (National Council of Teachers of Mathematics, 2006). *Curriculum Focal Points* does not, however, provide the kind of in-depth coverage of what children should know and can do that this report does.

In order to gain a more systematic understanding of the content of states' mathematics standards, the committee commissioned two content analyses of current standards for young children: one at the prekindergarten level (here termed "early learning standards") and one at the kindergarten level (Reys, Chval, and Switzer, 2008; Scott-Little, 2008).

## Early Learning Standards

Many states developed early learning standards to improve classroom instruction and professional development; they also serve as a component of accountability systems. The age levels addressed in the standards documents vary across states. In 17 states the standards targeted children ages 3 to 5, 12 states targeted 3- and 4-year-olds, and 11 states targeted children finishing prekindergarten or starting kindergarten.

State-funded prekindergarten programs are the most common target audience for the early learning standards (42 states), which are usually required to implement the standards (39 states) (Scott-Little et al., 2007).

Currently, 17 states have developed monitoring systems to ensure that standards are being implemented, and 4 others are in the process of developing such a system. States also report that they intend for the early learning standards to be used in child care (39 states), Head Start (38 states), the Individuals with Disabilities Education Act (26 states), and Even Start (27 states) programs, although the use of the standards in these programs is typically voluntary.

For the early learning standards it was possible to evaluate how much emphasis each state has given to mathematics across all of the standards as a whole. On average, states devoted 15 percent of the total number of early learning standards to mathematics, although there was wide variation across states (from a low in New Mexico of only 4 percent to a high in Colorado of 54 percent).

In the content analysis of the mathematics early learning standards (Scott-Little, 2008), each standard was first coded into 1 of the 10 mathematics content and process areas in the PSSM. These categories include the three content areas emphasized in this report and in the *Curriculum Focal Points*—number and operations and geometry and measurement. After the mathematics standards items from a state's document were coded, the total number of items in each area was summed. Because the total number of items varied from state to state, the total for each area was divided by the total number of mathematics items to produce a percentage that was comparable across documents. In effect, the percentage represents the relative emphasis given to each area of mathematics. Table 7-1 presents these results.

**TABLE 7-1** Percentages of States Early Learning Mathematics Standards That Fall in Each of the PSSM Areas

| PSSM Area | Mean | SD | Min. | Max. |
|---|---|---|---|---|
| Content | | | | |
| Numbers and operations | 32.3 | 9.8 | 9 | 50 |
| Algebra | 19.0 | 8.8 | 0 | 50 |
| Geometry | 17.8 | 7.9 | 0 | 44 |
| Measurement | 15.8 | 8.7 | 0 | 50 |
| Data analysis | 5.3 | 5.8 | 0 | 17 |
| Process | | | | |
| Problem solving | 3.7 | 6.2 | 0 | 25 |
| Communication | 1.4 | 3.6 | 0 | 4 |
| Reasoning | 1.3 | 3.1 | 0 | 13 |
| Representation | 0.6 | 1.8 | 0 | 11 |
| Connections | 0.4 | 1.3 | 0 | 7 |
| Other | 2.5 | 3.4 | 0 | 15 |

NOTE: PSSM = *Principles and Standards for School Mathematics*, n = 49 states.
SOURCE: Scott-Little (2008).

These data show a focus on the area of number and operations; on average, states devoted 32 percent of their mathematics standards to this area, and all states had at least some standards in this area. Geometry received less emphasis than number in the early learning standards (18 percent), and measurement accounted for 16 percent of standards in mathematics. In addition, there was much greater overall emphasis on the content standards areas than on the process standards areas (see Table 7-1).

A more detailed analysis was conducted of all standards in each of the three content areas that are the focus of this report (as well as the NCTM *Curriculum Focal Points*): (1) number and operations, (2) geometry, and (3) measurement. Table 7-2 provides the details of the results for each area.

In the area of number and operations, states have most often addressed number sense (an average of 24 percent of the number/operations standards); however, there is considerable variation among states—from 11 states with no standards in this area, to 4 states for which number sense accounted for 100 percent of their number and operations standards. Three other core areas of number were relatively frequent—the number word list, 1-to-1 counting correspondences, and written number symbols—and each is addressed by 11 to 14 percent of the standards. Cardinality and the three basic kinds of addition/subtraction situations received minimal attention.

In the geometry early learning standards, there was an emphasis on children's knowledge of properties of shapes (40 percent) and spatial reasoning (25 percent) (e.g., knowledge related to spatial location and direction), although, again, there was considerable variability among states. Some important aspects of geometry for young children receive little attention, including transformation and visualization of shapes.

In the measurement standards, areas most often emphasized are measurement of objects (34 percent of the standards), comparing objects (27 percent), and understanding of concepts related to time (27 percent). Again there was variability—for example, 2 states had no measurement standards at all, and 15 states had no standard related to comparisons of objects and the concept of time (see Table 7-2).

## Kindergarten Standards

The committee also commissioned an analysis of the 10 states with the largest student populations that publish kindergarten-specific mathematics standards: California, Florida, Georgia, Michigan, New Jersey, New York, North Carolina, Ohio, Texas, and Virginia (Reys, Chval, and Switzer, 2008). These states were selected for analysis because they represent approximately 50 percent of the U.S. school population and therefore influence the intended curriculum for a substantial population of students. Given their size, these 10 states are also likely to influence textbook development and materials that are produced by commercial curriculum publishers.

**TABLE 7-2** Classification of State Mathematics Early Learning Standards by Content Area and Focal Area

| Content/Focal Area | Mean% | SD | Minimum% | Maximum% |
|---|---|---|---|---|
| *Number and Operations* | | | | |
| Number sense | 24.1 | 26.6 | 0 | 100 |
| 1-to-1 correspondence | 13.8 | 10.3 | 0 | 43 |
| Number word list | 13.1 | 10.2 | 0 | 50 |
| Written number symbols | 11.4 | 11.6 | 0 | 40 |
| Perceptional comparisons | 9.6 | 10.3 | 0 | 50 |
| Combining/taking apart | 7.3 | 9.6 | 0 | 33 |
| Cardinality | 5.4 | 7.2 | 0 | 25 |
| Estimation | 4.7 | 8.4 | 0 | 33 |
| Change | 3.9 | 7.9 | 0 | 33 |
| Ordinal numbers | 3.8 | 6.6 | 0 | 25 |
| Counting comparisons | 2.2 | 9.0 | 0 | 60 |
| Additive comparisons | 0.6 | 2.1 | 0 | 11 |
| Place value | 0.2 | 1.6 | 0 | 11 |
| *Geometry* | | | | |
| Properties of shapes | 39.6 | 17.9 | 0 | 100 |
| Spatial reasoning | 25.3 | 23.2 | 0 | 100 |
| Analyzing and comparing shapes | 13.3 | 15.8 | 0 | 67 |
| Location and directionality | 12.2 | 15.5 | 0 | 50 |
| Composing/decomposing shapes | 6.6 | 10.7 | 0 | 40 |
| Symmetry | 1.6 | 5.3 | 0 | 25 |
| Transformation of shapes | 1.5 | 6.0 | 0 | 33 |
| Visualization of shapes | 0.0 | 0.0 | 0 | 0 |
| *Measurement* | | | | |
| Measurement of objects | 33.9 | 25.3 | 0 | 100 |
| Comparing objects | 27.1 | 26.0 | 0 | 100 |
| Time | 26.9 | 23.3 | 0 | 100 |
| Measurable attributes | 12.7 | 16.0 | 0 | 50 |
| Composing objects | 0.0 | 0.0 | 0 | 0 |

NOTE: For number and operations n = 49 states; for geometry n = 48 as one state had no geometry standards; for measurement n = 47 as two states had no measurement standards. Percentages represent the number of a state's standards in a focal area divided by the total number of standards in the content area (content areas are number and operation, geometry, and measurement).
SOURCE: Scott-Little (2008).

The kindergarten learning standards for each state were coded into the five PSSM mathematical content areas or strands: (1) number and operation, (2) geometry, (3) measurement, (4) algebra, and (5) data analysis/probability (Clements, 2004; National Council of Teachers of Mathematics, 2000). Results allow an examination of which of these mathematical strands are emphasized across and within states. Relative emphasis devoted to each strand was calculated as a percentage of standards in that strand within the total number of mathematics standards.

There was considerable variability across the 10 states studied. The total number of mathematics standards varied widely, from 11 in Florida to 74 in Virginia (average number of standards was 29). Of the "total set" of 103 specific standards identified in the analysis, only 1 standard was common to all 10 states (extending a pattern) and another 3 standards were common to 9 states. Only 20 percent of the 103 standards were common to 6 or more states.

In kindergarten (as with the early learning standards), the greatest emphasis across all the mathematics standards is placed on number and operations—40 percent of a state's mathematics standards on average (with a range from a low of 27 percent to a high of 56 percent among states). Geometry and measurement each receive less emphasis than number (19 and 21 percent, respectively), although, again, variability is high (from 9 to 45 percent across states for geometry and from 11 to 28 percent for measurement).

In the number strand, the heaviest emphasis is placed on counting. Areas of emphasis (meaning at least 6 of the 10 states had standards in this focal area) include counting objects, reading and writing numerals, identifying ordinal numbers, comparing the relative size of groups of objects, and modeling and solving problems using addition and subtraction. Consistent with the theme of state variability, however, no single number/operations standard appeared in all 10 state documents.

In both geometry and measurement, few learning standards were common across the states; only 6 topics (of 43 total across geometry and measurement) appeared in the documents of 6 or more states. In geometry, these topics were identifying and naming two-dimensional (2-D) shapes and knowing the relative position of objects. In measurement, the most common topics were comparing the weight of objects; sort, compare, and/or order objects; compare length of objects; and know days of the week.

Taken together, the three focal areas emphasized by the committee (number, geometry, and measurement) account for 80 percent of the content of the kindergarten standards across the 10 states. However, many states also include some specific standards that would not be considered core or primary mathematics by the committee—such as knowing the names of the months, parts of the day, seasons, ordering events by time, comparing time, understanding the concept of time, identifying the time of everyday events to the nearest hour, and measurement of weight, capacity, and temperature.

Process strands were addressed quite differently by different states, so no systematic analyses could be done. Specifically, three states make no mention of process standards at the kindergarten level (Florida, North Carolina, and Virginia), and three other states include identification of specific standards by process strand (Georgia, New York, and Texas). Notably, although these strands are specified for kindergarten, these process standards are very similar, if not identical, at each grade, K-8. Two states (Arizona and Massachusetts) provide a general description of process standards in the introductory material of their K-6 or K-8 document. These descriptions emphasize the importance of the process strands outlined in the PSSM (National Council of Teachers of Mathematics, 2000). The California and Ohio documents include process standards organized within one strand ("Mathematical Reasoning" in the California document and "Mathematical Process Standard" in the Ohio document) for each grade. The California document lists process standards that are common across kindergarten and Grade 1. Likewise, the Ohio document includes a list of process standards that are common to Grades K-2.

### Summary

A total of 49 states have early learning standards in mathematics; on average, states devote the greatest emphasis to the area of number (32 percent of the standards on average). Specific emphasis within the areas of number, geometry, and measurement showed considerable state-to-state variation. According to our analysis for the 10 largest states, the greatest emphasis in kindergarten is also placed on number (40 percent of the standards on average). However, there is also considerable variation in content of the specific standards across all of the areas. In fact, of the 103 total standards across the 10 states, 47 are unique to just 1 or 2 state documents.

A pattern of wide variation across states in emphasis given to mathematics as a whole and relative emphasis given to various topics in mathematics emerges from these analyses of standards. Thus, while some common topics could be identified, when taken as a whole, the state standards do not communicate a clear consensus about the most important mathematical ideas for young children to learn.

## THE CLASSROOM CONTEXT

We begin with a description of the classroom context in which mathematics instruction takes place. We then focus specifically on what is known about mathematics teaching and learning practices in preschool and kindergarten classrooms—when it occurs, how often, and in what contexts.

Results from several large studies of prekindergarten (pre-K) and kindergarten classrooms paint a detailed picture of how young children spend their time in these settings and the quality of their learning experiences. We draw particularly on two studies conducted by the National Center for

Early Development and Learning (NCEDL) and on the Early Childhood Longitudinal Study-Kindergarten[1] (ECLS-K).

The NCEDL conducted two major studies of state-funded pre-K and kindergarten classrooms: the six-state Multi-State Study of Preschool (MS) and the five-state State-wide Early Education Programs (SWEEP) Study (Early et al., 2005). While neither of these studies included a nationally representative sample, as of 2001-2002, almost 80 percent of all children in the United States who were participating in state-funded prekindergarten were in one of these 11 states (Early et al., 2005). When combined, these two studies provide observational data on over 700 preschool and 800 kindergarten classrooms across the United States and offer a unique window on children's classroom experiences.

It is important to note that classrooms were included in these studies only if they received state pre-K funding, so the results are not representative of the larger segment of schooling opportunities for 4-year-olds. State-funded pre-K classrooms are a small subset of early childhood classrooms, generally with greater funding and tighter regulation and monitoring, than the larger set of early childhood classrooms. The studies must be interpreted in this context.

In both studies, classrooms were observed using a variety of measures to capture the content and quality of learning opportunities and materials afforded to children, including the Early Childhood Environment Rating Scale (ECERS-R; Harms, Clifford, and Cryer, 1998), Classroom Assessment Scoring System (CLASS; Pianta, La Paro, and Hamre, 2008), and Emerging Academics' Snapshot (Ritchie et al., 2001).[2]

### How Children Spend Their Time in Prekindergarten and Kindergarten

Results from both of the NCEDL studies (the MS and the SWEEP) indicate that children in state pre-K programs spend a great deal of time *not* engaged in any type of instructional activity. Using the Emerging Academics Snapshot, both NCEDL studies recorded the proportion of time spent in all major areas of curriculum, assessing the amount of time students spent in

[1]Material in this section is based on a paper prepared for the committee by Hamre et al. (2008), which included a review of the published literature related to these studies as well as some reanalysis of the data conducted for this report.

[2]During pre-K, observation days lasted from the beginning of class until the end of class in part-day rooms and until nap in full-day rooms. In pre-K, observers stayed with the children all day, including lunch, outside time, and special activities. In kindergarten, the observations were slightly different because the days were generally longer. Snapshot and CLASS observations lasted the entire day, but no observations were made during lunch, recess, or nap. For this reason, pre-K and kindergarten Snapshot percentages of time spent are discussed separately. More information about these studies can be found on the NCEDL website (http://www.fpg.unc.edu/~ncedl/) and in several published articles (Clifford et al., 2005; Howes et al., 2008; Pianta et al., 2005).

reading, oral language and phonemic awareness activities, writing, mathematics, science, social studies, aesthetics, and fine and gross motor activities. Each area was broadly defined so that time spent in dramatic play, block areas, coloring with markers, talking with teachers about things outside school, and singing songs were included in one of these areas. During the preschool day, the average student spent 44 percent of the time engaged in none of these curriculum activities. Data from kindergarten classrooms revealed that the average student was not engaged in any instructional activity in 39 percent of the observed intervals.

What were children doing during this noninstructional time? In preschool classrooms, much of the time (22 percent) was spent engaged in routine activities, such as transitioning, waiting in line, or washing hands. Some time (11 percent) was also spent in meals and snacks (Early et al., 2005). Importantly, routine, meal, and snack times could be included as instructional time if, for example, teachers and children engaged in a conversation, sang a song, or played a number game during a transition. But few preschool or kindergarten teachers appeared to take advantage of the learning opportunities that arose during transitional periods or employed strategies for getting the most out of this time in the classroom.

Which types of instructional opportunities are young children exposed to most often? Of all content areas, young children spent more time in language and literacy activities than any other—14 percent of the observed day in preschool and 28 percent of the observed day in kindergarten (La Paro et al., 2008). None of the other major areas occurred much more than 10 percent, on average, in any given day. Pre-K children in the NCEDL studies were exposed to mathematics content in only 6 percent of the observations, and kindergarten children were exposed to mathematics an average of 11 percent of the day.

Another relevant question concerns the use of various instructional contexts, such as free choice/center time or whole-group instruction. Data from the NCEDL studies suggest there is a major shift in the preferred instructional context from preschool to kindergarten. Children in preschool classrooms spent an average of 33 percent of the school day in free choice or center time, compared with only 6 percent of the day in kindergarten classrooms. Once in kindergarten, both whole-group instruction and individual time, in which children work independently at desks, becomes much more frequent. Across kindergarten and preschool, teachers rarely made use of small-group instruction.

### Quality of Teacher-Child Interactions in Preschool and Kindergarten

The NCEDL data also provide a window into the quality of teacher-child interactions and instruction to which young children are exposed, using the *CLASS Framework for Children's Learning Opportunities in*

*Early Childhood and Elementary Classrooms* (CLASS framework; Pianta, La Paro, and Hamre, 2007). The CLASS framework captures three broad domains of classroom interactions—emotional supports, classroom organization, and instructional supports—as well as more specific dimensions in each domain. The CLASS framework was derived from basic, theory-driven research on classroom environments and research on effective teaching practices, and it aligns well with a variety of conceptualizations of effective teaching and empirical evidence on effective practices (see Hamre and Pianta, 2007, for a more detailed discussion).

### *Emotional Supports in Preschool and Kindergarten*

NCEDL results indicate that across preschool and kindergarten, children, on average, experienced moderately positive interactions with teachers in moderately well-managed classrooms (La Paro et al., 2008). Approximately one-third of children in this study were in classrooms characterized by high-quality emotional supports in both pre-K and kindergarten.

Teachers' emotional support may have direct links to students' learning (e.g., Connor, Son, and Hindman, 2005), as well as indirect links in which emotional support fosters engagement, which in turn leads to greater achievement (Rimm-Kaufman, Early, and Cox, 2002). Children's social and emotional functioning in the classroom is increasingly recognized as an indicator of school readiness (Blair, 2002; Denham and Weissberg, 2004; Raver, 2004) and a potential target for intervention (Greenberg, Weissberg, and O'Brien, 2003; Zins et al., 2004). Children who are more motivated and connected to others in the early years of schooling are much more likely to establish positive trajectories of development in both social and academic domains (Hamre and Pianta, 2001; Pianta, Steinberg, and Rollins, 1995; Silver et al., 2005). Furthermore, there is some evidence that emotional supports may be particularly important for supporting the academic development of students with social and emotional difficulties (Hamre and Pianta, 2005). Recent nonexperimental research in elementary classrooms suggests that there may be direct links between emotional supports and students' mathematics knowledge (Pianta et al., 2008).

### *Classroom Organization and Management*

Classrooms function best and provide the most opportunities to learn when students are well behaved, consistently have things to do, and are interested and engaged in learning tasks (Pianta et al., 2005). In short, children are better regulated in well-regulated classroom environments. In the NCEDL studies, this dimension of classrooms was measured using the CLASS.

In general, the quality of classroom organization and management in the early childhood classrooms observed in the NCEDL studies was moderately positive. The typical classroom was characterized by a mix of productive periods, with children engaged in learning, and other periods, in which significant behavior problems disrupted learning or teachers failed to actively engage children in learning opportunities.

### *Instructional Supports*

Of greatest concern are results suggesting very low levels of instructional supports across pre-K and kindergarten, as measured by both the CLASS and ECERS-R. In the CLASS, instructional supports include the three dimensions of support for concept development, quality of feedback, and language modeling. Interactions between adults and children are the key mechanism through which these instructional supports are provided to children in the early years of schooling. A child gets more out of an activity if the teacher is either directly interacting with the child in an intentional way or if the child's participation in the activity has been sufficiently supported by the teacher prior to the start of the activity, so that the child, in playing, is more intentional in the purpose of the activity (the section below called "Research on Effective Mathematics Pedagogy" is a more detailed discussion of instructional supports).

## Mathematics Practices in U.S. Preschools and Kindergartens

Little is known about the math-specific learning opportunities that are provided to children in early childhood settings. This may reflect, in part, the focus on early literacy and language development that has consumed much of early childhood policy and research attention for the past decade. Although more recent attention has focused on early childhood mathematics (Clements and Sarama, 2007a), there is not yet detailed, national-level information on the typical mathematical practices to which children are exposed. In this section, we again draw on the NCEDL MS Study and on data from the nationally representative ECLS-K cohort. We begin with a more detailed analysis of observational data from pre-K classrooms in the NCEDL MS Study and end with a description of kindergarten teachers' self-reported mathematics practices from the ECLS-K. Although the ECLS-K is nationally representative, the information about mathematics instruction is limited.

### *Mathematics Instruction in Prekindergarten*

The most relevant NCEDL MS data come from observations conducted during visits to pre-K classrooms in the fall and the spring. The average amount of time focused on mathematics content in the pre-K classrooms was minimal (6.5 percent in the fall, and 6.7 percent in the spring).[3]

More detailed analysis of the actual activities that took place during this mathematics time suggests that, for about half of the time, mathematics content occurred during whole-group activities (49 percent in the fall and 48 percent in the spring). Free choice/center time was the second most common mathematics setting (31 and 29 percent in the fall and spring respectively), with small group instruction third (11 and 12 percent).

Another important question is whether mathematics is taught alone or in conjunction with other content. Data indicate that mathematics content co-occurred with other academic content during the majority of the time (61 percent in the fall and 55 percent in the spring). About 20 percent of the time, when mathematics co-occurred with something else, it was with an art or music activity (aesthetics), and between 15 and 18 percent of the time it was with a fine motor activity. Other academic content occurred simultaneously with mathematics about 11 percent of the time for reading (a combination of being read to, prereading, and letter-sound), 13 percent for social studies, and 11 percent for science. These findings indicate that, when they do teach mathematics, early childhood education programs rely on integrated or embedded mathematics experiences a majority of the time, rather than including activities with a primary focus on mathematics. The selection of materials and activities such as puzzles, blocks, games, songs, and fingerplays seem to constitute mathematics for many teachers (Clements and Sarama, 2007a).

Using the Emerging Academics Snapshot, researchers found that teachers' interactions with children during mathematics content were likely to be either encouraging or didactic in nature. *Encouraging* was coded when teachers provided feedback about effort and persistence, including praise, personal comments, and general statements that helped children stay engaged in their work. *Didactic* was coded when teachers focused on giving instructions, asking questions with one correct answer, and engaging children in instruction focused on mastering a discrete set of materials. Less often, teachers spent time scaffolding while delivering mathematics content. *Scaffolding* was coded when teachers showed an awareness of an individual child's needs and responded in a manner that supported and expanded the child's learning.

[3]Note that *math* was coded when a child was verbally counting, counting with 1-to-1 correspondence, skip counting, identifying written numerals, matching numbers to pictures, making graphs, playing counting games (e.g., dice, dominoes, Candyland, Chutes and Ladders), keeping track of how many days until a special event, counting marbles in a jar, playing Concentration or Memory with numbers, working on mathematics worksheets, identifying shapes, talking about the properties of shapes (e.g., how many sides), finding shapes in the room, identifying same and different (e.g., big/little, biggest), sorting (by color, size, shape), discerning patterns (red, blue, red, blue), or measuring for cooking or size.

To summarize, in the state-funded pre-K classrooms observed in the NCEDL MS Study, mathematics was often taught in conjunction with art, music, and fine motor activities, suggesting that perhaps teachers were integrating mathematics with activities that they assumed would heighten children's engagement and were making use of manipulatives. However, the committee thinks that the integration of mathematics with other activities may or may not be effective in supporting children's development of mathematics knowledge, depending on the integrity of, and emphasis on, the mathematical ideas. It is also evident that mathematics, like literacy, was often taught in a manner in which teachers focused on student performance of a discrete skill or display of factual knowledge. Children were less often exposed to instruction that was conversational, interactive, and focused on understanding and problem solving.

### *Mathematics Instruction in Kindergarten*

The ECLS-K cohort is a nationally representative sample of 22,000 students in approximately 1,000 classrooms across the United States. This cohort of students entered the study in 1998-1999 as they began kindergarten and will be followed through eighth grade. The most relevant ECLS-K data for our purposes are items from a survey of kindergarten teachers who reported how often their students were exposed to classroom instruction in mathematics, including (1) broad exposure to mathematics, (2) instructional emphasis on specific mathematics concepts and skills, and (3) exposure to specific mathematics instructional strategies and activities.

The committee commissioned a reanalysis of these teacher survey items because existing published analyses did not provide sufficient detail on mathematics (Hamre et al., 2008a). For the purposes of our analysis, the items were organized conceptually according to the *NCTM Content Standards* (National Council of Teachers of Mathematics, 2000) into the areas of number and operations, geometry, measurement, algebra, and data analysis and probability.

The vast majority of teachers (81 percent) indicated that mathematics instruction is a part of their daily classroom routine, with over half of the teachers (65 percent) reporting that they provide more than 30 minutes of mathematics instruction each day. Teachers also indicated the frequency with which they taught a list of 27 specific mathematics concepts and skills. By far, teachers reported concepts and skills associated with number and operations to be the most common emphasis of mathematics instruction. However, in contrast to the recommendations in this report for focusing on learning paths in a few key areas, concepts and skills associated with all of the NCTM standards were the emphasis of mathematics instruction to some extent in a given academic week.

Specific to number and operations, the most common concepts and skills teachers reported teaching were correspondence between number and quantity, writing all numbers between 1 and 10, and reading two-digit numbers—all of these were frequently (77, 55, and 52 percent, respectively) reported to be the emphasis of instruction three or more times per week.

Counting by 2s, 5s, and 10s was fairly common, with 44 percent of teachers reporting this to occur at least three times per week. Instruction was slightly less often focused on ordinal numbers (35 percent reported at least three times per week), adding single-digit numbers (40 percent), and subtracting single-digit numbers (28 percent). Research on children's number and operations learning discussed in previous chapters suggests that such emphases are out of balance. For example, time dedicated to skip counting—especially if involving only verbal counting—might be better used to address concepts, strategies, and skills related to addition and subtraction.

As for measurement concepts and skills, the most commonly endorsed items were identification of relative quantity (e.g., most, least, more, less), ordering objects by size or other properties, and sorting objects into subgroups according to a rule, all of which were reported to be the emphasis of instruction once a week or more for 56-76 percent of teachers. Measurement concepts and skills received less frequent emphasis but still were reportedly the focus of instruction at least once a month for most classrooms, as were using measurement instruments accurately, telling time, estimating quantities, and recognizing the value of coins and currency.

Geometry, algebra, and data analysis/probability consisted of the fewest survey items. The lone geometry skill in the survey, recognizing and naming geometric shapes, was reported to be the emphasis of instruction once per week or more by more than 66 percent of teachers. Similarly, related to algebra, over two-thirds of teachers (72 percent) reported teaching copying, making, and extending a pattern at least once a week. Under data analysis and probability, over half of the teachers emphasized reading simple graphs once per week or more, while simple data collection and graphing was less often emphasized (54 percent reported doing this two to three times per month or less). The majority of teachers (59 percent) noted that estimating probability was a skill to be taught at a higher grade level.

Another set of survey items asked teachers about the extent to which they used various instructional activities or strategies. In numbers and operations, the most common math-related activity reported by teachers was verbal counting, which happened on a daily basis in more than 79 percent of the kindergarten classrooms. Another relatively common activity involved use of counting manipulatives to learn basic operations, with 66 percent of teachers reporting this to occur three or more times per week. The use of geometric manipulatives was also fairly common, with 45 percent of teachers reporting their use three or more times a week. In contrast, work with rulers, measuring cups, spoons, and other measuring instruments was fairly infrequent, with two-thirds of teachers (66 percent) reporting use of them three times per month or less.

Generalized teaching strategies and activities are defined as those that can apply to a variety of the NCTM mathematics standards. The most prominent generalized strategy was calendar-related activities, which occurred on a daily basis in over 90 percent of the classrooms surveyed, this despite the fact that mathematics educators do not consider most calendar activities to be useful early childhood mathematics instruction and have serious questions about the efficacy of "doing the calendar" every day (see Box 7-1).

More than half of the teachers reported using the following strategies and activities twice a week or more: playing mathematics-related games, explaining how a mathematical problem is solved, doing mathematical

**BOX 7-1**
**How Using the Calendar Does Not Emphasize Foundational Mathematics**

Many preschool and kindergarten teachers spend time each day on the calendar, in part because they think it is an efficient way to teach mathematics. Although the calendar may be useful in helping children begin to understand general concepts of time, such as "yesterday" and "today," or plan for important events, such as field trips or visitors, these are not core mathematical concepts. The main problem with the calendar is that the groups of seven days in the rows of a calendar have no useful mathematical relationship to the number 10, the building block of the number system. Therefore, the calendar is not useful for helping students learn the base 10 patterns; other visual and conceptual approaches using groups of 10 are needed because these patterns of groups of 10 are foundational.

Time spent on the calendar would be better used on more effective mathematics teaching and learning experiences. "Doing the calendar" is not a substitute for teaching foundational mathematics.

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 11 | 21 | 31 | 41 | 51 | 61 | 71 | 81 | 91 | 101 | 111 |
| 2 | 12 | 22 | 32 | 42 | 52 | 62 | 72 | 82 | 92 | 102 | 112 |
| 3 | 13 | 23 | 33 | 43 | 53 | 63 | 73 | 83 | 93 | 103 | 113 |
| 4 | 14 | 24 | 34 | 44 | 54 | 64 | 74 | 84 | 94 | 104 | 114 |
| 5 | 15 | 25 | 35 | 45 | 55 | 65 | 75 | 85 | 95 | 105 | 115 |
| 6 | 16 | 26 | 36 | 46 | 56 | 66 | 76 | 86 | 96 | 106 | 116 |
| 7 | 17 | 27 | 37 | 47 | 57 | 67 | 77 | 87 | 97 | 107 | 117 |
| 8 | 18 | 28 | 38 | 48 | 58 | 68 | 78 | 88 | 98 | 108 | 118 |
| 9 | 19 | 29 | 39 | 49 | 59 | 69 | 79 | 89 | 99 | 109 | 119 |
| 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 |

worksheets, solving mathematical problems in small groups or with a partner, working on mathematical problems that reflect real-life situations, working in mixed achievement groups on mathematics activities, and using computers to learn mathematics. A somewhat different pattern was evident for using music to understand mathematics, using creative movement or creative drama to understand mathematical concepts, completion of mathematical problems on the chalkboard, and engaging in peer tutoring. A quarter or more of the teachers indicated that they never asked students to do these activities, whereas another quarter or more used these activities at least one to two times per week.

### *Mathematics Practices Across Diverse Preschool Settings*

Findings from the few smaller scale studies that examined mathematics in early childhood settings show a similar pattern. In one study, teachers in two states from a range of preschool settings, including family and group child care providers, were surveyed about their mathematics instruction (Sarama, 2002; Sarama and DiBiase, 2004). Most teachers reported using manipulatives (95 percent), number songs (74 percent), and games (71 percent). Only 33 percent used software, and 16 percent reported using mathematical worksheets. Teachers reported a preference for children to explore mathematics activities and engage in free play rather than participate in large group lessons or do mathematical worksheets. The mathematics topics teachers reported were counting (67 percent), sorting (60 percent), numeral recognition (51 percent), patterning (46 percent), number concepts (34 percent), spatial relations (32 percent), making shapes (16 percent), and measuring (14 percent). The least popular topics were geometry and measurement.

In an observational study of New Jersey preschools, teachers were found to provide little support for children's mathematical skill development and seldom used mathematics terminology (Frede et al., 2007). Of particular interest is that over 40 percent of the classrooms in this study were rated as good to excellent quality on the ECERS-R measure of the environmental quality of early childhood programs. Apparently, mathematics teaching and learning is relatively rare even in classrooms that are otherwise judged to be of high quality.

## RESEARCH ON EFFECTIVE MATHEMATICS INSTRUCTION

The majority of research that is focused specifically on mathematics taught in early childhood examines the effectiveness of a particular mathematics curriculum (e.g., Clements and Sarama, 2008a; Sophian, 2004; Starkey, Klein, and Wakeley, 2004). Although much of this work meets very high empirical standards, it is often difficult to derive information about specific types of effective instructional practices from general information on whether or not a curriculum is successful. Nevertheless, this research base does provide some guidance on effective mathematics instruction. (Curriculum research is discussed later in the chapter.)

There is also a large body of research on effective instruction in early childhood that is not specific to mathematics. The general principles of effective instruction that emerge from this research can and should be taken into consideration when designing mathematics instruction for young children. Both of these bodies of research are briefly reviewed below. Taken together, they provide guidance on effective instruction, although further research on strategies specific to mathematics is needed.

The large body of research on effective instruction informed the development of the CLASS system for observation described briefly in the previous section. Since the domain of the CLASS most closely associated with the development of mathematics knowledge and skill is instructional supports, we begin with a discussion of various kinds of instructional supports as defined in the CLASS. We then move to discussion of other aspects of instruction that are important for supporting learning in mathematics.

## Instructional Supports

The theoretical foundation for the CLASS conceptualization of instructional supports comes primarily from research on children's cognitive and language development (e.g., Catts et al., 2002; Fujiki, Brinton, and Clarke, 2002; Romberg, Carpenter, and Dremlock, 2005; Taylor et al., 2003; Vygotsky, 1991; Wharton-McDonald and Pressley, 1998). This literature highlights the distinction between simply learning facts and gaining usable knowledge, which is built on learning how facts are interconnected, organized, and conditioned on one another (Mayer, 2002; National Research Council, 1999). A child's cognitive and language development is contingent on the opportunities adults provide to express existing skills and scaffold more complex ones (Davis and Miyake, 2004; Skibbe, Behnke, and Justice, 2004; Vygotsky, 1991). The development of metacognitive skills, including children's awareness and understanding of their own thinking processes as well as their executive function skills, are also critical to their academic development (Blair, 2002; Veenman, Kok, and Blote, 2005; Williams, Blythe, and White, 2002).

The CLASS assessment system has been validated, both in terms of its factor structure (Hamre et al., 2008b) and in relation to preschool children's language, literacy, and mathematics knowledge and social and emotional development (Burchinal et al., 2008; Howes et al., 2008; Mashburn et al., 2008). Children in classrooms that score higher on the instructional dimensions of concept development and quality of feedback, as measured by the CLASS, display greater gains in mathematics knowledge over the course of the year, although the effect sizes are small (between .10 and .20; Mashburn et al., 2008). These two dimensions of instructional support are discussed in greater detail below.

### *Promoting Conceptual Development*

Concept development describes the instructional behaviors, conversations, and activities that teachers use to help stimulate students' higher order thinking skills (Pianta et al., 2007), which refers not only to the acquisition of knowledge, but also to the ability to access and apply this knowledge in new situations (Mayer, 2002). The four key elements of high-quality concept development are (1) analysis and reasoning, (2) creating, (3) integration, and (4) connections to the real world.

In classrooms that fall at the high end of concept development, teachers not only plan activities in ways that will stimulate higher order thinking, but also they take advantage of the moment-to-moment opportunities in their daily interactions to push children toward deeper thinking. In contrast, classrooms that are low on conceptual development lack instructional opportunities or focus instruction solely on remembering facts or on simple tasks that require only recognition or recall.

### *Providing Scaffolding and Feedback*

In order for students to get the most benefit from instructional opportunities, they need feedback about their learning. Feedback refers to a broad range of interactions through which the teacher provides some information back to the students about their performance or effort. There are five major types of feedback interactions described in the CLASS: (1) scaffolding, (2) feedback loops, (3) prompting of thought processes, (4) provision of information, and (5) encouragement or affirmation. Feedback is a key element of formative assessment, which is discussed in greater detail later in this chapter.

***Scaffolding.*** Teachers scaffold children's learning by providing hints and assistance that enable them to perform at a higher level than they might be able to do on their own. This may occur during a whole-group or small-group discussion or individually during center time or children's play (scaffolding is also discussed in the section on formative assessment in this chapter).

***Feedback loops.*** Effective feedback is also characterized by sustained exchanges with a child (or group of children), leading them to a better or deeper understanding of a particular idea. This is in contrast to a teacher who might give a single hint to a child but then move on, even if the child does not seem to understand.

***Prompting thought processes.*** This feedback strategy asks students to explain their thinking or actions. Prompting thought processes helps to identify children who may have completed an activity or answered a question correctly, but who cannot yet clearly articulate their reasoning. By having a child articulate his or her thought process, the teacher discovers erroneous thinking and can intervene. This learning opportunity is in contrast to one in which the teacher just tells the child that he or she was correct or incorrect.

***Providing information.*** In the context of instructional interactions, children often give the wrong answer or action. Each instance provides an opportunity for effective feedback by expanding on children's answers and actions, clarifying incorrect answers, or providing very specific information about the correct answer. These are all in contrast to a teacher who simply tells students they are wrong.

***Encouragement and affirmation.*** Another form of feedback consists of strategies that can motivate children to sustain their efforts and engagement. Simple recognition statements, such as "You are working really hard on that puzzle" reinforce students' effort and encourage persistence. This may be especially important in the area of mathematics, in which older children in the United States have been found to assume that mathematics achievement is a product of ability rather than effort (National Mathematics Advisory Panel, 2008). Young children may need help to learn that effort leads to improved results in learning mathematics.

## The Importance of Math Talk

In a mathematics context, teachers' use of language can facilitate connections between numbers, words, and ideas. In an elegant demonstration of the importance of mathematical language for young children, Klibanoff and colleagues (2006) showed that children exposed to more math talk in their preschool classrooms displayed greater gains in mathematical knowledge from October to April. The authors transcribed an hour of teachers' utterances, including circle time, and coded the transcripts for the number of mathematical inputs in the following categories: counting, cardinality, equivalence, nonequivalence, number symbols, conventional nominative (as in naming an address or phone number), ordering, calculation, and placeholding. There was a wide range of mathematical inputs among the 26 classrooms (a range from 1 to 104, with an average of 28). References to cardinality were the most common, accounting for 48 percent of all inputs. Many of the inputs, such as equivalence, nonequivalence, ordering, calculation, and placeholding, were rare, each accounting for less than 5 percent of all inputs.

After controlling for children's prior performance, those in classrooms with a higher number of mathematics inputs displayed better performance in April on a short (15-item), multiple-choice test of general mathematical knowledge. Klibanoff et al. (2006) found only a small correlation between teachers' syntactic complexity and frequency of math talk ($r = .18$). And only math talk, not syntactic complexity, was associated with gains in mathematical skills. As the authors point out, this is the first study to examine the specific effects of math talk on children's knowledge, and research is needed to understand more about the direct role of math talk in early childhood classrooms.

In general, the amount and kind of language that occurs in the classroom among teachers and children is frequently related to outcomes for children. Correlational research with preschoolers demonstrates that, during large-group times, teachers' explanatory talk and use of cognitively challenging vocabulary are related to better learning outcomes for children (Dickinson and Tabors, 2001).

The use of open-ended questions also has the potential to increase the math talk in a classroom or in a home. Effective teachers make greater use of open-ended questions than less effective teachers. They ask children "Why?" and "How do you know?" They expect children, as young as preschool, to share strategies, explain their thinking, work together to solve problems, and listen to each other (Askew et al., 1997; Carpenter et al., 1998, 1999; Clarke et al., 2001; Clements and Sarama, 2007a, 2008a; Cobb et al., 1991; Thomson et al., 2005). As the questions become internal, children can increasingly become self-sustaining mathematical learners who carry and use a mathematical lens for seeing and understanding their world. Examples of such open-ended mathematical questions are

- Where do you see this (mathematical idea) in our classroom?
- Tell me how you figured out (this mathematical idea).
- What is (insert mathematical idea, such as adding or subtracting)?
- What happens when I break this apart/put these together?
- How does this compare with something else? (Which one is smaller/larger? Longer/shorter?)
- Where are the units? What are the units (that children are familiar with)?
- Do you see a pattern? What is the pattern?
- How can I describe this idea for myself or for someone else (such as, can you draw a picture, describe it in words, or use your body)?

## Grouping as an Instructional Strategy

As described previously, data from the NCEDL studies and the ECLS-K indicate that mathematics is taught in a whole group most of the time, especially in kindergartens, where little time is allocated for centers or small groups. The almost nonexistent use of small groups in early childhood programs, documented in these studies, is of concern given that small-group instruction has been found to be an effective context for enhancing young children's learning (Dickinson and Smith, 1993; Karweit and Wasik, 1996; Morrow, 1988).

Various mathematics curricula that use small groups as one of several or as the main instructional strategy have shown substantial positive effects (e.g., Clements, 2007; Clements and Sarama, 2008a; Preschool Curriculum Evaluation Research Consortium, 2008; Sarama et al., 2008; Starkey et al., 2006). The results suggest that small-group work can significantly increase children's scores on tests aligned with that work (Klein and Starkey, 2004; Klein, Starkey, and Wakeley, 1999), and can transfer to knowledge and abilities that have not been taught (Clements and Sarama, 2007a). Guidelines in these curricula generally suggest four children with a teacher as the small-group size, although teachers have been observed using group sizes of two (for low achievers, for children with special needs, or to introduce an idea or activity for the first time) to six (usually for efficiency's sake; often used for easily managed activities).

Whole groups can also be effective for supporting mathematics learning. In one program, children as young as kindergarten engaged in teacher- and peer-scaffolded mathematics learning, problem solving, and discussion during whole-class instruction (Fuson and Murata, 2007). Based on teaching-learning paths, the program successfully enabled teachers to individualize mathematics in large-group activities—a promising strategy to give mathematics needed attention in the already packed schedule of half-day kindergarten. French and Song (1998) document extensive use of whole-group instruction to good effect in Korean preschools. Effective whole-group interactions include brief demonstrations and discussions, problem solving in which children talk to and work with the person next to them (other children and possibly adults), and physically active activities, such as marching around the room while counting (Clements, 2007; Clements and Sarama, 2007a, 2008a). Box 7-2 provides an example.

## Play as a Teaching and Learning Context

A highly motivating learning context for young children is child-initiated play (Wiltz and Klein, 2001). Preschool children engage in different types of play that have potentially different benefits for learning and develop-

**BOX 7-2**
**Mathematics Activities with Different Size Groups**

The Building Blocks Program dedicates several weeks to shape composition. One theme is puzzles. In a whole-group setting, the teacher asks the children what puzzles they like to solve at home and at school. She discusses various types of puzzles and what puzzles are, showing some examples, telling the children she will put them all out in the mathematics centers. She then introduces a new kind of puzzle: outline puzzles that can be completed with geometric shapes (e.g., pattern blocks or tangram pieces). She solves a simple puzzle with the children, using their ideas as to solutions.

Later, with small groups of four children, the teacher introduces several of the outline puzzles. She carefully observes children's solutions to these, evaluating where each child is in the learning trajectory for shape composition. Based on these observations, she provides individuals with puzzles at different levels of the learning trajectory (or mathematics teaching-learning paths), individualizing the challenge for each child.

Meanwhile, the teacher's assistant observes and discusses children's work with the puzzles in the mathematics center, as well as supervising those in other centers, allowing the teacher to concentrate on the small-group work. One special center involves a series of computer activities, the Piece Puzzler series, in which children also solve puzzles by manipulating pattern blocks or tangram pieces to complete similar outline puzzles. They use icons of the geometric motions to slide, turn, and flip the shapes into place. Individualized help and feedback are offered to them immediately. For example, if they put on too large a shape, covering the puzzle and also other areas, the computer activity makes the shape transparent and shows them that it covers too much (something difficult to show with physical manipulatives). Also, the computer activity automatically adjusts the levels of the puzzle to match the children's development along the learning trajectory.

ment. Among the typically observed play experiences in an early childhood classroom are constructive play, such as block building; play with table toys (manipulatives, puzzles, Lego blocks); pretend play; mathematical play; and games, including ones in which mathematics is a secondary focus, as well as ones in which mathematics is the primary focus. (Of course children also engage in outdoor play, rough-and-tumble play, and other forms of play that have benefits as well.)

### *Block Building*

Play, especially block play, provides valuable opportunities for children to explore and engage in mathematical activity on their own (Ginsburg, 2006; Hirsch, 1996). Young children enjoy playing with blocks, and there is evidence that they naturally engage in mathematical play with them (Seo and Ginsburg, 2004). However, mathematics learning is enhanced if teachers engage children in a discussion of mathematical principles during block play (Clements and Sarama, 2007a), such as introducing new terminology (e.g., edges, faces) and commenting on children's rotation of objects during construction. The provision of these supports by teachers during play enhances children's learning during the specific interaction as well as in future play sessions, when the child may incorporate these new ideas.

Research also indicates that teachers should incorporate planned, systematic block building into their curriculum, which they rarely do (Kersh, Casey, and Young, 2008). Preschoolers who are provided such scaffolding display significant increases in the complexity of their block building (Gregory, Kim, and Whiren, 2003). Important to our teaching-learning paths approach (also called learning trajectories), the teachers' scaffolding was based on professional development aimed at helping them recognize developmental progressions in the levels of complexity of block building. Teachers learned to provide verbal scaffolding based on those levels but not to directly assist children or engage in any block building themselves. Interventions that incorporate full teaching-learning paths—that is, a goal, a developmental progression, and matched activities—appear to be effective in developing children's skills. Groups of kindergartners who experienced such a learning trajectory improved in block-building skill more than control groups who received an equivalent amount of block-building experience during unstructured free play sessions (Kersh et al., 2008).

One longitudinal study indicated that block building may help lay a foundation for mathematics achievement in later years (Wolfgang, Stannard, and Jones, 2001). More specifically, block building has been linked to improved spatial skills, although most of these studies are correlational (Brosnan, 1998; Serbin and Connor, 1979). Similarly, in a preschool population, two types of block-building skills were associated with two measures of spatial visualization: block design and analyzing and reproducing abstract patterns (Caldera et al., 1999). In an experimental study, children who received instruction on spatial-manipulation improved in spatial visualization skills, whereas the control group did not (Sprafkin et al., 1983).

### *Sociodramatic Play*

One particularly valuable form of play is mature sociodramatic play—pretend play that lasts 10 minutes or more and involves a theme, props, roles, rules for roles, and language interaction. An example would be four children playing grocery store with play food, a cash register, and shopping carts, and different children playing the roles of store manager, cashier, and customers. Rules restrict the behavior of each player—for example, only the cashier can use the register. A Vygotskian-based curriculum, Tools of the Mind, uses this type of mature sociodramatic play as a primary format for children's learning and development (Bodrova and Leong, 2007). In this approach, teachers scaffold children's play skills by engaging them in preparing written play plans and reflecting after play is finished. Teachers work with children to make play more complex over time and to encourage the use of sophisticated vocabulary.

Studies of Tools of the Mind show positive impacts on language and early literacy (Barnett et al., 2006, 2008) and on self-regulation and executive functioning (Diamond et al., 2007). The latter is relevant for mathematics learning, as executive function and self-regulation are important for academic success. Executive function has been found to predict academic outcomes in school independent of intelligence or family background (Blair and Razza, 2007). Importantly, the approaches used in Tools of the Mind have been shown to be effective with children from low-income families.

### *Practice During Play*

Learning many early mathematics skills, such as counting, requires large amounts of practice to become fluent. Play can be an excellent context for children to practice developing abilities. For example, 3- and 4-year-old children will repeatedly attempt to build a block tower or string a set of beads in a pattern until they have mastered the skill to their personal satisfaction. Many mathematics competencies, such as counting, require repeated, often massive amounts of experience, as well as demonstrations, modeling, or scaffolding from adults (Fuson, 1988). Practice is important for consolidating skills, but such practice can be done in the meaningful and motivating context of children's play and with teachers' assistance as needed. For example, after a walk in the park, children can return to the classroom and examine their collections of leaves, trying to find out who has the most. The teacher can help the children to count their leaf collections, which they choose to do again and again. After repeatedly counting the separate collections, they can work as a group to count the total.

Children's play and self-selected activities can provide valuable contexts for mathematics teaching and learning experiences. Capitalizing on their everyday experience is likely to motivate and help them see the relevance in mathematics, as well as lead to complex child-centered projects that include mathematics. Early childhood education has a strong tradition of teachers' observation of children's play for the purpose of determining how best to respond to support their learning. Teachers can and should be intentional in supporting and mathematizing children's play experiences. However, using only "teachable moments" during child-initiated play is unlikely to lead to an effective, comprehensive mathematics program (Ginsburg, Lee, and Boyd, 2008).

### *Mathematical Play*

These examples bring us to another type of play, *mathematical play*, or *play with mathematics itself* (Sarama and Clements, 2009; see Steffe and Wiegel, 1994). The following features of mathematical play may be important for supporting learning: (a) it is a solver-centered activity with the solver in charge of the process; (b) it uses the solver's current knowledge; (c) it develops links between the solver's current schemes while the play is occurring; (d) it will, via "c," reinforce current knowledge; (e) it will assist future problem solving/mathematical activity as it enhances future access to knowledge; and (f) these behaviors and advantages apply irrespective of the solver's age (Holton et al., 2001).

### *Games*

One recent study provides evidence that board games in which young children count on (1 or 2) along a number list (squares with numbers on them) can be an effective instructional tool for developing their numerical knowledge (Siegler and Ramani, in press). In an experiment conducted in a Head Start program, children played a board game, similar to Chutes and Ladders, four times (for 15 to 20 minutes) over a two-week period. The game used numbered squares for the experimental group and colored squares for the control group. Children using the numbered squares said the numbers on the squares as they moved their token one or two spaces. At the end of the intervention, children who played the number game demonstrated increased knowledge of four different number skills: numerical magnitude comparison, number line estimation, numeral identification, and counting. The gains were still apparent nine weeks later (Siegler and Ramani, in press). To achieve such gains through play, however, requires that important mathematical structures are used by children within the game.

## Using Concrete Materials and Manipulatives

Using concrete materials, such as puzzles and matching games, with task selection and scaffolding adjusted to children's strategies, is effective in moving them through mathematics teaching-learning paths (Clements and Sarama, 2007a). Manipulatives, such as small blocks, cubes, beads, and pegs, are ubiquitous in high-quality early childhood classroom environments. There is evidence suggesting that the use of manipulatives enhances mathematical knowledge for young children (Clements and Sarama, 2007a). This is an area in which there has been a fair amount of mathematics-specific research (Clements and McMillen, 1996), although most work in this area has focused on elementary school children (e.g., Greabell, 1978;

Prigge, 1978). Concrete objects are needed for preschoolers to learn nonverbal and counting strategies for addition and subtraction. In fact, children need objects to solve larger number problems until about age 5½ (Jordan, Huttenlocher, and Levine, 1992). The manipulatives give meaning to the task, count words, and order (Clements and Sarama, 2007a). That is, at a certain level, number is an adjective rather than a noun for children—"5 kittens" is meaningful, but "5" as an abstract quantity is not.

Pictures can be useful in several ways, such as to illustrate concepts, and young children can learn to interpret pictures (Scott and Neufeld, 1976). However, manipulatives can be more effective than pictures for teaching certain mathematical concepts, because pictures are not manipulable, that is, they cannot be acted on extensively and flexibly (Clements and McMillen, 1996; Gerhardt, 1973; Prigge, 1978; Sowell, 1989; Stevenson and McBee, 1958). For example, in one study children benefited more from using pipe cleaners than pictures to make nontriangles into triangles (Martin, Lukong, and Reaves, 2007). They merely drew on top of the pictures, but they transformed the pipe cleaners.

The suggestion that manipulatives and other materials are effective should not be interpreted to mean that young children should always be provided with manipulatives or that simply providing these manipulatives is sufficient. Rather, teachers should be thoughtful about the most appropriate manipulative for a specific lesson. Once children have mastered a task using manipulatives, they can often solve simple arithmetic tasks without them (Grupe and Bray, 1999).

## Using Computers

As all-purpose tools, computers can also constitute quite different environments that support mathematics teaching and learning. They can provide effective experiences, ranging from complex problem solving to practice with concepts and skills, managed at the children's level of thinking and at the level of individual tasks.

The computer aids the metacognitive aspect of spatial activity, enabling the child to go beyond the physical world limitations (Clements and Battista, 1991; Johnson-Gentile, Clements, and Battista, 1994). For example, children can cut shapes and put them together in new ways. They become aware of and describe the geometric motions they use to solve geometric puzzles (Sarama and Clements, 2009; Sarama, Clements, and Vukelic, 1996)—that is, doing physical puzzles, they move shapes intuitively. However, on the computer, they choose the geometric motion—slide, flip, or turn—that they need. This helps them become explicitly aware of those motions and intentional in their use.

Children as young as age 3 have been shown to benefit from focused computer activities (Clements, 2003). Connected representations in practice or tutorial computer environments help them form concepts that are interrelated and thus mutually reinforcing. Computer environments can also foster deeper conceptual thinking, including a valuable type of "cognitive play" (Steffe and Wiegel, 1994). That is, children will pose problems for themselves and explore the computer objects or shapes with the same playful attitude—and the same beneficial learning—found in other types of play.

Several characteristics of effective computer software can guide its creation and selection (Clements and Sarama, 2005, 2008b; Sarama and Clements, 2002a, 2006):

- Actions and graphics should provide a meaningful context for children.
- Reading level, assumed attention span, and way of responding should be appropriate for the age level. Instructions should be clear, such as simple choices in the form of a picture menu.
- After initial adult support, children should be able to use the software independently. There should be multiple opportunities for success.
- Feedback should be informative.
- Children should be in control. Software should provide as much manipulative power as possible.
- Software should allow children to create, program, or invent new activities. It should have the potential for independent use but should also challenge. It should be flexible and allow more than one correct response.

As with using manipulatives, initial adult support and active mentoring has significant positive effects on children's learning with computers (Sarama and Clements, 2002b). Effective teachers closely guide children's learning of basic tasks; then they encourage experimentation with open-ended problems. These teachers are frequently encouraging, questioning, prompting, and demonstrating, without offering unnecessary help or limiting children's opportunity to explore. The teachers redirect inappropriate behaviors, model strategies, and give children choices. Whole-group discussions that help children communicate about their solution strategies and reflect on what they've learned are also essential components of good teaching with computers.

## Using Movement

Another context for learning mathematics is teachers' use of movement to engage children. There is evidence suggesting that young children benefit from engaging in self-directed movement during instruction, particularly in learning spatial concepts (Poag, Cohen, and Weatherford, 1983; Rieser, Garing, and Young, 1994). During a mapping activity, for example, children are more likely to benefit from actually taking a tour around the classroom than simply thinking about the classroom and being asked to represent it abstractly (Ginsburg and Amit, 2008).

### Book Reading

Book reading is used frequently in early childhood settings. Earlier studies have produced equivocal results with relation to the effect of book reading on mathematics achievement (Hong, 1996). However, several recent studies provide evidence that this can be an effective learning context for mathematics instruction (Casey, Kersh, and Young, 2004; Casey et al., 2008; Young-Loveridge, 2004). Young-Loveridge (2004) provides evidence that children exposed to a seven-week pull-out[4] mathematics program, using storybooks, rhymes, and games, made greater gains pre- to posttest on mathematical knowledge than did children not receiving this program. Casey and colleagues (2008) provide evidence that mathematics content (spatial and number skills) delivered in a storytelling context produced greater mathematics learning than delivering the content alone. Notably, the approach, Storytelling Sagas, is based on a series of specially written mathematics storybooks for preschool through Grade 2 that are primarily mathematical and secondarily for literacy. However, the approach demonstrates the important role of language in children's mathematics learning. In this study, researchers compared an intervention that taught a specific set of geometry skills in a storytelling context and alone. Kindergarten children who learned the geometry content in a storytelling context appeared to gain more knowledge, as assessed on both near- and far-transfer tasks. The authors suggest that the storytelling context engages children in the content in ways that more decontextualized instruction does not.

## FORMATIVE ASSESSMENT

A core instructional principle of early childhood education is that teaching must be child-centered and "developmentally appropriate" (Copple and Bredekamp, 2009). To promote genuine and enthusiastic learning, the teacher must be sensitive to the individual child's emotions and must establish a trusting and supportive relationship with him or her. But child-centered and developmentally appropriate teaching requires cognitive as well as emotional sensitivity: to support mathematics learning, the teacher

[4]Pull-out programs remove children from the regular classroom for some portion of the day to give specialized instruction.

must acquire an understanding of the child's current mathematical performance and knowledge.

Formative assessment is the process of gaining insight into children's learning and thinking in the classroom and using that information to guide instruction (Black and Wiliam, 1998b) and improve it (Black and Wiliam, 2004). According to the National Mathematics Advisory Panel (2008), "Teachers' regular use of formative assessments improves their students' learning, especially if teachers have additional guidance on using the assessment results to design and individualize instruction" (p. 47).

Formative assessment does not involve formal testing conducted outside the classroom (with results usually left there); however, it can provide teachers a way in which to track children's progress toward high-quality early learning standards. Formative assessment entails the use of several methods—observation, task, and flexible interview—to collect information about children's thinking and learning and then adapt teaching methods to help them learn. It is often inseparable from teaching and usually not distinctly identified as assessment. Teachers assess children all the time, often unaware that they do so. But formative assessment can also be more deliberate and organized than is usually the case. This section provides guidance about how teachers can use formative assessment to improve classroom teaching practices so that students' learning needs are best met.

### Rationale for Formative Assessment

The need for sound formative assessment is evident from a variety of theoretical perspectives. Approaches that stress the need to capitalize on the teachable moment (Dodge, Colker, and Heroman, 2002) require teachers to understand when that moment occurs—that is, when the child is ready to learn—and then to exploit it so as to help the child undertake further learning. Using observation to identify the teachable moment is one use of formative assessment (Seo, 2003).

Early childhood educators often draw on Vygotsky's theory to advocate effective scaffolding. Scaffolding in turn involves first determining the child's "actual developmental level" so that one can help the child reach his or her potential "through problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Determining both actual and potential developmental level, as well as the scaffolding useful to help the child traverse this "zone of proximal development," requires formative assessment.

Piaget's theory stresses the distinction between overt performance and underlying thought (Piaget and Inhelder, 1969). To illustrate: A child says that the sum of 3 apples and 2 apples is 6 apples. The incorrect response is clearly important and needs to be corrected, but even more important is the method used to obtain the response. The child may have got it by faulty memory ("I just knew it"), faulty calculation (the child miscounts the objects in front of him or her), or faulty reasoning ("I know that 3 and 2 is more than 4 and 6 is 2 more than 4"). Identifying and promoting underlying thought requires formative assessment.

Contemporary cognitive theories often stress establishing a link between the child's informal knowledge and what is to be taught (Baroody, 1987; National Research Council, 1999; Resnick, 1989, 1992). The child brings to the task of learning a body of prior knowledge—an "everyday mathematics" that is often relatively powerful and sometimes a source of misconceptions. In either case, the teacher needs to understand the child's current cognitive state (the everyday mathematics) in order to adjust instruction to it. Sometimes the everyday mathematics can serve as a fruitful basis for further development; the child's learning may in part involve mathematizing what she or he already knows. Sometimes the teacher needs to employ methods to help the child abandon everyday concepts in favor of more accurate notions, as when the child believes that the symbol = means "get an answer" instead of an equivalence relation (Seo and Ginsburg, 2003), or that a long, skinny scalene triangle is not an acceptable triangle (Clements, 2004).

Those who practice behavior modification also need to employ formative assessment to acquire an accurate account of the child's current behavior so they know what to shape. Careful observation of behavior and decisions about appropriate reinforcement can also be conceptualized under the rubric of formative assessment.

In brief, many theoretical approaches advocate getting information about the child's current behavior, thinking, and learning so that effective teaching can be implemented. It is hard to imagine a theory of teaching that would advocate ignorance of the child's mind or behavior.

## Three Kinds of Formative Assessment

Formative assessment is a very natural and commonplace activity for teachers, who do it all the time without necessarily knowing that what they do is assessment. Here we discuss three major kinds of formative assessment: everyday observations, tasks, and interviews (see Box 7-3). These everyday practices of observation, presenting tasks, and interviewing involve an informal, often unplanned, implementation of formative assessment, which is so bound up with everyday teaching that it often goes unrecognized. Yet the three types of formative assessment can be rigorous, focused, and deliberate. The early childhood assessment systems discussed here include widely used integrated programs as well as mathematics-specific programs: Big Math for Little Kids, Building Blocks, Core Knowledge, Creative Curriculum, High/Scope, and Number Worlds.

**BOX 7-3**
**Formative Assessment**

Teachers often use *everyday observations* to make inferences about children's abilities. The teacher sees that Juanita often spontaneously names shapes as she places them on the table. She can identify large and small objects and red and green objects as rectangles, and she even knows the name for a trapezoid. The teacher concludes that she can see the differences among various shapes, understands that color and size are irrelevant, and even knows some shape names. She is now ready to learn to mathematize her knowledge—that is, to analyze the properties of shapes so that she will understand explicitly what defines a rectangle and other shapes.

Teachers also give children specific *tasks* to elicit their understanding. The teacher has seen that Juanita spontaneously names rectangles and trapezoids but has never seen her name a square. So the teacher shows her a large red square and a small green one and asks what they are. Juanita says that the red one is a square but that the green one is not. Having given this specific task, the teacher now concludes that Juanita knows the name for square but applies it in a rather unusual way. The teacher is puzzled because Juanita was able to identify small green rectangles as rectangles, but she cannot ignore size or color in the case of squares.

Why does she do this? To find out, the teacher goes a step further. She wants to know how Juanita thinks about squares. What makes something a square? What's the role of color and size? Can she talk about it? So she *interviews* Juanita. She asks her why she said this large red object was a square, whereas this small green one was not. Juanita says that color does not matter, but that squares have to have 4 sides the same length and have to be big. Why do they have to be big? She does not know.

The teacher concludes that her lessons on shape should include specific attention to issues of size and when it is relevant or not relevant. And the teacher has a clue about how to proceed. She will put Juanita in a situation of cognitive conflict, which, according to Piaget (1985), is a major impetus to cognitive growth (Limon, 2001). She points out to Juanita that color and size do not matter for rectangles. Why should they matter for squares? Juanita looks puzzled. But then she quickly agrees that of course squares can be small, too. Her expression says: How silly to think otherwise! Assessments like these take place in many classrooms. Teachers observe their children, set them brief everyday tasks, and question them about their thinking. They do these classroom assessments on the fly, spontaneously, and without special preparation. Sometimes children learn a good deal, and sometimes they don't.

### *Observation*

Observation involves several components. One is obtaining useful information. The teacher needs to observe relevant aspects of an individual child's mathematical behavior. For example, she needs to observe that, in free play, the child is not only comparing the lengths of two blocks but also makes the mistake of failing to use a common baseline. This is not easy to do when the teacher must observe and supervise a room full of young children who have many needs and who exhibit complex patterns of behavior. There is an enormous amount of behavior taking place at any one moment in the classroom day. Nevertheless, it is possible for teachers to focus observations on at least a few children in order to provide activities that promote further mathematics learning.

A second important component of observation is interpretation of the evidence. The observer needs to understand what the behavior means. In the example above, the child's failure to use a common baseline in comparing the length of blocks indicates a common misconception of a fundamental idea underlying measurement (Clements, 2004; Piaget and Inhelder, 1967). Teachers need to be aware of and understand this misconception in order to interpret behavior accurately. Observation is very theoretical. To interpret everyday behavior, the teacher needs to be familiar with the development of mathematical thinking, as well as with the mathematics about which the child is thinking. Teachers need to receive professional development about learning in early childhood to be able to effectively interpret their observations of children's mathematical thinking.

A third component of observation is careful evaluation of evidence. Suppose the teacher sees a child spontaneously place a red and a blue isosceles triangle into one collection. But the child does not place a red skinny irregular (scalene) triangle into that same collection. Does this evidence suggest that the child has an understanding of triangles? On one hand, maybe the child did not see the small triangle and, had she seen it, perhaps would have placed it with the others, thus demonstrating at least some understanding of what defines a triangle. On the other hand, maybe she did see it and decided not to include it with the others because it was so strange looking, not an isosceles triangle, thus revealing that she had a narrow concept of a triangle. The evidence is inconclusive, and one cannot make a firm conclusion; both interpretations are possible. Evaluation of evidence requires skills of critical thinking that do not come easily and often need to be taught (Kuhn, 2005).

How well do teachers observe mathematical behavior? Research on this issue appears to be lacking. But there are reasons to be pessimistic about the likelihood that they make useful and insightful observations. Teachers seldom have time to observe behavior during free play; they tend to have their hands full with management and discipline (Kontos, 1999). Also, teachers may not know what to look for. As Piaget said, observation requires knowledge of what is to be observed—in this case, mathematical thinking: "if they are not on the lookout for anything . . . they will never find anything" (Piaget, 1976a, p. 9). In addition, as Chapter 8 discusses, early childhood teachers have little training in either early mathematics education or mathematics, especially in the analysis of mathematical behavior.

***Organized systems of observation.*** Teachers need guidance in the observation of mathematical behavior. Their college or university education may have provided some useful experience in observation. And although popular textbooks on the subject (e.g., Boehm and Weinberg, 1997; Cohen, Stern, and Balaban, 1997) discuss general issues of observation, they do not discuss in any depth the observation of mathematical behavior in particular.

Several widely used curricula offer guidance in observation of mathematical behavior. They may provide checklists for observation of various topics with directions about which behaviors should be recorded. For example, one form instructs teachers to record their observations of a child's knowledge of number and operations. The checklist specifically focuses on counting aloud in the correct order and grouping objects. A checklist like this is broad and provides teachers with little guidance for assessing children's mathematical knowledge. Rather, teachers should use the checklist as a start to assessing where children are on the mathematics teaching-learning path. Ideally, teachers would use follow-up questions and various tasks in conjunction with observation to ascertain the child's level of mathematical knowledge. At best, the observations give only an extremely crude idea of the child's interests and provide very little information about his or her knowledge.

Other widely used early childhood mathematics assessment systems offer the opportunity for the teacher to collect interesting anecdotes about individual children. For example, teachers create a personal log of each child's actions and abilities with spaces for writing numerous anecdotes—brief reports on individual children's classroom behavior and work samples that highlight their developing abilities. Again, we note that observation of mathematical behavior requires training and theoretical background. These assessment systems do not seem to provide evidence concerning the quality of observations or their usefulness.

Other curricula and their related assessment systems stress the analysis of various products of learning activities. The Reggio Emilia group in Italy uses "pedagogical documentation to capture learning moments through observation, transcriptions and visual representations that provoke reflection and inspire teachers, children, and parents to consider the significance of the interactions taking place, and the next steps to be taken in teaching and learning" (MacDonald, 2007, p. 232).

***Strengths and weaknesses of observation.*** Observation can be an extremely powerful method. It may provide insight into the child's spontaneous interests and everyday competence in the absence of adult pressure or constraint. Observation deals with behavior in "authentic" situations, like block play or snack time. The teacher may learn from careful observation that the child possesses a competence that is not expressed when he or she is tested or given instruction.

At the same time, no single method of assessment is perfect, always accurate, or completely informative, and observation has some limitations. Sometimes, the teacher can wait indefinitely before observing truly important behavior. Sometimes, the child's behavior does not express the true extent of her or his competence. As Piaget said: "How many inexpressible thoughts must remain unknown so long as we restrict ourselves to observing the child without talking to him?" (Piaget, 1976a, pp. 6-7). Thus, observation may show that the child does not seem to sort objects by common shapes, putting triangles into one group and rectangles into another. Instead, he or she places them all into one messy collection and tells stories about them. Does this mean that the child does not understand the difference between triangles and rectangles? The observer will never know without explicitly asking the child to sort them. This is a *task*, the next type of assessment.

### *Task*

Sometimes to find out about a child's learning, thinking, or performance, one presents him or her with some kind of task, a simple problem to solve. The teacher may ask, "What do you see in this picture book? What is the clown doing?" Or the teacher may say, "What do you call this thing [a triangle]?" The child's response may give an indication of his or her competence. If the child says, "The clown is juggling three balls," then the teacher may learn something about the accuracy of his or her counting skills. If in response to the question about the triangle the child says, "I don't know," then the teacher has learned that the child may not be able to produce the correct word or apply it to at least a certain kind of triangle. Yet from Vygotsky's perspective, the child's response to this task may be an indication only of current developmental level. The teacher therefore goes on to provide a little scaffolding, asking, "What shape is it?" The child then answers, "a triangle," and this indicates his level of potential development.

In brief, tasks are initiated by the teacher to learn about the child's performance with respect to a particular topic of interest to the teacher. Basically, the teacher wants to know whether the child can do something—count, recognize a triangle, or make a pattern—perhaps with a little help.

Evidence about how well teachers employ tasks in the classroom appears to be lacking. Yet it may be relatively easy for the teacher to ask the child to respond to simple tasks with which the child is engaged during free play ("What is that block called?") or to ask questions about the topic of the teacher's instruction ("Which animal is first in line?"). Of course the teacher must interpret the child's response with accuracy and is therefore faced with some of the same difficulties as discussed in the case of observation. The teacher must understand the development of mathematical thinking to appreciate the meaning of the child's response.

***Organized systems of tasks.*** Some early childhood curricula present a series of tasks as the basis of their assessment system. For example, an item might instruct the teacher to use manipulative counters (e.g., blocks) to create different groups of objects containing between one and four items and to arrange the groups in different configurations (e.g., straight line, random grouping). The teacher would need to be sure that the child had several opportunities to correctly count, and she would record whether the child counted correctly and, for incorrect counts, the kinds of mistakes the child made.

The task employed, namely to count a given number of objects, is common both in the research literature and in some tests at this age level (e.g., Ginsburg and Baroody, 2003). What appears to be lacking, however, is any indication of how to interpret the results. What does it mean, for example, if the child can count a randomly arranged set of 3 but not 4, which he or she can count if it is placed in a line? Several primary mathematics curricula have a large collection of tasks with a clear theoretical basis (Case and Okamoto, 1996; Griffin, 2004).

***Strengths and weaknesses of tasks.*** The strength of this method is that it provides information about the child's performance on a task in which the teacher has an interest. The teacher is attempting to teach something about pattern and needs to know whether the child is "getting it" so that she can take the next appropriate instructional step. There is some evidence that, at least at the elementary school level, frequent monitoring of student behavior can improve performance (Fuchs et al., 1999).

But there are at least two basic weaknesses in using tasks. The first is also its strength, namely that the teacher's interests determine the choice of task. The teacher is trying to teach pattern, but the child may in fact be more interested in or dealing with another topic, like the shapes of the objects intended to comprise the pattern. Because children do not always learn what teachers teach, teachers' questions about what they are trying to teach do not necessarily reveal what the child is learning. Second, the child's behavior may indicate success or failure on the task but does not necessarily reveal how the child construes or solves the task. As Piaget pointed out, it is not enough to ascertain the child's answer; one must in addition learn how the child got it. It is possible for the correct answer to be the result of a mechanical process devoid of understanding and for an incorrect answer to be the result of insightful thinking.

### *Flexible Interview*

A constructivist and child-centered perspective demands that the teacher go beyond observation and tasks to probe the child's thinking. Observation and tasks can provide useful information about performance, but the flexible interview is needed to dig below the surface to learn what the child is thinking. A truly child-centered, cognitively sensitive approach requires asking how the child solved the problem, how she got the answer, and why she said what she did. This kind of questioning originated in Piaget's "clinical interview method" (Piaget, 1976a), which we term "flexible interviewing," so as to avoid any connotation of the "clinical interview" devoted to the investigation and cure of pathological phenomena.

Flexible interviewing involves several steps (Ginsburg, 1997). First, the interviewer notices what seems to be an important child behavior worthy of further investigation. Sometimes this occurs in the course of naturalistic observation of everyday classroom activities. More frequently it occurs when the child gives an interesting response to a task. In either event, the interviewer follows up in various ways. He may rephrase the initial question, ask the child to talk about how she or he solved the problem, or request that the child expand on an answer or justify it. Occasionally the interviewer may challenge a child's response and ask her to prove why it is not correct. The essential questions include: "How did you figure it out? How did you know? How did you get the answer? Tell me more about it. How do you know you are right?"

In general, the rationale is that, if the goal is to learn what the child is thinking, the teacher must engage in flexible interviewing, asking the child to elaborate on his or her ways of interpreting and approaching a problem. Note that the flexible interview involves elements of both the task and observation. The interviewer frequently begins with a simple task for the child to solve ("What do you call this figure?") and then follows up on the child's response ("Why do you think it is a triangle?"). And as the child seems to be thinking about the problem or provides an answer, the interviewer carefully observes the child's behavior to determine, for example, whether he points to a certain object or looks confused or seems to whisper his thought aloud. Indeed, Piaget maintained that the interview method combines the best of observation and task.

Flexible interviewing involves a good deal of skill and mental agility. It requires the same kind of observational sensitivity, critical thinking, and interpretive skills discussed in connection with observation and task. It also requires the interviewer to think on her feet, to improvise, and to come up with the right follow-up question on the spot.

How frequently and well do teachers employ the flexible interview in the classroom? Research on the issue seems to be lacking. At the same time, flexible interviewing, although difficult, is a natural form of human interaction in which the participants attempt to make sense of problems and how they can be solved—"clinical interviewing is a species of naturally occurring mutual inquiry" (diSessa, 2007, p. 534). Asking a person why he or she said or did something is an entirely familiar form of discourse and not necessarily artificial or lacking in ecological validity.

***Organized systems.*** Few curricula provide extensive guidance in flexible interviewing. D.M. Clarke and colleagues (Clarke et al., 2001) have used a developmental trajectory theory as the basis for development of an extensive collection of "task-based interviews" for children beginning at age 5.

The collection of interview items is intended to form the basis for a comprehensive program of professional development, as well as to serve as a formative assessment tool for the teacher. "The [theoretical] framework of growth points provides a means for *understanding* young children's mathematical thinking in general, the interview provides a tool for *assessing* this thinking for particular individuals and groups, and the professional development program is geared towards *developing* further such thinking" (p. 2). In many respects, the work is a model for what should be done in this area. To date, few early childhood curricula provide guidance on flexible interview. Big Math for Little Kids (Ginsburg, Greenes, and Balfanz, 2003), however, includes extensive guidance on flexible interviewing for each major topic. The Number Worlds curriculum (Griffin, 2007) offers an assessment system that largely involves a series of tasks (boldly called "tests"), some of which include flexible interview follow-ups. For example, "How many *more* smiley faces does the hexagon have than the triangle has? How did you figure that out?" (p. 72). After these instructions, an example of a possible child response is presented: "2 more; I counted to 3 and there were 2 left that I didn't count" (p. 72). In general, the focus on flexible interviewing, even though it is at the very heart of a child-centered approach, is limited in current curricula.

***Strengths and weaknesses.*** The flexible interview can provide basic and often surprising information about children's knowledge. It sometimes shows that the child who seems to know something really doesn't, and the child who doesn't seem to know something really does. This kind of information can help teachers overcome preconceptions they might have about children's abilities. For example, teachers may expect low-income children to be more capable of procedural than conceptual knowledge.

The results of a flexible interview may help to disabuse the teachers of this preconception.

The flexible interview allows the interviewer to make sense of puzzling observations of everyday behavior or responses to tasks. The benefits accruing from this knowledge may be considerable: Understanding the child's perspective can provide a sensitive guide to instruction. If the child's wrong response was the result of a misinterpretation of the question, the teaching solution is different from what is needed if the response resulted from a basic misunderstanding.

Also, use of the method entails secondary benefits. Flexible interviewing requires that teachers talk a great deal with children. Furthermore, flexible interviewing not only promotes the teacher's language but also requires it from the child. Flexible interviewing stresses to the child the importance of talking about one's thinking, justifying one's conclusions, and in general engaging in mathematical communication, which as we have seen is one of the main goals of mathematics education at all levels (National Council of Teachers of Mathematics, 2000; National Research Council, 2001a). Indeed, the very process of being interviewed may have a salutary effect on the child. There is some evidence with older children that self-explanation (providing an explanation of material recently studied) promotes increased understanding (Chi et al., 1994). Similarly, the requirement to explain one's thinking might help one to examine, organize, and in the process even improve it.

Interviewing can be hard to do well, especially when very young children are involved. As noted, it demands interpretative skill, creativity, and flexibility in questioning. It is easy to ask misleading or uninformative questions and distort results; it requires considerable skill and sophistication to do really well. It is hard for young children to be aware of their mental processes or to describe them in words (Flavell, Green, and Flavell, 1995; Kuhn, 2000; Piaget, 1976). The strength of the method—its flexibility and sensitivity to the individual—is at the same time its weakness.

## Some General Remarks

In general, children's developmental characteristics make it difficult, although not impossible, to assess their learning, thinking, or performance. They can be shy, uncooperative, nonverbal, impatient, noncommunicative, and so on. Their self-regulation skills are imperfectly developed (Bronson, 2000), and they are egocentric (Piaget, 1955). The result is that assessment of the child at any one point in time may be inaccurate. But that does not mean teachers should not attempt to assess. It means that assessment needs to be done as sensitively as possible. Similarly, it is hard to diagnose a 2-year-old child's hearing, but there is a moral obligation to do it as well as possible.

Similarly, because of the natural fluctuation and rapid development of children's behavior, a single assessment—whether done by observation, task, or interview—may not provide accurate information. It is necessary to assess young children frequently and to base educational decisions on multiple sources of information (National Research Council, 2001b, 2008). Formative assessment should be complementary to program evaluation, which is conducted outside the classroom (see Box 7-4). Also, it is possible and sometimes desirable to blend the three methods. Thus, the teacher can observe in the natural setting and at the same time give the children simple tasks and even interview them.

The importance of teachers' understanding of their students cannot be overemphasized. According to the National Research Council report *Adding It Up: Helping Children Learn Mathematics*, "information about students is crucial to a teacher's ability to calibrate tasks and lessons to students' current understanding. . . . In addition to tasks that reveal what students know and can do, the quality of instruction depends on how teachers interpret and use that information. Teachers' understanding of their students' work and the progress they are making relies on . . . their ability to use that understanding to make sense of what the students are doing" (National Research Council, 2001a, pp. 349-350). Teachers' understanding of their students is the key, or at least one key to successful teaching.

Finally, although formative assessment shows great promise, the methods of assessment have not been clearly linked to instructional interventions. In fact, there seem to be few if any research studies that investigate the power of formative assessment to improve student achievement (exceptions include Black and Wiliam, 1998a, 1998b; Heritage, Kim, and Vendlinkskil, 2008). One of these studies suggests that, although elementary school teachers are reasonably skilled in interpreting student behavior, they have difficulty linking the assessment to subsequent teaching (Heritage, Kim, and Vendlinski, 2008). Clearly, further research and development are required. Development is needed to create links between assessment and instruction, and research is needed to investigate the effectiveness of those links. All of this should be easier to do in the teaching-learning paths described in this report because they keep the teacher situated in an organized set of goals with directionality both for individual children and for the class.

## RESEARCH ON THE EFFECTIVENESS OF MATHEMATICS CURRICULUM

Although this chapter addresses the topics of pedagogy and curriculum separately, in practice there is often no clear distinction between the two. This is especially true in early childhood education. Early childhood curriculum has traditionally emphasized the process of teaching and learning rather than the content of what children are learning (National Association

**BOX 7-4**
**Comments on Program Evaluation**

Programs for young children, like those for older ones, need to be held accountable. People want their children to receive the best early childhood mathematics education possible. There is no dispute as to the necessity for evaluation of programs, but the evaluation has to be as fair, sound, and based on scientific evidence and theory as much as possible.

Current evaluations are informative but limited. Several obstacles need to be overcome to improve the quality of evaluation efforts. First, it is hard to assess young children. Just as in the case of formative assessment, observation alone is insufficient, and the adult must employ some form of task or interview. But as pointed out earlier, even when a friendly adult does the assessment on a 1-to-1 basis, young children can be shy, uninterested, uncooperative, or inconsistent. Conditions like these require highly trained adult assessors who can engage children and approach the assessment with sensitivity and intelligence. This in turn "creates significant feasibility issues for large-scale accountability initiatives. Relatively large numbers of assessors must be trained and supervised. Quality assurance is another major challenge: the consistency, credibility and integrity of child assessment reports must be established and monitored" (National Early Childhood Accountability Task Force, 2007, p. 23).

Second, and even more important, there are few psychometrically valid assessment instruments to use in the evaluation of early mathematics education programs. Current instruments either focus on a narrow aspect of early mathematics, like number (e.g., Ginsburg and Baroody, 2003), or lack extensive psychometric support. A useful assessment should cover a broad array of mathematical knowledge, from number to pattern to space. Also, it should examine the "productive disposition," that is, the "habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy" (National Research Council, 2001a, p. 5). And the evaluation it should be easy to administer and enjoyable to take. Such an instrument with sound psychometric qualities is not yet available. Because evaluations are only as valuable as the measures they employ, current evaluations must be considered of limited value.

Finally, it is as important to assess program quality, including teaching, to assess the children's performance. At present there are few psychometrically sound measures of early mathematics teaching or program quality (for an in-depth discussion on this topic, see National Research Council, 2008).

Just as early mathematics education has been neglected for many years, so have the methods needed to evaluate it. In view of the former, the latter should come as no surprise. As a result, considerable research and development need to be conducted to create evaluation methods appropriate for examining the quality of programs and their success in educating children.

for the Education of Young Children, 1997). Given this view of curriculum, research and debate have focused on which curriculum model is most effective in supporting children's short-term and long-term development (Epstein, Schweinhart, and McAdoo, 1996).

Many early childhood educators are not comfortable with defining curriculum as a written plan or specifying scope and sequence in advance. This concern grows out of the strong tradition of emergent curriculum in early childhood education (Jones and Nimmo, 1994). According to this perspective, the focus should be on children, not on curriculum. Advocates of emergent curriculum believe that children's interests and needs should determine what goes on in a classroom rather than a predetermined plan. They also assume that a planned scope and sequence cannot be responsive to children's individual and cultural variations.

Emergent curriculum is often implemented using the project approach (Katz and Chard, 1989), in which children and teachers engage in an intensive investigation of a topic of interest. Sometimes people refer to the project approach as a curriculum model, but it is more akin to a teaching strategy or context. In recent years, advocates of the project approach have been more specific about how state standards can be incorporated and met during the planning and implementation of a project (Helm and Katz, 2000). To help children achieve learning goals, educators have begun to emphasize intentional teaching in an emergent curriculum or project approach (Epstein, 2007).

During the past 15 years, early childhood practice in the United States (and throughout the world) has been influenced by the Reggio Emilia approach (Edwards, Gandini, and Forman, 1998). The approach is not a curriculum, nor is it a model. It is a coherent set of principles and practices that reflect a sociocultural perspective on learning and development. A key element of the approach is serious project work involving small groups of children collaborating with teachers to undertake investigations, theorizing, representing, revisiting experiences, and revising conceptualizations. Project work often arises from real rather than contrived situations. For example, one school needed a new table and the carpenter asked for measurements, a project documented in a book called *Shoe and Meter* (Reggio Children, 1997). The children worked together to figure out how to measure the table. They tried measuring using their various body parts but were dismayed to discover that each person's foot was a different length. Finally, they chose one child's foot to be their standard length. Then, they held his foot up to the ruler and determined how it compared, and so on.

In the past decade in the United States, there has been an explosion in commercially published early childhood curriculum resources. In 2007, the PreK Now website listed 27 research-based curricula for preschool children (see http://www.preknow.org). Some of these curricula are comprehensive—designed to address all domains of children's learning and development. These comprehensive programs tend to be organized into units, often called themes, based on children's predictable interests, but they are also broad enough to connect many different experiences and achieve multiple goals. Such themes usually include such topics as weather, animals, or construction. Comprehensive curricula are sometimes integrated curricula, in which one topic or experience is designed to meet goals across subject matter areas, such as reading a book that includes scientific information. Some comprehensive curricula have a limited number of themes, six to nine, allowing for more in-depth attention to the topic. Others change the topic weekly. In the past (and today as well), teacher-developed preschool "curriculum" was often theme-based, consisting of a series of activities related to the changing seasons, holidays, and events in children's lives, such as visits to the firehouse.

Often newly available curriculum resources are designed to provide instruction focusing on language, literacy, and/or mathematics. In some of these resources, learning and instruction are devoted to a single content domain, such as mathematics or literacy skills. Sometimes, a curriculum resource focuses on only one aspect of one domain rather than on an entire domain, such as phonological awareness or social-emotional development. These resources require teachers to figure out how to offer a coherent curriculum that covers all important learning goals.

Little research is available on the extent to which preschool programs use specific curriculum. The six-state study of prekindergarten conducted by the National Center for Early Development and Learning provides some evidence about curriculum use in state-funded preschool programs (Early et al., 2005). Only 4 percent of teachers reported having no curriculum, 14 percent used a locally developed curriculum, and 9 percent used a state curriculum. The most widely used curricula are High/Scope, with 38 percent of classrooms, and Creative Curriculum, accounting for 19 percent (National Center for Early Development and Learning Prekindergarten Study, 2005). These two curricula are also the most widely used in Head Start programs (U.S. Department of Health and Human Services, 2006).

There is increasing agreement over many early childhood teaching practices, often called developmentally appropriate practice (see previous section on effective instruction; see also Copple and Bredekamp, 2006, 2009). Developmentally appropriate practice as defined by the National Association for the Education of Young Children (Copple and Bredekamp, 2009) calls for teachers to make decisions that are informed by knowledge of child development and learning, knowledge about individual children, and knowledge about the social and cultural context in which they live. The concept is that teachers adapt the curriculum and teaching strategies for the age, experience, and abilities of individual children to help them make learning progress.

Despite the support for developmentally appropriate practice in the field, there is less acceptance of the need for a written curriculum, especially if that curriculum provides a planned sequence of teaching and learning opportunities (Lee and Ginsburg, 2007). Yet such a curriculum organized by research-based teaching-learning paths, such as those described in Chapters 5 and 6, or at least some learning path organization of the mathematics activities over the year, is needed to ensure that all children have a chance to learn the topics in the learning path. Such systematic opportunities are needed to help improve mathematical outcomes for all young children.

## Mathematics Curriculum

A limited amount of research is available on the effectiveness of specific mathematics curricula or curricular approaches. As described earlier, most early childhood programs do not include primary mathematics experiences or focused mathematics time but rather rely on integrated mathematics experiences in which mathematics is a secondary goal and often incidental (Preschool Curriculum Evaluation Research Consortium, 2008). However, incidental mathematics instruction appears to be less effective than activities with a primary focus on mathematics, although this evidence is only correlational (Starkey et al., 2006).

In addition, reliance on incidental or integrated mathematics may contribute to the fact that little time is spent on math. For example, in the Preschool Curriculum Evaluation Research (PCER) Study, conducted by the U.S. Department of Education, a literacy-oriented curriculum (Bright Beginnings, available at http://www.brightbeginningsinc.org/) and a developmentally focused one (Creative Curriculum, available at http://www.teachingstrategies.com/) engendered no more mathematics instruction than a control group (Aydogan et al., 2005). Other research (Farran et al., 2007) found a negligible time devoted to mathematics in a literacy-oriented comprehensive curriculum.

It is important to note, however, that in response to changing standards and current research, the developers of Creative Curriculum have recently added a mathematics component to their approach (Copley, Jones, and Dighe, 2007). In addition, the High/Scope curriculum (Hohmann and Weikart, 2002) is developing a more challenging focused mathematics component (Schweinhart, 2007).

Large effect sizes support the strategy of designing a mathematics curriculum built on comprehensive research-based principles, including an emphasis on hypothesized teaching-learning paths (Clarke, Clarke, and Horne, 2006; Clements and Sarama, 2007b, 2008a; Thomas and Ward, 2001; Wright et al., 2002). Most of these studies also emphasized key developmental milestones in the main teaching-learning paths, promoting deep, lasting learning of critical mathematical concepts and skills.

Teaching-learning paths or learning trajectories are useful instructional, as well as theoretical, constructs (Bredekamp, 2004; Clements and Sarama, 2004; Simon, 1995; Smith et al., 2006). The developmental progressions—levels of understanding and skill, each more sophisticated than the last—are essential for high-quality teaching based on understanding both mathematics and learning. Early childhood teachers' knowledge of young children's mathematical development is related to their students' achievement (Carpenter et al., 1988; Peterson, Carpenter, and Fennema, 1989). In one study, the few teachers that actually led in-depth discussions in reform mathematics classrooms saw themselves not as moving through a curriculum, but as helping students move through levels of understanding (Fuson, Carroll, and Drueck, 2000). Furthermore, research suggests that professional development focused on developmental progressions increases not only teachers' professional knowledge but also their students' motivation and achievement (Clarke, 2004; Clarke et al., 2001, 2002; Fennema et al., 1996; Kühne, van den Heuvel-Panhulzen, and Ensor, 2005; Thomas and Ward, 2001; Wright et al., 2002). Thus, teaching-learning paths can facilitate developmentally appropriate teaching and learning for all children (see Brown et al., 1995).

A few words of caution are in order in interpreting findings about mathematics curriculum research. In the early childhood context, randomized control trials in mathematics may tend to overstate effect sizes because teaching some mathematics will always be more effective than teaching no or almost no mathematics (which is usually what the control classrooms are doing). Comparing the large effect sizes of the mathematics PCER study (Starkey et al., 2006) with the results of no significant differences for most of the literacy PCER studies does not mean that mathematics curricula are effective while literacy curricula are not. Preschools have had a decade of focus on literacy, so the control groups in those studies were doing a lot of literacy as well as the experimental groups. Curricular research does have great potential to advance understanding of effective instructional strategies, but only if this research is conducted with this explicit goal in mind. The inclusion of observational measures, both of fidelity to the curriculum and generalized instructional processes, greatly enhances the ability of the research to speak to specific teaching strategies that may be most important for student learning.

For example, Clements and Sarama (2008a) included extensive observation using the Classroom Observation of Early Mathematics Environment and Teaching (COEMET) and Fidelity of Implementation during a randomized control trial of two mathematics curricula—Building Blocks and Preschool Mathematics Curriculum (PMC; Klein, Starkey, and Ramirez, 2002)—and a control condition. The results indicate that research-based mathematics preschool curricula can be implemented with good fidelity, if teachers are provided ongoing training and support.

Using data from the COEMET the researchers identified instructional strategies that significantly predicted gains in children's mathematical knowledge over the course of the year: (1) the percentage of time the teacher was actively engaged in activities, (2) the degree to which the teacher built on and elaborated children's mathematical ideas and strategies, and (3) the degree to which the teacher facilitated children's responding. Examples are provided in Box 7-5. In addition, the researchers' inclusion of multiple curricula also facilitates generalization beyond the effects of a specific curriculum to the broader approaches that may be embedded in it.

The ability of curricular research to inform effective practice would also be enhanced if individual curricula more clearly defined the instructional approaches embedded in them. Often curricula distinguish themselves in terms of content (e.g., covering geometry or not) and generalized approach (e.g., whole-group versus small-group instruction) more than in the instructional strategies that are endorsed and supported in the activities. Thus, any findings that one curriculum is more effective than another provides little knowledge about specific teaching strategies that may be useful.

### Improving Mathematics Outcomes for Children in Poverty

The limited amount of time devoted to the subject of mathematics may account for why Head Start children make little or no gain in mathematics. For example, using randomized assignment, the Head Start Impact Study found no significant impacts for the early mathematics skills of 3- or 4-year-olds (U.S. Department of Health and Human Services, 2005). Other examples include control groups from experiments (Clements and Sarama, 2007b; Clements and Lewis, 2009; Starkey et al., 2006). The control group in one study, for example, made small gains in number, but little or no gain in geometry (Clements and Sarama, 2007b) and Head Start children made no significant gain in any area of mathematics during the school year (control classrooms continued using their school's mathematics activities, which were informed by a mixture of influences ranging from commercially published curricula to homegrown materials based on state standards).

Research demonstrates that interventions with a primary focus on mathematics have the potential to increase the mathematics achievement of children living in poverty and those with special needs (Campbell and Silver, 1999; Clements and Lewis, 2009; Fuson, Smith, and Lo Cicero, 1997; Griffin, 2004; Griffin, Case, and Capodilupo, 1995; Ramey and Ramey, 1998), which can be sustained into first (Magnuson et al., 2004) to third grade (Gamel-McCormick and Amsden, 2002). For example, both the Building Blocks and Big Math for Little Kids curricula significantly and substantially increase the mathematical knowledge of children from low-income families (e.g., Clements and Lewis, 2009; Clements and Sarama, 2007b, 2008a). The success, even in comparison to other curricula, is probably due to the shared core of learning trajectories (teaching-learning paths) emphasized in the curriculum and the professional development that ensures that teachers spend time teaching appropriate mathematics topics during the year.

Another example, the Rightstart program[5] (Griffin, Case, and Siegler,

[5]Now published as Number Worlds (Griffin, 2004, 2007).

**BOX 7-5**
**Examples of Low-Quality and High-Quality Mathematics Teaching**

1. The teacher was actively engaged.

*Consider a situation in which the teacher has put out a mathematics center with play dough.*

| | |
|---|---|
| A nonengaged teacher talks for several minutes exclusively to another adult in the room. | An engaged teacher works with several children at the center until she observes they "have the idea" of the activity. She keeps her eye on the center and encourages children to keep building. |
| Another works with children in another area of the room, but neither she nor the aide visits the math center. | In another room, the teacher works with children in another area of the room, while the aide visits the center and helps or acknowledges the children's mathematics work. |

2. The teacher built on and elaborated children's mathematical ideas and strategies.

*Consider a situation in which children are to put some dinosaurs on a play scene and describe what they did. One child put out dinosaurs, but then just pointed.*

| | |
|---|---|
| A teacher who did not build on or elaborate children's ideas, merely says, "OK." | A teacher who does build on or elaborate children's ideas says, "What do you have there?" The child does not respond. |
| Another teacher asks, "What are the dinosaurs doing?" "Fighting!" says the child, as he picks up the dinosaurs and loudly demonstrates the fighting. The teacher says, "That's enough of that!" excuses this child and calls another child to the activity. | "What are these two dinosaurs doing?" "Fighting." "How many are in your pond?" "Two." "What are they going to see? On the hill here? "A T-rex. One T-rex." "Wow! Four dinosaurs, two here and two on the pond, are seeing that Tyrannosaurus Rex. I'll bet they are scared!" |

3. The teacher facilitated children's responding.

*Consider a situation in which the teacher asks one child to figure out how many 1 more than 3 is.*

| | |
|---|---|
| One teacher who does not facilitate children's responding says to a child who does not answer, "Someone else can answer." Once another child gave the correct answer the teacher moves on to the next task. | A teacher who does facilitate children's responding says, "Can you show me 3 to get started?" The child says "four." The teacher asks, "Can you teach us how you did that?" After the child explains, the teacher asks, "Did anybody do it a different way?" |

SOURCE: COEMET.

1994), which uses small-group games and active experiences with different models of number, led to substantial improvement in children's knowledge of number. Children in the program were better able to employ reasonable problem-solving strategies and solve arithmetic problems even more difficult than those in the program. (Core Knowledge includes a mathematics sequence developed by Sharon Griffin based on this work.)

Program children also passed five far-transfer tests that were hypothesized to depend on similar cognitive structures (e.g., balance beam, time, money). The foundation these children received supported their learning of new, more complex mathematics through Grade 1. In a 3-year longitudinal study in which children received consistent experiences from kindergarten through the primary grades, children gained and surpassed both a second low-income group and a mixed-income group that showed a higher initial level of performance and attended a magnet school with an enriched mathematics curriculum. The children also compared favorably with high-income groups from China and Japan (Case, Griffin, and Kelly, 1999). On a more limited scale, a study of 8 classrooms with 112 children found that a 6-week focused mathematics intervention was successful in improving Head Start children's mathematical skills as well as their interest in mathematics (Arnold et al., 2002). Teachers in experimental classrooms were provided with a choice of math-relevant activities to use during circle time, with small groups, and during routines and transitions, while the control classrooms did typical activities. Experimental group children scored significantly higher on the Test of Early Mathematics Ability (TEMA-2) and also reported that they enjoyed mathematics more than the control children. Teachers, too, reported that they increased their knowledge and enjoyment in implementing mathematics activities. Notably, boys showed substantial gains compared with girls, and African American and Puerto Rican children gained more than white children. Like other mathematics interventions, this study includes several variables, making it impossible to determine which particular teaching and learning experiences make the most difference to children. At the very least, the study indicates once again that more intentional teaching of mathematics leads to better mathematics outcomes.

## PRINCIPLES TO GUIDE MATHEMATICS CURRICULUM AND PEDAGOGY

Based on an extensive review of research on the current state of early mathematics education and effective practices, we present a set of principles to guide early childhood mathematics curriculum and instruction. Research points specifically to the following key indicators of an effective mathematics program at the preschool level (e.g., Clarke et al., 2002; Clements and Sarama, 2007b, 2008a; Thomson et al., 2005; Wood and Frid, 2005):

- Uses research to specify a comprehensive set of cognitive concepts, processes, and teaching-learning paths to design developmentally sequenced activities and help teachers collect data by observation and interaction with children and use those data to modify planning and teaching strategies. Tasks are sequenced, but teachers need to adapt for particular students' conceptual development rather than rigidly following a prescribed curriculum.
- Emphasizes mathematization of children's experiences, including redescribing (i.e., with more specific and often mathematical language), reorganizing, abstracting, generalizing, reflecting on, and giving language to what is first understood on an intuitive, informal level (premathematical foundations).
- Builds an awareness of the need for direct, formal development of children's concepts in mathematics together with an instructional focus on mathematics. This includes explicit plans for mathematics as a separate area of the program and ability to plan based on teaching-learning paths.
- Uses a variety of instructional methods, such as a combination of small groups, the whole group, play, routines and transitions, and computer activities. Uses teachable moments as they occur—in general, has the ability to make connections between mathematical ideas, between activities, between mathematics and other subjects, and everyday life.
- Uses an "assisted performance" approach to instruction that supports problem solving and inquiry processes in mathematics activities. Uses a variety of question types to encourage children to explain their thinking and to listen attentively to individual children and understand their level of thinking along mathematical teaching-learning paths.
- Engages and focuses children's thinking through introductions and activities. Draws out key mathematical ideas at the conclusion of an activity or period of study and helps children consolidate and connect their knowledge.
- Across the program, teachers show an interest in mathematics and have high but realistic expectations and clear goals and an ability to communicate these clearly. Engages and cultivates children's interests and motivation to learn mathematics.
- Uses classroom-based formative assessment to make adjustments to teachers' instructional practices so that they better understand children's learning needs and facilitate their mathematical development.

## SUMMARY

Young children in early childhood classrooms do not spend much time engaged in mathematics content. Time spent on mathematics increases somewhat in kindergarten. The time that is spent engaged in mathematics is typically of low instructional quality (La Paro et al., 2008) and, more often than not, is conducted as a part of whole-class activities or embedded in center time or free play. Early childhood teachers rarely teach mathematics in small groups. They report that they are much more likely to use embedded mathematical strategies or do the calendar, which they consider to be teaching mathematics, rather than provide experiences with a primary focus on mathematics in which they scaffold children's progress along important mathematics teaching-learning paths. Formative assessment has considerable potential to provide teachers with meaningful methods for assessing children's mathematical knowledge and improving their instruction to meet children's needs.

On a more optimistic note, the early childhood education field is actively working to improve the teaching of mathematics. The National Association for the Education of Young Children and the National Council of Teachers of Mathematics (2002) issued a joint position statement calling for more and better mathematics curriculum and teaching in early childhood programs. Head Start has launched a new mathematics professional development initiative. In addition, the reauthorization of Head Start calls for research-based curriculum and practices. The time is right to enhance young children's mathematics experiences not only to improve school readiness, but also to lay a foundation for lifelong understanding and enjoyment of mathematics. The challenges as well as the advances in research and policies aimed at improving young children's mathematics learning speak to the need for extensive professional development around young children's mathematics—the focus of the next chapter.

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# 8

# The Early Childhood Workforce and Its Professional Development

It is often said that the quality of any institution is based on the quality of its personnel. This is especially true of the array of institutions and programs that serve young children. The adults—early childhood teachers—who directly support the academic/intellectual, social, emotional, and physical development of preschoolers in the United States are pivotal to children's short-term development and their long-term outcomes. Early childhood teachers are an essential ingredient in achieving the intentions of this report, notably improved attention to and outcomes in early childhood mathematics. For these reasons, we address the early childhood workforce and their professional development.

Terminology regarding the early childhood workforce is often used inconsistently (Kagan, Kauerz, and Tarrant, 2008). In this discussion, the following terms are used:

- Early childhood education (ECE) teachers or the ECE teaching workforce includes all personnel whose primary role is to provide direct instructional services for young children. Included in this category are lead teachers, assistant teachers, aides, and family child care (FCC) providers.
- ECE workforce includes those who carry out both instructional and noninstructional roles in early childhood education settings. Thus, the term *workforce* is an inclusive one that embraces teachers, others who work in early childhood education settings and whose primary responsibility is not instructional (e.g., administrators), and indi-

viduals who work in settings that support early childhood education (e.g., resource and referral coordinators).

In this chapter, we begin by discussing the nature of the current early childhood workforce. We first present information on this workforce in general, discussing characteristics about the teachers themselves, including age, gender, ethnicity, educational experience, and background and key variables that influence their work, including compensation, turnover, work settings, and beliefs. We then turn to a more specific discussion of the early childhood workforce from a mathematical perspective. In the second section, we discuss the nature of the professional development of the workforce, first addressing the professional development of early childhood teachers in general and then turning to mathematics-specific professional development.

## BACKGROUND ON THE WORKFORCE

### Demographic Characteristics

Over 50 percent of U.S. families with children under the age of 5 rely on nonparental care (Chernoff et al., 2007), and thus the ECE workforce is responsible for the care and education of large numbers of the nation's young children. The early childhood workforce is fairly large, comprising 2.3 million individuals (Burton et al., 2002) and dispersed: About 24 percent work in centers, 28 percent in family child care, and 48 percent in informal family, friend, and neighbor (FFN) settings (Burton et al., 2002). It is important to note that although most early childhood care providers work in FFN settings the majority of children attend center-based programs in which the child-to-teacher ratio is higher (Burton et al., 2002). The focus of this section is on teachers in center-based and FCC settings.

According to national averages, the ECE teaching workforce is mainly comprised of white women in their late 30s and 40s (Saluja, Early, and Clifford, 2002); however, race/ethnicity varies across state and program type (see Table 8-1 for a breakdown of early childhood educators by race/ethnicity). For example, the Head Start and home-based early child care teaching workforce is more ethnically balanced than the prekindergarten workforce (Early et al., 2005; Hart and Schumacher, 2005). In addition, in certain parts of the country, for example, Alameda County, California, the early childhood education and care workforce is more ethnically diverse. Three-quarters of the family child care centers there are staffed by women of color (Whitebook and Bellm, 2004). Also, in the population as a whole, there are increasingly more children who speak English as a second language (as cited in Hart and Schumacher, 2005), and thus there is a need for

**TABLE 8-1** Early Childhood Educators by Race/Ethnicity (percentage)

| | Race/Ethnicity | | | | |
|---|---|---|---|---|---|
| Program Type | White | Black | Latino | Asian | Other |
| Prekindergarten | 64 | 13 | 15 | 2 | 8 |
| Head Start | 36 | 28 | 24 | 2 | — |
| Family child care | 20 | 27 | 26 | 23 | — |

NOTE: Family child care from Layzer and Goodson (2006); Head Start from Hart and Schumacher (2005); prekindergarten by Early et al. (2005). Prekindergarten refers to school or center-based programs that serve 4-year-olds, have an explicit purpose of improving school readiness, and are funded fully or partially by the state.
SOURCE: Kagan et al. (2008).

a more linguistically and ethnically/racially diverse ECE workforce (Howes, James, and Ritchie, 2003).

## Educational Experience and Background

ECE teachers are a diverse group of individuals, with some having formal education and holding degrees from institutions of higher education or community colleges and others receiving credentials of competence offered by the profession. Some have only very limited training that is delivered on the job. Not surprisingly, the amount of formal education and credentials varies by program type; prekindergarten programs generally have the highest percentage of teachers with degrees, while home-based or FCC providers have the lowest levels of formal education (Kagan et al., 2008). Table 8-2 shows the breakdown of percentages by program type. The specific nature of these variations and their relationship to teaching quality and effectiveness are elaborated in the section on the professional development of the workforce.

## Compensation

Compensation, defined as a combination of annual salary or hourly wages and benefits (e.g., health insurance, paid vacation, sick leave, retirement plan), is quite low for some segments of the early childhood workforce. In the United States, the average annual salary for preschool teachers, one group of early childhood educators, is $25,800; for child care workers including FCC providers, it is $19,670 (Bureau of Labor Statistics, 2007); and for Head Start teachers, it is $24,608 (Hamm, 2006). Distinctions exist in the salaries of individuals according to the settings in which they work.

A national survey conducted by the Bureau of Labor Statistics (BLS)

**TABLE 8-2** Level of Formal Education and Training of Early Childhood Education and Care Workforce (percentage)

| | Level of Education and/or Training | | | | |
|---|---|---|---|---|---|
| Program Type | High School or Less | Associate's Degree/Some College | B.A. or More | Child Development Associate | State License or Endorsement |
| Prekindergarten | 13 | 14 | 73 | 23 | 57 |
| Head Start | 31 | 33 | 36 | 22 | N/A |
| Center-based | 30 | 41 | 30 | 18 | 44 |
| Home-based (FCC) | 56 | 32 | 11 | 3 | 7 |

NOTE: Prekindergarten data from Gilliam and Marchesseault (2005); Head Start data from Hamm (2006); center-based data (includes teachers and directors) and home-based data on formal education are from Herzenberg, Price, and Bradley (2005), center-based and family child care data on credentials from Saluja, Early, and Clifford (2002).
SOURCE: Kagan et al. (2008).

characterizes the field in terms of two categories: child care workers and preschool teachers. Child care workers are adults who primarily perform such duties as feeding, dressing, and overseeing the play of children, and preschool teachers provide a more educational experience for the children in their care. Using these definitions, child care workers were near the bottom of the compensation ladder, earning more than only 22 of the 820 occupations that were assessed by BLS in 2004—their earned incomes were within 5 percent of short-order cooks and parking lot attendants and considerably less than preschool teachers (Center for the Child Care Workforce, 2006).

While there is little dispute regarding the wide salary differences that exist among early childhood teachers, most observers suggest that compensation differs according to the particular type of program and its attendant required credentials. For example, preschool teachers who work in settings in which teacher certification is required command higher salaries and compensation packages than teachers who work in settings in which lower levels or no certification is required. Setting and its attendant requirements are not the only variable that influences compensation; it also varies by geographic region, with early childhood educators in southern states receiving the lowest levels of compensation (Center for the Child Care Workforce, 2006).

In addition to low wages, many ECE teachers do not receive health insurance benefits from their employers. Specifically, 28 percent of center-based early childhood educators received health insurance benefits from their employer between 2002 and 2004, and 21 percent of ECE teachers reported that they had no health insurance during this time (Herzenberg, Price, and Bradley, 2005).[1] Lack of health insurance is a significant issue; it may influence early childhood educators' interactions at work, their overall financial status, and thus their ability to remain in the field over time, fueling heavy personnel turnover rates.

### Stability and Turnover

The turnover of early childhood teachers is quite high in some settings. A longitudinal study in California by Whitebook and colleagues (2001) found that 76 percent of the teachers employed by centers in 1996 and 82 percent of teachers employed by centers in 1994 had left these jobs by 2000 (Whitebook et al., 2001). Such high turnover rates have often been associated with low compensation (Whitebook and Sakai, 2003). For example, Whitebook and colleagues (2001) found that early childhood educators receiving higher than average wages were more likely to remain in their jobs, and those who left the field were more likely to go to higher paying jobs. Wage levels are often directly associated with the program type or sector in which the individual is employed.

One national study showed that, on average, center-based teachers were in their current programs for 6.8 years, teachers in programs in public schools and religious settings were working in their programs for 7.8 years, and teachers in for-profit centers were in their programs for 5.6 years (Saluja, Early, and Clifford, 2002). Confirming these data, a five-state study found that publicly operated prekindergarten programs were found to have lower turnover rates than privately operated programs (Bellm et al., 2002). On average, publicly operated prekindergarten programs offered higher wages than privately operated programs (Gilliam and Marchesseault, 2005), which may be an explanation for the difference in turnover. Moreover, when ECE teachers are compared with K-12 teachers, the salaries for K-12 teachers are significantly higher (Kagan et al., 2008) and turnover is lower (Provasnik and Dorfman, 2005).

Teacher turnover is relevant for all students, and it is particularly important for young children because of the impact on their development and learning. High levels of unpredictable turnover have been linked to poorer developmental outcomes for children, as well as to lower quality service (Helburn, 1995; Howes and Hamilton, 1993; Howes, Phillips, and Whitebook, 1992; Phillips et al., 2001; Whitebook, Sakai, and Howes, 1997, as cited in Kagan et al., 2008).

[1] Although health insurance data were not collected for the remaining 51 percent of early childhood teachers, some probably received health insurance through a spouse when a spouse was present and had health coverage, purchased it privately, or purchased it through Medicaid (Mark Price, personal communication, January 12, 2009).

For example, one study, *The Cost, Quality and Child Outcomes in Child Care Centers* (Cost, Quality, and Child Outcomes Study Team, 1995), found that higher quality programs, in which children demonstrated more advanced language and premathematical skills, were associated with lower turnover rates. Furthermore, the children showed better nonacademic outcomes than did children in high turnover programs. The children had more positive self-concepts, better relations with their teachers, and demonstrated more advanced social behavior and more positive attitudes toward child care situations. The effects of program quality are obvious for children of all socioeconomic backgrounds, but children from low-income backgrounds are especially influenced by the quality (or lack thereof) of their child care (Helburn, 1995). Finally, turnover is important in early childhood settings because many ECE teachers who leave the field are replaced by individuals with less training and experience; thus, turnover has long-term effects on teacher and program quality.

While high turnover is often associated with instability and poorer outcomes for children, it is important to note that turnover is not always a negative factor (Kagan et al., 2008; Whitebook and Sakai, 2003). It may be beneficial when individuals who enter the early childhood education field and find that it is a poor fit for their skills or occupational goals leave (Whitebook and Sakai, 2003). Also, many studies do not distinguish between job turnover, which is defined as the rate at which teachers leave programs to take new positions in the early childhood education field, and occupational turnover, which is defined as the rate at which teachers leave programs to retire or enter a new field of work (Kagan et al., 2008). Clearly, more data are needed on turnover in the early childhood field.

## The Work Environment

In any industry, the environment in which one works is likely to influence one's on-the-job attitude and performance. Work environment is defined as the physical setting, the reward system, clarity about expectations and roles, agency in decision making, supervisor support, and communication (Hatch, 2006; Stremmel, Benson, and Powell, 1993; Whitebook, Howes, and Phillips, 1990). While the measures of work environment vary for different studies, the research shows that the work environment of early childhood educators plays a role in teachers' quality and effectiveness (Kagan et al., 2008). For example, the Child Care Services Association (2003) found that 22 percent of preschool teachers throughout North Carolina planned to leave the field within three years, yet only half as many teachers who worked in supportive environments reported having the same plans. The supports that were presumably related to more positive work environments include: (1) orientation, (2) written job descriptions, (3) written personnel policies, (4) paid education and training expenses, (5) paid breaks, (6) compensatory time for training, and (7) paid preparation/planning time (Child Care Services Association, 2003). Not surprisingly, teachers were more likely to stay in their positions when they understood the responsibilities of their position and the expectations that their supervisors and colleagues had of them, and there were improvements in compensation.

Interestingly, improvements in the work environment have also been related to better psychological functioning, as defined by less emotional exhaustion (Stremmel, Benson, and Powell, 1993). Teachers are required to interact closely with children (Kagan et al., 2008); however, those who show lower levels of emotional well-being are less likely to spend time engaged with children (Hamre and Pianta, 2004). Children who have teachers who are less engaged may have fewer opportunities to learn from teacher-guided situations.

The supervision and leadership that are provided to ECE teachers also make a difference in the quality of the work environment and subsequently in teachers' quality and effectiveness (Kagan et al., 2008). Supervisors must have the management skills and leadership abilities necessary to support early childhood educators. Moreover, they must support teaching staff, but they also need support to continue to develop positive management styles and leadership abilities for themselves. Research by Jorde-Bloom and Sheerer (1992) suggests that professional development programs for supervisory staff improve the overall workplace climate and classroom quality. Fostering these skills in the supervisory staff has the potential to positively impact children's learning through classroom quality and workplace climate.

## Teachers' Beliefs About Early Childhood Education

Like the variables discussed above, early childhood educators' beliefs and values are important to understand. Teachers' beliefs and values about teaching and learning not only shape classroom practices (Fang, 1996; Kagan, 1992; Stipek et al., 2001), but also serve as a filter through which meaning is derived. As such, values and beliefs have a powerful influence on educational change and innovation. Attempting changes in pedagogy without considering teachers' pedagogical beliefs and values about education may lead to resistance in implementation of a new practice if teachers do not agree with the underlying educational value (Lee and Ginsburg, 2007b; Ryan, 2004). Thus, any effort to change educators' classroom practices must include consideration of how those teachers view their roles, the children they teach, and the purpose of the setting in which their interactions take place.

Historically, the field of early childhood education has placed great emphasis on supporting children's social and emotional development, with somewhat less of an emphasis on academic learning as an outcome of experiences in ECE settings (Kowalski, Pretti-Frontczak, and Johnson, 2001). Academic subjects were believed to be less important at this age because young children should investigate and explore their interests so that they develop a love of learning (Lee, 2006). However, in the past decade, there has been a groundswell of focus on academic learning as a legitimate, desirable, and appropriate outcome of preschool enrollment (particularly in publicly funded programs, such as Head Start or state-funded prekindergarten). This movement, challenging teachers' conventional beliefs, has created pressure on early childhood education systems and personnel to address academic achievement more focally and intentionally.

Preschool programs that provide children with social, emotional, physical, and academic learning opportunities are ideal learning environments. Educating the "whole child," including social and emotional development, and providing preschool children with opportunities to engage in developmentally appropriate mathematics[2] is essential to children's immediate and later school success (Duncan et al., 2007; National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002; National Mathematics Advisory Panel, 2008). It is important to note, in this regard, that the third edition of the National Association for the Education of Young Children's (NAEYC) (2009) guidelines for developmentally appropriate practice emphasize that pre-academic and cognitive skills, including those in mathematics, are essential to developmentally appropriate instruction.

Teachers' educational goals and pedagogical beliefs are also influenced by the backgrounds and characteristics of the children themselves. For example, socioeconomic status (SES) has been found to be related to ECE teachers' instructional practices (Lee and Ginsburg, 2007a, 2007b; Stipek and Byler, 1997). Children from low-SES backgrounds are often behind their more affluent peers in mathematics achievement as early as kindergarten (Clements, Sarama, and Gerber, 2005; Denton and West, 2002; Griffin and Case, 1997; Jordan, Huttenlocher, and Levine, 1994; Lee and Burkam, 2002; National Research Council, 2001b; Saxe, Guberman, and Gearhart, 1987; Starkey and Klein, 1992, 2008; Stipek and Ryan, 1997), and awareness of this disparity may influence teachers' educational goals, beliefs, and instructional practices with children from economically disadvantaged backgrounds. Children coming from low-SES homes, although increasingly enrolled in and benefiting from early childhood education, also require more intensive and appropriate educational interventions in

[2]Developmentally appropriate mathematics includes a child-centered and positive nonevaluative mathematics environment, developmentally appropriate mathematics activities and manipulatives, and authentic mathematics assessment (as cited in Lee, 2005).

order to perform at levels consistent with their more advantaged and skilled peers (Hamre and Pianta, 2005). In short, less advantaged children need programs that actually accelerate learning if they are to enter school not behind at the start. However, preschool and kindergarten teachers of low-SES children rate memorizing facts and rote tasks (procedural knowledge) as more important educational goals than problem solving and tasks involving reasoning (conceptual knowledge), and they tend to agree with a more basic skills teaching orientation than teachers of middle-SES children (Stipek and Byler, 1997).

## THE EARLY CHILDHOOD WORKFORCE AND MATHEMATICS

The teaching of mathematics has been considered a part of the early childhood educators' portfolio, along with many other developmental and disciplinary domains (e.g., social and emotional development, physical development, literacy, social studies) that they must address. To understand how the early childhood workforce currently views and addresses mathematics, we examine early childhood teachers' beliefs about mathematics, their mathematics knowledge, and how these beliefs and knowledge actually impact what they do in the classroom.

### Teachers' Values and Beliefs About Mathematics Education in Early Childhood

Generally, early childhood teachers believe that social-emotional and physical development are more important to young children's development and learning than academic activities, including mathematics (Ginsburg et al., 2006a; Lin, Lawrence, and Gorrell, 2003; Piotrkowski, Botsko, and Matthews, 2001). In a recent review of the research, Ginsburg and colleagues (2008) found that preschool teachers report social-emotional development, literacy, and then mathematics—in that order—as important educational goals for young children to achieve.

A second set of beliefs focuses on the nature of mathematics instruction. Early childhood educators generally believe that mathematics education should focus on numeracy and arithmetic through some direct instruction (Lee and Ginsburg, 2007b). They also tend to believe that young children should engage in games and other activities in which mathematics learning is fun and involves interesting toys or materials in small groups and that mathematics learning should not be highly demanding, nor should it be pushed on young children before they are "ready" (Lee and Ginsburg, 2007b).

Finally, a third set of beliefs regarding instructional practice is driven by children's characteristics, particularly SES. Research examining ECE teachers' beliefs about instructional practices as a function of SES is a nascent area; however, recent studies shed light on how this characteristic shapes beliefs about teaching practices. For example, one study showed that early childhood teachers of children from low-SES backgrounds believed that mathematics instruction was an excellent way of preparing children for kindergarten and that children should engage in mathematics activities, even if they initially showed little or no interest (Lee and Ginsburg, 2007b). Conversely, teachers of middle-SES prekindergarten children were more likely to state that, instead of having an academic focus, prekindergarten education should be child-centered and child-initiated and encourage children's social-emotional development (Lee, 2006; Lee and Ginsburg, 2007b). In large part, this belief was in response to the notion that middle-SES parents put significant academic pressure on their children at home (Lee and Ginsburg, 2007b). It should be noted that, while SES-related differences were found in both early childhood educators' beliefs about instructional practices and their educational goals, the field of early childhood education tends to stress social-emotional development rather than academic subjects.

There are multiple reasons that early childhood teachers may not be inclined to focus on mathematics. One explanation is related to ECE policies that put a premium on early literacy at the expense of other subject areas (which is discussed later in this chapter). Another reason stems from the education and training many ECE teachers receive, which has historically placed more emphasis on social-emotional development. Specifically, some researchers suggest that this focus on social-emotional development is rooted in misconceptions or limited knowledge of the young children's developmental capacities. For example, early childhood educators' beliefs that young children are too cognitively immature for mathematics learning may be based on Piagetian theory, which states that young children in the preoperational stage (ages 2 to 6) are not likely to use or understand abstract ideas to make sense of their experiences (Ginsburg, Pappas, and Seo, 2001; Lee and Ginsburg, 2007b). However, Gelman and Gallistel (1986) found that young children do think abstractly in regard to counting objects (e.g., the abstraction principle: any discrete object can be counted, from stones to unicorns). Heuvel-Panhuizen (1990) found that early childhood educators significantly underestimated 6-year-olds' mathematical capability. Specifically, teachers, counselors, and teacher trainers held significantly lower expectations for children's knowledge of symbols, the counting sequence, and adding and subtracting than what child outcomes showed (Heuvel-Panhuizen, 1990).

Others suggest that such beliefs may rest on mistaken assumptions that young children are neither interested in, nor capable of, learning mathematics. In fact, young children from birth to age 5 have informal mathematics knowledge (Clements and Sarama, 2007b; Ginsburg et al., 2006b) and, given developmentally appropriate experiences, enjoy mathematics learning (Gelman, 1980; Irwin and Burgham, 1992). This informal knowledge includes the ideas of more and less, shape, space, pattern, as well as number and operations, and several other important areas (Gelman, 2000).

Moreover, some researchers suggest that teachers' fundamental knowledge about mathematics and mathematics instruction may be limited. For example, most teachers in the United States believe that mathematics is a static body of knowledge that mainly involves manipulating rules and procedures. From this point of view, the main objective in mathematics is to learn about discrete knowledge and arrive at the correct answer (Ball, 1991). Little thought is given to mathematics as a problem-solving process; rather, the outcome (i.e., getting the correct answer) is seen as the most important part of learning mathematics (Thompson, 1992). This belief is reflected clearly in early education instruction that is rote and feedback processes that focus solely on right and wrong answers (Pianta et al., 2005). Traditionally, early childhood educators have been taught that mathematics is a subject that requires the use of instructional practices that are developmentally inappropriate for young children (Balfanz, 1999). In short, it is often the case that preschool teachers believe the content of meaningful mathematics is too difficult for themselves as well as for their students.

### The Impact of Teachers' Beliefs and Knowledge on Instruction

Given these beliefs and knowledge, we examine how early childhood teachers beliefs and understandings about mathematics impact mathematics instruction. Early childhood educators' beliefs are clearly associated with their teaching practices (Charlesworth et al., 1991, 1993; Pianta et al., 2005; Stipek and Byler, 1997; Stipek et al., 2001). Pianta and colleagues (2005), for example, in their multistate study, found that, even after adjusting for teachers' experience or degree status and program factors, such as teacher-student ratio or full-day/part-day classes, prekindergarten teachers' beliefs about children were the factor most related to global classroom quality as measured by the Early Childhood Environmental Rating Scale-Revised (ECERS-R) and the Classroom Assessment Scoring System (CLASS, which reported on two dimensions, instructional climate and emotional climate).

What instructional practices are teachers engaged in? Not only is emphasis on social and emotional development in early childhood settings a belief, but also it is borne out in reality. Pianta and La Paro (2003), characterizing findings from standardized observations in more than a thousand early education settings, note that many early childhood classrooms are socially positive yet instructionally passive. Generally speaking and not surprisingly, preschool teachers spend less instructional time on mathematics than they do on literacy (Clements and Sarama, 2007b; Early et al., 2005; Layzer, Goodson, and Moss, 1993), a finding not much different from what is observed in the early elementary grades (National Institute of Child Health and Human Development Network Early Child Care Research Network, 2002, 2005; and see Chapter 7 of this report for further discussion of instruction).

Early childhood educators' pedagogical beliefs direct and constrain their instructional practices, which subsequently shape children's academic and social environments. When addressed, early childhood mathematics is usually constrained to basic ideas in number and operations, such as 1-to-1 correspondence, simple addition and subtraction, and number symbols or numerals (Lee and Ginsburg, 2007b). Geometry and measurement are noted less frequently (Clements, 2004). In addition to rote memorization and basic skills, such as memorizing the first 10 or so counting words, young children are capable of understanding more sophisticated mathematical concepts, such as cardinality. The content of young children's mathematics can be both deep and broad, and, when provided with engaging and developmentally appropriate mathematics activities, their mathematics knowledge flourishes. Yet these research findings are largely not represented in practice.

## PROFESSIONAL DEVELOPMENT OF THE WORKFORCE

The professional development of early childhood teachers is nuanced and complicated. We begin our discussion with an overview of professional development, looking at the nature of quality professional development and the context for the delivery of professional development, both in-service and pre-service. We address the impact of professional development on teachers' performance generally. We then turn to a discussion of the professional development for teaching mathematics to young children, addressing the need for mathematics preparation; mathematics content and teacher preparation; efforts at in-service mathematics support, including the outcomes of such support; and efforts at pre-service preparation for teachers in mathematics.

To aid the discussion, we define key terms as follows:

- Professional development: an umbrella term that refers to both formal education and training.
- Formal education: refers to the amount of credit-bearing coursework a teacher has completed at an accredited institution, including two- or four-year colleges and universities.
- Training: refers to educational activities that take place outside the formal education process. Such efforts may include coaching, mentoring, and workshops.

- Pre-service education: refers to the formal education and training that one receives prior to having formal responsibility for a group of children.
- In-service education: refers to the formal education and training that one may receive while having formal responsibility for a group of children.
- Credentialing: refers to the process of demonstrating and receiving formal recognition from an organization for achieving a predefined level of expertise in education.

### The Nature and Quality of Successful Professional Development Efforts

An examination of the literature from the fields of elementary education, early childhood education, and early childhood mathematics education reveals some common principles that characterize high-quality professional development experiences. Research indicates that professional development efforts are most successful when they are focused on producing long-lasting change, longer in duration, focused on content knowledge rather than teaching strategies alone, involve active learning, and are part of a coherent set of professional development experiences (Birman et al., 2000). According to Clements (2004, p. 65), six themes related to professional development emerge from reviews of the research:

1. Professional development should be standards-based, ongoing, and embedded in the job (i.e., practical, concrete, immediate, gradually connecting research and theory).
2. Teachers must have time to learn and work with colleagues, especially a consistent group.
3. Teachers should be provided with stable, high-quality sources of professional development that includes observation, experimentation, and mentoring, with plenty of time for reflection.
4. Professional development experiences should be grounded in a sound theoretical and philosophical base and structured as a coherent and systematic program.
5. Professional development experiences should respond to each individual's background, experiences, and current context or role.
6. Professional development experiences should address mathematics knowledge as well as mathematics education. It should be grounded in particular curriculum materials that focus on children's mathematical thinking and learning, including learning trajectories.

These principles pertain to professional development of all types, including pre-service education and in-service professional development, because they reflect sound practices in adult learning, as well as data on the practices that ultimately lead to improved outcomes in the classroom. While their application may be tailored to a particular cohort or setting, these principles should guide development of personnel preparation programs in early childhood mathematics. The following section describes the overall context of professional development as it pertains to the early childhood workforce.

## The Context for the Delivery of Professional Development

The professional development of early childhood and elementary school teachers happens both prior to teachers' assuming classroom responsibilities through pre-service training and while they are teaching through in-service training. Unlike the professional development of most elementary school teachers, which occurs formally prior to their becoming teachers, many early childhood educators receive the majority of their professional development while they are already working. Moreover, for most elementary school teachers, there is a common entry floor into the profession, typically consisting of the achievement of a B.A. or B.S. degree and the successful completion of the Praxis exams. No such common entry floor for early educators exists. In fact, the range of entry-level requirements for early educators varies from the holding of a health clearance certificate and being 18 years of age to meeting requirements equivalent to those for elementary school teachers.

Although efforts are under way to elevate the quality and consistency of entry-level requirements and professional development opportunities for early educators, abundant variations of requirements and professional development delivery mechanisms exist. Moreover, there is considerable variation in what is required of, and offered to, early educators as professional development, depending on the program sponsor and funding stream or the state or locality in which the early educator practices her work. Complicating this picture, new public policies, some at the federal level but mostly at the state level, mean that early educator teacher and professional development requirements are in constant flux. In this section we elaborate on the unique sociopolitical context in which professional development for early educators exists.

On one hand, the news is quite promising. There is a broad consensus emerging that the professional development of the early childhood workforce is a priority (Kagan et al., 2008). Increasingly, policy makers and the public are recognizing the importance of early experiences on children's brain development, success in school, and general well-being (Center on the Developing Child at Harvard University, 2007; Martinez-Beck and Zaslow, 2006; National Research Council, 2000, 2001a). In addition, increasing attention has been given to closing the achievement gap between children from diverse economic and racial/ethnic backgrounds that has been documented prior to the start of school (Clements, Sarama, and Gerber, 2005; Starkey and Klein, 2008). Mounting evidence of the central role that teachers play in supporting children's development and learning through relationships and teaching interactions in general has added to a sense of urgency to improve the quality of professional development (National Research Council, 2001b).

To that end, a number of federal efforts have supported professional development. The Head Start Program, continuing its historical commitment to professional development, has expanded these efforts by calling for higher professional requirements for its teachers. Good Start, Grow Smart, a presidential initiative launched during the Bush administration, specifically charges all states with developing plans to offer education and training to preschool and child care personnel to receive Child Care Development Fund dollars. In addition, Title II of the No Child Left Behind Act provides competitive grants for the creation of training and educational opportunities for early educators through the Early Childhood Educator Professional Development Program.

At the state level, qualifications for teachers are being increased, as are support and incentives for teachers to seek additional professional development (Tout, Zaslow, and Berry, 2006). The creation of professional development systems and quality rating systems are now abundant nationally and are driving reform in pre-service and in-service education for early educators (Kagan et al., 2008). These changes and initiatives are occurring in a broader climate of increased accountability and standards in education (Kagan et al., 2008), further underscoring the need to provide the early childhood workforce with the knowledge and skills they will need to meet standards for early mathematics learning.

***Access.*** Not only have mandates for degrees expanded, but access to higher education has also expanded in many states. Scholarship programs, such as the Teacher Education Assistance for College and Higher Education Grant Program, online degree programs at both the associate and baccalaureate levels, better opportunities for working professionals to link or articulate their community-based training, Child Development Associate (CDA) programs, and other degree programs are all having an influence on the ability of early childhood educators to enter the higher education system and to convert their prior professional development into academic credit.

***The landscape of early childhood teacher education programs in general.*** An overview of the general landscape of early childhood teacher education provides a context for considering how ECE teachers are, and might be, prepared for their responsibilities in the domain of mathematics. According to estimates based on data collected in 2004, approximately

1,350 institutions of higher education offer some kind of degree program in early childhood education (Maxwell, Lim, and Early, 2006). Of these, roughly 44 percent offer a bachelor's and/or graduate degree and 56 percent offer an associate's degree, with some institutions offering both. Graduation rates in these programs produce at least 40,000 early childhood teachers per year.

***Associate degree programs.*** There are more than 750 early childhood associate degree programs in the United States (Maxwell, Lim, and Early, 2006). Most are located in community colleges, although some are under the umbrella of a university. In the early childhood degree program (sometimes called child development), a major influence on course offerings is whether the focus is on transfer to local baccalaureate programs in early childhood or elementary education (transfer programs) or whether students are primarily being prepared for work in child care, Head Start, and other settings immediately upon graduation (terminal programs). Although national organizations discourage classifying associate programs as transfer or terminal, in reality many still fall into these categories. Programs primarily aimed at transfer often have very few courses in early childhood curriculum and methods, aiming mainly at giving students a general education foundation with transfer potential. Programs with greater emphasis on immediate career opportunities include many more child development/ECE courses and field experiences.

***Bachelor's degree programs.*** Like associate degree programs, bachelor's degree programs that prepare future early childhood educators are diverse. Some of this diversity derives from state teacher certification categories, which for most programs serve to define the scope of their efforts. For example, some states define early childhood for licensure as birth to age 8; others birth to age 5; others ages 3 to 8; others preschool to Grade 2, and so on.

Programs' identities and the organizational features of the different higher education institutions in which these programs are situated also play a role in creating program diversity. For example, baccalaureate-level early childhood departments or programs may be part of a school or college of education, or the program may be in a different college entirely—for example, a college of human development or a child and family studies department. These institutional arrangements, along with state requirements, may influence what is expected of students in all areas, including mathematics.

On the other hand, despite these promising developments, the overall early childhood educator professional development context is hampered by intransigent workforce challenges. First, given the salary and compensation limitations of the field, those who have achieved professional degrees and teacher certification are often not attracted to early education. In an effort to remedy this situation, some new programs are compensating qualified early education teachers at rates comparable to elementary school teachers. Second, the rampant turnover rate in the field cannot be denied, and departing early educators are being replaced with individuals who are less qualified, making the need for professional development even more important. Third, there are serious questions regarding the quality of the professional development content itself. There are barely a handful of certifications for individuals who provide early childhood mentoring, coaching, or professional development. The few states that do have such credentials have remarkably low bars for those who deliver in-service professional development. Compounding these contextual challenges, there are few consistent delivery mechanisms, except institutions of higher education, that deliver high-quality early educator professional development. Finally, for those wishing to avail themselves of professional development experiences, either out of desire or mandate, there are serious issues of quality of educational opportunity and inequity in access to training. As this review suggests, the context for the professional development of early educators is complex.

### The Impact of General Professional Development on Teacher Quality and Effectiveness

What is the role of a teacher's education in her teaching? Several studies have found that the level and nature of early educators' formal education is related to the overall quality of their teaching (e.g., Barnett, 2003; Tout, Zaslow, and Berry, 2006; Whitebook, 2003). Teachers with higher levels of formal education have also been linked to higher quality programs and more positive teacher-child interactions (Howes, 1997; Tout, Zaslou, and Berry, 2006). However, more recent, multistate studies have found that the evidence on formal education and its link to teacher effectiveness is questionable (Early et al., 2006, 2007), with teacher knowledge, attitudes, and specific teaching practices more predictive of child outcomes. While Early and colleagues (2007) did not find a significant relationship between teachers' level of education and young children's academic outcomes, they suggest that their findings should not dissuade early childhood educators from pursuing postsecondary education. Early and colleagues (2007) did not examine the course content or rigor of early childhood education programs, which may be related to teachers' knowledge, skills, and behaviors. Thus, available data do not provide a comprehensive investigation of the host of variables that are likely to be related to teacher quality or effectiveness (Early et al., 2007; Kagan et al., 2008).

***Questions about degrees and child outcomes.*** Much attention has been focused on several recent studies that have renewed the controversy over the value of degrees as guarantees of quality in early childhood teaching or of positive child outcomes (Early et al., 2006, 2007). Although the results need to be interpreted in light of the limited measures available and other constraints (as discussed earlier in this chapter), these studies call into question the assumption that having a degree—especially an early childhood degree—must produce better developmental and learning outcomes for children. An important next step is to look carefully at the quality of early childhood degree programs (Hyson, Tomlinson, and Morris, 2008).

Another challenge in discerning the relationship between formal education and teacher outcomes is definitional in nature. Kagan et al. (2008) note that, often in the literature, the term "teacher quality" refers to the positive actions and behaviors of teachers, particularly with regard to their interactions with young children. To distinguish this definition from studies that focus on actual child outcomes, Kagan et al. (2008) use the term "teacher effectiveness" to refer to the impact of teachers' actions and behaviors on the accomplishments of the children they teach (Kagan et al., 2008). Given these distinctions, there is some evidence of a relationship between teacher effectiveness and formal education for FCC providers. Clarke-Stewart and colleagues (2002) found children in the care of providers who had not attended college scored lower on cognitive tests than children in the care of providers who had attended college. One explanation for these findings is that FCC homes usually have only one adult present to care for children, and this adult has a significant amount of influence on children's learning and development. Center-based settings, in contrast, have many adults with whom children interact and thus no single teacher will have as much influence on them. To gain a clearer understanding of teacher quality and effectiveness, it will be important to examine teacher preparation and support in preparation programs (Early et al., 2007; Kagan et al., 2008).

Although some early childhood educators receive a formal education to prepare to work with young children, others obtain preparation through general training. It is important to note that training can take place prior to their entering the classroom, but it often occurs after teachers have begun teaching. General training, defined as educational activities that take place outside the formal education system (Kagan et al., 2008), has also been found to impact teacher quality and the quality of classroom environments (Ghazvini and Mullis, 2002; Honig and Hirallal, 1998; Tout, Zaslow, and Berry, 2006). For example, Honig and Hirallal (1998) found that training, independent of education and experience, had a large impact on the quality of services that teachers provided (e.g., positive language interactions, greater support for concept learning). In addition to research linking training to teacher quality, one study suggests that training is linked to teacher effectiveness. Burchinal and colleagues (2002) found that teachers' attendance in workshops predicted global quality and children's receptive language.

In addition to the research examining the relationship between general training and teacher quality, several studies have shown that overall program quality improves when early childhood teachers have specialized training or education in child development (Blau, 2000; Phillips et al., 2001; Tout and Zaslow, 2004; Tout, Zaslow, and Berry, 2006). Furthermore, specialized formal education, defined by an emphasis on child development and early childhood education, has also been linked to improvements in teacher quality—specifically, that teachers who had more child development education were more sensitive, less harsh, and more responsive to children (Howes, 1997).

Separate studies have been conducted on FCC settings and the impact of training. Generally, training for FCC providers has shown similar trends to those found for center-based providers. That is, this training is related to higher scores on measures of global environmental classroom quality (Burchinal et al., 2002; Clarke-Stewart et al., 2002; Norris, 2001). Furthermore, providers who received more training were more likely to offer a variety of activities and toys for children, balance their time indoors and outdoors, and actively interact with them (Norris, 2001). Training was also linked to teacher effectiveness in FCC settings. Specifically, children in the care of individuals who had participated in training in the past year scored higher on cognitive tests (Clarke-Stewart et al., 2002).

Overall, these findings indicate that formal education and training generally have a positive impact on teacher quality and effectiveness. However, the studies on which these conclusions are based are largely correlational, preventing the ability to draw conclusions about a causal relationship between training and/or formal education and teacher quality and effectiveness. Furthermore, questions regarding the impact of certain types of training, hourly requirements for training, or specific formats or content are essentially not addressed (Tout, Zaslow, and Berry, 2006). Despite these limitations, the data indicate that, in general, teacher quality and effectiveness are measurably better when teachers have higher levels of education and training, which in turns lends support for using these pre-service and in-service preparation systems as a means for improving practices and outcomes related to early childhood mathematics.

## Professional Development and Mathematics Education for Young Children

The Joint Position Statement of the NAEYC and the National Council of Teachers of Mathematics (NCTM) on Early Childhood Mathematics

(2002) names five critical areas of knowledge that early childhood teachers must have to be effective in teaching mathematics to young children: (1) knowledge of the mathematical content that they will be teaching, (2) knowledge of children's learning and development, (3) knowledge of effective mathematics pedagogy, (4) knowledge of effective means for assessing children's development and learning, and (5) knowledge of the resources and tools available for teaching early childhood mathematics. In addition to acquiring these areas of knowledge, teachers also need to have a positive attitude toward mathematics (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002), believe that young children are competent mathematics learners, and believe that mathematics is appropriate in the early childhood classroom (Ginsburg et al., 2006a; Lee and Ginsburg, 2007b). Themes related to the need for, and the nature of, such preparation are discussed in the following sections.

Early childhood educators need preparation in mathematics for several reasons. Unlike their elementary school counterparts, most early childhood teachers, including those with degrees in early childhood education, have received no prior preparation in teaching mathematics (Copple, 2004; Ginsburg et al., 2006b) Therefore, virtually all early childhood teachers need professional development to build their knowledge and skills around mathematics. This is especially important in light of the recent attention that researchers, funding agencies, major early childhood professional organizations, and policy makers are focusing on targeting improved mathematics outcomes in early childhood, particularly for children from low-income backgrounds (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002; National Mathematics Advisory Panel, 2008). As stated by Copple (2004):

> Practically all teachers need to know more about mathematics—the nature of the beast—and how to work with children in mathematics. They need to know much more about what mathematics young children are interested in and capable of doing; many vastly underestimate the range of young children's interests and the extent of their capabilities. (pp. 86-87)

### *Mathematics Content and Early Childhood Teacher Preparation*

A good deal of the research in early childhood mathematics has focused on the content that is necessary to be taught in teacher preparation programs, including both in-service and pre-service programs. That is, this research has focused on (1) mathematics knowledge, (2) mathematics beliefs, and (3) children's mathematical development and curricula to support it.

***Mathematics knowledge.*** Virtually no empirical research exists directly examining teachers' mathematics knowledge (Ginsburg and Ertle, 2008;

National Mathematics Advisory Panel, 2008). However, Ginsburg and Ertle (2008) provide several key reasons that professional development should target teachers' mathematics knowledge. First, teachers need to understand the mathematics that children are learning and how they may be thinking. According to Ginsburg and Ertle (2008), "to understand . . . students' mathematical thinking and then build on it in a way that encourages continued enjoyment of the subject, the teacher must therefore understand the mathematics that the thinking involves" (p. 55).

Second, teachers will be more effective implementers of mathematics curricula, as recommended by NCTM and NAEYC, if they understand the mathematics well themselves. At the pre-service level in particular, this means that teachers may need coursework related to deeply understanding the important mathematical concepts of early childhood rather than simply general mathematics courses that might be appropriate for college students, such as calculus.

Third, teachers can take advantage of teachable moments in mathematics only if they carefully observe, accurately interpret, plan, and implement appropriate activities to further learning, all of which require deep mathematics knowledge. Given that, until recently, teachers may not have had to teach mathematics in early childhood settings, that few have received professional development in early childhood mathematics education, and that many early childhood educators have limited professional preparation in general, researchers and professional organizations have recommended that professional development address teachers' knowledge of mathematics (National Association for the Education of Young Children and National Council of Teachers of Mathematics, 2002).

***Mathematics beliefs.*** As noted earlier, teachers have quite strong beliefs about mathematics, with many feeling it lacks key significance in early childhood programs. Ginsburg and colleagues (2006a), in describing efforts to provide training to teachers using the curriculum, Big Math for Little Kids, stress the importance of directly addressing the emotionally charged beliefs that teachers may have around mathematics. In fact, many early childhood teachers report they are uncomfortable with mathematics (Copley, 1999) and identify it as their weakest subject (Schram et al., 1988). In the prekindergarten settings in which the Ginsburg et al. (2006b) study took place, there appeared to be more resistance to mathematics than is typically found in kindergarten and elementary school, in which mathematics has long been expected to be taught.

***Children's mathematical development and curriculum.*** Naturally, professional development in early childhood mathematics includes helping teachers learn about children's developmental progression in various areas of mathematics, the specific learning experiences they can plan, and the teaching strategies, materials, and supportive environment they can provide to promote mathematical development. A study with California elementary school teachers showed that those who received professional development in which teachers worked directly with curriculum materials associated with NCTM standards were more likely to report reform-oriented teaching practices in mathematics. Furthermore, results suggested that a professional development curriculum that overlaps with the curriculum of students improves instructional practices and student outcomes (Cohen and Hill, 2000).

In early childhood mathematics, few studies exist demonstrating the causal effects of professional development on children's outcomes. Nevertheless, two programs of research in early childhood mathematics have demonstrated a causal link between the delivery of professional development to implement a mathematics curriculum and positive child outcomes (Clements and Sarama, 2007a, 2008; Sarama et al., 2008). This research demonstrates the effectiveness of curriculum-based professional development methods at the early childhood level, which complements and extends the existing data on effective approaches at the elementary level (Cohen and Hill, 2000; Sarama and DiBiase, 2004). Because experimental research is quite limited in this area, no studies comparing alternative approaches to professional development (i.e., curriculum-based versus non-curriculum-based) have been conducted. However, there is a strong rationale for the use of a mathematics curriculum to provide young children with carefully sequenced mathematical experiences in the classroom. Thus, although additional research would broaden understanding of the best means for providing professional development in early childhood mathematics, the current curriculum-based research provides evidence to support the link between curriculum and professional development (Clements and Sarama, 2007a, 2008; Sarama et al., 2008).

### *In-Service Mathematics Support Efforts*

Research on early childhood mathematics has largely been focused on understanding children's mathematical development and the types of experiences that facilitate this learning. This work has also led to the development of an array of early childhood mathematics curricula. However, little research has been done to date on the best methods to prepare educators to support children's mathematical development or how to best provide training on mathematical curriculum implementation. As a result, questions about how to effectively scale up efforts to meet the needs of the early childhood workforce, as described in this chapter, have not yet been adequately addressed. The data that do exist can provide an example of effective practices and are presented below.

Research using the Technology-enhanced, Research-based, Instruction, Assessment, and professional Development (TRIAD) model (Sarama et al., 2008) provides the clearest evidence from the early childhood mathematics literature regarding specifically tested approaches to providing professional development to diverse groups of teachers from various types of programs serving diverse groups of children. TRIAD is a model for developing and scaling up a research-based curriculum. It is during the latter phases of this process that the focus of the research shifts from curriculum development and efficacy testing to the specific testing of the best methods for training and implementation, at first on a small scale and then to larger and more diverse populations (Clements, 2007). TRIAD is focused on successful change of classroom practices around mathematics for the long term. In that spirit, the professional development of teachers is just one component of the overall change process, and teachers are only one of the key players involved. Successful change requires the support not only of teachers, but also of administrators, parents, and children themselves (Clements, 2007).

Evaluations of the TRIAD model have proven it to be effective in improving the quality of the mathematical environment and child outcomes (Clements and Sarama, 2008; Sarama et al., 2008). For example, in one study, mathematics outcomes of children participating in the experimental group demonstrated significant gains over children in the control group (effect size, 1.07, Cohen's *d*) and comparison classrooms (effect size, .47, Cohen's *d*) (Clements and Sarama, 2008). Another TRIAD-based in-service training experiment provided evidence that teachers in the experimental group reported doing more mathematics in the classroom, rating mathematics as more important than did control teachers, and feeling more prepared to teach mathematics.

Key components of the in-service professional development as demonstrated by the TRIAD studies are (1) training is job-specific and tied directly to the use of a curriculum; (2) the training is extensive and ongoing, including an initial training at the outset of the school year, with follow-up sessions; (3) teachers are supported through onsite coaching once per month, aimed at helping with curriculum implementation and discussion of any problems or concerns that teachers have regarding its use; and (4) teachers have opportunities for hands-on practice, discussion, and collaboration with others, as well as for reflection on their practice. In-person coaching is the primary resource for teachers, in contrast with the combination of coaching and web media support offered through Building Blocks.

Two early childhood mathematics curricula, which include in-service professional development, that have been rigorously evaluated are SRA Real Math Building Blocks (Clements and Sarama, 2008) and Pre-K Mathematics (Starkey, Klein, and Wakeley, 2004). An intervention that combined elements of these two curricula has also been tested through experimental research (Sarama et al., 2008). Each is a research-based curriculum that has been evaluated using randomized control-group designs, and both curricula have met the What Works Clearinghouse criteria for inclusion, demonstrating their effectiveness in meaningfully improving child outcomes in mathematics (What Works Clearinghouse, 2007).

The documentation provided to programs adopting Building Blocks details elements of the training and support offered to teachers using the TRIAD model (Clements and Sarama, 2008; Sarama et al., 2008). Building Blocks training and support, which has been demonstrated to be effective through research, consists of three elements over the course of one school year: (1) 34 hours of focused group training, (2) 16 hours of in-class coaching and mentoring, and (3) electronic communications, including the use of an interactive project website (Clements and Sarama, 2008).

Understanding mathematical learning trajectories (which are called teaching-learning paths in this book) is a particular focus of the training, as a part of helping teachers learn the "conceptual storyline" (Clements and Sarama, 2008). In addition, trained coaches provide teachers with regular coaching and mentoring as well as individualized feedback and address any concerns or problems with implementation. The Building Blocks Learning Trajectory web application provides best practice exemplars, video-based illustration of children's mathematical thinking and development, and resources for lesson planning. Finally, teachers receive resources for documenting student progress. Thus, training is fairly extensive, ongoing, hands-on, specific, job-embedded, and tied to curriculum. Furthermore, training is provided by highly qualified trainers, and distance learning facilitates reaching participants in multiple locations. The documented gains in outcomes for teachers, classrooms, and children confirmed the efficacy of this approach to professional development (Clements and Sarama, 2008).

In sum, the research from these examples indicates that professional development in mathematics in early childhood settings is most successful when it is a component of an overall change process that is supported by all key players. They demonstrate that, although teachers can make highly significant improvements in children's mathematics outcomes, learning the knowledge and skills needed to do so requires an ongoing effort with support to achieve this success. Frequently, the number of contact hours in professional development that produces success is substantially greater than typically offered by curriculum publishers, an issue that should be addressed. Mentoring or coaching also appears to play an important role in helping teachers to solve problems as they learn to apply new knowledge and skills, as well as helping to sustain the change process over time. Evidence also shows that providing teachers with knowledge of mathematics and children's mathematical thinking and development, as well as how to apply this knowledge through the use of a particular curriculum, is highly effective at the early childhood level. These early efforts to bring professional development efforts to scale also indicate that technology may play an important role in overcoming logistical barriers to delivering high-quality training to a large, diverse workforce.

### *Outcomes of Mathematics In-Service Preparation in Elementary Education*

To date, there is not much research examining the relationship between in-service preparation and the effectiveness of mathematics teaching for preschool age children. However, one way to examine how formal in-service preparation in mathematics impacts the teaching of mathematics is to investigate the relationship between such preparation and K-12 mathematics outcomes. Research on the K-12 system has found effects between teacher content preparation and teacher effectiveness. For example, Monk (1994) found a positive relationship between mathematics and science secondary teachers who received content-specific preparation and their students' mathematics and science achievement. It should also be noted that the effects of content-specific preparation faded over time, suggesting that professional development opportunities throughout teachers' careers are necessary. It seems, then, that early childhood educators must have a deep knowledge of mathematics as it applies to young children and must have their learning periodically reinforced.

Research on mathematics preparation at the early elementary level also provides some useful implications for early childhood education, because the research is particularly focused on professional development itself, rather than on training as a component of curriculum implementation. A recent review of how professional development affects student achievement at the K-12 level examined over 1,300 research studies and identified only 9 that met the evidence criteria of the What Works Clearinghouse (Yoon et al., 2007). Five of the nine studies targeted mathematics outcomes, either solely or in combination with targeting outcomes in other learning domains. Studies that demonstrated effects on mathematics had an average effect size of 0.57 in mathematics outcomes, evidence of a significant impact on student mathematics learning outcomes. Together, they averaged slightly more than 53 contact hours of training over a period of four months to one year, which is substantially more hours than the typical elementary school teacher would have available for professional development (Yoon et al., 2007) or in which they would typically participate (Birman et al., 2007, as cited in Yoon et al., 2007). Across all nine studies, 14 contact hours or more produced gains in various other domains of student achievement, such as literacy, indicating that mathematics-focused efforts were more sustained or intensive (or both) than those targeting other domains.

Sarama and DiBiase (2004) described the effectiveness of several research-based professional development models for elementary school teachers in mathematics. The authors discuss three models in particular: Teaching to the Big Ideas (TBI), Cognitively Guided Instruction (CGI), and Project IMPACT. While these programs have a number of features, one key cross-cutting element is their emphasis on understanding children's mathematical thinking. There are a number of differences between early childhood and elementary school settings, such as expectations and beliefs about mathematics education and the educational levels of teachers, which make generalizations between them problematic. However, understanding how professional development can effectively help teachers understand the developmental progressions in children's mathematical thinking has important implications for professional development at the early childhood level. According to Sarama and DiBiase (2004), "starting with theory and research is not as effective as starting with practice, and then integrating theory and research into reflections on this practice" (p. 427). This emphasis on helping teachers to understand children's mathematical thinking can inform professional development efforts at the early childhood level, above and beyond adopting and learning a curriculum.

### *Pre-Service Teacher Preparation in Mathematics*

The examples of effective in-service professional development indicate the depth and breadth of preparation that all teachers need to address children's mathematics learning effectively, including those who pursue pre-service education. Specifically, teachers need preparation that (1) considers their beliefs about mathematics; (2) provides them with knowledge about mathematics, about children's mathematical development, and how to apply it in the classroom (mathematics education); and (3) affords them opportunities to practice these skills in a classroom setting. However, to date, most college and universities offer little by way of training teachers to effectively teach early childhood mathematics (Ginsburg et al., 2004, 2006a). Furthermore, many of today's early childhood educators completed their university training or general training when mathematics was deemphasized for young children's learning (Early et al., 2007). Thus, many early childhood educators, even the most qualified, degreed teachers, are not sufficiently well prepared to teach young children about mathematics.

To date, there are few if any empirical data sets that examine effective practices in pre-service preparation of early childhood teachers in mathematics. We consider data about the range of existing approaches to providing preparation in mathematics based on a preliminary review, which was conducted for this report, of recent college program submissions for accreditation with the National Council for Accreditation of Teacher Education (NCATE), at both the associate's and bachelor's degree levels.

In addition, we discuss the ways in which the pre-service teacher education could be affected by changes in other related systems. Clearly, more research is needed to determine the effects and the quality of early childhood pre-service mathematics preparation. The following section addresses: (1) issues affecting pre-service preparation for early childhood teachers, (2) the landscape of early childhood teacher education programs in general, (3) the ways in which these programs can address the needs of teachers to be prepared to promote young children's mathematical development, and (4) the ways in which other related credentialing systems can support the needed changes at the pre-service level for adequately preparing teachers in early childhood mathematics.

***Issues affecting pre-service preparation for early childhood teachers.*** Before focusing on the role of mathematics in pre-service teacher preparation, we examine some more general and potentially relevant trends and issues that affect early childhood educators' pre-service preparation. These trends include degree requirements, the academic content in teacher education courses, and assessment of the effectiveness of teacher preparation programs.

***Degree requirements.*** Policy makers at the federal and state levels continue to increase their requirements for early childhood educators to possess degrees—and, increasingly, the baccalaureate degree. Thus, one might expect an ever-higher percentage of early childhood educators to pass through the higher education system, creating more opportunities to enhance their mathematical competence through that system.

***Academic content.*** State and federal governments have placed greater emphasis on academic content in teacher education. This is reflected in some states' requirements for all education students to have an academic major and in states' limiting the number of credits that can be taken in more applied areas, such as pedagogy. This trend potentially expands opportunities to enhance mathematics content for future early childhood educators, but it may also limit students' opportunities to apply their content knowledge through field experiences and related pedagogical coursework. A related trend, prompted by concerns about the achievement gap in children's literacy skills, has been an increase in state and institutional requirements in the areas of literacy and reading. The potential for competition among literacy, mathematics, and other content areas creates dilemmas for the design of early childhood teacher preparation programs.

***Assessing competence.*** There is a growing tendency—spurred to a great extent by NCATE—to focus less on counting time for seatwork assignments and more on assessment of future teachers' competence (including their effects on children's learning), when judging whether a teacher preparation program is effective. This emphasis is posing new challenges for programs as they consider how to conduct standards-based, valid assessments in key areas.

### *Preparing Teachers to Promote Young Children's Mathematics Development*

No systematic national evaluation has been conducted to date of the nature and amount of preparation specifically in mathematics that these degree programs offer. However, a preliminary review of the NCATE submissions of both bachelor's and associate's degree programs conducted for this study indicates that programs currently address mathematics in a number of ways that involve required coursework and field experiences (Hyson, Tomlinson, and Morris, 2008).

***Coursework.*** The coursework that degree programs offer to prepare teachers to teach mathematics at the early childhood level may consist of general mathematics courses, courses on how to teach mathematics, or mathematics education. Pre-service teacher preparation programs have addressed this in a number of ways. Generally, if an early childhood mathematics course is offered, it often focuses on "math methods" (Ginsburg et al., 2006a). Some programs have general education mathematics requirements, either solely or in combination with course requirements in mathematics education. Associate's and bachelor's degree programs may require one or more general mathematics course, such as college algebra, while others offer students the choice of selecting a course in mathematics or in science as part of their degree requirements.

For mathematics education coursework, both associate's and bachelor's degree programs use a range of approaches, such as requiring one or more courses in teaching early childhood mathematics, embedding mathematics education in a general early childhood curriculum course, or combining mathematics and science education. Some offer mathematics education courses focused only on elementary mathematics. This broad range of approaches indicates that there is considerable variability in the depth and breadth of teachers' knowledge, exposure, and experiences in mathematics teaching, even among teachers with degrees, who represent the most qualified in their field.

Overall, the evidence shows that some programs offer in-depth, high-quality early mathematics education, and some programs provide almost no preparation. Pre-service programs should review their coursework in early childhood mathematics to ensure that they are preparing teachers to teach and support their students as effectively as possible. This involves preparing teachers in the following areas:

- **Mathematics.** A deep understanding of the mathematical concepts discussed in Chapter 2 and children's mathematical development as discussed in Chapters 5 and 6 is necessary for teachers to know what and how to teach mathematics effectively to young children.
- **Curriculum.** Teachers need to learn about the curriculum available to them for teaching mathematics to young children. They also need to study the different pedagogical arguments underlying different curriculum in order to be able to make informed choices when they have their own classrooms (see Chapter 7).
- **Assessment.** Programs need to prepare teachers to effectively assess young children's mathematical skills and thinking. Furthermore, teachers should be trained to use assessments to inform and improve on their instructional practices (see Chapter 7).
- **Beliefs.** Pre-service programs should provide teachers with an opportunity to discuss and explore their attitudes and beliefs about mathematics and the effects of those beliefs on their teaching.

***Faculty.*** Some programs, particularly those at the associate's level that rely heavily on adjunct faculty, may face challenges with having personnel qualified to teach early childhood mathematics courses. Because many teacher educators may have been prepared at a time when mathematics was deemphasized for young children, these personnel themselves may require some support to be adequately knowledgeable and prepared to teach the content. Alternatively, programs may take advantage of distance learning and web-based courses offered by mathematics educators at other universities and programs to fill gaps in the mathematics preparation of their students.

***Field experiences.*** Some programs require specific field experiences in mathematics associated with mathematics education coursework, others include it as one component of many in a general student teaching experience or simply do not require any practical mathematics teaching experience at all. Research is clear that effective approaches to professional development in early childhood mathematics require opportunities to practice and use new knowledge and skills and to receive meaningful feedback.

***Role of credentialing systems in preparing teachers in early childhood mathematics.*** To ensure that future degreed teachers have the knowledge and skills that they need to promote early childhood mathematics in the classroom, providers of pre-service preparation programs are likely to need to make changes to their offerings and requirements in early childhood mathematics. While some programs may initiate these changes on their own, in reality four key systems have a great deal of influence over the content and experiences of pre-service education programs in early childhood education. They should be updated to reflect current knowledge in early childhood mathematics, along the lines presented in this report. These systems include: state certification and licensure requirements, Praxis exams, NAEYC standards and other credentialing systems outside of states.

Currently, 48 states have arrangements such that they will give at least initial licensure to a teacher who has graduated from an NCATE-accredited institution, in a program that has been recognized by the appropriate national specialty professional association, such as NAEYC (Margie Crutchfield, NCATE, personal communication, April 2, 2008) rather than specifying particular coursework or credits. However, programs with both NCATE and NAEYC accreditation account for less than one third of all early childhood bachelor's programs (Maxwell, Lim, and Early, 2006).

The Praxis exams, which are used in NCATE's national accreditation/recognition of early childhood programs, include multiple-choice tests of students' basic skills (Praxis I) and tests of their competence in a specific teaching area (Praxis II). These exams serve as gatekeepers at various stages of students' progress through the pre-service program and entry into the profession.

The NAEYC standards are reportedly are used by faculty to guide design and improvement of associate's and bachelor's degree programs (Hyson, Tomlinson, and Lutton, 2007). Also, programs participating in NAEYC's national recognition and accreditation systems are likely to focus on the NAEYC standards. However, while mathematics is explicitly part of the standards (in Standard 4: Teaching and Learning), the current system for national recognition and accreditation does not require pre-service programs to specifically document their graduates competence in math, nor are the actual learning opportunities offered in mathematics explicitly evaluated.

Finally, the Child Development Associate (CDA) and the National Board for Professional Teaching Standards (NBPTS) certification are two credentialing systems that operate outside of state teacher licensure systems. They are important because much of the early childhood workforce obtains or extends their professional development through them. The CDA is obtained through a combination of fieldwork, coursework, and other reading, writing, and conferencing requirements and is the most frequently required qualification for child care center directors (National Child Care Information Center, 2005). A review of the key materials used in CDA training and assessment conducted for this report revealed a need for additional mathematics-related resources to increase the ability of advisers and instructors to support CDA candidates' understanding of and engagement in early childhood mathematics.

The NBPTS uses a rigorous review process to certify "accomplished teachers" in 26 fields, including the early childhood generalist category, which covers teachers with bachelor's degrees who educate and care for children ages 3 to 9. Candidates provide the national board with four portfolio entries that document teaching competence and accomplishments outside the classroom, and demonstrate their content knowledge in a set of "assessment center exercises" specific to their certificate area. NBPTS requires that early childhood candidates include a videotaped mathematics-related instructional sequence in their portfolios (with detailed justification and self-analysis) and that one of two challenging assessment center exercises be in the domain of mathematics. Recent research has linked national board certification with improved child outcomes (National Research Council, 2008).

***Summary of issues related to pre-service teacher preparation in mathematics.*** Early childhood educators who pursue pre-service education prior to their entry into the workforce participate in a range of types of associate's and bachelor's degree programs. These programs, in turn, address mathematics education preparation in a variety of ways, ranging from requiring general mathematics courses or specific mathematics coursework and fieldwork (or both), to combining mathematics with other disciplines, to hardly addressing it at all. While no data on the effects of pre-service mathematics programs on later teaching and outcomes exist, data from effective in-service preparation indicate the content and types of experiences in early childhood mathematics that lead to positive outcomes—specifically, to be prepared to teach mathematics to young children, teachers need knowledge of mathematics, mathematical development, effective pedagogy, including the use of curriculum, and assessment, as well as opportunities to use this knowledge in early childhood classrooms. In addition, beliefs that may hinder the acquisition and application of this knowledge should be addressed. The influence of systems, including licensure requirements, Praxis exams, NAEYC standards, and credentialing systems, is important to consider. These systems are potential levers for increasing the focus on mathematics in early childhood professional development.

## SUMMARY

The nature of the early childhood workforce is important to understand as perhaps one of the most critical contextual factors to improving the mathematical development of young children. As one of the primary vehicles through which children learn mathematics, teachers exert enormous influence. Yet in preparing teachers to take on this challenge, it is critical to face the realities of the workforce—namely that teachers present with a wide range of educational backgrounds, compensation, and work settings but tend to share beliefs and values that are generally less supportive of mathematics in the early childhood classroom than social-emotional development. Compounding the challenge, these teachers, despite their diverse qualifications, have typically received little, if any, preparation to teach early childhood mathematics.

Research on the effective delivery of mathematics-specific professional development is fairly new and there continues to be a need for more work in this area. Research indicates that professional development efforts at all levels are most effective when they address teachers' own mathematics knowledge, beliefs about mathematics, knowledge of children's mathematical thinking and development as well as mathematics pedagogy, knowledge of appropriate mathematical assessment practices, and knowledge of resources for supporting mathematics in their classrooms. Of these, a focus on understanding children's developmental progression in mathematics tied to specific activities through a curriculum is the most salient feature of effective professional development in mathematics.

Effective approaches to in-service professional development in mathematics are ongoing, grounded in theory, tied to a curriculum, job-embedded, at least partially onsite, delivered by a knowledgeable and prepared trainer, supported by administrators, and accompanied by supports for teachers during implementation through mentors, coaches, and technology, meaningful feedback, time for hands-on practice and reflection, and opportunities to work and solve problems collaboratively with other teachers and trainers. Professional development in mathematics may require extensive contact hours and a sustained effort. Furthermore, professional development is but one component of successful teacher/program change. This requires collaboration from administrators, teachers, parents, and children, as well as those from the outside helping to bring about change.

While few data are available regarding effective approaches to pre-service education in early childhood mathematics, the range of approaches to providing this preparation that currently exists demonstrates that many program graduates leave with minimal preparation to teach early childhood mathematics. To prepare early childhood educators at the pre-service level, programs need to require coursework and fieldwork in mathematics, focusing on the content areas described in this report that all teachers need in this domain. To support these changes in programs, teacher educators will require support. Furthermore, licensure and credentialing systems, assessments of teacher competence, and professional and state standards should reflect greater emphasis on mathematics.

Although more data are available at the in-service level than at the pre-service level, even the available studies represent relatively small-scale efforts, presenting considerable logistical challenges to meeting the needs of the field. While data indicate that the use of technology, such as interactive websites and distance learning, is effective in reaching large numbers of teachers in early childhood programs, both at the in-service and pre-service levels, more research and creative solutions will be needed for scale-up efforts.

This chapter describes the importance of the early childhood workforce in promoting children's mathematical development. The next chapter presents the committee's conclusions and recommendations to improve the teaching and learning of early childhood mathematics.

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# Part IV

# Future Directions for Policy, Practice, and Research

# 9

# Conclusions and Recommendations

Over the past several decades there has been an increased focus on the importance of the preschool period—between ages 3 and 5—in providing children with the opportunities they need to get off to a successful start in formal schooling. Many policy makers are now intent on implementing universal public preschool because of the mounting evidence that high-quality preschool can help ameliorate inequities in educational opportunity and begin to address achievement gaps. The importance of supporting literacy in these early childhood settings is widely accepted, but little attention is given to mathematics. However, research on children's capacity to learn mathematics, when combined with evidence that early success in mathematics is linked to later success in both mathematics and reading, makes it clear that basic literacy consists of both reading *and* mathematics. Improvements in early childhood mathematics education can provide young children with the foundational educational resources that are critical for school success. Furthermore, the increasing importance of science and technology in everyday life and for success in many careers highlights the need for a strong foundation in mathematics.

Historically, mathematics has been viewed by many as unimportant to or developmentally inappropriate for young children's learning experiences. However, the research synthesized in this report makes it clear that these beliefs are unfounded. In the course of normal development, young children develop key mathematical ideas and skills that include counting; adding and subtracting; finding which is more (or less); working with shapes by moving, combining, and comparing them to learn some of their properties; experiencing and labeling spatial terms (e.g., above, below);

and understanding length measurement as the number of length units that makes the total; as well as representing and communicating mathematics understanding to others.

Relying on a comprehensive review of the research, this report lays out the critical areas that should be the focus of young children's early mathematics education, explores the extent to which they are currently incorporated into early childhood settings, and identifies the changes needed to improve the quality of mathematics experiences for young children. The committee describes these critical areas of mathematics in terms of teaching-learning paths that can be used to promote optimal learning. Such a path describes the skills and knowledge that are foundational to later learning and lays out a likely sequence of the steps toward greater competence. One can look closely along the path to gauge what children will be able to do next and to design instructional activities that will help them move along the path. The notion of such teaching-learning paths is a framing assumption for the conclusions and recommendations of this report.

To ensure that all children enter elementary school with the mathematical foundation they need for success, the committee recommends a major national initiative in early childhood mathematics. The success of such an initiative requires that parents, early childhood teachers, policy makers, and communities reconceptualize the way they think about and understand young children's mathematics. The early childhood education system (e.g., workforce, early childhood programs, and policies) will need to work coherently together toward this goal. Furthermore, families and communities must also adopt this goal if they are serious about improving children's mathematics education.

In this chapter, the committee summarizes the major conclusions of the report organized around the chapters, articulates the key recommendations that flow from these conclusions, and lays out an agenda for future research.

## CHILDREN'S COMPETENCE AND POTENTIAL TO LEARN MATHEMATICS

The committee's review of developmental research with infants and toddlers demonstrates that the knowledge and competencies relevant to mathematics are present from early in life. As early as infancy, babies are curious about their world and are able to think about it in mathematical ways. Preverbal number knowledge is shared by humans from diverse cultural backgrounds as well as by other species. For example, by 10 months of age, young infants can distinguish a set of two items from a set of three items, and over time they are able to distinguish the number of items in sets with larger numbers. Building on this foundation, young children continue to expand their knowledge and competence and enjoy their early informal experiences with mathematics, such as spontaneously counting toys, excitedly asking who has more, or pointing out shapes.

**Conclusion 1: Young children have the capacity and interest to learn meaningful mathematics. Learning such mathematics enriches their current intellectual and social experiences and lays the foundation for later learning.**

Knowledge and competencies acquired through everyday experiences provide a starting point for mathematics learning. Infants' and toddlers' natural curiosity initially sparks their interest in understanding the world from a mathematical perspective, and the adults and communities that educate and care for them also provide experiences that serve as the basis for further mathematics learning. Children's everyday environments are rich with mathematics learning opportunities, for example, using relational words, such as more than/less than, and counting and sorting objects by shape or size. These foundational, everyday mathematics experiences can be built on to move children further along in their understanding of mathematical concepts.

**Conclusion 2: Children learn mathematics, in part, through everyday experiences in the home and the larger environment beginning in the first year of life.**

Children need rich mathematical interactions and guidance, both at home and school to be well prepared for the challenges they will meet in formal schooling. Parents, other caregivers, and teachers can play a fundamental role in the organization of learning experiences that support mathematics because they can expose children to mathematically rich environments and engage them in mathematics activities. For example, parents and caregivers can teach children to see and name small quantities, count, and point out shapes in the world, "Here are *two* crackers. You have *one* in each hand. These crackers are *square*."

One important way that young children's mathematics learning can be enhanced is through adult support and instruction that is connected to and extends their preexisting mathematics knowledge. For example, a situation in which a young child insists on having "more" teddy bears than his playmate provides an opportunity for the adult to engage the child with a mathematical question (e.g., who has more and how can you find out?). In this instance, the adult can use several key mathematical ideas to help the child understand who has more bears, such as using the number word list to count, 1-to-1 counting correspondence, cardinality (i.e., knowing the total number of items in the set), and comparing the number of bears in the two sets. These kinds of mathematics learning opportunities help children learn to *mathematize* or engage in processes that involve focusing on the mathematical aspects of an everyday situation, learn to represent and elaborate a model of the situation, and use that model to solve problems.

**Conclusion 3: Children need adult support and instruction to build and extend their early knowledge and learn to focus on and elaborate the mathematical aspects of everyday situations—*to mathematize*.**

The committee was keenly aware of the influence that developmental and contextual variations have on children's learning opportunities and the quality of their educational environments both inside and outside the classroom. Understanding individual differences in children's development—for example, in executive function or in opportunities to learn about mathematics in their everyday environments—is fundamental to supporting the development of competence in mathematics. Although all children need extensive exposure to mathematics, there is a wide range of individual variation across all domains of learning. This affects the kinds of learning experiences and instruction that individual children need. The need to support early childhood mathematics education in ways that are appropriate for diverse learners and contexts is a theme throughout the committee's discussion of early childhood mathematics.

**Conclusion 4: Due to individual variation, which is related to a combination of previous experiences, opportunities to learn, and innate ability, some children need more extensive support in mathematics than others.**

It is important to understand the sources of observed differences in children's competence and not confuse one source of individual variation for another. For example, low performance might be attributed to a deficit in a child's ability to learn mathematics, when it actually results from other factors, such as that child's lack of opportunities to learn mathematics or difficulties stemming from linguistic and cultural barriers between teacher and child.

Opportunities to explore the mathematics of everyday life differ depending on children's background, including their socioeconomic status (SES) and cultural group. Mathematics knowledge and skills vary within and between cultural groups due to a variety of factors, including language and relative emphasis placed on mathematics. Cultural, linguistic, and socioeconomic factors interact in complex ways that are difficult to tease apart.

The committee was particularly concerned about mathematics teaching and learning for children from low socioeconomic backgrounds because of the particular challenges they face that can have an impact on their knowledge and competence in mathematics. For example, they may be more likely to attend schools with fewer resources and have less support for mathematics at home. Thus, although children with very low and high mathematics knowledge and competence are found across all SES groups, those with low SES will need particular attention. Importantly, providing young children with high-quality mathematics instruction can help to ameliorate systematic inequities in educational outcomes and later career opportunities.

**Conclusion 5: Young children in lower socioeconomic groups enter school, on average, with less mathematics knowledge and skill than their higher socioeconomic status peers. Formal schooling has not been successful in closing this gap for low socioeconomic status children.**

In addition to needing instructional support in mathematics, evidence indicates that young children also need to be supported in their social-emotional development as an integral part of their education. Specifically, during the early education years, children develop general competencies and approaches to learning that include their capacity to regulate their emotions and behavior, to focus their attention, and to communicate effectively with others. In turn, mathematics learning can help to promote the development of these general competencies.

**Conclusion 6: All learning, including learning mathematics, is facilitated when young children also are developing skills to regulate their own learning, which includes regulating emotions and behavior, focusing their attention, and communicating effectively with others.**

## FOUNDATIONAL AND ACHIEVABLE MATHEMATICS FOR YOUNG CHILDREN

On the basis of research evidence about children's knowledge and competence during the early childhood years, as well as on the established consensus of the early childhood mathematics community (see, for example, the NCTM Curriculum Focal Points), the committee identified two areas of mathematics on which to focus: (1) number, including whole number, operations, and relations, and (2) geometry, spatial thinking, and measurement. In each of these areas, the committee offers guidance about the teaching-learning paths based on what is known from developmental and classroom-based research. Each child's progression along these mathematics teaching-learning paths is a function of his or her own level of development as well as opportunities and experiences, including instruction. The teaching-learning paths can provide the basis for curriculum and can be used by teachers to assess where each child is along the path.

Although it is true that young children are more competent in mathematics than many early childhood teachers, parents, and the general public believe, there are limits to what they can do in mathematics. The committee kept this in mind throughout the study process, and thus the teaching-learning paths presented in this report are both foundational and achievable.

*The first content area is number, including whole number, operations, and relations.* Working with number (e.g., learning to count) is the primary goal of many early childhood programs; however, when given the opportunity, children are capable of demonstrating competence in more sophisticated mathematics activities related to whole number, operations, and relations. For example, cardinality—knowing how many are in a set—is a key part of children's number learning. Relations and operations are extensions of understanding number. The relations core consists of such skills as constructing the relations more than, less than, and equal to. The operations core includes addition and subtraction.

*The second major content area is geometry, spatial thinking, and measurement.* Children's foundational mathematics involves geometry or learning about space and shapes in two and three dimensions (e.g., learning to recognize shapes in many different orientations, sizes, and shapes). A fundamental understanding of shape begins with experiences in which children are shown varied examples and nonexamples and understand attributes of shapes that are mathematically relevant as well as those (e.g., orientation, size) that are not. As children progress along the teaching-learning path, they need opportunities to discuss and describe shapes, and, on the basis of these experiences, they gain abilities to combine shapes into pictures and eventually learn to take apart and put together shapes to create new shapes. Young children also need instructional activities involving spatial orientation and spatial visualization. For example, they can use mental representations of their environment and, on the basis of the representation, model relationships between objects in their environment. Importantly, children's knowledge of measurement helps them connect number and geometry because measurement involves covering space and quantifying this coverage. Later, children can compare lengths by measuring objects with manipulable units, such as centimeter cubes.

Number is particularly important to later success in school mathematics, as number and related concepts make up the majority of mathematics content covered in later grades. However, it is important to point out that concepts related to number (and relations and operations) can also be explored through geometry and measurement. In addition, geometry and measurement provide rich contexts in which children can deepen their mathematical reasoning abilities.

**Conclusion 7: Two broad mathematical content areas are particularly important as a focus for mathematics instruction in the early years: (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial thinking, and measurement.**

In the context of these core content areas, young children should engage in both general and specific thinking processes that underpin all levels of mathematics. These include the general processes of representing, problem solving, reasoning, connecting, and communicating, as well as the more specific processes of unitizing, decomposing and composing, relating and ordering, looking for patterns and structures, and organizing and classifying information. In other words, children should learn to mathematize their world: focusing on the mathematical aspects of an everyday situation, learning to represent and elaborate the quantitative and spatial aspects of a situation to create a mathematical model of the situation, and using that model to solve problems.

**Conclusion 8: In the context of each of these content areas, young children should engage in both general and specific mathematical thinking processes as described above and in Chapter 2.**

## THE EARLY CHILDHOOD EDUCATION SYSTEM

The early childhood education "delivery system," which educates and cares for children before kindergarten entry, has a great deal of diversity and is best characterized as a loosely sewn-together patchwork of different kinds of programs and providers that vary widely in the extent to which they articulate and act on their educational missions or are explicitly designed to provide education services. Program types range from friends and relatives who care for children in the home through informal arrangements, to large centers staffed by teachers offering a structured curriculum.

This diversity in the early childhood education system characterizes the education and care arrangements of young children in the United States today. About 40 percent of young children spend their day in a home-based setting, either with a parent or some other caregiving adult (this percentage includes children in home-based relative and nonrelative care as well as children who do not have any regular early education and care arrangements), and about 60 percent are in some kind of center-based care (this includes children in center-based non-Head Start and Head Start settings).

Depending on the type of setting, different regulations regarding educational standards or expectations may be in place, which in turn influence the nature and quality of young children's learning experiences from setting to setting. Increasingly, policy makers are focused on how to provide high-quality preschool education for more children, especially to those whose families cannot afford to pay for it. A number of states are moving toward state-funded preschool education to provide early education and care for these children.

Across all settings, there is a need to increase the amount and quality of time devoted to mathematics. Formal settings with an educational agenda represent the greatest opportunity for implementing a coherent, sequenced set of learning experiences in mathematics. For this reason, the committee focused attention on the kind of curriculum and instruction that can be implemented in centers and preschools. The committee gave more limited attention to how to increase support for mathematics in informal settings. These approaches are discussed in the section "Beyond the Education System."

## Curriculum and Instruction

Having laid out a vision for optimal teaching-learning paths in early childhood mathematics, the committee turned to the evidence base related to curriculum and instruction. The committee first examined the extent to which the content and learning experiences embodied in the teaching-learning paths are represented in current curricula and preschool classrooms. Next, the committee explored what is known about effective mathematics instruction for young children and what might need to be done to improve existing practice. The committee looked for evidence to address two sets of questions: What is known about how much mathematics instruction is available currently to children in preschool settings and of what quality? What is known about the best methods of instruction and effective curriculum to teach mathematics to young children? Although few systematic data exist, the committee was able to identify some useful sources. We conducted original analyses of the standards documents pertaining to early childhood for 49 states and those pertaining to kindergarten for the 10 states with the largest student populations. On the basis of these analyses, the committee concludes:

> **Conclusion 9: Current state standards for early childhood do not, on average, include much mathematics. When mathematics is included, there is a pattern of wide variation among states in the content that is covered.**

Although standards represent broad guidance from the states regarding appropriate content for early childhood settings, they do not provide a window on what actually occurs in classrooms. For the latter, the committee examined data from a large-scale study of instruction in state-funded preschools drawn from 11 states as well as several, small-scale studies of curriculum. The results show that when mathematics activities are incorporated into early childhood classrooms, they are often presented as part of an integrated or embedded curriculum, in which the teaching of mathematics is secondary to other learning goals. This kind of integration occurs when, for example, a storybook has some mathematical content but is not designed to bring mathematics to the forefront, a teacher counts or does simple arithmetic during snack time, or points out the mathematical ideas children might encounter during play with blocks. However, data suggest that heavy reliance on integrated or embedded mathematics activities may contribute to too little time being spent on mathematics in early childhood classrooms. Furthermore, the time that is spent may be on activities in which the integrity and depth of the mathematics is questionable. Few of the existing comprehensive early childhood curriculum approaches provide enough focused mathematics instruction for children to progress along the teaching-learning paths recommended by the committee.

**Conclusion 10: Most early childhood programs spend little focused time on mathematics, and most of it is of low instructional quality. Many opportunities are therefore missed for learning mathematics over the course of the preschool day.**

Evidence examined by the committee suggests that instructional time focused on mathematics is potentially more effective than embedded mathematics. Emerging evidence from a few studies of rigorous mathematics curricula show that children who experience focused mathematics activities in which mathematics teaching is the major goal have higher gains in mathematics and report enjoying mathematics more than those who do not. Furthermore, these studies indicate that a planned, sequenced curriculum can support young children's mathematical development in a sensitive and responsive manner. Supplemental opportunities to use mathematics during mathematical play, sociodramatic play, and with concrete materials (e.g., blocks, puzzles, manipulatives, interactive computer software) can provide children with the opportunity to "practice" mathematics in a meaningful and engaging context.

**Conclusion 11: Children's mathematics learning can be improved if they experience a planned, sequenced curriculum that uses the research-based teaching-learning paths described in this report, as well as integrated mathematics experiences (e.g., mathematics in the context of a storybook) that extend mathematical thinking through play, exploration, creative activities, and practice.**

Effective mathematics curricula use a variety of instructional approaches, such as a combination of individual, small-group, and whole-group activities focused on mathematics that move children along the research-based teaching-learning paths described in this report. Furthermore, in all these contexts, intentional teaching enhances the mathematics learning of young children. Intentional teaching varies from teacher-guided activities to responsive feedback that builds on and extends the child's understanding. It is also important to engage children in *math talk*—discussion between adults and children that focuses on mathematics concepts, such as how many objects are in a set or how to arrive at an answer—as this facilitates their mathematical development by increasing the connections they make between mathematics concepts, words, and ideas. It should be noted that the committee does not endorse any specific model or curriculum; rather we hope to convey that the research-based principles described in this report should guide choices about development of early childhood mathematics curriculum and instruction.

**Conclusion 12: Effective early mathematics curricula use a variety of instructional approaches and incorporate intentional teaching.**

Evidence also indicates that instruction is more effective when it can build on information about the child's current level of understanding. Such responsive instruction can be accomplished when teachers know how to use formative assessment to guide instruction. Formative assessment is an important component of what teachers need to know to effectively guide children along the mathematics teaching-learning paths.

**Conclusion 13: Formative assessment provides teachers with information about children's current knowledge and skills to guide instruction and is an important element of effective mathematics teaching.**

Evidence from studies of early childhood education indicates that any approach to curriculum and pedagogy is more effective if undertaken in the context of a positive learning environment. Positive relationships between children and their teachers are a key aspect of high-quality early childhood education. In this kind of classroom, children are provided with a safe and nurturing environment that promotes learning and positive interactions between teachers and peers.

**Conclusion 14: Successful mathematics learning requires a positive learning environment that fully engages children and promotes their enthusiasm for learning.**

### Workforce and Professional Development

The early childhood workforce—those who serve both instructional and noninstructional roles in early childhood settings—is central to supporting the academic, social, emotional, and physical development of young children. This workforce consists of people who serve in a variety of roles, are located in a variety of settings, and have a wide range of education and training backgrounds. About 24 percent of early childhood workers are in center-based settings, 28 percent are in regulated home-based settings, and about 48 percent work in informal care arrangements outside both of these systems. Although the majority of early childhood professionals work in informal care settings, the majority of children are in center-based settings. Even in a single setting, individuals fill different roles, such as lead teacher, assistant teacher, classroom aide, or program administrator. Level and type of training can vary by both role and setting. For example, family child-care providers may have little or no specific training in early childhood education, a teachers' assistant may have some formal coursework, and center-based lead teachers may have a 4-year college degree (or even a graduate degree) with specialization in early childhood.

This diversity of roles and educational backgrounds creates challenges for addressing the workforce needs related to supporting early childhood mathematics. Individuals in different roles are likely to need different kinds of knowledge and training to support children's mathematics. Depending on level of education, there are also likely to be differences in individuals' knowledge of mathematics, of children's development in mathematics, and of how to support mathematics learning.

In addition, the field of early childhood has historically placed great emphasis on teaching its workforce to support children's social and emotional development, placing less attention on cognitive development and academic domains. Indeed, academic activities, such as mathematics learning, can be a context in which social-emotional development flourishes. In large part, the heavy emphasis on social-emotional development in early childhood is based on misinterpretations of cognitive development theories; that is, the notion of young children engaging in more abstract thinking, such as mathematics, was believed to be at odds with the development and learning of preschool-age children. Research on early childhood mathematics has disproved this notion, but the idea is still pervasive in the field and continues to be a challenge in moving from research to practice.

**Conclusion 15: Many in the early childhood workforce are not aware of what young children are capable of in mathematics and may not recognize their potential to learn mathematics.**

Professional development, which typically provides training to those already in the workforce, can be a vital mechanism for providing teachers with new or updated skills and knowledge that they need and for reaching those in the workforce who have little or no formal training. Based on studies at the K-12 level, effective approaches to in-service professional development in mathematics are ongoing, grounded in theory, tied to a curriculum, job-embedded, and delivered at least partially onsite by a knowledgeable trainer who allows teachers time for reflection. The committee reviewed emerging data from studies conducted in early childhood settings that support these findings. These studies indicate that professional development focused on understanding children's developmental progression in mathematics in the context of a research-based curricular sequence can improve teachers' instructional effectiveness. An effort to provide professional development to teachers is one important component of successfully improving instruction, but sustainable change will also require collaboration from administrators, teachers, and parents.

**Conclusion 16: In-service education of teachers and other staff to support mathematics teaching and learning is essential to effective implementation of early childhood mathematics education. Useful professional development will require a sustained effort that involves helping teachers to (a) understand the necessary mathematics, the crucial teaching-learning paths, and principles of intentional teaching and curriculum and (b) learn how to implement a curriculum.**

Evidence reviewed by the committee about the formal preparation of early childhood educators (courses taken as part of an associate or undergraduate degree) indicates that there are few opportunities to learn about children's development in mathematics or how to teach early childhood mathematics. To better prepare early childhood educators in mathematics, additional courses and additional materials in existing courses that cover children's development in mathematics and mathematics pedagogy are needed. Furthermore, licensure and credentialing systems exert a great deal of influence over the content and experience of pre-service education programs in early childhood, and few incorporate mathematics requirements.

**Conclusion 17: Pre-service preparation of early childhood educators typically includes few opportunities to learn about children's mathematical development or how to support it. Licensure and certification requirements for credentialing teachers and programs are both potential leverage points for increasing the amount of attention given to supporting mathematics.**

In addition to the challenges already outlined regarding the diverse training and settings of the workforce, attracting and retaining qualified individuals to work in early childhood is difficult due to poor compensation, lack of benefits, and high turnover rates in the field. This situation presents an additional challenge to designing pre-service and in-service experiences that can improve early childhood educators' knowledge of how to support young children's learning in mathematics.

**Conclusion 18: Improving the training and knowledge requirements for early childhood teachers will present significant challenges unless existing issues of recruitment, compensation, benefits, and high turnover are also addressed.**

## BEYOND THE EDUCATION SYSTEM

A significant number (about 40 percent) of children do not attend centers but instead are educated and cared for by a parent, relative, or another adult in homes. Parents or other caregivers serve as children's first teachers; evidence reviewed by the committee indicates that they can play a key role in shaping children's early mathematics learning through such activities as encouraging play with blocks and other manipulatives, teaching number words, playing counting and board games, sorting, classifying, writing, and viewing educational television programs while talking with children about what they are watching. Math talk has been shown to be a particularly effective way for adults to support the development of mathematical ideas. In fact, math talk beginning as early as infancy is related to children's mathematics knowledge at preschool entry. In addition, informal learning environments, such as libraries, museums, and community centers, have the potential to be resources that parents and caregivers can use to engage children in mathematics activities.

**Conclusion 19: Families can enhance the development of mathematical knowledge and skills as they set expectations and provide stimulating environments.**

Evidence indicates, however, that low-SES families are less likely than families from higher socioeconomic groups to engage in the kind of practices that promote language and mathematics competence. Although many types of educational programs have been designed to promote the use of these practices with low-SES parents, there is little evidence about the qualities that make such efforts successful. Educational programs for parents based on models that place parents in the traditional role of students learning from "experts" have difficulty sustaining family participation long enough to be successful.

**Conclusion 20: Educational programs for parents have the potential to enhance the mathematical experiences provided by parents; however, there is little evidence about how to design such programs to make them effective.**

The resources available to parents and other caregivers as well as those available through informal educational environments (e.g., libraries, museums, community centers) can also be an effective mechanism for supporting children's mathematics learning. Educational television programming and software, for example, can teach children about mathematics. The committee reviewed research on software and educational programs, as well as models of community-based programs that promote mathematics, and concludes:

**Conclusion 21: Given appropriate mathematical content and adult support, the media (e.g., television, computer software) as well as community-based learning opportunities (e.g., museums, libraries, community centers) can engage and educate young children in mathematics. Such resources can provide additional mathematics learning opportunities for young children, especially those who may not have access to high-quality early education programs.**

## RECOMMENDATIONS

As the committee's conclusions make clear, there is much work to be done to provide young children with the learning opportunities in mathematics that they need. Thus, the committee thinks it is critically important to begin an intensive national effort to enhance opportunities to learn mathematics in early childhood settings to ensure that all children enter school with the mathematical foundations they need for academic success. The research-based principles and mathematics teaching-learning paths described in this report can also reduce the disparity in educational outcomes between children from low-SES backgrounds and their higher SES peers.

The research to date about how young children learn key concepts in mathematics has clear implications for practice, yet these findings are not widely known or implemented by early childhood educators or even those who teach early childhood educators. This report has focused on synthesizing and translating this evidence base into a usable form that can be used to guide a national effort. Thus the committee recommends:

**Recommendation 1: A coordinated national early childhood mathematics initiative should be put in place to improve mathematics teaching and learning for all children ages 3 to 6.**

A number of specific recommendations for action follow from this overarching recommendation. The specific steps and the individuals or organizations that must be involved in enacting them are outlined below.

**Recommendation 2: Mathematics experiences in early childhood settings should concentrate on (1) number (which includes whole number, operations, and relations) and (2) geometry, spatial relations, and measurement, with more mathematics learning time devoted to number than to the other topics. The mathematical process goals should be integrated in these content areas. Children should understand the concepts and learn the skills exemplified in the teaching-learning paths described in this report.**

In both content areas, sufficient time should be devoted to instruction to allow children to become proficient with the concepts and skills outlined in the teaching-learning paths. In addition, the general and specific mathematical process goals (see Chapter 2) must be integrated with the content in order to allow children to make connections between mathematical ideas and deepen their mathematical reasoning abilities. This new content focus will require that everyone involved rethink how they view and understand the mathematics that is learned in early childhood. Early childhood learning goals, programs, curricula, and professional development will need to be informed by and adapted to the research-based teaching-learning paths laid out in this report. The committee therefore recommends:

**Recommendation 3: All early childhood programs should provide high-quality mathematics curricula and instruction as described in this report.**

Early childhood programs will each need to implement a thoughtfully planned curriculum that includes a sequence of teacher-guided mathematics activities as well as child-focused, teacher-supported experiences. Such curricula must be based on models of instruction that are appropriate for young children and support their emotional and social development as well as their cognitive development. As noted previously, effective mathematics curricula use a variety of instructional approaches and should incorporate opportunities for children to extend their mathematical thinking through play, exploration, creative activities, and practice.

Programs will need to review, revise, and align their existing standards, professional development, curriculum, and materials to achieve the teaching-learning paths for early childhood mathematics education presented in this report. It is especially important that children living in poverty receive such high-quality experiences so that they start first grade on a par with children from more advantaged backgrounds. This means that implementation of our recommendations by programs serving economically disadvantaged children, such as Head Start and publicly funded early childhood programs, is particularly urgent.

To make the recommended changes, early childhood programs will need explicit policy directives to do so. To encourage this, the committee recommends:

**Recommendation 4: States should develop or revise their early childhood learning standards or guidelines to reflect the teaching-learning paths described in this report.**

Given the fresh knowledge and perspectives this report affords, it is important that states review their early learning and development standards and guidelines to ensure that they reflect an appropriate emphasis on early mathematics. To that end, we call for all states to examine their early learning and development guidelines, first, to determine that sufficient emphasis is given to the importance of mathematics for young children's development and, second, to ensure that the mathematics content focuses on (1) number (including whole number, operations, and relations) and (2) geometry, spatial thinking, and measurement.

**Recommendation 5: Curriculum developers and publishers should base their materials on the principles and teaching-learning paths described in this report.**

Teachers and early childhood programs need appropriate materials in order to support children's mathematical development and learning. Curriculum developers and publishers who produce materials for curriculum, instruction, and assessment should revise and update them so that they reflect the principles articulated in this report.

The success of this overall effort will need to focus on reaching both the existing early childhood workforce and pre-service educators to provide them with skills and knowledge they need to teach mathematics. Thus, we make several recommendations related to teachers and the workforce.

**Recommendation 6: An essential component of a coordinated national early childhood mathematics initiative is the provision of professional development to early childhood in-service teachers that helps them**

**(a) to understand the necessary mathematics, the crucial teaching-learning paths, and principles of intentional teaching and curriculum and (b) to learn how to implement a curriculum.**

Applying teachers' theoretical knowledge to a curriculum with a strong mathematics component provides them with the opportunity to get feedback and reflect on the instructional practices that they will actually be implementing in the classroom. Professional development should also focus on teachers' beliefs about children's mathematics, the activities and resources in the classroom that can promote children's mathematical development, and their knowledge of curriculum-linked assessment practices. All of these important areas should be included in professional development delivered by a highly qualified teacher educator.

To implement high-quality mathematics instruction, the committee also recommends that early childhood educators be taught to use a range of effective instructional strategies in a variety of formats, including whole-group, pair/small-group, and individual work; exploration and practice; and play and focused activities.

Serious efforts to improve the preparation of early childhood teachers will need to include the state licensure/certification, accreditation and recognition, and credentialing systems that assess teachers' competence and program quality. The early childhood mathematics described in this report should be reflected in the core components of these systems and programs.

**Recommendation 7: Coursework and practicum requirements for early childhood educators should be changed to reflect an increased emphasis on children's mathematics as described in the report. These changes should also be made and enforced by early childhood organizations that oversee credentialing, accreditation, and recognition of teacher professional development programs.**

The committee also recognizes the need to go beyond the formal early childhood education system to reach families and communities—both of which have a strong impact on young children's learning. An important component of reaching all children will need to include strategies aimed at children who are in other settings, such as homes or family child care.

**Recommendation 8: Early childhood education partnerships should be formed between family and community programs so that they are equipped to work together in promoting children's mathematics.**

For example, family education and support programs, such as the Head Start Family and Community Partnerships Program, should include information that provides guidance to families and communities as to why they should and how they can help children develop key mathematical ideas and skills. Furthermore, professionals working with families should be given training focused on early mathematics knowledge and skills, as well as have access to programs and resources on home-based mathematics activities. To this end, there is a need for development of more resources that can support mathematics in informal settings and through media and technology.

**Recommendation 9: There is a need for increased informal programming, curricular resources, software, and other media that can be used to support young children's mathematics learning in such settings as homes, community centers, libraries, and museums.**

## FUTURE RESEARCH

In its work, the committee conducted a comprehensive review of the existing evidence related to mathematics development and learning in early childhood. As noted, we have determined that the evidence base is robust enough to guide a major national initiative in early mathematics. Yet gaps remain in the knowledge base about children's mathematics education. We think it is critical that the research base continue to advance in a number of key areas outlined below.

*Implications for English language learners.* Increasingly, early childhood classrooms serve significant numbers of children whose first language is not English; these children will be held to the same expectations for future achievement as children whose home language is English. To date, little published research has investigated the teaching and learning of mathematics with preschool age children who are simultaneously learning English. The committee recommends research be conducted that can help identify the best methods of enhancing the mathematical learning of young children who speak a first language other than English.

*Research on the role of teachers in providing effective instruction.* In recent years, researchers have made progress in understanding the process of teaching mathematics at the elementary school level. This research stresses the role of teachers' knowledge and skill including their knowledge of mathematics, their understanding of children's mathematical thinking and learning and their pedagogical content knowledge (i.e., their knowledge of how to structure the classroom and curriculum and to engage children in activities so that the mathematics is accessible). However, there has been much less attention to similar issues in early childhood settings. Research is needed to determine the extent to which the findings from research in the higher grades apply to mathematics instruction in early childhood and what might be unique to early childhood.

*Evaluation of curricula.* In the course of our review of early childhood mathematics, it became clear that many of the available curricula have not been rigorously evaluated for effectiveness. High-quality curriculum research is needed that tracks the effectiveness of curricula during implementation, using the theories and instructional models that were originally used to guide development of the curriculum. This research must also consider how diversity in children's backgrounds and across learning environments influences implementation and effectiveness. To achieve these goals, the committee recommends that curriculum research and development move through phases: from early reviews of relevant research to the creation of learning materials to help children along the teaching-learning paths in this report, to cycles of baseline evaluation, and finally to confirmatory evaluation using rigorous designs, with all phases integrating quantitative and qualitative methodologies. Research of this type will help ensure that early childhood programs can make informed, evidence-based choices among curricula.

*Effective teacher preparation.* Much of the recent research on the preparation of early childhood educators has focused on whether the bachelor's degree is an effective marker for teachers' competency. While this line of inquiry has been helpful in identifying some of teachers' skills that are related to positive child learning outcomes, research in the field needs to move beyond the B.A./non-B.A. distinction. The committee recommends that research on the effectiveness of early childhood teachers focus on the content and quality of teacher education programs rather than on whether or not teachers have a bachelor's degree.

*Parental involvement.* It is unclear why families from low SES backgrounds often do not participate in educational activities and what can be done to promote their involvement in these programs. The committee therefore recommends the conduct of better descriptive studies that examine what parents understand about supporting their children's mathematics learning and how to promote parents involvement in these efforts. Furthermore, if parents do have knowledge about how to support their children's mathematical development but are not putting this knowledge into practice, it is important that research examine the impediments that stand in the way of their actively promoting early childhood mathematics.

*Interventions for children with mathematics learning disabilities.* Exploration of learning difficulties or disabilities in mathematics is a nascent area of research that needs expansion. Further exploration is needed to better understand what early number competencies are predictive of future success in mathematics. Such research can help identify children at risk for learning difficulties or disabilities in mathematics during the preschool years, develop targeted interventions for such children, and test their effectiveness.

# Appendix A

# Glossary

**Accumulator mechanism** refers to the nonverbal counting mechanism of infants that generates mental magnitudes for sets by adding a fixed magnitude for each unit that is enumerated. This system is inherently inexact, and its inexactness increases with increasing number. It provides an approximate numerical representation that does not preserve any representation of the items. Hence, it does not provide a way to distinguish successive numbers, such as 10 and 11.

**Additive comparison** situations are those in which two quantities are compared to find out how much more or how much less one is than the other.

**Analog magnitude system** refers to approximate representations of large numbers beginning with toddler and preschool-age children.

**Attribute blocks** refer to collections of blocks in which attributes (e.g., color, shape, size, thickness) are systematically varied so that children can sort them in multiple ways.

**Cardinality** refers to the number of items in the set.

**Change plus/change minus** situations refer to addition and subtraction situations in which there are three quantitative steps over time, a start quantity, a change, and a result. Change plus situations can be formulated with an equation of the form start quantity + change quantity = result quantity. Change minus situations can be formulated with an equation of the form start quantity − change quantity = result quantity.

**Child-guided experiences** refer to experiences in which children acquire knowledge and skills through their own exploration and through interactions with objects and with peers.

**Composing/decomposing** refers to putting together and taking apart and applies to numbers as well as to geometry and measurement. For example, 10 ones are composed to form one group of 10 and 6 can be decomposed into 5 + 1. Two identical right triangles can be composed to form a rectangle, and a hexagon can be decomposed into six triangles. Measurement itself requires viewing the attribute to be measured as composed of units.

**Computational fluency** refers to accurate, efficient, and flexible computation with basic operations.

**Credentialing** refers to the process of demonstrating and receiving formal recognition from an organization for achieving a pre-defined level of expertise in education.

**Direct instruction** refers to situations in which teachers give information or present content directly to children.

**Early childhood education (ECE) teachers** refer to all personnel whose primary role is to provide direct instructional services for young children. Included in this category are lead teachers, assistant teachers, aides, and family child care providers.

**ECE teaching workforce** refers to those who carry out both instructional and noninstructional roles in ECE settings. The term is an inclusive one that embraces teachers, others who work in the ECE settings and whose primary responsibility is not instructional (e.g., administrators), and individuals who work in settings that support ECE (e.g., resource and referral coordinators).

**Encouragement and affirmation** refers to feedback that relates to teachers' abilities to motivate children to sustain their efforts and engagement.

**Explicit instruction** refers to all of a teachers' instructional actions and interactions that are not unplanned or incidental.

**Feedback loops** refer to sustained exchanges between a teacher and child (or group of children) that leads the child to a better or deeper understanding of a particular idea.

**Finding a pattern** refers to looking for structures and organizing and classifying information. It is a mathematical process used throughout mathematics.

**Focused curriculum (primary mathematics)** refers to a curriculum that is designed and has the primary goal to teach mathematics with meaningful connections to children's interest and prior knowledge.

**Formal education** refers to the amount of credit-bearing coursework a teacher has completed at an accredited institution, including two- or four-year colleges and universities.

**Formative assessment** refers to the process of gaining insight into children's learning and thinking in the classroom and of using that information to guide instruction. It entails the use of several methods—observation, task, and flexible interview—that help the teacher develop ideas about children's thinking and learning and about teaching methods that can help them learn. Formative assessment is often inseparable from teaching and usually not distinctly identified as assessment, but formative assessment can also be used in a deliberate and organized format.

**Geometry** refers to the study of shapes and space, including flat, two-dimensional space as well as three-dimensional space.

**In-service education** refers to the formal education and training that one may receive while having formal responsibility for a group of children.

**Instruction/pedagogy** refers to **intentional teaching**.

**Instructional feedback** refers to a response where the teacher provides students with specific information about the content or process of learning and provides the opportunity to practice and master knowledge and skill.

**Instructional supports** refer to concept development, quality of feedback, and language modeling.

**Integration** refers to the blending together of two or more content areas in one activity or learning experience with the purpose of making content meaningful and accessible but also allowing more content to be covered during the instructional period.

**Intentional teaching** refers to holding a clear learning target as a goal and adapting teaching to the content and type of learning experience for the individual child, along with the use of formative assessment to determine the child's development in relation to the goal.

**Language modeling** refers to a practice by adults when they converse with children, ask open-ended questions, repeat or extend children's responses, and use a variety of words, including more advanced language and building on words the children already know.

**Manipulatives** refer to concrete objects—including blocks, geometric shapes, and items for counting—to support children's mathematical thinking.

**Mathematics teaching-learning path** refers to the significant steps in learning a particular mathematical topic with each new step building on the earlier steps. Teaching-learning paths are often referred to as learning trajectories, a term that emphasizes the sequential and direct route from one skill level to the next. The sources of a teaching-learning path are: (1) the subject matter being taught—what skills and knowledge provide the foundation for later learning, and (2) what is achievable/understandable for children at a certain age given their prior knowledge. Teaching-learning paths also provide a basis for targeting the curriculum, assessing children's progress along the path, and adapting their instruction to help children make continued progress.

**Mathematizing** refers to reinventing, redescribing, reorganizing, quantifying, structuring, abstracting, and generalizing concepts and situations first understood on an intuitive and informal level in the context of everyday activity into mathematical terms. This process allows children to create models of situations using mathematical objects or actions and their relationships to solve problems, including the use of increasingly abstract representations.

**Measurement** refers to the process of determining the size of an object with respect to a chosen attribute (such as length, area, or volume) and a chosen unit of measure (such as an inch, a square foot, or a gallon).

**Morphological marker** refers to the word element that signifies quantity, such as whether the word is singular or plural. For example, the *s* on the end of dog*s*, which indicates that the word is plural, is the morphological marker. The term **quantifier morphology** is used interchangeably with morphological marker.

**Number competencies** refer equally to both the knowledge and skills concerning number and operations that can be taught and learned.

**Number sense** refers to the interconnected knowledge of numbers and operations. It is a combination of early preverbal number sense and the increasingly important influence of experience and instruction.

**Numeral** refers to the symbol used to represent a number.

**Numerosity** refers to the quantity of a set.

**Object file system** refers to the representation of each object in a set comprised of very small numbers, but no representation of set size. For this form of representation, the objects in a small set are in 1-to-1 correspondence with each mental symbol. Thus, a set of three items is represented as "this" "this" "this" rather than "a set of three things."

**One-to-one (1-to-1) correspondence** refers to correspondence between two collections if every member of each collection is paired with exactly one member of the other collection and no members of either collection is unpaired or is paired with more than one member.

**Place value** refers to the meaning of a digit in a written number as determined by its placement within the number.

**Pre-service education** refers to the formal education and training that one receives prior to having formal responsibility for a group of children.

**Primary mathematics/focused mathematics time** refers to a dedicated instructional time focused on mathematics as the primary goal.

**Professional development** is an umbrella term including both formal education and training.

**Prompting thought processes** refers to a particular feedback strategy for mathematics instruction that asks students to explain their thinking or actions.

**Providing information** refers to clarifying incorrect answers or providing very specific information about the correct answer.

**Put together** situations refer to addition/subtraction situations in which two quantities are put together to make a third quantity.

**Relating and ordering** refers to mathematical processes of comparing and placing in order.

**Relating parts and wholes level** refers to a level of thinking that occurs when children combine pattern block shapes to make composites that they recognize as new shapes and to fill puzzles, with growing intentionality and anticipation.

**Scaffolding** refers to an instructional strategy in which the teacher provides information and assistance that allow children to perform at a higher level than they might be able to do on their own. It extends knowledge rather than verifying prior or existing knowledge.

**Secondary (embedded) mathematics** refers to a form of integration through which teaching and exposure to mathematics content is an ancillary activity. One or more subjects other than mathematics, such as literacy or science, are the primary goals of the activity.

**Spatial orientation** refers to knowing where one is and how to get around in the world. Children have cognitive systems that are based on their own position and their movements through space, as well as external references. They can learn to represent spatial relations and movement through space using both of these systems, eventually mathematizing their knowledge.

**Spatial visualization/imagery** refers to the process that occurs when there is understanding and performing imagined movements of two-dimensional and three-dimensional objects. To do this requires creating a mental image and manipulating it, showing the close relationship between these two cognitive abilities.

**Subitizing** is the process of recognizing and naming the number of objects in a set.

**Conceptual subitizing** refers to using pattern recognition to quickly determine the number of objects in a set, such as seeing 2 things and 2 things and knowing this makes 4 things in all.

**Perceptual subitizing** refers to instantly recognizing and naming the number of objects in a set.

**Superposition** is the act of placing one item on top of another.

**Take apart** situations refer to addition/subtraction situations in which a total quantity is taken apart to make two quantities (which do not have to be equal). These situations generally have several solutions. For example: Joey has 5 marbles to put in his 2 pockets. How many can he put in his left pocket and how many in his right pocket?

**Tangram** is a puzzle consisting of seven flat shapes, called *tans*, which are put together in different ways to form distinct geometric shapes.

**Teacher effectiveness** refers to the impact of teachers' actions and behaviors on the accomplishments and/or learning outcomes of the children they teach.

**Teacher-guided instruction** refers to teachers' planning and implementing experiences in which they provide explicit information, model or demonstrate skills, and use other teaching strategies in which they take the lead.

**Teacher-initiated learning experiences** refer to classroom experiences that are determined by the teacher's goals and direction, but ideally also reflect children's active engagement.

**Teacher quality** refers to the positive actions and behaviors of teachers, particularly with regard to their interactions with young children.

**Thinking about parts level** refers to a level of thinking that occurs when preschoolers can place shapes contiguously to form pictures in which several shapes play a single role (e.g., a leg might be created from three contiguous squares) but use trial and error and do not anticipate creation of new geometric shapes.

**Training** refers to the educational activities that take place outside of the formal education process. Such efforts may include coaching, mentoring, or workshops.

**Unitizing** refers to finding or creating a mathematical unit as it occurs in numerical, geometric, and spatial contexts.

**Virtual manipulatives** refer to manipulatives accessed through learning software and composed of digital "objects" that resemble physical objects and can be manipulated, usually with a mouse, in the same ways as their authentic counterparts. Virtual versions of concrete manipulatives typically used in mathematics education include base 10 blocks, Cuisenaire rods, and tangrams. Many available virtual manipulatives are paired with structured activities or suggestions to aid implementation in the classroom.

**Visual/holistic level** refers to a level of thinking that occurs when children have formed schemes, or mental "patterns," for these shape categories. It refers to the ability of preschoolers to learn to recognize a wide variety of shapes, including shapes that are different sizes and are presented at different orientations. They also learn to name common three-dimensional shapes informally and with mathematical names ("ball"/sphere, "box" or rectangular prism, "rectangular block" or "triangular block," "can"/cylinder). They name and describe these shapes, first using their own descriptions and increasingly adopting mathematical language.

# Appendix B

# Concepts of Measurement

At least eight concepts form the foundation of children's understanding of length measurement. These concepts include understanding of the attribute, conservation, transitivity, equal partitioning, iteration of a standard unit, accumulation of distance, origin, and relation to number.

*Understanding of the attribute* of length includes understanding that lengths span fixed distances ("Euclidean" rather than "topological" conceptions in the Piagetian formulation).

*Conservation* of length includes understanding that lengths span fixed distances and the understanding that as an object is moved, its length does not change. For example, if children are shown two equal length rods aligned, they usually agree that they are the same length. If one is moved to project beyond the other, children 4½ to 6 years often state that the projecting rod is longer (at either end; some maintain, "both are longer"; the literature is replete with different interpretations of these data, but certainly children's notion of "length" is not mathematically accurate). At 5 to 7 years, many children hesitate or vacillate; beyond that, they quickly answer correctly. Conservation of length develops as the child learns to measure (Inhelder, Sinclair, and Bovet, 1974).

*Transitivity* is the understanding that if the length of object X is equal to (or greater/less than) the length of object Y and object Y is the same length as (or greater/less than) object Z, then object X is the same length as (or greater/less than) object Z. A child with this understanding can use an object as a referent by which to compare the heights or lengths of other objects.

*Equal partitioning* is the mental activity of slicing up an object into the same-sized units. This idea is not obvious to children. It involves mentally seeing the object as something that can be partitioned (or cut up) before even physically measuring. Asking children what the hash marks on a ruler mean can reveal how they understand partitioning of length (Clements and Barrett, 1996; Lehrer, 2003). Some children, for instance, may understand "five" as a hash mark, not as a space that is cut into five equal-sized units. As children come to understand that units can also be partitioned, they come to grips with the idea that length is continuous (e.g., any unit can itself be further partitioned).

*Units and unit iteration.* Unit iteration requires the ability to think of the length of a small unit, such as a block as part of the length of the object being measured, and to place the smaller block repeatedly along the length of the larger object (Kamii and Clark, 1997; Steffe, 1991), tiling the length without gaps or overlaps, and counting these iterations. Such tiling, or space filling, is implied by partitioning, but that is not well established for young children, who also must see the need for equal partitioning and thus the use of identical units.

*Accumulation of distance and additivity.* Accumulation of distance is the understanding that as one iterates a unit along the length of an object and count the iteration, the number words signify the space covered by all units counted up to that point (Petitto, 1990). Piaget, Inhelder, and Szeminska (1960) characterized children's measuring activity as an accumulation of distance when the result of iterating forms nesting relationships to each other. That is, the space covered by three units is nested in or contained in the space covered by four units. Additivity is the related notion that length can be decomposed and composed, so that the total distance between two points is equivalent to the sum of the distances of any arbitrary set of segments that subdivide the line segment connecting those points. This is, of course, closely related to the same concepts in composition in arithmetic, with the added complexities of the continuous nature of measurement.

*Origin* is the notion that any point on a ratio scale can be used as the origin. Young children often begin a measurement with "1" instead of zero. Because measures of Euclidean space are invariant under translation (the distance between 45 and 50 is the same as that between 100 and 105), any point can serve as the origin.

*Relation between number and measurement.* Children must reorganize their understanding of the items they are counting to measure continuous units. They make measurement judgments based on counting ideas, often based on experiences counting discrete objects. For example, Inhelder, Sinclair, and Bovet (1974) showed children two rows of matches, in which the rows were the same length but each row was comprised of a different number of matches as shown in Figure B-1. Although, from the adult perspective, the lengths of the rows are the same, many children argued that

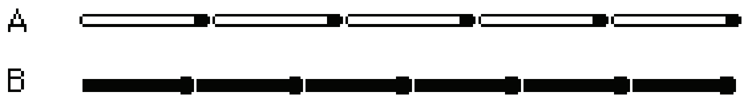


**FIGURE B-1** Relationship between number and measurement.

the row with 6 matches was longer because it had more matches. Thus, in measurement, there are situations that differ from the discrete cardinal situations. For example, when measuring with a ruler, the order-irrelevance principle does not apply and every element (e.g., each unit on a ruler) should not necessarily be counted (Fuson and Hall, 1982).

## Concepts of Area Measurement

Understanding of area measurement involves learning and coordinating many ideas (Clements and Stephan, 2004). Most of these ideas, such as transitivity, the relation between number and measurement, and unit iteration, operate in area measurement in a manner similar to length measurement. Two additional foundational concepts will be briefly described.

*Understanding of the attribute of area* involves giving a quantitative meaning to the amount of bounded two-dimensional surfaces.

*Equal partitioning* is the mental act of cutting two-dimensional space into parts, with equal partitioning requiring parts of equal area (usually congruent).

*Spatial structuring*. Children need to *structure an array* to understand area as truly two-dimensional. Spatial structuring is the mental operation of constructing an organization or form for an object or set of objects in space, a form of abstraction, the process of selecting, coordinating, unifying, and registering in memory a set of mental objects and actions. Based on Piaget and Inhelder's (1967) original formulation of coordinating dimensions, spatial structuring takes previously abstracted items as content and integrates them to form new structures. It creates stable patterns of mental actions that an individual uses to link sensory experiences, rather than the sensory input of the experiences themselves. Such spatial structuring precedes meaningful mathematical use of the structures, such as determining area or volume (Battista and Clements, 1996; Battista et al., 1998; Outhred and Mitchelmore, 1992). That is, children can be taught to multiply linear dimensions, but conceptual development demands this build on multiplicative thinking, which can develop first based on, for example, their thinking about a number of square units in a row times the number of rows (Nunes, Light, and Mason, 1993; note that children were less successful using rulers than square tiles).

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# Appendix C

# Biographical Sketches of Committee Members and Staff

**Christopher T. Cross** (*Chair*) is chairman of Cross & Joftus, LLC, an education policy consulting firm. He has been a senior fellow with the Center for Education Policy and a distinguished senior fellow with the Education Commission of the States. He also serves as a consultant to the Broad Foundation and the C.S. Mott Foundation and is on the board of directors of TeachFirst. From 1994 to 2002 he served as president and chief executive officer of the Council for Basic Education. Previously he served as director of the education initiative of the Business Roundtable and as assistant secretary for educational research and improvement in the U.S. Department of Education. He chaired the National Assessment of Title I Independent Review Panel on Evaluation for the U.S. Department of Education in 1995-2001 and the National Research Council Panel on Minority Representation in Special Education in 1997-2002 and was a member of the International Education and Foreign Language project in 2006-2007. He has written extensively in the education and public policy areas and has been published in numerous scholarly and technical publications. He has a B.A. from Whittier College and an M.A. in government from California State University, Los Angeles.

**Oscar Barbarin** is the L. Richardson and Emily Preyer bicentennial distinguished professor for strengthening families in the School of Social Work and a fellow of the Frank Porter Graham Child Development Institute at the University of North Carolina, Chapel Hill. His work has focused on understanding the roles that families play in preschool child competence, including the links between home and school, the early learning needs of

African American children and families, early childhood mental health, ethnic and gender-based achievement gaps, and the factors associated with and outcomes of preschool quality. He conducted a longitudinal study of child development in South Africa after the end of apartheid and authored *Mandela's Children: Child Development in Post-Apartheid South Africa*. Currently, he is leading studies targeting the academic needs of boys of color and their families. He recently organized and led the International Conference: Developmental Science and Early Schooling, sponsored by the Society for Research in Child Development, the Frank Porter Graham Child Development Center, and the Foundation for Child Development, which involved presentations and discussion of issues of translating research into practice. He has a B.A. from St. Joseph's Seminary College, an M.A. in counseling psychology from New York University, and M.S. and Ph.D. degrees in psychology from Rutgers University.

**Sybilla Beckmann** is professor of mathematics at the University of Georgia. Her mathematics research is focused on algebra/group theory, arithmetic geometry/algebraic number theory, commutative algebra/algebraic geometry, and tilings of the plane. She recently completed the second edition of *Mathematics for Elementary Teachers*, along with an activities guide and an instructor resource guide. Her recent work has focused on professional development of pre-service and in-service mathematics teachers, including preparing mathematicians to teach mathematics content to teachers and directly leading professional development workshops with teachers of mathematics. She was a member of the Curriculum Focal Points writing team conducted by the National Council of Teachers of Mathematics. In addition, she was a member of an expert panel on mathematics teacher preparation for the National Research Council Committee on Teacher Preparation. She also taught a daily class of sixth grade mathematics during the 2004-2005 school year. She has a Sc.B. in mathematics from Brown University and a Ph.D. in mathematics from the University of Pennsylvania.

**Sue Bredekamp** is director of research for the Council for Early Childhood Professional Recognition in Washington, DC. In her current role, she develops resources related to the administration of the Child Development Associate National Credentialing System. Previously, during her tenure at the National Association for the Education of Young Children, she developed the accreditation system for early childhood programs and coauthored the initial and revised edition of *Developmentally Appropriate Practice in Early Childhood Programs*. Throughout her career, she has focused on promoting the professional development of the early childhood workforce and developing standards for practice, also serving as a consultant to numerous programs and initiatives. She has a B.A. in English, an M.A. in early childhood education, and a Ph.D. in curriculum and instruction with concentrations in early childhood education and human development from the University of Maryland.

**Douglas H. Clements** is professor in the Department of Learning and Instruction at the State University of New York at Buffalo. He has led a number of initiatives aimed at identifying the key standards for early childhood mathematics, including participating in the writing group of the National Council of Teachers of Mathematics' *Curriculum Focal Points* to specify what mathematics should be taught at each grade level. In addition, he led a joint initiative between the National Association for the Education of Young Children and the National Council of Teachers of Mathematics to produce a joint position statement on the mathematics education of young children. He is also a member of the National Mathematics Advisory Panel created by President George W. Bush. His research and publications have focused on early childhood mathematics development, particularly children's development of geometry skills and the use of computers in mathematics education. He has also coauthored a number of curriculum products based on his Curriculum Research Framework, including a preschool curriculum, *Building Blocks*, which includes print, manipulatives, and the *Building Blocks* software, as well as extensions of that software up through the grades. He has a B.A. in sociology, an M.Ed. in elementary and remedial education, and a Ph.D. in elementary education from the State University of New York at Buffalo. He also has permanent certification to teach in the State of New York at the nursery, kindergarten, and first through sixth grade levels.

**Karen C. Fuson** is professor emerita at the School of Education and Social Policy at Northwestern University. Her recent work has focused on the continued development and revisions of *Children's Math Worlds*, a research-based program for students in kindergarten through fifth grade developed over 10 years in a wide range of classrooms and now published as *Math Expressions*. This research focused on developing a research-based coherent sequence of supportive representations and classroom structures through extensive classroom-based research and using analysis of curricula and strategies from a variety of countries. Through the years Fuson has devoted particular attention to the teaching of mathematical understandings and skills from age 2 to 8 and has also done extended research concerning the mathematics learning of Latino and urban children. She has studied and published widely on children's development of number concepts and arithmetic operations, word problem solving, as well as on mathematics education pedagogy. At the National Research Council, she was a member of the Mathematics Learning Study Committee. She has a B.A. in mathematics from Oberlin College and an M.A.T. in mathematics education and a Ph.D. in teacher education with emphases in mathematics and psychology from the University of Chicago.

**Yolanda Garcia** is director of the E3 Institute Advancing Excellence in Early Education at WestEd in San Jose, California. In this role, she supervises the Compensation and Retention Encourages Stability Program as well as other efforts to improve local early education in a variety of settings and program types through professional development, recruitment, and financial incentives. In addition, she is engaged in research to determine the impact of such programs on child outcomes. Her other research interests have focused on preschool English language learners and language development. Previously she served for 20 years as director of the Children's Services Department of the Office of Education of Santa Clara, California, overseeing services for more than 3,000 children in Head Start, state preschool, and other child care programs. She has served as a fellow in the U.S. Department of Health and Human Services and a senior program officer for the Charles Mott Foundation, focusing on strategies for grant programs on early education and family support. She was a member of the Head Start Quality and Improvement Panel and the National Research Council's Committee on Integrating the Science of Early Childhood Development. She has an M.A. in education administration from San Jose State University and an M.S. in social services administration with an emphasis in child welfare and public policy from the University of Chicago.

**Herbert Ginsburg** is the Jacob H. Schiff Foundation professor of psychology and education at Teachers' College, Columbia University. He is also professor in the Department of Mathematics Education and a Fulbright senior specialist. His research interests have focused on intellectual development and education, especially among poor and minority children, development of mathematical thinking, mathematics education and assessment, and the professional development of teachers. His current research involves evaluating *Big Math for Little Kids*, an early childhood mathematics curriculum he coauthored; examining the use of web-based video vignettes as a professional development tool; and studying computer-guided mathematics assessments for children. He is the author of numerous books, chapters, articles, and reviews, as well as several mathematics textbooks. He is a codeveloper of the Test of Early Mathematics Ability. He has a B.A. in social relations from Harvard University and M.S. and Ph.D. degrees in developmental psychology from the University of North Carolina, Chapel Hill.

**Nancy C. Jordan** is professor of education at the University of Delaware. Since 1998 she has been principal investigator of a federally funded project on children's mathematics difficulties and disabilities. She is the author or coauthor of many articles in mathematics learning difficulties and most recently has published articles in *Child Development*, the *Journal of Learning Disabilities*, *Developmental Science*, and the *Journal of Educational Psychology*. Her work focuses on early prediction and prevention of mathematics difficulties and connections between mathematics and reading difficulties. She has a B.A. from the University of Iowa (phi beta kappa), an M.A. from Northwestern University, and a Ph.D. in education from Harvard University. She completed a postdoctoral fellowship at the University of Chicago. Before beginning her doctoral studies, she taught elementary school children with special needs. She also taught and did clinical work in the Center for Development and Learning at the University of North Carolina, Chapel Hill.

**Sharon Lynn Kagan** is the Marx professor of early childhood and family policy, codirector of the National Center for Children and Families; and associate dean for policy and director of the Office of Policy and Research at Teachers College, Columbia University. She has examined the effects of policies and institutions on the development of children from birth to age 8 and their families, with particular interest in low-income children; private-public collaboration in service delivery; and standards, professional development, organizational change, and family support. Currently, she is working with UNICEF on the development, validation, and implementation of early learning standards in 40 countries. She is chair of the National Task Force on Early Childhood Accountability, coauthor of a recent book on the early childhood teaching workforce, director of the Policy Matters Project, and a consultant to states, foundations, and political leaders on early childhood pedagogy and practice. She was president of the National Association for the Education of Young Children and of Family Support America and chaired the National Education Goals Panel work on readiness. She has been a member of national panels on Head Start and Chapter I and was a member of the Committee on Early Childhood Pedagogy at the National Research Council. Early in her career, she was a Head Start teacher and director. She is the recipient of the Conant award from the Education Commission of the States, the Distinguished Services award from the Council of Chief State School Officers, and the McGraw Hill prize. She has a B.A. in English with a teaching certificate from the University of Michigan, an M.A. from Johns Hopkins University, and an Ed.D. in curriculum and teaching from Columbia University.

**Susan C. Levine** is professor of psychology and chair of the Developmental Psychology Program at the University of Chicago. She has studied early mathematical and cognitive development beginning in infancy, focusing most recently on the role of mathematical language and gesture inputs by parents and teachers. She is coauthor of *Quantitative Development Infancy and Early Childhood*. Her work has focused on basic cognitive developmental research to understand the nature of mathematical development in such areas as early numerical development, measurement, mental rotation, and proportional and spatial reasoning. In addition, she has examined the effects of brain injury and stroke on brain and cognitive development. She has a B.S. from Simmons College and a Ph.D. in psychology from the Massachusetts Institute of Technology.

**Kevin Miller** is professor and cochair of the Combined Program in Education and Psychology at the University of Michigan, where he is also professor in the Educational Studies and Psychology Departments, the Center for Human Growth and Development, and the Center for Chinese Studies. He has conducted extensive cross-cultural research between China and the United States in the areas of cognitive and mathematical development, specifically examining the role of culture, linguistics, and classroom practices in contributing to children's learning. More recently, he has been studying how video representations of teaching and learning can be used in understanding the relations between teaching and learning and improving the preparation of prospective teachers. He is chair of the Mathematics Education Review Panel for the Institute of Education Sciences at the U.S. Department of Education and a member of the Mathematical Sciences Education Board at the National Research Council. He has a B.A. in psychology from Haverford College and a Ph.D. in child and school psychology from the University of Minnesota.

**Robert C. Pianta** is the Novartis US Foundation professor of education and dean of the Curry School of Education at the University of Virginia, as well as professor in the Department of Psychology. He also serves as director of the National Center for Research in Early Childhood Education and the Center for Advanced Study of Teaching and Learning. His work has focused on the predictors of child outcomes and school readiness, particularly adult-child relationships, and the transition to kindergarten. His recent work has focused on better understanding the nature of teacher-child interactions, classroom quality, and child competence, through standardized observational assessment. He has also conducted research on professional development, at both the pre-service and in-service levels. He has recently begun work to develop a preschool mathematics curriculum, incorporating a web-based teacher support component. He has a B.S. and an M.A. in special education from the University of Connecticut and a Ph.D. in psychology from the University of Minnesota. He began his career as a special education teacher.

**Heidi Schweingruber** (*Senior Program Director*) is the deputy director of the Board on Science Education at the National Research Council. She codirected the study that produced the report *Taking Science to School: Learning and Teaching Science in Grades K-8* (2007) and served as research associate on the study that produced *America's Lab Report: Investigations in High School Science* (2005). She is currently directing a congressionally mandated review of precollege education programs at the National Aeronautics and Space Administration. Previously she worked as a senior research associate at the Institute of Education Sciences in the U.S. Department of Education, where she served as a program officer for the preschool curriculum evaluation program and for a grant program in mathematics education. She was also a liaison to the Department of Education's Mathematics and Science Initiative and an adviser to the Early Reading First Program. Previously, she was the director of research for the Rice University School Mathematics Project, an outreach program in K-12 mathematics education, and she taught in the psychology and education departments at Rice University. She has a Ph.D. in psychology (developmental) and anthropology and a certificate in culture and cognition from the University of Michigan.

**Taniesha A. Woods** (*Study Director*) is a senior program officer in the Center for Education and the Board on Children, Youth, and Families at the National Research Council and the Institute of Medicine. Her research interests include the examination of children's educational and social outcomes in an ecological systems framework. Her recent work investigates how school reform, particularly professional development, can improve the educational outcomes of children of color and those from low-income backgrounds. Previously she was a Society for Research in Child Development and American Association for the Advancement of Science congressional fellow assigned to the U.S. Senate Health, Education, Labor, and Pensions Committee and the Children and Families Subcommittee in the office of Senator Christopher Dodd, specializing in K-12 and postsecondary education issues. She has a B.A. in psychology and African and African American studies from the University of Oklahoma and a Ph.D. in developmental psychology, with a formal concentration in quantitative psychology, from the University of North Carolina, Chapel Hill.

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