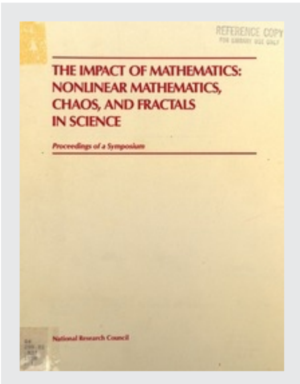

Impact of Mathematics: Nonlinear Mathematics, Chaos, and Fractals in Science: Proceedings of a Symposium

DETAILS

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Board on Mathematical Sciences; Commission on Physical Sciences, Mathematics, and Applications; National Research Council

THE IMPACT OF MATHEMATICS: NONLINEAR MATHEMATICS, CHAOS, AND FRACTALS IN SCIENCE

Proceedings of a Symposium

Board on Mathematical Sciences
Commission on Physical Sciences, Mathematics, and Applications
National Research Council

NATIONAL ACADEMY PRESS
Washington, D.C. 1990

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This report has been reviewed by a group other than the authors according to procedures approved by a Report Review Committee consisting of members of the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine.

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* The project that is the subject of this report was initiated under the predecessor group of the Commission on Physical Sciences, Mathematics, and Applications, which is the Commission on Physical Sciences, Mathematics, and Resources, whose members are listed in the appendix.

PREFACE

The Board on Mathematical Sciences held its third National Science and Technology Week Symposium at the National Academy of Sciences in Washington, DC on April 28, 1988. These symposia present significant mathematical developments of importance to science, technology, and national needs. The topics discussed at the 1988 symposium were nonlinear mathematics, chaos, and fractals. These concepts have gained prominence in the last one and one-half decades in concert with advancements in computer graphics. Nonlinear mathematics provides the machinery for mathematical models that are better approximations of some physical phenomena than linear models, and the new computer graphics capabilities enhance the researcher's ability to simulate and illustrate these phenomena.

The 1988 symposium featured presentations by the following scholars on the indicated topics:

Heinz-Otto Peitgen
Professor of Mathematics
University of Bremen and
University of California, Santa Cruz
"Fractals: Algorithms to Model Reality"

Fereydoon Family
Associate Professor of Physics
Emory University
"Mathematical Modelling of Snowflake Growth"

Francis C. Moon
Professor of Mechanical Engineering
Director, Sibley School of Mechanical and
Aerospace Engineering
Cornell University
"Fractals and Chaos in Mechanical Systems"

Phillip S. Marcus
Associate Professor of Mechanical Engineering
University of California, Berkeley
"Nonlinear Mathematics and Jupiter's Red Spot"

The symposium was moderated by James A. Yorke, Acting Director, Institute for Physical Science and Technology, University of Maryland. This volume contains summaries of the presentations and the ensuing discussions.

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National Research Council

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WELCOMING REMARKS

Lawrence H. Cox
Director
Board on Mathematical Sciences
National Research Council

Good afternoon. I am Larry Cox, Director of the Board on Mathematical Sciences which is sponsoring this event, the title of which is "The Impact of Mathematics: Nonlinear Mathematics, Chaos, and Fractals in Science." The Board on Mathematical Sciences is generally interested in what is new, important, and exciting in mathematics and, also, in its impact in the sciences and engineering. That is why we are here today. This event is supported by the National Science Foundation, the Air Force Office of Scientific Research, the Army Research Office, the Department of Energy, the National Security Agency, the Office of Naval Research, and the Conference Board of the Mathematical Sciences. Our moderator for today will be James Yorke of the University of Maryland. I will now turn the program over to Jim.

INTRODUCTORY REMARKS

James A. Yorke
Acting Director
Institute for Physical Science and Technology
University of Maryland

I would like to welcome you to a treat in mathematical concepts. You are going to encounter a wide variety of ideas that are unified mainly by their difficulty. The speakers today were chosen for their ability to present their material, but the material they are presenting is very complicated. The ideas to be presented today really came to the fore about a dozen years ago. For example, a dozen years ago the number of people who had high quality computer graphics was very small. The usual situation was as follows: One submitted a deck of cards and a day later it came back saying that the cards were in the wrong order. The cards were finally submitted in the right way and one obtained a form of graphics; big sheets with X's and O's, because that was the only way one could plot. While computers have been around in some sense since the 1950's, there has recently been a tremendous change in computer graphics. A number of the speakers today will make use of these improvements.

The title of today's conference mentions "nonlinear mathematics." Mathematics is an attempt to mirror the world. Linear models are not realistic in most cases. Thus, as our ability to handle more difficult situations increases, we must deal with a world that is not linear. To get a better approximation of reality, one must deal with new ideas that result from nonlinearity.

We will begin with Heinz-Otto Peitgen who is certainly a pioneer in the use of graphics. He is, also, independently, an excellent mathematician. The title of his talk is "Fractals: Algorithms to Model Reality."

FRACTALS: ALGORITHMS TO MODEL REALITY

Heinz-Otto Peitgen
Professor of Mathematics
University of Bremen and
University of California at Santa Cruz

Hartmut Jurgens
Director Graphics Laboratory
University of Bremen

"Fractal Geometry will make you see everything differently. There is a danger in reading further. You risk the loss of your childhood vision of clouds, forests, galaxies, leaves, feathers, flowers, mountains, torrents of water, carpets, bricks, and much else besides. Never again will your interpretation of these things be quite the same."

Michael Barnsley, 1988

1. Introduction

Our way to discuss and introduce fractals will be guided by the metaphor of languages. While Western languages, such as English, have a finite alphabet, Eastern languages, such as Chinese, have such a large number of characters that it is appropriate to say they have an infinite number of elements. Similarly, traditional geometry, i.e., Euclidean geometry, rests upon a few elements, such as the straight line, the circle and so on. Thus, Euclidean geometry compares very much to Western languages, while the new fractal geometry is more like an Eastern language, i.e., the number of its elements is unlimited.

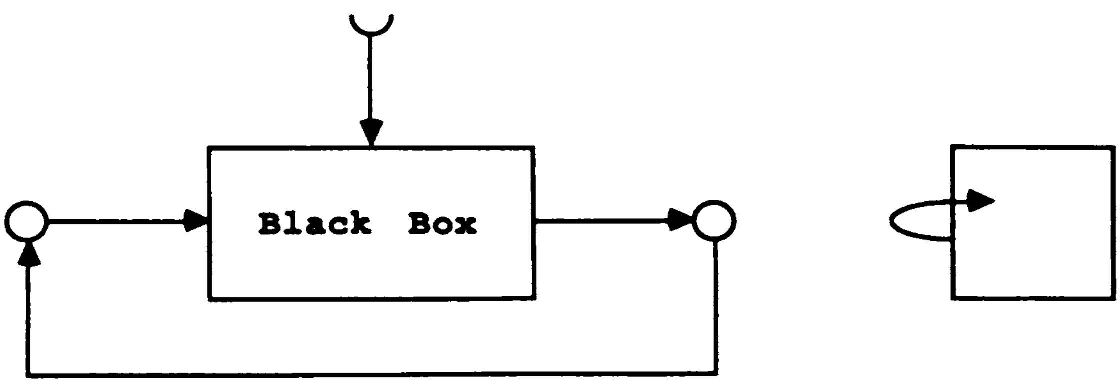

Figure 1: Algorithm/Feedback Loop

We will see that, in fact, there are even infinitely many dialects of fractal languages. What are they? The best way to describe them is to identify them as basic algorithms or feedback loops. The icon on the right of Figure 1 comprises a feedback loop with one processor. Figure 1 shows the typical scheme of a feedback loop. We use the icon on the right hand side as a symbol for feedback loops with one processor.

2. Linear Dialect

The most fundamental dialect is a version of the linear dialect. It has in turn, as all fractal dialects, an infinite number of elements (algorithms). We describe their typography in another metaphor: the multiple reduction copying machine (MRCM), see Figure 2.

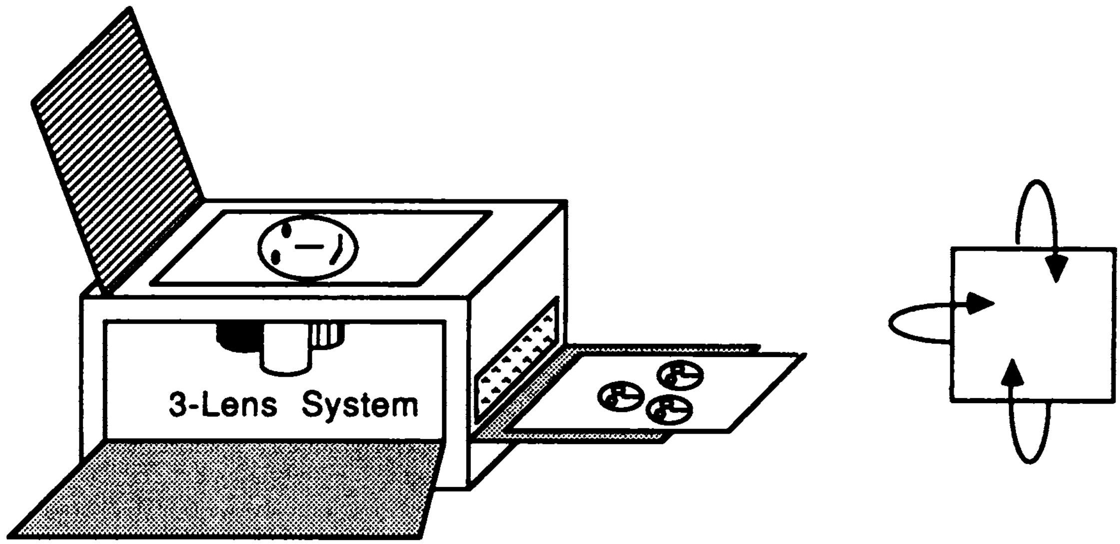

Figure 2: Multiple Reduction Copying Machine

The gist of the machine is this: It has several independent reduction lens systems (e.g., three in Figure 2), which reduce any given picture individually and displace the reduced copy. If we implement this machine into our general processor of Figure 1 we have an element of the linear fractal dialect and we have a different one for any choice of reductions and displacements.

Figure 3 shows an example. Step 0 identifies the initial picture, which is arbitrary. Step 1 displays the result after one application of the MRCM, thereby identifying the particular reductions and displacements. Incidentally, the reduction factor is 2 for each reduction lens here.

Figure 3 suggests that there is a final limit image, known in mathematics as the 'Sierpinsky triangle'. How does this element depend on the initial image fed into the MRCM? This is demonstrated in Figure 4, where we start the same MRCM as in Figure 3 with two more initial images, a triangle and the logo NRC.

Remarkably, the limit picture is always the same. Indeed, we have just illustrated a mathematical theorem, due to J. Hutchinson, which states that there is a unique limit picture for any MRCM and that limit picture can be obtained starting with an arbitrary image. Figure

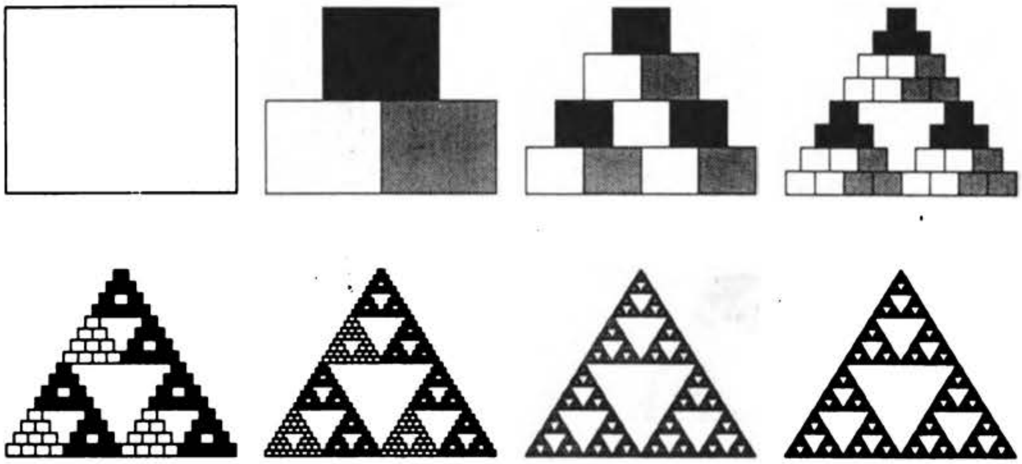

Figure 3: Development of final element

5 shows 3 more MRCMs, out of a variety of infinitely many choices, demonstrating the fascinating richness of the linear fractal dialect. Incidentally, reductions need not be uniform into all directions of the plane. For example, they may squeeze more horizontally than vertically, as demonstrated in the examples. Note that each element of the linear dialect is determined by $6 \cdot N$ numbers, where N is the number of reduction lenses in a MRCM.

Indeed, each reduction lens is given by a linear contraction transformation f of the plane which is usually described by a matrix and a displacement vector

$$f(x) = Ax + \mathbf{b}. \tag{1}$$

It is obvious that any black and white picture (given by some finite number of, say, black pixels on white background) can be coded in a trivial MRCM. Just select the trivial contraction for any black pixel for which the $a_{ij} = 0$ and the vector $\mathbf{b}$ identifies the pixel. M. Barnsley [2] has developed algorithms which approximate a given picture by a number N of contraction transformations which is small in comparison to the number of (black) pixels, i.e., the linear fractal dialect can be used for the task of image compression.

Once an image is compressed into N contractions of the form (1) the approximation which it yields with respect to the given image can be recovered in principal by applying the associated. Example 2 in Figure 5, however, shows that there is a computational problem. There we have 4 contractions and therefore the number of rectangles to be drawn when applying this MRCM grows like 4^k. Unfortunately, as one can see in Figure 5, one of the contractions reduces only slightly and therefore it would take some 50 applications of the MRCM to arrive close enough to the limit picture. Obviously $4^{50} \sim 10^{30}$ little rectangles

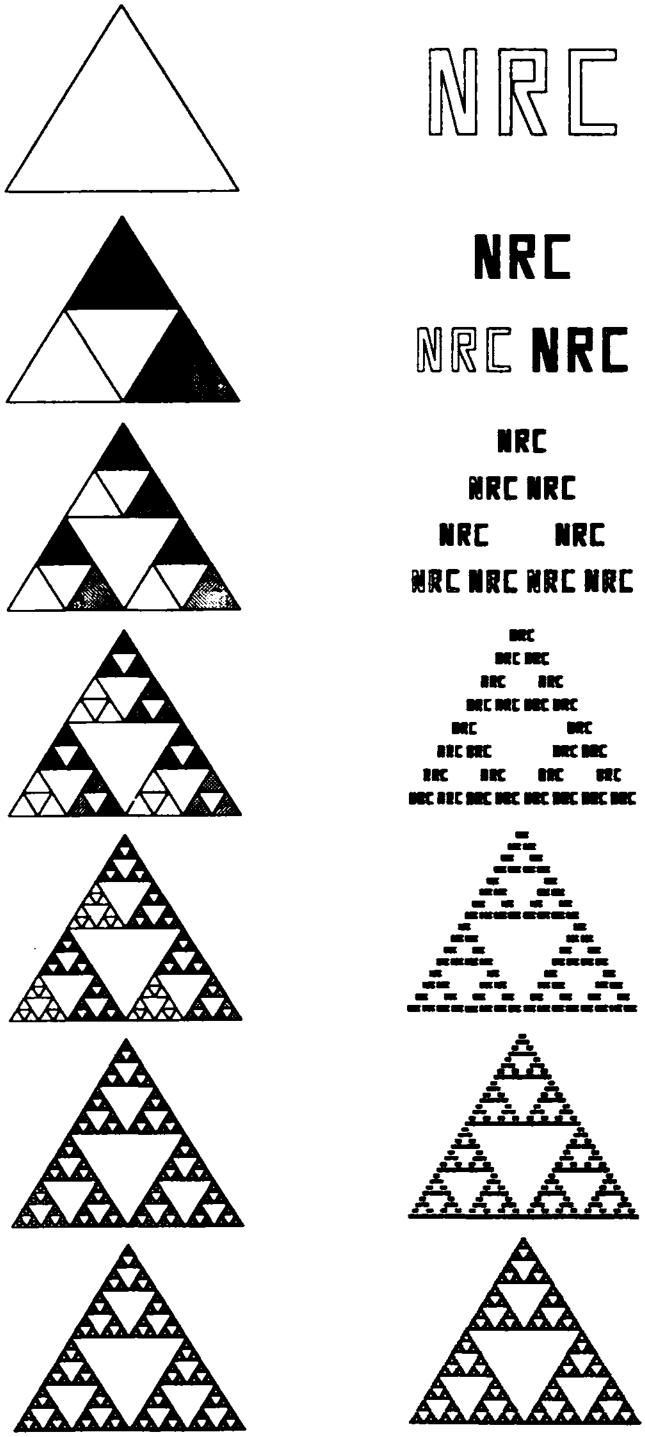

Figure 4: Independence of the initial image

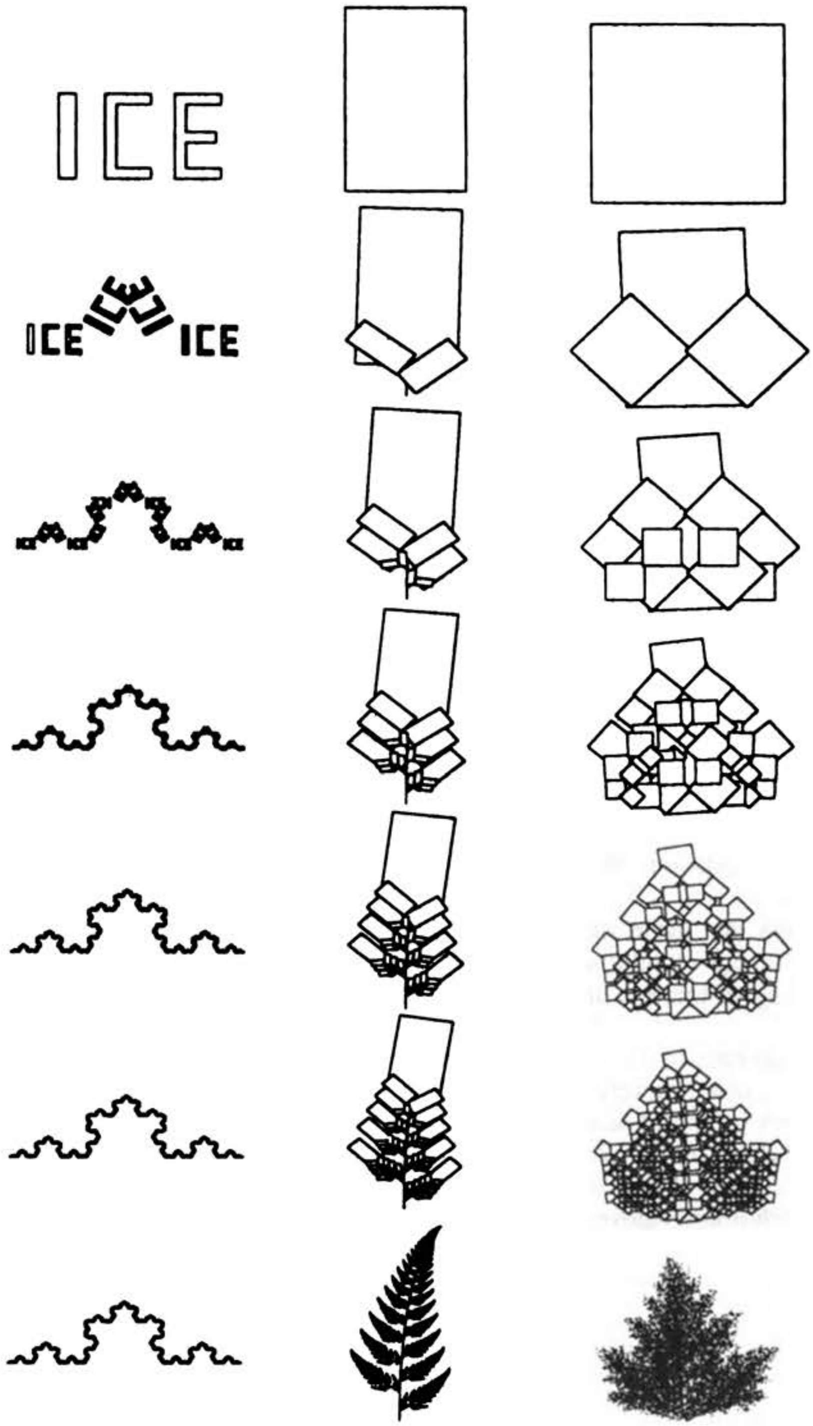

Figure 5: More elements of the linear dialect

are beyond the power of any reasonable computer. In other words, one needs a different algorithm to recover images from the set of N contractions, say $f_1, ..., f_N$, which have been chosen to compress a given image. This is the *random iteration algorithm, RIA*. One chooses N probabilities $p_1, ... p_N$, i.e., $p_i > 0$ for each i and their sum is 1, and then iterates the f_i randomly. More precisely, one selects an arbitrary starting point z_0 in the plane and then one generates the sequence

$$z_{\kappa+1} = f_{n(\kappa)} (z_\kappa), \ \kappa = 0, 1, ... \tag{2}$$

where in each step $n(\kappa) \in \{1, ..., N\}$ is chosen randomly with probability $p_{n(\kappa)}$. The sequence thus generated fills the final picture densely and it does so the more efficiently the better the p_i are adapted to the contractiveness of f_i, i.e. one chooses p_i larger if f_i contracts less.

3. Quadratic Dialect

Apart from the linear dialect there is an infinite number of nonlinear dialects. We will look into one of them: the *quadratic dialect*. This dialect is intimately connected with the recent development of chaos theory and is derived from the equation

$$x^2 + c = u \ , \tag{3}$$

where x, c, and u are complex numbers and c serves as a control parameter. Solving (3) for x we obtain

$$x_1 = + \sqrt{u - c} \text{ and } x_2 = - \sqrt{u - c}. \tag{4}$$

Thus, for each c there are two transformations

$$u \to + \sqrt{u - c} \text{ and } u \to - \sqrt{u - c}.$$

which we implement into our MRCM setting.

Figure 6 shows that again we have limit pictures. In fact, the existence of these pictures follows from the theory of Julia sets (see [12], [7], [3]). For all choices of c we find different limit pictures, so again there is an infinite number of *elements* in this dialect.

So far we have learned to speak in two different fractal dialects. As there are many more, we want to address now our final question. This is one of the most important questions in developing a theory of fractal languages. The question is: Is there a useful grammar for the dialect? Or mathematically spoken: Is there an order principle in the infinite variety of images obtained from (3). The answer is one of the most beautiful discoveries of modern mathematics. It rests upon a dichotomy, known as the Julia-Fatou dichotomy for quadratic polynomials. This establishes that for any choice of c the resulting image is classified by one of only two possible characteristic cases: It is (Figure 7)

- one piece (mathematically connected) or
- a cloud of infinitely many points (mathematically a Cantor set).

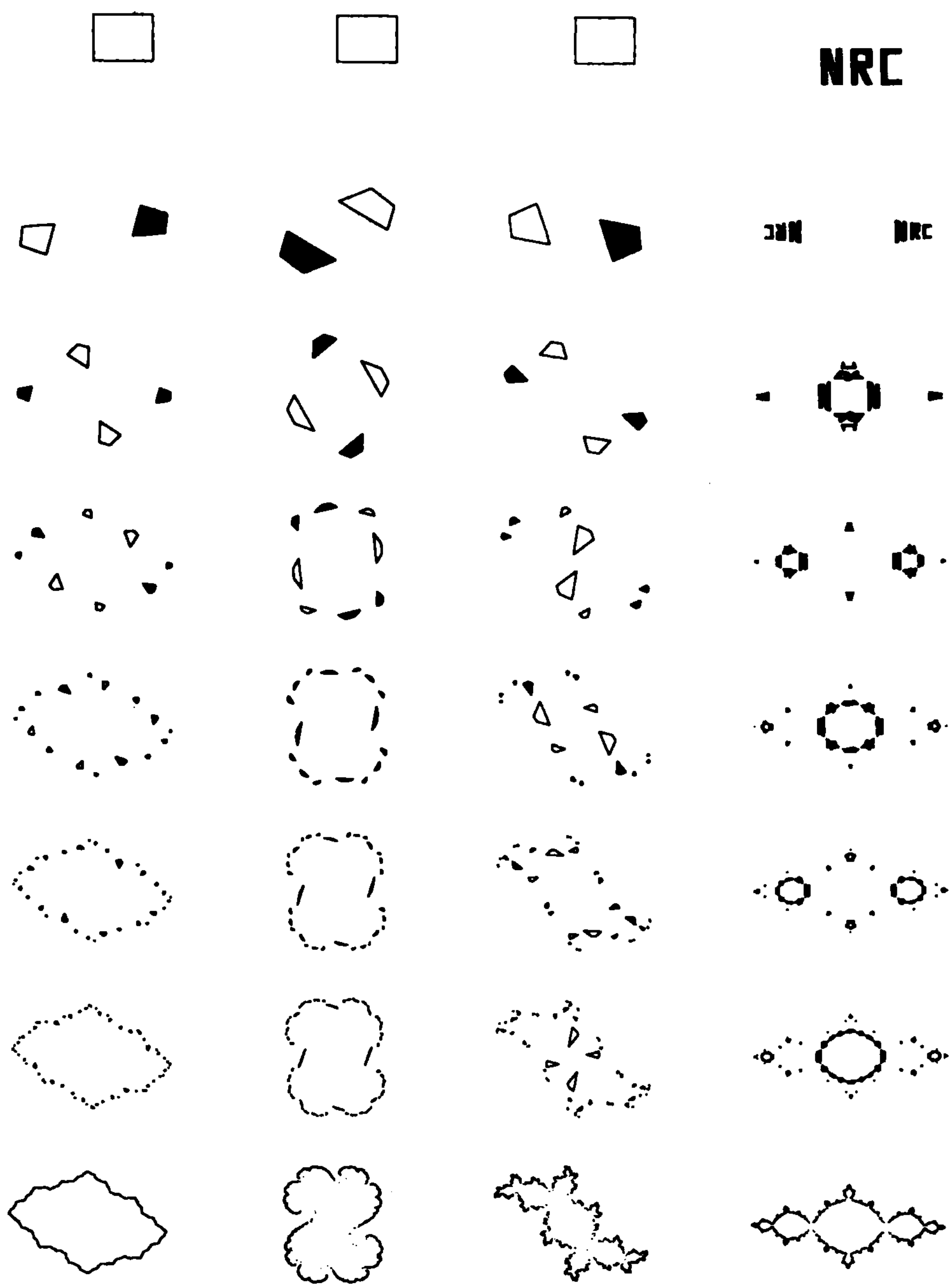

Figure 6: MRCM with c=(–0.4,0.2), c=(0.31,0.04), c=(–1.2,0.74), c=(–1.0,0.0) (from left to right)

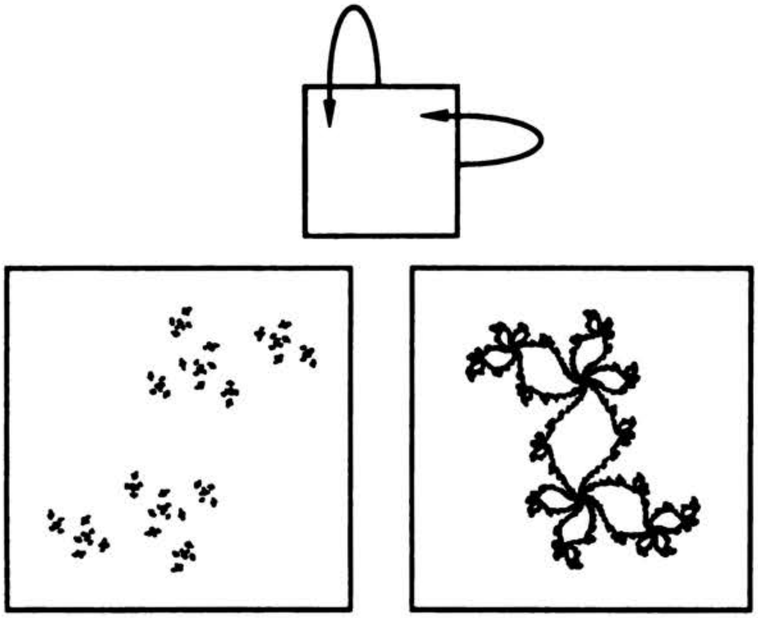

Figure 7: Dichotomy

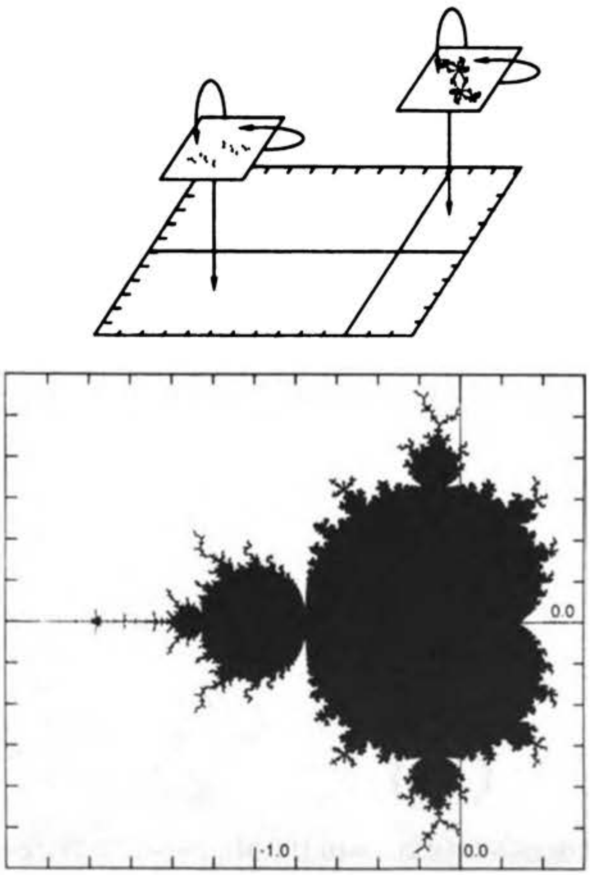

Figure 8: The Mandelbrot set as a grammar for the quadratic dialect

This leads us to the grammar of the quadratic dialect, the Mandelbrot set (Figure 8). It was obtained by a history-making experiment of B. Mandelbrot [13] in 1980, where each point c in the complex plane was colored black or white according to the dichotomy (a pixel representing a value c is set black provided the final image resulting from the corresponding MRCM for that c-value is connected).

The Mandelbrot set shows an amazingly complex pattern. Each of its parts can be identified with a distinctive paragraph of the grammar. Figure 9 shows a blow up sequence of the Mandelbrot set which reveals that little copies of itself can be found by magnifying. These identify subdialects of the quadratic dialect having similar grammars.

Our final point is a brief discussion of one of the most beautiful properties of the Mandelbrot set; i.e., it can be seen as a dictionary of all elements of the quadratic dialect. This is illustrated in Figure 10. Magnifying (by about 10^6) about the c-value identified by the crosshair in the Mandelbrot set we obtain a structure which we like to identify as a double-spiral. Looking up the corresponding MRCM-element for that c-value we obtain the structure in the lower right, which lives in the u-plane of complex numbers. Fixing c in that plane, which is the crosshair in that picture, and magnifying around that value by again about 10^6 we obtain the double spiral in the lower left. In this sense the Mandelbrot set can be seen as an image compressor of infinitely many fractal images, the entire quadratic dialect.

This property explains the unimaginable complexity of the Mandelbrot set and was already observed experimentally by its discoverer. Only recently it was established in all mathematical rigor by Tan Lei [18] for a subset of c-values which are dense in the boundary of the Mandelbrot set (the set of M. Misiurewicz points). Tan Lei's work rests upon a series of mathematical jewels by A. Douady and J. H. Hubbard [4], [5], [16] who have integrated one of the most beautiful shapes into some of the most beautiful mathematics.

Meanwhile the Mandelbrot set has been observed in many other nonlinear dialects as an order principle. This property has become known as the *universality* of the Mandelbrot set (see [6]).

The discovery of the Mandelbrot set has stimulated hope that fractal dialects (and even more generally, dynamical systems) can be understood through appropriate order principles as, for example, the Julia-Fatou dichotomy for the quadratic dialect.

4. Conclusion

This short note was intended to give a brief introduction to fractal languages. This is only one of many exciting interpretations and applications of the fractal approach (see [14]). Moreover, many current developments could not be discussed for reasons of length. Some should be mentioned at least.

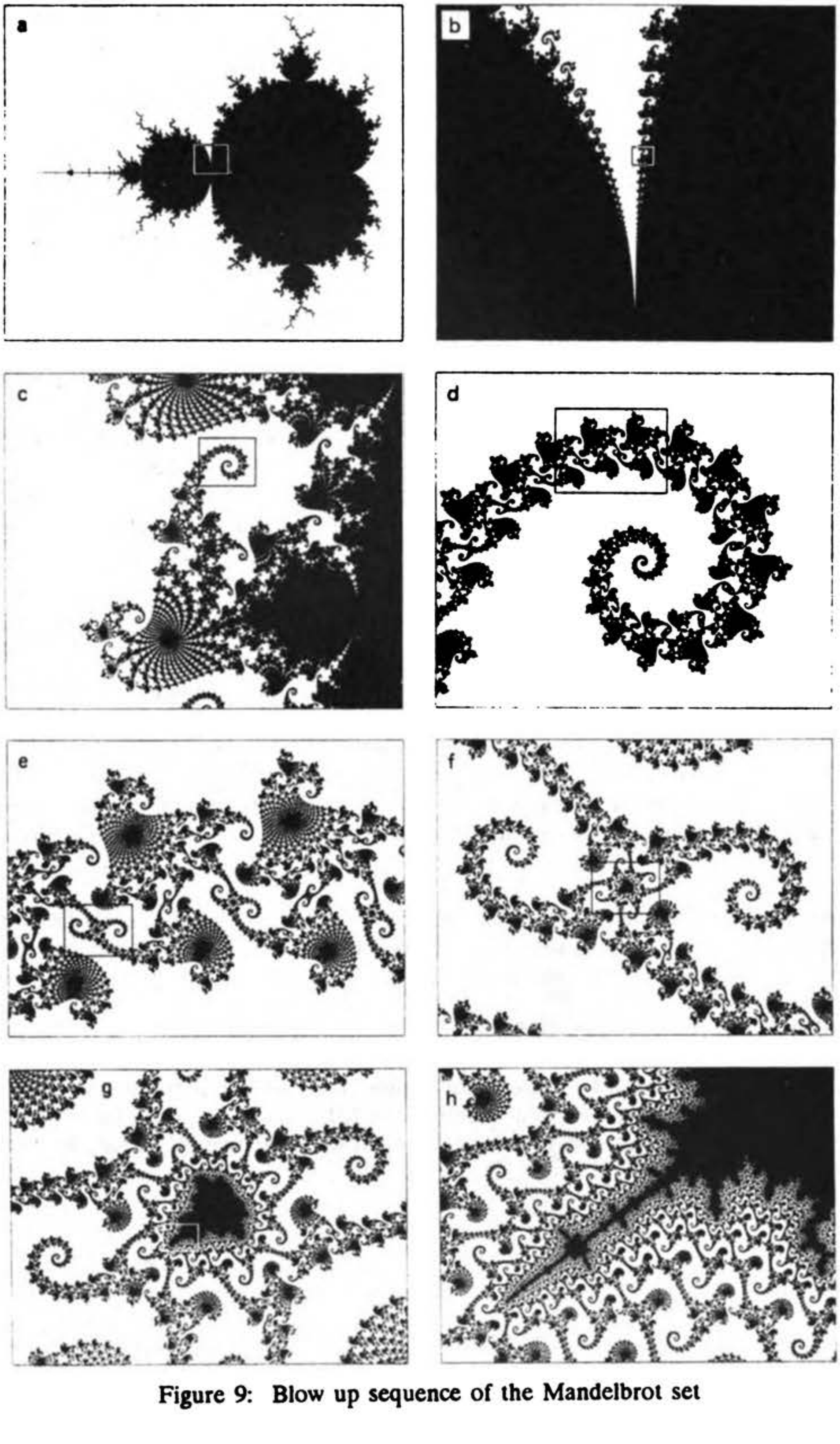

Figure 9: Blow up sequence of the Mandelbrot set

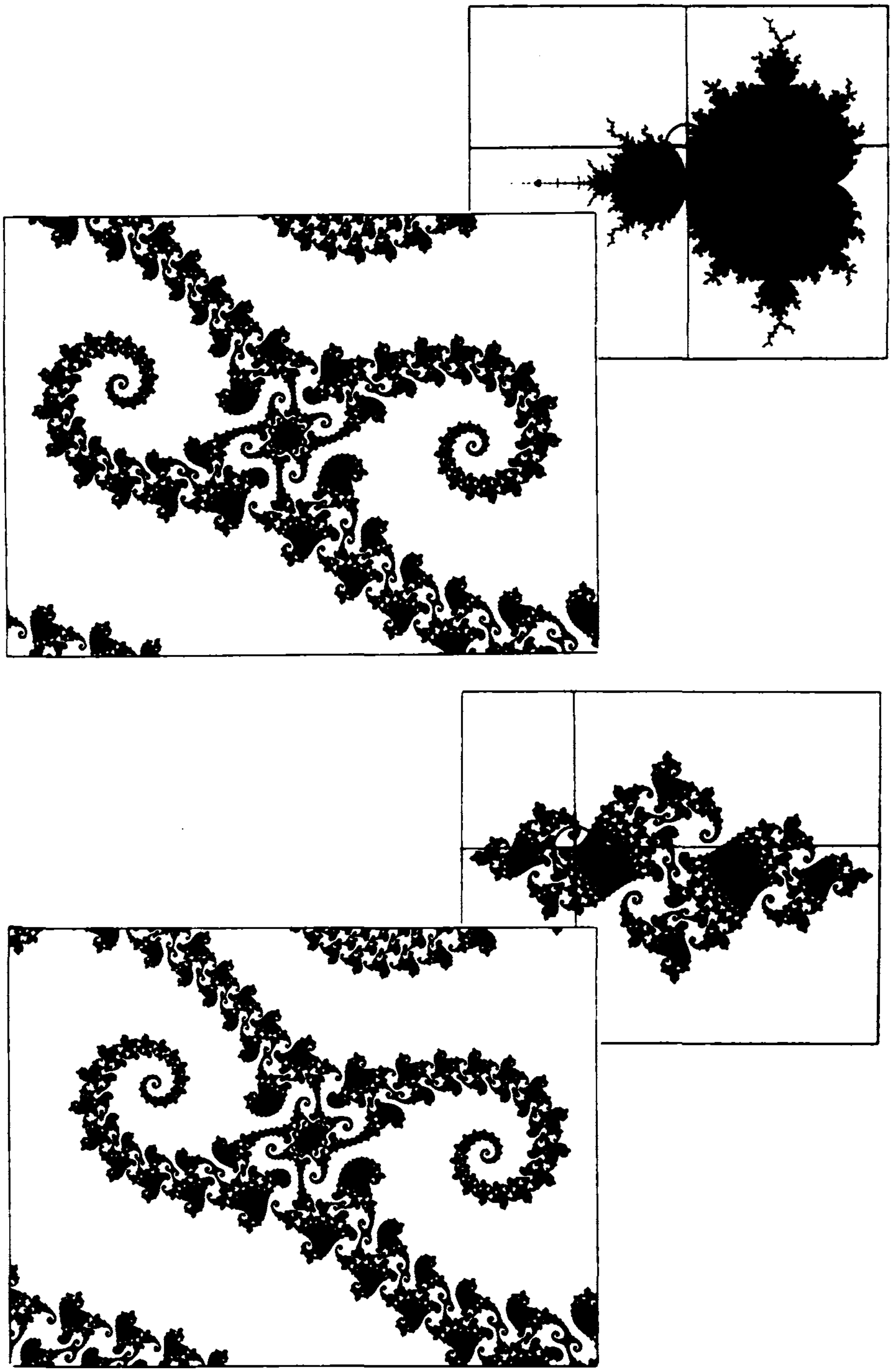

Figure 10: Similarity between Mandelbrot set and elements of the quadratic dialect

It may appear that the ideas for fractal image compression are restricted to black and white images. M. Barnsley et al. [2] have recently shown how the random iteration algorithm can be used for half tone and thus also for color images.

The term fractal in this note means *deterministic* fractal. There is a whole world of fractals which are best characterized as *random* fractals (see [14], [17]) and which have their own languages. Actually, the random iteration algorithm is not in that class. It merely reflects a generating method for a deterministic fractal. Indeed, if one uses the random iteration algorithm for any of the images in Figures 3 - 5 the final image does not depend on the choices of the probabilities p_i. They only have an impact on the efficiency of the algorithm.

A typical random fractal is obtained by the midpoint displacement method: Choose a line segment, pick its midpoint and lift/lower it by a randomly chosen amount. Then repeat for the resulting line segments, and so on. Obviously, the choice of the random process leaves its imprints on the final image (assume that the random lifts decrease according to some power law). R. Voss has demonstrated the power of related methods for the graphical construction of clouds, mountains, etc., very convincingly (see [17]).

A final remark is about dimensions. There are numbers attached to fractals which characterize their intrinsic complexity. This is the family of *fractal dimensions*. There is Hausdorff dimension, self-similarity dimension, capacity dimension, information dimension and so on (see [8] and [10]). Some people appear to be disappointed that there are that many dimensions. However, each of them characterizes one significant aspect of the complexity of fractal structures. Another exciting development is the *multifractal* approach. Here one tries to discuss a whole spectrum of characteristic fractal dimensions (see [15] and [9]).

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MATHEMATICAL MODELLING OF SNOWFLAKE GROWTH

Fereydoon Family
Associate Professor of Physics
Department of Physics
Emory University

1. Introduction

Snowflakes, with their poetic beauty and delicate geometrical complexity, have occupied the interests of generations of scientists and laymen alike. Snow crystals are solidified water droplets or ice, arranged in a limitless variety of shapes and patterns. The earliest recorded study of snow crystals dates back to the second century B.C. when a Chinese scholar, Han Yin [1], observed that while *'Flowers of plants and trees are generally five pointed, those of snow are always six-pointed.'* The first illustration of snowflakes (reproduced in Figure 1a) is found in a book published in 1555 on natural phenomena by Olaus Magnus [2], Archbishop of Uppsala. A long standing puzzle is that among the 23 patterns in this book only one of them is a six-sided star and the rest bear no resemblance to snowflakes. Systematic and serious scientific study of snow crystals began with the works of Kepler [3] in 1611 and Descartes [4] in 1635. They clearly recognized the hexagonal symmetry of the snow crystal (see Figure 1b) and made serious attempts to explain how they are formed and why they are six-sided. Since then, the greatest advances have been in the qualitative and quantitative illustrations of the variety of snow crystals. From an aesthetic, rather than scientific, point of view, the most fascinating and familiar collections of snowflakes are found in ***Snow Crystals*** by Bentley and Humphreys [5]. This book contains 2,000 photographs of snow crystals taken by Bentley in Vermont in a 50 year period. Even a small sample of these patterns, such as those reproduced in Figure 1c, gives one a feeling of the beauty and variety of snow crystal shapes. In more modern times, understanding the growth of snow crystals has been a problem of considerable experimental and theoretical interest [6].

Despite many attempts, no theoretical approach had ever produced the variety of intricate snow crystal shapes that are found in nature. Thus, the recent development of the deterministic growth model [7] by my colleague Tamás Vicsek, my graduate student Daniel Platt, and myself, which for the first time could produce a limitless variety of snowflake patterns, has been a step forward in understanding the complexity of this remarkable phenomenon. The goal of the model is to generate snow crystal patterns using the fundamental mathematical and physical laws that govern the solidification of ice molecules under conditions appropriate for the growth of snowflakes and other dendritic solidification processes.

One of the novel features of the deterministic growth model is that in this model we solve the governing mathematical equations for the growth of snow crystals using a computer simulation method based on an aggregation process. An aggregation approach lends itself very naturally to modelling of diffusion-limited growth processes, such as the growth of snow crystals. This is in contrast to methods based on numerically solving the thermal diffusion equation, such as the boundary integral method, which are not intuitive and consequently have not been successful in generating realistic snowflake patterns. In this talk I will describe the deterministic growth model [7] and I will attempt to demonstrate the effectiveness of this approach in modelling the growth of snow crystals. I believe that our ability to model the formation of snow crystals using computer simulations will provide us with a deeper understanding of a variety of other solidification problems and will enhance our admiration of the beauty of snowflakes and strengthen our appreciation of natural phenomena in general.

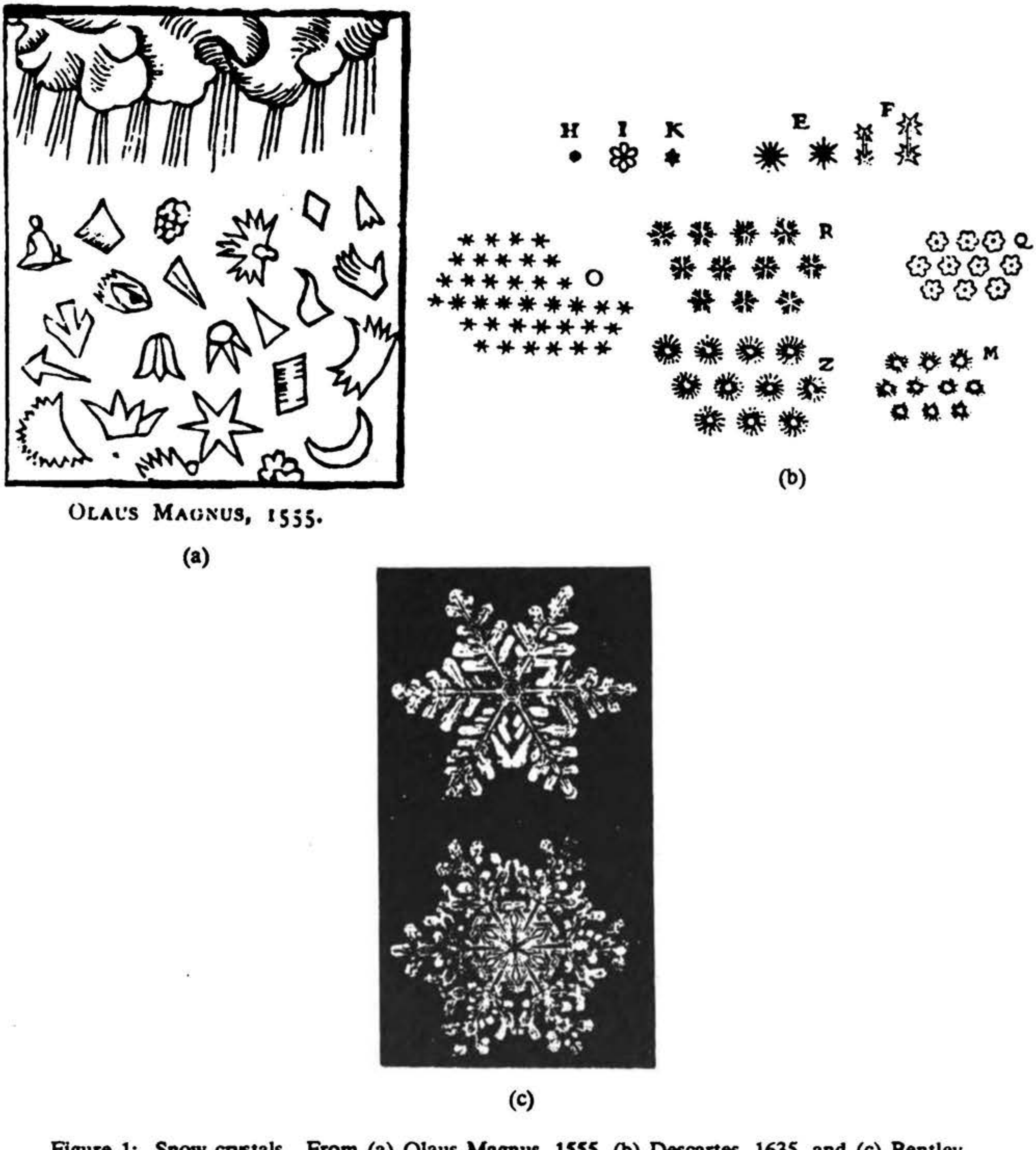


Figure 1: Snow crystals. From (a) Olaus Magnus, 1555, (b) Descartes, 1635, and (c) Bentley and Humphreys, 1931.

2. Why Study Snow Crystal Growth?

The aesthetic beauty and the complex processes involved in the formation of snowflakes would, in most cases, be considered overwhelmingly sufficient justification for attempting to answer the question: How are snowflakes formed? It turns out that the problem of the growth of snow crystals is related to a host of other equally perplexing phenomena of considerable scientific and practical interest. These problems arise naturally in many diverse fields ranging from pure mathematics and physics to meteorology and chemical engineering. I will discuss three of these areas of interest. I will emphasize how understanding the growth of snow crystals could lead to significant developments in solving other problems of both scientific and practical importance.

2.1 Meteorology

The severe effects of snow on conditions of human life needs little elaboration and discussion. This alone is a compelling reason for wanting to know how snow is formed and how it can be controlled. There are, however, many less obvious reasons for wanting to understand snow crystal growth in meteorology. Snow crystals are unique among all atmospheric phenomena in that they arrive on earth carrying with them a complete record of all the environmental conditions that they have passed through. Rather precise phase diagrams have been developed relating the qualitative shapes of snow crystals to such environmental conditions as temperature and humidity. Therefore, snow crystal patterns can be used as precise scientific tools for studying atmospheric phenomena far from the observation point. From another practical point of view, understanding how snow crystals are formed will eventually lead to a better understanding of snow clouds and snow storms. This will in turn make the flight of aircraft and communications through snow clouds safer and more effective, respectively.

2.2 Metallurgy: Dendritic Solidification

Almost all metallic and most other types of materials used in industrial and scientific applications are formed by casting, which is the method of shaping a material by first melting it and then allowing it to solidify into a specific shape. Many properties of materials are determined by the detail of the processes that it undergoes during casting. For many centuries, casting has been an art, rather than a science, because solidification is one of the least understood processes in materials science. One of the most complex problems during a solidification process is the formation of tiny crystalline protrusions that look like pine trees. These are called *dendrites*, from the Greek word for tree. Dendrites form during any process where a small piece of a material grows out into an undercooled liquid. The fact that this protrusion extends beyond the rest of the solid enhances its ability to release heat, and, therefore, it tends to solidify faster than the rest of the interface.

Dendritic solidification commonly occurs in castings and its presence has significant effects on the properties of materials. The cause of unstable dendritic growth in casting is the presence of impurities from which tree-like projections form in the liquid. The formation of dendrites during commercial production of metals is one of the long-standing problems in metallurgy. The growth of snow crystals is the classic example of dendritic solidification. Therefore, understanding the complex morphology and dendritic structure of snow crystals is of fundamental importance in solving the problem of how and why dendrites are formed and how they can be used to produce man-made materials with specific properties.

2.3 Pattern Formation: Self-Organized Complexity

One of the most exciting and challenging problems of current interest is to understand how matter and energy possess the innate ability to spontaneously form complex structures and patterns. The study of self-organized pattern formation processes encompasses all areas of science, from physics and astronomy to chemistry and biology. The classic example of self-organization is the formation of snow crystals [8]. Here, billions of water molecules join together to produce patterns of immense complexity and symmetry on scales that far exceed the range of the forces that hold the molecules together. The initial state of the water molecules before agglomerating to form the snowflake is completely chaotic. Yet the final product is a snowflake possessing immense geometrical intricacy and mathematical symmetry. This is an example of forming order from chaos, a process which points to the inadequacy of the second law of thermodynamics in describing such complex processes.

In recent years it has been recognized that the formation of a wide variety of complex patterns, both fractal [9] and regular [10], is governed by the diffusion equation which also controls the growth of snowflakes. This implies that understanding how snow crystal patterns are formed could open the door for understanding other processes in which complex patterns are spontaneously generated. In particular, it would be of both scientific and industrial interest to understand such related processes as dendritic solidification, viscous fingering, electrodeposition, and dielectric breakdown. To indicate the richness and variety of the patterns, a few examples are shown in Figure 2. In addition to physical processes, the formation of such biological structures as the networks of blood vessels and nerves (Figure 2) are also governed by the diffusion equation. The possibility that the same mathematical models developed for understanding snow crystals could be useful in understanding biological development is a significant prospect that can not be overemphasized.

3. The Deterministic Growth Model

Snow crystals are formed by the freezing of water molecules. We know very well that this is not sufficient to explain the formation of snowflakes, because when we make ice by freezing water in an ice tray we do not get snowflakes. To see why, let us consider the two types of solidification processes shown in Figure 3. Both of these systems consist of a container that is filled with a liquid, such as water, and heat is removed through the walls. The main difference between the two systems is that in the system in Figure 3a the liquid is initially at a temperature above the melting temperature. In this case, solidification starts at the walls and proceeds uniformly toward the center of the container. This type of growth is completely stable and no dendritic patterns are formed. On the other hand, in the system Figure 3b the liquid is initially cooled to a temperature well below the melting temperature. If the solidification is initiated from a seed at the center of the vessel, then the heat generated at the surface of the solid must be carried away by the surrounding liquid. The most efficient way to remove heat is to increase the surface area. Thus, the liquid-solid interface breaks up into more and more wrinkles to allow for more heat to be removed. This leads to an unstable growth because if a little wrinkle bulges out from the surface, the temperature gradient there will increase. Thus, more heat can diffuse away, which in turn will lead to faster growth at that tip. This effect -- known as the Mullins-Sekerka instability [8] -- is responsible for the dendritic shapes found in snowflakes.

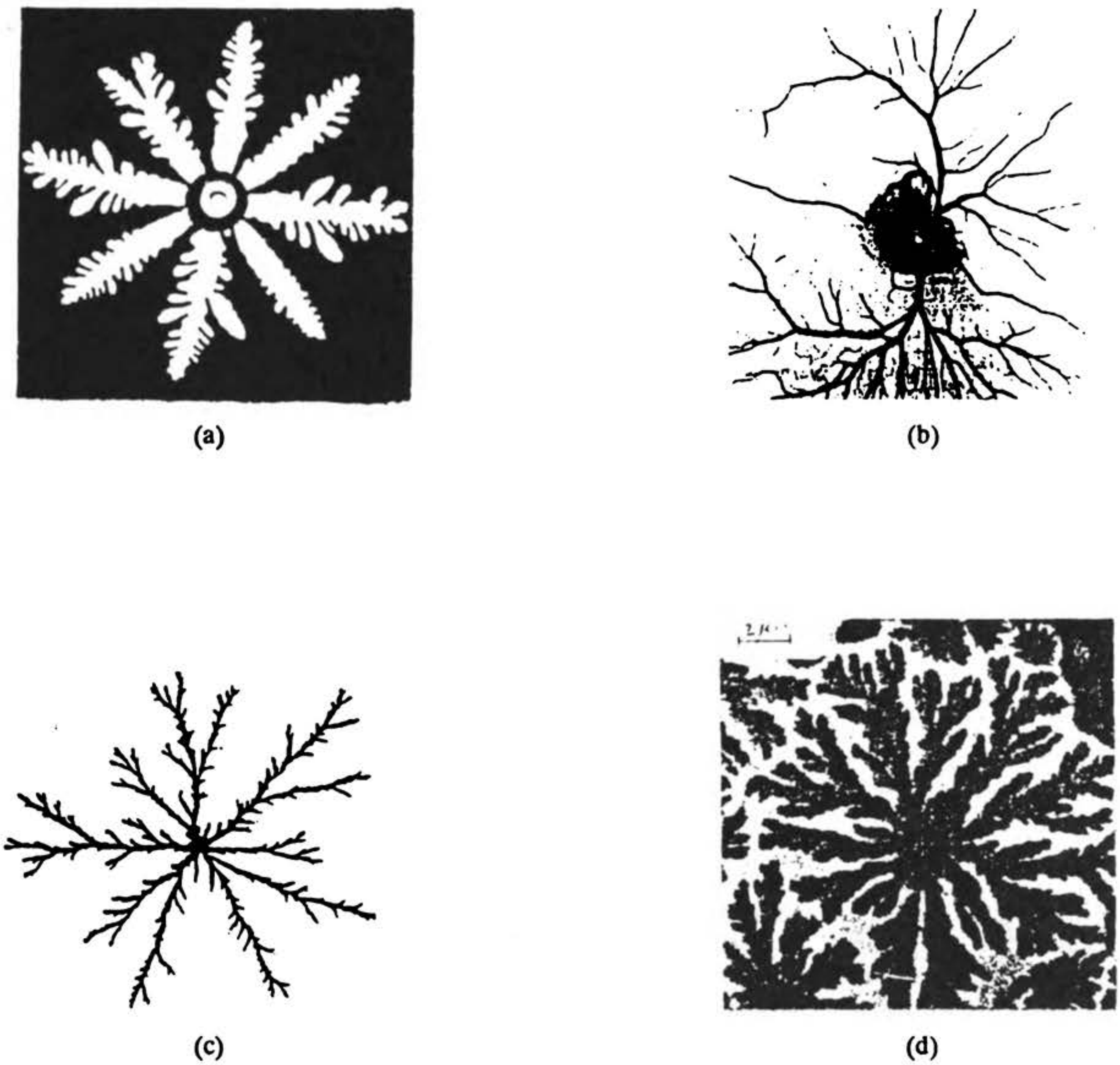


Figure 2: Examples of diffusion-limited pattern formation. (a) Viscous fingering patterns in nematic liquid crystals (From Buka, Vicsek and Kertész, *Nature* **323**, 424, 1986), (b) blood vessels in a chick embryo (From Tsonis and Tsonis, *Perspect. Biol. Med.* **30**, 355, 1986), (c) Fractal viscous fingering pattern (From Daccord, Nitmann and Stanley, *Phys. Rev. Lett.* **56**, 336, 1986), (d) Amorphous Al-Ge crystals (From Deutscher and Lareah, Physica **140A**, 191, 1986)

Snow crystals are formed under conditions similar to Figure 3b, except that the growth occurs in a low-density vapor rather than a liquid. Water vapor in the air diffuses into the surface of the growing crystal and freezes there. Mass conservation implies that the speed with which the interface advances is proportional to the gradient of the vapor density. Similarly, the latent heat generated in the solidification process diffuses away by escaping to the atmosphere. Again, heat conservation imposes the condition that the normal velocity at a given point on the interface is proportional to the temperature gradient there. Thus, one can treat the growth of snow crystals as either aggregation of water molecules, diffusion of heat, or both. It turns out that, mathematically, these are all equivalent formulations of the problem, because they all lead to the same set of mathematical equations, namely the diffusion equation

$$D\nabla^2 u = \partial u/\partial t, \tag{1}$$

with the boundary condition at the surface,

$$v_n \propto |\nabla u| \hat{n}. \tag{2}$$

Here, $u(r,t)$ is the normalized diffusion field, D is the diffusion coefficient, and v_n is the component of the velocity normal to the interface. In addition to (1) and (2), the field u is assumed to be constant at a distance much larger than the size of the growing crystal, which for dimensions greater than two can be taken to be infinity.

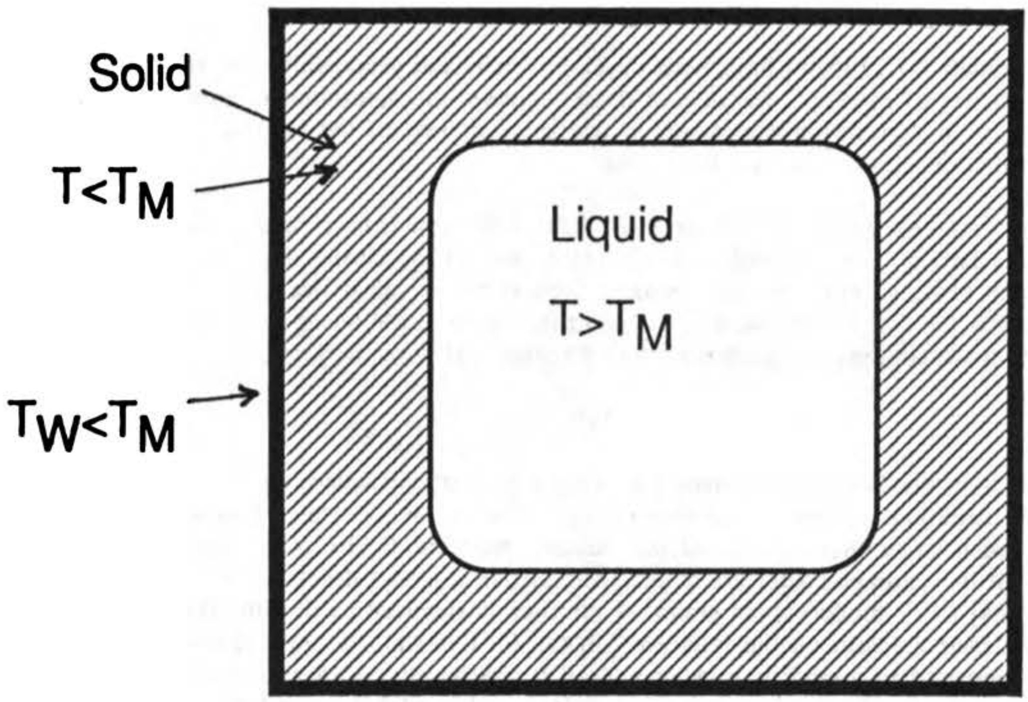
The Mullins-Sekerka instability [8], implies that under conditions (1) and (2) a smooth surface is unstable against small perturbations of all wavelengths, such as noise. Therefore, if a part of the surface moves faster, then the gradient at the tip of this protrusion is increased, leading to further growth at this tip. This effect results in continuous branching of tips. Thus, any wrinkling at the interface is amplified, leading to more wrinkling at the next length scale. Under these conditions, the resulting structures are fractal patterns.

If (1) and (2) were the only factors controlling the growth, all snowflakes would be stringy and tenuous fractal objects. The factor that stabilizes this instability is the surface tension. The effect of surface tension is to smooth out the perturbations at the surface so that the instability due to the increased gradient at sharp tips is reduced. Surface tension is introduced by the Gibbs-Thomson condition which specifies the value of the diffusion field u at the interface Γ,

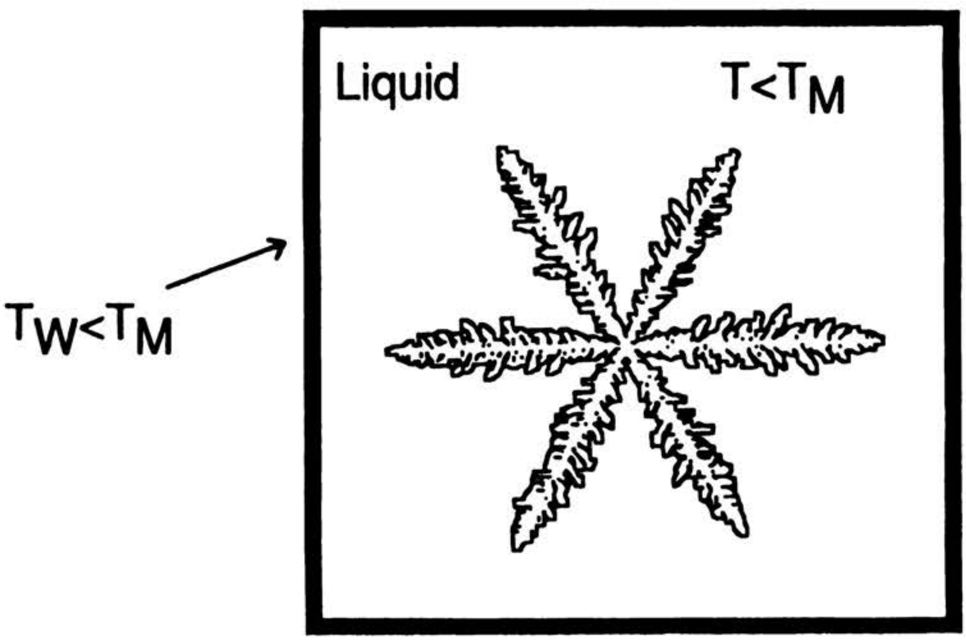
$$u_\Gamma = u_0(1 - d_0\kappa), \tag{3}$$

where u_0 is the value of the field at the interface in the absence of surface tension, the capillary length d_0 is proportional to surface tension, and κ is the surface curvature. Let us assume that u is proportional to the temperature. Suppose a bump appears at the interface. Then by (3) the temperature at that point is slightly reduced because of positive curvature, and heat will flow there causing the bump to melt back. Thus, the effect of surface tension is to reduce the gradient at sharp tips and stabilize smooth surfaces.

In addition to describing the growth of snow crystals, (1) - (3) contain the essential mathematics of a wide class of non-local growth processes with moving boundary conditions, including aggregation [11,12], electrodeposition [13,14], dielectric breakdown [15], and viscous fingering [16,17]. In the last two cases the field u satisfies the Laplace equation. But despite their generality and intensive efforts, little progress has been made in generating snowflake patterns by analytically solving the equations. On the other hand, numerical methods based on



(a)



(b)

Figure 3: Conditions which produce (a) stable and (b) unstable solidification.

aggregation models [11,12,15,18] have been developed in recent years which are very effective in producing a variety of patterns. Because of these successes the deterministic growth model [7] is also based on an aggregation-type model in which the motion of the interface is simulated by addition of single particles to a growing cluster.

In the deterministic growth model the cluster starts to grow from a seed particle placed at the center of a lattice. Since according to (2) the growth of the cluster depends on the gradient of the diffusion field, we must first determine u everywhere on the lattice by solving (1). In order to simplify the calculation, we first assume that the interface advances so slowly that we can make the quasi-stationary approximation and replace (1) with the Laplace equation,

$$\nabla^2 u = 0. \tag{4}$$

At each time step, the field u is calculated by solving the lattice Laplace equation (4) subject to the boundary conditions (2) and (3) at the surface, and $u=0$ on a circle of radius R, which is taken to be much larger than the size of the cluster. After the field u is calculated, the gradients at all the perimeter sites are determined. As in a model developed by Family, Vicsek, and Taggett [18] (FVT model) these gradients are then normalized by dividing them by the value of the largest gradient on the interface. In the FVT model, the normalized gradient at each site is compared to a random number and all perimeter sites with a gradient larger than the random number are filled. In the deterministic growth model this condition is generalized by comparing the gradients with a parameter, p, whose choice is dictated by the physics of the problem. In this model only those sites having a normalized gradient larger than the parameter p are filled. When p is a random number, we recover the FVT model [18]. The correct choice of p for crystal growth is the one which satisfies boundary conditions (2) and (3).

4. Random Fractals: FVT Model

Before I discuss the form of p which would be appropriate for the solidification problem, I would like to show you what happens when p is a random variable, *i.e.*, its value varies from one perimeter site to the next, and the surface tension is zero. This is equivalent to the FVT model [18]. In the simulation, the value of the normalized gradient at a perimeter site is compared with a random number and the site is occupied if the value of the gradient is larger than the random number. The process of picking a new random number for each surface site and testing and occupying the site is repeated until all the perimeter sites have been checked. After this process is finished, the field u is determined everywhere using a relaxation method and the above steps are repeated again. Clearly, in this approach a finite number of particles are added to the growing cluster before the field u is relaxed.

A typical fractal pattern obtained from the FVT model on the square lattice is shown in Figure 4 [18]. One of the main features of this figure is the anisotropic shape of the cluster. This four-fold anisotropy was not imposed by the growth process. It is due to the inherent anisotropy of the underlying square lattice. Thus, in the growth of snowflakes we have used the six-fold anisotropy of the triangular lattice to simulate the effects of the six-fold anisotropy in their surface tension.

5. Regular Fractals: Laplace Carpets

Instead of choosing a different random number for each perimeter site we now let p be a fixed constant throughout the growth process and let $d_0 = 0$, i.e., no surface tension. Unlike

Figure 4: Random Laplace Fractal pattern generated using the FVT model.

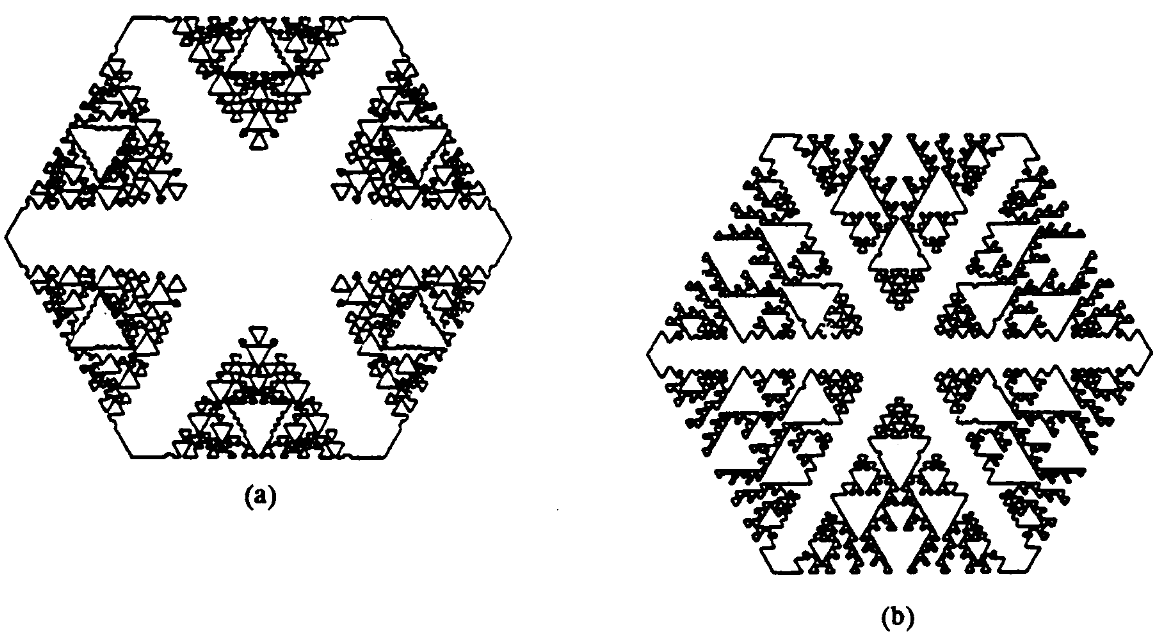


Figure 5: Regular Fractals: Laplace carpets generated on a triangular lattice by the deterministic growth model with no surface tension.

the previous model and most other growth models, there is no noise in this process, *i.e.*, this is a *deterministic* growth model [7]. Consequently, the patterns generated in this process are regular fractals [7]. The resulting patterns on a triangular lattice for $p = 0.30$ and $p = 0.40$ are shown in Figure 5. In the limit $p = 0$, all perimeter sites are filled and the result is a dense polygon having the symmetry of the underlying lattice. For finite p, the patterns are regular fractals with a fractal dimension which varies from 2 to 1 as p is increased from 0 to 1.

The Laplace carpets closely resemble the type of patterns that are often found in a cellular automaton [19]. It would be instructive to investigate the possibility of a deeper connection between these two processes than visual similarity. This would provide the first direct connection between a local growth model – the cellular automaton – and a nonlocal diffusive process.

6. Dendritic Solidification: Snowflakes

As shown in Figure 6a, when p is a constant, as in the previous model, v_n is constant at surface sites having a normalized gradient larger than p and zero otherwise. In contrast, dendritic growth is governed by (2), and the interface growth velocity must be proportional to the local gradient, as shown in Figure 6b. This implies that within a time interval Δt, p must vary linearly with the time so that sites having maximum gradient are always filled, while those with smaller gradients are filled less frequently, depending on the local gradient. In order to implement this boundary condition in our model, we discretize the time interval into c steps and assume that during this time interval p is given by [18]

$$p = a + b \cdot t_{\mathrm{mod}\ c}, \tag{5}$$

which is a piecewise function with c steps, approximating a straight line with slope b.

The above method can produce practically all types of observed two-dimensional dendritic patterns by changing the surface tension in (3) and parameters a and b in (5). In order to take surface tension into account, we first solve for u everywhere outside the cluster. We then apply boundary condition (3) by numerically determining κ using a method developed by Vicsek [20]. After the new values of u at the surface sites are determined, we calculate the gradients and proceed to normalize them as discussed before. The parameter a in (5) can be varied in order to simulate the effects of changing the environmental conditions during the growth. By changing a, we can effectively compensate for the quasi-stationary approximation made in replacing (1) with (4). Large and positive values of a lead to the formation of highly branched and dendritic patterns, while for negative values of a we obtain smooth, almost equilibrium hexagonal crystal shapes. For intermediate values of the parameters, combinations of these patterns are obtained. Changing a also accounts for such factors as surface migration and rearrangements that occur with varying degrees under different environmental conditions.

The effectiveness of the deterministic growth method is best demonstrated by the great variety of morphologies it generates. In Figure 7, we show only a few examples of six-fold symmetric snowflakes generated by varying the environmental conditions during the growth. All of these figures were generated by the stepwise function (5), with $c = 5$ and varying a. To obtain a faceted, near equilibrium interface pattern, a was made negative ($0 > a > -0.1$), and to obtain a boundary with a sharp, dendritic shape, a was made greater than zero ($0.5 > a > 0$). As a was changed, b was adjusted to account for the fact that at the end of each time interval the straight line must pass through the point $p = 1$.

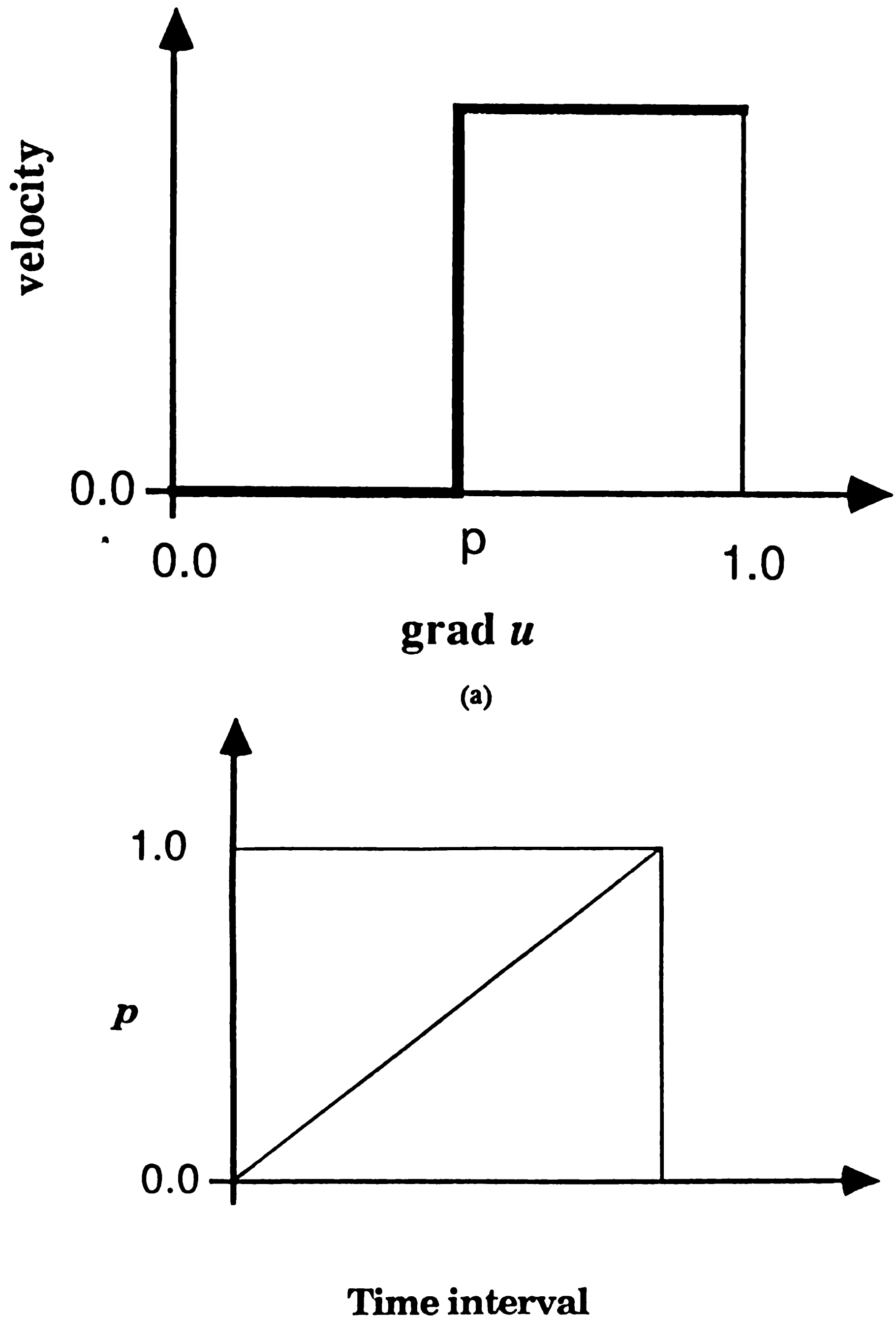
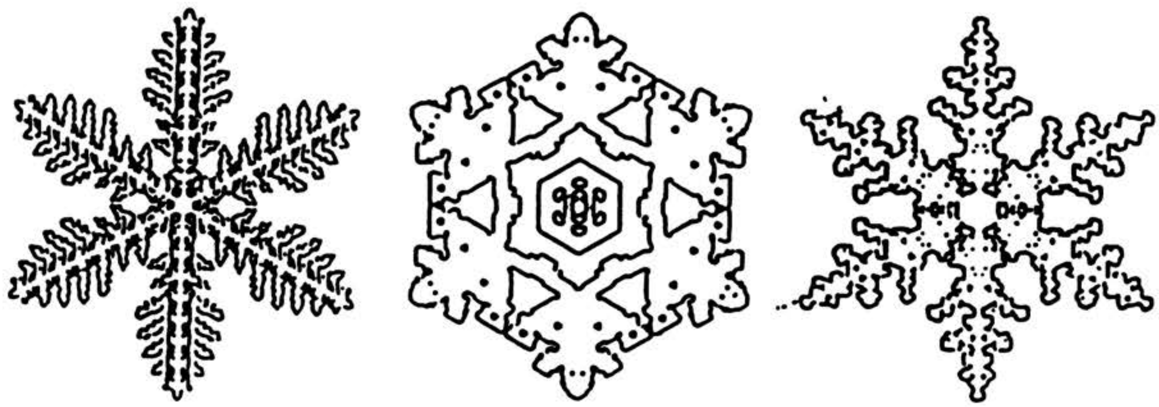
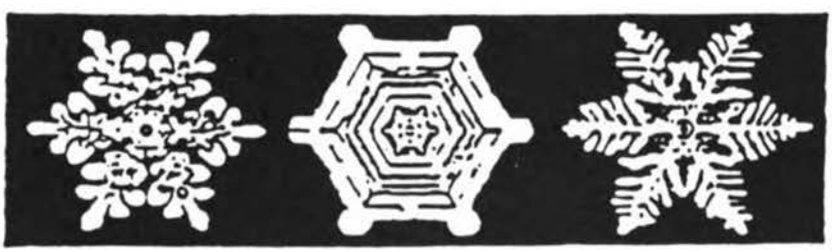


Figure 6: (a) When p is a constant, the interface velocity is a step function and resulting patterns are regular Laplace carpets. (b) In dendritic solidifcation, the fact that (2) must be satisfied implies that p must be a linear function of time.



(a)



(b)

Figure 7: (a) Three examples of snow crystal patterns are shown which were generated by the deterministic grown model, (b) three patterns of natural snowflakes. From [7].

7. **Conclusions**

In this brief talk I have discussed simulations of the growth of snow crystals by the deterministic growth model [7] based on aggregation-type processes. This approach appears to be much more effective in producing complex dendritic structures than the previous methods based on numerical solutions to the solidification equations. The most rewarding feature of the model is that despite its simplicity, it can generate an infinite variety of snowflake patterns that are in striking resemblance to those found in nature. I hope that this has convinced you that the mathematics behind the deterministic growth model contains the essential ingredients for describing the growth of snow crystals. This model is going to be a useful tool for sorting out various long-standing and puzzling aspects of the growth and formation of complex snow crystal patterns.

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Color Plates

Plate 1: Fractals and Chaos in Mechanical Systems (Francis C. Moon) (Discussed on page 44)

Plate 2: Jupiter's Great Red Spot (Phillip S. Marcus) (Discussed on page 47)

Plate 3: An Example of Initial Conditions for the Spectral Numerical Method (Phillip S. Marcus) (Discussed on page 49)

Plates 4-10: Temporal Evaluation of the Flow in a Spectral Numerical Simulation (Phillip S. Marcus) (Discussed on page 50)

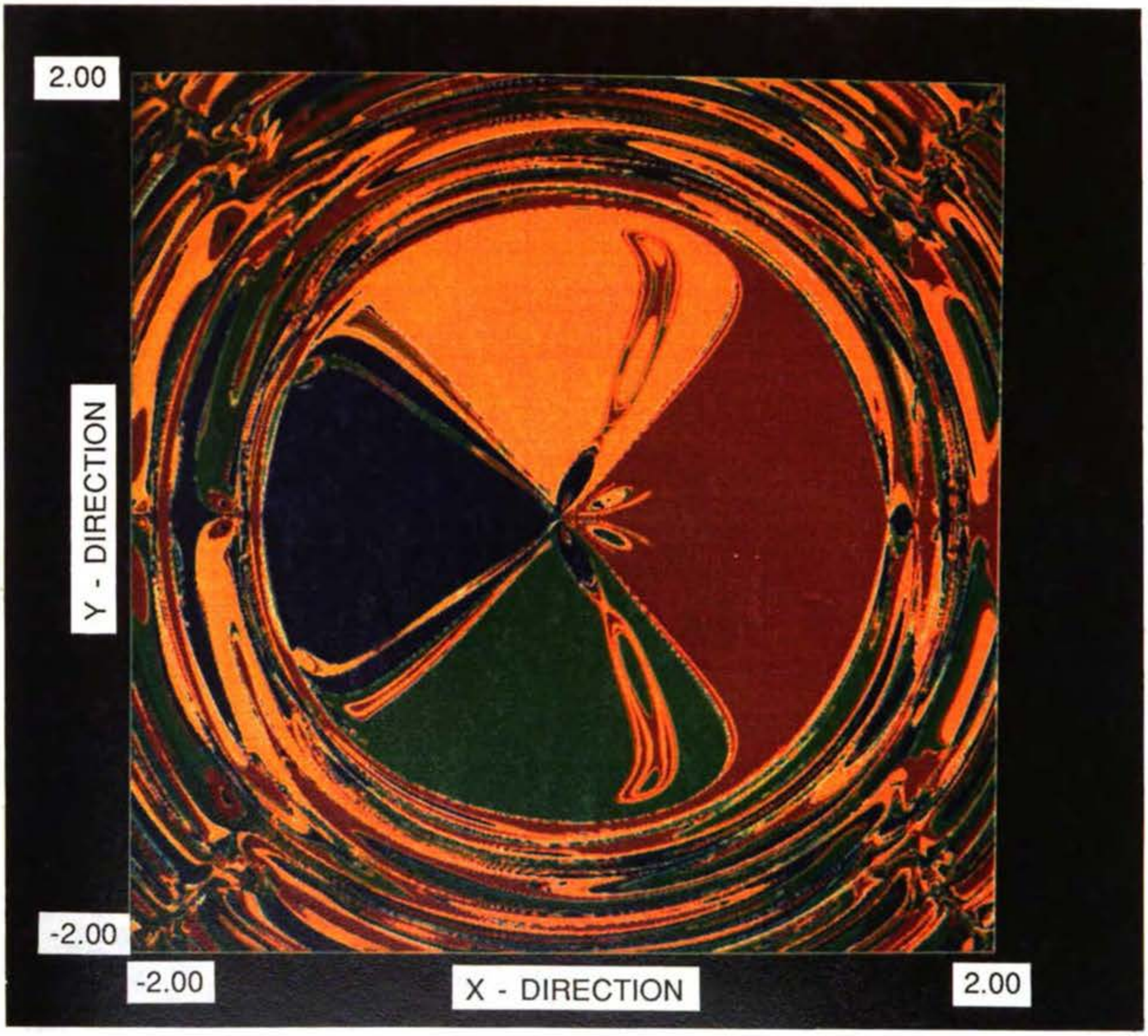


Plate 1

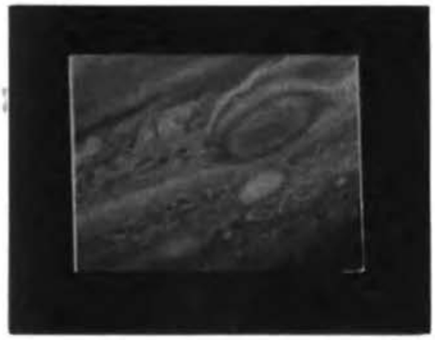
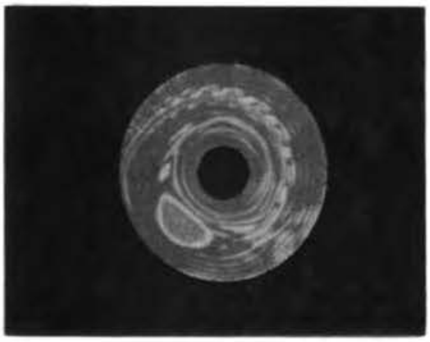


Plate 2

Plate 3

Plate 4



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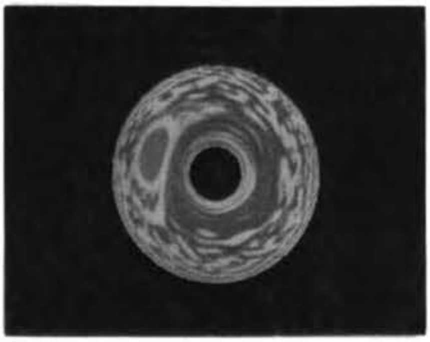


Plate 6

Plate 7

Plate 8

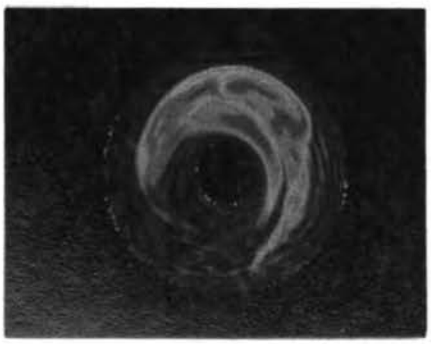


Plate 9

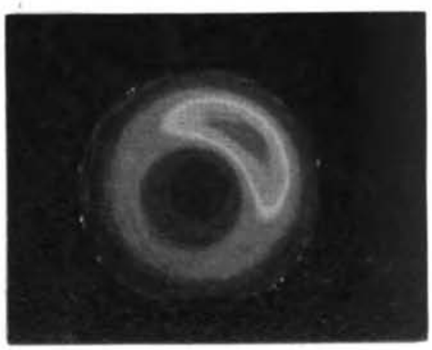


Plate 10

FRACTALS AND CHAOS IN MECHANICAL SYSTEMS

Francis C. Moon
Director
Professor of Mechanical Engineering
Sibley School of Mechanical and Aerospace Engineering
Cornell University

1. Introduction

I like to collect quotes on chaos. Here is one from Milton: "In the beginning, how the heavens and earth rose out of chaos." This quote illustrates the idea of chaos before the current revolution in physics and mathematics. This idea was that chaos existed before there was order. It goes back to the Greek concept of the original state of the universe as some primeval state which was in disorder out of which came our present world which is rather ordered. But, the current thinking of chaos is just the opposite, as illustrated by the flow of fluid past a cylinder. In front of the cylinder there is ordered flow, while in the cylinder's wake there results a rather disordered flow or turbulence. Here the idea of chaos is that one can start with a very deterministic world, a world described by deterministic differential equations, and when the system is evolved over a long period of time one can end up with a state of chaos and disorder.

2. Chaotic Vibrations

What are chaotic vibrations? They are random-like motions in deterministic systems. They occur in systems with a sensitive dependence on initial conditions. They are motions with fractal properties in the phase space, and a loss of absolute predictability. Of course, they occur in systems with strong nonlinearities.

An example of one system which we have constructed in the laboratory is a particle in a two-well potential (Figure 1). If we imagine a little ball that we put in one of the two potential wells, then the ball can sit in one well or the other; so, when there is no applied motion there are two stable equilibrium positions. But, if we begin to vibrate this two-well potential gently back and forth with some periodic motion (not a random input, but a deterministic input) then the ball will begin to rock back and forth. It will begin to jump back and forth between one well and the other. The question one asks is: If the input is periodic, will the output be periodic? Will the hopping back and forth occur in a periodic way? The answer is that it doesn't always jump back and forth in a periodic way. In fact, if one does a Fourier analysis of this jumping back and forth, then one can see a broad spectrum of frequencies, usually of subharmonics.

A historical note I like to make is from the front page of Newton's *Principia*. Note the date. The imprimatur is 1686. We have passed the tricentennial of the publication of the *Principia*, with almost no recognition of Newton's contribution to physics. I should say until about a dozen years ago, it was thought that classical or Newtonian physics was dead. There was not much interest in classical physics. Anything in physics that had any surprises was in quantum mechanics or nuclear physics. Yet, we are learning now that we didn't completely understand classical physics. The so-called "deterministic models of Newton" can result in rather complex behavior. Hence, simple physical laws do not necessarily imply a simple physical outcome.

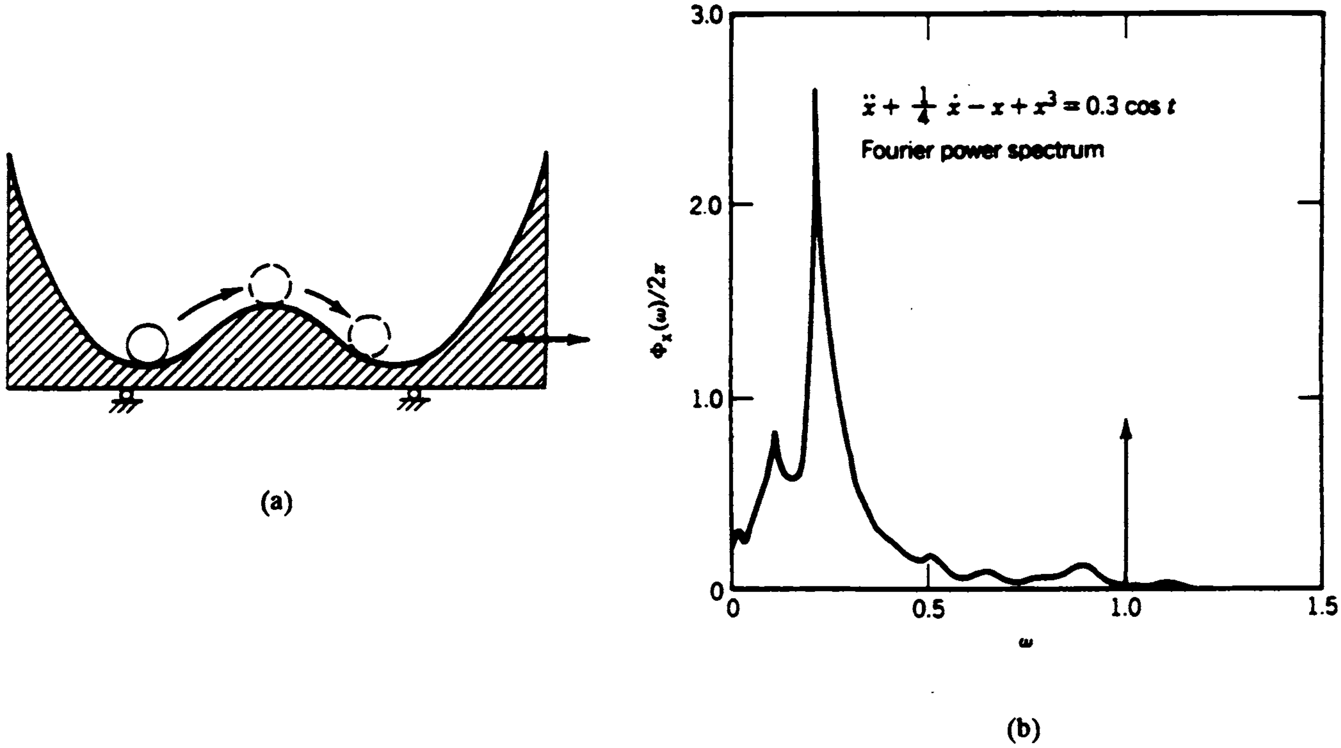


Figure 1

Here is a quote that epitomizes the thoughts about classical mechanics or physics from Mark Twain. He was no physicist, but he said, "By the terms of the law of periodic repetition, nothing whatsoever happens a single time only. Everything happens again, and yet again, and yet again, monotonously. Nature has no originality." That was more or less the view of dynamics and Newtonian physics before 1975. Of course, all of that has changed.

In the field of mechanical engineering we are looking at problems in space structures; large structures that might be very flexible and could be put in space, and may be subject to various kinds of vibrations. We have built a model of a satellite (Figure 2) which consists of a shell-like structure with some beam-like connectors and two plate-like structures which simulate solar panels. When we apply a steady periodic input to the structure, we see rather complex dynamic behavior. For example, one of the things that we do is look at motions in the phase plane by plotting the displacement versus the velocity (Figure 3a). This is experimental output. One can see a rather elliptical motion for the vibration of that structure for small forcing amplitude. Yet, if we push the amplitude just a little harder, we get complex behavior which is not periodic (Figure 3b). However, the input is highly periodic.

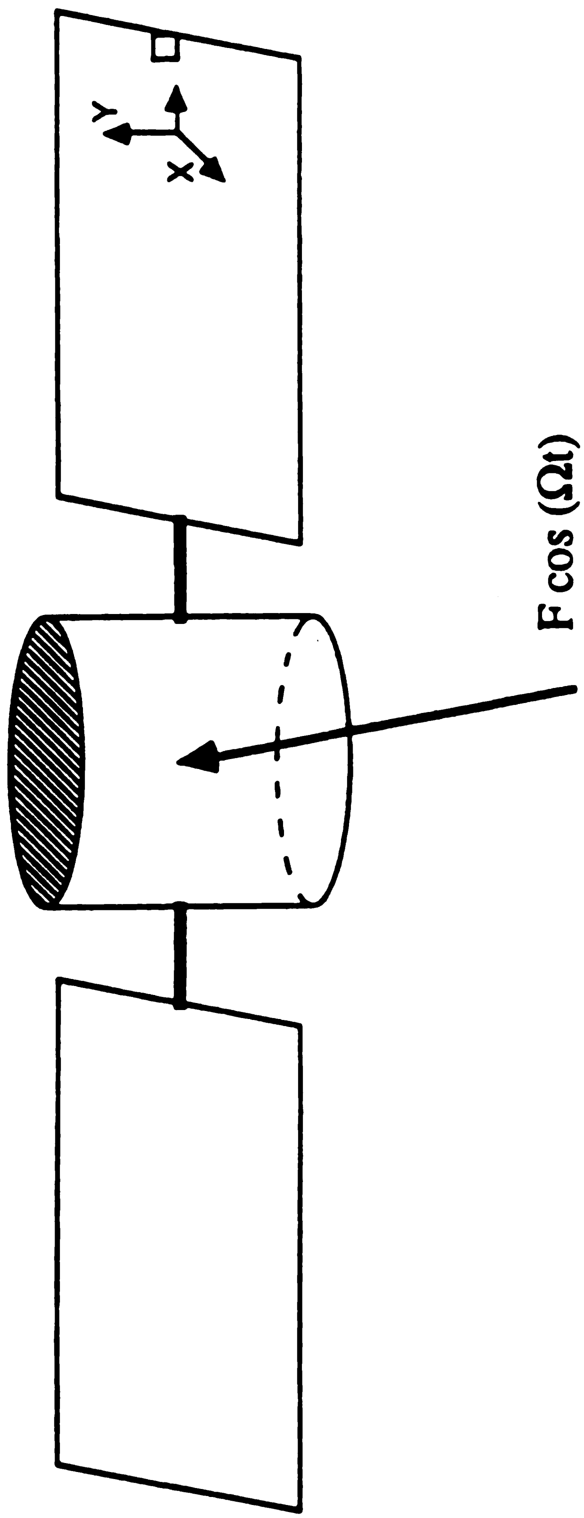
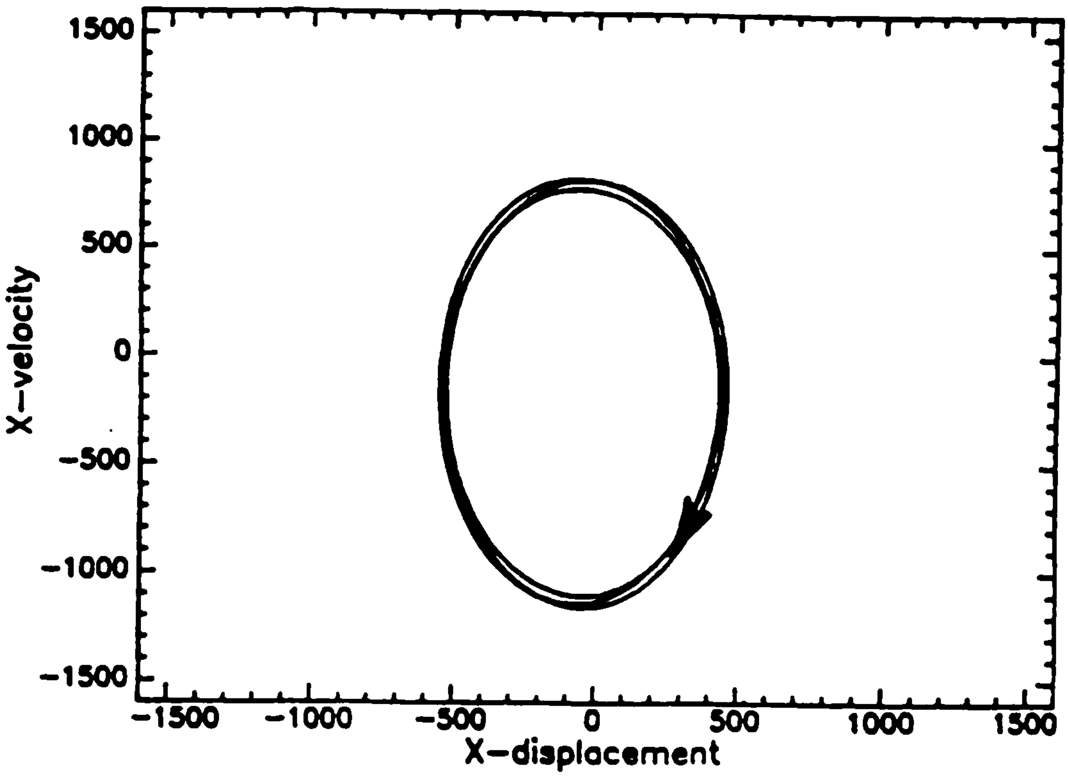
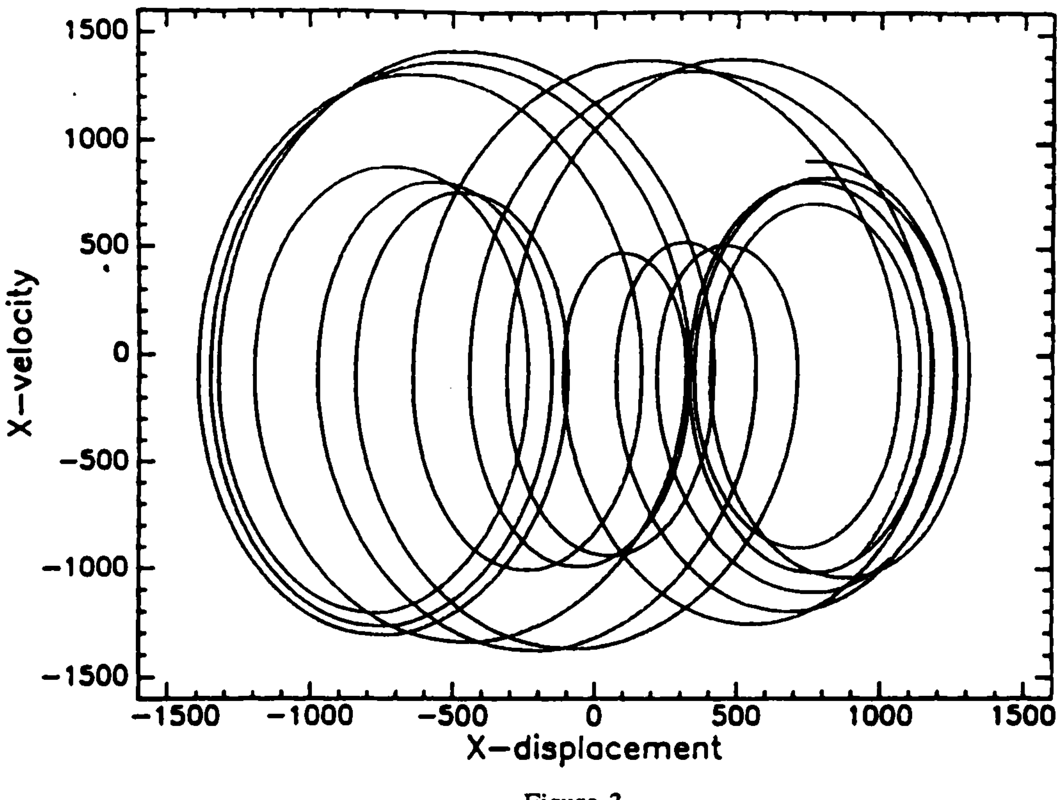


Figure 2



1500
1000
500
0
−500
−1000
−1500
X−velocity
−1500
−1000
−500
0
500
1000
1500
X−displacement

(a)



1500
1000
500
0
−500
−1000
−1500
X−velocity
−1500
−1000
−500
0
500
1000
1500
X−displacement

(b)

Figure 3

Now, there are some simple examples that one can do in the laboratory where one can see chaotic vibrations. One of them has to do with problems with multiequilibrium states. In the case of structural mechanics one creates the buckling or multiequilibria by placing a very thin, slender beam under a compressive load. Beyond a critical value of the force, there are two deformed states. If this structure is vibrated, then one can observe chaotic dynamics. If it is observed in the phase plane, one can see orbits about one equilibrium position, orbits about a second equilibrium position, and orbits about both equilibrium positions. There is very little information in this picture. But, if one looks stroboscopically at the motion, in other words, instead of looking at things continuously in time, one looks at the motion at a particular phase of the forcing motion, then the result is a set of dots or a Poincaré map. We have learned to take some of the new mathematical ideas about nonlinear maps and fractals and implement them in the vibrations laboratory to describe these Poincaré maps.

These pictures were taken from an analog oscilloscope (Figure 4). One can see a kind of fingerprint of chaos. The pattern shows a fractal-like structure. This pattern is about 4,000 points. If we go out to lunch, come back, and take another 4,000 points, we would get the same pattern.

This fractal picture is an experimental measurement. It is not generated by a computer. One can think of the Poincaré map by imagining an abstract picture of the motion. If we think of the buckled beam problem, and look at the position, the velocity, and time as a third variable in three-dimensional space, then we can think of the motion as taking place on a torus; if you are from New York City, a motion in a bagel, where time is represented as a periodic variable around the bagel. Then we can think of the Poincaré section as slicing the bagel. This continuous orbit is converted into a series of points or maps, and if the beam motion is periodic, one gets these fractal looking patterns. Not only that, we can slice the bagel at different points around this attractor and obtain a series of Poincaré maps. These fractal objects can be seen in a laboratory, and one does not need exotic equipment to see them.

In addition, there is structure in these Poincaré maps. If this pattern is observed at zero time phase and again at 180° around the torus, one will see a structure; by rotating the map at zero-phase, the pattern at the 180° phase results. But, the rotation doesn't occur in a rigid body way. Note that one of the arms in the Poincaré deforms into another arm. There is tremendous structure in these maps. So, with chaotic dynamics, there is a kind of double paradox. One begins with Newton's laws. Years ago our physics professors said that all we have to do is write down Newton's laws for all particles, and if we had a big enough computer, we could predict what was going to happen. I know people are laughing because they remember that. Untrue! The biggest super computer in the world cannot predict what is going to happen. Yet, out of this chaos, there is an order. There is structure that can be observed in the laboratory.

We see fractal-looking patterns in other experiments. For example, we experimented with a stepper motor, or a little rotor, with a permanent magnet on the axis and four stator coils. If you are interested in differential equations, this motor is described by a second order equation with sines and cosines. We also get some interesting Poincaré maps. The Poincaré map in a polar coordinate form is shown in Figure 5. We have over a dozen different types of experiments which have been constructed in the laboratory; control system experiments, fluid running through flexible tubes, objects bouncing on one another, and flexible structures. Every time we look for chaos, we find it.

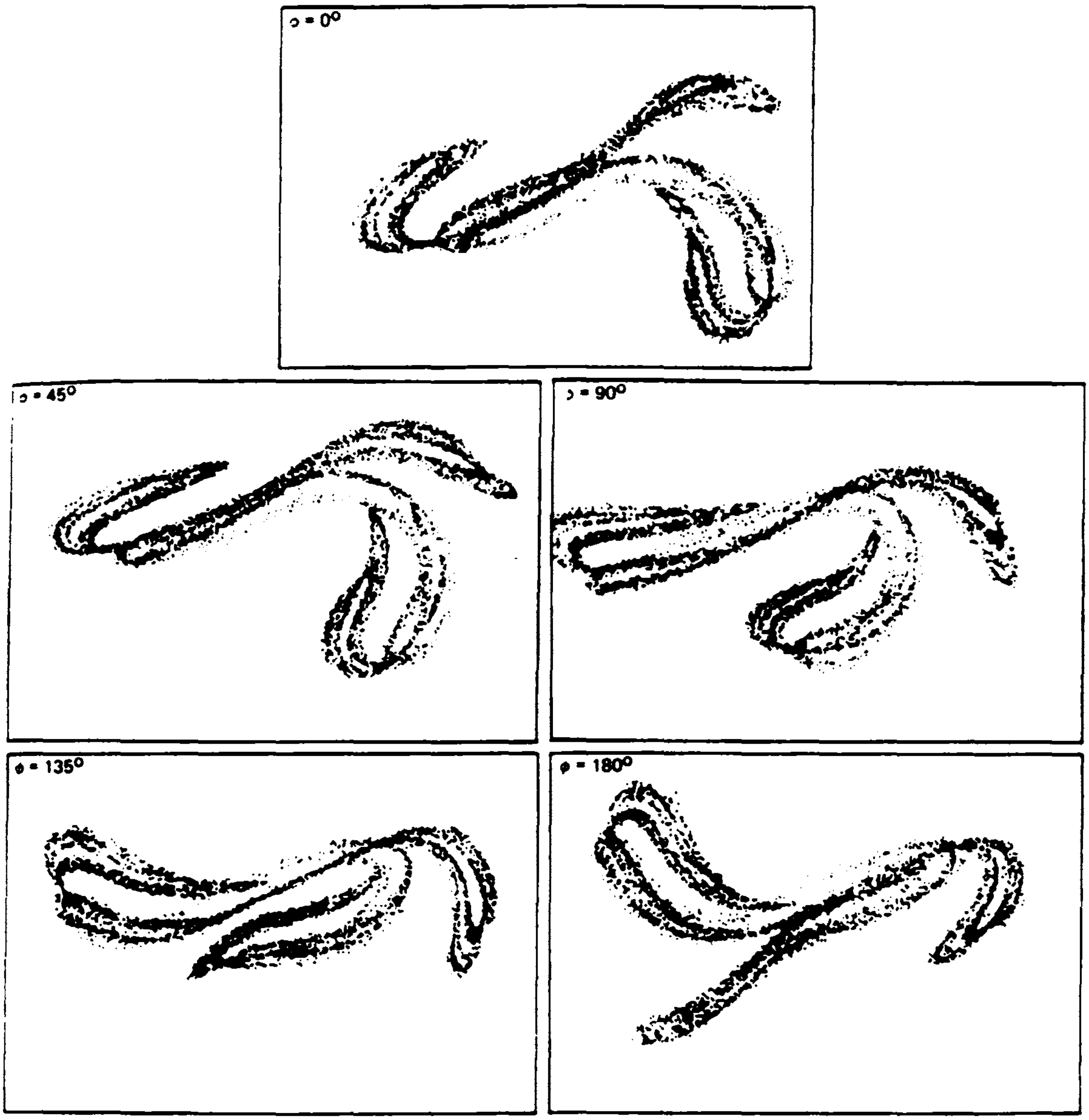


Figure 4: Experimental Poincaré Maps

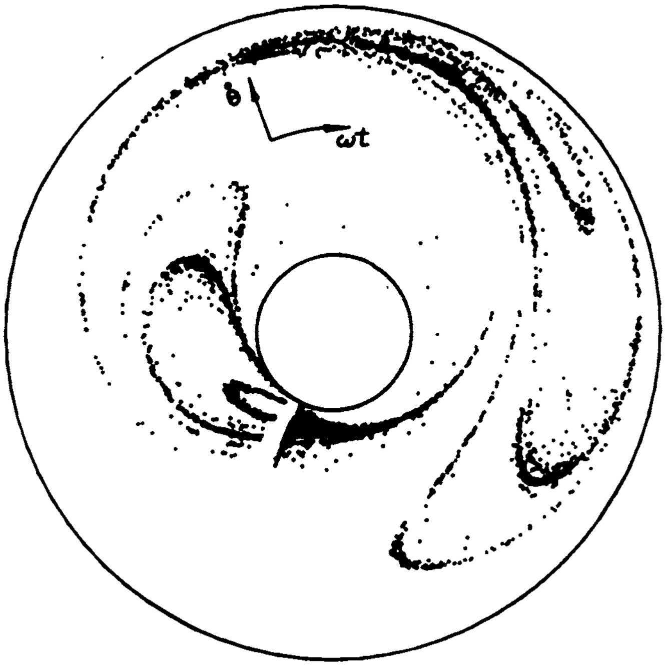


Figure 5: Experimental Poincaré Maps

3. Fractals and Nonlinear Vibrations

One can ask, "Are fractal concepts useful in nonlinear vibrations?" We are using fractal mathematics to describe these Poincaré maps in the laboratory and to identify and categorize classes of motions. We are also using fractals to calculate chaos criteria. We are looking for those parameters in the system for which the system will behave in a regular, predictable way, and for parameters for which the system behaves in a chaotic manner. Also, we are calculating the fractal dimension of chaotic motions using the concepts of fractals. So, the message here is that this is a tool that is being used in engineering research laboratories. It is one of the examples of how fast mathematics can go from chalk dust on a mathematician's blackboard to the laboratory and to potential applications.

Now, let me ask another question, "Are fractals practical?" We will see. Certainly right now it is exciting to be able to go into the laboratory and take ordinary mechanical-type things and to be able to see this beautiful mathematics embodied in the laboratory data. How useful fractals will be 10 to 15 years from now to practicing engineers and scientists I don't know. But right now, it is very exciting to be able to explore the potential for applying this new mathematical tool to engineering problems.

Here is a quote from the physicist Brillouin: "Determinism is dead." Brillouin published a book on scientific uncertainty in 1963, on the work of Poincaré, a mathematician around the turn of the century. The ideas put forward by Brillouin in the sixties were largely ignored; that is, that classical mechanics is a great source of unpredictability. We have only begun to appreciate this idea in the last decade.

Let us look at the two-well potential problem again. Consider the case where, when the particle is excited with a periodic force, one observes whether the ball ends up vibrating in the right or the left well. I am going to adjust the forcing so that it is not chaotic. I just want to know when it is going to oscillate in one well or the other. So, going back to my differential equations, I adjust the sinusoidal forcing function at a fixed frequency and see whether the particle vibrates in the right well or in the left. I was inspired to do this by a lecture that Jim Yorke of the University of Maryland gave at Cornell to our applied mathematics seminar. The idea is the following. I ask, "What happens to the system when I start with different initial positions and velocities?" I can watch the system physically or I can use the computer to integrate the differential equations that describe the system. This set of computer experiments is known as a basin boundary calculation. One looks for all the initial conditions (initial position, initial velocity) that lead to one attractor or the other. Points which lead to motion of the particle in the left potential well are colored blue. Those that lead to motion of the particle in the right well are colored red. The region of the two-dimensional plane of position and velocity with the same color is called a basin of attraction, while the boundary between the two colored regions is called the basin boundary.

Now, when the level of periodic forcing on the particle is low, the boundary appears to be smooth. But, as the forcing amplitude is increased, this boundary appears to become fractal. That is, the two colors become mixed up. Our pictures were calculated at the National Supercomputer Center at Cornell University and involve 640,000 pairs of initial conditions. For each initial condition, the differential equations are integrated numerically to see in which well the particle ends up after a long time.

So before the particle motion becomes chaotic, there are fractal properties in the basin boundary. Fractal basin boundaries imply unpredictability in deterministic systems. To understand how unpredictability arises in this two-well potential problem, imagine that I choose some initial condition near the basin boundary. Now, all physical systems have some uncertainties in measurement or calculation. So my initial point is really a disc whose size is a measure of the uncertainty. If the uncertainty is large, then the disc will straddle the boundary and I only have a knowledge of the probability of the particle going to the left or right well. If the boundary is smooth, I can always shrink the size of the uncertainty so that the disc lies completely in one well or the other. When the boundary is fractal, however, the disc of uncertainty may still have points in each well no matter how small I make the uncertainty. This I call deterministic uncertainty or unpredictability. It usually occurs in systems that are on the threshold of chaotic behavior.

We have performed these computer experiments for other physical problems. The pictures are a different type from Poincaré maps. We look at initial condition space instead of physical space.

In these computer experiments we looked at a magnetic mass as it vibrates near magnets; similar to a kind of roulette game. Except in these experiments we are exciting the roulette table with periodic motion and want to know if we start at a certain position and velocity, will we end up orbiting about one magnet or another. There is a different color assigned to each basin of attraction. As the forcing is increased, there is some mixing of the colors.

Here is a four-color game where we have four magnets. This plane consists of initial horizontal position and vertical position (Plate 1). The velocities are set to zero initially. When the level of periodic forcing is small, the basins of attraction show up as four quadrants of different colors with smooth boundaries between each color. As the forcing increases, the basin boundary becomes fractal-looking and the colors get mixed up as in a paint commercial. The pattern resembles an insect shape. We call this a butterfly attractor. These beautiful fractal pictures illustrate how small uncertainties in deterministic systems can lead to unpredictability in classical physics.

4. Conclusion

In some ways, engineers have always known about unpredictability. When the machine wasn't working right, they would give it a kick. I have a feeling that what they knew instinctively was that there was another attractor somewhere, and all they had to do was change the initial conditions. I hope I have given you some examples to show how research in modern mathematics has changed the way we look at dynamical processes in engineering.

Discussion

QUESTION: In a very simple limit there are Matthieu-type equations with the forcing condition $A\cos(\gamma t)$ because the old problem 34, years ago, was the inverse pendulum. Oscillate it, and you get stability. Is that related in some way?

DR. MOON: If you push it hard enough, you will see chaotic behavior and fractal basin boundaries. I think James Yorke may have done work along those lines. One can get chaos out of that problem.

QUESTION: How does the Mandelbrot set relate to this Poincaré map?

DR. MOON: The Mandelbrot set is related to a complex two-dimensional map. The Poincaré maps that I showed you are two-dimensional, but they are not complex in the sense that they are not analytic. The Mandelbrot set results from analytic maps. These physical Poincaré maps are not necessarily analytic maps.

QUESTION: Can one get analytic maps out of mechanical systems?

DR. MOON: Probably in very special cases. We don't know the mathematical form of these maps. We know the differential equations, but we do not have rules for the maps. We can observe them computationally. One of the gaps left in the mathematics is to go from the differential equation to the maps.

QUESTION: It appears that the physical device you were modeling is symmetric, mechanically symmetric. The four-magnet problem appears to be symmetric, but the maps are not symmetric.

DR. MOON: We were forcing it along one dimension. That destroyed the symmetry. However, it does not have to be symmetric for the experiment. One can also make the problems non-symmetric and still see chaos and fractal basin boundaries.

DR. YORKE: Also, it depends on the phase of the forcing. One must choose it. The picture is being made at a given instant in time. Change the given instant, and the opposite picture results.

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NONLINEAR MATHEMATICS AND JUPITER'S GREAT RED SPOT

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1.0 Introduction

The planet Jupiter, unlike Earth, is almost entirely a fluid. Jupiter's solid surface lies deep beneath the upper atmosphere which is the only part that can be seen from Earth-based optical telescopes or from those aboard the Voyager satellites. The Jovian "weather" that we can observe is therefore very different from terrestrial weather which is strongly coupled to the Earth's surface. The most striking feature of the Jovian weather is that it is dominated by a set of strong east-west winds that encircle the planet. Through a telescope, these belts and zones of winds appear as an alternating series of light and dark bands. We do not yet have a good theoretical understanding of how the belts and zones were created or how they are maintained.

The second most striking Jovian feature is that the belts and zones contain spots that look superficially like hurricanes. Some are large enough to be observable from Earth, but there are many small ones as well, and over 100 were catalogued from photographs made during the Voyager spacecraft fly-by of the planet during the 1970's. The best known is the Great Red Spot which lies at 23° S latitude (Plate 2). It is huge - the diameter is over 26,000 kilometers, and the continental United States would fit into it over 200 times. The Great Red Spot is old compared with the average lifetime of terrestrial storms and hurricanes, and it has been observed and documented in the scientific literature for over 300 years.

We would like to understand the physics that created and maintains the Great Red Spot (and its smaller cousins). The physics is especially fascinating because the Red Spot thrives among the chaos and turbulence in its surrounding east-west wind. Unlike a hurricane which derives its energy from the warm surface water of an ocean, the Red Spot obtains its energy directly from the turbulence of the surrounding winds. It merges with and appends smaller nearby spots, suggesting that it was created by the self-organization of small scale chaos into a large coherent feature. In particular, we would like to know if the Great Red Spot and its self-organization can be explained by the usual simple equations of motion that govern terrestrial fluids, from the movements of oceans to the waves in a bathtub.

2.0 The Physics of Bathtubs and Jupiter

We have all been told that if you let the tub water go down the drain it will rotate counter-clockwise in the northern hemisphere and clockwise in the southern. It is doubtful whether any of us has actually seen this happen because even the smallest perturbation or splash will overwhelm this phenomenon. However, if we had a 26,000-kilometer bathtub on Jupiter, the effect would be spectacular. To understand this, consider an infinitesimal particle or element of water in the bathtub. As the tub containing the water rotates about the planet, each spinning element of water conserves its own angular momentum or spin. (We are ignoring viscosity which is very small.) The total spin of a fluid element as seen by an inertial observer far from the rotating planet is equal to the sum of the local spin seen by an observer sitting in (and rotating with) the bathtub and the spin of the rotating planet. As on Earth, Jupiter's spin is counter-clockwise when viewed by an observer perched above the North Pole or anywhere else in the northern hemisphere and clockwise above the South Pole or elsewhere in the southern hemisphere. Let the observer in the bathtub pour a small closed circle of blue ink onto the surface of the water. The total integrated spin of all the water inside this blue boundary as measured by the initial observer is conserved in time - no matter where the boundary moves or how its shape distorts. This total spin is the amount that the observer in the tub measures plus the angular velocity or spin of the planet times the area inside the blue boundary. If the bathtub drain beneath the blue boundary is opened, the area enclosed by the blue ink decreases. To keep the total spin as measured by the inertial observer constant, the local spin as seen by the observer in the tub must increase. If the fluid, as seen by the bathtub observer, were initially motionless, the bathtub observer would see the fluid begin to spin as soon as the drain was opened. The direction of the spin is the same as the planet's spin and hence is counter-clockwise in the northern hemisphere and clockwise in the southern.

Most tubs have sloping bottoms, so consider an experiment in which the blue ink that is poured into a circle on the surface diffuses downward to the bottom of the tub. The boundary of ink now encloses a cylinder of fluid. Suppose this column of fluid drifts from the shallow end of the tub to the deep end. The volume of fluid inside the blue boundary must be conserved; because the height of the column increases, the area of its top surface must decrease. However, we already know that if the area decreases, the local spin of the fluid as observed by the tub observer increases to keep the total spin inside the blue boundary constant.

The same principle applies to the Red Spot and other Jovian spots. Imagine the Red Spot to be confined to a spherical shell of atmosphere where the shell has a small but finite radial thickness (for example, the layer of atmosphere between 70,000 and 70,200 kilometers above the planet's center). If the Red Spot were to move from the north to the south it would act as a gyroscope because it is rapidly rotating, and therefore it would keep its own spin axis parallel to the north-south spin axis of the planet. If the radial thickness of the spherical shell that contains the Red Spot were constant, then its effective *depth* is a function of latitude. The depth is defined as the distance between the top and bottom boundaries of the shell as measured parallel to the north-south spin axis of the planet. At the poles, the depth is equal to the radial thickness of the shell, while at the equator it is much larger. Because the effective depth of the shell of atmosphere changes with location, it acts like the depth of a bathtub. A column of fluid that moves away from the north pole towards the equator moves into "deeper water" and spins-up. Mathematicians express this equation which governs a fluid in a rotating tub or planet as a conservation law: although the local spin of a fluid element as measured by an observer on the rotating planet is not conserved, the fluid's "potential vorticity" which is the sum of its spin plus the spin due to the rotating tub or planet (all multiplied by 2) is conserved. Thus, each fluid element on Jupiter carries with it its own value of potential vorticity which never changes.

In a homogeneous turbulent flow, fluid elements are well mixed in space, and every fluid element visits every location equally often. Because each element carries its value of potential vorticity with it, a parcel of fluid that contains *many* elements will on average have the same mean value of potential vorticity regardless of where that parcel is located. Thus, if the zonal wind that contains the Great Red Spot is well mixed, its mean potential vorticity is uniform throughout the zone. We shall assume that Jupiter's zonal winds have uniform potential vorticity. (By measuring the zonal wind velocity we can determine the effective depth of the shell of atmosphere that contains the zone.) The Great Red Spot is a deviation of the potential vorticity from the mean value of the zonal wind. We shall call this deviation or excess spin q, so mathematically the Great Red Spot is just a region of large q. The total velocity of the atmosphere of the planet is the sum of the zonal wind plus the velocity due to q.

3.0 N-Body Dynamics

The equation that governs q and the winds it produces is highly nonlinear and therefore difficult to solve. However, because the equation represents a conservation law, its meaning is easy to convey. We now present a technique to demonstrate the dynamics associated with q that can be used on a computer (with a little modification) to solve the equation for q. The potential vorticity q is a continuous function of position and time, so to use a digital computer we must first discretize it. Imagine dividing the fluid in the Jovian layer that contains the Red Spot into squares and labeling each square with the average value of q or excess spin of the fluid in that square. A zone with one Great Red Spot would have $q = 0$ in all of the squares not inside the Spot. To solve for q, the computer need consider only those N interacting squares or bodies where $q \neq 0$. Each body produces a velocity clockwise or counter-clockwise around itself (but never radial to itself) where the magnitude is proportional to q. The sign of the velocity depends on the sign of q, and the velocity's strength decreases as one goes away from the body. However, the velocity produced by a body on itself is always zero. The total velocity of the fluid is the sum of the N contributions *plus* the velocity of the zonal wind. Each body "goes with the flow;" that is, it moves with the total velocity at the position of the body.

First, consider the one-body problem. If only one body has $q \neq 0$, then that one body produces no velocity at its own location, so it just moves with the zonal wind. If the zonal wind were zero, the body's position remains fixed for all time.

There are two types of two-body problems. Let there be no zonal flow, and let the two bodies with $q \neq 0$ have q with the same sign and magnitude. They orbit about themselves. When there is no zonal flow and when the two bodies have q with opposite sign and the same magnitude, they move in a straight line perpendicular to the vector that points from one body to the other. This movement of opposite-signed bodies can be seen in the contrails of airplanes and in the cross-sections of smoke rings. When there is a zonal flow, these motions are superimposed onto the zonal velocity. With $N > 2$ bodies, the equations for the N particles of fluid must be solved numerically, but there is an analogy between N particles with excess spin q and N infinitely-long, co-parallel rods with electric charge q. An ensemble of rods or bodies that all have the same sign of q has a "self energy" that is large and positive when the ensemble is compact or when the spacing between the rods or bodies is small. That is, it requires energy to push distant bodies with the same sign together.

When the zonal wind is not zero, there is an additional term in the energy. This "interaction energy" is due to the interaction between the bodies and zonal wind. It is analogous to the interaction energy between charged rods and an applied external electric field. We shall discuss its form and consequences in the next section.

4.0 Numerical Simulations

Using a very accurate spectral numerical method, we have solved the equation for q with a variety of different starting conditions. Plate 3 shows q in one example. Here, as in all of the other figures, the colored annulus represents Jupiter's atmosphere between 20° and 26° south latitude. This figure is the view that one would have perched high above the South Pole looking down onto the atmosphere (and disregarding all of the atmosphere not between 20° and 26°). The azimuthal angle around the annulus corresponds to the planetary longitude and the distance from the center to the latitude. The green color corresponds to $q = 0$. Where $q = 0$, there is only a zonal wind. The zonal wind is clockwise (to the east) at the inner boundary, counter-clockwise at the outer boundary, and is zero halfway between the boundaries. Notice that the shear of the zonal wind (the direction that an infinitesimal paddle wheel would turn if it were placed in the wind) is counter-clockwise. Superimposed on the zonal wind are vortices where $q \neq 0$. We use the colors as ordered in the spectrum to represent q, with the reddest color representing fluid elements spinning more counter-clockwise than the zonal wind (like the Red Spot) and bluest for fluid spinning clockwise with respect to the zonal wind. Plates 4-6 show the temporal evolution of the flow and clearly demonstrate that the zonal wind first pulls the blue q into a thin spiral and then pulls it toward the inner and outer boundaries. In contrast, the red q is focussed midway between the two boundaries. This effect can be understood by using the rules stated in the previous section. For example, follow the motion of the fluid element represented by the blue pixel in the large blue spot that is closest to the outer boundary of the annulus (at the top of the large blue spot). At first its motion is dominated by the zonal velocity which pushes it in a counter-clockwise direction. Once the pixel is to the left of the large blue spot, it feels the velocity produced by the blue spot. This component of the velocity is clockwise about the center of the blue spot and therefore pushes the blue pixel towards the outer boundary. A similar analysis of the fluid element represented by the blue pixel in the large blue spot that is closest to the inner boundary of the annulus (at the bottom of the large blue spot), shows that it is pushed first to the right and then towards the inner boundary. Thus, the blue spot is pulled apart. The same type of analysis shows that the red spot is pushed together; the subsequent evolution shows that the blue spot is ripped into small pieces and in fact becomes a small-scale, chaotic component of the zonal wind. The red spot stays intact.

A second numerical simulation is shown in Plates 7-10. In Plate 7, the flow begins as an approximately axisymmetric disturbance in the zonal wind. (Here $q = 0$ corresponds to blue; there are no initial motions with clockwise q.) The red band breaks up into three vortices that then merge together. It is this merging process that is responsible for both the formation and stability of the Great Red Spot; Jupiter's turbulent atmosphere is constantly producing small vortices of both signs in the zonal wind. In the language presented here and shown in Plates 4-6, turbulence produces fluid elements with both red and blue q. The small pieces of blue q do not merge together; in fact, they break into even smaller pieces. The red pieces merge together and form a large spot. If the turbulent zonal flow were to break the Red Spot into two or more pieces, they would merge back together.

Detailed numerical solutions have been carried out to find the conditions needed for two spots to merge together. We have found that two spots of q merge together if and only if the energy of the coherent, non-chaotic, component of the flow (that is, the energy of the large coherent spots) decreases. We saw in the last section that if two infinitesimal fluid elements with the same sign q were to merge, their "self energy" would increase. It can be shown [1,2] that the "interaction energy" of spots with the zonal wind *decreases* when two finite-size elements with the same sign q merge *if* q has the same sign or spin as the shear of the zonal wind. If they have the opposite sign, the "interaction energy" increases. Thus, both the "self energy" and the "interaction energy" prevent blue elements of q from merging. The red elements merge together if and only if the decrease in the "interaction energy" is greater than the increase in the "self energy." Arguments based on energy have allowed us to make predictions about vortex mergers. In particular, we have carried out numerical experiments with two or more vortices initially placed in the same zonal wind, and we have been able to predict successfully how many merges will occur and how many vortices will remain after the merging is finished. The Great

Red Spot merges with all other spots in its zonal wind, and so it is unique.

5.0 **Conclusion**

Any explanation of the behavior of the Great Red Spot and the other smaller, long-lived spots on Jupiter requires a direct comparison to the detailed observations of the Voyager satellites to be convincing. Much of the observed behavior of the Jovian spots, such as their mergers, shapes, and wind speeds can be accounted for in a refinement of the theory sketched here. However, some of the details will require more sophisticated models that account for variations in the atmosphere as a function of depth and a better understanding of the physics that maintains the zonal winds. One of the most interesting results of this study of Jupiter's Great Red Spot has been the demonstration of self-organization of small-scale, chaotic motions into large-scale coherent ones. This nonlinear behavior is not limited to Jupiter's atmosphere, and some of the most challenging problems in nonlinear mathematics are in the understanding of self-organization observed in terrestrial flows in the oceans, atmosphere, and laboratory.

Discussion

QUESTION: What chance is there that this model has something to do with the formation of planets?

DR. MARCUS: Probably none. The physics is very different. Perhaps if one couched both problems in the same language one would find that the underlying mathematics is the same, but it is not obvious to me.

QUESTION: One exciting thing about fractals is that they unite completely, intuitively, disparately.

DR. MARCUS: It is not obvious that fractals have anything to do with this process.

QUESTION: Do you have any prediction of whether or not the Red Spot of Jupiter will break up?

DR. MARCUS: If the zone in which the Red Spot sits does not break up, I predict the spot wouldn't break up. It is constantly losing pieces of itself, but constantly reamalgamating other pieces.

QUESTION: Why doesn't one see this type of phenomenon on others of the large gaseous fast-rotating planets?

DR. MARCUS: One does see long-lived spots on Saturn.

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CLOSING REMARKS

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The program now calls for me to make some comments on all of this. I should first point out that the people you heard talk have been working on these concepts in some cases for five years. They have compressed their talks into 25 minutes. I am supposed to compress them even more. Instead, I would like to discuss the nature of fractal pictures.

Fractal pictures are somewhat akin to music. A great deal of music consists of variations on a theme. So, one has a linear string of tones. One hears patterns of tones ranging from a scale of less than a second, to a duration of several minutes. The composer places similar types of patterns through the piece and one's interest is in looking at the variations on those regular patterns. People who study music say that it has the characteristics of "1 over F noise." That is a scientist's way of saying that there is a lot of persistence and repetition in music. Of course, simply having regular or irregular variations on a theme does not result in music. As one looks at scales varying over perhaps a factor of 1,000 one can see that little patterns are repeated over a longer and longer period of time. That, in fact, is what one sees in fractal pictures. These pictures are much more like music than they are like standard western art. People sometimes say these pictures remind them of the work of Escher. Now, why is that? Escher may show pictures of birds changing into fish. There may be a repetition of perhaps ten fish vertically, and ten such left to right, one order of magnitude. When we see pictures of chaos, we see repetitions on all scales. Artists don't seem to be able to do this. I believe fractal pictures are something totally new in western art simply because western artists have not tried to produce variations on a theme on so many different scales of distance. Perhaps they cannot produce such a large number of scales while maintaining the unity of form.

DISCUSSION

QUESTION: Dr. Marcus, do you anticipate that the present computational facilities would permit you to look at vortex rings, in other words, three-dimensional versions of this type of behavior?

DR. MARCUS: Yes, that can be done on the current version of, say, Cray XMP in a matter of an hour to an hour and one-half. Some models have been developed to show how the Red Spot would actually change if one really did a three-dimensional simulation with vertical flows. The model shows that the spot would only exist in certain zones, give itself a natural height, and look like a pancake type object.

QUESTION: Dr. Marcus, have you tried to carry out this two-dimensional analogy between vorticity and electromagnetism in experiments, possibly with wires: separate wires attracting and repelling and so forth?

DR. MARCUS: There are enough differences, particularly in the subtle dissipation mechanisms which we really want to understand that we don't want to actually do that experiment.

QUESTION: Dr. Peitgen, I am intrigued on the fundamental understanding of the Mandelbrot set. What is the parameter that produces the Cantor sets? I am puzzled as to how controlling it is, and why you constantly find this set as a representation in all the patterns at all scales.

DR. PEITGEN: The point about the Mandelbrot set is that it is not just beautiful, but rather a paradigm of organization and order in dynamical systems. Of course, as already pointed out by Francis Moon, it will never be a paradigm for all possible dynamical systems. It does occur in analytic dynamical systems. The reason that it occurs is related to the fact that in any dynamical system the first approximation one would do by a more simple dynamical system would be one which just carries quadratic second-degree terms.

QUESTION: Are Feigenbaum numbers connected?

DR. PEITGEN: Of course. A Feigenbaum number is in the Mandelbrot picture if you look at the Mandelbrot set. There is the main body, and attached to it are many disk-like components. If one goes along the rear axis in that complex plane and measures the diameter of those disk-like components and looks at their ratio in the proper sense, that is exactly the Feigenbaum number. There are other paths one could take in the Mandelbrot set to other disk-like components and get other numbers. All of them would be universal in the sense that the Feigenbaum number is universal. That is a nice point of view because it says that the Mandelbrot set incorporates infinitely many different routes to chaos.

DR. YORKE: The origins of the work you heard about today date back 100 years in most cases, but much of this work is based on computer power and specifically on computer graphics. This graphics capability has only become available to the scientist in the last dozen years. Innovators such as today's speakers are looking for what is new and surprising, and, as such, they cannot predict the course of the future in their field. I would like to ask them, contrary to their natures, to predict the future of their fields in the next ten years.

DR. MOON: Maybe I can speak about vibration engineering. Back in the late fifties as a student aide in the New York Naval Shipyard I was a spectrum analyst. The engineers would come in with a roll of charts showing the vibration of submarines or ships and I would take a caliper and measure the periods of the vibrations. Now, of course, the Fourier transform is on a chip, and it is done thousands of times per second. I think some of these tools, such as fractal dimensions and Poincaré maps, will find their way into electronic chips so that scientists and engineers will press a button labeled FD for fractal dimension or PM for Poincaré map.

DR. FAMILY: In my area of condensed matter physics, or materials science, it will provide a much deeper understanding of disordered systems. We now are learning how they can be described by their fractal properties. Chemists can put together all kinds of polymers, but there is very little really known about their properties from a fundamental point of view. As we develop techniques where we can do more and more mathematical modeling with the help of computers we will have a better understanding of this kind of material. There is very little known about most of the kinds of pattern-forming phenomena that I was talking about. We are now, also, moving toward understanding pattern formation in biology. Some of that will come from some of these extensive numerical analyses of complicated mathematical equations.

DR. MOON: Until recently it was thought that physics was concerned with discovering the law, the rule, and that the rest was simply computation. I think that what we have been discovering in the last dozen years is that the patterns of change, the evolution from those laws, are just as important as the law itself. One could have very simple laws which lead to complex patterns of behavior. I think that is a kind of physics. It involves mathematics and physics, but it is, in some ways, just as important.

DR. MARCUS: I would repeat that. Basically, I consider myself sort of a computational fluid mechanic. So, I am very optimistic about computation over the next ten years. Numerical algorithms have exponentially grown over the last ten years. I think that both the onset of the exploitation of computers, and the fact that very soon everyone will have a PC on his/her desk that has the power of a present Cray are going to make a big difference. There will be a lot more phase space computations.

DR. PEITGEN: I would like to support Francis' point of view, but let me answer the question as a mathematician and say how I see changes in mathematics. I feel lots of tension in mathematics about these issues. Experimental mathematics of the kind we do is not considered mainstream mathematics. It is a big challenge to mathematicians to see the popularity of this kind of mathematics. It is a challenge of most mathematicians to accept that all of a sudden there is an interesting topic in mathematics which can be understood to some degree by the general public. I think these forces on mathematics are tremendous. They have to be thought out and will bring changes for the good or the worse. I am not sure what is going to happen. So, I am very pleased about occasions, like this today, to somehow bring out the potential.

DR. YORKE: May I pick up on that point? The greatest inventions of scientists are not individual laws. The greatest invention of science is actually science and how one proceeds. We understand how to reproduce and continue in the directions that scientists have led in the past, but science is still in its creative stages. There are imperfections in the structure of science as it has been created. For example, there is an immense gulf between mathematics and physics or the other sciences. This gap is man-made. We create it by giving tenure to people who are only on the correct side of the line, and then the line broadens. Dr. Peitgen is talking about people who are trying to cross the line or straddle the line. These individuals are confronting on a daily basis the question, "What is science?" They are trying to change what is perceived as science through their work. The individual experiments may be the least of their contributions.

Rather, they are leading the way for other people to follow with totally new definitions of science.

QUESTION: Actually, I have comments. Dr. Moon may wish to respond to one of them. I am not quite sure about the other panel members. This is for the engineers in the audience. The classical approach to control systems was, also, either deterministic or with stochastic terms added. As a control system designer, I can tell you that the hunting behavior that is seen in actual control systems, especially in those that have any so-called "dead band" and feedback in them, what we called limit cycles, occurring down here when you were close to the desired aim points, were really bounded strange attractors in some cases. These occur in everyday control systems. We simply did not pay much attention to them. This is closely related to a second point. The question has come up as to whether chaotic signals may be related, in some sense, to stochastic process modeling. There is now a result due to Jonathan Victor which says the following: If one wishes to identify nonlinear systems, one must have an input noise which has high fractal dimension or something which random number generators and computer modeling had not looked at until very recently.

DR. MOON: I will comment on the first one. I could have put up a set of view-graphs on an experiment we did on precisely that. We had a control system in which there was a dead band. We were looking at a "pick and place" type of robotic system. We wanted to shuttle it back and forth between two positions. It turned out that if we made the control system go slowly, it was predictable. If we tried to speed it up, it would become schizophrenic. So, we believe that there is such a thing as schizophrenic robots, and the schizophrenia is a kind of chaos. So, yes, it is in control systems. If one goes back to some papers in nonlinear vibrations in the past one finds little notes. Chaos has always been there, and it has been seen. Technicians know about noise. Why didn't we recognize it as something more meaningful than that? I think it had to do with the way we were taught. Both scientists and engineers, and in some ways mathematicians, are to blame. They gave us what they thought were the most beautiful examples of mathematics. They called it linear mathematics. It had a structure that was complete and could be presented in one or two years. So, all engineers and scientists have been taught linear mathematics. In fact, even now, with chaos theory in the last ten years, there are very few changes in that mathematics education. I think that we really have to broaden the kind of mathematics that we teach scientists and engineers if they are going to recognize these complexities in dynamical systems.

APPENDIX

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