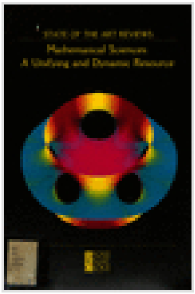

Mathematical Sciences: A Unifying and Dynamic Resource (1986)

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State of the Art Reviews

Mathematical Sciences: A Unifying and Dynamic Resource

Panel on Mathematical Sciences

Board on Mathematical Sciences

Commission on Physical Sciences, Mathematics, and Resources

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This report has been reviewed by a group other than the authors according to procedures approved by a Report Review Committee consisting of members of the National Academy of Sciences, the National Academy of Engineering, and the Institute of Medicine.

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Cover: The cover picture is a computer-generated image of a new minimal surface, one of a family of surfaces discovered in 1984 by David Hoffman and William Meeks III of the University of Massachusetts, Amherst. Computer graphics methods developed by James T. Hoffman (also at Massachusetts) were critical in obtaining qualitative clues to the behavior of these surfaces and their mathematical determination.

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PREFACE

During the summer of 1985, a group of state-of-the-art reviews was initiated by the National Research Council (NRC) at the request of the National Science Foundation. The purpose of these reviews is to assess and monitor world trends, relative strengths, and competitiveness of the United States in rapidly evolving areas of science and technology. Particular emphasis is to be placed on developments that influence the rate at which these fields evolve. Three study areas—cell biology, pure and applied mathematics, and materials science—were chosen for review.

The study on mathematics was conducted by the Panel on Mathematical Sciences under the auspices of the Board on Mathematical Sciences of the NRC's Commission on Physical Sciences, Mathematics, and Resources. The Panel has described major trends in modern mathematics and then illustrated them with a few "vignettes." Through this case-study approach, the Panel illustrates its assertions that we are in a dynamic period of mathematical discovery and that mathematics is the fundamental discipline of science and hence a critical U.S. resource.

International competition and competitiveness are concepts that traditionally are alien to the study of mathematics, in fact, cooperation among individuals of different nations is a vital part of the pursuit of solutions to mathematical problems. Mathematics is intrinsically international, with its own language cutting across barriers of geography and culture. Research and applications in engineering and the physical sciences and, more recently, in business and the social sciences rely increasingly on mathematics for their basic structure, for the modeling of phenomena, and as the basis of new computational directions. Mathematics is a major field of application of the recent powerful computer advances and is the fundamental discipline underpinning both the development of the computer techniques themselves and the applications of these techniques in all fields.

Mathematics has become, even more directly, the language and the foundation of science, technology, and social organization. Indeed, it is a fundamental driving force in the worldwide progress that is altering the economic, political, and social balance among nations. It is essential for the United States to maintain momentum in mathematics if we are to maintain our overall competitiveness in other areas.

The National Research Council, the scientific community, and I are deeply indebted to the panel members, the many colleagues who assisted them, and the able and conscientious reviewers. Thanks also go to Patricia Kenschaft, a mathematician at Montclair State College, for her excellent mathematical editorial work. The chairman is especially grateful to the staff of the Board on Mathematical Sciences, which under the leadership of Frank L. Gilfeather provided throughout the project the support so essential to its successful completion.

Those responsible for guiding science policy in the Congress and the Administration are the primary audience for this report. The report will be useful to other audiences, too: leaders of universities, the mathematical sciences research community, and also those who are inquisitive about the mathematical sciences, about its structure and its current directions.

Phillip A. Griffiths, *Chairman*
Panel on Mathematical Sciences

EXECUTIVE SUMMARY

Trends

This report emphasizes four major trends in the mathematical sciences:

- Mathematical sciences research is strong worldwide, and the United States is maintaining the leading role.
- Mathematics is unifying internally.
- Applications of mathematics in both traditional and new areas are flourishing and involving more central areas of mathematics.
- Mathematics is the driving force behind new areas of computational science and is in turn profoundly influenced by high-speed computing.

These trends, which demonstrate the vitality of the mathematical sciences as well as the changing nature of this critical discipline are illustrated in a series of six vignettes given in Chapter 3.

There are several corollaries of the trends, which are especially notable in view of the declining U.S. Ph.D. production in mathematical sciences, as seen in the table below. The expanding number and sophistication of tools needed for successful research as areas of mathematics become intertwined now require protracted study often beyond the Ph.D. degree. This corollary development is especially critical for those working at high levels of applications of mathematics. The difficulty in reaching mathematical research frontiers with the requisite deep, broad range of knowledge, without the opportunity for extended study, may partially explain the decreasing Ph.D. production. This decrease will continue if the potential talent does not feel that continuing, as well as entry level, research support will be available.

Some of the vignettes point out that the number of young researchers is of great concern and that considerable talent must be imported to keep the United States in the forefront of mathematical research fields. With the continued resurgence of European mathematics, this cannot be a permanent solution to manpower shortages. While the United States maintains leadership in most areas of mathematics, in some important fields new talent is only sparsely available.

Ph.D. Production and Support

These issues of manpower and infrastructure relate directly to how the United States stands in production of new Ph.D.s and in support of postdoctoral fellowships and graduate students in the sciences generally. The results as summarized below and reported in Chapter 4 (Table 1) are dramatic.

Comparisons of Federal Support for Postdoctoral and Graduate Students in Three Fields of Science

	Chemistry		Physics		Mathematical Sciences	
	1980	1984	1980	1984	1980	1984
Annual Ph.D. Production	1538	1765	862	962	744	698
Postdoctorals (P.D.) federally supported	2255	2473	1210	1150	57	132
Graduate Students (G.S.) federally supported	3700	4118	2900	3348	200	411
Ratio P.D./Ph.D.	1.47	1.40	1.40	1.20	0.08	0.19
Ratio G.S./Ph.D.	2.41	2.33	3.36	3.48	0.27	0.59

Sources: NRC Survey of Doctorate Recipients.
Survey of Graduate Science and Engineering Students and Postdocs, NSF.
Survey of Doctorate Recipients, NSF, unpublished, Table D-32.

Challenges

Continued concern should be expressed over the decline in the number of Ph.D.s in the mathematical sciences in the United States, and in view of the critical role of mathematics this trend must be reversed. It is essential that the long-range attractiveness of the field and the prospects for success within it be such as to attract able researchers. The competition for students among the various fields of science is keen, and it is critical that mathematics be able to provide sufficient inducements to candidates to maintain its vitality. The figures cited above for Ph.D. production reveal a precarious situation, and we believe that it may be worsening. In addition, if we are to assure the quality of the mathematics enterprise in the United States, the opportunities for postdoctoral training must be significantly expanded.

As we provide sufficient opportunities for young investigators and students, we must also maintain the strength and vitality of senior leadership. That the United States maintains mathematical pre-eminence is due in part to the commitment and investment made in the past to an outstanding group of researchers. Unfortunately, this commitment has been considerably weakened over the past decade as was documented in the 1984 NRC report *Renewing U.S. Mathematics: Critical Resource for the Future* (National Academy Press, Washington, D.C., 1984). The challenge of current policy is to provide greater opportunities for the young while maintaining our current strength in leadership. The two goals are inextricably related.

1. INTRODUCTION

This expository report reviews world trends in mathematics and assesses the position of American mathematics in the world mathematical community. It was written in response to a request from the National Science Foundation to the Board on Mathematical Sciences. Chapter 2 delineates four current trends in the development of the mathematical sciences. Chapter 3 contains six short vignettes, illustrating some of the major efforts in current mathematical research. The topics of the vignettes are:

- D-modules
- Computational Complexity
- Nonlinear Hyperbolic Conservation Laws
- Yang-Mills Equations
- Operator Algebras
- Survival Analysis

The vignettes substantiate the trends and illustrate the excitement and struggle of mathematical research, indicating more accurately than could any compilation of areas and results the emphasis and vitality of the discipline. Moreover, since specific research results are the substance of living mathematics, our general observations are made more meaningful through the six vignettes. The report closes with a brief epilogue, which records the key recommendations of the NRC report *Renewing U.S. Mathematics: Critical Resource for the Future* and remarks on the response to and impact of this important report.

2. TRENDS

Mathematics is unifying, and it continues to be a dynamic resource in science. Of the many indications of this, the following are outstanding:

- Difficult problems that have been unsolved for many years are now being solved with amazing frequency—a strong confirmation of the vitality of mathematics.

- Mathematics is unifying internally. The division between pure and applied mathematics that developed in the first part of the twentieth century and allowed for the rapid development of new fields is now disappearing. Moveover, the distinctions between traditional areas of specialization have become blurred.

- Traditional areas of applied mathematics are flourishing. Further, significant, deep interactions are occurring directly between core mathematics and other fields, including the natural sciences, engineering, and the social sciences. Increasing interactions are occurring not only with the traditional areas of physics, chemistry, and engineering but also with newer fields such as molecular genetics, business, sociology, information sciences, and studies in policy analysis.

- Mathematics is the underpinning of the revolutionary changes taking place in all scientific and engineering fields as a result of the advent of powerful computers. The development of scientific computing has not only highlighted a host of critical new mathematical problems it has introduced new tools for mathematicians.

With the necessary support, mathematics will continue to flourish, to attract excellent minds, and in the coming years to produce much essential new mathematics on an international basis. The role of any particular country in this development is hard to predict. Mathematical leadership will depend on many factors, primary among them the support that individual nations give to basic sciences in general and mathematical research in particular.

Recent Breakthroughs

Among recurring solutions to previously unsolved problems, several examples merit mentioning. One of them is the program to classify finite groups, which generated a long list of important new efforts. The recent solution of the deep Bieberbach Conjecture in classical function theory showed that the tenacity of a senior individual investigator is still a vital force in mathematics. Finally, the solution of Mordell's Conjecture places mathematicians on the threshold of proving the enticing Fermat's Last Theorem.

Within all areas of mathematics a spectacular number of major but lesser known problems have been solved recently, including problems illustrated in the vignettes in Chapter 3. Further documentation is presented in the forthcoming report of the Board on Mathematical Sciences, *Survey of U.S. Mathematical Sciences*.

Unity of Mathematics

The unification that is taking place within mathematics is obvious to people in the field and will be apparent in the vignettes about D-modules, computational complexity, the Yang-Mills equations, and operator algebras. Unification occurs when there is a confluence of seemingly independent phenomena, motivating cooperative study of the significant underlying patterns. One symptom of this accelerating unification is the increasing difficulty that agencies are having in assigning proposals to their discipline programs, which now substantially overlap. The trend burdens young investigators with a need to pursue increasingly broad training, as is noted in several of the vignettes. In mathematics it is becoming critical to lengthen the training period substantially.

An example of this confluence of areas is the Korteweg-de Vries (KdV) equation $u_t + uu_x + u_{xxx} = 0$ [where u is an unknown function $u(x, t)$ in one space dimension and time], which arises both as the simplest nonlinear dispersive equation in shallow-water wave theory and as the equation of isospectral evolution of the potential in the Schrodinger equation $\psi_{xx} + (k^2 - u)\psi = 0$ of quantum mechanics. The intensive study of the KdV equation during the past quarter century has affected many major areas of mathematics. For example, about a year ago a young Japanese mathematician, using a development in the study of KdV equations initiated a decade ago by the Moscow school, solved a major problem in algebraic geometry that was first discussed more than a century ago by Riemann, a German.

Mathematics and Other Sciences

The symbiotic relationship between mathematics and its areas of application is ever deepening as more areas of science and engineering become almost indistinguishable from subareas of mathematics and this relationship is producing exciting and intriguing new mathematics. Cross-disciplinary collaboration between mathematicians and professionals in other fields is accelerating and deserves encouragement. An important number of interactions between mathematics and science and engineering are not exactly interdisciplinary, but might be more accurately described as resonance phenomena in that advances in one field spur development in another. Examples of this important trend are described in the vignettes on Yang-Mills Equations and Operator Algebras in Chapter 3. The broadening and deepening of these applications, as noted in the vignette on Nonlinear Hyperbolic Conservation Laws, create pressure for mathematicians to pursue significant postdoctoral study. An insightful and illuminating discussion of the interaction of mathematics with science and engineering is given in the 1984 NRC report *Computational Modeling and Mathematics Applied to the Physical Sciences* (National Academy Press, Washington, D.C., 1984).

New Opportunities

The third trend includes the tendency for mathematics to be incorporated into the language not only of science and technology, as has been traditional, but also into new fields of social science. Mathematical models (descriptions of real-world events that use mathematical language) form the basis of econometrics and health policy analysis. The Survival Analysis vignette in Chapter 3 demonstrates this; it examines statistical and mathematical methods used to provide a realistic analysis for problems in medical research, reliability theory, actuarial computations, and demographic studies. Mathematical analyses contribute substantially to decisions about economic and health policies, which in turn have enormous financial and social consequences.

There is no longer any question as to whether mathematical analysis will substantially influence discussions of public policy but only whether it will be used appropriately and effectively. It is essential that those making the decisions understand and influence the assumptions used to form the mathematical model and that mathematicians comprehend the applications sufficiently well that they address and solve the correct problem. In fields where mathematical models are not subject to experimental verification—such as those with the most drastic consequences—it is especially essential that the mathematics be critically scrutinized.

The Role of Computation

Computational mathematics is an integral part of the mathematics discussed in the vignettes about complexity and nonlinear hyperbolic equations, and there are two important observations worthy of emphasis. First, computational methods pervade almost all aspects of science, and mathematics is the foundation of these methods. Today's complex problems, involving computational solutions, range from the design of computer architecture itself through the mathematical modeling of physical, chemical, biological, and engineering processes. Mathematics, the intellectual basis of computational science, has been and will continue to be the key to the dynamic revolution being created by the computer in science and engineering. Second, computational results provide insight for the development of mathematical theory. For example, the behavior of the solutions to the KdV equation mentioned above was first discovered numerically. The mathematical theory, in turn, provides a deeper understanding of the models, revealing phenomena that enable people to analyze and test previous computational results and conceive of new computations that will facilitate further theory.

Core mathematics now consists of three basic operations—computation, abstraction, and generalization. Raw information leading to mathematical discovery comes from concrete examples, and it is increasingly the role of computation to provide such examples, although they are frequently formulated mathematically. Abstraction is the process of distilling the essential features from such examples. Number and space are, respectively, abstractions of the process of counting and of our experience in the physical world, and the mathematical idea of functions similarly abstracts human ideas of measurement and motion. In these contexts, ideas from one manifestation of the abstraction are often relevant in solving problems in seemingly dissimilar situations. Generalization uncovers hidden analogies between abstract patterns of mathematics and frequently extends the range of applicability of such patterns.

Thus in the development of mathematics, periods of computation often alternate with periods of theorizing. During the former, new raw data are generated and horizons expanded. Eventually there is a plethora of information in need of an intellectual framework on the basis of which masses of material can be comprehended simultaneously. As reunification occurs, seemingly disparate examples are often revealed as different aspects of the same phenomenon.

Computation, abstraction, and generalization need each other to be meaningful. Recent mathematics has focused more on concrete problems than on abstractions as it revels in the computing power suddenly available to it, and evidence of a new unification is now appearing. At the same time, our computing power has matured to the point where it can be an enormous asset in the investigation of the still more complex mathematical examples that will inevitably be suggested by pending research. A beautiful amplification of this theme on the development of mathematics is given by Arthur Jaffe in an Appendix to the NRC report, *Renewing U.S. Mathematics: Critical Resource for the Future* (National Academy Press, Washington, D.C., 1984).

A dramatic example of using the computer to gain valuable mathematical insight can be seen on the cover of this report, which represents a problem in the geometry of surfaces of constant mean curvature, a subject with applications to mathematical physics, polymer chemistry, architecture, and other sciences. The existence of a new complete embedded minimal surface with finite topology, the first to be found in more than 200 years, was established using computer-aided graphics. The complex representation of a potential example was analyzed by numerical approximation and three-dimensional computer graphics. An unexpected symmetry was revealed in the surface, which was shown to exist in the equations themselves. Further analysis using both traditional and computational methods established the existence of families of new examples as well as a rich new theory to explain their existence. The ability to create accurate computer images has greatly facilitated communication among scientists. The computer methods developed in this research are proving to be useful in solving problems in related mathematical subjects.

Organization of Mathematics

In addition to these four major directions in mathematics itself, what might be called "The Sociology of Mathematics" reveals how mathematics develops and accentuates the unification within mathematics and its growing interaction with other fields. More and more the mathematical community is now organizing itself by areas of interest instead of fields of traditional study. For example, the Yang-Mills vignette describes how a variety of mathematicians educated in the disparate fields of topology, algebra, differential geometry, algebraic geometry, several complex variables, and partial differential equations is cooperating to solve some exciting equations.

It is well known that some areas of science have become "big science," where research is accomplished by teams frequently gathered around major pieces of scientific equipment or laboratories. Each team consists of at least one senior scientist and many, sometimes a great many, junior investigators, working on related problems. Mathematicians are also often informally grouped around a common research interest such as Yang-Mills equations, operator algebras, or computational complexity. However, such mathematical groups are usually geographically separated, and the analogue of access to a common major piece of scientific equipment is their ability to gather together for sustained periods of collaboration. Furthermore, mathematicians share the scientific community's needs for a means to prepare and motivate potential researchers, from undergraduate through postdoctoral levels. Institutes such as the two National Science Foundation Mathematical Sciences Institutes (Mathematical Sciences Research Institute and Institute for Mathematics and Its Analysis), as well as other institutes, including those supported by Department of Defense agencies and special year projects provide essential opportunities for mathematicians, undergraduate and graduate students, and postgraduates to collaborate.

The vignettes in Chapter 3 illustrate the general patterns discussed above. However, these can give but a small sample of the many endeavors of the entire mathematical community. A more exhaustive, although still selective, survey of the mathematical sciences, *Survey of U.S. Mathematical Sciences*, is currently being prepared under the auspices of the NRC Board on Mathematical Sciences. Together these reports can provide only a glimpse of the scope of current mathematical research. Our choice of specific mathematical topics reflects our effort to select a representative sample of the entire mathematical enterprise.

Language of Mathematics

There is another significant challenge that we must face in our endeavor to explain mathematics to those without years of study in the field. At least three centuries ago, mathematics developed a language of its own, which has become thoroughly distinctive and international. Just as it takes years for an American or Japanese youngster, for example, to become fluent in the spoken language of the other, any aspiring American or Japanese mathematician spends years studying the common language of mathematics. As a result, each can open the other's mathematics books and recognize the topic under discussion (even if totally ignorant of the other's verbal language).

The complex and dynamic language of mathematics that has developed over the past three centuries brings great satisfaction in terms of international understanding for those who are fluent in it, but it presents substantial obstacles to those who have not spent years in its study. On the one hand, the history of mathematics is one of progressively less translatability for nonmathematicians, while, on the other hand, the past few centuries have seen ever-broadening human intellectual endeavors subsumed into the language and models of mathematics. Because of the language barrier between mathematicians and nonmathematicians, the following vignettes cannot describe the technical essence of the fields discussed but will focus only on a field's development and interaction with others.

U.S. Competitive Position

We conclude with a few remarks about the competitive position of the United States with regard to the field of mathematics. First, we emphasize that international cooperation is a long-standing tradition in mathematics. When the vignettes use the words "a Soviet mathematician" or "in the United States," it should be understood that the results being described often depend on a background developed by many investigators in diverse parts of the world.

Although mathematicians, with their specialized international language, view their community as an international one and think relatively little about the comparative ranking of national contributions to their discipline, the writers of this report believe that few would disagree that the United States was dominant mathematically in the decades following World War II. Just as mathematics has become the language of science, technology, and a widening circle of other fields, English has become the verbal language in which most mathematics is explained.

When combined with a federal policy of auxiliary support, the unique American system of combining academic teaching and research at all levels has been a key to our strength. The United States remains attractive to foreign mathematicians because of our relatively high employment opportunities and, more recently, our powerful, relatively accessible computers. We still have a favorable balance of trade from the wonderfully dynamic infusion of immigrants to American mathematics. However, Europe is regaining its former mathematical liveliness, now more on a continental than a national basis. Separately, the Soviet Union, despite its professional isolation and its loss of numerous superb mathematicians, continues its tradition of excellence. The strength of Japan's mathematical community is surging, and there is little doubt that in time major mathematical activity will develop in China.

It is still true that the United States is the world leader in mathematics, but this is surely less so than it was ten or even five years ago. The enthusiastic mathematical activity in other countries has created a tendency toward parity among the major mathematical communities. The fact that half as many U.S. citizens received mathematical sciences Ph.D.s in 1985 as in 1973 bodes ill for our future ability to compete in the world mathematical community (see Table 1 in Chapter 4). Certain areas of mathematics are becoming dependent on foreign-trained researchers as noted, for example, in the vignette on Operator Algebras. It is certain that during the next decade much excellent mathematics will be created. The major questions are, where and by whom.

3. VIGNETTES

D-Modules

Core mathematics is broadly divided into analysis, algebra, and geometry/topology, although all three subfields include extensive applied mathematics components. (Geometry and topology are not synonymous, but each term is often used for areas in their broad confluence.) However, as the interplay of analysis, algebra, and geometry/topology becomes ever more complicated, even the division of core mathematics into subfields (not to mention its distinction from applied mathematics) seems artificial. Indeed, the essential unity of mathematics is vivid as we review some important recently discovered relationships among these traditional subfields.

Algebraic geometry has been one of the most lively areas of research in algebra during recent decades. It is the study of geometric objects that are the loci of points satisfying polynomial equations in two or more variables, such as the familiar conics from classical geometry. Meanwhile, algebraic topology has become a leading area of geometry. Considerably more general geometric objects than loci of polynomial equations are studied in algebraic topology; in the 1950s and 1960s this was an especially active field and was discussed on a number of occasions in the popular scientific literature.

Any geometric object has a group of symmetries. For example, a cube is invariant under a finite set of rotations. Similarly, a sphere is symmetric under (the infinite set of) all rotations and reflections around its center. A continuous symmetry group, such as the latter example, is called a Lie group. Lie groups can also be viewed as certain groups of matrices with their usual matrix multiplication. This multiplication is not commutative; that is, in general $XY \neq YX$. The set of derivatives along curves through the identity matrix of a particular Lie group can be viewed as an additive set of matrices and is called a Lie algebra. The theory of Lie groups and algebras, one of the great achievements of modern mathematics, originated from questions in differential equations, which has always been central in analysis. But "group" and (of course) "algebra" are quintessentially algebraic notions. Thus Lie theory, based on geometric symmetries and increasingly useful for physicists, incorporates aspects of algebra, analysis, and geometry.

Another place where these three fields meet is in D-modules, which have been developed recently in Japan. A module (with or without the D) is an algebraic structure consisting of a group such as a vector space whose elements can be multiplied by another set of mathematical objects such as matrices. D-modules are modules whose vectors can be multiplied by partial differential operators with analytic coefficients. One motivation for their study was to focus attention on the equations themselves rather than solutions to differential equations. They were investigated using methods of algebraic geometry invented in France in the early 1960s.

The theory of D-modules is also related to the Riemann-Hilbert problem, posed in Germany around the turn of the century. Suppose we have a linear, homogeneous system of n first-order ordinary differential equations for n functions $f_1, \ldots, f_n$ of one complex variable z. It might be written $f' = Af$, where the coefficient A is an n by n matrix whose elements are analytic functions of z. In general the system will have n independent solutions $(f_{11}, \ldots, f_{1n}), (f_{21}, \ldots, f_{2n}), \ldots, (f_{n1}, \ldots, f_{nn})$ such that each solution $(g_1, \ldots, g_n)$ can be written uniquely as a linear combination $g_i = c_1 f_{1i} + c_2 f_{2i} + \cdots + c_n f_{ni}$ for some constants $c_1, \ldots, c_n$. But there may be singularities, points at which one or more of the elements in $A(z)$ tend to infinity or are otherwise irregular. Not only may solutions be undefined at the singularity, but when we follow a curve around the singularity, one solution may be transformed into a different one. Going around the given singularity (and no others) once in a clockwise direction has the effect of transforming a solution by multiplication by an invertable matrix, depending only on the system and not the path traversed. The Riemann-Hilbert problem, a question in analysis, was to show a converse for the theorem just presented—that for any admissible map from a set of singularities into invertible matrices there is a system of n differential equations, as described, such that encircling each singularity changes the solution by the corresponding matrix. The solution, now several decades old, has been extensively generalized.

Mathematicians working in this country and France were investigating ostensibly a quite different area—an area at the intersection of algebraic geometry and algebraic topology that focuses on the concept of duality in geometry. In 1977 researchers in the United States conjectured the existence of a relation between this geometric theory and a fundamental problem from Lie groups. Their conjecture was then proved independently by two pairs of mathematicians, one in France and the other in the Soviet Union. Their proof surprised the mathematical world by using the generalized Riemann-Hilbert results, thus again linking algebra with geometry through differential equations.

Papers about D-modules are impossible to classify into the three traditional fields of analysis, algebra, and geometry/topology, a problem for the editors of *Mathematical Reviews*. It also suggests how very tentative any division of mathematics into subfields may be.

Computational Complexity

An algorithm is a procedure for solving a given class of problems with a specified set of mathematical tools. In classical geometry the tools were straight edge and compass, and the ancient Greeks provided a simple algorithm for trisecting any segment using only these. Their extensive attempts to trisect a general angle were proved futile in the nineteenth century, when it was shown that there can be no such algorithm.

If the tools in algebra are addition, multiplication, division, and taking radicals (square roots, cube roots, etc.), then there is a familiar algorithm for solving any quadratic equation, $ax^2 + bx + c = 0$. However, a question posed by Renaissance Italians was answered when it was proved by Abel, a Norwegian in the early nineteenth century, that there can be no such algorithm for solving equations of degree five or more. The tenth problem posed by Hilbert at the 1900 International Congress of Mathematicians asked whether there is an algorithm for deciding if a general polynomial equation, $f(x_1, \ldots, x_n) = 0$, with integer coefficients has a solution in the positive integers. In the 1960s a Soviet mathematician, strongly incorporating the work of two Americans, proved that there is no such algorithm.

Such decision questions were formerly addressed primarily by logicians, but computers have given new urgency to algorithmic questions. Computers after all operate with only a few primitive tools, and the programs which instruct them are essentially algorithms. For most problems that computers are asked to solve the existence of some algorithm is usually evident; instead, the problem is to find algorithms that are efficient and reliable. The mathematical analysis of these practical considerations has spawned the field now called "complexity theory."

This field measures the complexity of an algorithm by the maximum number $f(n)$ of basic steps that the algorithm needs to solve those cases of the problem requiring n digits in their statement. The theoretically tractable problems, called type P, are those for which there is an algorithm with $f(n)$ growing no faster than some polynomial in n. There are adjustments to this assertion, most notably in the well-publicized linear programming problem discussed below, where the symplex algorithm, though exponentially long in the worst cases, is remarkably efficient in the vast majority of its application.

Each generation of computer scientists has quickly encountered pressing problems that overwhelm available computational resources. Unable to obtain exact answers, they resort to simulation, approximation, and sampling, perhaps by using Monte Carlo techniques. Such approaches are not always appropriate, as when a massive computation is used to make a momentous yes/no decision. In the development of antiballistic missile software during the early 1960s scientists tried to cope with such situations by the simultaneous use of many processors in a multiprocessor environment. It was found that this could result in unpredictable and often serious deterioration in the performance of the system as a whole. This discovery generated a serious study of such anomalies. The resulting work in the mid-1960s provided rigorous bounds for these deleterious effects. Performance guarantees of this type, usually called "worst-case bounds," remain a major focus of algorithmic analysis.

The fundamental class of NP-complete problems was defined in 1971 independently by a Canadian and by a former Soviet citizen who now resides in the United States. This class includes thousands of basic computational problems arising in computer science, mathematics, physics, biology, economics, business, and the social sciences. The NP-complete problems are of equivalent complexity in the sense that if one of them submits to an algorithm of polynomial complexity, then so can all the others. Proving the widespread conviction that no such algorithm exists (that $P \neq NP$) is considered the most fundamental of the open problems in theoretical computer science. It is known that any algorithm for solving an NP-complete problem can be modeled by an appropriate circuit. Until recently the only established lower bounds on circuits for NP-complete problems were linear; however, last year a Taiwanese mathematician now living in the United States made a dramatic breakthrough by establishing the conjectured exponential lower bound, but under the assumption that the number of "levels" of the circuit is bounded. Since then a Swedish graduate student in the United States simplified and strengthened these arguments. Almost simultaneously, two Soviet mathematicians independently established an exponential lower bound assuming that the circuit is "monotone," a related but distinct development. This work has been extended by an Israeli doing postdoctoral work in the United States, and many complexity specialists now believe that the tools are finally becoming available to prove the corresponding results for unrestricted circuits, thereby finally settling this fundamental problem.

As the difficulty of *NP*-complete problems was realized, attention shifted to other approaches. These included approximation algorithms, average-case instead of worst-case performance analysis, and randomized algorithms that give a confident guess rather than a firm answer. Timely examples of the latter are the twin problems of deciding if a large integer n is prime or, if it is not, of factoring n. One such algorithm randomly selects an integer k less than n, performs a simple test, and announces either that n is definitely composite or that the problem is still undecided. About three quarters of the possible k's will establish that a composite n is not prime. Thus, performing the test with 100 independent k's that do not prove n to be composite justifies the conclusion that n is prime with a mere one in 4^{100} chance of error.

The study of primes has long been central to number theory, a field that has been pursued for its own splendid beauty but traditionally was considered to have few pragmatic consequences. Recent innovations in cryptography have completely reversed the latter perception. The security of important cryptosystems depends crucially on the belief that the problem of finding the prime factors of a random number with a thousand decimal digits or more is, and will be for decades, a computationally infeasible problem. However faith in this belief is beginning to erode because of some recent unexpected advances in primality testing and factoring. Essentially overnight, a Dutch mathematician produced the currently best factoring algorithm by using the theory of elliptic curves from algebraic geometry, a field mentioned in the previous vignette. It is not yet clear whether this will lead to even more effective algorithms that will undermine the security of these cryptosystems.

This vignette concludes with a discussion of linear programming (LP), a subject that occurs widely in discrete optimization. For example, a linear profit function, possibly depending on a large number of variables, is to be maximized in a region defined by linear constraints. This question is geometrically equivalent to finding the highest point in some high-dimensional convex solid or polyhedron in n-dimensional space, where n can be quite large. The importance of these problems, which have many applications in such varied areas as airline scheduling, meteorology, portfolio management, and telephone traffic routing, was first observed more than 40 years ago by Dantzig in the United States and Kantorovitch in the Soviet Union. Dantzig developed the very effective "simplex algorithm" for solving LP problems, which examines first one vertex and then another, moving along the outside edges of the high-dimensional polyhedron in such a way as to improve the function that is to be optimized at each step until the optimal vertex is reached. Even though these n-dimensional polyhedrons, which typically arise in practical problems, can have exponentially many vertices, over 40 years of experience with the simplex algorithm indicates that only rarely are more than $4n$ or $5n$ vertices tested before the optimum point is attained, despite the fact that pathological examples can be constructed that do indeed require that all vertices be tested.

After the concept of NP-completeness was introduced in 1971, researchers struggled without success to find either a polynomial-time algorithm for LP or a proof that it was NP-complete. In 1979 the polynomial "ellipsoid algorithm" was produced in the Soviet Union, but the bound on this method grows as n^6, rendering it impractical for large problems, e.g., those having hundreds of thousands of variables. Subsequently, however, the ellipsoid algorithm has had a significant impact in the theory of combinatorial optimization.

In 1984 a young Indian mathematician in the United States made a striking breakthrough when he discovered an iterative method that plunges through the interior of the polyhedron, transforming it nonlinearly at each step so as to stay as far as possible from the boundaries of the changing solid. The number of steps required by this method is on the order of $n^{3.5}$, a significant improvement over the earlier n^6, and a variety of applications have shown that when implemented cleverly, this algorithm seems to perform significantly faster than the simplex algorithm.

Very recent efforts in understanding this new method indicate that an n-dimensional polyhedron can be equipped with a coordinate system that transforms its interior into a certain quasi-hyperbolic geometry so that the trajectories to the optimal vertex form geodesics in this space. Obviously, this area is extremely active worldwide, and much more work is needed before a full understanding can be achieved. Nevertheless, it is already apparent that complexity theory addresses many practical problems and that sophisticated core mathematics, including algebraic geometry, number theory, and geometry, is vital to complexity theory.

Nonlinear Hyperbolic Conservation Laws

The development of both theory and numerical methods for solving nonlinear hyperbolic conservation laws (NLHCLs) has been an exciting area of recent mathematical research. There are many applications of NLHCLs because they describe many important physical systems, including some in aerodynamics, meteorology, water waves, plasma physics, and combustion. In gas dynamics these are the laws of conservation of mass, momentum, and energy.

The major technical obstacle to both solving and analyzing these systems is the fact that their solutions are not smooth, that is, they do not have derivatives of all orders. Many standard approximation procedures require smoothness, and, furthermore, solutions tend to be unique only if certain physical constraints (called entropy conditions) are satisfied. Since there are other important equations in hydrodynamics, relativity, and optics that do not have smooth solutions, NLHCLs are providing a testing ground for innovations with potentially wide applicability.

The first systematic computational attempts to solve NLHCLs, motivated by problems of jet propulsion and in the Manhattan Project, were made in the United States during World War II. John von Neumann and others provided a clear formulation of the problem and introduced several crucial ideas, in particular artificial viscosity (a justifiable technique for smoothing the problem) and a first analysis of stability. These ideas were greatly extended after the war and gave rise to an elegant theory of weak solutions that treated the lack of smoothness without smoothing. In particular, the Lax-Wendroff scheme and its many variants yielded acceptable solutions for many practical problems. However, many other problems remained unsolved, and the theory was incomplete.

In the 1950s the relevance of the Riemann problem, well known to chemists and engineers working with the Riemann shock tube, became fully recognized. The Riemann problem contains the pathology of the general NLHCL problem but is in a more tractable form. An American mathematician gave a mathematical analysis of the Riemann problem, and then a Russian mathematician incorporated the American's analysis into a numerical construction that described the misbehavior of the solutions in a natural way. These developments led to a proof, developed in the United States, that solutions exist for one-dimensional NLHCLs subject to some technical restrictions. This result gave rise to practical algorithms that in turn generated sharper existence results.

The Russian's construction has been generalized in several ways, and dramatic progress has arisen in the last three years from a combination of these ideas. In particular, reliable and efficient solutions of the equations of gas dynamics in any number of space dimensions are now available, and they reveal and explain intricate physical phenomena that previously had been only dimly comprehended. Examples include the disclosure of unexpected complexity in flows involving interacting discontinuities, the discovery of transition criteria from regular to Mach reflection for waves impinging on a surface, and the revelation of instability mechanisms for supersonic jets. Both theory and experiments in NLHCL have been aided by elaborate computations from experiments using new laser technologies.

Recently, much computational activity has been directed toward solving systems of NLHCLs that are structurally more complex than those of gas dynamics, in particular those that arise in combustion theory or in the analysis of flow of porous media—a subject of great relevancy for oil recovery. New methodologies have appeared involving front tracking, mesh refinement, and piecewise parabolic approximations. Some of these are related to higher-order versions of the aforementioned Russian construction.

Many important practical problems remain open. For example, there is as yet no reliable numerical method for solving the equations of combustion theory in more than one space dimension except in the low-Mach-number limit. The newer numerical methods are so complex that computer science questions regarding their implementation, similar to those discussed in the previous vignette, have become crucial. Also, perturbations of NLHCLs, for example by boundary layers, are beginning to be considered.

Why numerical methods fail and why they sometimes succeed so spectacularly are questions that have been studied successfully in recent years, especially through the re-examination of the precise role and possible forms of artificial viscosity. New theoretical tools for understanding practical algorithms and new ideas, such as the notion of variation diminishing schemes, are producing a slow confluence of algorithms rooted in disparate a priori notions of what is important for practical calculations.

New techniques of functional analysis, in particular the compensated compactness method that originated in France, have given new impetus to existence theory, removing some of the earlier limitations. The compensated compactness analysis is significant to the broader context of homogenization and order/disorder phenomena, two other areas in which the French have been involved. In addition, partial existence and stability results have appeared for nonsmooth solutions in more than one space dimension for both convex and nonconvex systems.

The strong interaction between mathematics and practical applications in NLHCL is clear from this account. All the major advances in computation have been anchored in theoretical developments. Indeed, most of the more innovative practical algorithms are due to mathematicians, and more mathematicians have been involved than is apparent. An explosion of knowledge could at this time be safely forecast if there were more high-caliber people active in the field.

However, there are too few people with a combined understanding of the abstruse physical, mathematical, computer science, and related aspects of NLHCLs. Such an understanding requires a broad education in several fields that is not easily available in the United States. One explanation for the strong role played by Israelis in this field may be that in Israel many mathematicians are exposed to engineering problems and to programming during a lengthy military service. In other countries, such as Japan, China, the Soviet Union, and in much of Europe, most students complete algebra by the seventh grade and soon begin calculus, leaving time during secondary school for those with mathematical and technical talent to pursue advanced topics. Given that changes in precollegiate education will take a long time to evolve, the most immediate solutions in the United States would seem to be an extension of the graduate student years through supporting young investigators with adequate postdoctoral fellowships and providing enrichment for talented undergraduate students.

Yang-Mills Equations

In 1954, Yang in the United States and Mills in England constructed a nonlinear version of Maxwell's equations that incorporated a non-Abelian group, typically SU(2), the group of two by two unitary complex matrices with a determinant one. [SU(2) is a three-dimensional Lie group.] This was first conceived as a classical theory transplanted to Lorentz space-time; but when it is used in quantum theory, it is convenient to use Euclidean space-time. The theory has been incorporated into nearly every model of particle physics since the construction in 1975 of instanton solutions and the more recent construction of multi-instanton solutions in four-dimensional space. Shortly thereafter, the Penrose twistor theory was shown to transform an apparently very messy nonlinear system of partial differential equations into an elegant problem in algebraic geometry. Then the equations themselves were noticed to be natural geometric objects.

The Yang-Mills equations depict the curvatures (fields) of connections (potentials) as a principal bundle over a Riemannian manifold. By now, Yang-Mills theory is prominent in pure mathematics. It has already affected subjects as diverse as differential geometry, algebraic geometry, the topology of four-dimensional manifolds, the calculus of variations, nonlinear partial differential equations, index theory (or anomalies), and even the representation of infinite-dimensional groups, and it remains fertile research ground.

The extended impact of Yang-Mills theory does not yet involve the complete equations but concentrates on their role as nonlinear extensions of Laplace's equations, which are well known to be fundamental to earlier mathematics, physics, and engineering. In two variables Laplace's equation is related to the Cauchy-Riemann equations, part of the foundation of complex analysis. One form of the Yang-Mills equations, called the self-dual Yang-Mills equations (SDYM), is a four-variable analogue of the Cauchy-Riemann equations. The instantons are solutions of these equations, and they appear to have properties nearly as basic as solutions of the Cauchy-Riemann equations. The SDYM equations are in turn important in algebra, geometry/topology, and analysis, respectively.

At first glance, the SDYM equations seem absolutely intractable for writing explicit solutions. Even for SU(2), a small essentially non-Abelian Lie group, there are nine first-order equations with twelve unknown dependent variables as well as the four independent variables on the space. There are three extra degrees of freedom, due to gauge symmetries, so it is not surprising that insight from algebraic geometry had to precede progress in topology. The Penrose twistor methods were used to transform these equations in four-dimensional space to a problem concerning holomorphic bundles on a six-dimensional manifold.

These methods from algebraic geometry are surprisingly general and can lead in many directions, for example, toward Kac-Moody Lie algebras and models with loop groups.

Learning about the structure of the space of instantons over four-dimensional manifolds has generated profound insight into the topological structure of general four-dimensional manifolds, demonstrating the fundamental value of an equation specific to a low dimension like four. This was helpful to physicists, who have since developed the fundamental intuition of instantons as solitons, the wavelike solutions of KdV and Sine-Gordon that superimpose nonlinearly.

It is interesting to note that mathematicians had been able to deal with the structure of manifolds in dimensions two, three, five, and greater. The Cauchy-Riemann equations are used in two dimensions, geometric methods are employed in three, and mathematicians find five and more dimensions amenable to standard methods of algebraic topology. The gap of the fourth dimension appears to be filled by SDYM theory.

In analysis, one of the key properties of Yang-Mills theory is its conformal invariance. Some of the basic equations about instantons can be formulated in the context of the calculus of variations. The conformal invariance of the theory implies that the variational problem does not satisfy the conditions that are required in order to use the method of direct steepest descent, so helpful in three-dimensional work. The attempt to understand the failure of the steepest descent method for the Yang-Mills problem has led to the development of new variational techniques, which are useful on a variety of problems. It is interesting, and possibly significant, that the three fundamental scale-invariant geometric problems coincide with three basic models of quantum fields: the Yamabe problem (phi-four theory), harmonic maps (sigma models), and the Yang-Mills equations.

In any case, Yang-Mills theory is a beautiful example of the intense bonds between current theoretical physics and all subfields of core mathematics. Yang-Mills theory is a young discipline that will undoubtedly attract many more mathematicians in the near future. Its results to date are primarily due to the efforts of English, American, and Soviet researchers. Although Americans cannot claim to dominate the field, they have certainly contributed significantly to its development. The necessity of extremely broad training, mentioned in the final paragraph of the preceding vignette, also applies to successful research in this field.

Increasingly, postdoctoral or protracted study is necessary in order to become a successful researcher in many areas of mathematics.

Operator Algebras

The area of operator algebras is currently very active and provides another excellent example of unexpected interactions between areas of core mathematics and the natural sciences. Quantum physics originated the Heisenberg Uncertainty Principle, which forces consideration of quantities P and Q satisfying $PQ - QP = h/2\pi i$. It motivated a search for appropriate mathematical systems containing infinitely many such noncommuting variables. In the 1920s, M. H. Stone and J. von Neumann demonstrated that such systems require a general theory of algebras of operators on Hilbert spaces, generalizations of finite-dimensional vector spaces. These were studied extensively in the 1930s by F. Murray and von Neumann.

The mathematics of operator algebras is characterized by a profound blend of noncommutative algebra and infinite-dimensional analysis. Although operator algebras exhibit a very rich structure, no serious work following that of Murray and von Neumann appeared until after World War II.

The simplest example of an operator algebra is the entire set of n by n dimensional matrices for a specific n. A general operator algebra is a subset of the bounded linear transformations on a (generally infinite-dimensional) Hilbert space that is closed under addition, multiplication, an adjoint operation, and a suitable limiting process. If the system is closed only for the strongest limiting process, it is called a C^*-algebra. If it also contains the most general limits, it is said to be a von Neumann algebra. The full algebras of all linear transformations on Hilbert spaces of arbitrary dimensions are the building blocks of the simplest operator algebras. However, consideration of proper subalgebras of the universal operator algebra over some Hilbert space reveals far more complex situations. This complexity can be slightly relieved and the study reduced to three basic types of von Neumann algebras simply called Types I, II, and III.

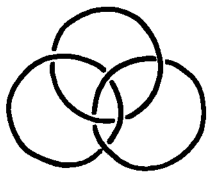
During the early postwar years, both American and French mathematicians made substantial progress in operator algebras, including some important applications to the theory of infinite-dimensional group representations. The French school became relatively inactive by the early 1960s and re-emerged in the mid-1970s when a young mathematician, subsequently a Fields Medalist, began working in the area. During the late 1950s and early 1960s, some Japanese mathematicians entered the field. Simultaneously, numerous theoretical physicists became involved and provided valuable insights derived from their physical applications. It was shown that there exist infinitely many distinct Type II and Type III factors. The latter, especially, remained mysterious because they did not possess the special functionals called traces that were so fundamental to studying the structure of the other two types. When it was discovered that the physically interesting algebras are of Type III and certain classes were explicitly parameterized by physicists, parts of the mystery began to unravel.

Major international conferences began having a crucial influence on operator algebra theory in the mid-1960s. In the late 1960s, a conference held in the United States removed a key obstacle to analyzing Type III factors by displaying results proved independently by a Japanese mathematician and some Dutch and German physicists. At a subsequent conference, the French Fields Medalist, then a student, was motivated to work on the subject. He opened entirely new vistas by investigations that combined algebra and analysis in new ways and led to the classification of the algebraic structure of Type III factors. In the 1970s, researchers turned toward the geometric aspects of the subject. A new extension theory of C^*-algebras generated a successful synthesis of geometry and algebra, resulting in a unified view of features from these two subjects. At about the same time, the fundamental Atiyah-Singer Index Theorem was extended from locally trivial families to the more general foliations by using operator algebras. Additionally, a Soviet mathematician solved specific cases of a long-standing problem in differential topology concerning smooth deformations by developing powerful new techniques in C^*-algebras.

Recently, an American mathematician, born in New Zealand and educated in Switzerland, has found a totally unexpected connection between three apparently diverse fields–knot theory, the classification of subfactors of Type II factors, and the theory of Hecke algebras. This activity has occurred within the past two years, a catalyst being the fortuitous meeting of the Mathematical Science Research Institute at Berkeley in 1985, sponsored by the National Science Foundation, where experts in operator algebras were meeting concurrently with those in low-dimensional topology, the field that contains knot theory. The excitement of the connection between von Neumann algebras and knot theory may be overshadowed by investigations of its utility to biologists in describing large-scale structures of DNA. A knot is a closed curve in three-space, and a link is a (possibly interlocking) system of knots. Although they can be surprisingly complicated, knots and links can be adequately represented by a projection onto the plane. Two relatively simple examples are:

The knot 6_2

The link 6_2^3
(also called the Borromean rings)



Although people probably have always used knots, the first known attempt to list and classify them mathematically (as opposed to mechanically) resulted from an erroneous hypothesis of Lord Kelvin in the late nineteenth century that atoms were knotted vortices in the ether. His vain hope of deriving the periodic table by classifying knots stimulated scientific advances entirely different from his vision—in the characteristic but unpredictable manner that pure research, stimulated by attractive ideas, often yields unanticipated harvests quite different from their original intent.

Knot theory considers two knots or links to be the same if one may be deformed without crossing strings until it is identical to the other. It is exceedingly difficult (and obviously fundamental) to decide when two given links are the same. Trial and error methods are rarely satisfactory. One approach to classifying two mathematical entities in some class is to assign each entity in the class some mathematical label (called an invariant) that coincides on the two entities considered to be the same. Thus a search for appropriate invariants for links began. In the 1920s polynomial invariants were developed by studying the topology of the space remaining when the link is removed from ordinary three-space. The corresponding polynomial for the knot 6_2 above is $1 - 3x + 3x^2 - 3x^3 + x^4$.

Another approach to knots first studied in the 1920s was a mathematical structure called the braid group. Braids can be spliced together thus forming a group, however splicing the top of a braid to its bottom forms a knot or link, and indeed all knots and links may be so formed (in possibly many ways).

Braids and their invariants have rewarded investigators with considerable valuable insight over the years. However, their connection with operator algebras remained unnoticed until the proof of a deep result about subfactors of Type II factors required an analysis of one representation of the braid group. Coincidentally, almost the same representation, arising from a special case of the Hecke algebras, had been discovered by mathematical physicists in the 1960s as they partially solved the Potts model of statistical mechanics. The connection with operator algebras generated a trace function for the braids that could be used to recover numerical information. The trace of a braid then provided a new (Laurent) polynomial invariant for the associated knot or link. It was more sensitive than the earlier polynomial in that it separated knots that were previously indistinguishable. It could even distinguish knots from their mirror images. Computers can be used to compute this polynomial for many links. For the knot 6_2, it is $x^{-1} - 1 + 2x - 2x^3 - 2x^4 + x^5$ and, for the link 6_2^3, it is $-x^{-3} + 3x^{-2} - 2x^{-1} + 4 - 2x + 3x^2 - x^3$.

Both operator algebra and topology already have produced substantial generalizations of this new invariant that have been used to solve many venerable problems of knot theory. Similar work is expected to shed new light soon on the Potts model, von Neumann algebras, knot theory, statistical mechanics, quantum physics, and possibly even basic structures of life via the DNA application. In any case, these developments emphasize again the eternity of a good mathematical result, the harmony of all mathematics, the unpredictable relationships between fields, and the many bridges across the humanly created gap between core mathematics and basic science.

At present, only a few institutions in the United States offer training in operator algebras. In this area, as with other areas of mathematics, the United States has become increasingly dependent on hiring foreign-trained students and scientists. In addition, some related areas with potential for interaction with engineering have been underemphasized in the United States. This is an ideal time to put a greater effort into providing the correct conditions for new young researchers and strong leadership in the United States in the subject.

Survival Analysis

Since problems in collecting, analyzing, and interpreting data are universal, it is not surprising that people in many countries (including England, France, Scandinavia, the Soviet Union, and the United States) have made significant contributions both to statistics and also to probability, the branch of core mathematics that has until recently provided the major theoretical support for statistics. Expansions in technology have both motivated and enabled the most striking progress in statistics during the past decade. Advances in instrumentation and communication have generated enormously complex sets of data, and the growth of computing power permits the collection and management of such data. The United States has been the unquestioned world leader in statistical computing, mainly because the availability and sophistication of its equipment has been unmatched elsewhere. A further discussion of these developments can be found in the NRC report *Computational Modeling and Mathematics Applied to the Physical Sciences*, and in the forthcoming NRC Board on Mathematical Sciences report, *A Survey of U.S. Mathematical Sciences.*

However, other countries have made significant contributions; a notable example from England is David Cox's proportional hazards model for life history data. Life history analysis is a body of statistical techniques useful in medical research, reliability theory, actuarial computations, and demographic studies. John Graunt initiated this field in 1662 with his invention of life tables for analyzing English mortality data, but recent clinical studies are far more complex. Typically, these begin with patients who have an unpleasant disease (possibly at different stages) being assigned randomly to two or more different treatments; they are then followed until they die or disappear or the study ends. Observers record variables, called covariates, that might affect the survival of the patients, including some that do not change, such as sex and age at diagnosis, and some that do, such as blood pressure and glucose level. Statisticians use these observations to study the relationships between the covariates and the survival time of patients after contracting the disease, and especially the comparative merits of the treatments. One goal may be to predict the effect of different treatment on life expectancy of future sufferers of the disease, another may be to determine which factors prolong the patients' normal functioning as long as possible. Subtleties such as the role of the individuals who are still alive at the end of the study and those who withdraw or disappear complicate the analysis.

One model for the life history of a patient (or piece of equipment) includes a vector $Z(t)$ of covariates that vary (possibly randomly) with time, from zero, when the study begins, to the time T when the study is terminated, and a finite-valued process $Y(t)$ designating the state of the dependent variable. These states might be "alive-and-functioning," "alive-not-functioning," "dead," or "lost." The treatments appear as coordinates of $Z(t)$. Analysis of these models concentrates on the intensity of changes; these intensities correspond to a stochastic process $J(t, y_0, y)$, where $J(t, y_0, y)\,dt$ represents the probability that Y changes from y_0 to y between time t and $t + dt$ assuming a past of $Z(t)$. The life histories of the n patients are regarded as a set of independent observations that can be used to estimate J.

Until the early 1970s, research concentrated on the extremes of either relatively simple situations with few assumptions or vastly more complicated parametric models with heavy assumptions on J. An example of the former is the setting for the product limit estimator for the probability of surviving beyond time t, when no covariants are measured and only questions of life expectancy are of interest. The assumptions of the parametric models were found to be too unrealistic by experts in biostatistics.

In 1973 Cox proposed his "proportional hazards model" for survival-time data, basing his work in part on a model developed during the 1960s at the National Cancer Institute. In this model

$$J(t, \text{Alive}, \text{Dead}) = \exp[\theta^T Z(t)]\lambda(t)$$

where $\lambda \geq 0$ is a (nonrandom) function and θ is a vector of unknown parameters. $J(t, \text{Alive}, \text{Dead})$ can be an essentially arbitrary function of $Z(s)$, $0 \leq s \leq t$.

The attractive features of the Cox model are easily seen if we specialize to fixed covariates. The model permits an arbitrary lifetime distribution for a control population and postulates a linear approximation for the log of the ratio of intensities corresponding to two different values of Z. The effects of the covariates can then be measured in terms of the components of θ.

The inferential procedures proposed by Cox were quickly applied to heart transplant and other data by statisticians and biomedical scientists in the United States. Evidence of these applications appears in numerous citations of the model, mostly in the medical literature. Although the model seemed to give sensible answers in applications, its highly nonlinear nature and the complex probabilistic structure of the data prevented rigorous theoretical analysis for some time.

The theory for the case of time-independent covariates was independently developed in the United States and Denmark in the mid to late 1970s, but these methods could not handle the much more difficult case of time-varying covariates. In 1975, the thesis of a Norwegian student studying in the United States with a United States statistician who emigrated from France showed how to attack the analytic problems in this area. He applied a multivariate counting process framework to both survival analysis models and more general life history models. Most significantly, he exhibited applications to these models of the deep results in continuous time martingales and stochastic integrals, which were introduced by researchers in Japan, the United States, and France. Then others, primarily in The Netherlands, Norway, Denmark, England, and the United States, used these techniques to analyze the heuristic suggestions of Cox and to devise and investigate new inferential procedures in more complicated life history models. Despite their flexibility, the Cox models are still burdened by the questionable assumption that the ratio of intensities for two individuals can be modeled parametrically. It is not clear that the counting process techniques will prove adequate for the analysis of the newer, more flexible, semiparametric models that have been proposed, but they should provide a good starting point.

The analysis of life history data is a rapidly expanding field widely used in a variety of disciplines, including biomedical science, demography, and sociology, fields that reciprocate by continually presenting statisticians with data for which previous methods are inadequate. Although the interaction of theory and application and the international nature of this work are hardly new features of statistics, they have become more prominent recently.

The United States and England are the primary world centers of statistical activity. An important pattern in the field, which is illustrated here, is that foreign scientists and students come to study, lecture, and meet in the United States, and subsequently many of the best remain as citizens or permanent residents.

4. REVIEW OF THE DAVID REPORT

The David Report, a several-year effort by a broad cross section of industry and university scientists, has considerably raised the consciousness of the science community. The principal finding of this 1984 National Research Council report, Renewing U.S. Mathematics: Critical Resource for the Future (National Academy Press, Washington, D.C., 1984), was that "federal support for mathematical research is markedly out of balance with support for related fields of science" The Panel on Mathematical Sciences believes that it is important to review the major findings, the subsequent support, and the effect of the critical David Report.

The major findings were as follows:

- Over a 15-year span from 1968 to 1982 constant dollar federal support of mathematics research dropped by over 33 percent.
- During the same period, the number of people in the field doubled in size (the influx from the late 1960s).
- The effects of the resulting paucity of funding are being severely felt at the research universities where almost all basic mathematical research is done.
- Unless corrective action is taken, the health of the nation's mathematical research effort, now still the strongest in the world, will be seriously weakened.
- The central importance of mathematics to science and technology, hence to the economy and defense, makes its continued strength in the United States imperative.
- To address this imbalance, basic needs by 1990 include substantial increases in support for graduate and postdoctoral students, in addition to establishing a greater base of support among young and senior researchers.

Since the David Report appeared in 1984 it has received wide acceptance in the universities and at the federal policy level. The problem seems to be understood. Evidence of this includes:

- The National Science Board (representing all disciplines of science and engineering) passed a resolution "that a concerted effort should be made by all funding agencies to increase support for the mathematical sciences for several years until a proper level of sustaining support has been achieved."
- A subcommittee of the Department of Defense (DOD)-University Forum reported that there had been significant erosion in mathematics support, partially because the growth of computer science had masked the mathematics funding situation during the 1970s.
- The University Research Initiative, part of the fiscal year 1986 DOD appropriation, states the critical need to address the imbalance that currently exists with respect to mathematics funding.

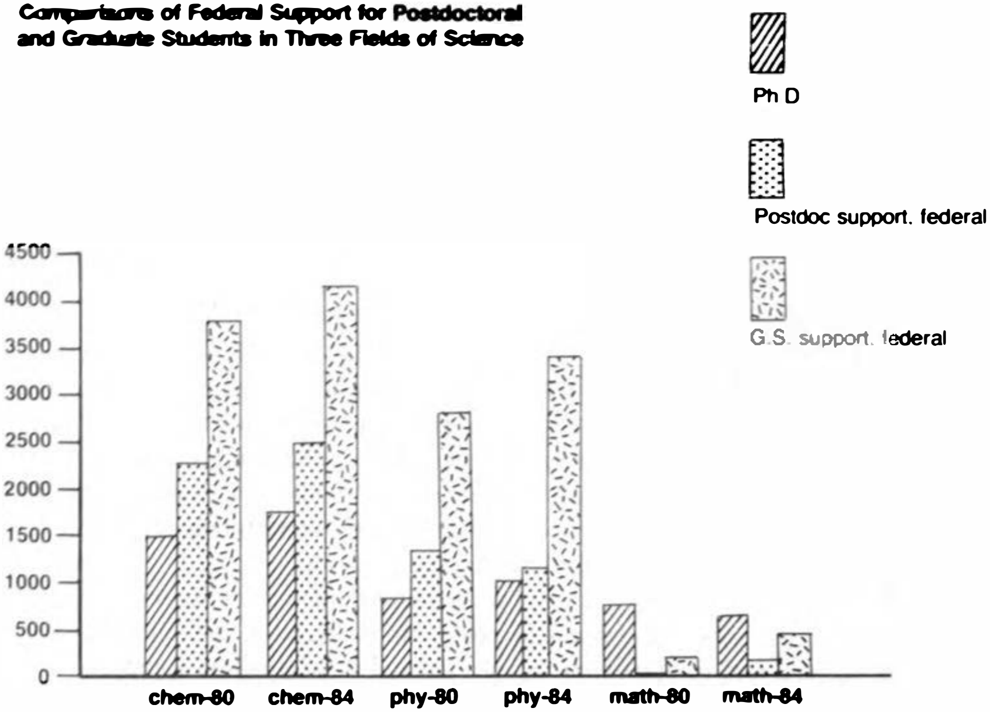
Currently, the federal agencies primarily responsible for mathematics funding are making an effort to redress the problem. However, recent budget constraints severely threaten their efforts. The issue is how to reach this balance of resources between related fields in a climate of overall limited budget enhancements. The difficult burden falls on the agencies which must set priorities in funding research.

Movement toward a proper level of sustaining support in the mathematical sciences for graduate and postdoctoral students, established investigators, and equipment is varied. The only real progress appears to be moderate increases in support for graduate and postdoctoral students, while other investigator support appears static. Even graduate student support progress leaves the discipline in a vastly unfavorable position at a critical time, see Table 1.

The National Science Foundation continues to advocate for additional resources for the mathematical sciences, primarily with a significant cross-disciplinary effort in computational science and engineering. A new mathematics program at Defense Advanced Research Projects Agency (DARPA) is potentially a bright spot. Without this new DARPA program, the current 1986 and 1987 budgets for mathematical sciences research at the DOD agencies would be severely depressed (Table 2). Even with these enhancements, if they remain, and the best intentions of the policy makers, the renewal of U.S. mathematics will require continued resolute effort.

TABLE 1

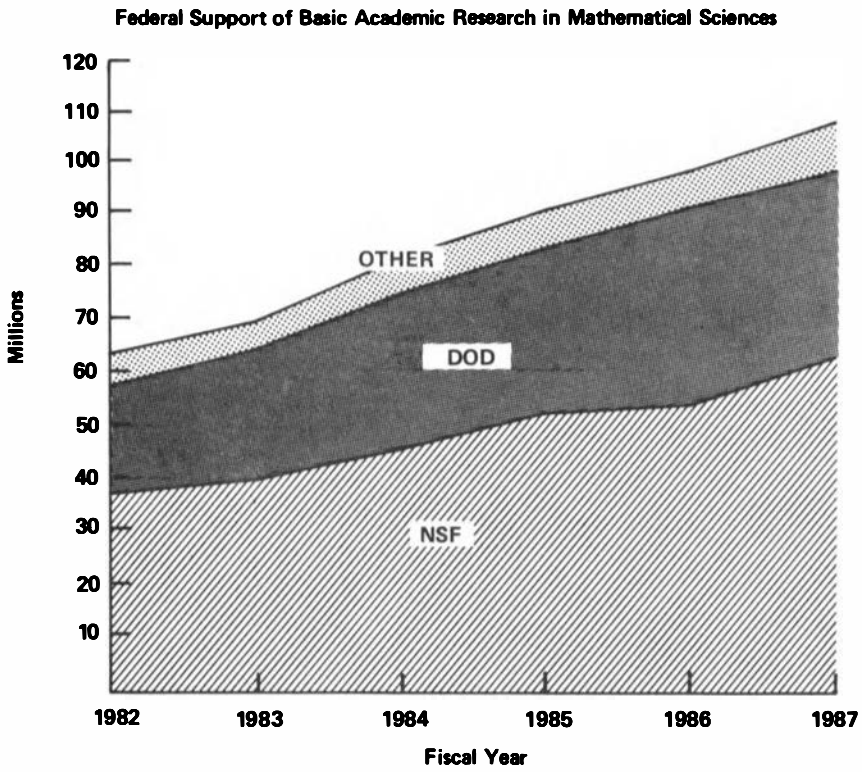
Comparisons of Federal Support for Postdoctoral and Graduate Students in Three Fields of Science



	Chemistry		Physics		Mathematical Sciences	
	1980	1984	1980	1984	1980	1984
University faculty and staff with primary or secondary activity in R&D	9800	9200	9200	9200	7400	7200
Annual Ph.D. Production						
U.S. citizens	1169	1332	611	659	520	407
Total Ph.D.s	1538	1765	862	962	744	698
Postdoctorals federally supported	2255	2473	1210	1150	57	132
Graduate Students federally supported	3700	4118	2900	3348	200	411
Ratio P.D./Ph.D.	1.47	1.40	1.40	1.20	0.08	0.19
Ratio G.S./Ph.D.	2.41	2.33	3.36	3.48	0.27	0.59

Sources: NRC Survey of Doctoral Recipients.
Survey of Graduate Science and Engineering Students and Postdocs, NSF.
Survey of Doctorate Recipients, NSF, unpublished, Table D-32.

TABLE 2



Federal Support of Basic Academic Research in Mathematical Sciences (FY)

	82	83	84	85	86[3]	87[4]
NSF[1]	34.2	37.1	45.6	52.7	57.2	65.5
DOD[2] (AFOSR, ARO, ONR, DARPA)	23.3	26.5	29.9	32.3	42.4	39.1
Other[2] (DOE, NASA, NIH)	4.3	4.8	4.9	5.5	5.9	4.8
Total	61.8	68.4	80.4	90.5	104.5	109.4

1) DMS represents about 90%.
2) This is based on estimates of the mathematics extramural component of some programs.
3) These are pre Gramm-Rudman, SDI and University Research Initiative figures. Their eventual effects may cancel each other.
4) The President's budget. Estimates of ultimate program emphasis (see 2) and future effects of URI and SDI make these tentative, especially outside NSF.