# CK-12 FlexBook

# CK-12 PreCalculus Concepts

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# CHAPTER 1 Functions and Graphs

## Chapter Outline



Here you will review and extend concepts about functions and graphing. You will learn how to transform basic functions and write these transformations using the correct notation. You will learn to describe a function in terms of its domain, range, extrema, symmetry, intercepts, asymptotes, and continuity. Finally, you will learn about function composition and inverses of functions.

# 1.1 Function Families

Here you will learn to identify primary function families by their equations and graphs. This will set the stage for analyzing all types of functions.

Functions come in all different shapes. A few are very closely related and others are very different, but often confused. For example, what is the difference between $x^2$ and $2^x$? They both have an $x$ and a 2 and they both equal 4 when $x = 2$, but one eventually becomes much bigger than the other.

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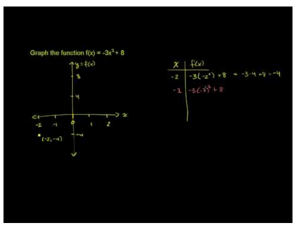

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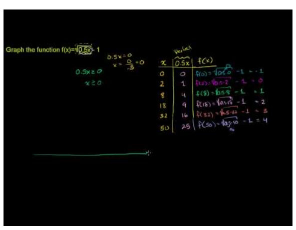

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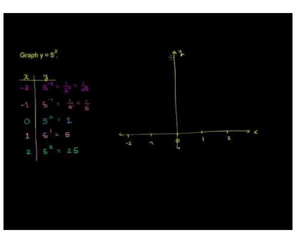

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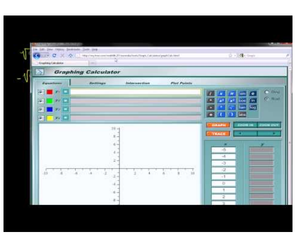
http://www.youtube.com/watch?v=9SOSfRNCQZQ Khan Academy: Graphing Exponential Functions



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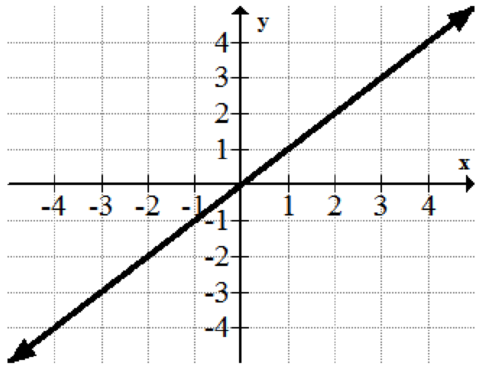
http://www.youtube.com/watch?v=outcfkh69U0 Khan Academy: Graphs of Square Root Functions

### Guidance

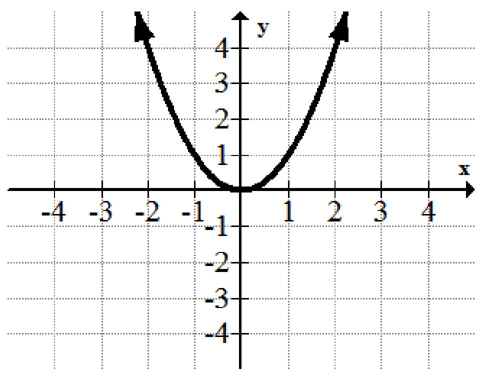
If mathematicians are cooks, then families of functions are their ingredients. Each family of functions has its own flavor and personality. Before you learn to combine functions to create an infinite number of potential models, you need to get a clear idea of the name of each function family and how it acts.

**The Identity Function:** $f(x) = x$



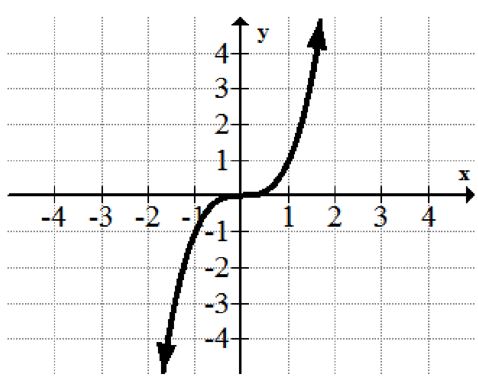
The identity function is the simplest function and all straight lines are transformations of the identity function family.

**The Squaring Function:** $f(x) = x^2$



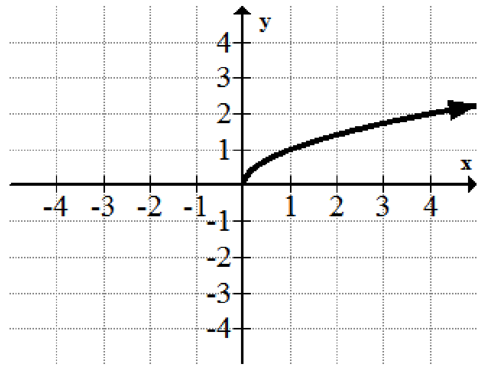
The squaring function is commonly called a parabola and is useful for modeling the motion of falling objects. All parabolas are transformations of this squaring function.

**The Cubing Function:** $f(x) = x^3$



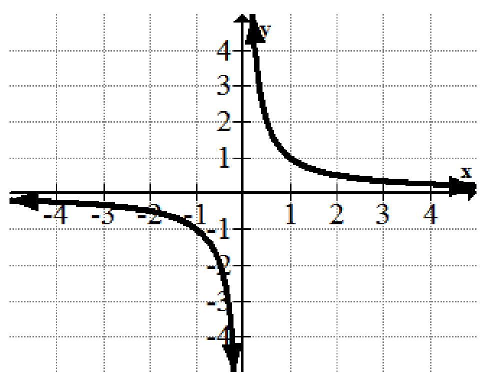
The cubing function has a different kind of symmetry than the squaring function. Since volume is measured in cubic units, many physics applications use the cubic function.

**The Square Root Function:** $f(x) = \sqrt{x} = x^{\frac{1}{2}}$




The square root function is not defined over all real numbers. It introduces the possibility of complex numbers and is also closely related to the squaring function.

**The Reciprocal Function:** $f(x) = x^{-1} = \frac{1}{x}$



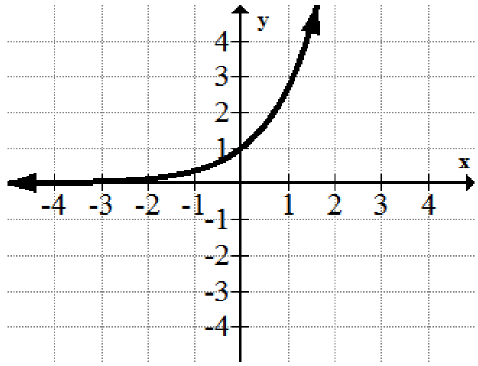

The reciprocal function is also known as a hyperbola and a rational function. It has two parts that are disconnected and is not defined at zero. Simple electric circuits are modeled with the reciprocal function.

So far all the functions can be grouped together into an even larger function family called the power function family.

**The Power Function Family:** $f(x) = cx^a$

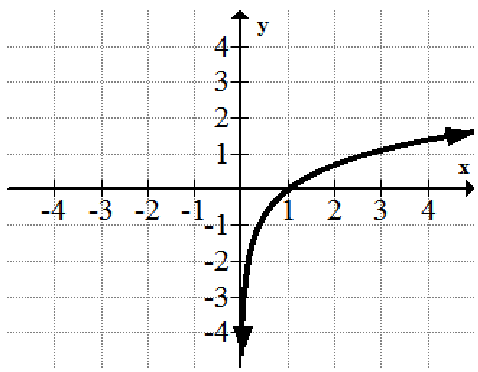
The power function family has two parameters. The parameter $c$ is a vertical scale factor. The parameter $a$ controls everything about the shape. The reason why all the functions so far are subsets of the larger power function family is because they only differ in their value of $a$. The power function family also shows you that there are an infinite number of other functions like quartics $(f(x) = x^4)$ and quintics $(f(x) = x^5)$ that don't really need a whole category of their own. The power function family can be extended to create polynomials and rational functions.

**The Exponential Function Family:** $f(x) = e^x$



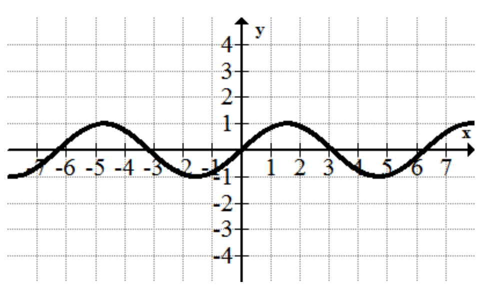

The exponential function family is one of the first functions you see where $x$ is not the base of the exponent. This function eventually grows much faster than any power function. $f(x) = 2^x$ is a very common exponential function as well. Many applications like biology and finance require the use of exponential growth.

**The Logarithm Function:** $f(x) = \ln x$



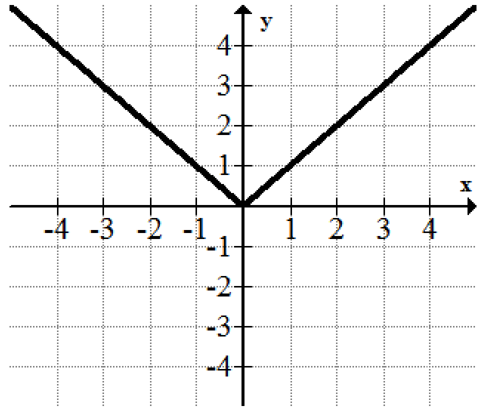

The log function is closely related to the exponential function family. Many people confuse the graph of the log function with the square root function. Careful analysis will show several important differences. The log function is the basis for the Richter Scale which is how earthquakes are measured.

**The Periodic Function Family:** $f(x) = \sin x$



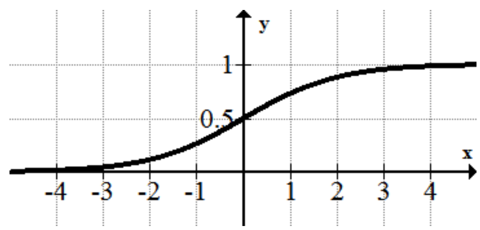

The sine graph is one of many periodic functions. Periodic refers to the fact that the sine wave repeats a cycle every period of time. Periodic functions are extremely important for modeling tides and other real world phenomena.

**The Absolute Value Function:** $f(x) = |x|$




The absolute value function is one of the few basic functions that is not totally smooth.

**The Logistic Function:** $f(x) = \frac{1}{1+e^{-x}}$




The logistic function is a combination of the exponential function and the reciprocal function. This curve is very powerful because it models population growths where the maximum population is limited by environmental resources.

**Example A**

Compare and contrast the graphs of the two functions: $f(x) = \ln x$ and $h(x) = \sqrt{x}$

**Solution:**

*Similarities:* Both functions increase without bound as $x$ gets larger. Both functions are not defined for negative numbers.

*Differences:* The log function approaches negative infinity as $x$ approaches 0. The square root function, on the other hand, just ends at the point (0, 0).

**Example B**

Describe the symmetry among the function families discussed in this concept. Consider both reflection symmetry and rotation symmetry.

**Solution:**

Some function families have reflective symmetry with themselves:

$y = x, y = x^2, y = \frac{1}{x}, y = |x|$

Some function families are rotationally symmetric:

$y = x, y = x^3, y = \frac{1}{x}, y = \sin x, y = \frac{1}{1+e^{-x}}$

Some pairs of function families are full or partial reflections of other function families:

$y = x^2, y = \sqrt{x}$

$y = e^x, y = \ln x$

**Example C**

Which function families are unbounded above and below?

**Solution:**

Look for the function families that don't have an overall maximum or minimum value.

$y = x, y = x^3, y = \frac{1}{x}, y = \ln x$

**Concept Problem Revisited**

While $x^2$ and $2^x$ have similar ingredients, they have very different graphical features. The squaring function is symmetric about the line $x = 0$ while the exponential function is not. When $x = 0$, the squaring function has a height of zero and the exponential function has a height of one. The squaring function has a slope that becomes steeper as $x$ gets further from the origin while the exponential function flattens as $x$ gets very small. All of these differences are important and not obvious at first glance.

## Vocabulary

A ***function family*** is a group of functions that all have the same basic shape.

A ***parameter*** is a constant embedded in a function that affects the shape of the function in a limited and specific way.

***Unbounded above*** means that the function gets bigger than any specific number you can choose.

***Unbounded below*** means that the function can get smaller than any specific number you can choose.

***Continuous*** means that the function can be drawn entirely without lifting your pencil.

## Guided Practice

1. Which functions are discontinuous?
2. Which functions always have a positive slope over the entire real line?
3. Which functions are defined for all $x$ values?

**Answers:**

1. The reciprocal function is the only function included here that is discontinuous.
2. $y = x, y = e^x, y = \frac{1}{1+e^{-x}}$. Some functions that are close but not quite: $y = x^3, y = \sqrt{x}$
3. $y = x, y = x^2, y = x^3, y = e^x, y = \sin x, y = |x|, y = \frac{1}{1+e^{-x}}$

## Practice

For 1-10, sketch a graph of the function from memory.

1. $y = e^x$
2. $y = \ln(x)$
3. $y = \sin(x)$
4. $y = x^2$
5. $y = |x|$
6. $y = \frac{1}{x}$

7. $y = \frac{1}{1+e^{-x}}$
8. $y = \sqrt{x}$
9. $y = x^3$
10. $y = x$
11. Which function is not defined at 0? Why?
12. Which functions are bounded below but not above?
13. What are the differences between $y = x^2$ and $y = x^3$?
14. What is a similarity between $y = e^x$ and $y = \ln(x)$?
15. Explain why $y = \sqrt{x}$ is not defined for all values of $x$.

# 1.2 Graphical Transformations

Here you will learn how a graph changes when you change its equation by adding, subtracting, and multiplying by constants.

The basic functions are powerful, but they are extremely limited until you can change them to match any given situation. Transformation means that you can change the**equation** of a basic function by adding, subtracting, and/or multiplying by constants and thus cause a corresponding change in the**graph**. What are the effects of the following transformations?

1) $f(x) \rightarrow f(x+3)$

2) $h(x) \rightarrow h(x)-5$

3) $g(x) \rightarrow -g(2x)$

4) $j(x) \rightarrow j\left(-\frac{x}{2}\right)$

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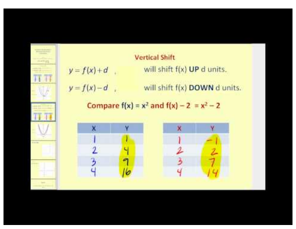

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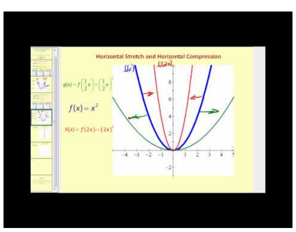
http://www.youtube.com/watch?v=CESXLJaq6Mk James Sousa: Function Transformations: Horizontal and Vertical Translations



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http://www.youtube.com/watch?v=2S9LUinJ8-w James Sousa: Function Transformations: Horizontal and Vertical Stretch and Compression

### Guidance

A function is a rule that takes any input $x$ and gives a specific output. When you use letters like $f$, $g$, $h$, or $j$ to describe the rule, this is called function notation. In order to interpret what effect the algebraic change in the equation will have on the graph, it is important to be able to read those changes in general function notation and then apply them to specific cases.

When transforming a function, you can transform the argument (the part inside the parentheses with the $x$), or the function itself. There are two ways to linearly transform the argument. You can multiply the $x$ by a constant and/or add a constant to the $x$ as shown below:

$$f(x) \to f(bx+c)$$

The function itself can also be linearly transformed in the same ways:

$$f(x) \to af(x)+d$$

Each of the letters $a,b,c$, and $d$ corresponds to a very specific change. Some of these changes are straightforward, while others may be the opposite of what you might expect.

- $a$ is a vertical stretch. If $a$ is negative, there is also a reflection across the $x$ axis.
- $d$ is a vertical shift. If $d$ is positive, then the shift is up. If $d$ is negative, then the shift is down.

When transforming the argument of the function things are much trickier.

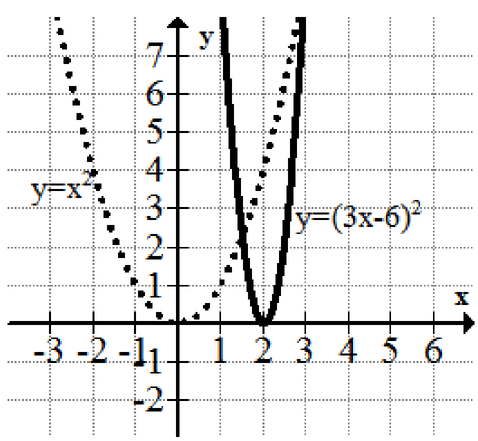
- $\frac{1}{b}$ is a horizontal stretch. If $b$ is negative, there is also a reflection across the $y$ axis.
- $c$ is a horizontal shift. If $c$ is positive, then the shift is to the left. If $c$ is negative, then the shift is to the right. *Note that this is the opposite of what most people think at first.*

The trickiest part with transforming the argument of a function is the order in which you carry out the transformations. See Example A.

**Example A**

Describe the following transformation in words and show an example with a picture: $f(x) \to f(3x-6)$

**Solution:** Often it makes sense to apply the transformation to a specific function that is known and then describe the transformation that you see.




Clearly the graph is narrower and to the right, but in order to be specific you must look closer. First, note that the transformation is entirely within the argument of the function. This affects only the horizontal values. This means while the graph seems like it was stretched vertically, you must keep your perspective focused on a horizontal compression.

Look carefully at the vertex of the parabola. It has moved to the right two units. This is because first the entire graph was shifted entirely to the right 6 units. Then the function was horizontally compressed by a factor of 3 which means the point (6, 0) became (2, 0) and the $x$ value of every other point was also compressed by a factor of 3 towards the line $x = 0$. This method is counter-intuitive because it requires reading the transformations backwards (the opposite of the way the order of operations tells you to).

Alternatively, the argument can be factored and each component of the transformation will present itself. This time the stretch occurs from the center of the transformed graph, not the origin. This method is ultimately the preferred method.

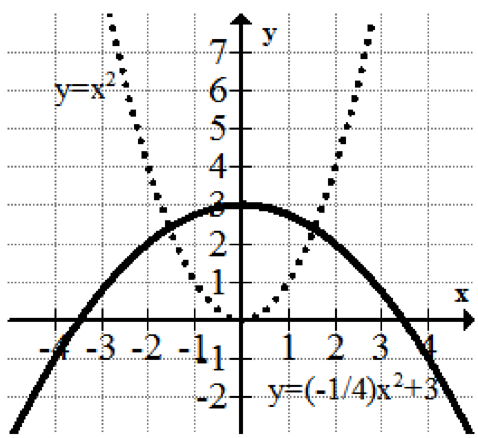
$f(3(x-2))$

**Either way, this is a horizontal compression by a factor of 3 and a horizontal shift to the right by 2 units.**

**Example B**

Describe the transformation in words and show an example with a picture: $f(x) \to -\frac{1}{4}f(x)+3$

**Solution:** This is a vertical stretch by a factor of $\frac{1}{4}$, a reflection over the $x$ axis, and a vertical shift 3 units up. As opposed to what you saw in Example A, the order of the transformations for anything outside of the argument is directly what the order of operations dictates.




**First, the parabola is reflected over the $x$ axis and compressed vertically so it appears wider. Then, every point is moved up 3 units.**

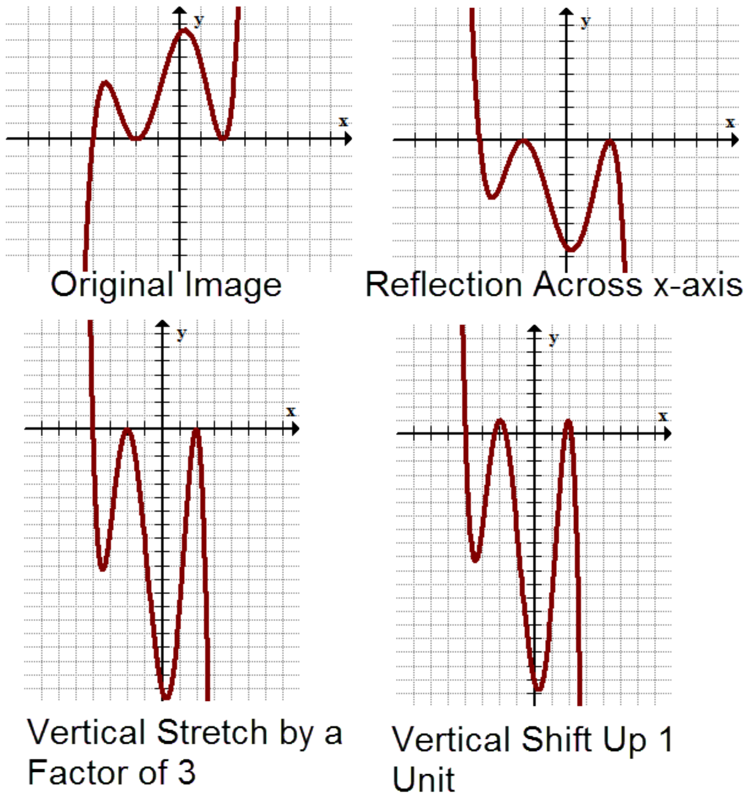
**Example C**

Describe the transformation in words and show an example with a picture:

$f(x) \to -3f\left(-\frac{1}{2}x-1\right)+1$

**Solution:** Every possible transformation is occurring in this example. The horizontal and the vertical components do not interact with each other and so your description of the transformation can begin with either component. Here, start by describing the vertical components of the transformation:

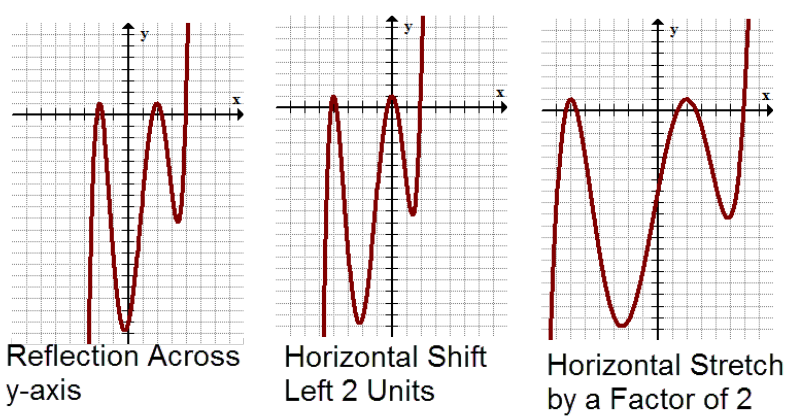
*There is reflection across the x axis and a vertical stretch by a factor of 3. Then, there is a vertical shift up 1 unit.* Below is an image of a non-specific function going through the vertical transformations.




In order to figure out the horizontal components of the transformation, start by factoring the inside of the parentheses (the argument):

$f\left(-\frac{1}{2}x-1\right)=f\left(-\frac{1}{2}(x+2)\right)$

*Factoring reveals a reflection across the y axis, a horizontal shift left 2 units and a horizontal stretch by a factor of 2.* Below is an image of the same function going through the horizontal transformations.




**Concept Problem Revisited**

1) $f(x)\rightarrow f(x+3)$

This transformation shifts the entire graph left 3 units. A common misconception is to shift right because the three is positive.

2) $h(x)\rightarrow h(x)-5$

This transformation shifts the entire graph down 5 units.

3) $g(x) \rightarrow -g(2x)$

This transformation is a vertical reflection across the $x$ axis and a horizontal compression by a factor of 2.

4) $j(x) = j\left(-\frac{x}{2}\right)$

This transformation is a horizontal reflection across the $y$ axis and a horizontal stretch by a factor of 2. A common misconception is to see the $\frac{1}{2}$ and believe that the $x$ values will be half as big which is a horizontal compression. However, the $x$ values need to be twice as big to counteract this factor of $\frac{1}{2}$.

## Vocabulary

***Horizontal*** comes from the word horizon and means flat.

***Vertical*** means up and down.

***Shift*** is a rigid transformation that means the shape keeps the exact shape.

***Stretch*** is a scaled transformation.

## Guided Practice

1. Describe the following transformation in words: $g(x) \rightarrow 2g(-x)$

2. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 3.
- Vertical shift down 4 units.
- Horizontal shift right 5 units.

3. Describe the transformation that would change $f(x)$ in the following ways:

- Horizontal stretch by a factor of 4 and a horizontal shift 3 units to the right.
- Vertical reflection across the $x$ axis and a shift down 2 units.

**Answers:**

1. Vertical stretch by a factor of 2 and a reflection across the $y$ axis.

2. $\frac{1}{3}h(x-5)-4$

3.$-f\left(\frac{1}{4}(x-3)\right)-2$ or $-f\left(\frac{1}{4}x-\frac{3}{4}\right)-2$

## Practice

Describe the following transformations in words.

1. $g(x) \rightarrow -g(-x)$

2. $f(x) \rightarrow -f(x+3)$

3. $h(x) \rightarrow h(x+1)-2$

4. $j(x) \rightarrow j(-x+3)$

5. $k(x) \rightarrow -k(2x)$

6. $f(x) \rightarrow 4f\left(\frac{1}{2}x+1\right)$

7. $g(x) \rightarrow -3g(x-2)-2$
8. $h(x) \rightarrow 5h(x+1)$
9. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical stretch by a factor of 2
- Vertical shift up 3 units.
- Horizontal shift right 2 units.

10. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical reflection across the $x$ axis.
- Vertical shift down 1 unit.
- Horizontal shift left 2 units.

11. Describe the transformation that would change $g(x)$ in the following ways:

- Vertical compression by a factor of 4.
- Reflection across the $y$ axis.

12. Describe the transformation that would change $j(x)$ in the following ways:

- Horizontal compression by a factor of 3.
- Vertical shift up 3 units.
- Horizontal shift right 2 units.

13. Describe the transformation that would change $k(x)$ in the following ways:

- Horizontal stretch by a factor of 4.
- Vertical shift up 3 units.
- Horizontal shift left 1 unit.

14. Describe the transformation that would change $h(x)$ in the following ways:

- Vertical compression by a factor of 2.
- Horizontal shift right 3 units.
- Reflection across the $y$ axis.

15. Describe the transformation that would change $f(x)$ in the following ways:

- Vertical stretch by a factor of 5.
- Reflection across the $x$ axis.

# 1.3 Point Notation and Function Notation

Here you will learn about the notation conventions involved with transformations.

When performing multiple transformations, it is very easy to make a small error. This is especially true when you try to do every step mentally. Point notation is a useful tool for concentrating your efforts on a single point and helps you to avoid making small mistakes.

What would $f(3x)+7$ look like in point notation and why is it useful?

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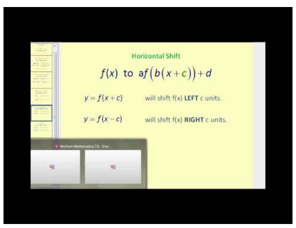


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http://www.youtube.com/watch?v=An29CALYjAA James Sousa: Function Transformations: A Summary

### Guidance

A transformation can be written in function notation and in point notation. Function notation is very common and practical because it allows you to graph any function using the same basic thought process it takes to graph a parabola in vertex form.

Another way to graph a function is to transform each point one at a time. This method works well when a table of $x, y$ values is available or easily identified from the graph.

Essentially, it takes each coordinate $(x, y)$ and assigns a new coordinate based on the transformation.

$(x, y) \rightarrow (new\ x, new\ y)$

The new $y$ coordinate is straightforward and is directly from what takes place outside $f(x)$ because $f(x)$ is just another way to write $y$. For example, $f(x) \rightarrow 2f(x)-1$ would have a new $y$ coordinate of $2y-1$.

The new $x$ coordinate is trickier. It comes from undoing the operations that affect $x$.

For example, $f(x) \rightarrow f(2x-1)$ would have a new $x$ coordinate of $\frac{x+1}{2}$.

**Example A**

Convert the following transformation into function notation and point notation. Then, apply the transformation to the three points in the table. Transformation: Horizontal shift right three units, vertical shift up 4 units.

TABLE 1.1:

| $x$ | $y$ |
|---|---|
| 1 | 3 |

**TABLE 1.1:** (continued)

| | |
|---|---|
| 2 | 5 |
| 8 | -11 |

**Solution:**

$f(x) \rightarrow f(x-3)+4$

$(x,y) \rightarrow (x+3,y+4)$

| $x$ | $y$ |
|---|---|
| 1 | 3 |
| 2 | 5 |
| 8 | -11 |

→

| $x$ | $y$ |
|---|---|
| 4 | 7 |
| 5 | 9 |
| 11 | -7 |

Note that the operations with the $x$ are different. The point notation is a straightforward approach to apply the transformation.

**Example B**

Convert the following function notation into point notation and apply it to the included table of points.

$f(x) \rightarrow \frac{1}{4}f(-x-3)-1$

**TABLE 1.2:**

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |

**Solution:** The $y$ component can be directly observed. For the $x$ component you need to undo the argument. $(x,y) \rightarrow \left(-x-3, \frac{1}{4}y-1\right)$

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 4 |
| 2 | 8 |

→

| $x$ | $y$ |
|---|---|
| -3 | -1 |
| -4 | 0 |
| -5 | 1 |

*Note: Point notation greatly reduces the mental visualization required to keep all the transformations straight at once.*

**Example C**

Convert the following point notation to words and to function notation and then apply the transformation to the included table of points.

$(x+3, y-1) \rightarrow (2x+6, -y)$

### TABLE 1.3:

| $x$ | $y$ |
|---|---|
| 10 | 8 |
| 12 | 7 |
| 14 | 6 |

**Solution:** This problem is different because it seems like there is a transformation happening to the original left point. This is an added layer of challenge because the transformation of interest is just the difference between the two points. Notice that the $x$ coordinate has simply doubled and the $y$ coordinate has gotten bigger by one and turned negative. This problem can be rewritten as:

$(x, y) \rightarrow (2x, -(y+1)) = (2x, -y-1)$

$f(x) \rightarrow -f\left(\frac{x}{2}\right) - 1$

| $x$ | $y$ |
|---|---|
| 10 | 8 |
| 12 | 7 |
| 14 | 6 |

→

| $x$ | $y$ |
|---|---|
| 5 | -9 |
| 6 | -8 |
| 7 | -7 |

**Concept Problem Revisited**

The function $f(3x)+7$ would be written in point notation as $(x, y) \rightarrow \left(\frac{x}{3}, y+7\right)$. This is useful because it becomes obvious that the $x$ values are all divided by three and the $y$ values all increase by 7.

### Vocabulary

***Notation*** is a mathematical convention that helps others read your work. It is supposed to be designed to help aid your thinking, but when misunderstood can cause great confusion.

### Guided Practice

1. Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

$f(x) \rightarrow -2f(x-1)+4$

2. Convert the following function in point notation to words and then function notation.

$(x, y) \rightarrow (3x+1, -y+7)$

3. Convert the following function in function notation to words and then point notation.

$f(x) \to -\frac{1}{2}f(x-1)+3$

**Answers:**

1. $f(x) \to -2f(x-1)+4$

Vertical reflection across the $x$ axis. Vertical stretch by a factor of 2. Vertical shift 4 units. Horizontal shift right one unit.

$(x,y) \to (x+1,-2y+4)$

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 6 |
| 2 | 7 |

→

| $x$ | $y$ |
|---|---|
| 1 | -6 |
| 2 | -8 |
| 3 | -10 |

2. $(x,y) \to (3x+1,-y+7)$

Horizontal stretch by a factor of 3 and then a horizontal shift right one unit.

Vertical reflection over the $x$ axis and then a vertical shift 7 units up.

$f(x) \to -f\left(\frac{1}{3}x-\frac{1}{3}\right)+7$

3. $f(x) \to -\frac{1}{2}f(x-1)+3$

Vertical reflection across the $x$ axis, stretch by a factor of $\frac{1}{2}$ and shift up 3. Horizontal shift right 1 unit.

$(x,y) \to \left(x+1,-\frac{1}{2}y+3\right)$

## Practice

Convert the following function notation into words and then point notation. Finally, apply the transformation to three example points.

**TABLE 1.4:**

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 6 |
| 2 | 7 |

1. $f(x) \to -\frac{1}{2}f(x+1)$
2. $g(x) \to 2g(3x)+2$
3. $h(x) \to -h(x-4)-3$
4. $j(x) \to 3j(2x-4)+1$
5. $k(x) \to -k(x-3)$

Convert the following functions in point notation to function notation.

6. $(x,y) \rightarrow \left(\frac{1}{2}x+3, y-4\right)$
7. $(x,y) \rightarrow (2x+4, -y+1)$
8. $(x,y) \rightarrow (4x, 3y-5)$
9. $(2x,y) \rightarrow (4x, -y+1)$
10. $(x+1, y-2) \rightarrow (3x+3, -y+3)$

Convert the following functions in function notation to point notation.

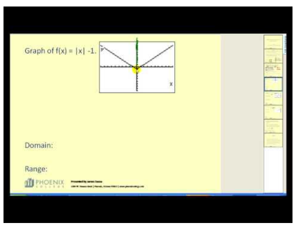
11. $f(x) \rightarrow 3f(x-2)+1$
12. $g(x) \rightarrow -4g(x-1)+3$
13. $h(x) \rightarrow \frac{1}{2}h(2x+2)-5$
14. $j(x) \rightarrow 5j\left(\frac{1}{2}x-2\right)-1$
15. $k(x) \rightarrow \frac{1}{4}k(2x-4)$

# 1.4 Domain and Range

Here you will refine your understanding of domain and range from Algebra 2 by exploring tables, basic functions and irregular graphs.

Analyze means to examine methodically and in detail. One way to analyze functions is by looking at possible inputs (domain) and possible outputs (range). Which of the basic functions have limited domains and why?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=FtJRstFMdhA James Sousa: Determining Domain and Range

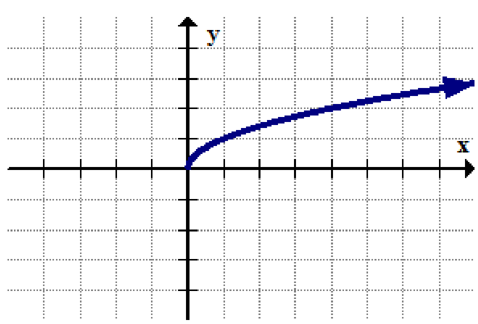
### Guidance

**Domain** is the possible inputs to a function. Many functions allow any kind of numbers to be inputted. This includes numbers that are positive, negative, zero, fractions or decimals. The squaring function $y = x^2$ is an example that has a domain of all possible real numbers. Three functions have very specific restrictions:

1) The square root function: $y = \sqrt{x}$

Domain restriction: $x \geq 0$

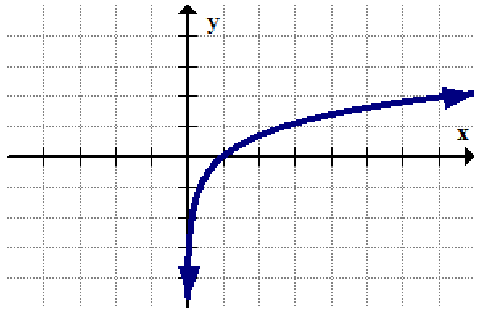
This is because the square root of a negative number is not a real number. This restriction can be observed in the graph because the curve ends at the point (0, 0) and is not defined anywhere where $x$ is negative.




2) The logarithmic function: $y = \ln x$

Domain restriction: $x > 0$

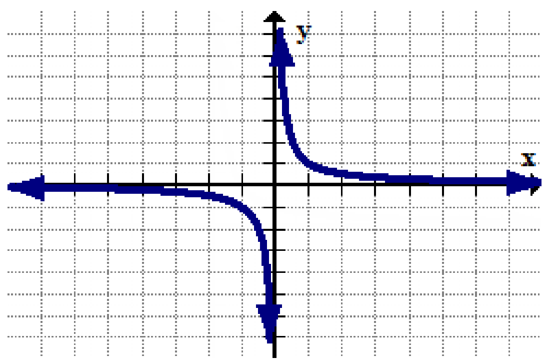
The log function is only defined on numbers that are strictly bigger than zero. This is because the logarithmic function is a different way of writing exponents. One property of exponents is that any positive number raised to any power will never produce a negative number or zero. The restriction can be observed in the graph by the way the log function approaches the vertical line $x = 0$ and shoots down to infinity.




3) The reciprocal function: $y = \frac{1}{x}$

Domain Restriction: $x \neq 0$

The reciprocal function is restricted because you cannot divide numbers by zero. Any $x$ values that make the denominator of a function zero are outside of the domain. This restriction can be observed in the graph by the way the reciprocal function never touches the vertical line $x = 0$.




**Range** is the possible outputs of a function. Just about any function can produce any output through the use of transformations and so determining the range of a function is significantly less procedural than determining the domain. Use what you know about the shape of each function and their equations to decide which $y$ values are possible to produce and which $y$ values are impossible to produce.

**Example A**

Domain and range are described in interval notation. Convert the following descriptions of numbers into interval notation.

a. All numbers.
b. All negative numbers not including 0.
c. All positive numbers including 0.
d. Every number between 1 and 4 including 1 and 4.
e. Every number between 5 and 6 not including 5 or 6.
f. The numbers 1 through 2 including 1 but not including 2 and the numbers 10 through 25 including both 10 and 25.

**Solution:** Parentheses, ( ), mean non-inclusive. Brackets, [], mean inclusive.

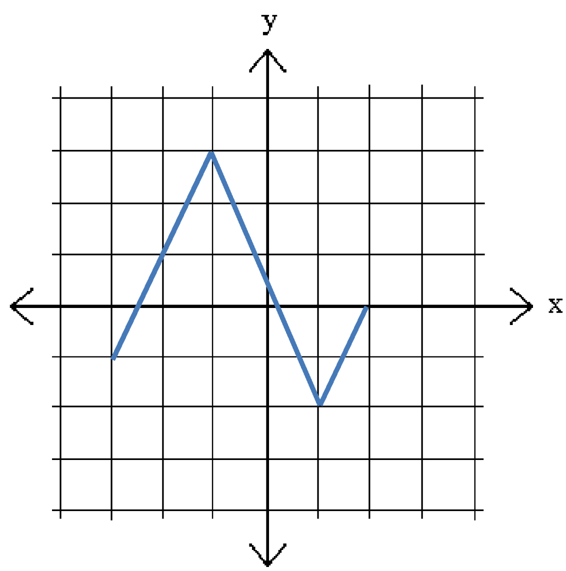
a. $(-\infty, \infty)$ Note: Parentheses are always used with infinity.

b. $(-\infty,0)$
c. $[0,\infty)$
d. $[1,4]$
e. $(5,6)$
f. $[1,2)\cup[10,25]$

Note: The $\cup$ symbol means Union and refers to the fact that if some number $x$ is in this union, then it is either in the first group or it is in the second group. This symbol is associated with the OR statement. While it is true that the Union symbol seems to bring one group and another group together, the symbol for AND is $\cap$ which means intersection. Intersection is different from union because intersection means all numbers that are in both the first group and second group at the same time.

**Example B**

Identify the domain and range for the following function.




**Solution:**

Domain: $x \in [-3,2]$

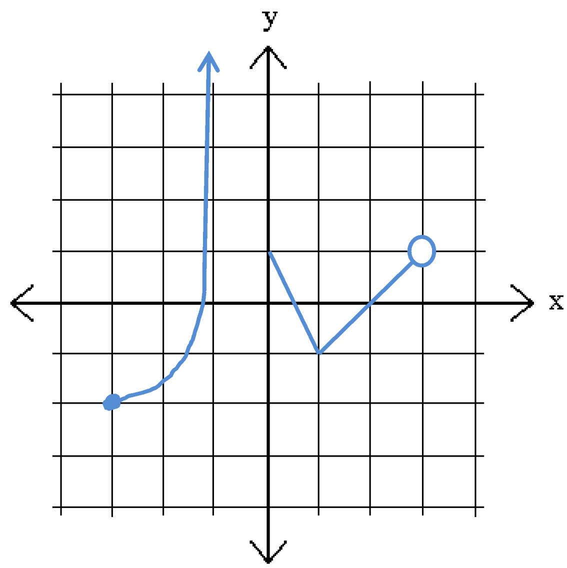
Range: $y \in [-2,3]$

Note that the $\in$ symbol means "is an element of" and means that the $x$ or the $y$ is in that interval. Also note that the numbers in the interval are always written in increasing order. [3, -2] is considered improper.

Lastly, note that even though the [-3, 2] may look similar to the ordered pair that represents the point where $x = -3$ and $y = 2$, this is not the case. Both the -3 and the 2 are $x$ values. This misconception is why you should always write $x \in$ because it reminds you of this fact. Many people get very confused when they see something like $x \in (-2,1)$ because they see the parenthesis and immediately see a point when they should see an interval on the $x$ axis.

**Example C**

Identify the domain and range of the following function.




**Solution:**

Domain: $x \in [-3,-1) \cup [0,3)$

Range: $y \in [-2,\infty)$

Note that the function seems to approach the vertical line $x = -1$ without actually reaching it. Also note the empty hole at the point (3, 1) which is why the domain excludes the $x$ value of 3.

**Concept Problem Revisited**

The three functions that have limited domains are the square root function, the log function and the reciprocal function. The square root function has a restricted domain because you cannot take square roots of negative numbers and produce real numbers. The log function is restricted because the log function is not defined to operate on non-positive numbers. The reciprocal function is restricted because numbers that are divided by zero are not defined.

### Vocabulary

***Domain*** is the possible $x$ values or inputs of a function.

***Range*** is the possible $y$ values or outputs of a function.

***Interval notation*** is a tool for describing groups of numbers in the domain and range. Intervals are either open or closed or both.

***Open intervals*** use parentheses, (), and refer to intervals that do not include the end points. They are always used with infinity.

***Closed intervals*** use brackets, [], and refer to intervals that do include the end points.

***Curly brackets***, { }, are used when the domain or range are distinct numbers and not an interval of values.

$\in$ is a symbol that stands for "***is an element of***" and tells you what kind of values an interval describes.

$\cup$ is a symbol that stands for ***Union*** and is used to connect two groups together. It is associated with the logical term OR.

### Guided Practice

1. Identify the domain and range of the following function written in a table:

**TABLE 1.5:**

| $x$ | $y$ |
|---|---|
| 0 | 5 |
| 1 | 6 |
| 2 | 7 |
| $\frac{1}{2}$ | 6 |
| $\pi$ | $\frac{\pi}{2}$ |

2. Identify the domain of the following three transformed functions.

a. $y = 10\sqrt{2-x} - 3$

b. $y = \frac{3x}{x^2+7x+12}$

c. $y = -4\ln(3x-9) + 11$

3. What is the domain and range of the sine wave?

**Answers:**

1. The specific equation of the function may be hidden, but from the table you can determine the domain and range directly from the $x$ and $y$ values. It may be tempting to guess that other values could potentially work in the table, especially if the pattern is obvious, but this is not the kind of question that asks what the function could be. Instead this question just asks what is the stated domain and range.

Domain: $x \in \{0, 1, 2, \frac{1}{2}, \pi\}$

Range: $y \in \{5, 6, 7, \frac{\pi}{2}\}$

Note that the two 6's that appear in the table do not need to be written twice in the range.

2. Each of the functions requires knowledge of one of the three domain restrictions.

a. $y = 10\sqrt{2-x} - 3$

The argument of the function must be greater than or equal to 0.

$$\begin{aligned} 2 - x &\geq 0 \\ -x &\geq -2 \\ x &\leq 2 \end{aligned}$$

Domain: $x \in (-\infty, 2]$

b. $y = \frac{3x}{x^2+7x+12}$

The denominator cannot be equal to 0. First find what values of $x$ would make it equal to zero and then you can exclude those values.

$$\begin{aligned} x^2 + 7x + 12 &= 0 \\ (x+4)(x+3) &= 0 \\ x &= -4, -3 \end{aligned}$$

Domain: $x \in (-\infty,-4) \cup (-4,-3) \cup (-3,\infty)$

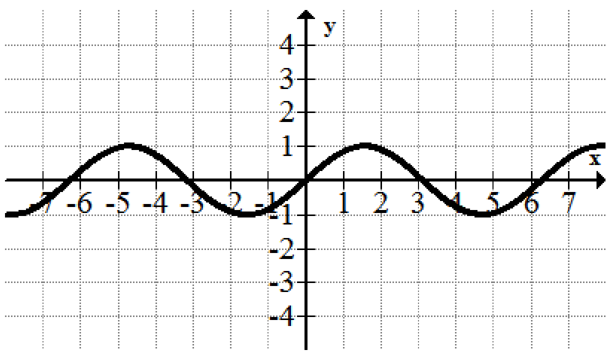
c. $y = -4\ln(3x-9)+11$

The argument must be strictly greater than 0.

$$\begin{aligned} 3x-9 &> 0 \\ 3x &> 9 \\ x &> 3 \end{aligned}$$

Domain: $x \in (3,\infty)$

3.




Domain: $x \in (-\infty,\infty)$

Range: $y \in [-1,1]$

## Practice

Convert the following descriptions of numbers into interval notation.

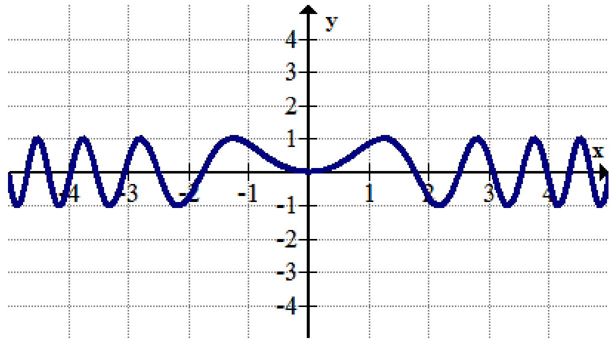
1. All positive numbers not including 0.
2. Every number between -1 and 1 including -1 but not 1.
3. Every number between 1 and 5 not including 2 or 3, but including 1 and 5.
4. Every number greater than 5, not including 5.
5. All real numbers except 1.

Translate the following inequalities into interval notation.

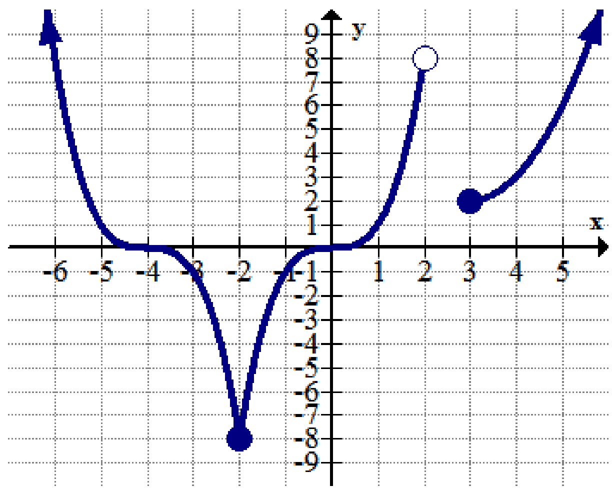
6. $-4 < x \leq 5$
7. $x > 0$
8. $-\infty < x \leq 4$ *or* $5 < x < \infty$

Find the domain and range of each graph below.

9.




10.




Given the stated domain and range, draw a possible graph.

11. Domain: $x \in [0, \infty)$ Range: $y \in (-2, 2]$

12. Domain: $x \in [-4, 1) \cup (1, \infty)$ Range: $y \in (-\infty, \infty)$

Given the table, find the domain and range.

13.

### TABLE 1.6:

| $x$ | $y$ |
|---|---|
| -2 | 7 |
| 3 | 7 |
| 2 | 1 |
| $\frac{3}{4}$ | 5 |
| $\frac{\pi}{2}$ | $\pi$ |

Find the domain for the following functions.

14. $y = -3\sqrt{x+4} - 1$

15. $y = \frac{7}{x+6} - 1$

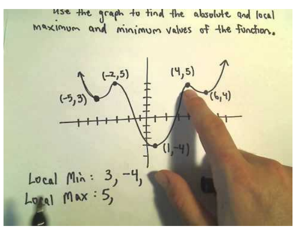
16. $y = 5\ln(x^2 - 1) + 4$

# 1.5 Maximums and Minimums

Here you will learn to identify the maximums and minimums in various graphs and be able to differentiate between global and relative extreme values.

When riding a roller coaster there is always one point that is the absolute highest off the ground. There are usually many other places that reach fairly high, just not as high as the first. How do you identify and distinguish between these different peaks in a precise way?

### Watch This

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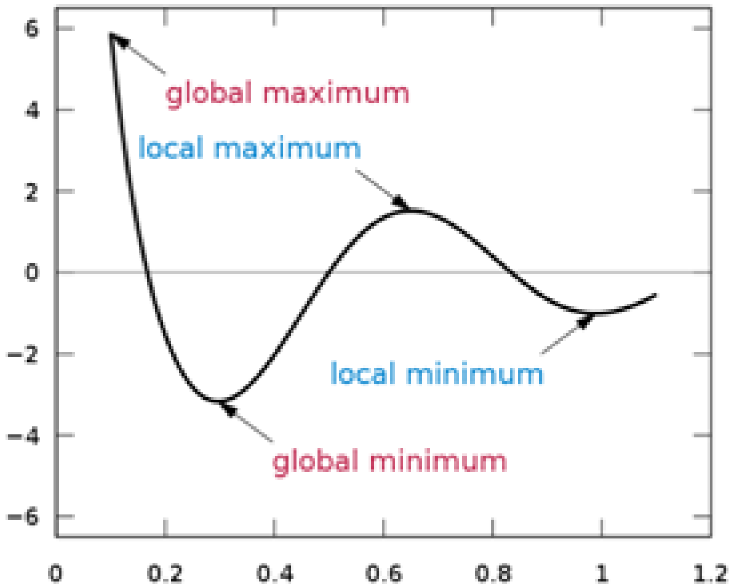
http://www.youtube.com/watch?v=votVWz-wKeI

### Guidance

A global maximum refers to the point with the largest $y$ value possible on a function. A global minimum refers to the point with the smallest $y$ value possible. Together these two values are referred to as global extrema. There can only be one global maximum and only one global minimum. Global refers to entire space where the function is defined. Global extrema are also called absolute extrema.

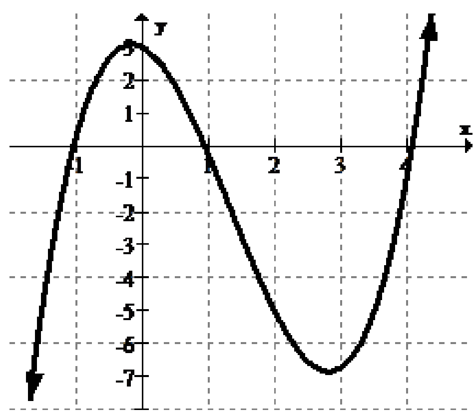
In addition to global maximums and global minimums, there are also local extrema or relative maximums and relative minimums. The word relative is used because in relation to some neighborhood, these values stand out as being the highest or the lowest.




Calculus uses advanced analytic tools to compute extreme values, but for the purposes of PreCalculus it is sufficient to be able to identify and categorize extreme values graphically or through the use of technology. For example, the TI-84 has a maximum finder when you select <2$^{nd}$>then <trace>.

**Example A**

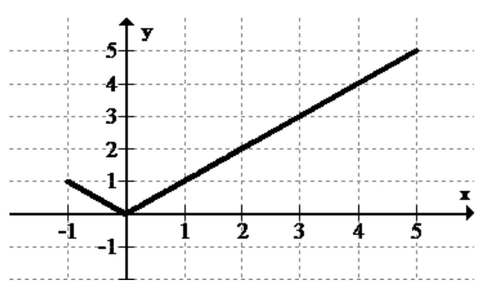
Identify and categorize all extrema:




**Solution:** Since the function appears from the arrows to increase and decrease beyond the display, there are no global extrema. There is a local maximum at approximately (0, 3) and a local minimum at approximately (2.8, -7).

**Example B**

Identify and categorize all extrema:

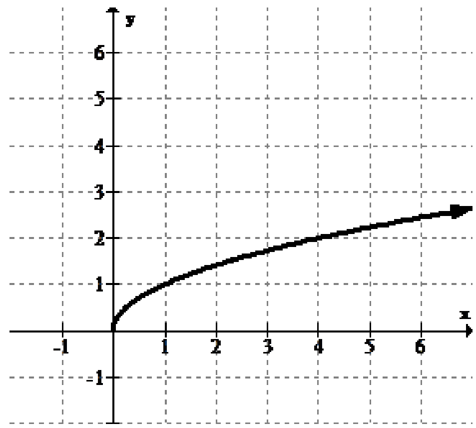


**Solution:** Since the function seems to abruptly end at the end points and does not go beyond the display, the endpoints are important.

There is a global minimum at (0, 0). There is a local maximum at (-1, 1) and a global maximum at (5, 5).

**Example C**

Identify and categorize all extrema.




**Solution:** Since this function appears to increase to the right as indicated by the arrow there is no global maximum. There are not any other high points either, so there are no local maximums. There is only the end point at (0, 0) which is a global minimum.

**Concept Problem Revisited**

Maximums and minimums should be intuitive because they simply identify the highest points and the lowest points, or the peaks and the valleys, in a graph. There is a formal distinction about whether a maximum is the highest on some local open interval (does not matter how small), or whether it is simply the highest overall.

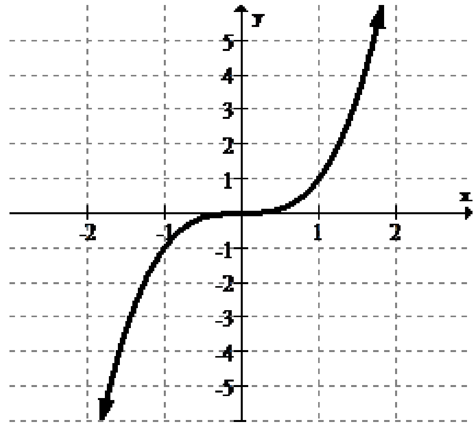
## Vocabulary

***Global extrema*** and ***absolute extrema*** are synonyms that refer to the points with the $y$ values that are either the highest or the lowest of the entire function.

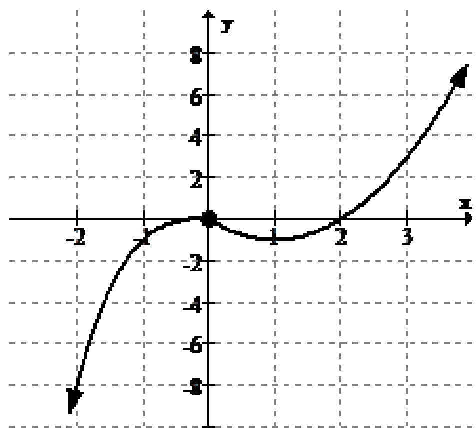
***Local extrema*** and ***relative extrema*** are synonyms that refer to the points with the $y$ values that are the highest or lowest of a local neighborhood of the function.

## Guided Practice

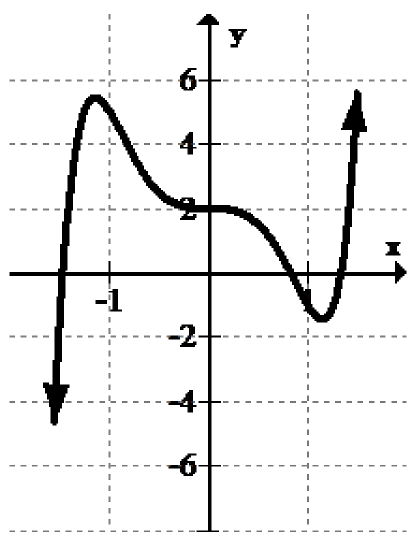
1. Identify and categorize all extrema.




2. Identify and categorize the extrema.




3. Identify and categorize the extrema.




**Answers:**

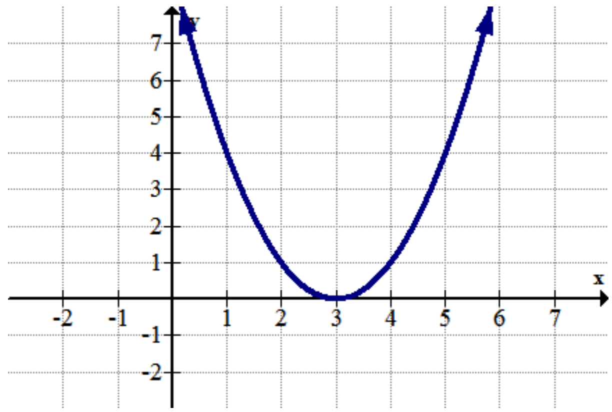
1. There are no global or local maximums or minimums. The function flattens, but does not actually reach a peak or a valley.

2. There are no global extrema. There appears to be a local maximum at (0, 0) and a local minimum at (1, -1).

3. There are no global extrema. There appears to be a local maximum at (-1.2, 5.3) and a local minimum at (1.2, -1.8). These values are approximated. If a function was given, you would need to graph the function on a calculator and use the maximum and minimum features to identify more exact points.

### Practice

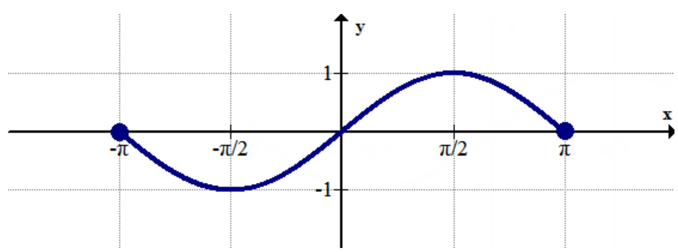
Use the graph below for 1-2.




1. Identify any global extrema.

2. Identify any local extrema.

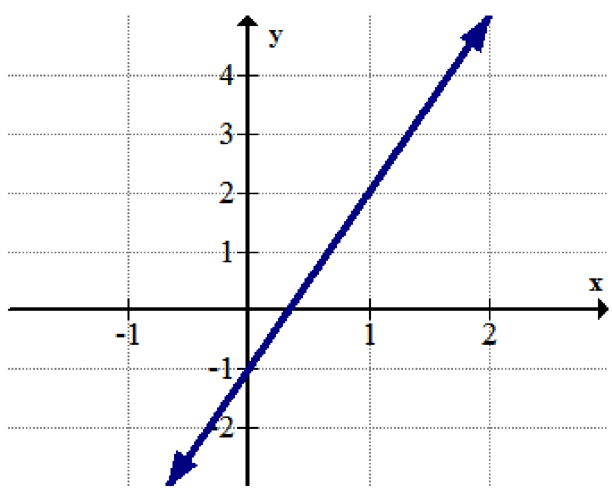
Use the graph below for 3-4.




3. Identify any global extrema.

4. Identify any local extrema.

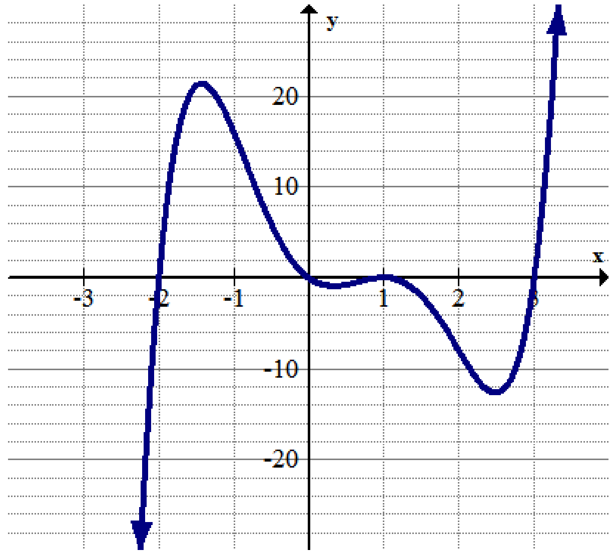
Use the graph below for 5-6.



5. Identify any global extrema.

6. Identify any local extrema.

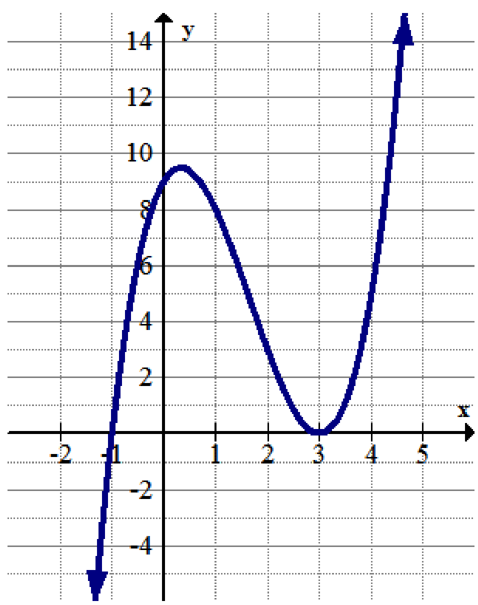
Use the graph below for 7-8.




7. Identify any global extrema.

8. Identify any local extrema.

Use the graph below for 9-10.




9. Identify any global extrema.

10. Identify any local extrema.

11. Explain the difference between a global maximum and a local maximum.

12. Draw an example of a graph with a global minimum and a local maximum, but no global maximum.

13. Draw an example of a graph with local maximums and minimums, but no global extrema.

14. Use your graphing calculator to identify and categorize the extrema of:
    $f(x) = \frac{1}{2}x^4 + 2x^3 - 6.5x^2 - 20x + 24.$
15. Use your graphing calculator to identify and categorize the extrema of:
    $g(x) = -x^4 + 2x^3 + 4x^2 - 2x - 3.$

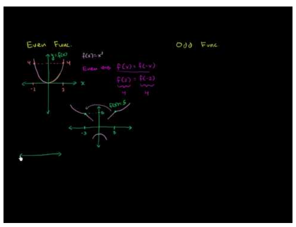
# 1.6 Symmetry

Here you will review rotation and reflection symmetry as well as explore how algebra accomplishes both.

Some functions, like the sine function, the absolute value function and the squaring function, have reflection symmetry across the line$x = 0$. Other functions like the cubing function and the reciprocal function have rotational symmetry about the origin.

Why is the first group categorized as even functions while the second group is categorized as odd functions?

### Watch This



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http://www.youtube.com/watch?v=8VgmBe3ulb8 Khan Academy: Recognizing Odd and Even Functions

### Guidance

Functions symmetrical across the line $x = 0$ (the $y$ axis) are called even. Even functions have the property that when a negative value is substituted for $x$, it produces the same value as when the positive value is substituted for the $x$.

$f(-x) = f(x)$

Functions that have rotational symmetry about the origin are called odd functions. When a negative $x$ value is substituted into the function, it produces a negative version of the function evaluated at a positive value.

$f(-x) = -f(x)$

This property becomes increasingly important in problems and proofs of Calculus and beyond, but for now it is sufficient to identify functions that are even, odd or neither and show why.

**Example A**

Show that $f(x) = 3x^4 - 5x^2 + 1$ is even.

**Solution:**

$$\begin{aligned} f(-x) &= 3(-x)^4 - 5(-x)^2 + 1 \\ &= 3x^4 - 5x^2 + 1 \\ &= f(x) \end{aligned}$$

The property that both positive and negative numbers raised to an even power are always positive is the reason why the term even is used. It does not matter that the coefficients are even or odd, just the exponents.

**Example B**

Show that $f(x) = 4x^3 - x$ is odd.

**Solution:**

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 + x \\ &= -(4x^3 + x) \\ &= -f(x) \end{aligned}$$

Just like even functions are named, odd functions are named because negative signs don't disappear and can always be factored out of odd functions.

**Example C**

Identify whether the function is even, odd or neither and explain why.

$f(x) = 4x^3 - |x|$

**Solution:**

$$\begin{aligned} f(-x) &= 4(-x)^3 - x \\ &= -4x^3 - x \end{aligned}$$

This does not seem to match either $f(x) = 4x^3 - |x|$ or $-f(x) = -4x^3 + |x|$. Therefore, this function is neither even nor odd.

*Note that this function is a difference of an odd function and an even function. This should be a clue that the resulting function is neither even nor odd.*

**Concept Problem Revisited**

Even and odd functions describe different types of symmetry, but both derive their name from the properties of exponents. A negative number raised to an even number will always be positive. A negative number raised to an odd number will always be negative.

## Vocabulary

An ***even function*** means $f(-x) = f(x)$. Even functions have **reflection symmetry** across the line $x = 0$.

An ***odd function*** means $f(-x) = -f(x)$. Odd functions have ***rotation symmetry*** about the origin.

## Guided Practice

1. Which of the basic functions are even, which are odd and which are neither?

2. Suppose $h(x)$ is an even function and $g(x)$ is an odd function. $f(x) = h(x) + g(x)$. Is $f(x)$ even or odd?

3. Determine whether the following function is even, odd, or neither.

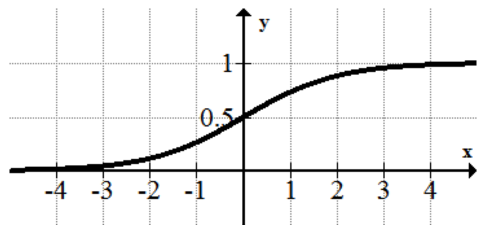
$f(x) = x(x^2 - 1)(x^4 + 1)$

**Answers:**

1. Even Functions: The squaring function, the absolute value function.

Odd Functions: The identity function, the cubing function, the reciprocal function, the sine function.

Neither: The square root function, the exponential function and the log function. The logistic function is also neither because it is rotationally symmetric about the point $\left(0, \frac{1}{2}\right)$ as opposed to the origin.




2. If $h(x)$ is even then $h(-x) = h(x)$. If $g(x)$ is odd then $g(-x) = -g(x)$.

Therefore: $f(-x) = h(-x) + g(-x) = h(x) - g(x)$

This does not match $f(x) = h(x) + g(x)$ nor does it match $-f(x) = -h(x) - g(x)$.

This is a proof that shows the sum of an even function and an odd function will never itself be even or odd.

3.

$$\begin{aligned} f(x) &= x(x^2-1)(x^4+1) \\ f(-x) &= (-x)((-x)^2-1)((-x)^4+1) \\ &= -x(x^2-1)(x^4+1) \\ &= -f(x) \end{aligned}$$

The function is odd.

## Practice

Determine whether the following functions are even, odd, or neither.

1. $f(x) = -4x^2 + 1$
2. $g(x) = 5x^3 - 3x$
3. $h(x) = 2x^2 - x$
4. $j(x) = (x-4)(x-3)^3$
5. $k(x) = x(x^2-1)^2$
6. $f(x) = 2x^3 - 5x^2 - 2x + 1$
7. $g(x) = 2x^2 - 4x + 2$
8. $h(x) = -5x^4 + x^2 + 2$
9. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) - g(x)$ is neither even nor odd.
10. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = \frac{h(x)}{g(x)}$ is odd.
11. Suppose $h(x)$ is even and $g(x)$ is odd. Show that $f(x) = h(x) \cdot g(x)$ is odd.

12. Is the sum of two even functions always an even function? Explain.
13. Is the sum of two odd functions always an odd function? Explain.
14. Why are some functions neither even nor odd?
15. If you know that a function is even or odd, what does that tell you about the symmetry of the function?

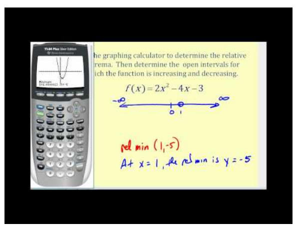
# 1.7 Increasing and Decreasing

Here you will apply interval notation to identify when functions are increasing and decreasing.

It is important to be able to distinguish between when functions are increasing and when they are decreasing. In business this could mean the difference between making money and losing money. In physics it could mean the difference between speeding up and slowing down.

How do you decide when a function is increasing or decreasing?

### Watch This



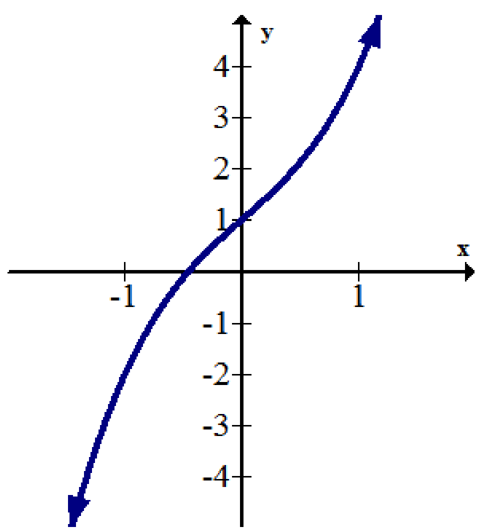
MEDIA

Click image to the left for more content.

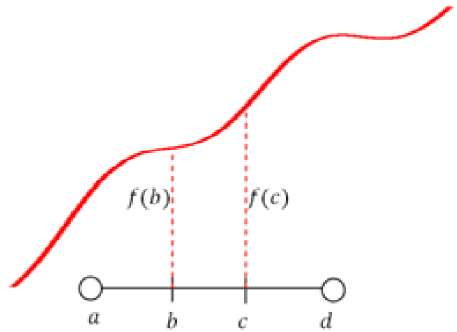
http://www.youtube.com/watch?v=78b4HOMVcKM James Sousa: Determine Where a Function is Increasing or Decreasing

### Guidance

**Increasing** means places on the graph where the slope is positive.

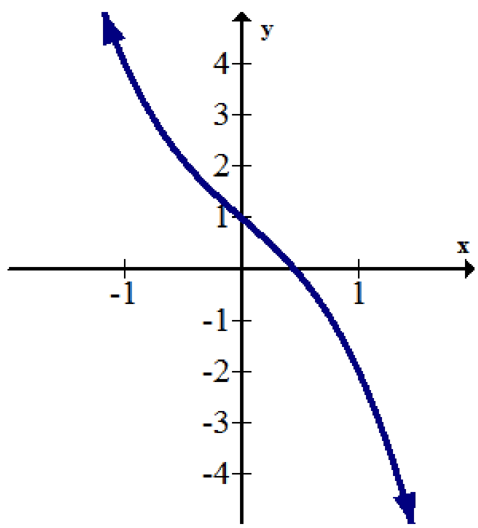



The formal definition of an increasing interval is: an open interval on the $x$ axis of $(a,d)$ where every $b,c \in (a,d)$ with $b < c$ has $f(b) \leq f(c)$.




A interval is said to be ***strictly*** increasing if $f(b) < f(c)$ is substituted into the definition.

**Decreasing** means places on the graph where the slope is negative. The formal definition of decreasing and strictly decreasing are identical to the definition of increasing with the inequality sign reversed.




A function is called **monotonic** if the function only goes in one direction and never switches between increasing and decreasing.

Identifying analytically where functions are increasing and decreasing often requires Calculus. For PreCalculus, it will be sufficient to be able to identify intervals graphically and through your knowledge of what the parent functions look like.

**Example A**

Identify which of the basic functions are monotonically increasing.

**Solution:** Of the basic functions, the monotonically increasing functions are:

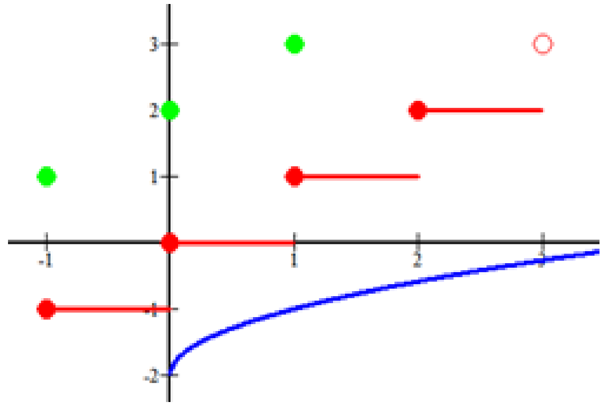
$f(x) = x, f(x) = x^3, f(x) = \sqrt{x}, f(x) = e^x, f(x) = \ln x, f(x) = \frac{1}{1+e^{-x}}$

The only basic functions that are not monotonically increasing are:

$f(x) = x^2, f(x) = |x|, f(x) = \frac{1}{x}, f(x) = \sin x$

**Example B**

Identify whether the green, red or blue function is monotonically increasing and explain why.



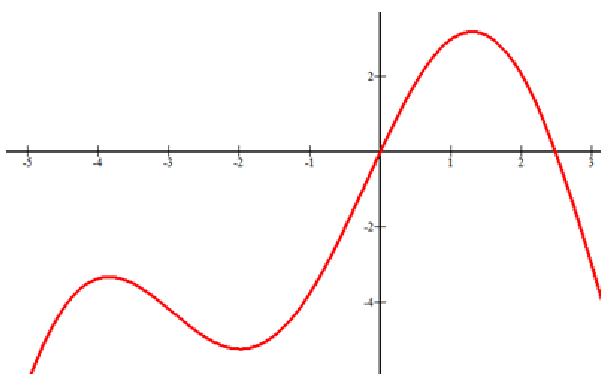
**Solution:** The green function seems to be discrete values along the line $y = x+2$. While the discrete values clearly increase and the line would be monotonically increasing, these values are missing a key part of what it means to be monotonic. The green function does not have a positive slope and is therefore not monotonic.

The red function also seems to be increasing, but the slope at every $x$ value is zero. In Calculus the definition of monotonic will be refined to handle special cases like this. For now, this function is not monotonic.

The blue function seems to be $y = \sqrt{x}-2$, is increasing everywhere that is visible, and probably extends to the right. This function is monotonic where the function is defined for $x \in (0,\infty)$.

**Example C**

Estimate the intervals where the function is increasing and decreasing.



**Solution:**

Increasing: $x \in (-\infty,-4) \cup (-2,1.5)$

Decreasing: $x \in (-4,-2) \cup (1.5,\infty)$

Note that open intervals are used because at $x = -4,-2,1.5$ the slope of the function is zero. This is where the slope transitions from being positive to negative. The reason why open parentheses are used is because the function is not actually increasing or decreasing at those specific points.

**Concept Problem Revisited**

Increasing is where the function has a positive slope and decreasing is where the function has a negative slope. A common misconception is to look at the squaring function and see two curves that symmetrically increase away from zero. Instead, you should always read functions from left to right and draw slope lines and decide if they are positive or negative.

### Vocabulary

***Increasing*** over an interval means to have a positive slope over that interval.

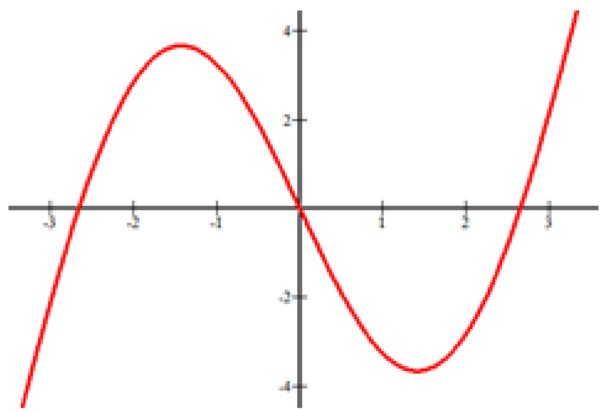
**Decreasing** over an interval means to have a negative slope over that interval.

***Monotonic*** means that the function doesn't switch between increasing and decreasing at any point.

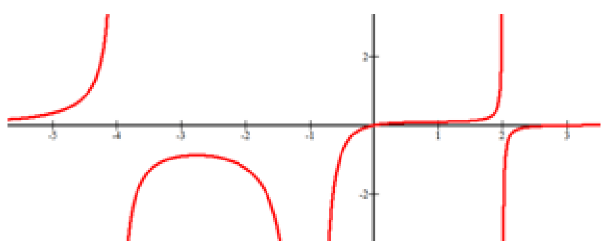
***Strictly*** is an adjective that alters increasing and decreasing to exclude any flatness.

### Guided Practice

1. Estimate where the following function is increasing and decreasing.



2. Estimate where the following function is increasing and decreasing.



3. A continuous function has a global maximum at the point (3, 2), a global minimum at (5, -12) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

**Answers:**

1. Increasing: $x \in (-\infty, -1.5) \cup (1.5, \infty)$. Decreasing: $x \in (-1.5, 1.5)$

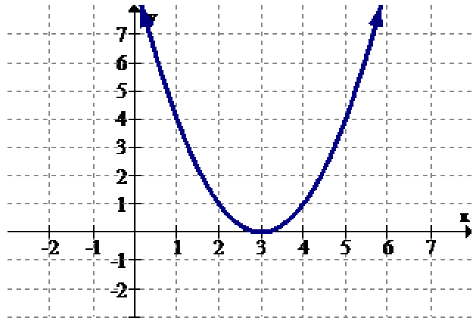
2. Increasing $x \in (-\infty, -4) \cup (-4, -2.7) \cup (-1, 2) \cup (2, \infty)$. Decreasing $x \in (-2.7, -1)$

3. Increasing $x \in (-\infty, 3) \cup (5, \infty)$. Decreasing $x \in (3, 5)$

*Notice that the y coordinates are not used in the intervals. A common mistake is to want to use the y coordinates.*

### Practice

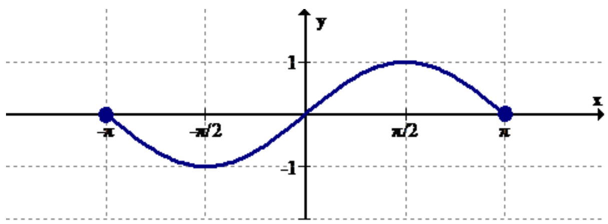
Use the graph below for 1-2.




1. Identify the intervals (if any) where the function is increasing.

2. Identify the intervals (if any) where the function is decreasing.

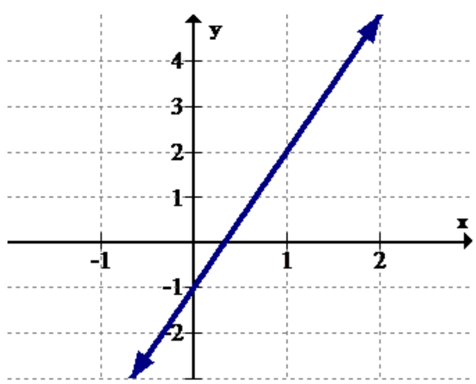
Use the graph below for 3-4.




3. Identify the intervals (if any) where the function is increasing.

4. Identify the intervals (if any) where the function is decreasing.

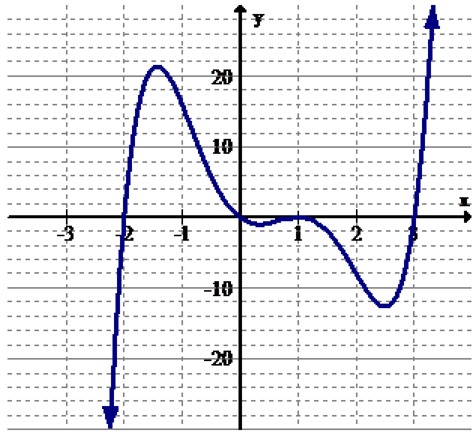
Use the graph below for 5-6.




5. Identify the intervals (if any) where the function is increasing.

6. Identify the intervals (if any) where the function is decreasing.

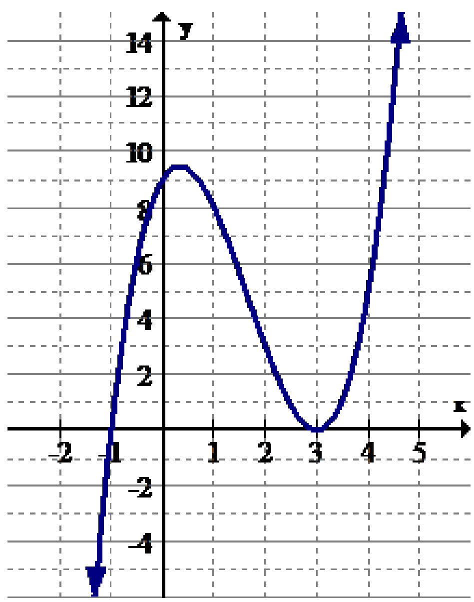
Use the graph below for 7-8.




7. Identify the intervals (if any) where the function is increasing.

8. Identify the intervals (if any) where the function is decreasing.

Use the graph below for 9-10.




9. Identify the intervals (if any) where the function is increasing.

10. Identify the intervals (if any) where the function is decreasing.

11. Give an example of a monotonically increasing function.

12. Give an example of a monotonically decreasing function.

13. A continuous function has a global maximum at the point (1, 4), a global minimum at (3, -6) and has no relative extrema or other places with a slope of zero. What are the increasing and decreasing intervals for this function?

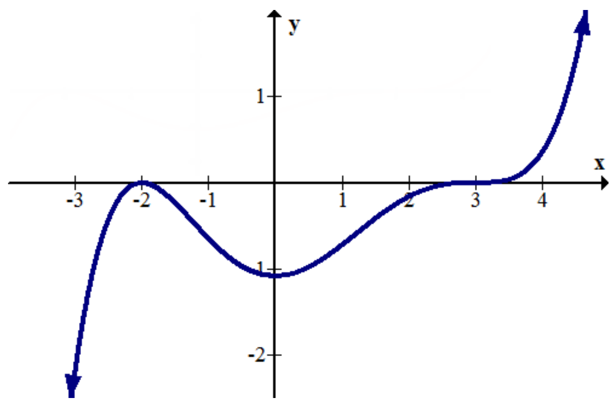
14. A continuous function has a global maximum at the point (1, 1) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

15. A continuous function has a global minimum at the point (5, -15) and has no other extrema or places with a slope of zero. What are the increasing and decreasing intervals for this function?

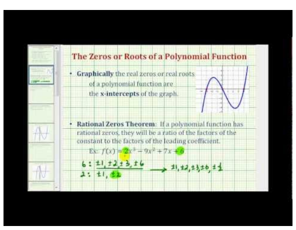
# 1.8 Zeroes and Intercepts of Functions

Here you will learn about $x$ and $y$ intercepts. You will learn to approximate them graphically and solve for them exactly using algebra.

An intercept in mathematics is where a function crosses the $x$or $y$axis. What are the intercepts of this function?




### Watch This

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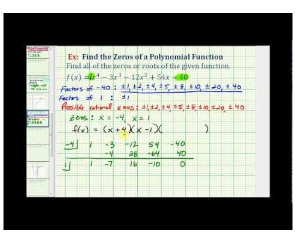


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### Guidance

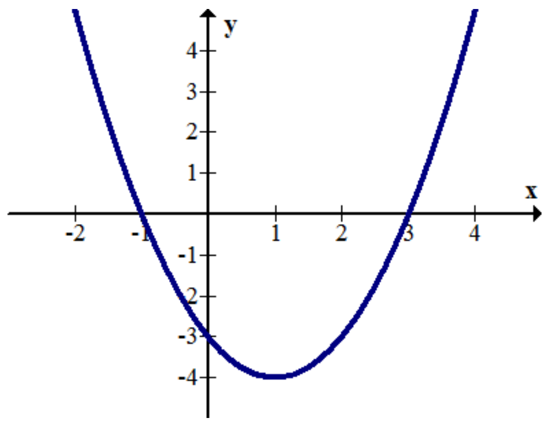
The first type of intercept you may have learned is the $y$-intercept when you learned the slope intercept form of a line: $y = mx + b$. A $y$-intercept is the unique point where a function crosses the $y$ axis. It can be found algebraically by setting $x = 0$ and solving for $y$.

$x$-intercepts are where functions cross the $x$ axis and where the height of the function is zero. They are also called roots, solutions and zeroes of a function. They are found algebraically by setting $y = 0$ and solving for $x$.

**Example A**

What are the zeroes and $y$-intercepts of the parabola $y = x^2 - 2x - 3$?

**Solution using Graph:**




The zeroes are at (-1, 0) and (3, 0). The $y$-intercept is at (0, -3).

**Solution using Algebra:**

Substitute 0 for $y$ to find zeroes.

$0 = x^2 - 2x - 3 = (x-3)(x+1)$

$y = 0, x = 3, -1$

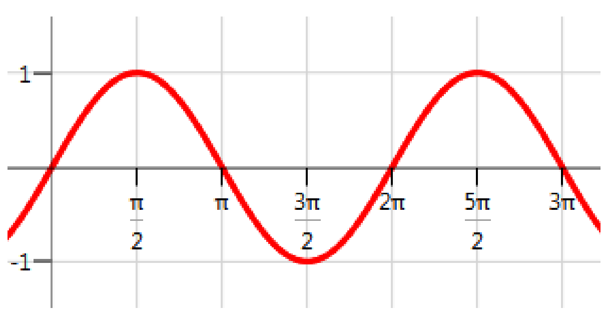
Substitute 0 for $x$ to find the $y$-intercept.

$y = (0)^2 - 2(0) - 3 = -3$

$x = 0, y = -3$

**Example B**

Identify the zeroes and $y$-intercepts for the sine function.




**Solution:** The $y$-intercept is (0, 0). There are four zeroes visible on this portion of the graph. One thing you know about the sine graph is that it is periodic and repeats forever in both directions. In order to capture every $x$-intercept, you must identify a pattern instead of trying to write out every single one.

The visible $x$-intercepts are $0, \pi, 2\pi, 3\pi$. The pattern is that there is an $x$-intercept every multiple of $\pi$ including negative multiples. In order to describe all of these values you should write:

The $x$-intercepts are $\pm n\pi$ where $n$ is an integer $\{0, \pm 1, \pm 2, \dots\}$.

**Example C**

Identify the intercepts and zeroes of the function: $f(x) = \frac{1}{100}(x-3)^3(x+2)^2$.

**Solution:**

To find the $y$-intercept, substitute 0 for $x$:

$y = \frac{1}{100}(0-3)^3(0+2)^2 = \frac{1}{100}(-27)(4) = -\frac{108}{100} = -1.08$

To find the $x$-intercepts, substitute 0 for $y$:

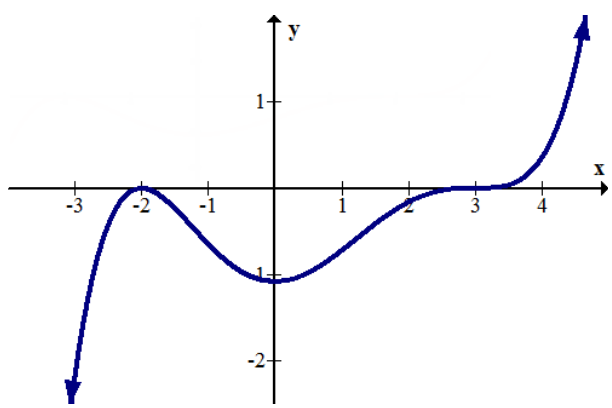
$0 = \frac{1}{100}(x-3)^3(x+2)^2$

$x = 3, -2$

Thus the $y$-intercept is (0, -1.08) and the $x$-intercepts are (3, 0) and (-2, 0).

**Concept Problem Revisited**

Graphically the function has zeroes at -2 and 3 with a $y$ intercept at about -1.1. The algebraic solution is demonstrated in Example C.




### Vocabulary

***Zeroes, roots, solutions*** and $x$**-intercepts** are synonyms for the points where a function crosses the $x$ axis.

A ***y-intercept*** is the point where a function crosses the $y$ axis.

*Note that in order for a function to pass the vertical line test, it must only have one y-intercept, but it may have multiple x-intercepts.*

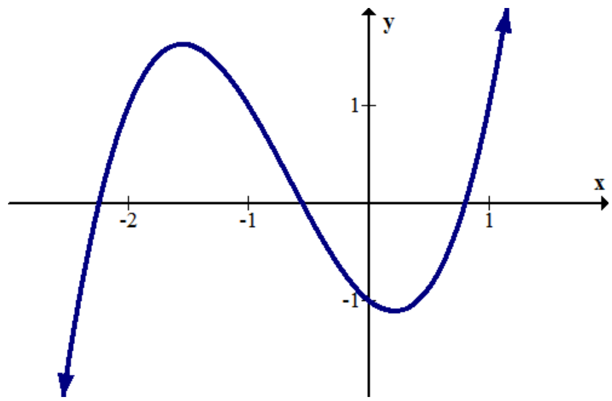
### Guided Practice

1. Determine the zeroes and $y$-intercept of the following function using algebra: $f(x) = (x+3)^2(x-2)$

2. Determine the roots and $y$-intercept of the following function using algebra or a graph:

$f(x) = x^4 + 3x^3 - 7x^2 - 15x + 18$

3. Determine the intercepts of the following function graphically.




**Answers:**

1. The $y$-intercept is (0, -18). The zeroes ($x$-intercepts) are (-3, 0) and (2, 0).

2. The $y$-intercept is (0, 18). The roots ($x$-intercepts) are (2, 0), (1, 0) and (-3, 0).

3. The $y$-intercept is approximately (0, -1). The $x$-intercepts are approximately (-2.3, 0), (-0.4, 0) and (0.7, 0). When finding values graphically, answers are always approximate. Exact answers need to be found analytically.

## Practice

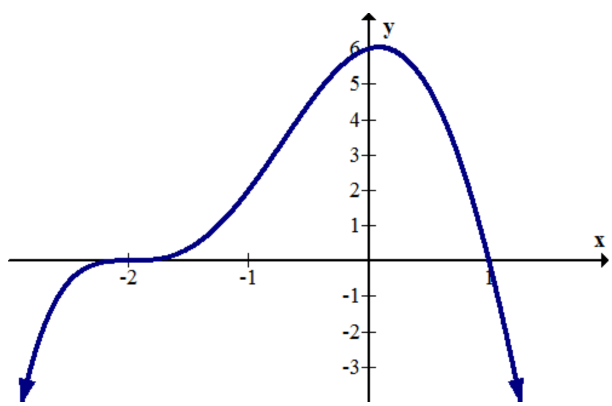
1. Determine the zeroes and $y$-intercept of the following function using algebra:

   $f(x) = (x+1)^3(x-4)$

2. Determine the roots and $y$-intercept of the following function using algebra or a graph:

   $g(x) = x^4 - 2x^3 - 7x^2 + 20x - 12$

3. Determine the intercepts of the following function graphically:




Find the intercepts for each of the following functions.

4. $y = x^2$

5. $y = x^3$

6. $y = \ln(x)$

7. $y = \frac{1}{x}$

8. $y = e^x$

9. $y = \sqrt{x}$

10. Are there any functions without a $y$-intercept? Explain.
11. Are there any functions without an $x$-intercept? Explain.
12. Explain why it makes sense that an $x$-intercept of a function is also called a "zero" of the function.

Determine the intercepts of the following functions using algebra or a graph.

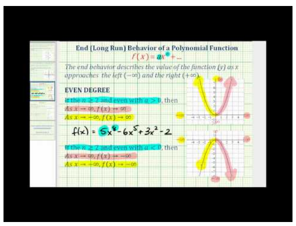
13. $h(x) = x^3 - 6x^2 + 3x + 10$
14. $j(x) = x^2 - 6x - 7$
15. $k(x) = 4x^4 - 20x^3 - 3x^2 + 14x + 5$

# 1.9 Asymptotes and End Behavior

Here you will get a conceptual and graphical understanding of what is meant by asymptotes and end behavior. This will lay the groundwork for future concepts.

Most functions continue beyond the viewing window in our calculator or computer. People often draw an arrow next to a dotted line to indicate the pattern specifically. How can you recognize these asymptotes?

### Watch This



MEDIA

Click image to the left for more content.

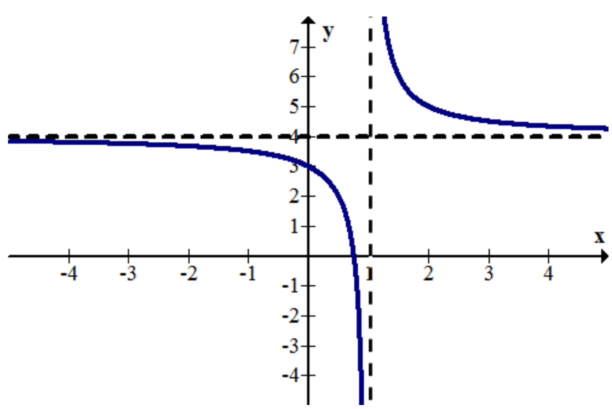
http://www.youtube.com/watch?v=y78Dpr9LLN0 James Sousa: Summary of End Behavior or Long Run Behavior of Polynomial Functions

### Guidance

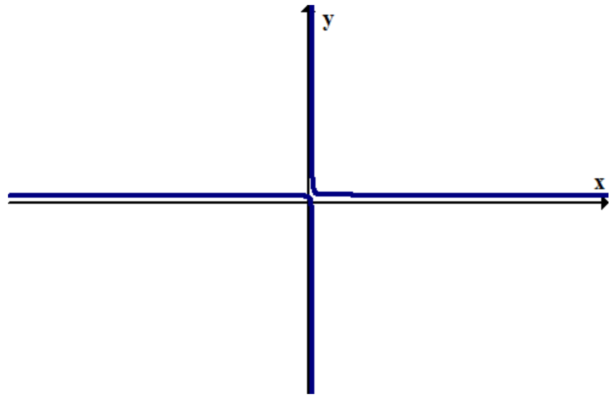
A vertical asymptote is a vertical line such as $x = 1$ that indicates where a function is not defined and yet gets infinitely close to.

A horizontal asymptote is a horizontal line such as $y = 4$ that indicates where a function flattens out as $x$ gets very large or very small. *A function may touch or pass through a horizontal asymptote.*

The reciprocal function has two asymptotes, one vertical and one horizontal. Most computers and calculators do not draw the asymptotes and so they must be inserted by hand as dotted lines.

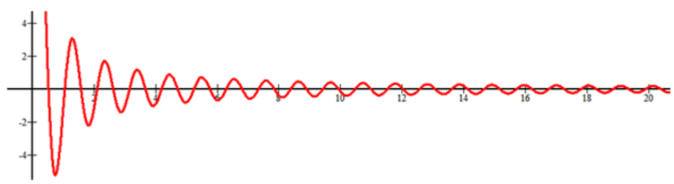



The reason why asymptotes are important is because when your perspective is zoomed way out, the asymptotes essentially become the graph.




**Example A**

Many students have the misconception that an asymptote is a line that a function gets infinitely close to *but does not touch*. This is not true. Identify the horizontal asymptote for the following function.

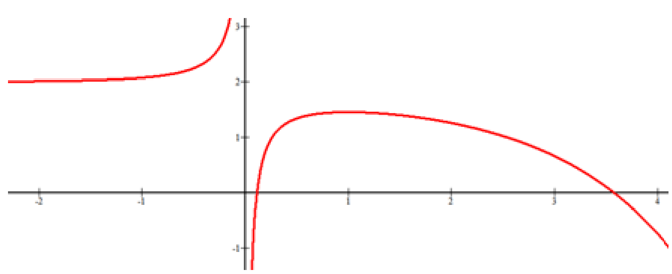


**Solution:**

The graph appears to flatten as $x$ grows larger. The horizontal asymptote is $y = 0$ even though the function clearly passes through this line an infinite number of times.

**Example B**

Identify the asymptotes and end behavior of the following function.

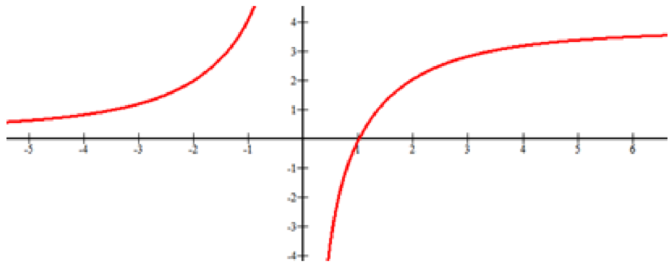


**Solution:**

The function has a horizontal asymptote $y = 2$ as $x$ approaches negative infinity. There is a vertical asymptote at $x = 0$. The right hand side seems to decrease forever and has no asymptote. *Note that slant asymptotes do exist and are called oblique asymptotes.*

**Example C**

Identify the asymptotes and end behavior of the following function.



**Solution:**

There is a vertical asymptote at $x = 0$. The end behavior of the right and left side of this function does not match. The horizontal asymptote as $x$ approaches negative infinity is $y = 0$ and the horizontal asymptote as $x$ approaches positive infinity is $y = 4$. At this point you can only estimate these heights because you were not given the function or the tools to find these values analytically.

**Concept Problem Revisited**

Asymptotes written by hand are usually identified with dotted lines next to the function that indicate how the function will behave outside the viewing window. The equations of these vertical and horizontal dotted lines are of the form $x =$ ____ and $y =$ ____. When problems ask you to find the asymptotes of a function, they are asking for the equations of these horizontal and vertical lines.

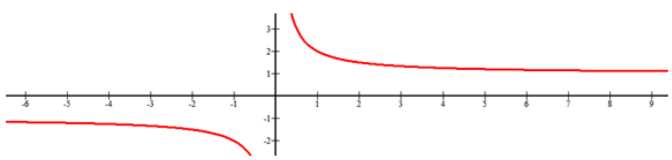
## Vocabulary

An ***asymptote*** is a line that a function gets arbitrarily close to. The function may touch horizontal asymptotes, but never touch vertical asymptotes.

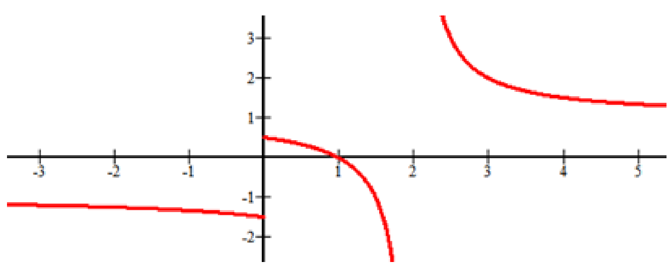
A ***piecewise function*** is a function defined to be part one function and part another function.

## Guided Practice

1. Identify the horizontal and vertical asymptotes of the following function.

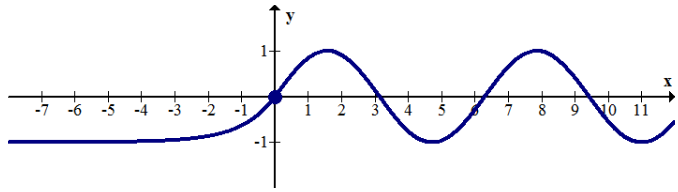


2. Identify the horizontal and vertical asymptotes of the following function.



3. Identify the horizontal and vertical asymptotes of the following piecewise function:

$$f(x) = \begin{cases} e^x - 1 & x \leq 0 \\ \sin x & 0 < x \end{cases}$$




**Answers:**

1. There is a vertical asymptote at $x = 0$. As $x$ gets infinitely small, there is a horizontal asymptote at $y = -1$. As $x$ gets infinitely large, there is another horizontal asymptote at $y = 1$.

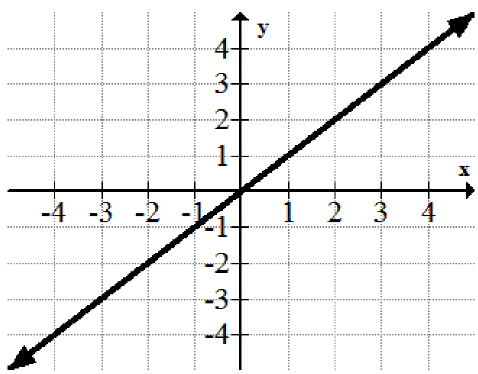
2. There is a vertical asymptote at $x = 2$. As $x$ gets infinitely small there is a horizontal asymptote at $y = -1$. As $x$ gets infinitely large, there is a horizontal asymptote at $y = 1$.

3. There is a horizontal asymptote at $y = -1$ as $x$ gets infinitely small. This is because $e$ raised to the power of a very small number becomes 0.000000... and basically becomes zero.

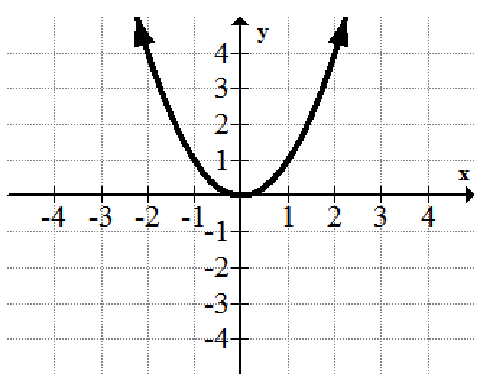
## Practice

Identify the asymptotes and end behavior of the following functions.

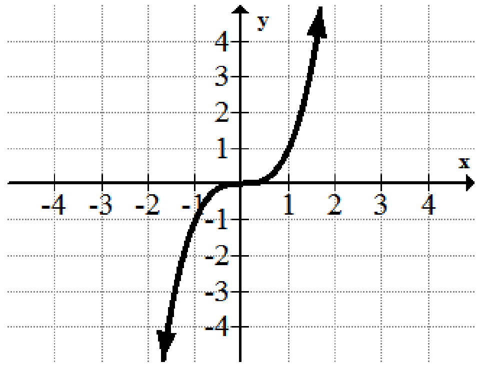
1. $y = x$



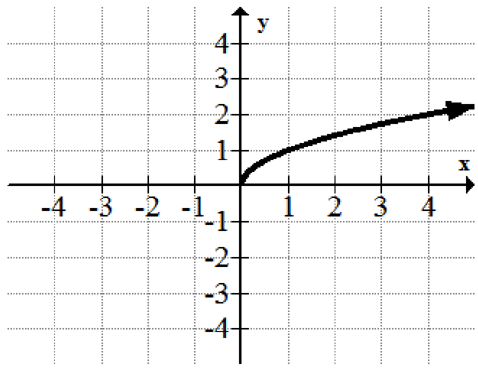

2. $y = x^2$



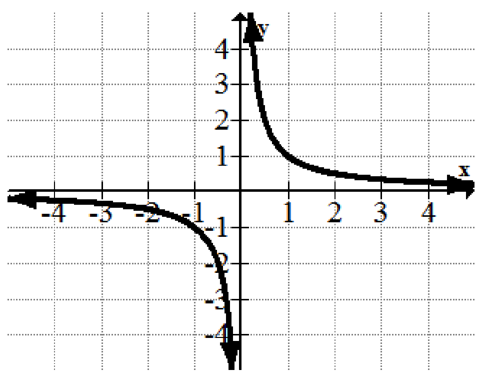

3. $y = x^3$



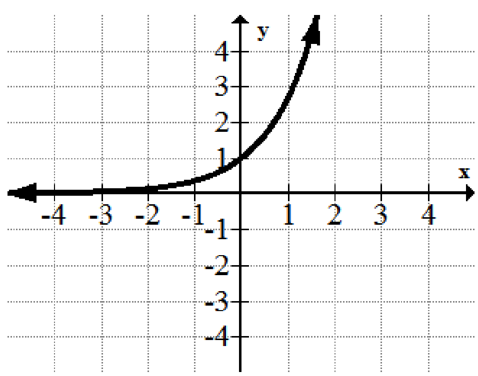

4. $y = \sqrt{x}$



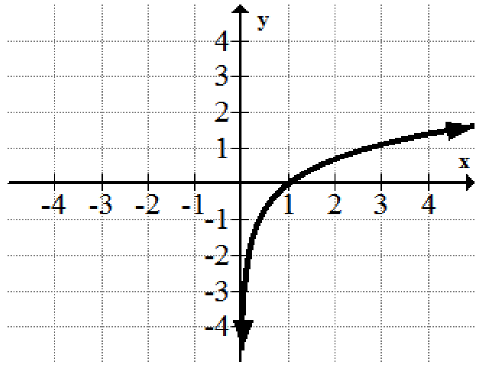

5. $y = \frac{1}{x}$



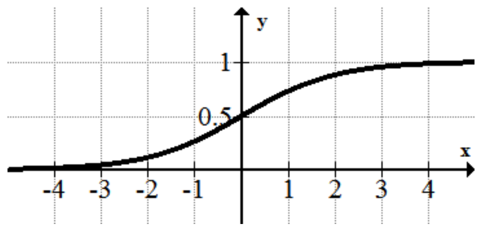

6. $y = e^x$



7. $y = \ln(x)$




8. $y = \frac{1}{1+e^{-x}}$

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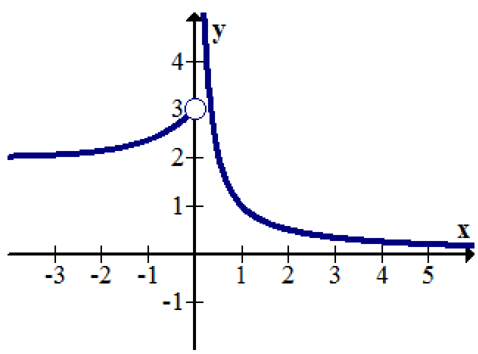


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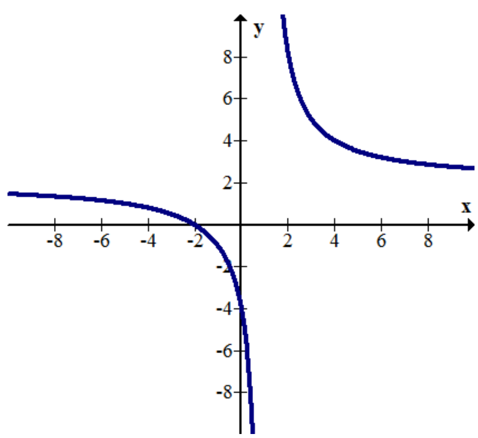


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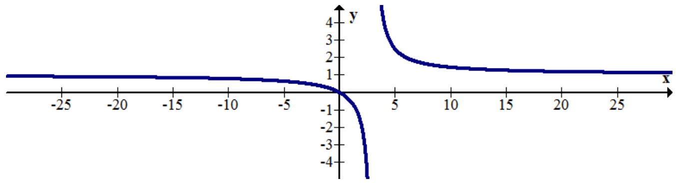



12. Vertical asymptotes occur at $x$ values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x}$ has a vertical asymptote at $x = 0$.

13. Vertical asymptotes occur at $x$ values where a function is not defined. Explain why it makes sense that $y = \frac{1}{x+3}$ has a vertical asymptote at $x = -3$.

14. Use the technique from the previous problem to determine the vertical asymptote for the function $y = \frac{1}{x-2}$.

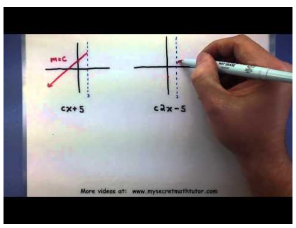
15. Use the technique from problem #13 to determine the vertical asymptote for the function $y = \frac{2}{x+4}$.

# 1.10 Continuity and Discontinuity

Here you will learn the formal definition of continuity, the three types of discontinuities and more about piecewise functions.

Continuity is a property of functions that can be drawn without lifting your pencil. Some functions, like the reciprocal functions, have two distinct parts that are unconnected. Functions that are unconnected are discontinuous. What are the three ways functions can be discontinuous and how do they come about?

### Watch This



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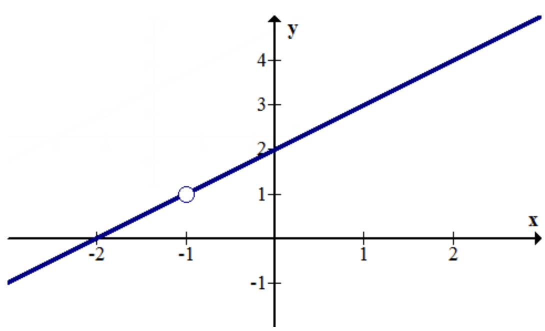
http://www.youtube.com/watch?v=sNWTxomEMEE

### Guidance

Functions that can be drawn without lifting up your pencil are called continuous functions. You will define continuous in a more mathematically rigorous way after you study limits.

There are three types of discontinuities: Removable, Jump and Infinite.

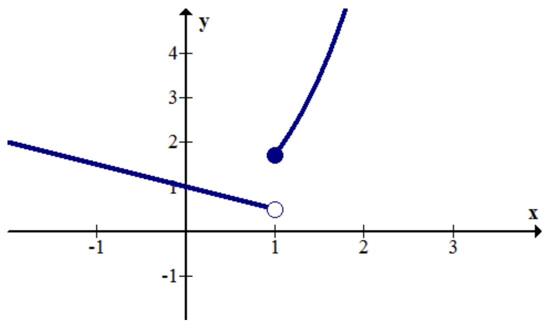
**Removable discontinuities** occur when a rational function has a factor with an $x$ that exists in both the numerator and the denominator. Removable discontinuities are shown in a graph by a hollow circle that is also known as a hole. Below is the graph for $f(x) = \frac{(x+2)(x+1)}{x+1}$. Notice that it looks just like $y = x+2$ except for the hole at $x = -1$. There is a hole at $x = -1$ because when $x = -1, f(x) = \frac{0}{0}$.



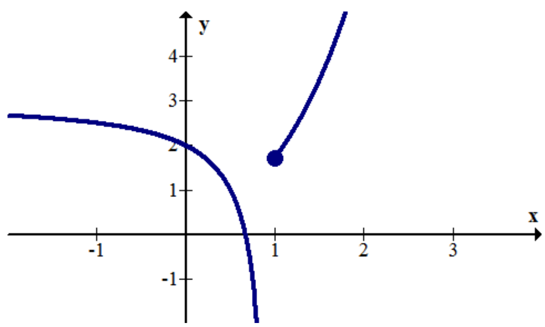

Removable discontinuities can be "filled in" if you make the function a piecewise function and define a part of the function at the point where the hole is. In the example above, to make $f(x)$ continuous you could redefine it as:

$$f(x) = \begin{cases} \frac{(x+2)(x+1)}{x+1}, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

**Jump discontinuities** occur when a function has two ends that don't meet even if the hole is filled in. In order to satisfy the vertical line test and make sure the graph is truly that of a function, only one of the end points may be filled. Below is an example of a function with a jump discontinuity.




**Infinite discontinuities** occur when a function has a vertical asymptote on one or both sides. This is shown in the graph of the function below at $x = 1$.



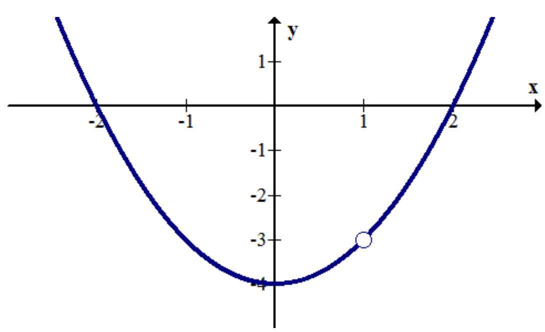

**Example A**

Identify the discontinuity of the function algebraically and then graph the function.

$f(x) = \frac{(x-2)(x+2)(x-1)}{(x-1)}$

**Solution:** Since the factor $x-1$ is in both the numerator and the denominator, there is a removable discontinuity at $x = 1$.

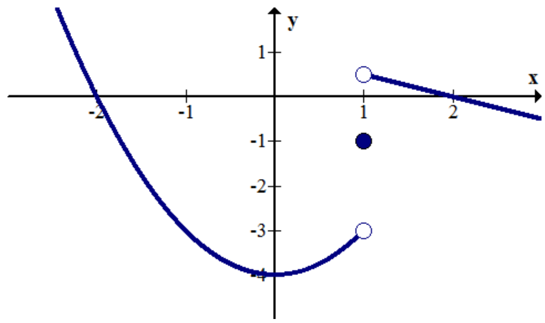
When graphing the function, you should cancel the removable factor, graph like usual and then insert a hole in the appropriate spot at the end.




**Example B**

Identify the discontinuity of the piecewise function graphically.

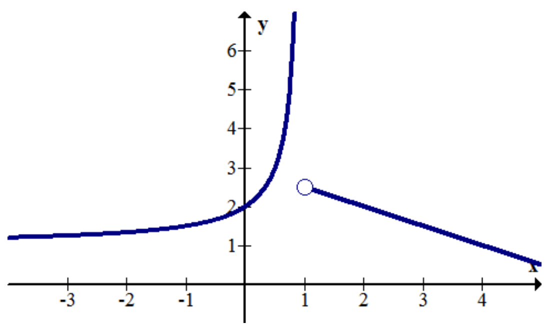
$$f(x) = \begin{cases} x^2 - 4 & x < 1 \\ -1 & x = 1 \\ -\frac{1}{2}x + 1 & x > 1 \end{cases}$$




**Solution:** There is a jump discontinuity at $x = 1$. The piecewise function describes a function in three parts; a parabola on the left, a single point in the middle and a line on the right.

**Example C**

Identify the discontinuity of the function below.




**Solution:** Since there is a vertical asymptote at $x = 1$, this is an infinite discontinuity.

**Concept Problem Revisited**

There are three ways that functions can be discontinuous. When a rational function has a vertical asymptote as a result of the denominator being equal to zero at some point, it will have an infinite discontinuity at that point. When the numerator and denominator of a rational function have one or more of the same factors, there will be removable discontinuities corresponding to each of these factors. Finally, when the different parts of a piecewise function don't "match", there will be a jump discontinuity.

## Vocabulary

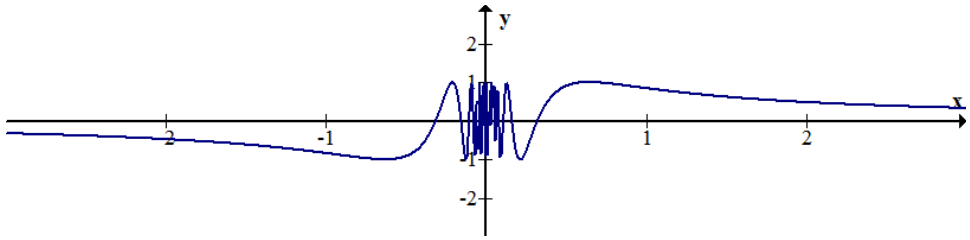
***Removable discontinuities*** are also known as holes. They occur when factors can be algebraically canceled from rational functions.

***Jump discontinuities*** occur most often with piecewise functions when the pieces don't match up.

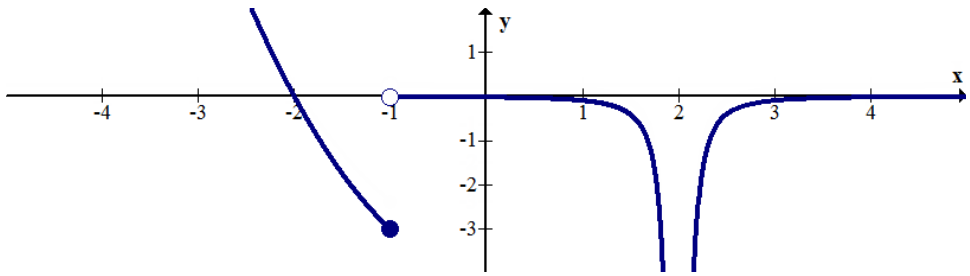
***Infinite discontinuities*** occur when a factor in the denominator of the function is zero.

## Guided Practice

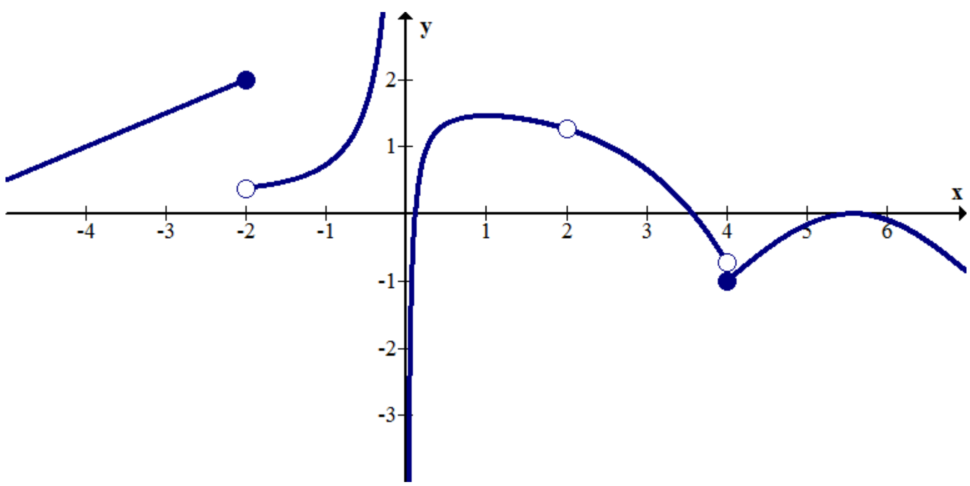
1. Describe the continuity or discontinuity of the function $f(x) = \sin\left(\frac{1}{x}\right)$.




2. Describe the discontinuities of the function below.




3. Describe the discontinuities of the function below.




**Answers:**

1. The function seems to oscillate infinitely as $x$ approaches zero. One thing that the graph fails to show is that 0 is clearly not in the domain. The graph does not shoot to infinity, nor does it have a simple hole or jump discontinuity. Calculus and Real Analysis are required to state more precisely what is going on.

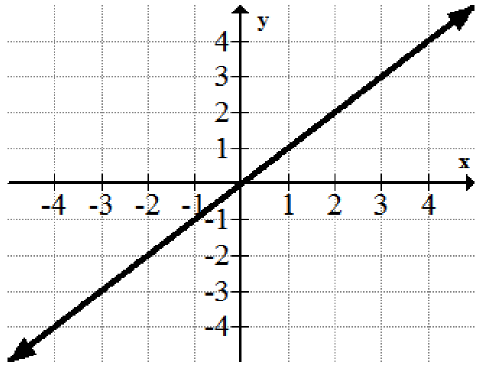
2. There is a jump discontinuity at $x = -1$ and an infinite discontinuity at $x = 2$.

3. There are jump discontinuities at $x = -2$ and $x = 4$. There is a removable discontinuity at $x = 2$. There is an infinite discontinuity at $x = 0$.

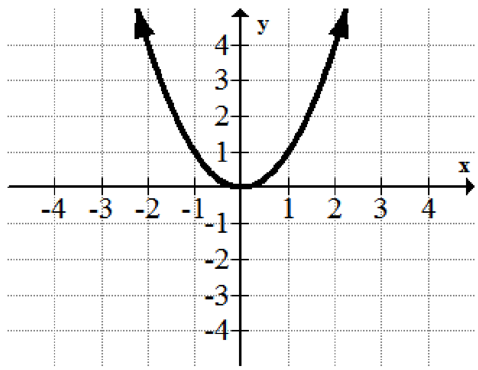
## Practice

Describe any discontinuities in the functions below:

1. $y = x$




2. $y = x^2$

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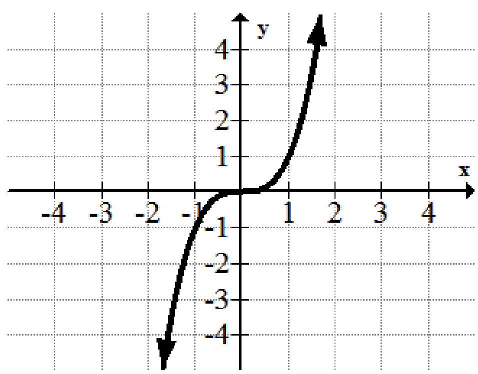
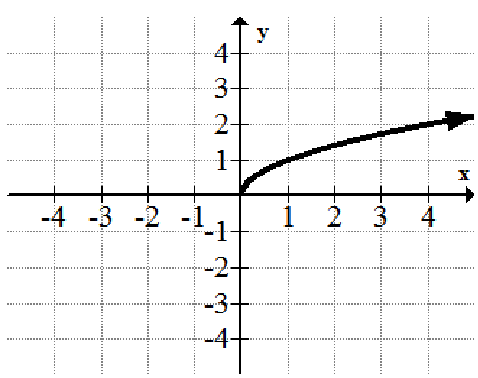
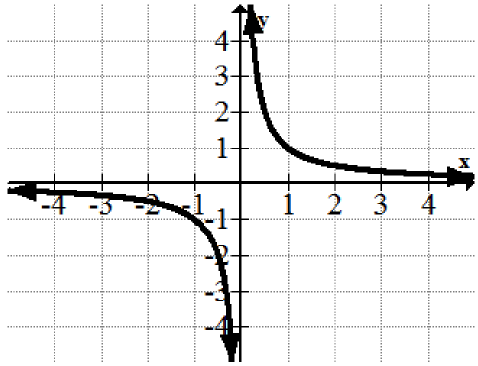


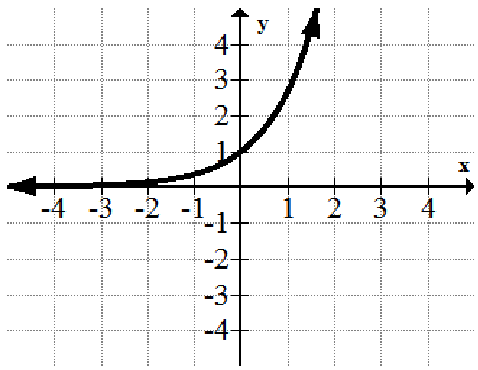

4. $y = \sqrt{x}$



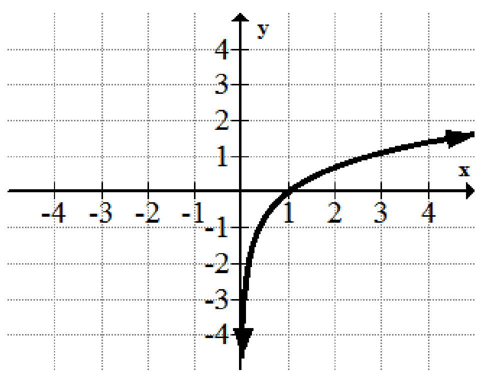
5. $y = \frac{1}{x}$



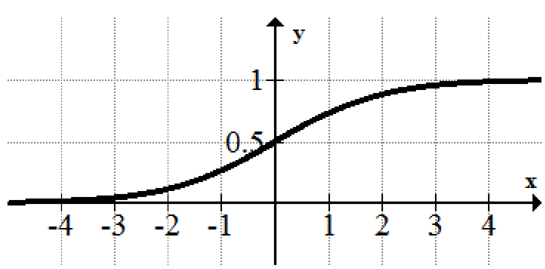

6. $y = e^x$




7. $y = \ln(x)$




8. $y = \frac{1}{1+e^{-x}}$



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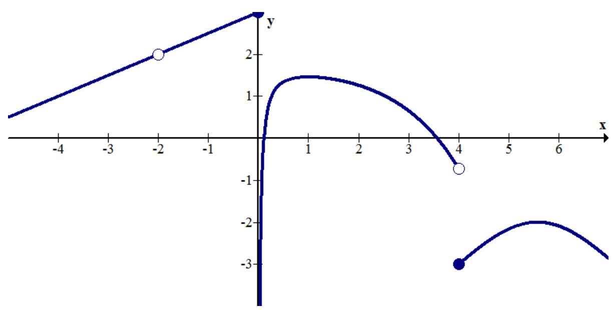


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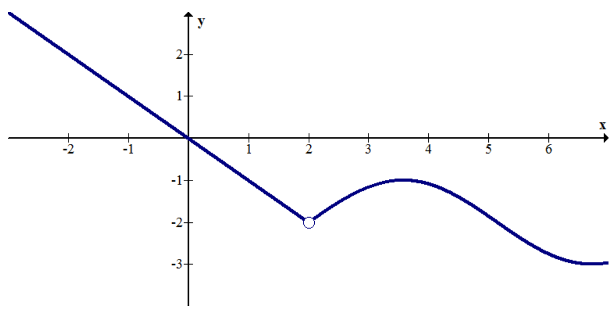


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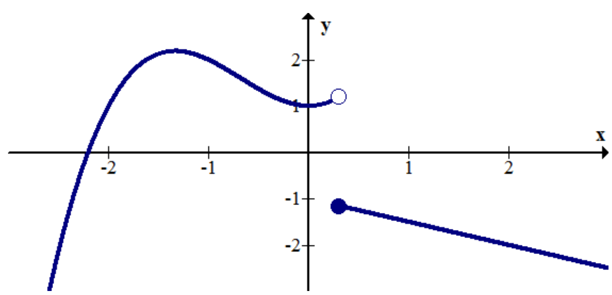


12. $f(x)$ has a jump discontinuity at $x = 3$, a removable discontinuity at $x = 5$, and another jump discontinuity at $x = 6$. Draw a picture of a graph that could be $f(x)$.

13. $g(x)$ has a jump discontinuity at $x = -2$, an infinite discontinuity at $x = 1$, and another jump discontinuity at $x = 3$. Draw a picture of a graph that could be $g(x)$.

14. $h(x)$ has a removable discontinuity at $x = -4$, a jump discontinuity at $x = 1$, and another jump discontinuity at $x = 7$. Draw a picture of a graph that could be $h(x)$.

15. $j(x)$ has an infinite discontinuity at $x = 0$, a removable discontinuity at $x = 1$, and a jump discontinuity at $x = 4$. Draw a picture of a graph that could be $j(x)$.

# 1.11 Function Composition

Here you will learn a new type of transformation called composition. Composing functions means having one function inside the argument of another function. This creates a brand new function that may not look like a regular transformation of any of the basic functions.

Functions can be added, subtracted, multiplied and divided creating new functions and graphs that are complicated combinations of the various original functions. One important way to transform functions is through function composition. Function composition allows you to line up two or more functions that act on an input in tandem.

Is function composition essentially the same as multiplying the two functions together?

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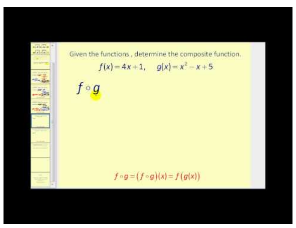

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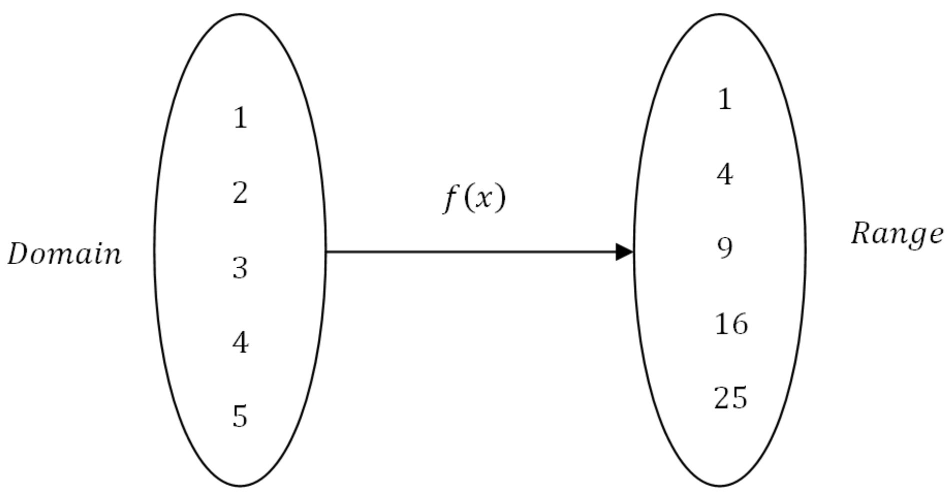
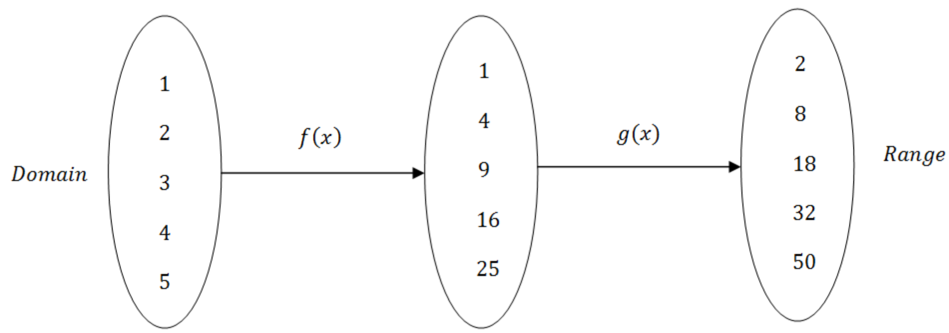
http://www.youtube.com/watch?v=qxBmISCJSME James Sousa: Composite Functions

### Guidance

A common way to describe functions is a mapping from the domain space to the range space:




Function composition means that you have two or more functions and the range of the first function becomes the domain of the second function.




There are two notations used to describe function composition. In each case the order of the functions matters because arithmetically the outcomes will be different. Squaring a number and then doubling the result will be different from doubling a number and then squaring the result. In the diagram above, $f(x)$ occurs first and $g(x)$ occurs second. This can be written as:

$g(f(x))$ or $(g \circ f)(x)$

You should read this "$g$ of $f$ of $x$." In both cases notice that the $f$ is closer to the $x$ and operates on the $x$ values first.

In the following three examples you will practice function composition with these functions:

$f(x) = x^2 - 1$

$h(x) = \frac{x-1}{x+5}$

$g(x) = 3e^x - x$

$j(x) = \sqrt{x+1}$

**Example A**

Show $f(h(x)) \neq h(f(x))$

**Solution:**

$f(h(x)) = f\left(\frac{x-1}{x+5}\right) = \left(\frac{x-1}{x+5}\right)^2 - 1$

$h(f(x)) = h(x^2-1) = \frac{(x^2-1)-1}{(x^2-1)+5} = \frac{x^2-2}{x^2+4}$

In order to truly show they are not equal it is best to find a specific counter example of a number where they differ. Sometimes algebraic expressions may look different, but are actually the same. You should notice that $f(h(x))$is undefined when $x = -5$ because then there would be zero in the denominator. $h(f(x))$ on the other hand is defined at $x = -5$. Since the two function compositions differ, you can conclude:

$f(h(x)) \neq h(f(x))$

**Example B**

What is $g(h(x))$?

$$\begin{aligned} g(h(x)) = g\left(\frac{x-1}{x+5}\right) &= 3e^{\left(\frac{x-1}{x+5}\right)} - \left(\frac{x-1}{x+5}\right) \\ &= 3\ \exp\left(\frac{x-1}{x+5}\right) - \left(\frac{x-1}{x+5}\right) \end{aligned}$$

*Note that it is difficult to write and read an exponential function with a large fraction in the exponent. In order to make it easier to work with you can use exp*$(x)$ *instead of* $e^x$ *which allows more space and easier readability.*

**Example C**

What is $f\Big(j(h(g(x)))\Big)$?

**Solution:** These functions are *nested* within the arguments of the other functions. Sometimes functions simplify significantly when composed together, as $f$ and $j$ do in this case. It makes sense to evaluate those two functions first together and keep them on the outside of the argument.

$f(x) = x^2 - 1; h(x) = \frac{x-1}{x+5}; g(x) = 3e^x - x; j(x) = \sqrt{x+1}$

$f(j(y)) = f\left(\sqrt{y+1}\right) = \left(\sqrt{y+1}\right)^2 - 1 = y + 1 - 1 = y$

Notice how the composition of $f$ and $j$ produced just the argument itself?

Thus,

$$\begin{aligned} f\Big(j(h(g(x)))\Big) = h(g(x)) &= h(3e^x - x) \\ &= \frac{(3e^x - x) - 1}{(3e^x - x) + 5} \\ &= \frac{3e^x - x - 1}{3e^x - x + 5} \end{aligned}$$

**Concept Problem Revisited**

Function composition is not the same as multiplying two functions together. With function composition there is an outside function and an inside function. Suppose the two functions were doubling and squaring. It is clear just by looking at the example input of the number 5 that 50 (squaring then doubling) is different from 100 (doubling then squaring). Both 50 and 100 are examples of function composition, while 250 (five doubled multiplied by five squared) is an example of the product of two separate functions happening simultaneously.

### Vocabulary

***Function composition*** is when there are two or more functions and the range of the first function becomes the domain of the second function.

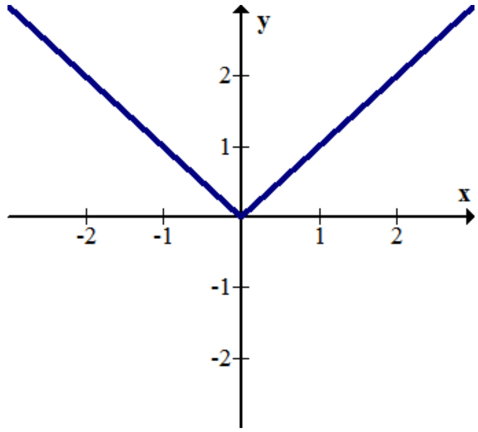
***Nesting*** refers to a function being operated on or in the argument of another function.

A ***counterexample*** is a specific instance that contradicts a statement. When you are asked to show a statement is not true, it is best to find a counterexample to the statement.

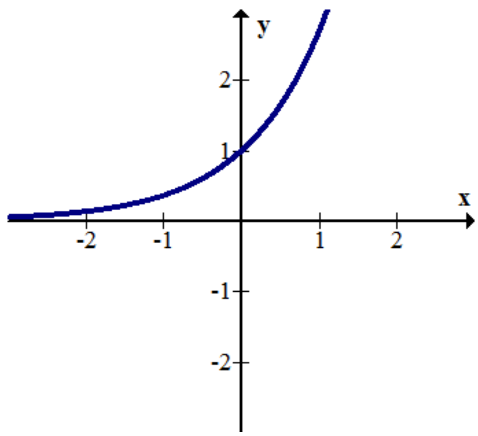
### Guided Practice

For the three guided practice problems use the following functions.

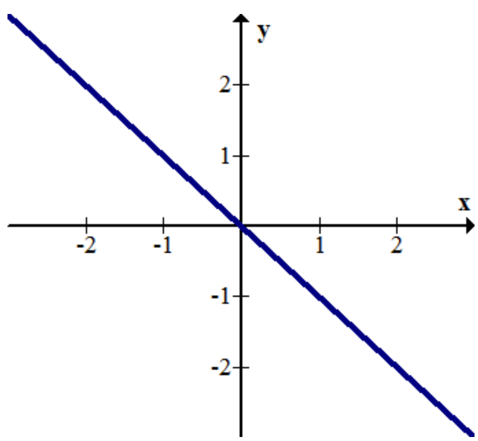
$f(x) = |x|$




$g(x) = e^x$



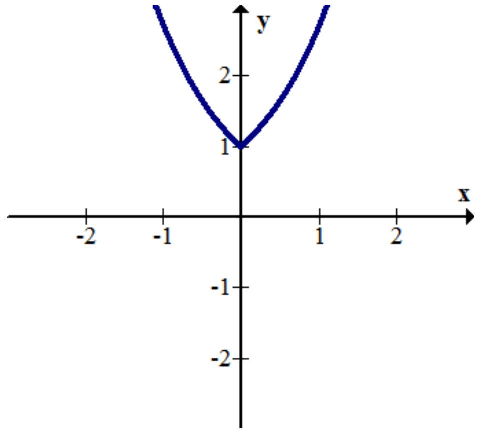

$h(x) = -x$




1. Compose $g(f(x))$ and graph the result. Describe the transformation.
2. Compose $h(g(x))$ and graph the result. Describe the transformation.
3. Compose $g\Big(h(f(x))\Big)$ and graph the result. Describe the transformation.

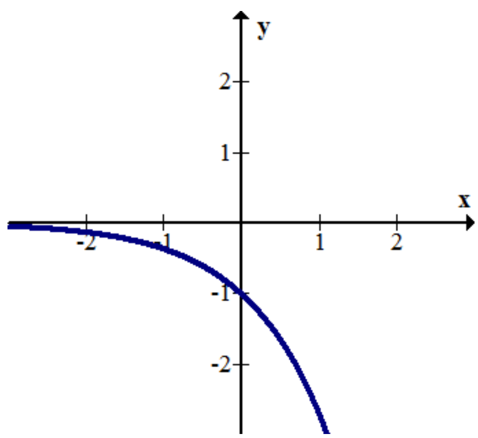
**Answers:**

1. $g(f(x)) = g(|x|) = e^{|x|}$



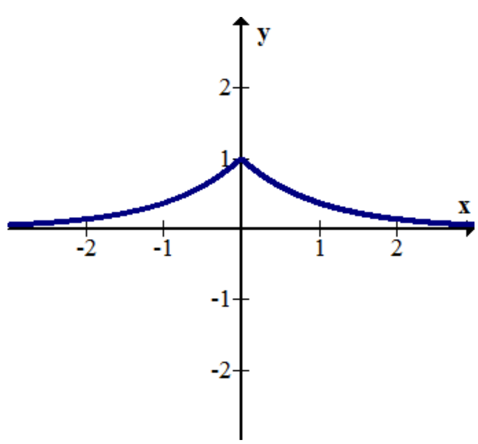

The positive portion of the exponential graph has been mirrored over the $y$ axis and the negative portion of the exponential graph has been entirely truncated.

2. $h(g(x)) = h(e^x) = -e^x$




The exponential graph has been reflected over the $x$-axis.

3. $g\Big(h(f(x))\Big) = g(h(|x|)) = g(-|x|) = e^{-|x|}$




The negative portion of the exponential graph has been mirrored over the $y$-axis and the positive portion of the exponential graph has been truncated.

### Practice

For questions 1-9, use the following three functions: $f(x) = |x|, h(x) = -x, g(x) = (x-2)^2 - 3$.

1. Graph $f(x), h(x)$ and $g(x)$.

2. Find $f(g(x))$ algebraically.

3. Graph $f(g(x))$ and describe the transformation.

4. Find $g(f(x))$ algebraically.

5. Graph $g(f(x))$ and describe the transformation.

6. Find $h(g(x))$ algebraically.

7. Graph $h(g(x))$ and describe the transformation.

8. Find $g(h(x))$ algebraically.

9. Graph $g(h(x))$ and describe the transformation.

For 10-16, use the following three functions: $j(x) = x^2, k(x) = |x|, m(x) = \sqrt{x}$.

10. Graph $j(x), k(x)$ and $m(x)$.

11. Find $j(k(x))$ algebraically.

12. Graph $j(k(x))$ and describe the transformation.

13. Find $k(m(x))$ algebraically.

14. Graph $k(m(x))$ and describe the transformation.

15. Find $m(k(x))$ algebraically.

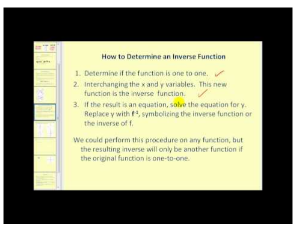
16. Graph $m(k(x))$ and describe the transformation.

# 1.12 Inverses of Functions

Here you will learn about inverse functions and what they look like graphically and numerically. You will also learn how to find and verify inverse functions algebraically.

Functions are commonly known as rules that take inputs and produce outputs. An inverse function does exactly the reverse, undoing what the original function does. How can you tell if two functions are inverses?

### Watch This



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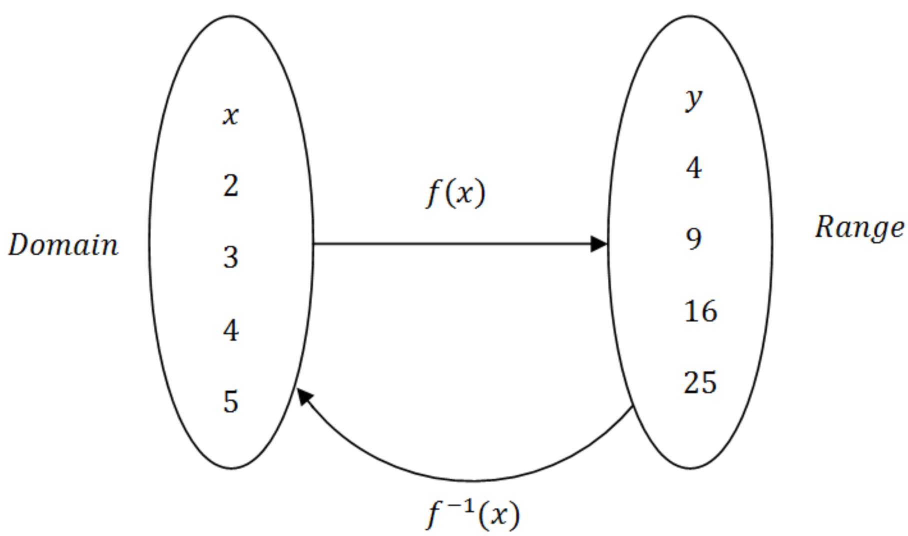
Click image to the left for more content.

http://www.youtube.com/watch?v=qgezKpQYH2w James Sousa: Inverse Functions

### Guidance

A function is written as $f(x)$ and its inverse is written as $f^{-1}(x)$. A common misconception is to see the -1 and interpret it as an exponent and write $\frac{1}{f(x)}$, but this is not correct. Instead, $f^{-1}(x)$ should be viewed as a new function from the range of $f(x)$ back to the domain.

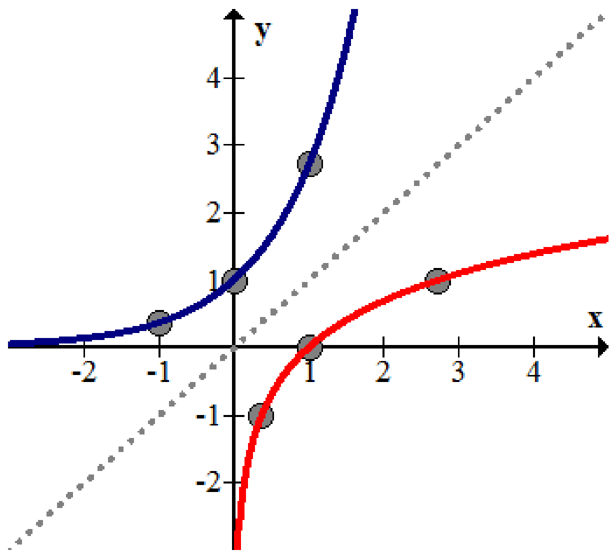



It is important to see the cycle that starts with $x$, becomes $y$ and then goes back to $x$. In order for two functions to truly be inverses of each other, this cycle must hold algebraically.

$f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$

When given a function there are two steps to take to find its inverse. In the original function, first switch the variables $x$ and $y$. Next, solve the function for $y$. This will give you the inverse function. After finding the inverse, it is important to check both directions of compositions to make sure that together the function and its inverse produce the value $x$. In other words, verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Graphically, inverses are reflections across the line $y = x$. Below you see inverses $y = e^x$ and $y = \ln x$. Notice how the $(x, y)$ coordinates in one graph become $(y, x)$ coordinates in the other graph.




In order to decide whether an inverse function is also actually a function you can use the vertical line test on the inverse function like usual. You can also use the horizontal line test on the original function. The horizontal line test is exactly like the vertical line test except the lines simply travel horizontally.

**Example A**

Find the inverse, then verify the inverse algebraically. $f(x) = y = (x+1)^2 + 4$

**Solution:** To find the inverse, switch $x$ and $y$ then solve for $y$.

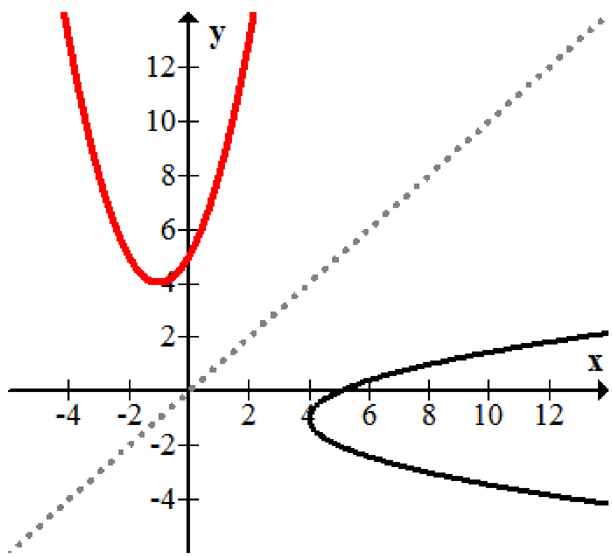
$$\begin{aligned} x &= (y+1)^2 + 4 \\ x - 4 &= (y+1)^2 \\ \pm\sqrt{x-4} &= y + 1 \\ -1 \pm \sqrt{x-4} &= y = f^{-1}(x) \end{aligned}$$

To verify algebraically, you must show $x = f(f^{-1}(x)) = f^{-1}(f(x))$:

$$\begin{aligned} f(f^{-1}(x)) &= f\left(-1 \pm \sqrt{x-4}\right) \\ &= ((-1 \pm \sqrt{x-4}) + 1)^2 + 4 \\ &= \left(\pm\sqrt{x-4}\right)^2 + 4 \\ &= x - 4 + 4 = x \end{aligned}$$

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}((x+1)^2+4) \\
&= -1 \pm \sqrt{((x+1)^2+4)-4} \\
&= -1 \pm \sqrt{(x+1)^2} \\
&= -1+x+1 = x
\end{aligned}$$

As you can see from the graph, the $\pm$ causes the inverse to be a relation instead of a function. This can be observed in the graph because the original function does not pass the horizontal line test and the inverse does not pass the vertical line test.




**Example B**

Find the inverse of the function and then verify that $x = f(f^{-1}(x)) = f^{-1}(f(x))$.

$f(x) = y = \frac{x+1}{x-1}$

**Solution:** Sometimes it is quite challenging to switch $x$ and $y$ and then solve for $y$. You must be careful with your algebra.

$$\begin{aligned}
x &= \frac{y+1}{y-1} \\
x(y-1) &= y+1 \\
xy - x &= y+1 \\
xy - y &= x+1 \\
y(x-1) &= x+1 \\
y &= \frac{x+1}{x-1}
\end{aligned}$$

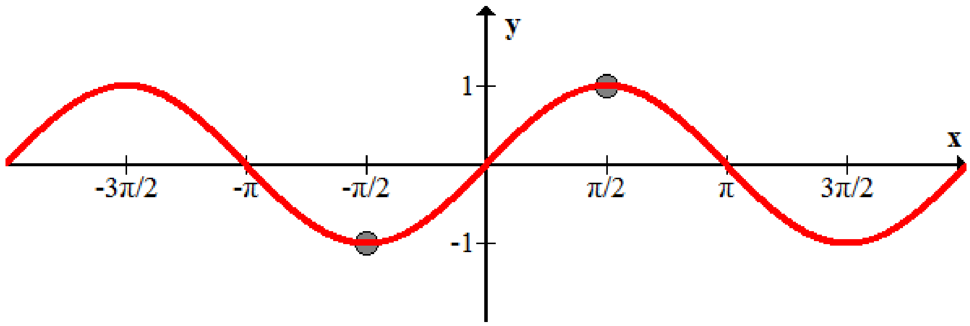
This function turns out to be its own inverse. Since they are identical, you only need to show that $x = f(f^{-1}(x))$.

$f\left(\frac{x+1}{x-1}\right) = \frac{\left(\frac{x+1}{x-1}\right)+1}{\left(\frac{x+1}{x-1}\right)-1} = \frac{x+1+x-1}{x+1-(x-1)} = \frac{2x}{2} = x$

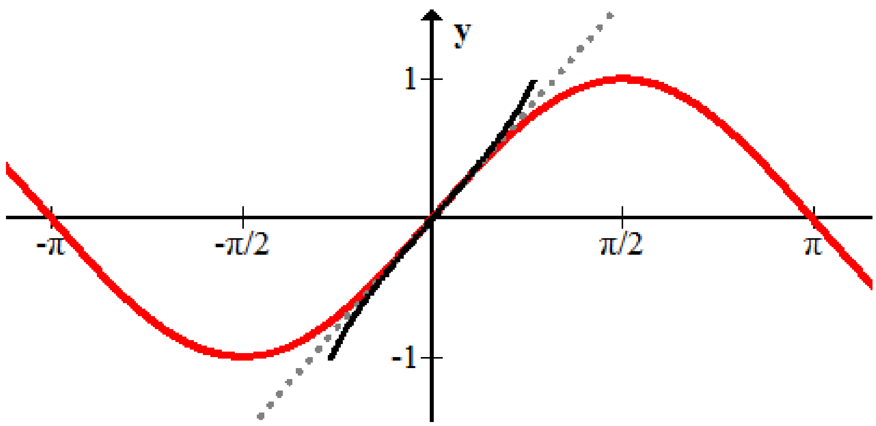
**Example C**

What is the inverse of $f(x) = y = \sin x$?

**Solution:** The sine function does not pass the horizontal line test and so its true inverse is not a function.




However, if you restrict the domain to just the part of the $x$-axis between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ then it will pass the horizontal line test and the inverse will be a function.




The inverse of the sine function is called the arcsine function, $f(x) = \sin^{-1}(x)$, and is shown in black. It is truncated so that it only inverts a part of the whole sine wave. You will study periodic functions and their inverses in more detail later.

**Concept Problem Revisited**

You can tell that two functions are inverses if each undoes the other, always leaving the original $x$.

### Vocabulary

A **relation** is a general term for functions and non-functions that relate two variables and may or may not pass the vertical line test.

Two relations are ***inverses*** of each other if they are reflections across the line $y = x$.

The **horizontal line test** is a means of discovering whether the inverse of a function is also a function.

### Guided Practice

1. Determine the inverse of $f(x) = 5 + \frac{x}{2}$. Verify that the inverse is actually the inverse.

2. Determine if $f(x) = \frac{3}{7}x - 21$ and $g(x) = \frac{7}{3}x + 21$ are inverses of one another.

3. Determine the inverse of $f(x)=\frac{x}{x+4}$.

**Answers:**

1. To find the inverse,

$$\begin{aligned}y &= 5+\frac{x}{2}\\ x &= 5+\frac{y}{2}\\ x-5 &= \frac{y}{2}\\ 2(x-5) &= y = f^{-1}(x)\end{aligned}$$

Verification:

$2\left(5+\frac{x}{2}-5\right)=2\left(\frac{x}{2}\right)=x$

$5+\frac{2(x-5)}{2}=5+x-5=x$

They are truly inverses of each other.

2. Even though $f(x)=\frac{3}{7}x-21$ and $g(x)=\frac{7}{3}x+21$ have some inverted pieces, they are not inverses of each other. In order to show this, you must show that the composition does not simplify to $x$. $\frac{3}{7}\left(\frac{7}{3}x+21\right)-21=x+9-21=x-12\neq x$

3. To find the inverse, switch x and y.

$$\begin{aligned}f(x)=y &= \frac{x}{x+4}\\ x &= \frac{y}{y+4}\\ x(y+4) &= y\\ xy+4x &= y\\ xy-y &= -4x\\ y(x-1) &= -4x\\ f^{-1}(x)=y &= -\frac{4x}{x-1}\end{aligned}$$

## Practice

Consider $f(x)=x^3$.

1. Sketch $f(x)$ and $f^{-1}(x)$.

2. Find $f^{-1}(x)$ algebraically. It is actually a function?

3. Verify algebraically that $f(x)$ and $f^{-1}(x)$ are inverses.

Consider $g(x)=\sqrt{x}$.

4. Sketch $g(x)$ and $g^{-1}(x)$.

5. Find $g^{-1}(x)$ algebraically. It is actually a function?

6. Verify algebraically that $g(x)$ and $g^{-1}(x)$ are inverses.

Consider $h(x)=|x|$.

7. Sketch $h(x)$ and $h^{-1}(x)$.

8. Find $h^{-1}(x)$ algebraically. It is actually a function?

9. Verify graphically that $h(x)$ and $h^{-1}(x)$ are inverses.

Consider $j(x) = 2x - 5$.

10. Sketch $j(x)$ and $j^{-1}(x)$.

11. Find $j^{-1}(x)$ algebraically. It is actually a function?

12. Verify algebraically that $j(x)$ and $j^{-1}(x)$ are inverses.

13. Use the horizontal line test to determine whether or not the inverse of $f(x) = x^3 - 2x^2 + 1$ is also a function.

14. Are $g(x) = \ln(x+1)$ and $h(x) = e^{x-1}$ inverses? Explain.

15. If you were given a table of values for a function, how could you create a table of values for the inverse of the function?

Key features of functions were explored through the use of ten basic function families. Transforming functions and finding inverses of functions were also considered.

# 1.13 References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# CHAPTER 2 Polynomials and Rational Functions

## Chapter Outline



Here you will deepen you knowledge about polynomials from your work in algebra. You will review factoring and learn more advanced factoring techniques. You will learn how to divide polynomials and how polynomials and rational expressions are related. You will then focus on rational functions and learn about their unique features including three different types of asymptotes. Finally, you will put it all together and learn how to graph rational functions by hand.

# 2.1 Factoring Review

Here you will review factoring techniques from Algebra 1 and 2 in preparation for more advanced factoring techniques. The review will include factoring out a greatest common factor, factoring into binomials and the difference of squares.

To factor means to write an expression as a product instead of a sum. Factoring is particularly useful when solving equations set equal to zero because then logically at least one factor must be equal to zero. In PreCalculus, you should be able to factor even when there is no obvious greatest common factor or the difference is not between two perfect squares.

How do you use the difference of perfect squares factoring technique on polynomials that don't contain perfect squares and why would this be useful?

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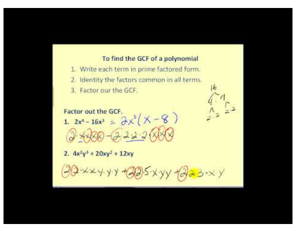


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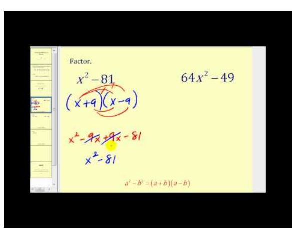


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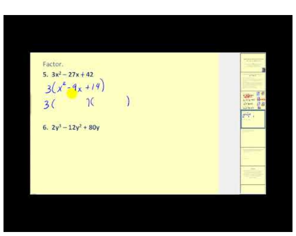


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http://www.youtube.com/watch?v=cWapvAWXdoY James Sousa: Factoring: Basic Trinomials with a=1

### Guidance

A polynomial is a sum of a finite number of terms. Each term consists of a constant that multiplies a variable. The variable may only be raised to a non-negative exponent. The letters $a, b, c \dots$ in the following general polynomial expression stand for regular numbers like $0, 5, -\frac{1}{4}, \sqrt{2}$ and the $x$ represents the variable.

$ax^n + bx^{n-1} + \ldots + fx^2 + gx + h$

You have already learned many properties of polynomials. For example, you know the commutative property which states that terms of a polynomial can be rearranged to create an equivalent polynomial. When two polynomials are added, subtracted or multiplied the result is always a polynomial. This means polynomials are closed under addition, and is one of the properties that makes the factoring of polynomials possible. Polynomials are not closed under division because dividing two polynomials could result in a variable in the denominator, which is a rational expression (not a polynomial).

There are three methods for factoring that are essential to master. The first method you should always try is to factor out the greatest common factor (GCF) of the expression (see Example A). The second method you should implement after factoring out a GCF is to see if you can factor the expression into the product of two binomials (see Example B). This type of factoring is usually recognizable as a trinomial where $x^2$ has a coefficient of 1. The third type of basic factoring is the difference of squares. It is recognizable as one square monomial being subtracted from another square monomial.

The rigor of the following factoring examples and exercises is greater than an introductory level factoring lesson but the techniques are the same.

**Example A**

Use the GCF technique to factor the following expression. Check that the factored expression matches the original.

$-\frac{1}{2}x^4 + \frac{7}{2}x^2 - 6$

**Solution:** Many students just learning factoring may conclude that the three terms share no factors besides one. However, the name GCF is deceiving because this expression has an infinite number of equivalent expressions many of which are more useful. In order to find these alternative expressions you must strategically factor numbers that are neither the greatest factor nor common to all three terms. In this case, $-\frac{1}{2}$ is an excellent choice.

$-\frac{1}{2}x^4 + \frac{7}{2}x^2 - 6 = -\frac{1}{2}(x^4 - 7x^2 + 12)$

In order to check to see that this is an equivalent expression, you need to distribute the $-\frac{1}{2}$. When you distribute, the first coefficient matches because it just gets multiplied by 1, the second term becomes $\frac{7}{2}$ and the third term becomes -6.

**Example B**

Factor the expression from Example A into the product of two binomials and a constant.

$-\frac{1}{2}(x^4 - 7x^2 + 12)$

**Solution:** Many students familiar with basic factoring may be initially stuck on a problem like this. However, you should recognize that beneath the $4^{th}$ degree and the $-\frac{1}{2}$ the problem boils down to being able to factor $u^2 - 7u + 12$ which is just $(u-3)(u-4)$.

*Start by rewriting the problem:* $-\frac{1}{2}(x^4 - 7x^2 + 12)$

*Then choose a temporary substitution:* Let $u = x^2$.

*Then substitute and factor away. Remember to substitute back at the end.*

$$-\frac{1}{2}(u^2-7u+12) = -\frac{1}{2}(u-3)(u-4)$$
$$= -\frac{1}{2}(x^2-3)(x^2-4)$$

This type of temporary substitution that enables you to see the underlying structure of an expression is very common in calculus.

**Example C**

Factor the resulting expression from Example B into four linear factors and a constant.

$-\frac{1}{2}(x^2-3)(x^2-4)$

**Solution:** Many students may recognize that $x^2-4$ immediately factors by the difference of squares method to be $(x-2)(x+2)$. This problem asks for more because sometimes the difference of squares method can be applied to expressions like $x^2-3$ where each term is not a perfect square. The number 3 actually is a square.

$3 = \left(\sqrt{3}\right)^2$

So the expression may be factored to be:

$-\frac{1}{2}\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)(x-2)(x+2)$

**Concept Problem Revisited**

One reason why it might be useful to completely factor an expression like $-\frac{1}{2}(x^4-7x^2+12)$ into linear factors is if you wanted to find the roots of the function $f(x) = -\frac{1}{2}(x^4-7x^2+12)$. The roots are $x = \pm\sqrt{3}, \pm 2$.

You should recognize that $x^2-3$ can still be thought of as the difference of perfect squares because the number 3 can be expressed as $\left(\sqrt{3}\right)^2$. Rewriting the number 3 to fit a factoring pattern that you already know is an example of using the basic factoring techniques at a PreCalculus level.

### Vocabulary

A ***polynomial*** is a mathematical expression that is often represented as a sum of terms or a product of factors.

***To factor*** means to rewrite a polynomial expression given as a sum of terms into a product of factors.

***Linear factors*** are expressions of the form $ax+b$ where $a$ and $b$ are real numbers.

### Guided Practice

1. Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$\frac{x^5}{3} - \frac{11x^3}{3} + 6x$

2. Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$-\frac{2}{7}x^4 + \frac{74}{63}x^2 - \frac{8}{63}$

3. Factor the following expression into strictly linear factors if possible. If not possible, explain why.

$x^4 + x^2 - 72$

**Answers:**

1. $\frac{x^5}{3} - \frac{11x^3}{3} + 6x$

$$= \frac{1}{3}x(x^4 - 11x^2 + 18)$$
$$= \frac{1}{3}x(x^2 - 2)(x^2 - 9)$$
$$= \frac{1}{3}x(x + \sqrt{2})(x - \sqrt{2})(x + 3)(x - 3)$$

2. $-\frac{2}{7}x^4 + \frac{74}{63}x^2 - \frac{8}{63}$. Let $u = x^2$.

$$= -\frac{2}{7}u^2 + \frac{74}{63}u - \frac{8}{63}$$
$$= -\frac{2}{7}\left(u^2 - \frac{37}{9}u + \frac{4}{9}\right)$$

Factoring through fractions like this can be extremely tricky. You must recognize that $-\frac{1}{9}$ and -4 sum to $-\frac{37}{9}$ and multiply to $\frac{4}{9}$.

$$= -\frac{2}{7}\left(u - \frac{1}{9}\right)(u - 4)$$
$$= -\frac{2}{7}\left(x^2 - \frac{1}{9}\right)(x^2 - 4)$$
$$= -\frac{2}{7}\left(x - \frac{1}{3}\right)\left(x + \frac{1}{3}\right)(x - 2)(x + 2)$$

3. $x^4 + x^2 - 72$
$= (x^2 - 8)(x^2 + 9)$

Notice that $(x^2 - 8)$ can be written as the difference of perfect squares because $8 = \left(\sqrt{8}\right)^2 = \left(2\sqrt{2}\right)^2$. On the other hand, $x^2 + 9$ cannot be written as the difference between squares because the $x^2$ and the 9 are being added not subtracted. This polynomial cannot be factored into strictly linear factors.
$= (x - 2\sqrt{2})(x + 2\sqrt{2})(x^2 + 9)$

### Practice

Factor each polynomial into strictly linear factors if possible. If not possible, explain why not.

1. $x^2 + 5x + 6$
2. $x^4 + 5x^2 + 6$
3. $x^4 - 16$
4. $2x^2 - 20$
5. $3x^2 + 9x + 6$
6. $\frac{x^4}{2} - 5x^2 + \frac{9}{2}$
7. $\frac{2x^4}{3} - \frac{34x^2}{3} + \frac{32}{3}$
8. $x^2 - \frac{1}{4}$
9. $x^4 - \frac{37x^2}{4} + \frac{9}{4}$
10. $\frac{3}{4}x^4 - \frac{87}{4}x^2 + 75$

11. $\frac{1}{2}x^4 - \frac{29}{2}x^2 + 50$
12. $\frac{x^4}{2} - \frac{5x^2}{9} + \frac{1}{18}$
13. $x^4 - \frac{13}{36}x^2 + \frac{1}{36}$
14. How does the degree of a polynomial relate to the number of linear factors?
15. If a polynomial does not have strictly linear factors, what does this imply about the type of roots that the polynomial has?

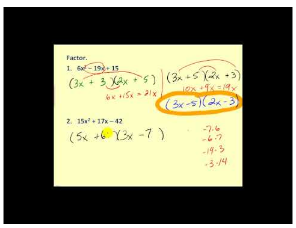
# 2.2 Advanced Factoring

Here you will be exposed to a variety of factoring techniques for special situations. Additionally, you will see alternatives to trial and error for factoring.

The difference of perfect squares can be generalized as a factoring technique. By extension, any difference between terms that are raised to an even power like $a^6 - b^6$ can be factored using the difference of perfect squares technique. This is because even powers can always be written as perfect squares: $a^6 - b^6 = (a^3)^2 - (b^3)^2$.

What about the sum or difference of terms with matching odd powers? How can those be factored?

### Watch This



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http://www.youtube.com/watch?v=55wm2c1xkp0 James Sousa: Factoring: Trinomials using Trial and Error and Grouping

### Guidance

Factoring a trinomial of the form $ax^2 + bx + c$ is much more difficult when $a \neq 1$. In Examples A and B, you will see how the following expression can be factored using educated guessing and checking and the quadratic formula. Additionally, you will see an algorithm (a step by step procedure) for factoring these types of polynomials without guessing. The proof of the algorithm is beyond the scope of this book, but is a reliable technique for getting a handle on tricky factoring questions of the form:

$6x^2 - 13x - 28$

When you compare the computational difficulty of the three methods, you will see that the algorithm described in Example A is the most efficient.

A second type of advanced factoring technique is factoring by grouping. Suppose you start with an expression already in factored form:

$(4x + y)(3x + z) = 12x^2 + 4xz + 3xy + yz$

Usually when you multiply the factored form of a polynomial, two terms can be combined because they are like terms. In this case, there are no like terms that can be combined. In Example C, you will see how to factor by grouping.

The last method of advanced factoring is the sum or difference of terms with matching odd powers. The pattern is:

$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

This method is shown in the guided practice and the pattern is fully explored in the exercises.

**Example A**

Factor the following expression: $6x^2 - 13x - 28$

**Solution:** While this trinomial can be factored by using the quadratic formula or by guessing and checking, it can also be factored using a factoring algorithm. Here, you will learn how this algorithm works.

$6x^2 - 13x - 28$

First, multiply the first coefficient with the last coefficient:

$x^2 - 13x - 168$

Second, factor as you normally would with $a = 1$:

$(x - 21)(x + 8)$

Third, divide the second half of each binomial by the coefficient that was multiplied originally:

$\left(x - \frac{21}{6}\right)\left(x + \frac{8}{6}\right)$

Fourth, simplify each fraction completely:

$\left(x - \frac{7}{2}\right)\left(x + \frac{4}{3}\right)$

Lastly, move the denominator of each fraction to become the coefficient of $x$:

$(2x - 7)(3x + 4)$

Note that this is a procedural algorithm that has not been proved in this text. It does work and can be a great time saving tool.

**Example B**

Factor the following expression using two methods different from the method used in Example A: $6x^2 - 13x - 28$

**Solution:** The educated guess and check method can be time consuming, but since there are a finite number of possibilities, it is still possible to check them all. The 6 can be factored into the following four pairs:

1, 6

2, 3

-1, -6

-2, -3

The -28 can be factored into the following twelve pairs:

1, -28 or -28, 1

-1, 28 or 28, -1

2, -14 or -14, 2

-2, 14 or 14, -2

4, -7 or -7, 4

-4, -7 or -7, -4

The correctly factored expression will need a pair from the top list and a pair from the bottom list. This is 48 possible combinations to try.

If you try the first pair from each list and multiply out you will see that the first and the last coefficients are correct but the $b$ coefficient does not.

$(1x + 1)(6x - 28) = 6x - 28x + 6x - 28$

A systematic approach to every one of the 48 possible combinations is the best way to avoid missing the correct pair. In this case it is:

$(2x-7)(3x+4) = 6x^2 + 8x - 21x - 28 = 6x^2 - 13x - 28$

This method can be extremely long and rely heavily on good guessing which is why the algorithm in Example A is provided and preferable.

An alternative method is using the quadratic formula as a clue even though this is not an equation set equal to zero.

$6x^2 - 13x - 28$

$a = 6, b = -13, c = -28$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{13 \pm \sqrt{169-4\cdot 6 \cdot -28}}{2\cdot 6} = \frac{13 \pm 29}{12} = \frac{42}{12}\ or - \frac{16}{12} = \frac{7}{2}\ or\ -\frac{4}{3}$

This means that when set equal to zero, this expression is equivalent to

$\left(x-\frac{7}{2}\right)\left(x+\frac{4}{3}\right) = 0$

Multiplying by 2 and multiplying by 3 only changes the left hand side of the equation because the right hand side will remain 0. This has the effect of shifting the coefficient from the denominator of the fraction to be in front of the $x$ just like in Example A.

$6x^2 - 13x - 28 = (2x-7)(3x+4)$

**Example C**

Factor the following expression using grouping: $12x^2 + 4xz + 3xy + yz$

**Solution:** Notice that the first two terms are divisible by both 4 and $x$ and the last two terms are divisible by $y$. First, factor out these common factors and then notice that there emerges a second layer of common factors. The binomial $(3x+z)$ is now common to both terms and can be factored out just as before.

$$\begin{aligned} 12x^2 + 4xz + 3xy + yz &= 4x(3x+z) + y(3x+z) \\ &= (3x+z)(4x+y) \end{aligned}$$

**Concept Problem Revisited**

The sum or difference of terms with matching odd powers can be factored in a precise pattern because when multiplied out, all intermediate terms cancel each other out.

$$a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$$

When $a$ is distributed: $a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4$

When $b$ is distributed: $+a^4b - a^3b^2 + a^2b^3 - ab^4 + b^5$

Notice all the inside terms cancel: $a^5 + b^5$

## Vocabulary

***To factor*** means to rewrite a polynomial expression given as a sum of terms into a product of factors.

## Guided Practice

1. Show how $a^3 - b^3$ factors by checking the result given in the guidance section.

2. Show how $a^3+b^3$ factors by checking the result given in the guidance section.

3. Factor the following expression without using the quadratic formula or trial and error:

$8x^2+30x+27$

**Answers:**

1. Factoring,

$$\begin{aligned} a^3-b^3 &= (a-b)(a^2+ab+b^2) \\ &= a^3+a^2b+ab^2-a^2b-ab^2-b^3 \\ &= a^3-b^3 \end{aligned}$$

2. Factoring,

$$\begin{aligned} a^3+b^3 &= (a+b)(a^2-ab+b^2) \\ &= a^3-a^2b+ab^2+ba^2-ab^2+b^3 \\ &= a^3+b^3 \end{aligned}$$

3. Use the algorithm described in Example A.

$$\begin{aligned} 8x^2+30x+27 &\to x^2+30x+216 \\ &\to (x+12)(x+18) \\ &\to \left(x+\frac{12}{8}\right)\left(x+\frac{18}{8}\right) \\ &\to \left(x+\frac{3}{2}\right)\left(x+\frac{9}{4}\right) \\ &\to (2x+3)(4x+9) \end{aligned}$$

## Practice

Factor each expression completely.

1. $2x^2-5x-12$
2. $12x^2+5x-3$
3. $10x^2+13x-3$
4. $18x^2+9x-2$
5. $6x^2+7x+2$
6. $8x^2+34x+35$
7. $5x^2+23x+12$
8. $12x^2-11x+2$

Expand the following expressions. What do you notice?

9. $(a+b)(a^8-a^7b+a^6b^2-a^5b^3+a^4b^4-a^3b^5+a^2b^6-ab^7+b^8)$

10. $(a-b)(a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5+b^6)$

11. Describe in words the pattern of the signs for factoring the difference of two terms with matching odd powers.

12. Describe in words the pattern of the signs for factoring the sum of two terms with matching odd powers.

Factor each expression completely.

13. $27x^3-64$

14. $x^5-y^5$

15. $32a^5-b^5$

16. $32x^5+y^5$

17. $8x^3+27$

18. $2x^2+2xy+x+y$

19. $8x^3+12x^2+2x+3$

20. $3x^2+3xy-4x-4y$

# 2.3 Polynomial Expansion and Pascal's Triangle

Here you will explore patterns with binomial and polynomial expansion and find out how to get coefficients using Pascal's Triangle.

The expression $(2x+3)^5$ would take a while to multiply out. Is there a pattern you can use?

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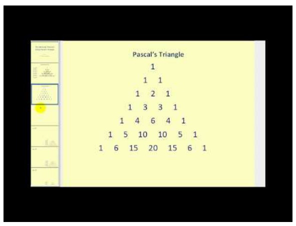


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http://www.youtube.com/watch?v=NLQmQGA4a3M James Sousa: The Binomial Theorem Using Pascal's Triangle

### Guidance

Pascal was a French mathematician in the $17^{th}$ century, but the triangle now named Pascal's Triangle was studied long before Pascal used it. The pattern was used around the $10^{th}$ century in Persia, India and China as well as many other places.

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
\vdots
\end{array}
$$

The primary purpose for using this triangle is to introduce how to expand binomials.

$$
\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2y+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3
\end{aligned}
$$

Notice that the coefficients for the $x$ and $y$ terms on the right hand side line up exactly with the numbers from Pascal's triangle. This means that given $(x+y)^n$ for any power $n$ you can write out the expansion using the coefficients from the triangle. When you study how to count with combinations then you will be able to calculate the value of any coefficient without writing out the whole triangle.

There are many patterns in the triangle. Here are just a few.

1

1 1

1 2 1

1 3 3 1

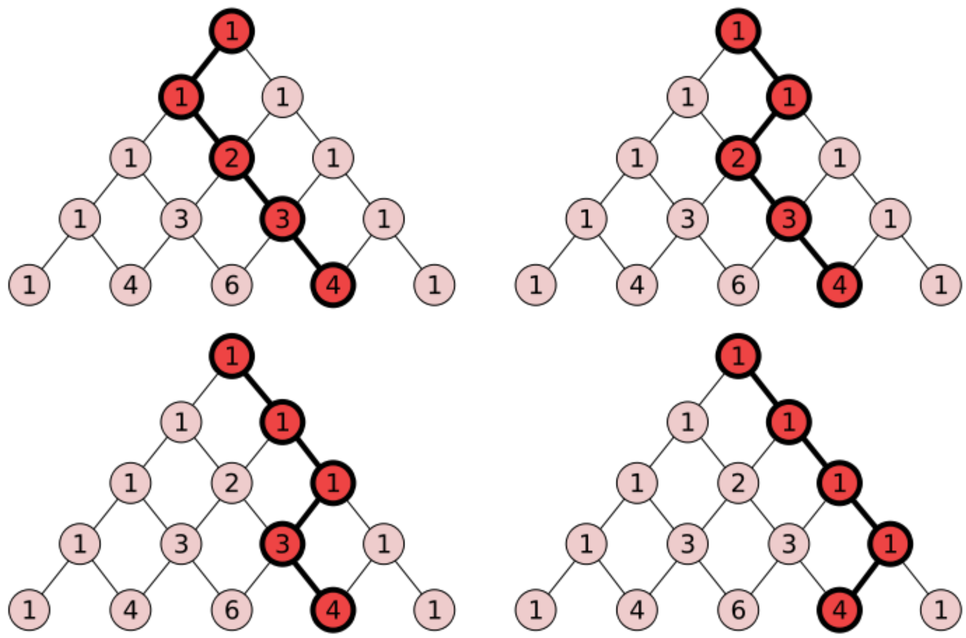
1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

⋮

1. Notice the way each number is created by summing the two numbers above on the left and right hand side.
2. As you go further down the triangle the values in a row approach a bell curve. This is closely related to the normal distribution in statistics.
3. For any row that has a second term that is prime, all the numbers besides 1 in that row are divisible by that prime number.
4. In the game Plinko where an object is dropped through a triangular array of pegs, the probability (which corresponds proportionally to the values in the triangle) of landing towards the center is greater than landing towards the edge. This is because every number in the triangle indicates the number of ways a falling object can get to that space through the preceding numbers.




**Example A**

Expand the following binomial using Pascal's Triangle: $(3x-2)^4$

**Solution:** The coefficients will be 1, 4, 6, 4, 1; however, since there are already coefficients with the $x$ and the constant term you must be particularly careful.

$1\cdot(3x)^4+4\cdot(3x)^3\cdot(-2)+6\cdot(3x)^2\cdot(-2)^2+4\cdot(3x)\cdot(-2)^3+1\cdot(-2)^4$

Then it is only a matter of multiplying out and keeping track of negative signs.

$81x^4-216x^3+216x^2-96x+16$

**Example B**

Expand the following trinomial: $(x+y+z)^4$

**Solution:** Unfortunately, Pascal's triangle does not apply to trinomials. Instead of thinking of a two dimensional triangle, you would need to calculate a three dimensional pyramid which is called Pascal's Pyramid. The sum of all the terms below is your answer.

$$\begin{aligned}
&1x^4+4x^3z+6x^2z^2+4xz^3+1z^4\\
&4x^3y+12x^2yz+12xyz^2+4yz^3\\
&6x^2y^2+12xy^2z+6y^2z^2\\
&4xy^3+4y^3z\\
&1y^4
\end{aligned}$$

Notice how many patterns exist in the coefficients of this layer of the pyramid.

**Example C**

Expand the following binomial: $\left(\frac{1}{2}x-3\right)^5$

**Solution:** You know that the coefficients will be 1, 5, 10, 10, 5, 1.

$$1\left(\frac{1}{2}x\right)^5+5\left(\frac{1}{2}x\right)^4(-3)+10\left(\frac{1}{2}x\right)^3(-3)^2+10\left(\frac{1}{2}x\right)^2(-3)^3+5\left(\frac{1}{2}x\right)(-3)^4+1\cdot(-3)^5$$

$$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{90x^3}{8}-\frac{270x^2}{4}+\frac{405x}{2}-243$$

Remember to simplify fractions.

$$=\frac{x^5}{32}-\frac{15x^4}{16}+\frac{45x^3}{4}-\frac{135x^2}{2}+\frac{405x}{2}-243$$

**Concept Problem Revisited**

Pascal's triangle allows you to identify that the coefficients of $(2x+3)^5$ will be 1, 5, 10, 10, 5, 1 like in Example C. By carefully substituting, the expansion will be:

$$1\cdot(2x)^5+5\cdot(2x)^4\cdot 3+10\cdot(2x)^3\cdot 3^2+10\cdot(2x^2)\cdot 3^3+5(2x)^1\cdot 3^4+3^5$$

Simplifying is a matter of arithmetic, but most of the work is done thanks to the patterns of Pascal's Triangle.

### Vocabulary

A ***binomial expansion*** is a polynomial that can be factored as the power of a binomial.

***Pascal's Triangle*** is a triangular array of numbers that describes the coefficients in a binomial expansion.

### Guided Practice

1. Factor the following polynomial by recognizing the coefficients.

$x^4+4x^3+6x^2+4x+1$

2. Factor the following polynomial by recognizing the coefficients.

$8x^3-12x^2+6x-1$

3. Expand the following binomial using Pascal's Triangle.

$(A-B)^6$

**Answers:**

1. $(x+1)^4$

2. Notice that the first term of the binomial must be $2x$, the last term must be -1 and the power must be 3. Now all that remains is to check.

$(2x-1)^3=(2x)^3+3(2x)^2\cdot(-1)+3(2x)^1(-1)^2+(-1)^3=8x^3-12x^2+6x-1$

3. $(A-B)^6=A^6-6A^5B+15A^4B^2-20A^3B^3+15A^2B^4-6AB^5+B^6$

### Practice

Factor the following polynomials by recognizing the coefficients.

1. $x^2+2xy+y^2$
2. $x^3+3x^2+3x+1$
3. $x^5+5x^4+10x^3+10x^2+5x+1$
4. $27x^3-27x^2+9x-1$
5. $x^3+12x^2+48x+64$

Expand the following binomials using Pascal's Triangle.

6. $(2x-3)^3$
7. $(3x+4)^4$
8. $(x-y)^7$
9. $(a+b)^{10}$
10. $(2x+5)^5$
11. $(4x-1)^4$
12. $(5x+2)^3$
13. $(x+y)^6$
14. $(3x+2y)^3$
15. $(5x-2y)^4$

# 2.4 Rational Expressions

Here you will add, subtract, multiply and divide rational expressions in order to help you solve and graph rational expressions in the future.

A rational expression is a ratio just like a fraction. Instead of a ratio between numbers, a rational expression is a ratio between two expressions. One driving question to ask is:

Are the rules for simplifying and operating on rational expressions are the same as the rules for simplifying and operating on fractions?

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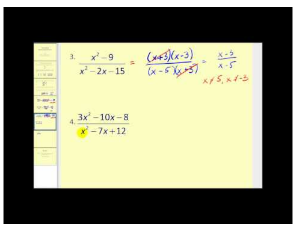

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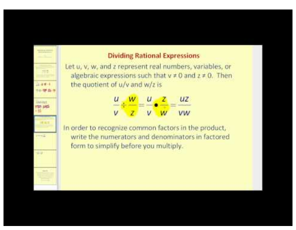
http://www.youtube.com/watch?v=TxbWaDUrYls James Sousa: Simplifying Rational Expressions



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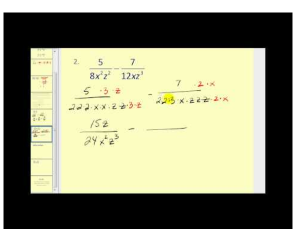
http://www.youtube.com/watch?v=5mcwdhoRSOc James Sousa: Multiplying and Dividing Rational Expressions



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http://www.youtube.com/watch?v=mJ6qOMno4-g James Sousa: Adding and Subtracting Rational Expressions

### Guidance

A rational expression is a ratio of two polynomial expressions. When simplifying or operating on rational expressions, it is vital that each polynomial be fully factored. Once all expressions are factored, identical factors in the numerator and denominator may be canceled. The reason they can be "canceled" is that any expression divided by itself is equal to 1. An identical expression in the numerator and denominator is just an expression being divided by itself, and so equals 1.

- To multiply rational expressions, you should write the product of all the numerator factors over the product of all the denominator factors and then cancel identical factors.
- To divide rational expressions, you should rewrite the division problem as a multiplication problem. Multiply the first rational expression by the reciprocal of the second rational expression. Follow the steps above for multiplying.

To add or subtract rational expressions, it is essential to first find a common denominator. While any common denominator will work, using the least common denominator is a means of keeping the number of additional factors under control. Look at each rational expression you are working with and identify your desired common denominator. Multiply each expression by an appropriate clever form of 1 and then you should have your common denominator.

In both multiplication and division problems answers are most commonly left entirely factored to demonstrate everything has been reduced appropriately. In addition and subtraction problems the numerator must be multiplied, combined and then re-factored. Example B shows you how to finish an addition and subtraction problem appropriately.

**Example A**

Simplify the following rational expression.

$$\frac{x^2+7x+12}{x^2+4x+3} \cdot \frac{x^2+9x+8}{2x^2-128} \div \frac{x+4}{x-8} \cdot \frac{14}{1}$$

**Solution:** First factor everything. Second, turn division into multiplication (only one term). Third, cancel appropriately which will leave the answer.

$$\begin{aligned}
&= \frac{(x+3)(x+4)}{(x+3)(x+1)} \cdot \frac{(x+8)(x+1)}{2(x+8)(x-8)} \cdot \frac{(x-8)}{(x+4)} \cdot \frac{14}{1} \\
&= \frac{\cancel{(x+3)}\cancel{(x+4)}}{\cancel{(x+3)}\cancel{(x+1)}} \cdot \frac{\cancel{(x+8)}\cancel{(x+1)}}{2\cancel{(x+8)}\cancel{(x-8)}} \cdot \frac{\cancel{(x-8)}}{\cancel{(x+4)}} \cdot \frac{14}{1} \\
&= \frac{14}{2} \\
&= 7
\end{aligned}$$

In this example, the strike through is shown. You should use this technique to match up factors in the numerator and the denominator when simplifying.

**Example B**

Combine the following rational expressions.

$$\frac{x^2-9}{x^4-81} - \frac{4x}{x^2+9} \div \frac{x-3}{2}$$

**Solution:**

$$\frac{\cancel{(x+3)(x-3)}}{(x^2+9)\cancel{(x+3)(x-3)}} - \frac{4x}{(x^2+9)} \cdot \frac{2}{(x-3)}$$
$$= \frac{1}{(x^2+9)} - \frac{8x}{(x^2+9)(x-3)}$$
$$= \frac{(x-3)}{(x^2+9)(x-3)} - \frac{8x}{(x^2+9)(x-3)}$$
$$= \frac{(-7x-3)}{(x^2+9)(x-3)}$$

The numerator cannot factor at this point, so in this example there is not a factor that cancels at the end. Remember that the sum of perfect squares does not factor.

**Example C**

Combine the following rational expressions.

$$\frac{1}{x^2+5x+6} - \frac{1}{x^2-4} + \frac{(x-7)(x+5)+5}{(x+2)(x-2)(x+3)(x-4)}$$

**Solution:** First factor everything and decide on a common denominator. While the numerators do not really need to be factored, it is sometimes helpful in simplifying individual expressions before combining them. Note that the numerator of the expression on the right hand seems factored but it really is not. Since the 5 is not connected to the rest of the numerator through multiplication, that part of the expression needs to be multiplied out and like terms need to be combined.

$$= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+2)(x-2)} + \frac{x^2-2x-35+5}{(x+2)(x-2)(x+3)(x-4)}$$
$$= \frac{1}{(x+2)(x+3)} - \frac{1}{(x+2)(x-2)} + \frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$$

Note that the right expression has 4 factors in the denominator while each of the left expressions have two that match and two that are missing from those four factors. This tells you what you need to multiply each expression by in order to have the denominators match up.

$$= \frac{(x-2)(x-4)}{(x+2)(x-2)(x+3)(x-4)} - \frac{(x+3)(x-4)}{(x+2)(x-2)(x+3)(x-4)} + \frac{x^2-2x-30}{(x+2)(x-2)(x+3)(x-4)}$$

Now since the rational expressions have a common denominator, the numerators may be multiplied out and combined. Sometimes instead of rewriting an expression repeatedly in mathematics you can use an abbreviation. In this case, you can replace the denominator with the letter $D$ and then replace it at the end.

$$= \frac{(x-2)(x-4)-(x+3)(x-4)+x^2-2x-30}{D}$$
$$= \frac{x^2-6x+8-[x^2-x-12]+x^2-2x-30}{D}$$

Notice how it is extremely important to use brackets to indicate that the subtraction applies to all the terms of the middle expression not just $x^2$. This is one of the most common mistakes.

$$= \frac{x^2-6x+8-x^2+x+12+x^2-2x-30}{D}$$
$$= \frac{x^2-8x-10}{D}$$

After the numerator has been entirely simplified try to factor the remaining expression and see if anything cancels with the denominator which you now need to replace.

$$= \frac{\cancel{(x+2)}(x-5)}{\cancel{(x+2)}(x-2)(x+3)(x-4)}$$
$$= \frac{(x-5)}{(x-2)(x+3)(x-4)}$$

**Concept Problem Revisited**

Rational expressions are an extension of fractions and the operations of simplifying, adding, subtracting, multiplying and dividing work in exactly the same way.

### Vocabulary

A ***rational expression*** is a ratio of two polynomial expressions.

### Guided Practice

1. Simplify the following expression.

$\frac{\frac{1}{x+1}-\frac{1}{x+2}}{\frac{1}{x-2}+\frac{1}{x+1}}$

2. Subtract the following rational expressions.

$\frac{x-2}{x+3} - \frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)}$

3. Simplify the following expression which has an infinite number of fractions nested within other fractions.

$\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\dots}}}}}$

**Answers:**

1. The expression itself does not look like a rational expression, but it can be rewritten so it is more recognizable. Also working with fractions within fractions is an important skill.

$$\begin{aligned}
&= \left(\frac{1}{x+1} - \frac{1}{x+2}\right) \div \left(\frac{1}{x-2} + \frac{1}{x+1}\right) \\
&= \left[\frac{(x+2)}{(x+1)(x+2)} - \frac{(x+1)}{(x+1)(x+2)}\right] \div \left[\frac{(x+1)}{(x+1)(x-2)} - \frac{(x-2)}{(x+1)(x-2)}\right] \\
&= \left[\frac{1}{(x+1)(x+2)}\right] \div \left[\frac{3}{(x+1)(x-2)}\right] \\
&= \frac{1}{(x+1)(x+2)} \cdot \frac{(x+1)(x-2)}{3} \\
&= \frac{(x-2)}{3(x+2)}
\end{aligned}$$

2. Being able to factor effectively is of paramount importance.

$$\begin{aligned}
&= \frac{x-2}{x+3} - \frac{x^3-3x^2+8x-24}{2(x+2)(x^2-9)} \\
&= \frac{(x-2)}{(x+3)} - \frac{x^2(x-3)+8(x-3)}{2(x+2)(x^2-9)} \\
&= \frac{(x-2)}{(x+3)} - \frac{(x-3)(x^2+8)}{2(x+2)(x+3)(x-3)}
\end{aligned}$$

Before subtracting, simplify where possible so you don't contribute to unnecessarily complicated denominators.

$$= \frac{(x-2)}{(x+3)} - \frac{x^2+8}{2(x+2)(x+3)}$$

The left expression lacks $2(x+2)$, so multiply both its numerator and denominator by $2(x+2)$.

$$\begin{aligned}
&= \frac{2(x+2)(x-2)}{2(x+2)(x+3)} - \frac{(x^2+8)}{2(x+2)(x+3)} \\
&= \frac{2(x^2-4)-x^2-8}{2(x+2)(x+3)} \\
&= \frac{x^2-16}{2(x+2)(x+3)}
\end{aligned}$$

3. It would be an exercise in futility to attempt to try to compute this expression directly. Instead, notice that the repeating nature of the expression lends itself to an extremely nice substitution.

Let $\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\frac{-1}{2+\dots}}}}} = x$

Notice that the red portion of the fraction is exactly the same as the rest of the fraction and so $x$ may be substituted there and solved.

$$\frac{-1}{2+x} = x$$
$$-1 = x(2+x)$$
$$-1 = x^2+2x$$
$$0 = x^2+2x+1$$
$$0 = (x+1)^2$$
$$x = -1$$

The reason why this expression is included in this concept is because it highlights one problem solving aspect of simplifying expressions that distinguishes PreCalculus from Algebra 1 and Algebra 2.

### Practice

Perform the indicated operation and simplify as much as possible.

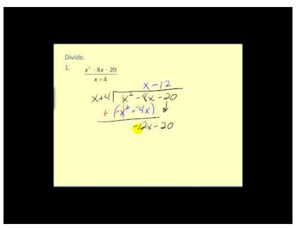
1. $\frac{x^2+5x+4}{x^2+4x+3} \cdot \frac{5x^2+15x}{x+4}$
2. $\frac{x^2-4}{x^2+4x+4} \cdot \frac{7}{x-2}$
3. $\frac{4x^2-12x}{5x+10} \cdot \frac{x+2}{x} \div \frac{x-3}{1}$
4. $\frac{4x^3-4x}{x} \div \frac{2x-2}{x-4}$
5. $\frac{2x^3+8x}{x+1} \div \frac{x}{2x^2-2}$
6. $\frac{3x-1}{x^2+2x-15} - \frac{2}{x+5}$
7. $\frac{x^2-8x+7}{x^2-4x-21} \cdot \frac{x^2-9}{1-x^2}$
8. $\frac{2}{x+7} + \frac{1}{x-7}$
9. $\frac{6}{x-7} - \frac{6}{x+7}$
10. $\frac{3x+35}{x^2-25} + \frac{2}{x+5}$
11. $\frac{2x+20}{x^2-4x-12} + \frac{2}{x+2}$
12. $\frac{2}{x+6} - \frac{x-9}{x^2-3x-18}$
13. $-\frac{5x+30}{x^2+11x+30} + \frac{2}{x+5}$
14. $\frac{x+3}{x+2} + \frac{x^3+4x^2+5x+20}{2x^4+2x^2-40}$
15. $\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\frac{-4}{2+\ldots}}}}}$

# 2.5 Polynomial Long Division and Synthetic Division

Here you will learn how to perform long division with polynomials. You will see how synthetic division abbreviates this process. In addition to mastering this procedure, you will see the how the remainder root theorem and the rational root theorem operate.

While you may be experienced in factoring, there will always be polynomials that do not readily factor using basic or advanced techniques. How can you identify the roots of these polynomials?

### Watch This



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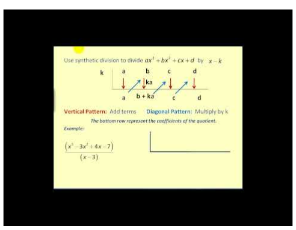
http://www.youtube.com/watch?v=brpNxPAkv1c James Sousa: Dividing Polynomials-Long Division



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http://www.youtube.com/watch?v=5dBAdzl2Mns James Sousa: Dividing Polynomials-Synthetic Division

### Guidance

There are numerous theorems that point out relationships between polynomials and their factors. For example there is a theorem that a polynomial of degree $n$ must have exactly $n$ solutions/factors that may or may not be real numbers. The **Rational Root Theorem** and the **Remainder Theorem** are two theorems that are particularly useful starting places when manipulating polynomials.

- The **Rational Root Theorem** states that in a polynomial, every rational solution can be written as a reduced fraction $\left(x = \frac{p}{q}\right)$, where $p$ is an integer factor of the constant term and $q$ is an integer factor of the leading coefficient. Example A shows how all the possible rational solutions can be listed using the Rational Root Theorem.
- The **Remainder Theorem** states that the remainder of a polynomial $f(x)$ divided by a linear divisor $(x-a)$ is equal to $f(a)$. The Remainder Theorem is only useful after you have performed polynomial long division because you are usually never given the divisor and the remainder to start. The main purpose of the Remainder Theorem in this setting is a means of double checking your application of polynomial long division. Example B shows how the Remainder Theorem is used.

Polynomial long division is identical to regular long division. Synthetic division is a condensed version of regular long division where only the coefficients are kept track of. In Example B polynomial long division is used and in Example C synthetic long division is used.

**Example A**

Identify all possible rational solutions of the following polynomial using the Rational Root Theorem.

$12x^{18}-91x^{17}+x^{16}+\cdots+2x^2-14x+5=0$

**Solution:** The integer factors of 12 are and 1, 2, 3, 4, 6 and 12. The integer factors of 5 are 1, 5. Since pairs of factors could both be negative, remember to include $\pm$.

$\pm\frac{p}{q}=\frac{1}{1},\frac{1}{5},\frac{2}{1},\frac{2}{5},\frac{3}{1},\frac{3}{5},\frac{4}{1},\frac{4}{5},\frac{6}{1},\frac{6}{5},\frac{12}{1},\frac{12}{5}$

While narrowing the possible solutions down to 24 possible rational answers may not seem like a big improvement, it surely is. This is especially true considering there are only a handful of integer solutions. If this question required you to find a solution, then the Rational Root Theorem would give you a great starting place.

**Example B**

Use Polynomial Long Division to divide:

$\frac{x^3+2x^2-5x+7}{x-3}$

**Solution:** First note that it is clear that 3 is not a root of the polynomial because of the Rational Root Theorem and so there will definitely be a remainder. Start a polynomial long division question by writing the problem like a long division problem with regular numbers:

$x-3\overline{)x^3+2x^2-5x+7}$

Just like with regular numbers ask yourself "how many times does $x$ go into $x^3$?" which in this case is $x^2$.

$$\begin{array}{r} x^2 \\ x-3\overline{)x^3+2x^2-5x+7} \end{array}$$

Now multiply the $x^2$ by $x-3$ and copy below. Remember to subtract the entire quantity.

$$\begin{array}{r} x^2\phantom{+2x^2-5x+7} \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)}\phantom{-5x+7} \end{array}$$

Combine the rows, bring down the next number and repeat.

$$\begin{array}{r} x^2+5x+10 \\ x-3\overline{)x^3+2x^2-5x+7} \\ \underline{-(x^3-3x^2)}\phantom{-5x+7} \\ 5x^2-5x\phantom{+7} \\ \underline{-(5x^2-15x)}\phantom{+7} \\ 10x+7 \\ \underline{-(10x-30)} \\ 37 \end{array}$$

The number 37 is the remainder. There are two things to think about at this point. First, interpret in an equation:

$\frac{x^3+2x^2-5x+7}{x-3}=(x^2+5x+10)+\frac{37}{x-3}$

Second, check your result with the Remainder Theorem which states that the original function evaluated at 3 must be 37. Notice the notation indicating to substitute 3 in for $x$.

$(x^3+2x^2-5x+7)|_{x=3} = 3^3+2\cdot 3^2-5\cdot 3+7 = 27+18-15+7 = 37$

**Example C**

Use Synthetic Division to divide the same rational expression as the previous example.

**Solution:** Synthetic division is mostly used when the leading coefficients of the numerator and denominator are equal to 1 and the divisor is a first degree binomial.

$\frac{x^3+2x^2-5x+7}{x-3}$

Instead of continually writing and rewriting the $x$ symbols, synthetic division relies on an ordered spacing.

$+3\overline{)1 \quad 2 \quad -5 \quad 7}$

Notice how only the coefficients for the denominator are used and the divisor includes a positive three rather than a negative three. The first coefficient is brought down and then multiplied by the three to produce the value which goes beneath the 2.

$$\begin{array}{l} +3\overline{)1 \quad 2 \quad -5 \quad 7} \\ \underline{\downarrow \; 3} \\ 1 \end{array}$$

Next the new column is added. $2+3=5$, which goes beneath the $2^{nd}$ column. Now, multiply $5\cdot +3=15$, which goes underneath the -5 in the $3^{rd}$ column. And the process repeats...

$$\begin{array}{rrrrr} +3) & 1 & 2 & -5 & 7 \\ & \downarrow & 3 & 15 & 30 \\ \hline & 1 & 5 & 10 & 37 \end{array}$$

The last number, 37, is the remainder. The three other numbers represent the quadratic that is identical to the solution to Example B.

$1x^2+5x+10$

**Concept Problem Revisited**

Identifying roots of polynomials by hand can be tricky business. The best way to identify roots is to use the rational root theorem to quickly identify likely candidates for solutions and then use synthetic or polynomial long division to quickly and effectively test them to see if their remainders are truly zero.

## Vocabulary

***Polynomial long division*** is a procedure with rules identical to regular long division. The only difference is the dividend and divisor are polynomials.

***Synthetic division*** is an abbreviated version of polynomial long division where only coefficients are used.

## Guided Practice

1. Divide the following polynomials.

$\frac{x^3+2x^2-4x+8}{x-2}$

2. Completely factor the following polynomial.

$x^4+6x^3+3x^2-26x-24$

3. Divide the following polynomials.

$\frac{3x^5-2x^2+10x-5}{x-1}$

**Answers:**

1. $\frac{x^3+2x^2-4x+8}{x-2} = x^2+4x+4+\frac{16}{x-2}$

2. Notice that possible roots are $\pm 1, 2, 3, 4, 6, 8, 24$. Of these 14 possibilities, four will yield a remainder of zero. When you find one, repeat the process.

$$\begin{aligned} & x^4+6x^3+3x^2-26x-24 \\ & = (x+1)(x^3+5x^2-2x-4) \\ & = (x+1)(x-2)(x^2+7x+12) \\ & = (x+1)(x-2)(x+3)(x+4) \end{aligned}$$

3. $\frac{3x^5-2x^2+10x-5}{x-1} = 3x^4+3x^3+3x^2+x+11+\frac{6}{x-1}$

### Practice

Identify all possible rational solutions of the following polynomials using the Rational Root Theorem.

1. $15x^{14}-12x^{13}+x^{12}+\cdots+2x^2-5x+5=0$
2. $18x^{11}+42x^{10}+x^9+\cdots+x^2-3x+7=0$
3. $12x^{16}+11x^{15}+3x^{14}+\cdots+6x^2-2x+11=0$
4. $14x^7-7x^6+x^5+\cdots+x^2+6x+3=0$
5. $9x^9-10x^8+3x^7+\cdots+4x^2-2x+2=0$

Completely factor the following polynomials.

6. $2x^4-x^3-21x^2-26x-8$
7. $x^4+7x^3+5x^2-31x-30$
8. $x^4+3x^3-8x^2-12x+16$
9. $4x^4+19x^3-48x^2-117x-54$
10. $2x^4+17x^3-8x^2-173x+210$

Divide the following polynomials.

11. $\frac{x^4+7x^3+5x^2-31x-30}{x+4}$
12. $\frac{x^4+7x^3+5x^2-31x-30}{x+2}$
13. $\frac{x^4+3x^3-8x^2-12x+16}{x+3}$
14. $\frac{2x^4-x^3-21x^2-26x-8}{x^3-x^2-10x-8}$
15. $\frac{x^4+8x^3+3x^2-32x-28}{x^3+10x^2+23x+14}$

# 2.6 Solving Rational Equations

Here you will extend your knowledge of linear and quadratic equations to rational equations in general. You will gain insight as to what extraneous solutions are and how to identify them.

The techniques for solving rational equations are extensions of techniques you already know. Recall that when there are fractions in an equation you can multiply through by the denominator to clear the fraction. The same technique helps turn rational expressions into polynomials that you already know how to solve. When you multiply by a constant there is no problem, but when you multiply through by a value that varies and could possibly be zero interesting things happen.

Since every equation is trivially true when both sides are multiplied by zero, how do you account for this when solving rational equations?

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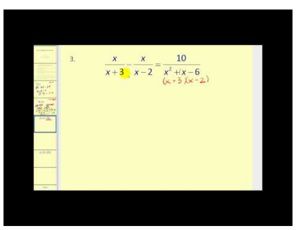

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http://www.youtube.com/watch?v=MMORnvhR4wA James Sousa: Solving Rational Equations

### Guidance

The first step in solving rational equations is to transform the equation into a polynomial equation. This is accomplished by clearing the fraction which means multiplying the entire equation by the common denominator of all the rational expressions. Then you should solve using what you already know. The last thing to check once you have the solutions is that they do not make the denominators of any part of the equation equal to zero when substituted back into the original equation. If so, that solution is called **extraneous** and is a "fake" solution that was introduced when both sides of the equation were multiplied by a number that happened to be zero.

**Example A**

Solve the following rational equation.

$x-\frac{5}{x+3}=12$

**Solution:** Multiply all parts of the equation by $(x+3)$, the common denominator.

$$
\begin{aligned}
x(x+3)-5 &= 12(x+3) \\
x^2+3x-5-12x-36 &= 0 \\
x^2-9x-41 &= 0 \\
x &= \frac{-(-9) \pm \sqrt{(-9)^2-4\cdot 1\cdot(-41)}}{2\cdot 1} \\
x &= \frac{9 \pm 7\sqrt{5}}{2}
\end{aligned}
$$

The only potential extraneous solution would have been -3, so both answers are possible.

**Example B**

Solve the following rational equation

$\frac{3x}{x+4} - \frac{1}{x+2} = -\frac{2}{x^2+6x+8}$

**Solution:** Multiply each part of the equation by the common denominator of $x^2+6x+8 = (x+2)(x+4)$.

$$
\begin{aligned}
(x+2)(x+4)\left[\frac{3x}{x+4} - \frac{1}{x+2}\right] &= \left[\frac{-2}{(x+2)(x+4)}\right](x+2)(x+4) \\
3x(x+2)-(x+4) &= -2 \\
3x^2+6x-x-2 &= 0 \\
3x^2+5x-2 &= 0 \\
(3x-1)(x+2) &= 0 \\
x &= \frac{1}{3}, -2
\end{aligned}
$$

Note that -2 is an extraneous solution. The only actual solution is $x = \frac{1}{3}$.

**Example C**

Solve the following rational equation for $y$.

$x = 2 + \frac{1}{2+\frac{1}{y+1}}$

**Solution:** This question can be done multiple ways. You can use the clearing fractions technique twice.

$$
\begin{aligned}
\left(2+\frac{1}{y+1}\right)x &= \left[2+\frac{1}{2+\frac{1}{y+1}}\right]\left(2+\frac{1}{y+1}\right) \\
2x+\frac{x}{y+1} &= 2\left(2+\frac{1}{y+1}\right)+1 \\
2x+\frac{x}{y+1} &= 4+\frac{2}{y+1}+1 \\
(y+1)\left[2x+\frac{x}{y+1}\right] &= \left[5+\frac{2}{y+1}\right](y+1) \\
2x(y+1)+x &= 5(y+1)+2 \\
2xy+2x+x &= 5y+5+2
\end{aligned}
$$

Now just get the $y$ variable to one side of the equation and everything else to the other side.

$$\begin{aligned}2xy-5y&=-3x+7\\ y(2x-5)&=-3x+7\\ y&=\frac{-3x+7}{2x-5}\end{aligned}$$

**Concept Problem Revisited**

In order to deal with extra solutions introduced when both sides of an equation are multiplied by a variable, you must check each solution to see if it makes the denominator of any fraction in the original equation zero. If it does, it is called an extraneous solution.

## Vocabulary

An ***extraneous solution*** is a "fake" solution to a rational equation that is introduced when both sides of an equation are multiplied through by zero.

A ***rational equation*** is an equation with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

## Guided Practice

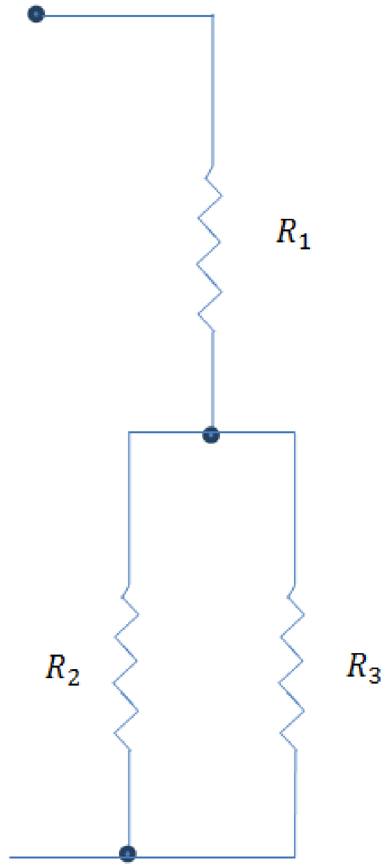
1. Solve the following rational equation.

$\frac{3x}{x-5}+4=x$

2. In electrical circuits, resistance can be solved for using rational expressions. This is an electric circuit diagram with three resistors. The first resistor $R_1$ is run in series to the other two resistors $R_2$ and $R_3$ which are run in parallel. If the total resistance $R$ is 100 ohms and $R_1$ and $R_3$ are each 22 ohms, what is the resistance of $R_2$?

The equation of value is:

$$R=R_1+\frac{R_2R_3}{R_2+R_3}$$




3. Solve the following rational equation.

$$\frac{x+2}{x} - \frac{3}{x+2} = \frac{6}{x^2+2x}$$

**Answers:**

1. $\frac{3x}{x-5} + 4 = x$

$$\begin{aligned} 3x + 4x - 20 &= x^2 - 5x \\ 0 &= x^2 - 12x + 20 \\ 0 &= (x-2)(x-10) \\ x &= 2, 10 \end{aligned}$$

Neither solution is extraneous.

2. $R = R_1 + \frac{R_2 R_3}{R_2 + R_3}$

$$\begin{aligned} 100 &= 22 + \frac{x \cdot 22}{x+22} \\ 78(x+22) &= 22x \\ 78x + 1716 &= 22x \\ 56x &= -1716 \\ x &= -30.65 \end{aligned}$$

A follow up question would be to ask whether or not ohms can be negative which is beyond the scope of this text.

3. $\frac{x+2}{x} - \frac{3}{x+2} = \frac{6}{x^2+2x}$

$$\begin{aligned}(x+2)(x+2) - 3x &= 6 \\ x^2 + 4x + 4 - 3x - 6 &= 0 \\ x^2 + x - 2 &= 0 \\ (x+2)(x-1) &= 0 \\ x &= -2, 1\end{aligned}$$

Note that -2 is an extraneous solution.

### Practice

Solve the following rational equations. Identify any extraneous solutions.

1. $\frac{2x-4}{x} = \frac{16}{x}$
2. $\frac{4}{x+1} - \frac{x}{x+1} = 2$
3. $\frac{5}{x+3} + \frac{2}{x-3} = 1$
4. $\frac{3}{x-4} - \frac{5}{x+4} = 6$
5. $\frac{x}{x+1} - \frac{6}{x+2} = 4$
6. $\frac{x}{x-4} - \frac{4}{x-4} = 8$
7. $\frac{4x}{x-2} + 3 = 1$
8. $\frac{-2x}{x+1} + 6 = -x$
9. $\frac{1}{x+2} + 1 = -2x$
10. $\frac{-6x-3}{x+1} - 3 = -4x$
11. $\frac{x+3}{x} - \frac{3}{x+3} = \frac{6}{x^2+3x}$
12. $\frac{x-4}{x} - \frac{2}{x-4} = \frac{8}{x^2-4x}$
13. $\frac{x+6}{x} - \frac{2}{x+6} = \frac{12}{x^2+6x}$
14. $\frac{x+5}{x} - \frac{3}{x+5} = \frac{15}{x^2+5x}$
15. Explain what it means for a solution to be extraneous.

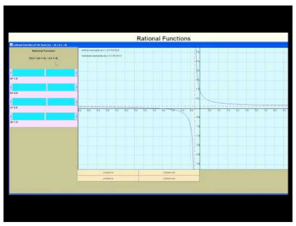
# 2.7 Holes in Rational Functions

Here you will start factoring rational expressions that have holes known as removable discontinuities.

In a function like $f(x) = \frac{(3x+1)(x-1)}{(x-1)}$, you should note that the factor $(x-1)$ clearly cancels leaving only $3x-1$. This appears to be a regular line. What happens to this line at $x = 1$?

### Watch This

Watch the first part of this video. Focus on how to identify the holes.



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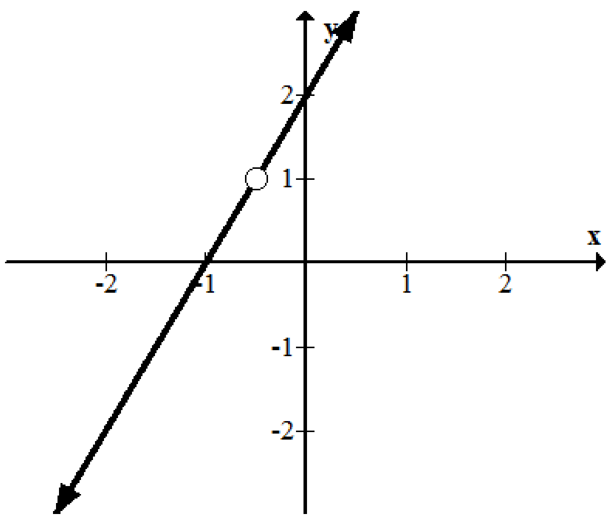
http://www.youtube.com/watch?v=OEQnQNvJtG0 James Sousa: Graphing Rational Functions

### Guidance

A hole on a graph looks like a hollow circle. It represents the fact that the function approaches the point, but is not actually defined on that precise $x$ value.




The reason why this function is not defined at $-\frac{1}{2}$ is because $-\frac{1}{2}$ is not in the domain of the function.

$f(x) = (2x+2) \cdot \frac{\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)}$

As you can see, $f\left(-\frac{1}{2}\right)$ is undefined because it makes the denominator of the rational part of the function zero which makes the whole function undefined. Also notice that once the factors are canceled/removed then you are left with a normal function which in this case is $2x+2$.

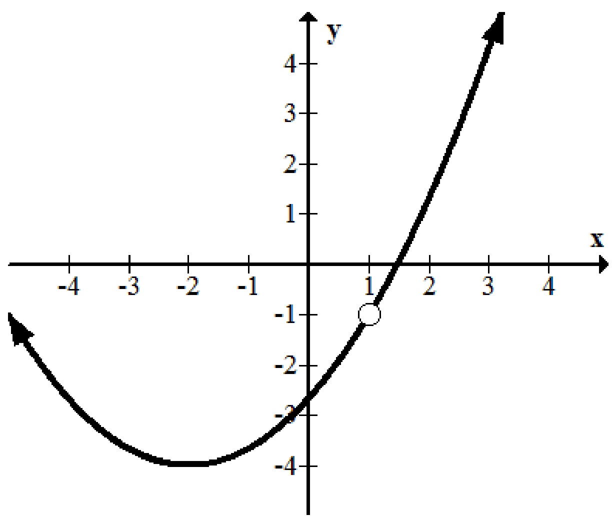
This is the essence of dealing with holes in rational functions. You should cancel what you can and graph the function like normal making sure to note what $x$ values make the function undefined. Once the function is graphed without holes go back and insert the hollow circles indicating what $x$ values are removed from the domain. This is why holes are called removable discontinuities.

**Example A**

Graph the following rational function and identify any removable discontinuities.

$f(x) = \left(\frac{1}{3}(x+2)^2 - 4\right) \cdot \frac{(x-1)}{(x-1)}$

**Solution:** This function is already separated into two parts, the rational part and a parabola. To graph the function, simply graph the parabola and then insert a hollow circle at the appropriate height at $x = 1$.




The hole is at (1, -1) because after removing the factors that cancel, $f(1) = -1$.

Note that most problems will require significant algebraic steps to reach this point. This example emphasizes that the backbone of the function is essentially a parabola with only one difference.

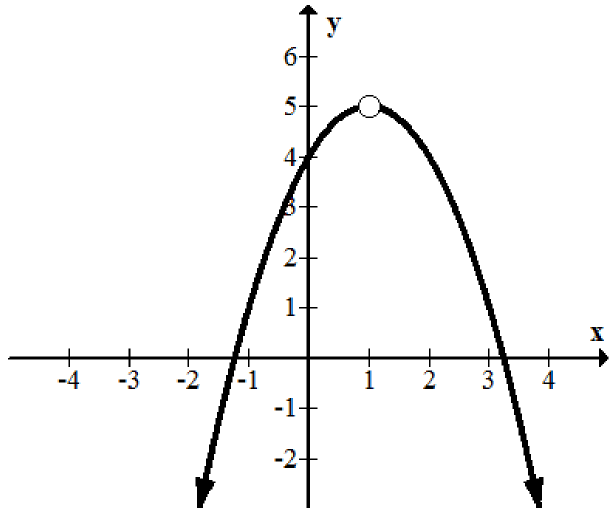
**Example B**

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{-x^3+3x^2+2x-4}{x-1}$

**Solution:** This function requires some algebra to change it so that the removable factors become obvious. You should suspect that $(x-1)$ is a factor of the numerator and try polynomial or synthetic division to factor. When you do, the function becomes:

$f(x) = \frac{(-x^2+2x+4)(x-1)}{(x-1)}$




The removable discontinuity occurs at (1, 5).

**Example C**

Graph the following rational function and identify any removable discontinuities.

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{x^3-7x-6}$

**Solution:** This is probably one of the most challenging rational expressions with only holes that people ever try to graph by hand. There are multiple ways to start, but a good habit to get into is to factor everything you possibly can initially. The denominator seems less complicated with possible factors $(x \pm 1), (x \pm 2), (x \pm 3), (x \pm 6)$. Using polynomial division, you will find the denominator becomes:

$f(x) = \frac{x^6-6x^5+5x^4+27x^3-48x^2-9x+54}{(x+1)(x+2)(x-3)}$

The factors of the denominator are strong hints as to the factors of the numerator so use polynomial division and try each. When you fully factor the numerator you will have:

$f(x) = \frac{(x^3-6x^2+12x-9)(x+1)(x+2)(x-3)}{(x+1)(x+2)(x-3)}$

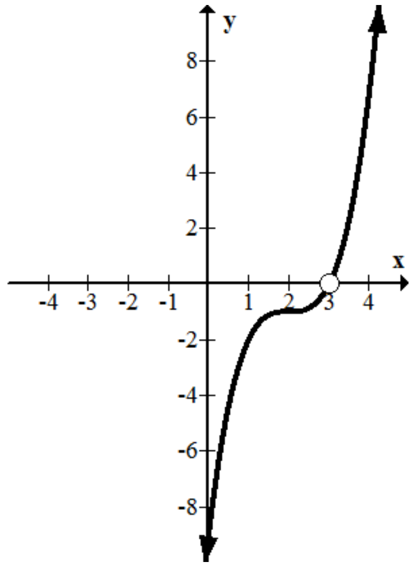
Note the factors that cancel $(x = -1, -2, 3)$ and then work with the cubic function that remains.

$f(x) = x^3 - 6x^2 + 12x - 9$

At this point it is probably reasonable to make a table and plot points to get a sense of where this cubic function lives. You also could notice that the coefficients are almost of the pattern 1 3 3 1 which is the binomial expansion. By separating the -9 into -8 -1 you can factor the first four terms.

$f(x) = x^3 - 6x^2 + 12x - 8 - 1 = (x-2)^3 - 1$

This is a cubic function that has been shifted right by two units and down one unit.




Note that there are two holes that do not fit in the graph window. When this happens you still need to note where they would appear given a properly sized window. To do this, substitute the invalid $x$ values: $x = -1, -2, 3$ into the factored cubic that remained after canceling.

$f(x) = (x-2)^3 - 1$

Holes: (3, 0); (-1, -28); (-2, -65)

**Concept Problem Revisited**

$f(x) = \frac{(3x+1)(x-1)}{(x-1)}$

For this function that is not defined at $x = 1$ there is a removable discontinuity that is represented as a hollow circle on the graph. Otherwise the function behaves precisely as $3x+1$.

### Vocabulary

A ***removable discontinuity,*** also known as a hole, is a point on a function that occurs because a factor can be canceled from the numerator and the denominator of the rational function.

A ***rational function*** is a function with at least one rational expression.

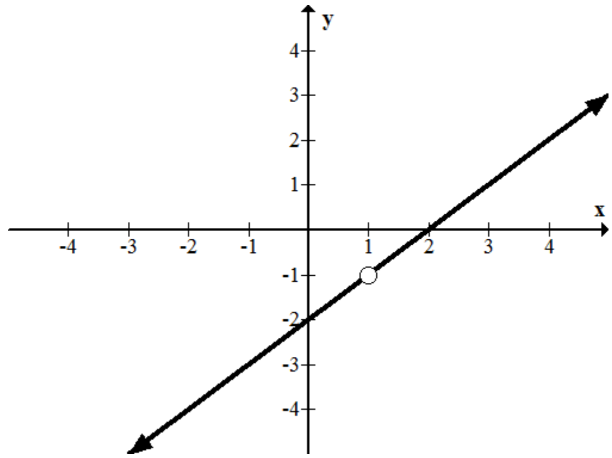
A ***rational expression*** is a ratio of two polynomial expressions.

### Guided Practice

1. Without graphing, identify the location of the holes of the following function.

$f(x) = \frac{x^3+4x^2+x-6}{x^2+5x+6}$

2. What is a possible equation for the following rational function?




3. Identify the holes of the following function.

$f(x) = x \cdot \frac{\sin x}{\sin x}$

**Answers:**

1. First factor everything. Then, identify the $x$ values that make the denominator zero and use those values to find the exact location of the holes.

$f(x) = \frac{(x+2)(x+3)(x-1)}{(x+3)(x+2)}$

Holes: (-3, -4); (-2, -3)

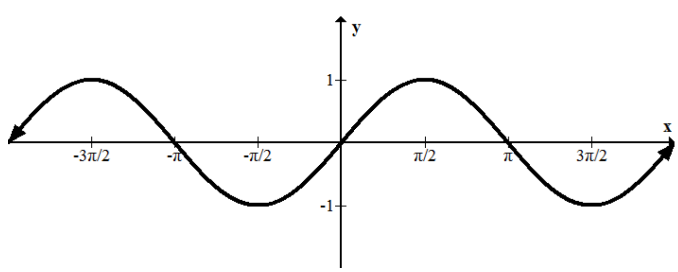
2. The function seems to be a line with a removable discontinuity at (1, -1). The line is has slope 1 and $y$-intercept of -2 and so has the equation:

$f(x) = x - 2$

The removable discontinuity must not allow the $x$ to be 1 which implies that it is of the form $\frac{x-1}{x-1}$. Therefore, the function is:

$f(x) = \frac{(x-2)(x-1)}{x-1}$

3. While the function is not a rational function because it includes a trigonometric expression, the exact same tools apply. You should ask yourself: when is the sine function equal to zero? Since the sine function is one of the basic functions you can sketch the function and note that it has a height of 0 at $0, \pm\pi, \pm 2\pi \ldots$




Since the function is just the line $f(x) = x$ with holes everywhere the sine function is zero, there are an infinite number of holes. The holes occur at $(0,0), (\pi,\pi), (-\pi,-\pi), (2\pi,2\pi)\ldots$

## Practice

1. How do you find the holes of a rational function?

2. What's the difference between a hole and a removable discontinuity?

3. If you see a hollow circle on a graph, what does that mean?

Without graphing, identify the location of the holes of the following functions.

4. $f(x) = \frac{x^2+3x-4}{x-1}$

5. $g(x) = \frac{x^2+8x+15}{x+3}$

6. $h(x) = \frac{x^3+6x^2+2x-8}{x^2+x-2}$

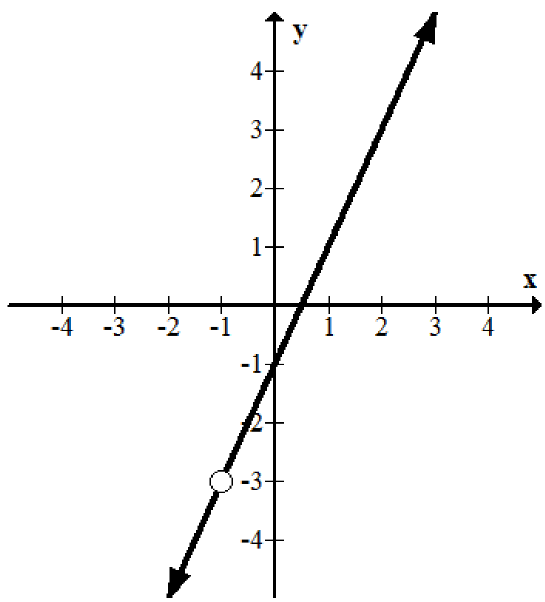
7. $k(x) = \frac{x^3+6x^2+2x-8}{x^2-3x+2}$

8. $j(x) = \frac{x^3+4x^2-17x-60}{x^2-9}$

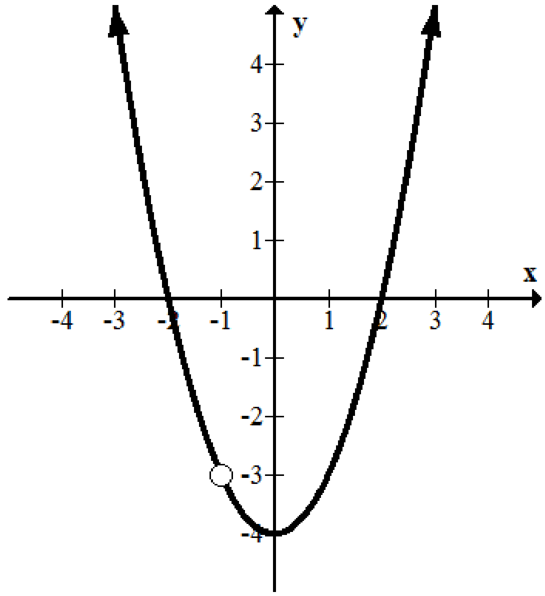
9. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-5x+4}$

10. $g(x) = \frac{x^3-4x^2-19x-14}{x^2-8x+7}$

11. What is a possible equation for the following rational function?




12. What is a possible equation for the following rational function?




Sketch the following rational functions.

13. $f(x) = \frac{x^3+4x^2-17x-60}{x^2-x-12}$

14. $g(x) = \frac{x^3+4x^2-17x-60}{x^2+8x+15}$

15. $h(x) = \frac{x^3-4x^2-19x-14}{x^2-6x-7}$

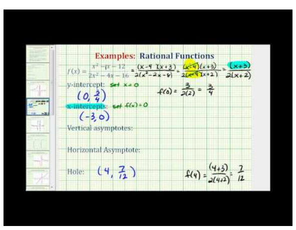
# 2.8 Zeroes of Rational Functions

Here you will revisit how to find zeroes of functions by focusing on rational expressions and what to do in special cases where the zeroes and holes seem to overlap.

The zeroes of a function are the collection of $x$values where the height of the function is zero. How do you find these values for a rational function and what happens if the zero turns out to be a hole?

### Watch This

Focus on the portion of this video discussing holes and $x$-intercepts.




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=UnVZs2EaEjI James Sousa: Find the Intercepts, Asymptotes, and Hole of a Rational Function

### Guidance

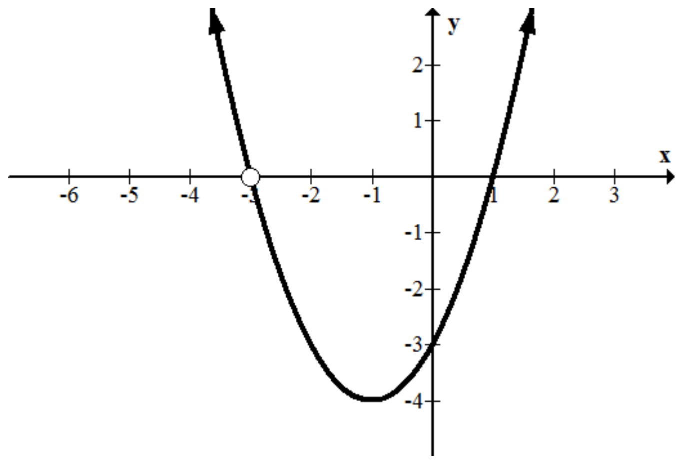
Zeroes are also known as $x$-intercepts, solutions or roots of functions. They are the $x$ values where the height of the function is zero. For rational functions, you need to set the numerator of the function equal to zero and solve for the possible $x$ values. If a hole occurs on the $x$ value, then it is not considered a zero because the function is not truly defined at that point.

**Example A**

Identify the zeroes and holes of the following rational function.

$f(x) = \frac{(x-1)(x+3)(x+3)}{x+3}$

**Solution:** Notice how one of the $x+3$ factors seems to cancel and indicate a removable discontinuity. Even though there are two $x+3$ factors, the only zero occurs at $x=1$ and the hole occurs at (-3, 0).




**Example B**

Identify the zeroes, holes and $y$ intercepts of the following rational function without graphing.

$f(x) = \frac{x(x-2)(x-1)(x+1)(x+1)(x+2)}{(x-1)(x+1)}$

**Solution:** The holes occur at $x = -1, 1$. To get the exact points, these values must be substituted into the function with the factors canceled.

$$f(x) = x(x-2)(x+1)(x+2)$$
$$f(-1) = 0, f(1) = -6$$

The holes are (-1, 0); (1, 6). The zeroes occur at $x = 0, 2, -2$. The zero that is supposed to occur at $x = -1$ has already been demonstrated to be a hole instead.

**Example C**

Create a function with holes at $x = 1, 2, 3$ and zeroes at $x = 0, 4$.

**Solution:** There are an infinite number of possible functions that fit this description because the function can be multiplied by any constant. One possible function could be:

$f(x) = \frac{(x-1)(x-2)(x-3)x(x-4)}{x(x-4)}$

**Concept Problem Revisited**

To find the zeroes of a rational function, set the numerator equal to zero and solve for the $x$ values. When a hole and a zero occur at the same point, the hole wins and there is no zero at that point.

## Vocabulary

A ***zero*** is where a function crosses the $x$-axis. It is also known as a root, solution or $x$-intercept.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

## Guided Practice

1. Create a function with holes instead of zeroes.
2. Identify the $y$ intercepts, holes, and zeroes of the following rational function.

$f(x) = \frac{6x^3-7x^2-x+2}{x-1}$

3. Identify the zeroes and holes of the following rational function.

$f(x) = \frac{2(x+1)(x+1)(x+1)}{2(x+1)}$

**Answers:**

1. There are an infinite number of functions that meet the requirements. An illustrative example would be:

$f(x) = (x-1)(x+1) \cdot \frac{(x-1)(x+1)}{(x-1)(x+1)}$

The two $x$ values that are holes are identical to the two $x$ values that would be zeroes. Therefore, this function has no zeroes because holes exist in their place.

2. After noticing that a possible hole occurs at $x = 1$ and using polynomial long division on the numerator you should get:

$f(x) = (6x^2-x-2) \cdot \frac{x-1}{x-1}$

A hole occurs at $x = 1$ which turns out to be the point (1, 3) because $6 \cdot 1^2 - 1 - 2 = 3$.

The $y$-intercept always occurs where $x = 0$ which turns out to be the point (0, -2) because $f(0) = -2$.

To find the $x$-intercepts you need to factor the remaining part of the function:

$(2x+1)(3x-2)$

Thus the zeroes ($x$-intercepts) are $x = -\frac{1}{2}, \frac{2}{3}$.

3. The hole occurs at $x = -1$ which turns out to be a double zero. The hole still wins so the point (-1, 0) is a hole. There are no zeroes. The constant 2 in front of the numerator and the denominator serves to illustrate the fact that constant scalars do not impact the $x$ values of either the zeroes or holes of a function.

## Practice

Identify the intercepts and holes of each of the following rational functions.

1. $f(x) = \frac{x^3+x^2-10x+8}{x-2}$
2. $g(x) = \frac{6x^3-17x^2-5x+6}{x-3}$
3. $h(x) = \frac{(x+2)(1-x)}{x-1}$
4. $j(x) = \frac{(x-4)(x+2)(x+2)}{x+2}$
5. $k(x) = \frac{x(x-3)(x-4)(x+4)(x+4)(x+2)}{(x-3)(x+4)}$
6. $f(x) = \frac{x(x+1)(x+1)(x-1)}{(x-1)(x+1)}$
7. $g(x) = \frac{x^3-x^2-x+1}{x^2-1}$
8. $h(x) = \frac{4-x^2}{x-2}$
9. Create a function with holes at $x = 3, 5, 9$ and zeroes at $x = 1, 2$.
10. Create a function with holes at $x = -1, 4$ and zeroes at $x = 1$.
11. Create a function with holes at $x = 0, 5$ and zeroes at $x = 2, 3$.
12. Create a function with holes at $x = -3, 5$ and zeroes at $x = 4$.
13. Create a function with holes at $x = -2, 6$ and zeroes at $x = 0, 3$.
14. Create a function with holes at $x = 1, 5$ and zeroes at $x = 0, 6$.
15. Create a function with holes at $x = 2, 7$ and zeroes at $x = 3$.

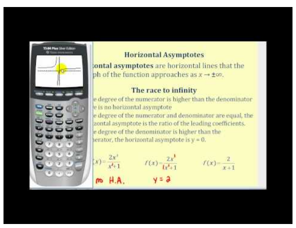
# 2.9 Vertical Asymptotes

Here you will learn to recognize when vertical asymptotes occur and what makes them different from removable discontinuities.

The basic rational function $f(x) = \frac{1}{x}$ is a hyperbola with a vertical asymptote at $x = 0$. More complicated rational functions may have multiple vertical asymptotes. These asymptotes are very important characteristics of the function just like holes. Both holes and vertical asymptotes occur at $x$ values that make the denominator of the function zero. A driving question is: what makes vertical asymptotes different from holes?

## Watch This

Watch the following video, focusing on the parts about vertical asymptotes.




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=wBZxVxiJS9I James Sousa: Determining Horizontal and Vertical Asymptotes of Rational Functions

## Guidance

Vertical asymptotes occur when a factor of the denominator of a rational expression does not cancel with a factor from the numerator. When you have a factor that does not cancel, instead of making a hole at that $x$ value, there exists a vertical asymptote. The vertical asymptote is represented by a dotted vertical line. Most calculators will not identify vertical asymptotes and some will incorrectly draw a steep line as part of a function where the asymptote actually exists.

Your job is to be able to identify vertical asymptotes from a function and describe each asymptote using the equation of a vertical line.

**Example A**

Identify the holes and the equations of the vertical asymptotes for the following rational function.

$f(x) = \frac{(2x-3)(x+1)(x-2)}{(x+2)(x+1)}$

**Solution:** The factor that cancels represents the removable discontinuity. There is a hole at (-1, 15). The vertical asymptote occurs at $x = -2$ because the factor $x+2$ does not cancel.

**Example B**

Identify the domain of the following function and then identify the holes and vertical asymptotes.

$f(x) = \frac{(3x-4)(1-x)(x^2+4)}{(3x-2)(x-1)}$

**Solution:** The domain of the function written in interval notation is: $\left(-\infty, \frac{2}{3}\right) \cup \left(\frac{2}{3}, 1\right) \cup (1, \infty)$

There are two discontinuities: one is a hole and one is a vertical asymptote. The hole occurs at (1, 5). The vertical asymptote occurs at $x = \frac{2}{3}$.

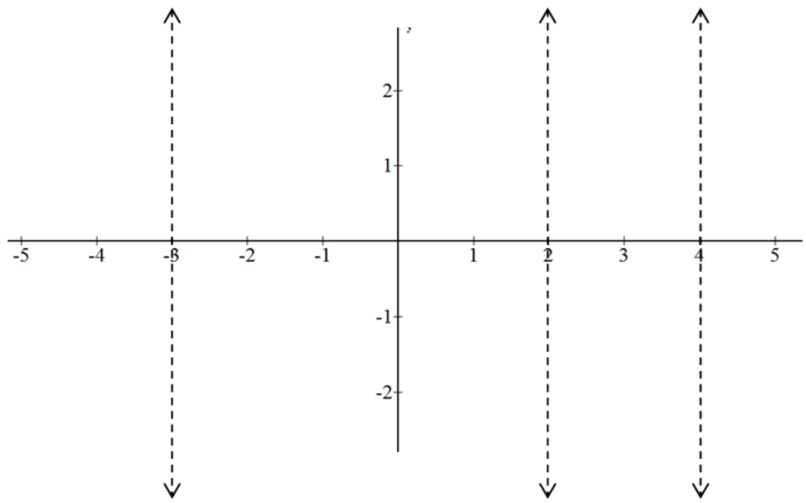
Notice that holes are identified as points while vertical asymptotes are identified as lines of the form $x = a$ where $a$ is some constant.

**Example C**

Draw the vertical asymptotes for the following function.

$f(x) = \frac{1}{(x-4)(x-2)(x+3)}$

**Solution:**




Note that you may not know the characteristics of what the function does inside these vertical lines. You will soon learn how to use sign tests as well as techniques you've already learned to fill in the four sections that this function is divided into.

**Concept Problem Revisited**

Holes occur when factors from the numerator and the denominator cancel. When a factor in the denominator does not cancel, it produces a vertical asymptote. Both holes and vertical asymptotes restrict the domain of a rational function.

## Vocabulary

A ***vertical asymptote*** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

## Guided Practice

1. Write a function that fits the following criteria:

- Vertical asymptotes at 0 and 3
- Zeroes at 2 and 5
- Hole at (4, 2)

2. Draw the vertical asymptotes for the following function.

$f(x) = 3 - \frac{x}{(x+1)(x-4)}$

3. Identify the holes and equations of the vertical asymptotes of the following rational function.

$f(x) = \frac{3(x-1)(x+2)(x-3)(x+4)}{5\left(x+\frac{1}{2}\right)(2+x)(3-x)(x-8)}$

**Answers:**

1. Each criteria helps build the function. The vertical asymptotes imply that the denominator has two factors that do not cancel with the numerator:

$\frac{1}{x \cdot (x-3)}$

The zeroes at 2 and 5 imply the numerator has two factors that do not cancel.

$\frac{(x-2)(x-5)}{x \cdot (x-3)}$

The hole at (4, 2) implies that there is a factor $x-4$ that cancels on the numerator and the denominator.

$\frac{(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$

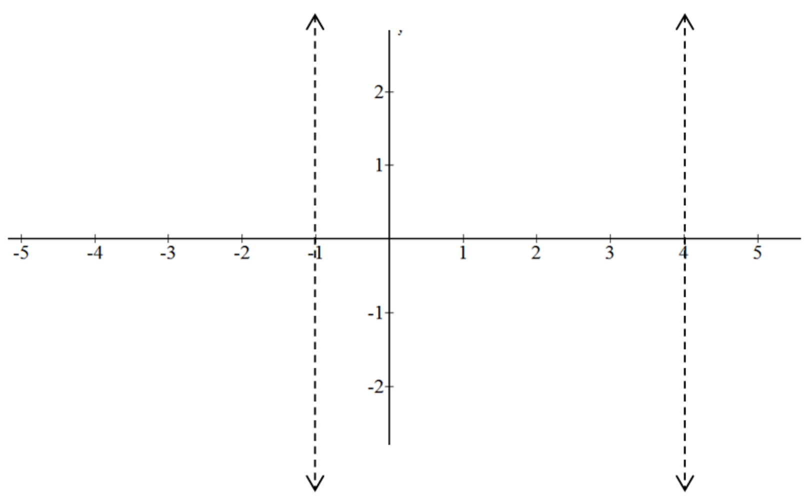
The tricky part is that the height of the function must be 2 after the $x-4$ factor has been canceled and the 4 is substituted in. Currently it is $-\frac{1}{2}$.

$\frac{(4-2)(4-5)}{4 \cdot (4-3)} = -\frac{1}{2}$

In order to make the hole exist at a height of 2, you need to multiply the function by a scalar of -4.

$f(x) = \frac{-4(x-2)(x-5)(x-4)}{x \cdot (x-3)(x-4)}$

2. The vertical asymptotes occur at $x = -1, x = 4$.




3. The vertical asymptotes occur at $x = -\frac{1}{2}, x = 8$. Holes occur when $x$ is -2 and 3. To get the height of the holes at these points, remember to cancel what can be canceled and then substitute the values. A very common mistake is to forget to cancel $\frac{x-3}{3-x} = -1$.

$$g(x) = \frac{-3(x-1)(x+4)}{5\left(x+\frac{1}{2}\right)(x-8)}$$
$$g(-2) = \frac{6}{25}$$
$$g(3) = \frac{12}{25}$$

The holes are at $\left(-2, \frac{6}{25}\right), \left(3, \frac{12}{25}\right)$.

### Practice

1. Write a function that fits the following criteria:
   - Vertical asymptotes at 1 and 4
   - Zeroes at 3 and 5
   - Hole at (6, 3)
2. Write a function that fits the following criteria:
   - Vertical asymptotes at -2 and 2
   - Zeroes at 1 and 5
   - Hole at (3, -4)
3. Write a function that fits the following criteria:
   - Vertical asymptotes at 0 and 3
   - Zeroes at 1 and 2
   - Hole at (8, 21)
4. Write a function that fits the following criteria:
   - Vertical asymptotes at 2 and 6
   - Zero at 5
   - Hole at (4, 1)
5. Write a function that fits the following criteria:
   - Vertical asymptote at 4
   - Zeroes at 0 and 3
   - Hole at (5, 10)

Give the equations of the vertical asymptotes for the following functions.

6. $f(x) = \frac{(2-x)}{(x-2)(x-4)}$

7. $g(x) = \frac{-x}{(x+1)(x-3)}$

8. $h(x) = 6 - \frac{x+2}{(x+1)(x-5)}$

9. $j(x) = \frac{10}{x-3} - \frac{x}{(x+2)(x-3)}$

10. $k(x) = 2 - \frac{(4-x)}{(x+3)(x-4)}$

Identify the holes and equations of the vertical asymptotes of the following rational functions.

11. $f(x) = \frac{3(x-1)(x+1)(x-4)(x+4)}{4(x+4)(2+x)(4-x)(x+1)}$

12. $g(x) = \frac{x(x-3)(x-8)(x-3)(x+4)}{7(x+1)(1+x)(3-x)(x-8)}$

State the domain of the following rational functions.

13. $h(x) = \frac{x(x+1)(x-3)(x+4)}{x(3-x)(x-1)}$

14. $j(x) = \frac{x^2+3x-4}{x^2-6x-16}$
15. $k(x) = \frac{2x-10}{x^3+4x^2+3x}$

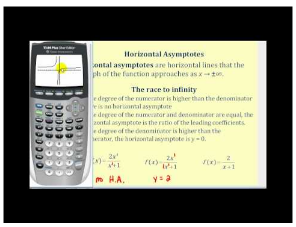
# 2.10 Horizontal Asymptotes

Here you will learn to identify when horizontal asymptotes exist, specify their height and write their equation.

Vertical asymptotes describe the behavior of a function as the values of $x$approach a specific number. Horizontal asymptotes describe the behavior of a function as the values of $x$become infinitely large and infinitely small. Since functions cannot touch vertical asymptotes, are they not allowed to touch horizontal asymptotes either?

### Watch This

Watch the following video, focusing on the parts about horizontal asymptotes.

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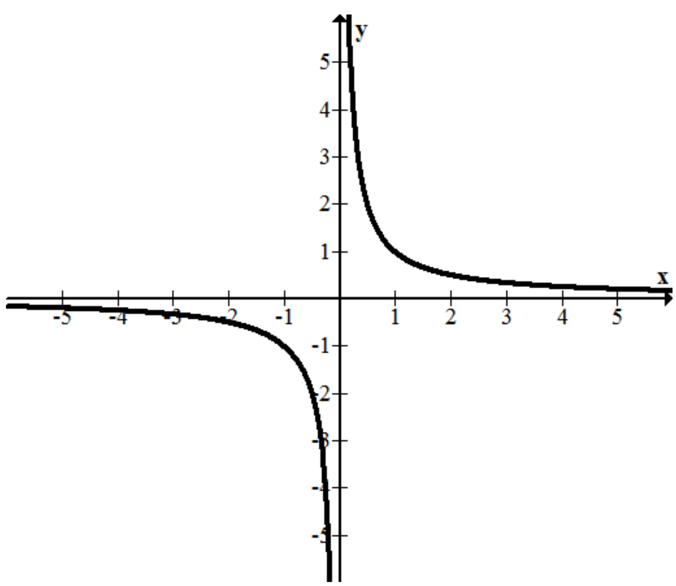
Click image to the left for more content.

http://www.youtube.com/watch?v=wBZxVxiJS9I James Sousa: Determining Horizontal and Vertical Asymptotes of Rational Functions

### Guidance

**Horizontal asymptotes** are a means of describing end behavior of a function. End behavior essentially is a description of what happens on either side of the graph as the function continues to the right and left infinitely. When you are determining the horizontal asymptotes, it is important to consider both the right and the left hand sides, because the horizontal asymptotes will not necessarily be the same in both places. Consider the reciprocal function and note how as $x$ goes to the right and left it flattens to the line $y = 0$.



Sometimes functions flatten out and other times functions increase or decrease without bound. There are basically three cases.

**Case 1:** The first case is the function flattens out to 0 as $x$ gets infinitely large or infinitely small. This happens when the degree of the numerator is less than the degree of the denominator. The degree is determined by the greatest exponent of $x$.

$$f(x) = \frac{2x^8 + 3x^2 + 100}{x^9 - 12}$$

One way to reason through why this makes sense is because when $x$ is a ridiculously large number then most parts of the function hardly make any impact. The 100 for example is nothing in comparison and neither is the $3x^2$. The two important terms to compare are $x^8$ and $x^9$. The 2 isn't even important now because if $x$ is even just a million than the $x^9$ will be a million times bigger than the $x^8$ and the 2 hardly matters again. Essentially, when $x$ gets big enough, this function acts like $\frac{1}{x}$ which has a horizontal asymptote of 0.

**Case 2:** The degree of the numerator is equal to the degree of the denominator. In this case, the horizontal asymptote is equal to the ratio of the leading coefficients.

$$f(x) = \frac{6x^4 - 3x^3 + 12x^2 - 9}{3x^4 + 144x - 0.001}$$

Notice how the degree of both the numerator and the denominator is 4. This means that the horizontal asymptote is $y = \frac{6}{3} = 2$. One way to reason through why this makes sense is because when $x$ gets to be a very large number all the smaller powers will not really make much of an impact. The biggest contributors are only the biggest powers. Then the value of the numerator will be about twice that of the denominator. As $x$ gets even bigger, then the function will get even closer to 2.

**Case 3:** The degree of the numerator is greater than the degree of the denominator. In this case there does not exist a horizontal asymptote and you must determine if the function increases or decreases without bound in both the left and right directions.

**Example A**

Identify the horizontal asymptotes of the following 3 functions:

a. $f(x) = \frac{4x^3 + 99}{3x^4 - 99}$
b. $h(x) = \frac{234x^{45} - 45x^{234} + 100}{x^{235}}$
c. $g(x) = \frac{x^3 + 3x^6}{x^3 - 6x^6}$

**Solution:**

a. $y = 0$ because the degree of the numerator is 3 and the degree of the denominator is 4, so the denominator gets bigger eventually and the fraction approaches 0.
b. $y = 0$ because the degree of the numerator is 234 which is smaller than the degree of the denominator (235).
c. $y = -\frac{1}{2}$ because the degree of both the numerator and the denominator is 6 so the horizontal asymptote is the ratio of the leading coefficients. Note that leading refers to the coefficients of the greatest powers of $x$ not the coefficients that happen to be written out front. Convention usually tells you to write the powers of $x$ in descending order.

**Example B**

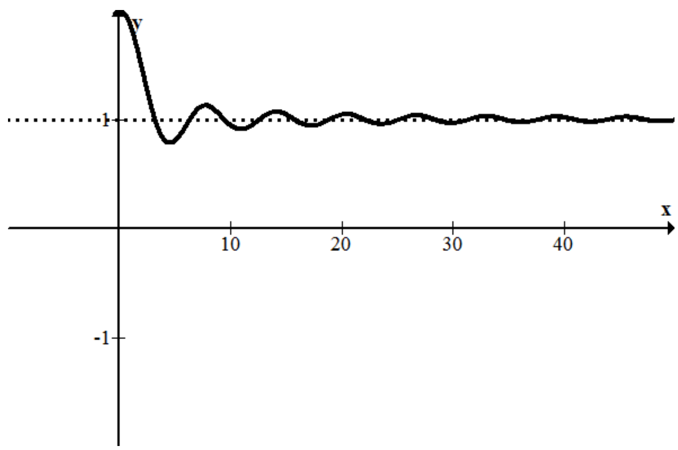
Identify the vertical and horizontal asymptotes of the following rational function.

$$f(x) = \frac{(x-2)(4x+3)(x-4)}{(x-1)(4x+3)(x-6)}$$

**Solution:** There is factor that cancels that is neither a horizontal or vertical asymptote. The vertical asymptotes occur at $x=1$ and $x=6$. To obtain the horizontal asymptote you could methodically multiply out each binomial, however since most of those terms do not matter, it is more efficient to first determine the relative powers of the numerator and the denominator. In this case they both happen to be 3. Next determine the coefficient of the cubic terms only. The numerator will have $4x^3$ and the denominator will have $4x^3$ and so the horizontal asymptote will occur at $y = \frac{4}{4} = 1$.

**Example C**

Describe the right hand end behavior of the following function.




**Solution:** Notice how quickly this dampening wave function settles down. There seems to be an obvious horizontal axis on the right at $y=1$

**Concept Problem Revisited**

As in Example C, functions may touch and pass through horizontal asymptotes without limit. This is a difference between vertical and horizontal asymptotes. In calculus, there are rigorous proofs to show that functions like the one in Example C do become arbitrarily close to the asymptote.

## Vocabulary

A ***horizontal asymptote*** is a flat dotted line that indicates where a function goes as $x$ get infinitely large or infinitely small.

***End behavior*** is a term that asks you to describe the horizontal asymptotes.

A ***vertical asymptote*** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

### Guided Practice

1. Identify the horizontal asymptotes of the following function.

$$f(x) = \frac{(x-3)(x+2)}{|(x-5)| \cdot (x-1)}$$

2. Identify the vertical and horizontal asymptotes and the holes of the following function.

$$f(x) = \frac{(x^4-9)(x-1)}{(x^2+3)(x-3)}$$

3. Identify the horizontal asymptotes if they exist for the following 3 functions.

a. $f(x) = \frac{3x^6-72x}{x^6+999}$
b. $h(x) = \frac{ax^4+bx^3+cx^2+dx+e}{fx^4+gx^3+hx^2}$
c. $g(x) = \frac{f(x)}{h(x)}$

**Answers:**

1. First notice the absolute value surrounding one of the terms in the denominator. The degrees of both the numerator and the denominator will be 2 which means that the horizontal asymptote will occur at a number. As $x$ gets infinitely large, the function is approximately:

$$f(x) = \frac{x^2}{x^2}$$

So the horizontal asymptote is $y = -1$ as $x$ gets infinitely large.

On the other hand, as $x$ gets infinitely small the function is approximately:

$$f(x) = \frac{x^2}{-x^2}$$

So the horizontal asymptote is $y = -1$ as $x$ gets infinitely small.

In this case, you cannot blindly use the leading coefficient rule because the absolute value changes the sign.

2. The numerator of the function factors to be:

$$f(x) = \frac{(x^2+3)(x^2-3)(x-1)}{(x^2+3)(x-3)} = \frac{(x^2-3)(x-1)}{x-3}$$

Note that a factor does cancel and also notice that this factor is never equal to zero. Not all factors that cancel indicate a hole. A horizontal asymptote does not exist because the degree of the numerator is greater than the degree of the denominator. The vertical asymptote is at $x = 3$.

3.

a. The degrees of the numerator and the denominator are equal so the horizontal asymptote is $y = 3$.
b. The degrees of the numerator and the denominators are equal again so the horizontal asymptote is $y = \frac{a}{f}$.
c. As $x$ gets infinitely large,

$$g(x) = \frac{f(x)}{h(x)} = \frac{\frac{3x^6-72x}{x^6+999}}{\frac{ax^4+bx^3+cx^2+dx+e}{fx^4+gx^3+hx^2}} \approx \frac{3}{\frac{a}{f}} = \frac{3f}{a}$$

When you study calculus, you will learn the rigorous techniques that enable you to feel more confident about results like this.

### Practice

Identify the horizontal asymptotes, if they exist, for the following functions.

1. $f(x) = \frac{5x^4-2x}{x^4+32}$
2. $g(x) = \frac{3x^4-2x^6}{-x^4+2}$
3. $h(x) = \frac{3x^4-5x}{8x^3+3x^4}$
4. $j(x) = \frac{2x^3-15x}{-x^4+3}$
5. $k(x) = \frac{2x^5-3x}{5x^2+3x^4+2x-7x^5}$
6. $f(x) = \frac{ax^{14}+bx^{23}+cx^{12}+dx+e}{fx^{24}+gx^{23}+hx^{21}}$
7. $g(x) = \frac{(x-1)(x+4)}{|(x-2)|\cdot(x-1)}$
8. Write a function that fits the following criteria:
   - Vertical asymptotes at $x = 1$ and $x = 4$
   - Zeroes at 3 and 5
   - Hole when $x = 6$
   - Horizontal asymptote at $y = \frac{2}{3}$
9. Write a function that fits the following criteria:
   - Vertical asymptotes at $x = -2$ and $x = 2$
   - Zeroes at 1 and 5
   - Hole when $x = 3$
   - Horizontal asymptote at $y = 1$
10. Write a function that fits the following criteria:
    - Vertical asymptotes at $x = 0$ and $x = 3$
    - Zeroes at 1 and 2
    - Hole when $x = 8$
    - Horizontal asymptote at $y = 2$
11. Write a function that fits the following criteria:
    - Vertical asymptotes at 2 and 6
    - Zero at 5
    - Hole when $x = 4$

- Horizontal asymptote at $y = 0$

12. Write a function that fits the following criteria:
    - Vertical asymptote at 4
    - Zeroes at 0 and 3
    - Hole at when $x = 5$
    - No horizontal asymptotes

Identify the vertical and horizontal asymptotes of the following rational functions.

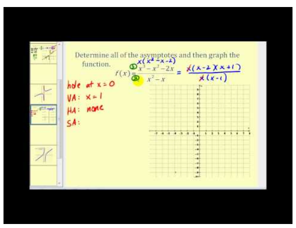
13. $f(x) = \frac{(x-5)(2x+1)(x-3)}{(x-3)(4x+5)(x-6)}$
14. $g(x) = \frac{x(x-1)(x+3)(x-5)}{3x(x-1)(4x+3)}$
15. $h(x) = \frac{(x-2)(x+3)(x-6)}{(x-4)(x+3)^2(x+2)}$

# 2.11 Oblique Asymptotes

Here you will extend your knowledge of horizontal and vertical asymptotes and learn to identify oblique (slanted) asymptotes. You will also be able to apply your knowledge of polynomial long division.

When the degree of the numerator of a rational function exceeds the degree of the denominator by one then the function has oblique asymptotes. In order to find these asymptotes, you need to use polynomial long division and the non-remainder portion of the function becomes the oblique asymptote. A natural question to ask is: what happens when the degree of the numerator exceeds that of the denominator by more than one?

### Watch This



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Click image to the left for more content.

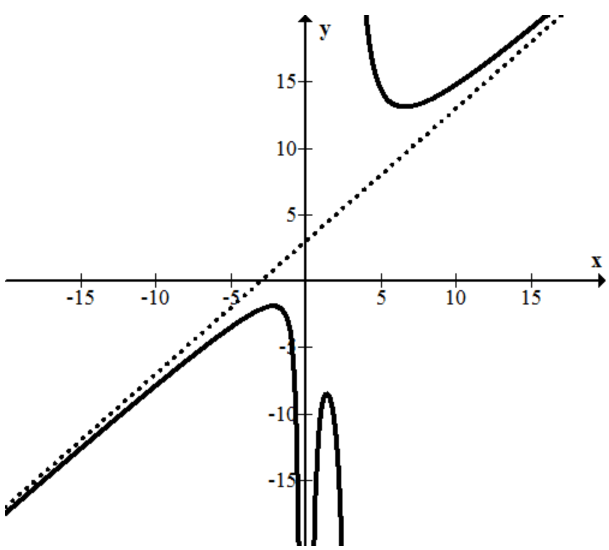
http://www.youtube.com/watch?v=W8ASTRfEMVo James Sousa: Determining Slant Asymptotes of Rational Functions

### Guidance

The following function is shown before and after polynomial long division is performed.

$f(x) = \frac{x^4+3x^2+2x+14}{x^3-3x^2} = x+3+\frac{12x^2+2x+14}{x^3-3x^2}$

Notice that the remainder portion will go to zero when $x$ gets extremely large or extremely small because the power of the numerator is smaller than the power of the denominator. This means that while this function might go haywire with small absolute values of $x$, large absolute values of $x$ are extremely close to the line $y = x+3$.



**Oblique asymptotes** are these slanted asymptotes that show exactly how a function increases or decreases without bound.

**Example A**

Identify the oblique asymptotes of the following rational function.

$f(x)=\frac{x^3-x-33}{x^2+3x-4}=x-3+\frac{12x-45}{(x-1)(x+4)}$

**Solution:** Since this function has been rewritten after long division has been performed, the oblique asymptote is the line that remains:

$y=x-3$

**Example B**

Identify the vertical and oblique asymptotes of the following rational function.

$f(x)=\frac{x^3-x^2-x-1}{(x-3)(x+4)}$

**Solution:** After using polynomial long division and rewriting the function with a remainder and non-remainder portion it looks like this:

$f(x)=x-2+\frac{13x-25}{x^2+x-12}=x-2+\frac{13x-25}{(x-3)(x+4)}$

The oblique asymptote is $y=x-2$. The vertical asymptotes are at $x=3$ and $x=-4$ which are easier to observe in last form of the function because they clearly don't cancel to become holes.

**Example C**

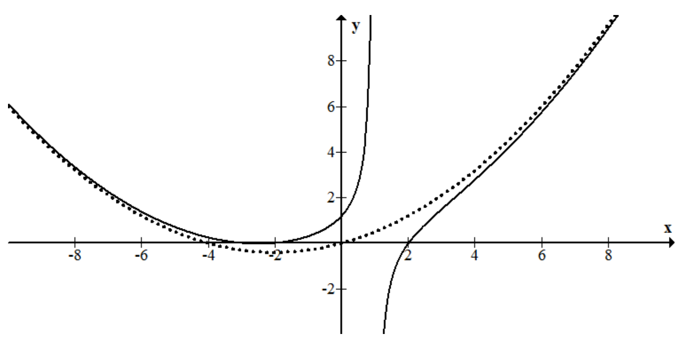
Identify the oblique asymptotes of the following rational function.

$f(x)=\frac{(x^2-4)(x+3)}{10(x-1)}$

**Solution:** The degree of the numerator is 3 so the slant asymptote will not be a line. However when the graph is observed, there is still a clear pattern as to how this function increases without bound as $x$ approaches very large and very small numbers.

$f(x)=\frac{1}{10}(x^2+4x)-\frac{12}{10(x-1)}$

As you can see, this looks like a parabola with a remainder. This rational function has a parabola backbone. This is not technically an oblique asymptote because it is not a line.




**Concept Problem Revisited**

When the numerator exceeds the denominator by more than one, the function develops a backbone as in Example C that can be shaped like any polynomial. Oblique asymptotes are always lines.

## Vocabulary

***Oblique asymptotes*** are asymptotes that occur at a slant. They are always lines.

A ***horizontal asymptote*** is a flat dotted line that indicates where a function goes as $x$ get infinitely large or infinitely small.

***End behavior*** is a term that asks you to describe the horizontal asymptotes.

A ***vertical asymptote*** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

## Guided Practice

1. Find the asymptotes and intercepts of the function:

$f(x) = \frac{x^3}{x^2-4}$

2. Create a function with an oblique asymptote at $y = 3x - 1$, vertical asymptotes at $x = 2, -4$ and includes a hole where $x$ is 7.

3. Identify the backbone of the following function and explain why the function does not have an oblique asymptote.

$f(x) = \frac{5x^5+27}{x^3}$

**Answers:**

1. The function has vertical asymptotes at $x = \pm 2$.

After long division, the function becomes:

$f(x) = x + \frac{4}{x^2-4}$

This makes the oblique asymptote at $y = x$

2. While there are an infinite number of functions that match these criteria, one example is:

$f(x) = 3x - 1 + \frac{(x-7)}{(x-2)(x+4)(x-7)}$

3. While polynomial long division is possible, it is also possible to just divide each term by $x^3$.

$f(x) = \frac{5x^5+27}{x^3} = \frac{5x^5}{x^3} + \frac{27}{x^3} = 5x^2 + \frac{27}{x^3}$

The backbone of this function is the parabola $y = 5x^2$. This is not an oblique asymptote because it is not a line.

## Practice

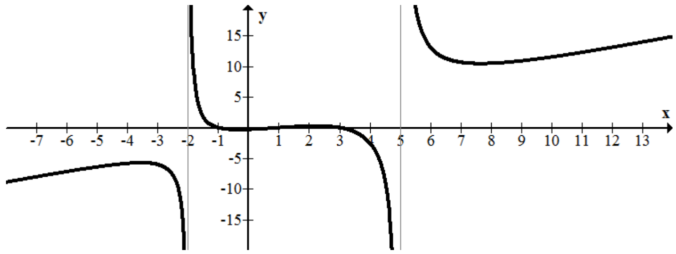
1. What is an oblique asymptote?

2. How can you tell by looking at the equation of a function if it will have an oblique asymptote or not?

3. Can a function have both an oblique asymptote and a horizontal asymptote? Explain.

For each of the following graphs, sketch the graph and then sketch in the oblique asymptote if it exists. If it doesn't exist, explain why not.

4.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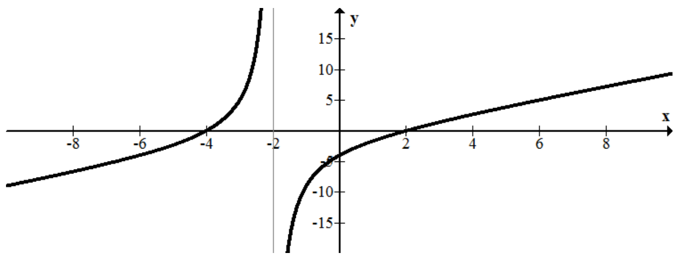
5.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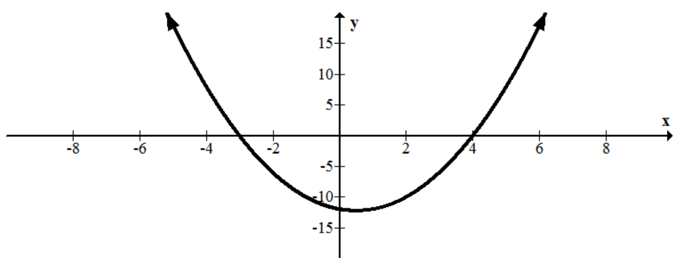
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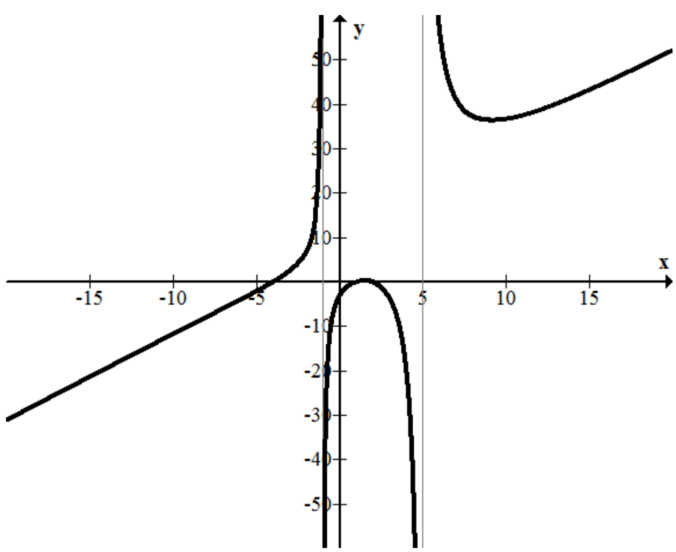
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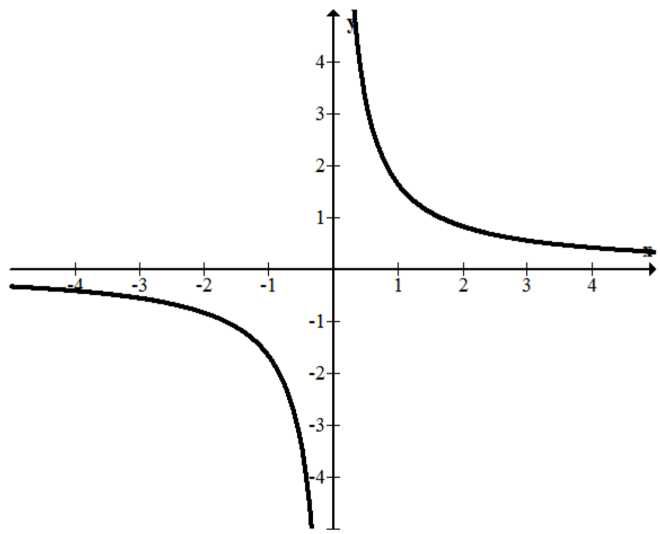



Find the equation of the oblique asymptote for each of the following rational functions. If there is not an oblique asymptote, explain why not and give an equation of the backbone of the function if one exists.

9. $f(x) = \frac{x^3-7x-6}{x^2-2x-15}$

10. $g(x) = \frac{x^3-7x-6}{x^4-3x^2-10}$

11. $h(x) = \frac{x^2+5x+6}{x^2+2x+1}$

12. $k(x) = \frac{x^4+9x^3+21x^2-x-30}{x^2+2x+1}$

13. Create a function with an oblique asymptotes at $y = 2x - 1$, a vertical asymptote at $x = 3$ and a hole where $x$ is 7.

14. Create a function with an oblique asymptote at $y = x$, vertical asymptotes at $x = 1, -3$ and no holes.

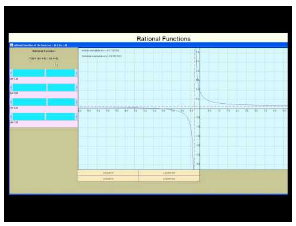
15. Does a parabola have an oblique asymptote? What about a cubic function?

# 2.12 Sign Test for Rational Function Graphs

Here you will learn to predict the nature of a rational function near the asymptotes.

The asymptotes of a rational function provide a very rigid structure in which the function must live. Once the asymptotes are known you must use the sign testing procedure to see if the function becomes increasingly positive or increasingly negative near the asymptotes. A driving question then becomes how close does near need to be in order for the sign test to work?

### Watch This



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Click image to the left for more content.

http://www.youtube.com/watch?v=OEQnQNvJtG0 James Sousa: Graphing Rational Functions

### Guidance

Consider mentally substituting the number 2.99999 into the following rational expression.

$$f(x) = \frac{(x-1)(x+3)(x-5)(x+10)}{(x+2)(x-4)(x-3)}$$

Without doing any of the arithmetic, simply note the sign of each term:

$$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(-)}$$

The only term where the value is close to zero is $(x-3)$ but careful subtraction still indicates a negative sign. The product of all of these signs is negative. This is strong evidence that this function approaches negative infinity as $x$ approaches 3 from the left.

Next consider mentally substituting 3.00001 and going through the same process.

$$f(x) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)\cdot(+)}$$

The product of all of these signs is positive which means that from the right this function approaches positive infinity instead.

**Example A**

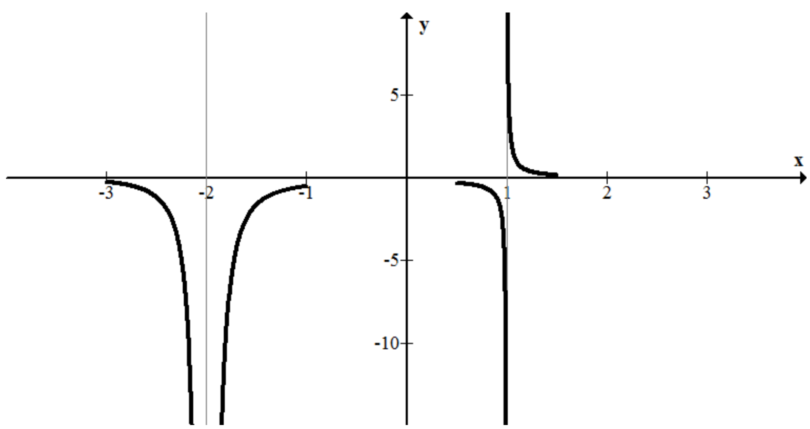
Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{1}{(x+2)^2 \cdot (x-1)}$$

**Solution:** The vertical asymptotes occur at $x = -2$ and $x = 1$. The points to use the sign testing procedure with are -2.001, -1.9999, 0.9999, 1.00001. The number of decimals does matter so long as the number is sufficiently close to the asymptote. Note that any real number squared is positive.

$$f(-2.001) = \frac{(+)}{(+) \cdot (-)} = -$$
$$f(-1.9999) = \frac{(+)}{(+) \cdot (-)} = -$$
$$f(0.9999) = \frac{(+)}{(+) \cdot (-)} = -$$
$$f(1.0001) = \frac{(+)}{(+) \cdot (+)} = +$$

Later when you sketch everything you will use your knowledge of zeroes and intercepts. *For now, focus on just the portions of the graph near the asymptotes.* Note that the graph below is NOT complete.




**Example B**

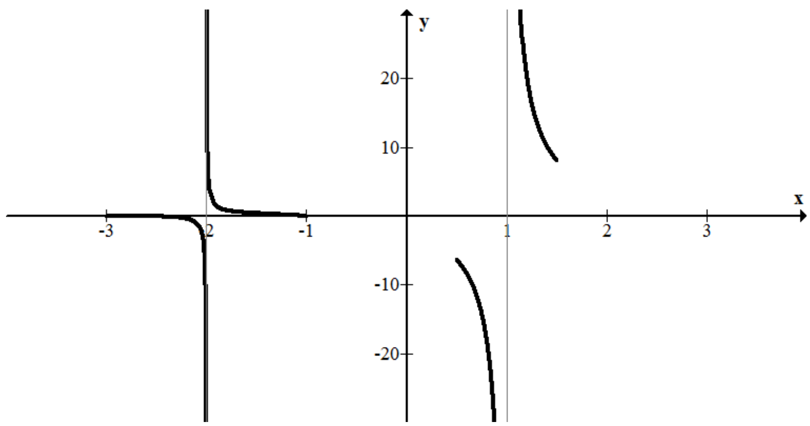
Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x+1)(x-4)^2(x-1)(x+3)^3}{100(x-1)^2(x+2)}$$

**Solution:** Note that $x = -2$ is clearly an asymptote. It may be initially unclear whether $x = 1$ is an asymptote or a hole. Just like holes have priority over zeroes, asymptotes have priority over holes. The four values to use the sign testing procedure are -2.001, -1.9999, 0.9999, 1.00001.

$$f(-2.001) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(-)} = -$$
$$f(-1.9999) = \frac{(-)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = +$$
$$f(0.9999) = \frac{(+)\cdot(+)\cdot(-)\cdot(+)}{(+)\cdot(+)} = -$$
$$f(1.0001) = \frac{(+)\cdot(+)\cdot(+)\cdot(+)}{(+)\cdot(+)} = +$$

A sketch of the behavior of this function near the asymptotes is:

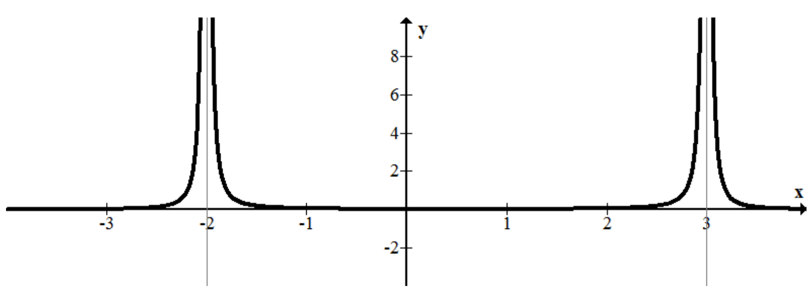



### Example C

Create a function with two vertical asymptotes at 3 and -2 such that the function approaches positive infinity from both directions at both vertical asymptotes.

**Solution:** In Example A there was a function that approached negative infinity from both sides of the asymptote. This occurred because the term was squared in the denominator.

$$f(x) = \frac{1}{(x-3)^2(x+2)^2}$$




### Concept Problem Revisited

In order to truly answer the question about how close the numbers need to be, calculus should be used. For the purposes of PreCalculus, the testing number should be closer to the vertical asymptote than any other number in the problem. If the vertical asymptote occurs at 3 and 3.01 is in the problem elsewhere, do not choose 3.1 as a sign test number.

### Vocabulary

The ***sign test*** is a procedure for determining only whether a function is above or below the $x$ axis at a particular $x$ value. The specific height is not calculated.

**Oblique asymptotes** are asymptotes that occur at a slant. They are always lines.

A ***horizontal asymptote*** is a flat dotted line that indicates where a function goes as $x$ get infinitely large or infinitely small.

***End behavior*** is a term that asks you to describe the horizontal asymptotes.

A ***vertical asymptote*** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

### Guided Practice

1. Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x-1.01)}{(x-3)^3x^4(x-1)^2}$$

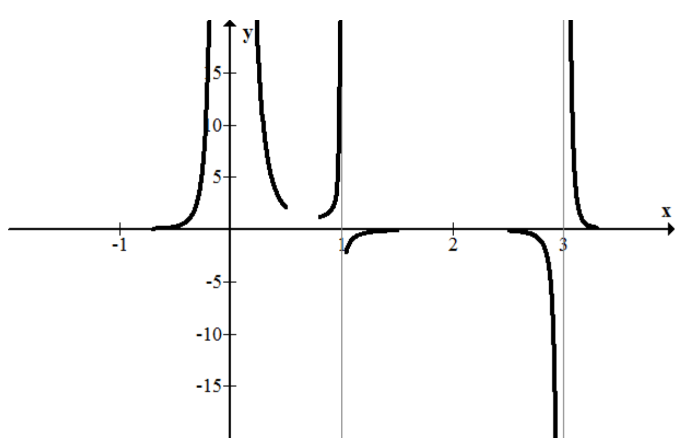
2. Create a function with three vertical asymptotes such that the function approaches negative infinity for large and small values of $x$ and has an oblique asymptote.

3. Identify the vertical asymptotes and use the sign testing procedure to roughly sketch the nature of the function near the vertical asymptotes.

$$f(x) = \frac{(x-2)^3(x-1)^2(x+1)(x+3)}{x^3\left(x+\frac{1}{2}\right)(x-1)(x-2)^2}$$

**Answers:**

1. The vertical asymptotes occur at $x = 0, 1, 3$.

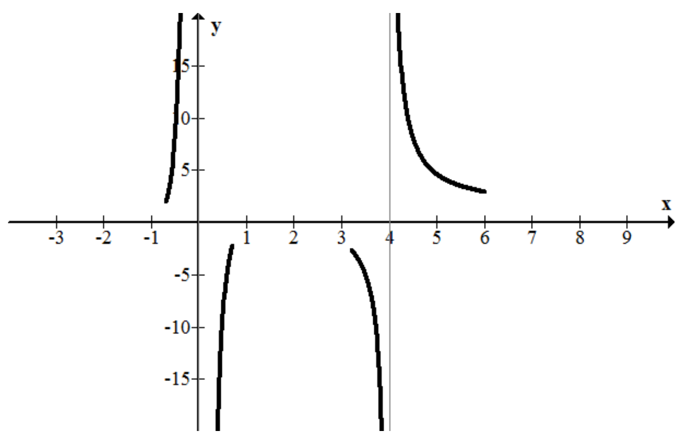


2. There are an infinite number of possible solutions. The key is to create a function that may work and then use the sign testing procedure to check. Here is one possibility.

$$f(x) = \frac{-x^7}{10(x-1)^2(x-2)^2(x-4)^2}$$

3. The vertical asymptotes occur at $x = 0, -\frac{1}{2}$. Therefore the $x$ values to sign test are -.001, 0.001, 3.999, 4.0001.

$$f(-0.001) = +$$
$$f(0.001) = -$$
$$f(3.999) = -$$
$$f(4.0001) = +$$




## Practice

Consider the function below for questions 1-4.

$$f(x) = \frac{(x-2)^4(x+1)(x+3)}{x^3(x+3)(x-4)}$$

1. Identify the vertical asymptotes.
2. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
3. What values will you need to use the sign test with in order to help you make a sketch of the graph?
4. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 5-8.

$$g(x) = \frac{3(x-2)^2(x-1)^2(x+1)(x+3)}{15x^2(x+5)(x+1)(x-3)^2}$$

5. Identify the vertical asymptotes.
6. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
7. What values will you need to use the sign test with in order to help you make a sketch of the graph?
8. Use the sign test and sketch the graph near the vertical asymptote(s).

Consider the function below for questions 9-12.

$$h(x) = \frac{9x^4 - 102x^3 + 349x^2 - 340x + 100}{x^3 - 9x^2 + 24x - 16}$$

9. Identify the vertical asymptotes.
10. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
11. What values will you need to use the sign test with in order to help you make a sketch of the graph?
12. Use the sign test and sketch the graph near the vertical asymptotes.

Consider the function below for questions 13-16.

$$k(x) = \frac{2x^3 - 5x^2 - 11x - 4}{3x^3 + 11x^2 + 5x - 3}$$

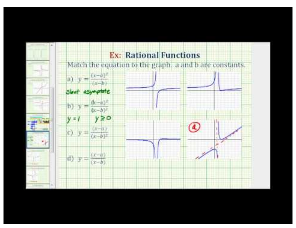
13. Identify the vertical asymptotes.
14. Will this function have an oblique asymptote? A horizontal asymptote? If so, where?
15. What values will you need to use the sign test with in order to help you make a sketch of the graph?
16. Use the sign test and sketch the graph near the vertical asymptotes.

# 2.13 Graphs of Rational Functions by Hand

Here you will use your knowledge of zeroes, intercepts, holes and asymptotes to sketch rational functions by hand.

Sketching rational functions by hand is a mental workout because it combines so many different specific skills to produce a single coherent image. It will require you to closely examine the equation of the function in a variety of different ways in order to find clues as to the shape of the overall function. Since computers can graph these complicated functions much more accurately than people can, why is sketching by hand important?

### Watch This

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http://www.youtube.com/watch?v=vMVYaFptvkk&feature=youtu.be James Sousa: Ex. Match Equations of Rational Functions to Graphs

### Guidance

While there is no strict procedure for graphing rational functions by hand there is a flow of clues to look for in the function. In general, it will make sense to identify different pieces of information in this order and record them on a sketch.

**STEPS FOR GRAPHING BY HAND**

1. Examine the denominator of the rational function to determine the domain of the function. Distinguish between holes which are factors that can be canceled and vertical asymptotes that cannot. Plot the vertical asymptotes.
2. Identify the end behavior of the function by comparing the degrees of the numerator and denominator and determine if there exists a horizontal or oblique asymptote. Plot the horizontal or oblique asymptotes.
3. Identify the holes of the function and plot them.
4. Identify the zeroes and intercepts of the function and plot them.
5. Use the sign test to determine the behavior of the function near the vertical asymptotes.
6. Connect everything as best you can.

**Example A**

Completely plot the following rational function.

$$f(x) = \frac{4x^3 - 2x^2 + 3x - 1}{8(x-1)^2(x+2)}$$

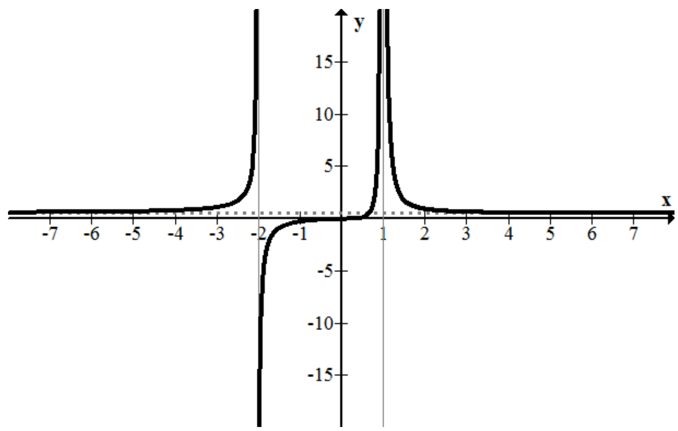
**Solution:** After attempting to factor the numerator you may realize that both $x = 1$ and $x = -2$ are vertical asymptotes rather than holes. The horizontal asymptote is $y = \frac{1}{2}$. There are no holes. The $y$-intercept is:

$f(0) = -\frac{1}{8 \cdot 2} = \frac{1}{16}$

The numerator is not factorable, but there is a zero between 0 and 1. You know this because there are no holes or asymptotes between 0 and 1 and the function switches from negative to positive in this region.

$$4(0)^3 - 2(0)^2 + 3(0) - 1 = -1$$
$$4(1)^3 - 2(1)^2 + 3(1) - 1 = 4$$

Putting all of this together in a sketch:

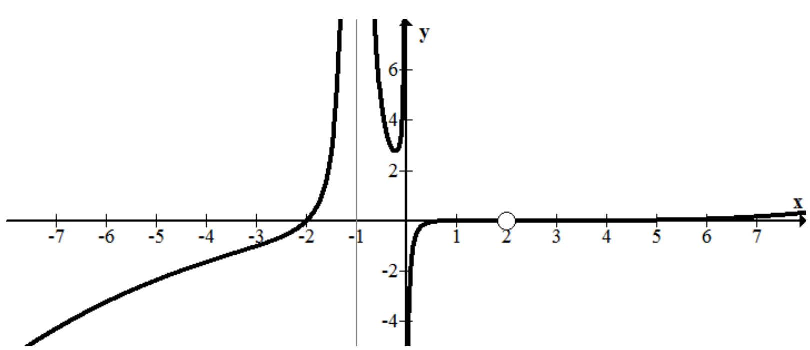



**Example B**

Completely plot the following rational function.

$$f(x) = \frac{(x-3)^2(x-2)^3(x-1)(x+2)}{300(x+1)^2(x-2)x}$$

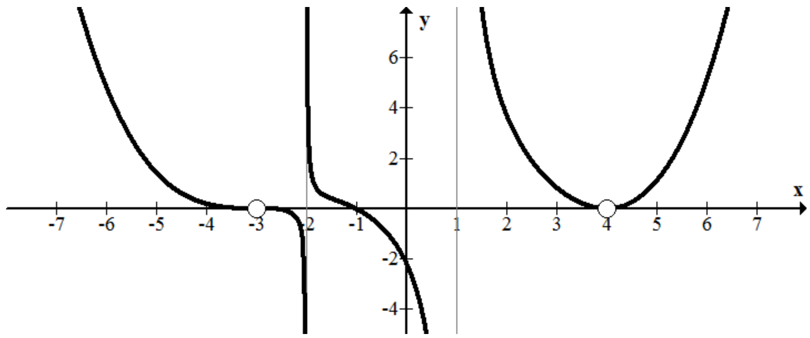
**Solution:** Since this function is already factored, much of the work is already done. There is a hole at (2, 0). There are two vertical asymptotes at $x = -1, 0$. There are no horizontal or oblique asymptotes because the degree of the numerator is much bigger than the degree of the denominator. As $x$ gets large this function grows without bound. As $x$ get very small, this function decreases without bound. The function has no $y$ intercept because that is where a vertical asymptote is. Besides the hole at (2, 0), there are zeroes at (-2, 0), (1, 0) and (3, 0). This is what the graph ends up looking like.



Notice on the right portion of the graph the curve seems to stay on the $x$-axis. In fact it does go slightly above and below $x$-axis, crossing through it at (1, 0), (2, 0) and (3, 0) before starting to increase.

**Example C**

Estimate a function that would have the following graphical characteristics:




**Solution:** First think about the vertical asymptotes and how they affect the equation of the function. Then consider zeroes and holes, and the way the graph looks at these places. Finally, use the $y$-intercept to refine your equation.

- The function has two vertical asymptotes at $x = -2, 1$ so the denominator must have the factors $(x+2)(x-1)$.
- There is one zero at $x = -1$, so the numerator must have a factor of $(x+1)$.
- There are two holes that appear to override zeroes which means the numerator and denominator must have the factors $(x+3)$ and $(x-4)$.
- Because the graph goes from above the $x$-axis to below the $x$-axis at $x = -3$, the degree of the exponent of the $(x+3)$ factor must be ultimately odd.
- Because the graph stays above the $x$-axis before and after $x = 4$, the degree of the $(x-4)$ factor must be ultimately even.

A good estimate for the function is:

$$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{(x+2)(x-1)(x+3)(x-4)}$$

This function has all the basic characteristics, however it isn't scaled properly. When $x = 0$ this function has a $y$-intercept of -216 when it should be about -2. Thus you must divide by 108 so that the $y$ intercept matches. Here is a better estimate for the function:

$$f(x) = \frac{(x+1)(x+3)^4(x-4)^3}{108(x+2)(x-1)(x+3)(x-4)}$$

**Concept Problem Revisited**

Computers can graph rational functions more accurately than people. However, computers may not be able to explain why a function behaves in certain ways. By being a detective and looking for clues in the equation of a function, you are applying high level analytical skills and powers of deduction. These analytical skills are vastly more important and transferrable than the specific techniques involved with rational functions.

### Vocabulary

The ***sign test*** is a procedure for determining only whether a function is above or below the $x$ axis at a particular $x$ value. The specific height is not calculated.

***Oblique asymptotes*** are asymptotes that occur at a slant. They are always lines.

A ***horizontal asymptote*** is a flat dotted line that indicates where a function goes as $x$ get infinitely large or infinitely small.

***End behavior*** is a term that asks you to describe the horizontal asymptotes.

A ***vertical asymptote*** is a dashed vertical line that indicates that as a function approaches, it shoots off to positive or negative infinity without ever actually touching the line.

A ***rational function*** is a function with at least one rational expression.

A ***rational expression*** is a ratio of two polynomial expressions.

### Guided Practice

1. Graph the following rational function:

$$f(x)=\frac{1}{x^2+3x+2}$$

2. Graph the following rational function:

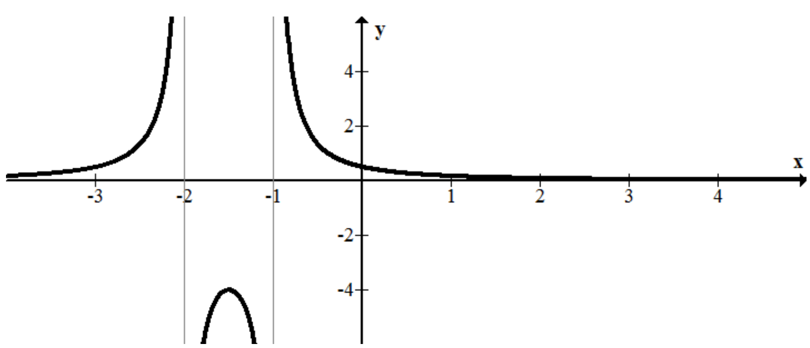
$$f(x)=\frac{-x^4}{10(x-1)^2(x-2)^2(x-4)^2}$$

3. Graph the following rational function:

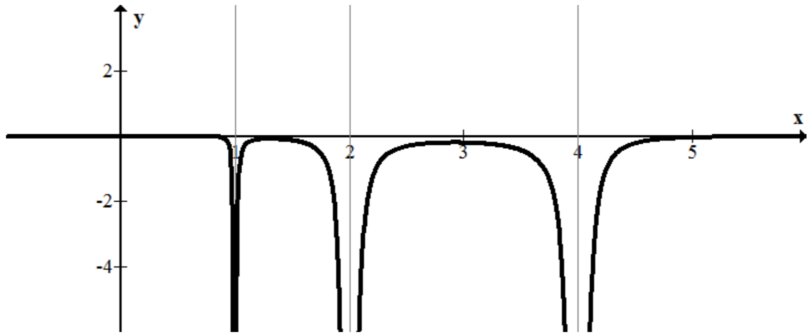
$$f(x)=\frac{3x^3+4x-2x-4}{x^2-7x+12}$$

**Answers:**

1. Here is the graph:

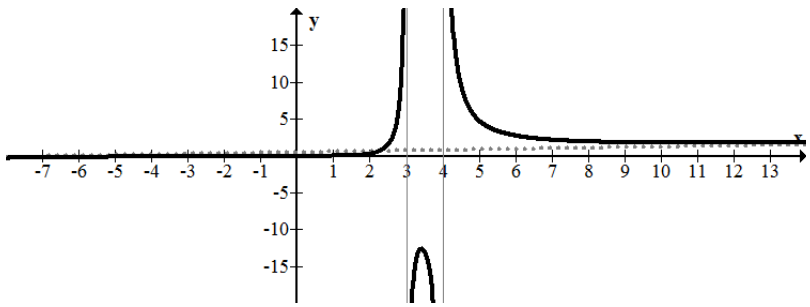



2. Here is the graph:




3.

$$f(x)=\frac{3x^3+4x-2x-4}{40(x-3)(x-4)}=\frac{3}{40}x+\frac{21}{40}+\frac{113x-256}{40(x-3)(x-4)}$$




### Practice

Use the function below for 1-7.

$$f(x)=\frac{2(x+4)(x-3)(x+1)}{8(x-1)^2(x+2)}$$

1. Identify the vertical asymptotes and holes for the function.
2. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.
3. Identify any horizontal or oblique asymptotes for the function.
4. Describe the end behavior of the function.
5. Find the zeroes of the function.
6. Find the $y$-intercept of the function.
7. Use the information from 1-6 to sketch the function.

Use the function below for 8-14.

$$g(x)=\frac{(x^2-9)(x^2-4)}{5(x-2)^2(x+1)^2}$$

8. Identify the vertical asymptotes and holes for the function.

9. What values will you use the sign test with in order to accurately sketch around the vertical asymptotes? Complete the sign test for these values.
10. Identify any horizontal or oblique asymptotes for the function.
11. Describe the end behavior of the function.
12. Find the zeroes of the function.
13. Find the $y$-intercept of the function.
14. Use the information from 8-13 to sketch the function.
15. Graph the function below by hand.

$$h(x) = \frac{x^3 + 5x^2 + 2x - 8}{x^2 - 3x - 10}$$

Polynomials and rational functions were explored. Special features of rational functions such as holes and asymptotes were introduced. Graphing rational functions by hand, which utilized work from many concepts, concluded the chapter.

# 2.14 References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# Chapter 3 Logs and Exponents

## Chapter Outline



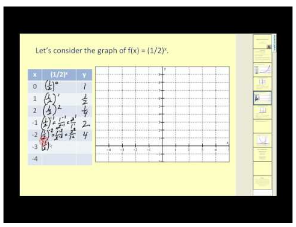
Exponential growth has been called one of the most powerful forces in the universe. You may already know the basic rules of exponents. Here you will explore their relationship with logarithms and their application to different types of growth and decay.

# 3.1 Exponential Functions

Here you will explore exponential functions as a way to model a special kind of growth or decay and you will learn more about the number $e$.

Exponential growth is one of the most powerful forces in nature. A famous legend states that the inventor of chess was asked to state his own reward from the king. The man asked for a single grain of rice for the first square of the chessboard, two grains of rice for the second square and four grains of rice for the third. He asked for the entire 64 squares to be filled in this way and that would be his reward. Did the man ask for too little, or too much?

### Watch This



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http://www.youtube.com/watch?v=7fpazNs1ZRE James Sousa: Graph Exponential Functions

### Guidance

Exponential functions take the form $f(x) = a \cdot b^x$ where $a$ and $b$ are constants. $a$ is the starting amount when $x = 0$. $b$ tells the story about the growth. If the growth is doubling then $b$ is 2. If the growth is halving (which would be decay), then $b$ is $\frac{1}{2}$. If the growth is increasing by 6% then $b$ is 1.06.

Exponential growth is everywhere. Money grows exponentially in banks. Populations of people, bacteria and animals grow exponentially when their food and space aren't limited.

Radioactive isotopes like Carbon 14 have something called a half-life that indicates how long it takes for half of the molecules present to decay into other more stable molecules. It takes about 5,730 years for this process to occur which is how scientists can date artifacts of ancient humans.

**Example A**

A mummified animal is found preserved on the slopes of an ice covered mountain. After testing, you see that exactly one fourth of the carbon-14 has yet to decay. How long ago was this animal alive?

**Solution:** Since this problem does not give specific amounts of carbon, it can be inferred that the time will not depend on the specific amounts. One technique that makes the problem easier to work with could be to create an example scenario that fits the one fourth ratio. Suppose 60 units were present when the animal was alive at time zero. This means that 15 units must be present today.

$$15 = a \cdot \left(\frac{1}{2}\right)^x$$

$$60 = a \cdot \left(\frac{1}{2}\right)^0$$

The second equation yields $a = 60$ and then the first equation becomes:

$15 = 60 \cdot \left(\frac{1}{2}\right)^x$

Although you may not yet have the algebraic tools to solve for $x$, you should still be able to see that $x$ is 2. This does not mean that two years ago the animal was alive, it means that two half life cycles ago the animal was alive. The half life cycle for carbon 14 is 5,730 years so this animal was alive over 11,000 years ago.

**Example B**

Suppose you invested \$100 the day you were born and it grew by 6% every year until you were 100 years old. How much would this investment be worth then?

**Solution:** The starting amount is 100 and the growth is 1.06. This is enough information to write an equation. The $x$ stands for time in years and the $f(x)$ stands for the amount of money in the account. After writing an equation you need to substitute in 100 years for $x$.

$$f(x) = 100 \cdot 1.06^x$$
$$f(x) = 100 \cdot 1.06^{100}$$
$$f(x) \approx 33,930.21$$

After a century, there will be almost \$34,000 in the account. Interest has greatly increased the \$100 initial investment.

**Example C**

Suppose forty rabbits are released on an island. The rabbits mate once every four months and produce up to 4 offspring who also produce more offspring four months later. Estimate the number of rabbits on the island in 3 years if their population grows exponentially. Assume half the population is female.

**Solution:** Even though parts of this problem are unrealistic, it serves to illustrate how quickly exponential growth works. Forty is the initial amount so $a = 40$. At the end of the first 4 month period 20 female rabbits could have their litters and up to 80 newborn rabbits could be born. The population has grown from 40 to 120 which means tripled. Thus, $b = 3$. The last thing to remember is that the time period is in 4 month periods. Three years must be 9 periods.

$f(x) = 40 \cdot 3^9 = 787,320$

So after three years, there could be up to 787,320 rabbits!

**Concept Problem Revisited**

The number of grains of rice on the last square would be almost ten quintillion (million million million). That is more rice than is produced in the world in an entire year.

$2^{63} = 9,223,372,036,854,775,808$

## Vocabulary

***Exponential growth*** is growth by repeated multiplication.

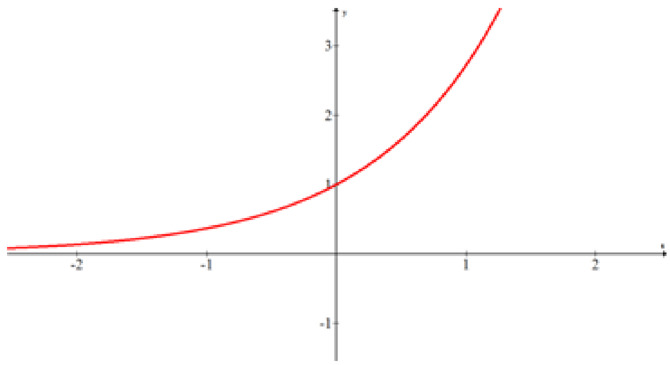
***Arithmetic growth*** is growth by repeated addition.

***Half-life*** is the amount of time it takes for the isotope carbon-14 to decay into more stable molecules. Since carbon 14 is naturally present in predictable amounts in all organic life, scientists can predict how old something is by how much carbon 14 is left.

### Guided Practice

1. Completely analyze the following exponential function.

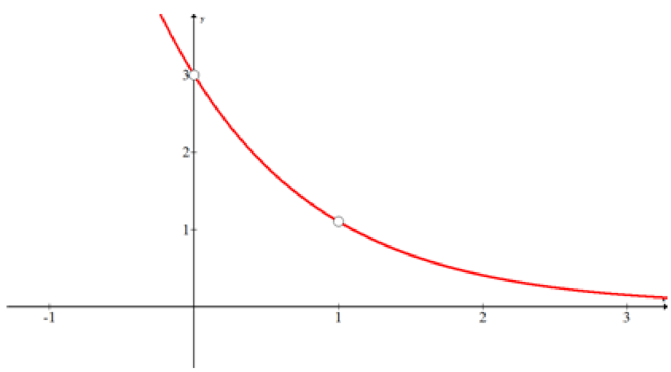
$f(x) = e^x$



2. Identify which of the following functions are exponential functions and which are not.

a. $y = x^6$
b. $y = 5^x$
c. $y = 1^x$
d. $y = x^x$
e. $y = x^{\frac{1}{2}}$

3. Write the exponential function that passes through the following points: $(0,3), \left(1, \frac{3}{e}\right)$.



**Answers:**

1. Analyze in this context means to define all the characteristics of a function.

Domain: $x \, \varepsilon \, (-\infty, \infty)$

Range: $y \, \varepsilon \, (0, \infty)$

Increasing: $x \, \varepsilon \, (-\infty, \infty)$

Decreasing: NA

Zeroes: None

Intercepts: (0, 1)

Maximums: None

Minimums: None

Asymptotes: $y = 0$ as $x$ gets infinitely small

Holes: None

2. Exponential functions are of the form $y = a \cdot b^x$

a. $y = x^6$ is not an exponential function because $x$ is not in the exponent.

b. $y = 5^x$ Exponential function.

c. $y = 1^x$ Not a true exponential function because $y$ is always 1 which is a constant function.

d. $y = x^x$ Not an exponential function because $x$ is both the base and power of the exponent.

e. $y = x^{\frac{1}{2}}$ Not an exponential function.

3. The starting number is $a = 3$. This number is changed by a factor of $\frac{1}{e}$ which is $b$.

$f(x) = 3\left(\frac{1}{e}\right)^x = 3e^{-x}$

## Practice

1. Explain what makes a function an exponential function. What does its equation look like?
2. Is the domain for all exponential functions all real numbers?
3. How can you tell from its equation whether or not the graph of an exponential function will be increasing?
4. How can you tell from its equation whether or not the graph of an exponential function will be decreasing?
5. What type of asymptotes do exponential functions have? Explain.
6. Suppose you invested \$4,500 and it grew by 4% every year for 30 years. How much would this investment be worth after 30 years?
7. Suppose you invested \$10,000 and it grew by 12% every year for 40 years. How much would this investment be worth after 40 years?

Write the exponential function that passes through the following points.

8. (0, 5) and (1, 25)
9. (0, 2) and (1, 8)
10. (0, 16) and (2, 144)
11. (1, 4) and (3, 36)
12. (0, 16) and (3, 2)
13. (0, 81) and (2, 9)
14. (1, 144) and (3, 12)
15. Explain why for exponential functions of the form $y = a \cdot b^x$ the $y$-intercept is always the value of $a$.

# 3.2 Properties of Exponents

Here you will learn how exponents interact in a variety of algebraic situations including addition, subtraction, multiplication and exponentiation.

It is important to quickly and effectively manipulate algebraic expressions involving exponents. One simplification that comes up often is that expressions and numbers raised to the 0 power are always equal to 1. Why is this true and is it always true?

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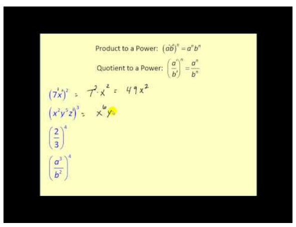


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http://www.youtube.com/watch?v=0GAMbuPJGOY James Sousa: Properties of Exponents

### Guidance

Consider the following exponential expressions with the same base and what happens through the algebraic operations. You should feel comfortable with all of these types of manipulations.

$b^y, b^x$

**Addition and subtraction:** $b^x \pm b^y = b^x \pm b^y$

Only in the special case when $x = y$ can the terms be combined. This is a basic property of combining like terms.

**Multiplication:** $b^x \cdot b^y = b^{x+y}$

When the bases are the same then exponents can be added.

**Division:** $\frac{b^x}{b^y} = b^{x-y}$

The division rule is an extension of the multiplication rule with the possibility of a negative in the exponent.

**Negative exponent:** $b^{-x} = \frac{1}{b^x}$

A negative exponent means reciprocal.

**Fractional exponent:** $(b)^{\frac{1}{x}} = \sqrt[x]{b}$

Square roots are what most people think of when they think of roots, but roots can be taken with any real number using fractional exponents.

**Powers of Powers:** $(b^x)^y = b^{x \cdot y}$

**Example A**

Simplify the following expression until all exponents are positive.

$$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0}$$

**Solution:**

$$\frac{(a^{-2}b^3)^{-3}}{ab^2c^0} = \frac{a^6b^{-9}}{ab^2 \cdot 1} = \frac{a^5}{b^{11}}$$

**Example B**

Simplify the following expression until all exponents are positive.

$$(2x)^5 \cdot \frac{4^2}{2^{-3}} \cdot \frac{a^3b^2c^4}{a^2b^{-4}c^0}$$

**Solution:**

$$(2x)^5 \cdot \frac{4^2}{2^{-3}} = \frac{2^5x^52^4}{2^{-3}} = \frac{2^9x^5}{2^{-3}} = 2^{12}x^5$$

**Example C**

Simplify the following expression until all exponents are positive.

$$\frac{(a^{-3}b^2c^4)^{-1}}{(a^2b^{-4}c^0)^3}$$

**Solution:**

$$\frac{(a^{-3}b^2c^4)^{-1}}{(a^2b^{-4}c^0)^3} = \frac{a^3b^{-2}c^{-4}}{a^6b^{-12}} = \frac{b^{10}}{a^3c^4}$$

**Concept Problem Revisited**

Consider the following pattern and decide what the next term in the sequence should be:

16, 8, 4, 2, ___

It makes sense that the next term is 1 because each successive term is half that of the previous term. These numbers correspond to powers of 2.

$2^4, 2^3, 2^2, 2^1,$ ___

In this case you could decide that the next term must be $2^0$. This is a useful technique for remembering what happens when a number is raised to the 0 power.

One question that extends this idea is what is the value of $0^0$? People have argued about this for centuries. Euler argued that it should be 1 and many other mathematicians like Cauchy and Möbius argued as well. If you search today you will still find people discussing what makes sense.

In practice, it is convenient for mathematicians to rely on $0^0 = 1$.

### Vocabulary

***Exponents*** are repeated multiplication.

***Power*** refers to the number in the exponent

***Power to a Power*** is a number raised to an exponent which in turn is raised to another exponent.

### Guided Practice

1. Simplify the following expression using positive exponents.

$$\frac{(2^6 \cdot 8^3)^{-3}}{4^2\left(\frac{1}{2}\right)^4 64^{\frac{1}{3}}}$$

2. Simplify the following expression using only positive exponents.

$((a^3b^2c)^{-1}a^2b^7c)^2$

3. Solve the following equation using properties of exponents.

$(32^{0.6})^2 = x^3$

**Answers:**

1. Rewrite every exponent as a power of 2.

For example $8^3 = (2^3)^3 = 2^9$ and $64^{\frac{1}{3}} = (2^6)^{\frac{1}{3}} = 2^2$

$$\frac{(2^6 \cdot 8^3)^{-3}}{4^2\left(\frac{1}{2}\right)^4 64^{\frac{1}{3}}} = \frac{(2^6 \cdot 2^9)^{-3}}{2^4 2^{-4} 2^2} = \frac{(2^{15})^{-3}}{2^2} = \frac{2^{-45}}{2^2} = \frac{1}{2^{47}}$$

2. $((a^3b^2c)^{-1}a^2b^7c)^2 = (a^{-3}b^{-2}c^{-1}a^2b^7c)^2 = (a^{-1}b^5)^2 = \frac{b^{10}}{a^2}$

3. First work with the left hand side of the equation.

$$\begin{aligned}
(32^{0.6})^2 = \left((2^5)^{\frac{3}{5}}\right)^2 &= 2^6 \\
2^6 &= x^3 \\
(2^6)^{\frac{1}{3}} &= (x^3)^{\frac{1}{3}} \\
2^2 &= x \\
4 &= x
\end{aligned}$$

### Practice

Simplify each expression using positive exponents.

1. $81^{-\frac{1}{4}}$

2. $64^{\frac{2}{3}}$

3. $\left(\frac{1}{32}\right)^{-\frac{2}{5}}$

4. $(-125)^{\frac{1}{3}}$

5. $(4x^3y)(3x^5y^2)^4$

6. $(5x^3y^2)^2(7x^3y)^2$

7. $\frac{8a^3b^{-2}}{(-4a^2b^4)^{-2}}$

8. $\frac{5x^2y^{-3}}{(-2x^3y^2)^{-4}}$

9. $\left(\frac{3m^3n^{-4}}{2m^{-5}n^{-2}}\right)^{-4}$

10. $\left(\frac{4m^{-3}n^{-4}}{5m^5n^{-4}}\right)^{-3}$

11. $\left(\frac{a^{-1}b}{a^5b^4}\right)^{-3}$

12. $\frac{15c^{-2}d^{-6}}{3c^{-4}d^{-2}}$

13. $\frac{12e^5f}{(-2ef^3)^{-2}}$

Solve the following equations using properties of exponents.

14. $(81^{0.75})^2 = x^3$

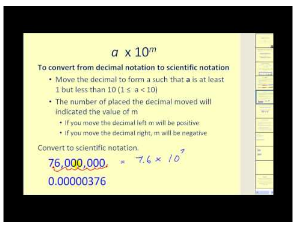
15. $\left(64^{\frac{1}{6}}\right)^{-3} = x^3$

# 3.3 Scientific Notation

Here you will review how to write very large and very small numbers in scientific notation and how to use scientific notation in arithmetic operations.

In science, measurements are often extremely small or extremely large. It is inefficient to write the many zeroes in very small numbers like 0.00000000000000523. Usually, the order of magnitude and the first few digits of the number are what people are interested in. How should you represent these extreme numbers?

### Watch This

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Click image to the left for more content.

http://www.youtube.com/watch?v=hY-ecKyZ244 James Sousa: Scientific Notation

### Guidance

Scientific notation is a means of representing very large and very small numbers in a more efficient way. The general form of scientific notation is $a \cdot 10^b$

The $a$ is a number between 1 and 10 and most often includes a decimal. The integer $b$ is called the order of magnitude and is a measure of the general size of the number. If $b$ is negative then the number is small and if $b$ is positive then the number is large.

$$1,240,000 = 1.24 \cdot 10^6$$
$$0.0000354 = 3.54 \cdot 10^{-5}$$

Note that when switching to and from scientific notation the sign of $b$ indicates which direction and how many places to move the decimal point.

Multiplying and dividing numbers that are in scientific notation is just an exercise in exponent rules:

$$(a \cdot 10^x) \cdot (b \cdot 10^y) = a \cdot b \cdot 10^{x+y}$$
$$(a \cdot 10^x) \div (b \cdot 10^y) = \frac{a}{b} \cdot 10^{x-y}$$

Addition and subtraction require the numbers to have identical order of magnitudes.

$1.2 \cdot 10^6 - 5.5 \cdot 10^5 = 12 \cdot 10^5 - 5.5 \cdot 10^5 = 6.5 \cdot 10^5$

**Example A**

An electron's mass is about 0.000 000 000 000 000 000 000 000 000 000 910 938 22 kg.

Write this number in scientific notation.

**Solution:** $9.109\ 3822 \cdot 10^{-31}$

**Example B**

The Earth's circumference is approximately 40,000,000 meters. What is the radius of the earth in scientific notation?

**Solution:** The relationship between circumference and radius is $C = 2\pi r$.

$$\begin{aligned} 4.0 \cdot 10^7 &= 2\pi r \\ r &= \frac{4.0}{2\pi} \cdot 10^7 \approx 0.6366\ldots \cdot 10^7 = 6.366 \cdot 10^6 \end{aligned}$$

Note that the number of significant digits required depends on the context.

**Example C**

Simplify the following expression.

$x = (4.56 \cdot 10^7) \cdot (2.89 \cdot 10^8) \div (7.15 \cdot 10^{-15}) + 216$

**Solution:** $x = 4.56 \cdot 2.89 \cdot 7.15 \cdot 10^0 + 216 = 310.22556$

**Concept Problem Revisited**

In order to represent an extremely large or small number you should count the number of moves necessary for the decimal point to be directly after the first non-zero digit. This count will be the order of magnitude and will be used as the exponent of 10 as a means of representing how large or small the number is.

## Vocabulary

***Order of magnitude*** is formally the exponent in scientific notation. Informally it refers to size. Two objects or numbers are of the same order of magnitude are relatively similar sizes. A marble and a planet are not of the same order of magnitude, but Earth and Venus are.

***Scientific notation*** is a means of representing a number as a product of a number between 1 and 10 and a power of 10.

## Guided Practice

1. Order the following numbers from least to greatest.

$5.411 \cdot 10^{-3}$ $7.837 \cdot 10^{-4}$ $9.999 \cdot 10^3$ $9.5983 \cdot 10^{-7}$ $8.0984 \cdot 10^3$

2. Compute the following number and use scientific notation.

$2,000,000^3 \cdot 3,000^4$

3. Simplify the following expression.

$(4.713 \cdot 10^7) + (8.985 \cdot 10^5) - (4.987 \cdot 10^2) \cdot (7.3 \cdot 10^{-6}) \div (6.74 \cdot 10^{-9})$

**Answers:**

1. First consider the order of magnitude of each number. Small numbers have negative exponents. If two numbers have the same order of magnitude, then compare the actual digits.

$9.5983 \cdot 10^{-7} < 7.837 \cdot 10^{-4} < 5.411 \cdot 10^{-3} < 8.0984 \cdot 10^3 < 9.999 \cdot 10^3$

2. First convert each number to scientific notation individually, then process the exponent and multiplication.

$$\begin{aligned} 2,000,000^3 \cdot 3,000^4 &= (2 \cdot 10^6)^3 \cdot (3 \cdot 10^3)^4 \\ &= 8 \cdot 10^{18} \cdot 81 \cdot 10^{12} \\ &= 648 \cdot 10^{30} \\ &= 6.48 \cdot 10^{32} \end{aligned}$$

3. Resolve in order of standard order of operations

$(4.713 \cdot 10^7) + (8.985 \cdot 10^5) - (4.987 \cdot 10^2) \cdot (7.3 \cdot 10^{-6}) \div (6.74 \cdot 10^{-9})$

$$\begin{aligned} &= (4.713 \cdot 10^7) + (8.985 \cdot 10^5) - (5.40135 \cdot 10^5) \\ &= (471.3 \cdot 10^5) + (8.985 \cdot 10^5) - (5.40135 \cdot 10^5) \\ &= 474.8836499 \cdot 10^5 \\ &= 4.748836499 \cdot 10^7 \end{aligned}$$

## Practice

Write the following numbers in scientific notation.

1. 152,780
2. 0.00003256
3. 56, 320
4. 0.0821
5. 1, 000, 000, 000, 000, 000, 000, 000
6. 7.32
7. If the federal budget is $1.5 trillion, how much does it cost each individual, on average, if there are 300,000,000 people?
8. The Library of Congress has about 60,000,000 items. How could you express this number in scientific notation?
9. The sun develops $5 \times 10^{23}$ horsepower per second. How much horsepower is developed in a day? In a year with 365 days?
10. A light-year is about 5,869,713,600 miles. A spacecraft travels $8.23 \times 10^4$ miles per hour. How long will it take the spacecraft to travel a light–year?
11. Compute the following number and use scientific notation: $324,000 \cdot 30,000^3$.
12. Compute the following number and use scientific notation: $14,300 \cdot 20,200^2$.

Simplify the following expressions.

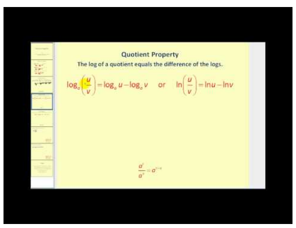
13. $(3.29 \cdot 10^4) - (3.295 \cdot 10^5) + (1.25 \cdot 10^2) \cdot (3.97 \cdot 10^{15}) \cdot (5.8 \cdot 10^{-6})$
14. $(1.95 \cdot 10^2) + (6.798 \cdot 10^6) + (2.896 \cdot 10^3) \cdot (5.6 \cdot 10^{-3}) \div (2.89 \cdot 10^4)$
15. $(2.158 \cdot 10^7) \cdot (1.679 \cdot 10^6) - (9.98 \cdot 10^4) \cdot (3.4 \cdot 10^{-2})$

# 3.4 Properties of Logs

Here you will be introduced to logarithmic expressions and will learn how they can be combined using properties of arithmetic.

Log functions are inverses of exponential functions. This means the domain of one is the range of the other. This is extremely helpful when solving an equation and the unknown is in an exponent. Before solving equations, you must be able to simplify expressions containing logs. The rules of exponents are applied, but in non-obvious ways. In order to get a conceptual handle on the properties of logs, it may be helpful to continually ask, what does a log expression represent? For example, what does$\log_{10} 1,000$ represent?

### Watch This

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http://www.youtube.com/watch?v=SxF44olWTyk James Sousa: Properties of Logarithms

### Guidance

Exponential and logarithmic expressions have the same 3 components. They are each written in a different way so that a different variable is isolated. The following two equations are equivalent to one another.

$b^x = a \leftrightarrow \log_b a = x$

The exponential equation is read "*b* to the power *x* is *a*." The logarithmic equation is read "log base *b* of *a* is *x*".

The two most common bases for logs are 10 and *e*. At the PreCalculus level *log* by itself implies log base 10 and *ln* implies base *e*. *ln* is called the natural log. One important restriction for all log functions is that they must have strictly positive numbers in their arguments. So, if you press log -2 or log 0 on your calculator, it will give an error.

There are four basic properties of logs that correlate to properties of exponents.

**Addition/Multiplication:**

$\log_b x + \log_b y = \log_b (x \cdot y)$

$b^{w+z} = b^w \cdot b^z$

**Subtraction/Division:**

$\log_b x - \log_b y = \log_b \left(\frac{x}{y}\right)$

$b^{w-z} = \frac{b^w}{b^z}$

**Exponentiation:**

$\log_b (x^n) = n \cdot \log_b x$

$(b^w)^n = b^{w \cdot n}$

There are a few standard results that should be memorized and should serve as baseline reference tools.

- $\log_b 1 = 0$
- $\log_b b = 1$
- $\log_b (b^x) = x$
- $b^{\log_b x} = x$

**Example A**

Simplify the following expressions:

a. $\log_4 64$
b. $\log_{\frac{1}{2}} 32$
c. $\log_3 3^5$
d. $\log_2 128$

**Solution:**

a. $\log_4 64 = \log_4 4^3 = 3 \cdot \log_4 4 = 3 \cdot 1 = 3$
b.

$$\begin{aligned} \log_{\frac{1}{2}} 32 = x \text{ can be rewritten as } \left(\frac{1}{2}\right)^x &= 32. \\ 2^{-x} &= 32 \\ x &= -5 \end{aligned}$$

c. $\log_3 3^5 = 5 \cdot \log_3 3 = 5$
d. $\log_2 128 = \log_2 2^7 = 7$

**Example B**

Write the expression as a logarithm of a single argument.

$\log_2 12 + \log_4 6 - \log_2 24$

**Solution:** Note that the center expression is of a different base. First change it to base 2 by switching back to exponential form.

$$\begin{aligned} \log_4 6 = x &\leftrightarrow 4^x = 6 \\ 2^{2x} = 6 &\leftrightarrow \log_2 6 = 2x \\ x &= \frac{1}{2} \log_2 6 = \log_2 6^{\frac{1}{2}} \end{aligned}$$

Thus the expression with the same base is:

$$\begin{aligned} \log_2 12 + \log_2 6^{\frac{1}{2}} - \log_2 24 &= \log_2 \left(\frac{12 \cdot \sqrt{6}}{24}\right) \\ &= \log_2 \left(\frac{\sqrt{6}}{2}\right) \end{aligned}$$

**Example C**

Simplify the following expression: $2\log_{12} 144^{-4}$.

**Solution:** $2\log_{12} 144^{-4} = -8 \cdot \log_{12} 12^2 = -16 \cdot \log_{12} 12 = -16$

**Concept Problem Revisited**

A log expression represents an exponent. The expression $\log_{10} 1,000$ represents the number 3. The reason to keep this in mind is that it can solidify the properties of logs. For example, adding exponents implies bases are multiplied. Thus adding logs means the bases of the exponents are multiplied.

## Vocabulary

A ***logarithm*** is a way of rewriting exponential equations to isolate the exponent.

## Guided Practice

1. Prove the following log identity:

$\log_a b = \frac{1}{\log_b a}$

2. Rewrite the following expression under a single log.

$\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$

3. True or false:

$(\log_3 4x) \cdot (\log_3 5y) = \log_3(4x + 5y)$

**Answers:**

1. Start by letting the left side of the equation be equal to $x$. Then, rewrite in exponential form, manipulate, and rewrite back in logarithmic form until you get the expression from the left side of the equation.

$$\begin{aligned}
\log_a b &= x \\
b^x &= a \\
b &= a^{\frac{1}{x}} \\
\log_b a &= \frac{1}{x} \\
x &= \frac{1}{\log_b a}
\end{aligned}$$

Therefore, $\frac{1}{\log_b a} = \log_a b$ because both expressions are equal to $x$.

2. $\ln e - \ln 4x + 2(e^{\ln x} \cdot \ln 5)$

$$\begin{aligned}
&= \ln\left(\frac{e}{4x}\right) + 2x \cdot \ln 5 \\
&= \ln\left(\frac{e}{4x}\right) + \ln(5^{2x}) \\
&= \ln\left(\frac{e \cdot 5^{2x}}{4x}\right)
\end{aligned}$$

3. Note, it may be very tempting to make errors in this practice problem.

False. It is true that the log of a product is the sum of logs. It is not true that the product of logs is the log of a sum.

### Practice

Decide whether each of the following statements are true or false. Explain.

1. $\frac{\log x}{\log y} = \log\left(\frac{x}{y}\right)$

2. $(\log x)^n = n \log x$

3. $\log x + \log y = \log xy$

Rewrite each of the following expressions under a single log and simplify.

4. $\log 4x + \log(2x+4)$

5. $5\log x + \log x$

6. $4\log_2 x + \frac{1}{2}\log_2 9 - \log_2 y$

7. $6\log_3 z^2 + \frac{1}{4}\log_3 y^8 - 2\log_3 z^4 y$

Expand the expression as much as possible.

8. $\log_4\left(\frac{2x^3}{5}\right)$

9. $\ln\left(\frac{4xy^2}{15}\right)$

10. $\log\left(\frac{x^2(yz)^3}{3}\right)$

Translate from exponential form to logarithmic form.

11. $2^{x+1} + 4 = 14$

Translate from logarithmic form to exponential form.

12. $\log_2(x-1) = 12$

Prove the following properties of logarithms.

13. $\log_{b^n} x = \frac{1}{n}\log_b x$

14. $\log_{b^n} x^n = \log_b x$

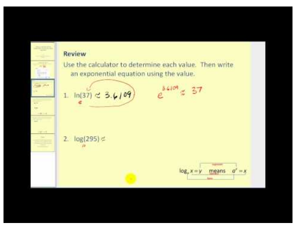
15. $\log_{\frac{1}{b}} \frac{1}{x} = \log_b x$

# 3.5 Change of Base

Here you will extend your knowledge of log properties to a simple way to change the base of a logarithm.

While it is possible to change bases by always going back to exponential form, it is more efficient to find out how to change the base of logarithms in general. Since there are only base $e$ and base 10 logarithms on a calculator, how would you evaluate an expression like $\log_3 12$?

### Watch This



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Click image to the left for more content.

http://www.youtube.com/watch?v=9kKg19s5b78 James Sousa: Logarithms Change of Base Formula

### Guidance

The change of base property states:

$\log_b a = \frac{\log_x a}{\log_x b}$

You can derive this formula by converting $\log_b a$ to exponential form and then taking the log base $x$ of both sides. This is shown below.

$$\begin{aligned}
\log_b a &= y \\
\rightarrow b^y &= a \\
\rightarrow \log_x b^y &= \log_x a \\
\rightarrow y \log_x b &= \log_x a \\
\rightarrow y &= \frac{\log_x a}{\log_x b}
\end{aligned}$$

Therefore, $\log_b a = \frac{\log_x a}{\log_x b}$.

**Example A**

Evaluate $\log_3 4$ by changing the base and using your calculator.

**Solution:**

$\log_3 4 = \frac{\log_{10} 4}{\log_{10} 3} = \frac{\ln 4}{\ln 3} \approx 1.262$

**Example B**

Prove the following log identity.

$\log_a b = \frac{1}{\log_b a}$

**Solution:**

$\log_a b = \frac{\log_x b}{\log_x a} = \frac{1}{\frac{\log_x a}{\log_x b}} = \frac{1}{\log_b a}$

**Example C**

Simplify to an exact result: $(\log_4 5) \cdot (\log_3 4) \cdot (\log_5 81) \cdot (\log_5 25)$

**Solution:**

$\frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{\log 3^4}{\log 5} \cdot \frac{\log 5^2}{\log 5} = \frac{\log 5}{\log 4} \cdot \frac{\log 4}{\log 3} \cdot \frac{4 \cdot \log 3}{\log 5} \cdot \frac{2 \cdot \log 5}{\log 5} 4 \cdot 2 = 8$

**Concept Problem Revisited**

In order to evaluate an expression like $\log_3 12$ you have two options on your calculator:

$\frac{\ln 12}{\ln 3} = \frac{\log 12}{\log 3} \approx 2.26$

## Vocabulary

***Change of base*** refers to the formula that allows you to rewrite a logarithm with a new base that you choose. Common bases to use are 10 and $e$.

## Guided Practice

1. Simplify the following expression: $(\log_{2013} 7 + \log_{2013} 9 - \log_{2013} 11) \log 2013$

2. Evaluate: $\log_2 48 - \log_4 36$

3. Given $\log_3 5 \approx 1.465$ find $\log_{25} 27$ without using a log button on the calculator.

**Answers:**

1. $\left[\frac{\log 7}{\log 2013} + \frac{\log 9}{\log 2013} - \frac{\log 11}{\log 2013}\right] \cdot \log 2013 = \log 7 + \log 9 - \log 11 = \log\left(\frac{7 \cdot 9}{11}\right)$

2. $\log_2 48 - \log_4 36$

$$\begin{aligned}
&= \frac{\log 48}{\log 2} - \frac{\log 36}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{\log 6^2}{\log 2^2} \\
&= \frac{\log 48}{\log 2} - \frac{2 \cdot \log 6}{2 \cdot \log 2} \\
&= \frac{\log 48 - \log 6}{\log 2} \\
&= \frac{\log\left(\frac{48}{6}\right)}{\log 2} \\
&= \frac{\log 8}{\log 2} \\
&= \frac{\log 2^3}{\log 2} \\
&= \frac{3 \cdot \log 2}{\log 2} \\
&= 3
\end{aligned}$$

3. $\log_{25} 27 = \frac{\log 3^3}{\log 5^2} = \frac{3}{2} \cdot \frac{1}{\left(\frac{\log 5}{\log 3}\right)} = \frac{3}{2} \cdot \frac{1}{\log_3 5} \approx \frac{3}{2} \cdot \frac{1}{1.465} = 1.0239$

## Practice

Evaluate each expression by changing the base and using your calculator.

1. $\log_6 15$
2. $\log_9 12$
3. $\log_5 25$

Evaluate each expression.

4. $\log_8(\log_4(\log_3 81))$
5. $\log_2 3 \cdot \log_3 4 \cdot \log_6 16 \cdot \log_4 6$
6. $\log 125 \cdot \log_9 4 \cdot \log_4 81 \cdot \log_5 10$
7. $\log_5(5^{\log_5 125})$
8. $\log(\log_6(\log_2 64))$
9. $10^{\log_{100} 9}$
10. $(\log_4 x)(\log_x 16)$
11. $\log_{49} 49^5$
12. $3\log_{24} 24^8$
13. $4^{\log_2 3}$

Prove the following properties of logarithms.

14. $(\log_a b)(\log_b c) = \log_a c$
15. $(\log_a b)(\log_b c)(\log_c d) = \log_a d$

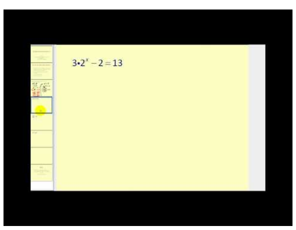
# 3.6 Exponential Equations

Here you will apply the new algebraic techniques associated with logs to solve equations.

When you were first learning equations, you learned the rule that whatever you do to one side of an equation, you must also do to the other side so that the equation stays in balance. The basic techniques of adding, subtracting, multiplying and dividing both sides of an equation worked to solve almost all equations up until now. With logarithms, you have more tools to isolate a variable. Consider the following equation and ask yourself: why is$x = 3$? Logically it makes sense that if the bases match, then the exponents must match as well, but how can it be shown?

$$1.79898^{2x} = 1.79898^6$$

### Watch This

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http://www.youtube.com/watch?v=5R5mKpLsfYg James Sousa: Solving Exponential Equations II

### Guidance

A common technique for solving equations with unknown variables in exponents is to take the log of the desired base of both sides of the equation. Then, you can use properties of logs to simplify and solve the equation. See the examples below.

**Example A**

Solve the following equation for $t$. *Note: This type of equation is common in financial mathematics. This example represents the unknown amount of time it will take you to save $9,000 in a savings account if you save $300 at the end of each year in an account that earns 6% annual compound interest.*

$9,000 = 300 \cdot \frac{(1.06)^t - 1}{0.06}$

**Solution:**

$$\begin{aligned}30 &= \frac{(1.06)^t - 1}{0.06}\\1.8 &= (1.06)^t - 1\\2.8 &= 1.06^t\\\ln 2.8 &= \ln(1.06^t) = t \cdot \ln(1.06)\\t &= \frac{\ln 2.8}{\ln 1.06} \approx 17.67 \text{ years}\end{aligned}$$

**Example B**

Solve the following equation for $x$: $16^x = 25$

**Solution:** First take the log of both sides. Then, use log properties and your calculator to help.

$$\begin{aligned}16^x &= 25\\\log 16^x &= \log 25\\x \log 16 &= \log 25\\x &= \frac{\log 25}{\log 16}\\x &= 1.61\end{aligned}$$

**Example C**

Solve the following equation for all possible values of $x$: $(\log_2 x)^2 - \log_2 x^7 = -12$.

**Solution:** In calculus it is common to use a small substitution to simplify the problem and then substitute back later. In this case let $u = \log_2 x$ after the 7 has been brought down and the 12 brought over.

$$\begin{aligned}(\log_2 x)^2 - 7\log_2 x + 12 &= 0\\u^2 - 7u + 12 &= 0\\(u-3)(u-4) &= 0\\u &= 3, 4\end{aligned}$$

Now substitute back and solve for $x$ in each case.

$$\begin{aligned}\log_2 x &= 3 \leftrightarrow 2^3 = x = 8\\\log_2 x &= 4 \leftrightarrow 2^4 = x = 16\end{aligned}$$

**Concept Problem Revisited**

In the equation, logs can be used to reduce the equation to $2x = 6$.

$1.79898^{2x} = 1.79898^6$

Take the log of both sides and use the property of exponentiation of logs to bring the exponent out front.

$$\log 1.79898^{2x} = \log 1.79898^{6}$$
$$2x \cdot \log 1.79898 = 6 \cdot \log 1.79898$$
$$2x = 6$$
$$x = 3$$

### Vocabulary

***Taking the log of both sides*** is an expression that refers to the action of writing log in front of the entire right hand side of an equation and the entire left hand side of the equation. As long as neither side is negative or equal to zero it maintains the equality of the two sides of the equation.

### Guided Practice

1. Solve the following equation for all possible values of $x$: $(x+1)^{x-4} - 1 = 0$

2. Light intensity as it travels at specific depths of water in a swimming pool can be described by the relationship between $i$ for intensity and $d$ for depth in feet. What is the intensity of light at 10 feet?

$\log\left(\frac{i}{12}\right) = -0.0145 \cdot d$

3. Solve the following equation for all possible values of $x$.

$$\frac{e^x - e^{-x}}{3} = 14$$

**Answers:**

1. $(x+1)^{x-4} - 1 = 0$

$(x+1)^{x-4} = 1$

Case 1 is that $x+1$ is positive in which case you can take the log of both sides.

$$(x-4) \cdot \log(x+1) = 0$$
$$x = 4, 0$$

Note that $\log 1 = 0$

Case 2 is that $(x+1)$ is negative 1 and raised to an even power. This happens when $x = -2$.

The reason why this exercise is included is because you should not fall into the habit of assuming that you can take the log of both sides of an equation. It is only valid when the argument is strictly positive.

2. Given $d = 10$, solve for $i$ measured in lumens.

$$\log\left(\frac{i}{12}\right) = -0.0145 \cdot d$$
$$\log\left(\frac{i}{12}\right) = -0.0145 \cdot 10$$
$$\log\left(\frac{i}{12}\right) = -0.145$$
$$\left(\frac{i}{12}\right) = e^{-0.145}$$
$$i = 12 \cdot e^{-0.145} \approx 10.380$$

3. First solve for $e^x$,

$$\frac{e^x - e^{-x}}{3} = 14$$
$$e^x - e^{-x} = 42$$
$$e^{2x} - 1 = 42e^x$$
$$(e^x)^2 - 42e^x - 1 = 0$$

Let $u = e^x$.

$$u^2 - 42u - 1 = 0$$
$$u = \frac{-(-42) \pm \sqrt{(-42)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} = \frac{42 \pm \sqrt{1768}}{2} \approx 42.023796, -0.0237960$$

Note that the negative result is extraneous so you only proceed in solving for $x$ for one result.

$$e^x \approx 42.023796$$
$$x \approx \ln 42.023796 \approx 3.738$$

## Practice

Solve each equation for $x$. Round each answer to three decimal places.

1. $4^x = 6$
2. $5^x = 2$
3. $12^{4x} = 1020$
4. $7^{3x} = 2400$
5. $2^{x+1} - 5 = 22$
6. $5x + 12^x = 5x + 7$
7. $2^{x+1} = 2^{2x+3}$
8. $3^{x+3} = 9^{x+1}$

9. $2^{x+4} = 5^x$

10. $13 \cdot 8^{0.2x} = 546$

11. $b^x = c + a$

12. $32^x = 0.94 - .12$

Solve each log equation by using log properties and rewriting as an exponential equation.

13. $\log_3 x + \log_3 5 = 2$

14. $2\log x = \log 8 + \log 5 - \log 10$

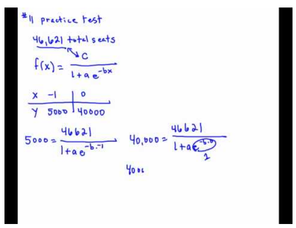
15. $\log_9 x = \frac{3}{2}$

# 3.7 Logistic Functions

Here you will explore the graph and equation of the logistic function.

Exponential growth increases without bound. This is reasonable for some situations; however, for populations there is usually some type of upper bound. This can be caused by limitations on food, space or other scarce resources. The effect of this limiting upper bound is a curve that grows exponentially at first and then slows down and hardly grows at all. What are some other situations in which logistic growth would be an appropriate model?

### Watch This



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Click image to the left for more content.

http://www.youtube.com/watch?v=OSMPeY354cU Finding a Logistic Function

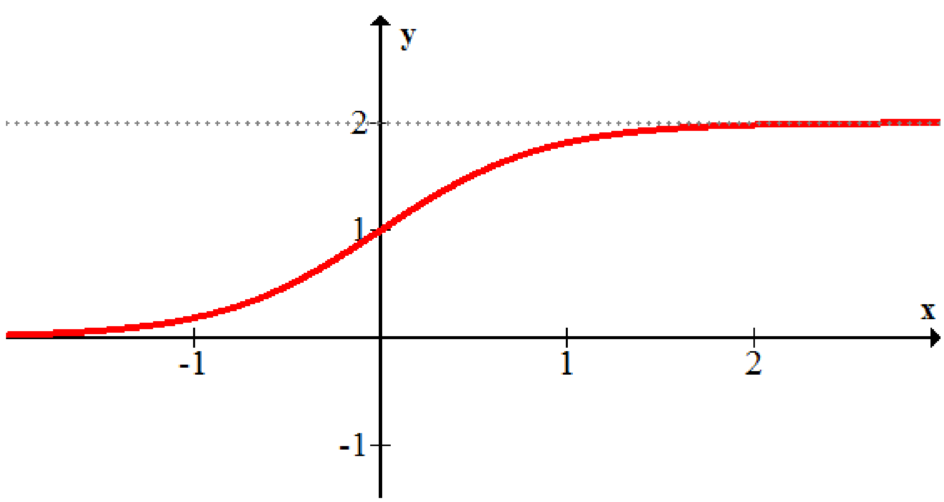
### Guidance

The logistic equation is of the form: $f(x) = \frac{c}{1+a \cdot b^x}$

The letters $a, b$ and $c$ are constants that can be changed to match the situation being modeled. The constant $c$ is particularly important because it is the limit to growth. This is also known as the carrying capacity.

The following logistic function has a carrying capacity of 2 which can be directly observed from its graph.

$f(x) = \frac{2}{1+0.1^x}$




An important note about the logistic function is that it has an inflection point. From the previous graph you can observe that at the point (0, 1) the graph transitions from curving up (concave up) to curving down (concave down). This change in curvature will be studied more in calculus, but for now it is important to know that the inflection point occurs halfway between the carrying capacity and the $x$ axis.

**Example A**

A rumor is spreading at a school that has a total student population of 1200. Four people know the rumor when it starts and three days later three hundred people know the rumor. About how many people at the school know the rumor by the fourth day?

**Solution:** In a limited population, the count of people who know a rumor is an example of a situation that can be modeled using the logistic function. The population is 1200 so this will be the carrying capacity.

Identifying information: $c = 1200; (0,4); (3,300)$. First, use the point (0, 4) so solve for $a$.

$$\begin{aligned}\frac{1200}{1+a\cdot b^0} &= 4\\ a &= 299\end{aligned}$$

Next, use the point (3, 300) to solve for $b$.

$$\begin{aligned}\frac{1200}{1+299\cdot b^3} &= 300\\ 4 &= 1+299\, b^3\\ \frac{3}{299} &= b^3\\ 0.21568 &\approx b\end{aligned}$$

The modeling equation at $x = 4$:

$f(x) = \frac{1200}{1+299\cdot 0.21568^x} \rightarrow f(4) \approx 729\ people$

A similar growth pattern will exist with any kind of infectious disease that spreads quickly and can only infect a person or animal once.

**Example B**

Long Island has roughly 8 million people. A hundred years ago, it had 2 million people. Suppose that the resources and infrastructure of the island could only support 20 million people. When will the population reach ten million inhabitants?

**Solution:** Identify known points and the carrying capacity. (0, 8,000,000) and (-100, 2,000,000). $c = 20,000,000$. Use the first point to solve for $a$.

$$\begin{aligned}8,000,000 &= \frac{20,000,000}{1+a\cdot b^0}\\ 1+a &= \frac{20,000,000}{8,000,000} = 2.5\\ a &= 1.5\end{aligned}$$

Now use the other point to solve for $b$.

$$\begin{aligned}
2,000,000 &= \frac{20,000,000}{1+1.5\cdot b^{-100}}\\
1+1.5\cdot b^{-100} &= 10\\
b^{-100} &= \frac{9}{1.5} = 6\\
b &\approx 0.98224
\end{aligned}$$

The question asks for the $x$ value when $f(x) = 10,000,000$.

$$\begin{aligned}
10,000,000 &= \frac{20,000,000}{1+1.5\cdot(0.98224)^x}\\
1+1.5\cdot(0.98224)^x &= 2\\
(0.98224)^x &= \frac{2}{3}\\
x\cdot\ln(0.98224) &= \ln\left(\frac{2}{3}\right)\\
x &= \frac{\ln\left(\frac{2}{3}\right)}{\ln(0.98224)} \approx 22.629
\end{aligned}$$

This means that according to your assumption and the two population data points you used, the predicted time from now that the population of Long Island will reach 10 million inhabitants is about 22.6 years.

**Example C**

A special kind of algae is grown in giant clear plastic tanks and can be harvested to make biofuel. The algae are given plenty of food, water and sunlight to grow rapidly and the only limiting resource is space in the tank. The algae are harvested when 95% of the tank is full leaving the tank 5% full of algae to reproduce and refill the tank. Currently the time between harvests is twenty days and the payoff is 90% harvest. Would you recommend a more optimal harvest schedule?

**Solution:** Identify known quantities and model the growth of the algae.

Known quantities: $(0,0.05);(20,0.95);c=1\ or\ 100\%$

$$\begin{aligned}
0.05 &= \frac{1}{1+a\cdot b^0}\\
a &= 19\\
0.95 &= \frac{1}{1+19\cdot b^{20}}\\
1+19\cdot b^{20} &= \frac{1}{0.95}\\
b^{20} &= \frac{\left(\frac{1}{0.95}-1\right)}{19}\\
b &\approx 0.74495
\end{aligned}$$

The model for the algae growth is:

$f(x) = \frac{1}{1+19\cdot(0.74495)^x}$

The question asks about optimal harvest schedule. Currently the harvest is 90% per 20 day or a unit rate of 4.5% per day. If you shorten the time between harvests where the algae are growing the most efficiently, then potentially this unit rate might be higher. Suppose you leave 15% of the algae in the tank and harvest when it reaches 85%. How much time will that take to yield 70%?

$$\begin{aligned} 0.15 &= \frac{1}{1+19\cdot(0.74495)^x} \\ x_1 &\approx 4.10897 \\ 0.85 &= \frac{1}{1+19\cdot(0.74495)^x} \\ x_2 &\approx 15.8914 \end{aligned}$$

It takes about 12 days for the batches to yield 70% harvest which is a unit rate of about 6% per day. This is a significant increase in efficiency. A harvest schedule that maximizes the time where the logistic curve is steepest creates the fastest overall algae growth.

**Concept Problem Revisited**

The logistic model is appropriate whenever the total count has an upper limit and the initial growth is exponential. Examples are the spread of rumors and disease in a limited population and the growth of bacteria or human population when resources are limited.

## Vocabulary

***Carrying capacity*** is the maximum sustainable population that the environmental factors will support. In other words, it is the population limit.

## Guided Practice

1. Given the following logistic model, predict the $x$ value that will produce a height of 14 and then predict the height when $x$ is 4.

$f(x) = \frac{20}{1+4\cdot(0.9)^x}$

2. Determine the logistic model given $c = 12$ and the points (0, 9) and (1, 11).

3. Determine the logistic model given $c = 7$ and the points (0, 2) and (3, 5).

**Answers:**

1. The first part involves solving for $x$ with a known height.

$$\begin{aligned} 14 &= \frac{20}{1+4\cdot(0.9)^x} \\ 1+4\cdot(0.9)^x &= \frac{20}{14} \\ (0.9)^x &= \frac{\left(\frac{20}{14}-1\right)}{4} \\ x = \log_{0.9}\left(\frac{\left(\frac{20}{14}-1\right)}{4}\right) &= \frac{\ln\left(\frac{\left(\frac{20}{14}-1\right)}{4}\right)}{\ln 0.9} \approx 21.1995 \end{aligned}$$

The second part requires a substitution for $x = 4$.

$f(x) = \frac{20}{1+4\cdot(0.9)^4} = 5.51815$

2. The two points give two equations, and the logistic model has two variables.

$$\begin{aligned} 9 &= \frac{12}{1+a\cdot b^0} \\ 1+a &= \frac{12}{9} \\ a &= \frac{1}{3} \\ 11 &= \frac{12}{1+\left(\frac{1}{3}\right)\cdot b^1} \\ 1+\left(\frac{1}{3}\right)\cdot b &= \frac{12}{11} \\ b &= 0.\overline{27} \end{aligned}$$

Thus the approximate model is:

$f(x) = \frac{12}{1+\left(\frac{1}{3}\right)\cdot(0.27273)^x}$

3. The two points give two equations, and the logistic model has two variables.

$$\begin{aligned} 2 &= \frac{7}{1+a} \\ a &= 1.5 \\ 5 &= \frac{7}{1+(1.5)\cdot b^3} \\ b^3 &= 0.4 \\ b &\approx 0.7368 \end{aligned}$$

Thus the approximate model is:

$f(x) = \frac{7}{1+(1.5)\cdot(0.7368)^x}$

### Practice

For 1-5, determine the logistic model given the carrying capacity and two points.

1. $c = 12; (0,5); (1,7)$
2. $c = 200; (0,150); (5,180)$
3. $c = 1500; (0,150); (10,1000)$
4. $c = 1000000; (0,100000); (-40,20000)$
5. $c = 30000000; (-60,10000); (0,8000000)$

For 6-8, use the logistic function $f(x) = \frac{32}{1+3e^{-x}}$.

6. What is the carrying capacity of the function?
7. What is the $y$-intercept of the function?
8. Use your answers to 6 and 7 along with at least two points on the graph to make a sketch of the function.

For 9-11, use the logistic function $g(x) = \frac{25}{1+4\cdot 0.2^x}$.

9. What is the carrying capacity of the function?

10. What is the $y$-intercept of the function?

11. Use your answers to 9 and 10 along with at least two points on the graph to make a sketch of the function.

For 12-14, use the logistic function $h(x) = \frac{4}{1+2\cdot 0.68^x}$.

12. What is the carrying capacity of the function?

13. What is the $y$-intercept of the function?

14. Use your answers to 12 and 13 along with at least two points on the graph to make a sketch of the function.

15. Give an example of a logistic function that is decreasing (models decay). In general, how can you tell from the equation if the logistic function is increasing or decreasing?

Exponential functions demonstrate applications of geometric growth and decay in the real world. After practicing with the rules and procedures to gain fluency with exponents and scientific notation, you transferred your knowledge to logarithms and their properties. Lastly, you explored how a new type of function, the logistic function, improves on exponential growth models for real world application.

# 3.8 References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA

# CHAPTER 4 Basic Triangle Trigonometry

## Chapter Outline



Trigonometry is the study of triangles and the relationships between their sides, angles and areas. Using what you know from Geometry like the Pythagorean theorem and the formula for the area of a triangle, you will be able to derive more powerful formulas and learn to apply them appropriately in different situations.

# 4.1 Angles in Radians and Degrees

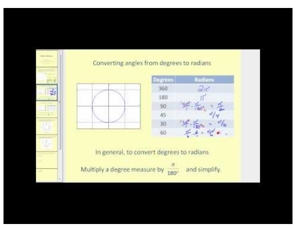
Here you will learn how to translate between different ways of measuring angles.

Most people are familiar with measuring angles in degrees. It is easy to picture angles like $30^\circ$, $45^\circ$ or $90^\circ$ and the fact that $360^\circ$ makes up an entire circle. Over 2000 years ago the Babylonians used a base 60 number system and divided up a circle into 360 equal parts. This became the standard and it is how most people think of angles today.

However, there are many units with which to measure angles. For example, the gradian was invented along with the metric system and it divides a circle into 400 equal parts. The sizes of these different units are very arbitrary.

A radian is a unit of measuring angles that is based on the properties of circles. This makes it more meaningful than gradians or degrees. How many radians make up a circle?

### Watch This



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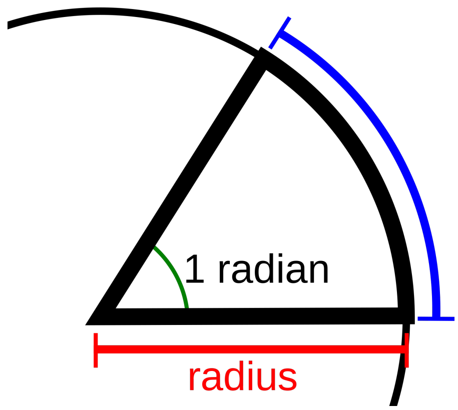
http://www.youtube.com/watch?v=nAJqXtzwpXQ James Sousa: Radian Measure

### Guidance

A radian is defined to be the central angle where the subtended arc length is the same length as the radius.




Another way to think about radians is through the circumference of a circle. The circumference of a circle with radius $r$ is $2\pi r$. Just over six radii (exactly $2\pi$ radii) would stretch around any circle.

To define a radian in terms of degrees, equate a circle measured in degrees to a circle measured in radians.

$360 \ degrees = 2\pi \ radians$, so $\frac{180}{\pi} \ degrees = 1 \ radian$

Alternatively; $360 \ degrees = 2\pi \ radians$, so $1 \ degree = \frac{\pi}{180} \ radians$

The conversion factor to convert degrees to radians is: $\frac{\pi}{180^\circ}$

The conversion factor to convert radians to degrees is: $\frac{180^\circ}{\pi}$

If an angle has no units, it is assumed to be in radians.

**Example A**

Convert $150^\circ$ into radians.

**Solution:** $150^\circ \cdot \frac{\pi}{180^\circ} = \frac{15\pi}{18} = \frac{5\pi}{6} \ radians$

Make sure the degree units cancel.

**Example B**

Convert $\frac{\pi}{6}$ radians into degrees.

**Solution:** $\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{180^\circ}{6} = 30^\circ$

Often the $\pi$'s will cancel.

**Example C**

Convert $(6\pi)^\circ$ into radians.

**Solution:** Don't be fooled just because this has $\pi$. This number is about $19^\circ$

$(6\pi)^\circ \cdot \frac{\pi}{180^\circ} = \frac{6\pi^2}{180} = \frac{\pi^2}{3}$

It is very unusual to ever have a $\pi^2$ term, but it can happen.

**Concept Problem Revisited**

Exactly $2\pi$ radians describe a circular arc. This is because $2\pi$ radiuses wrap around the circumference of any circle.

## Vocabulary

A ***radian*** is defined to be the central angle where the subtended arc length is the same length as the radius.

A ***subtended arc*** is the part of the circle in between the two rays that make the central angle.

## Guided Practice

1. Convert $\frac{5\pi}{6}$ into degrees.

2. Convert $210^\circ$ into radians.

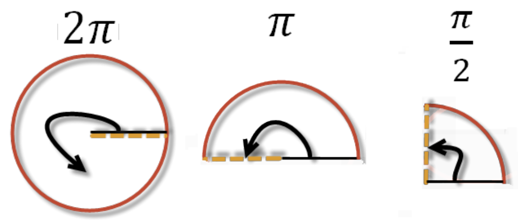
3. Draw a $\frac{\pi}{2}$ angle by first drawing a $2\pi$ angle, halving it and halving the result.

**Answers:**

1. $\frac{5\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{5 \cdot 30^\circ}{1} = 150^\circ$

2. $210^\circ \cdot \frac{\pi}{180^\circ} = \frac{7 \cdot 30 \cdot \pi}{6 \cdot 30} = \frac{7\pi}{6}$

3. $\frac{\pi}{2} = 90^\circ$




### Practice

Find the radian measure of each angle.

1. $120^\circ$
2. $300^\circ$
3. $90^\circ$
4. $330^\circ$
5. $270^\circ$
6. $45^\circ$
7. $(5\pi)^\circ$

Find the degree measure of each angle.

8. $\frac{7\pi}{6}$
9. $\frac{5\pi}{4}$
10. $\frac{3\pi}{2}$
11. $\frac{5\pi}{3}$
12. $\pi$
13. $\frac{\pi}{6}$
14. 3
15. Explain why if you are given an angle in degrees and you multiply it by $\frac{\pi}{180}$ you will get the same angle in radians.

# 4.2 Circular Motion and Dimensional Analysis

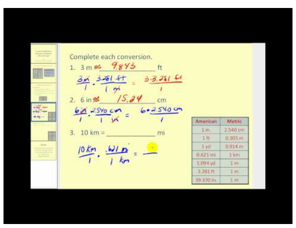
Here you'll review converting between linear and angular speeds using radians and circumference.

Converting between units is essential for mathematics and science in general. Radians are very powerful because they provide a link between linear and angular speed. One radian is an angle that always corresponds to an arc length of one radius. This will allow you to convert the rate at which you pedal a bike to the actual speed you can travel.

$$1 \ revolution = 2\pi r$$

The gear near the pedals on a bike has a radius of 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

### Watch This



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http://www.youtube.com/watch?v=sn8Y7qpYLCY&list=PL67FBCE6482808280 James Sousa: American and Metric Conversions

### Guidance

Dimensional analysis just means converting from one unit to another. Sometimes it must be done in several steps in which case it is best to write the original amount on the left and then multiply it by all the different required conversions. To convert 3 miles to inches you write:

$\frac{3 \ mile}{1} \cdot \frac{5280 \ feet}{1 \ mile} \cdot \frac{12 \ inches}{1 \ foot} = \frac{3 \cdot 5280 \cdot 12 \ inches}{1} = 190080 \ inches$

Notice how miles and feet/foot cancel leaving the desired unit of inches. Also notice each conversion factor is the same distance on the numerator and denominator, just written with different units.

Circular motion refers to the fact that on a spinning wheel points close to the center of the wheel actually travel very slowly and points near the edge of the wheel actually travel much quicker. While the two points have the same angular speed, their linear speed is very different.

**Example A**

Suppose Summit High School has a circular track with two lanes for running. The interior lane is 30 meters from the center of the circle and the lane towards the exterior is 32 meters from the center of the circle. If two people run 4 laps together, how much further does the person on the outside go?

**Solution:** Calculate the distance each person ran separately using 1 lap to be 1 circumference and find the difference.

$$\frac{4\ laps}{1} \cdot \frac{2\pi \cdot 30\ meters}{1\ lap} \approx 754\ meters$$
$$\frac{4\ laps}{1} \cdot \frac{2\pi \cdot 32\ meters}{1\ lap} \approx 804\ meters$$

The person running on the outside of the track ran about 50 more meters.

**Example B**

Andres races on a bicycle with tires that have a 17 inch radius. When he is traveling at a speed of 30 feet per second, how fast are the wheels spinning in revolutions per minute?

**Solution:** Look for ways to convert feet to revolutions and seconds to minutes.

$\frac{30\ feet}{1\ second} \cdot \frac{60\ seconds}{1\ minute} \cdot \frac{12\ inches}{1\ foot} \cdot \frac{1\ revolution}{2\pi \cdot 17\ inches} = \frac{30 \cdot 60 \cdot 12\ revolutions}{2\pi \cdot 17\ minute} \approx 202.2 \frac{rev}{min}$

**Example C**

When a car travels at 60 miles per hour, how fast are the tires spinning if they have 30 inch diameters?

$$\frac{60\ miles}{1\ hour} \cdot \frac{5280\ feet}{1\ mile} \cdot \frac{12\ inches}{1\ foot} \cdot \frac{1\ revolution}{2\pi \cdot 15\ inches} \cdot \frac{1\ hour}{60\ minute}$$
$$= \frac{60 \cdot 5280 \cdot 12\ revolutions}{2\pi \cdot 15 \cdot 60\ minute} \approx 672.3 \frac{rev}{min}$$

**Concept Problem Revisited**

The gear near the pedals on the bike has radius 5 inches and spins once every second. It is connected by a chain to a second gear that has a 3 inch radius. If the second wheel is connected to a tire with a 17 inch radius, how fast is the bike moving in miles per hour?

A bike has pedals that rotate a gear at a circular speed. The gear translates this speed to a linear speed on the chain. The chain then moves a second gear, which is a conversion to angular speed for the rear tire. This tire then converts the angular speed back to linear speed which is how fast you are moving. Instead of doing all these calculations in one step, it is easier to do each conversion in small pieces.

First convert the original gear into the linear speed of the chain.

$\frac{1\ revolution}{1\ second} \cdot \frac{2\pi \cdot 5\ inches}{1\ revolution} = 10\pi \frac{in}{sec}$

Then convert the speed of the chain into angular speed of the back gear which is the same as the angular speed of the rear tire.

$\frac{10\pi\ inches}{1\ second} \cdot \frac{1\ revolution}{2\pi \cdot 3\ inches} = \frac{10}{6} \frac{rev}{sec}$

Lastly convert the angular speed of the rear tire to the linear speed of the tire in miles per hour.

$$\frac{10\ rev}{6\ sec} \cdot \frac{2\pi \cdot 17\ in}{1\ rev} \cdot \frac{1\ ft}{12\ in} \cdot \frac{1\ mile}{5280\ ft} \cdot \frac{60\ sec}{1\ min} \cdot \frac{60\ min}{1\ hour}$$
$$= \frac{10 \cdot 2 \cdot \pi \cdot 17 \cdot 60 \cdot 60}{6 \cdot 12 \cdot 5280} \frac{miles}{hour}$$
$$\approx 10.1 \frac{miles}{hour}$$

### Vocabulary

***Angular speed*** is the ratio of revolutions that occur per unit of time.

***Linear speed*** is the ratio of distance per unit of time.

***Dimensional analysis*** means converting from one unit to another.

### Guided Practice

1. Does a tire with radius 4 inches need to spin twice as fast as a tire with radius 8 inches to keep up?

2. An engine spins a wheel with radius 4 inches at 1200 rpm. How fast is this wheel spinning in miles per hour?

3. Mike rides a bike with tires that have a radius of 15 inches. How many revolutions must Mike make to ride a mile?

**Answers:**

1. If the small wheel spins at 2 revolutions per minute, then the linear speed is:

$$\frac{2\ rev}{1\ min} \cdot \frac{2\pi \cdot 4\ in}{1\ rev} = 16\pi \frac{in}{min}$$

If the large wheel spins at 1 revolution per minute, then the linear speed is:

$$\frac{1\ rev}{1\ min} \cdot \frac{2\pi \cdot 8\ in}{1\ rev} = 16\pi \frac{in}{min}$$

Yes, the small wheel does need to spin at twice the rate of the larger wheel to keep up.

2. $\frac{1200\ rev}{1\ min} \cdot \frac{60\ min}{1\ hour} \cdot \frac{2\pi \cdot 4\ in}{1\ rev} \cdot \frac{1\ ft}{12\ in} \cdot \frac{1\ mi}{5280\ ft} \approx 28.6\ mph$

3. $\frac{1\ rev}{2\pi \cdot 15\ in} \cdot \frac{12\ in}{1\ ft} \cdot \frac{5280\ ft}{1\ mi} \approx 672.3 \frac{rev}{mile}$

### Practice

For 1-10, use the given values in each row to find the unknown value $(x)$ in the specified units in the row.

**TABLE 4.1:**

| Problem Number | Radius | Angular Speed | Linear Speed |
|---|---|---|---|
| 1. | $5\ inches$ | $60\ rpm$ | $x \frac{in}{min}$ |
| 2. | $x\ feet$ | $20\ rpm$ | $2 \frac{in}{sec}$ |
| 3. | $15\ cm$ | $x\ rpm$ | $12 \frac{cm}{sec}$ |
| 4. | $x\ feet$ | $40\ rpm$ | $8 \frac{ft}{sec}$ |
| 5. | $12\ inches$ | $32\ rpm$ | $x \frac{in}{sec}$ |
| 6. | $8\ cm$ | $x\ rpm$ | $12 \frac{cm}{min}$ |
| 7. | $18\ feet$ | $4\ rpm$ | $x \frac{mi}{hr}$ |
| 8. | $x\ feet$ | $800\ rpm$ | $60 \frac{mi}{hr}$ |
| 9. | $15\ in$ | $x\ rpm$ | $60 \frac{mi}{hr}$ |
| 10. | $2\ in$ | $x\ rpm$ | $13 \frac{in}{sec}$ |

11. An engine spins a wheel with radius 5 inches at 800 rpm. How fast is this wheel spinning in miles per hour?

12. A bike has tires with a radius of 10 inches. How many revolutions must the tire make to ride a mile?

13. An engine spins a wheel with radius 6 inches at 600 rpm. How fast is this wheel spinning in inches per second?

14. Bob has a car with tires that have a 15 inch radius. When he is traveling at a speed of 30 miles per hour, how fast are the wheels spinning in revolutions per minute?

15. A circular track has two lanes. The interior lane is 25 feet from the center of the circle and the lane towards the exterior is 30 feet from the center of the circle. If you jog 6 laps, how much further will you jog in the exterior lane as opposed to the interior lane?

# 4.3 Special Right Triangles

Here you will review properties of 30-60-90 and 45-45-90 right triangles.

The Pythagorean Theorem is great for finding the third side of a right triangle when you already know two other sides. There are some triangles like 30-60-90 and 45-45-90 triangles that are so common that it is useful to know the side ratios without doing the Pythagorean Theorem each time. Using these patterns also allows you to totally solve for the missing sides of these special triangles when you only know one side length.

Given a 45-45-90 right triangle with sides 6 inches, 6 inches and $x$ inches, what is the value of $x$?

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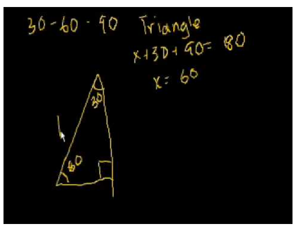


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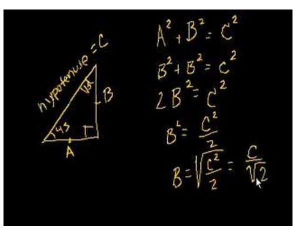
http://www.youtube.com/watch?v=681WWiNxMIQ Khan Academy: Intro to 30 60 90 Triangles Geometry



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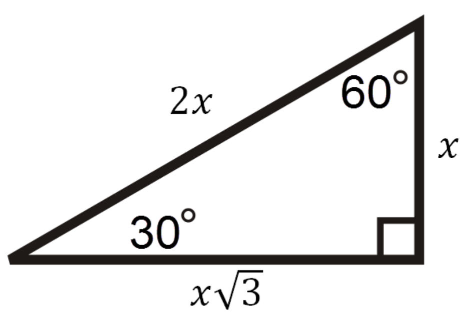
http://www.youtube.com/watch?v=tSHitjFIjd8 Khan Academy: 45 45 90 Triangles

### Guidance

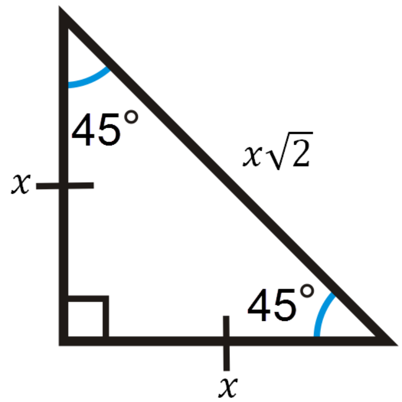
A 30-60-90 right triangle has side ratios $x, x\sqrt{3}, 2x$.



Confirm with Pythagorean Theorem:

$$x^2 + \left(x\sqrt{3}\right)^2 = (2x)^2$$
$$x^2 + 3x^2 = 4x^2$$
$$4x^2 = 4x^2$$

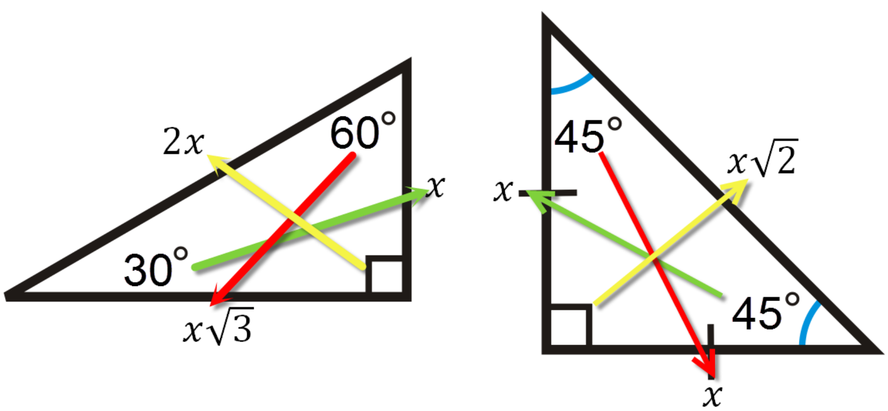
A 45-45-90 right triangle has side ratios $x, x, x\sqrt{2}$.




Confirm with Pythagorean Theorem:

$$x^2 + x^2 = \left(x\sqrt{2}\right)^2$$
$$2x^2 = 2x^2$$

Note that the order of the side ratios $x, x\sqrt{3}, 2x$ and $x, x, x\sqrt{2}$ is important because each side ratio has a corresponding angle. In all triangles, the smallest sides correspond to smallest angles and largest sides always correspond to the largest angles.




Pythagorean number triples are special right triangles with integer sides. While the angles are not integers, the side ratios are very useful to know because they show up everywhere. Knowing these number triples also saves a lot of time from doing the Pythagorean Theorem repeatedly. Here are some examples of Pythagorean number triples:

- 3, 4, 5
- 5, 12, 13
- 7, 24, 25
- 8, 15, 17
- 9, 40, 41

More Pythagorean number triples can be found by scaling any other Pythagorean number triple. For example:

$3,4,5 \rightarrow 6,8,10$ (scaled by a factor of 2)

Even more Pythagorean number triples can be found by taking any odd integer like 11, squaring it to get 121, halving the result to get 60.5. The original number 11 and the two numbers that are 0.5 above and below (60 and 61) will always be a Pythagorean number triple.

$11^2 + 60^2 = 61^2$

**Example A**

A right triangle has two sides that are 3 inches. What is the length of the third side?

**Solution:** Since it is a right triangle and it has two sides of equal length then it must be a 45-45-90 right triangle. The third side is $3\sqrt{2}$ inches.

**Example B**

A 30-60-90 right triangle has hypotenuse of length 10. What are the lengths of the other two sides?

**Solution:** The hypotenuse is the side opposite 90. Sometimes it is helpful to draw a picture or make a table.

### TABLE 4.2:

| 30 | 60 | 90 |
|---|---|---|
| $x$ | $x\sqrt{3}$ | $2x$ |
| | | 10 |

From the table you can write very small subsequent equations to solve for the missing sides.

$$10 = 2x$$
$$x = 5$$
$$x\sqrt{3} = 5\sqrt{3}$$

**Example C**

A 30-60-90 right triangle has a side length of 18 inches corresponding to 60 degrees. What are the lengths of the other two sides?

**Solution:** Make a table with the side ratios and the information given, then write equations and solve for the missing side lengths.

### TABLE 4.3:

| 30 | 60 | 90 |
|---|---|---|

**TABLE 4.3:** (continued)

| $x$ | $x\sqrt{3}$ | $2x$ |
|---|---|---|
| | 18 | |

$$18 = x\sqrt{3}$$
$$\frac{18}{\sqrt{3}} = x$$
$$x = \frac{18}{\sqrt{3}} = \frac{18}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{18\sqrt{3}}{3} = 6\sqrt{3}$$

Note that you need to rationalize denominators.

**Concept Problem Revisited**

If you can recognize the pattern for 45-45-90 right triangles, a right triangle with legs 6 inches and 6 inches has a hypotenuse that is $6\sqrt{2}$ inches. $x = 6\sqrt{2}$.

### Vocabulary

***Corresponding angles and sides*** are angles and sides that are on opposite sides of each other in a triangle. Capital letters like $A, B, C$ are often used for the angles in a triangle and the lower case letters $a, b, c$ are used for their corresponding sides (angle $A$ corresponds to side a etc).

***Pythagorean number triples*** are special right triangles with integer sides.

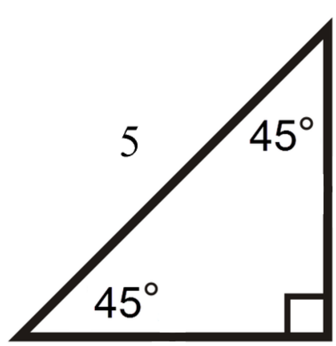
A ***45-45-90 triangle*** is a special right triangle with angles of $45^\circ, 45^\circ$, and $90^\circ$.

A ***30-60-90 triangle*** is a special right triangle with angles of $30^\circ, 60^\circ$, and $90^\circ$.

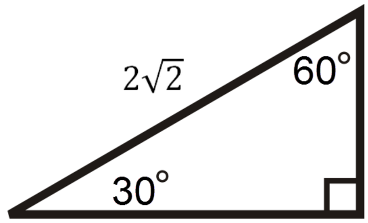
### Guided Practice

Using your knowledge of special right triangle ratios, solve for the missing sides of the following right triangles.

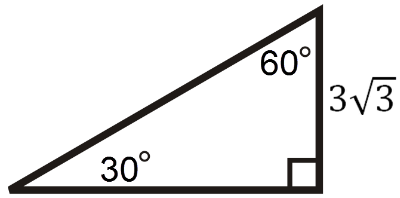
1.




2.




3.




**Answers:**

1. The other sides are each $\frac{5\sqrt{2}}{2}$.

### TABLE 4.4:

| | | |
|---|---|---|
| 45 | 45 | 90 |
| $x$ | $x$ | $x\sqrt{2}$ |
| | | 5 |

$$x\sqrt{2} = 5$$
$$x = \frac{5}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

2. The other sides are $\sqrt{2}$ and $\sqrt{6}$.

### TABLE 4.5:

| | | |
|---|---|---|
| 30 | 60 | 90 |
| $x$ | $x\sqrt{3}$ | $2x$ |
| | | $2\sqrt{2}$ |

$$2x = 2\sqrt{2}$$
$$x = \sqrt{2}$$
$$x\sqrt{3} = \sqrt{2} \cdot \sqrt{3} = \sqrt{6}$$

3. The other sides are 9 and $6\sqrt{3}$.

**TABLE 4.6:**

| | | |
|---|---|---|
| 30 | 60 | 90 |
| $x$ | $x\sqrt{3}$ | $2x$ |
| $3\sqrt{3}$ | | |

$$x = 3\sqrt{3}$$
$$2x = 6\sqrt{3}$$
$$x\sqrt{3} = 3\sqrt{3} \cdot \sqrt{3} = 9$$

### Practice

For 1-4, find the missing sides of the 45-45-90 triangle based on the information given in each row.

**TABLE 4.7:**

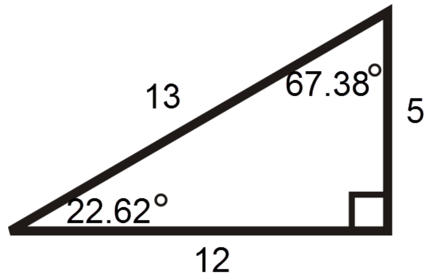
| Problem Number | Side Opposite 45° | Side Opposite 45° | Side Opposite 90° |
|---|---|---|---|
| 1. | 3 | | |
| 2. | | 7.2 | |
| 3. | | | 16 |
| 4. | $5\sqrt{2}$ | | |

For 5-8, find the missing sides of the 30-60-90 triangle based on the information given in each row.

**TABLE 4.8:**

| Problem Number | Side Opposite 30° | Side Opposite 60° | Side Opposite 90° |
|---|---|---|---|
| 5. | $3\sqrt{2}$ | | |
| 6. | | 4 | |
| 7. | | | 15 |
| 8. | | | $12\sqrt{3}$ |

Use the picture below for 9-11.




9. Which angle corresponds to the side that is 12 units?

10. Which side corresponds to the right angle?

11. Which angle corresponds to the side that is 5 units?

12. A right triangle has an angle of $\frac{\pi}{6}$ radians and a hypotenuse of 20 inches. What are the lengths of the other two sides of the triangle?

13. A triangle has two angles that measure $\frac{\pi}{4}$ radians. The longest side is 3 inches long. What are the lengths of the other two sides?

For 14-19, verify the Pythagorean Number Triple using the Pythagorean Theorem.

14. 3, 4, 5

15. 5, 12, 13

16. 7, 24, 25

17. 8, 15, 17

18. 9, 40, 41

19. 6, 8, 10

20. Find another Pythagorean Number Triple using the method explained for finding “11, 60, 61”.

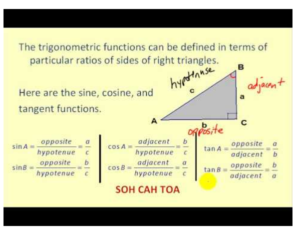
# 4.4 Right Triangle Trigonometry

Here you will learn the six right triangle ratios and how to use them to completely solve for the missing sides and angles of any right triangle.

Trigonometry is the study of triangles. If you know the angles of a triangle and one side length, you can use the properties of similar triangles and proportions to completely solve for the missing sides.

Imagine trying to measure the height of a flag pole. It would be very difficult to measure vertically because it could be several stories tall. Instead walk 10 feet away and notice that the flag pole makes a 65 degree angle with your feet. Using this information, what is the height of the flag pole?

### Watch This




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=Ujyl_zQw2zE James Sousa: Introduction to Trigonometric Functions Using Triangles

### Guidance

The six trigonometric functions are sine, cosine, tangent, cotangent, secant and cosecant. *Opp* stands for the side opposite of the angle $\theta$, *hyp* stands for hypotenuse and *adj* stands for side adjacent to the angle $\theta$.

$\sin\theta = \frac{opp}{hyp}$

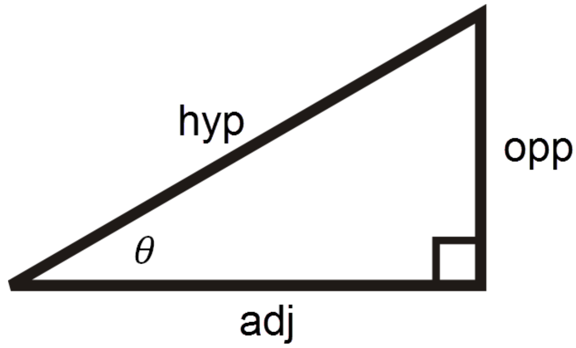
$\cos\theta = \frac{adj}{hyp}$

$\tan\theta = \frac{opp}{adj}$

$\cot\theta = \frac{adj}{opp}$

$\sec\theta = \frac{hyp}{adj}$

$\csc\theta = \frac{hyp}{opp}$




The reason why these trigonometric functions exist is because two triangles with the same interior angles will have side lengths that are always proportional. Trigonometric functions are used by identifying two known pieces of information on a triangle and one unknown, setting up and solving for the unknown. Calculators are important because the operations of sin, cos and tan are already programmed in. The other three (cot, sec and csc) are not usually in calculators because there is a reciprocal relationship between them and tan, cos and sec.

$\sin\theta = \frac{opp}{hyp} = \frac{1}{\csc\theta}$

$\cos\theta = \frac{adj}{hyp} = \frac{1}{\sec\theta}$

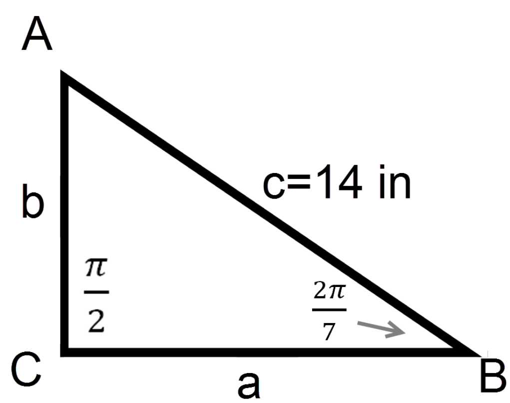
$\tan\theta = \frac{opp}{adj} = \frac{1}{\cot\theta}$

Keep in mind that your calculator can be in degree mode or radian mode. Be sure you can toggle back and forth so that you are always in the appropriate units for each problem.

***Note: The images throughout this concept are not drawn to scale.***

**Example A**

Solve for side $b$.



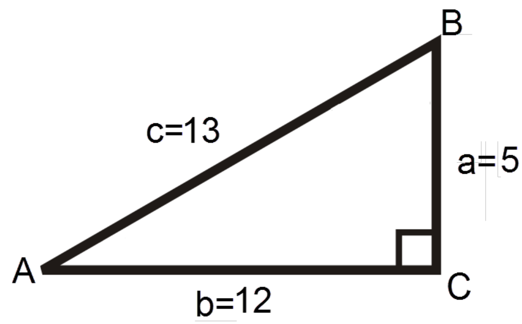

**Solution:**

$$\sin\left(\frac{2\pi}{7}\right) = \frac{b}{14}$$

$$b = 14 \cdot \sin\left(\frac{2\pi}{7}\right) \approx 10.9 \ in$$

**Example B**

Solve for angle $A$.



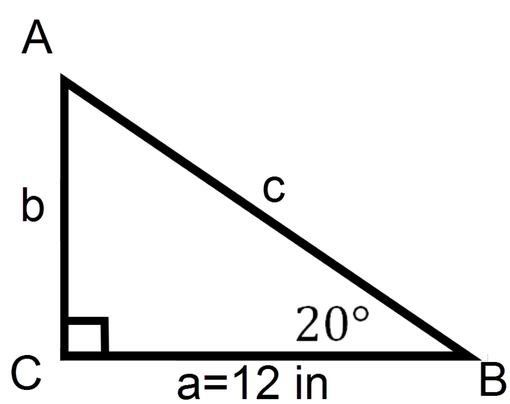

**Solution:** This problem can be solved using sin, cos or tan because the opposite, adjacent and hypotenuse lengths are all given.

The argument of a sin function is always an angle. The arcsin or $\sin^{-1}\theta$ function on the calculator on the other hand has an argument that is a side ratio. It is useful for finding angles that have that side ratio.

$$\begin{aligned}\sin A &= \frac{5}{13}\\ A &= \sin^{-1}\left(\frac{5}{13}\right) \approx 0.39\ radian \approx 22.6^\circ\end{aligned}$$

**Example C**

Given a right triangle with $a = 12\ in$, $m\angle B = 20^\circ$, and $m\angle C = 90^\circ$, find the length of the hypotenuse.



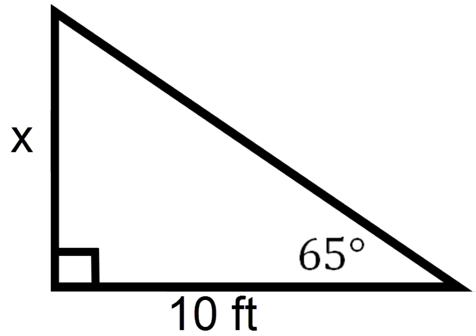

**Solution:**

$$\begin{aligned}\cos 20^\circ &= \frac{12}{c}\\ c &= \frac{12}{\cos 20^\circ} \approx 12.77\ in\end{aligned}$$

**Concept Problem Revisited**

Instead walk 10 feet away and notice that the flag pole makes a 65° angle with your feet.

If you walk 10 feet from the base of a flagpole and assume that the flagpole makes a 90° angle with the ground.




$$\tan 65^\circ = \frac{x}{10}$$
$$x = 10 \tan 65^\circ \approx 30.8\ ft$$

### Vocabulary

The ***six trigonometric ratios*** are universal proportions that are always true of similar triangles (triangles with congruent corresponding angles).

θ ***(theta)*** is a Greek letter and is just a letter used in math to stand for an unknown angle.

### Guided Practice

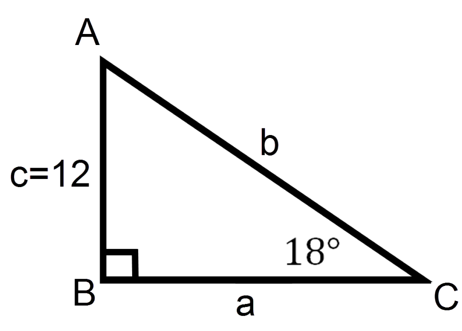
1. Given $\triangle ABC$ where $B$ is a right angle, $m\angle C = 18^\circ$, and $c = 12$. What is $a$?

2. Given $\triangle XYZ$ where $Z$ is a right angle, $m\angle X = 1\ radian$, and $x = 3$. What is $z$?

3. Given $\triangle MNO$ where $O$ is a right angle, $m = 12$, and $n = 14$. What is the measure of angle $M$?

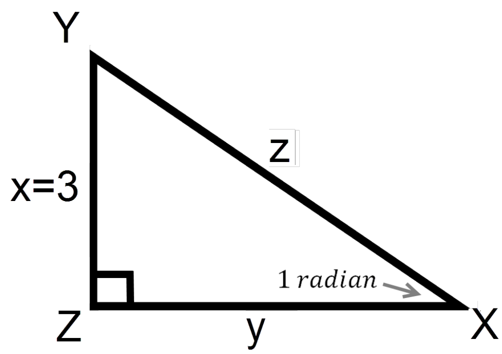
**Answers:**

1. Drawing out this triangle, it looks like:



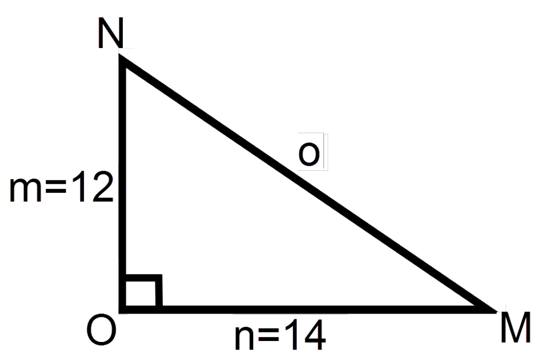

$$\tan 18^\circ = \frac{12}{a}$$
$$a = \frac{12}{\tan 18^\circ} \approx 36.9$$

2. Drawing out the triangle, it looks like:




$$\sin 1 = \frac{3}{z}$$
$$z = \frac{3}{\sin 1} \approx 3.6$$

3. Drawing out the triangle, it looks like:




$$\tan M = \frac{12}{14}$$
$$M = \tan^{-1}\left(\frac{12}{14}\right) \approx 0.7\ radian \approx 40.6^\circ$$

### Practice

For 1-15, information about the sides and/or angles of right triangle $ABC$ is given. Completely solve the triangle (find all missing sides and angles) to 1 decimal place.

**TABLE 4.9:**

| Problem Number | $A$ | $B$ | $C$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|---|---|

**TABLE 4.9:** (continued)

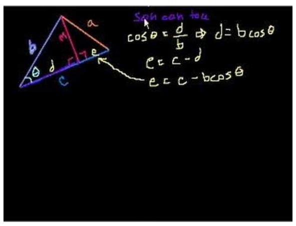
| | | | | | | |
|---|---|---|---|---|---|---|
| 1. | 90° | | | | 4 | 7 |
| 2. | 90° | | 37° | 18 | | |
| 3. | | 90° | 15° | | 32 | |
| 4. | | | 90° | 6 | | 11 |
| 5. | 90° | 12° | | 19 | | |
| 6. | | 90° | | | 17 | 10 |
| 7. | 90° | 10° | | | 2 | |
| 8. | 4° | 90° | | 0.3 | | |
| 9. | $\frac{\pi}{2}$ radian | | 1 radian | | | 15 |
| 10. | | $\frac{\pi}{2}$ radian | | 12 | 15 | |
| 11. | | | $\frac{\pi}{2}$ radian | | 9 | 14 |
| 12. | $\frac{\pi}{4}$ radian | $\frac{\pi}{4}$ radian | | | 5 | |
| 13. | $\frac{\pi}{2}$ radian | | | 26 | 13 | |
| 14. | | $\frac{\pi}{2}$ radian | | | 19 | 16 |
| 15. | | | $\frac{\pi}{2}$ radian | 10 | | $10\sqrt{2}$ |

# 4.5 Law of Cosines

Here you will solve non-right triangles with the Law of Cosines.

The Law of Cosines is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle. Suppose you have a triangle with sides 11, 12 and 13. What is the measure of the angle opposite the 11?

### Watch This



MEDIA

Click image to the left for more content.

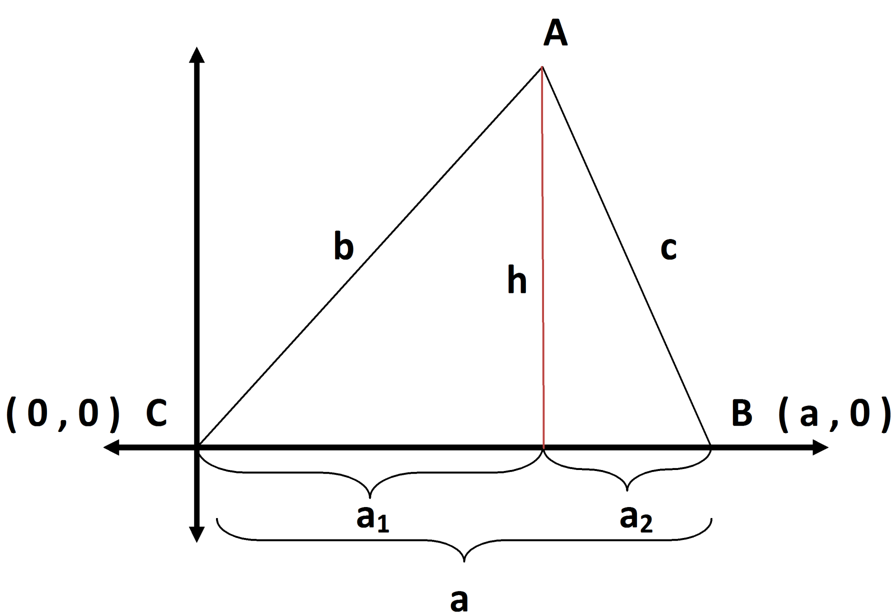
http://www.youtube.com/watch?v=pGaDcOMdw48 Khan Academy: Law of Cosines

### Guidance

The Law of Cosines is:

$c^2 = a^2 + b^2 - 2ab \cdot \cos C$

It is important to understand the proof:




You know four facts from the picture:

$a = a_1 + a_2 \qquad (1)$

$b^2 = a_1^2 + h^2 \qquad (2)$

$c^2 = a_2^2 + h^2 \qquad (3)$

$\cos C = \frac{a_1}{b} \qquad (4)$

Once you verify for yourself that you agree with each of these facts, check algebraically that these next two facts must be true.

$a_2 = a - a_1 \qquad (5, \text{ from } 1)$

$a_1 = b \cdot \cos C \qquad (6, \text{ from } 4)$

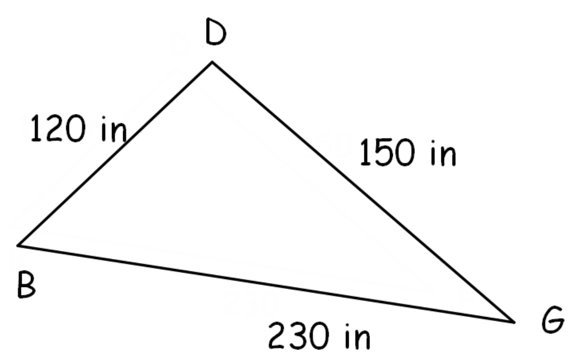
Now the Law of Cosines is ready to be proved using substitution, FOIL, more substitution and rewriting to get the order of terms right.

$$\begin{aligned}
c^2 &= a_2^2 + h^2 && \text{(3 again)}\\
c^2 &= (a - a_1)^2 + h^2 && \text{(substitute using 5)}\\
c^2 &= a^2 - 2a \cdot a_1 + a_1^2 + h^2 && \text{(FOIL)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + a_1^2 + h^2 && \text{(substitute using 6)}\\
c^2 &= a^2 - 2a \cdot b \cdot \cos C + b^2 && \text{(substitute using 2)}\\
c^2 &= a^2 + b^2 - 2ab \cdot \cos C && \text{(rearrange terms)}
\end{aligned}$$

There are only two types of problems in which it is appropriate to use the Law of Cosines. The first is when you are given all three sides of a triangle and asked to find an unknown angle. This is called SSS like in geometry. The second situation where you will use the Law of Cosines is when you are given two sides and the included angle and you need to find the third side. This is called SAS.

**Example A**

Determine the measure of angle *D*.

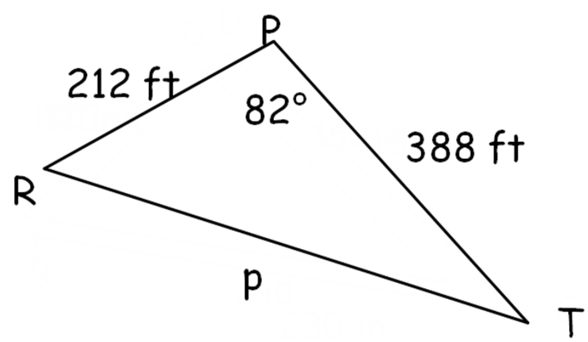



**Solution:** It is necessary to set up the Law of Cosines equation very carefully with *D* corresponding to the opposite side of 230. The letters are not *ABC* like in the proof, but those letters can always be changed to match the problem as long as the angle in the cosine corresponds to the side used in the left side of the equation.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\
230^2 &= 120^2 + 150^2 - 2 \cdot 120 \cdot 150 \cdot \cos D \\
230^2 - 120^2 - 150^2 &= -2 \cdot 120 \cdot 150 \cdot \cos D \\
\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150} &= \cos D \\
D &= \cos^{-1}\left(\frac{230^2 - 120^2 - 150^2}{-2 \cdot 120 \cdot 150}\right) \approx 116.4^\circ \approx 2.03 \textit{ radians}
\end{aligned}
$$

**Example B**

Determine the length of side $p$.

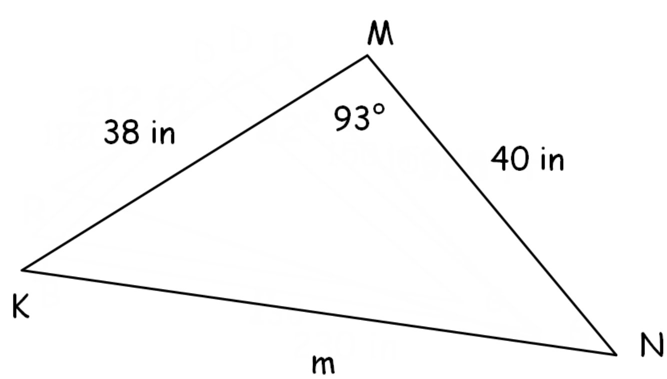



**Solution:**

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab \cdot \cos C \\
p^2 &= 212^2 + 388^2 - 2 \cdot 212 \cdot 388 \cdot \cos 82^\circ \\
p^2 &\approx 194192.02\ldots \\
p &\approx 440.7
\end{aligned}
$$

**Example C**

Determine the degree measure of angle $N$.



**Solution:** This problem must be done in two parts. First apply the Law of Cosines to determine the length of side $m$. This is a SAS situation like Example B. Once you have all three sides you will be in the SSS situation like in Example A and can apply the Law of Cosines again to find the unknown angle $N$.

$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cdot\cos C\\
m^2 &= 38^2+40^2-2\cdot 38\cdot 40\cdot\cos 93^\circ\\
m^2 &\approx 3203.1\ldots\\
m &\approx 56.59\ldots
\end{aligned}$$

Now that you have all three sides you can apply the Law of Cosines again to find the unknown angle $N$. Remember to match angle $N$ with the corresponding side length of 38 inches. It is also best to store $m$ into your calculator and use the unrounded number in your future calculations.

$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cdot\cos C\\
38^2 &= 40^2+(56.59\ldots)^2-2\cdot 40\cdot(56.59\ldots)\cdot\cos N\\
38^2-40^2-(56.59\ldots)^2 &= -2\cdot 40\cdot(56.59\ldots)\cdot\cos N\\
\frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)} &= \cos N\\
N &= \cos^{-1}\left(\frac{38^2-40^2-(56.59\ldots)^2}{-2\cdot 40\cdot(56.59\ldots)}\right)\approx 42.1^\circ
\end{aligned}$$

**Concept Problem Revisited**

A triangle that has sides 11, 12 and 13 is not going to be a right triangle. In order to solve for the missing angle you need to use the Law of Cosines because this is a SSS situation.

$$\begin{aligned}
c^2 &= a^2+b^2-2ab\cdot\cos C\\
11^2 &= 12^2+(13)^2-2\cdot 12\cdot 13\cdot\cos C\\
C &= \cos^{-1}\left(\frac{11^2-12^2-13^2}{-2\cdot 12\cdot 13}\right)\approx 52.02\ldots^\circ
\end{aligned}$$

### Vocabulary

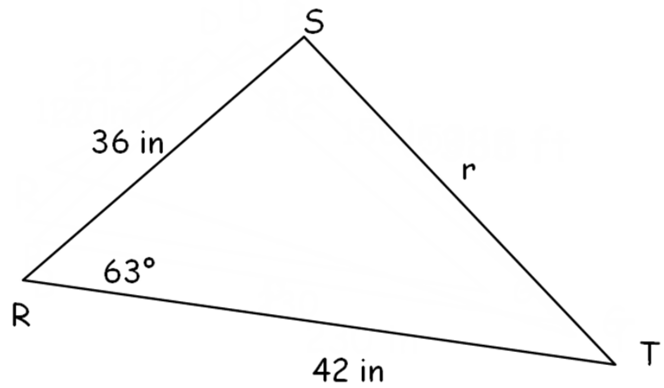
The **Law of Cosines** is a generalized Pythagorean Theorem that allows you to solve for the missing sides and angles of a triangle even if it is not a right triangle.

**SSS** refers to Side, Side, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that all three sides are known in the problem.

**SAS** refers to Side, Angle, Side and refers to a property of congruent triangles in geometry. In this case it refers to the fact that the known quantities of a triangle are two sides and the included angle.

**Included angle** is the angle between two sides.

### Guided Practice




1. Determine the length of side $r$.

2. Determine the measure of angle $T$ in degrees.

3. Determine the measure of angle $S$ in radians.

**Answers:**

1. $r^2 = 36^2 + 42^2 - 2 \cdot 36 \cdot 42 \cdot \cos 63$

$r = 41.07\ldots$

2. $36^2 = (41.07\ldots)^2 + 42^2 - 2 \cdot (41.07\ldots) \cdot 42 \cdot \cos T$

$T \approx 51.34\ldots°$

3. You could repeat the process from the previous question, or use the knowledge that the three angles in a triangle add up to 180.

$$63 + 51.34\ldots + S = 180$$
$$S \approx 65.65° \cdot \frac{\pi}{180°} \approx 1.145\ldots radians$$

### Practice

For all problems, find angles in degrees rounded to one decimal place.

In $\Delta ABC, a = 12, b = 15$, and $c = 20$.

1. Find the measure of angle $A$.

2. Find the measure of angle $B$.

3. Find the measure of angle $C$.

4. Find the measure of angle $C$ in a different way.

In $\Delta DEF, d = 20, e = 10$, and $f = 16$.

5. Find the measure of angle $D$.

6. Find the measure of angle $E$.

7. Find the measure of angle $F$.

In $\Delta GHI, g = 19, \angle H = 55°$, and $i = 12$.

8. Find the length of $h$.

9. Find the measure of angle $G$.

10. Find the measure of angle $I$.

11. Explain why the Law of Cosines is connected to the Pythagorean Theorem.

12. What are the two types of problems where you might use the Law of Cosines?

Determine whether or not each triangle is possible.

13. $a = 5, b = 6, c = 15$

14. $a = 1, b = 5, c = 4$

15. $a = 5, b = 6, c = 10$

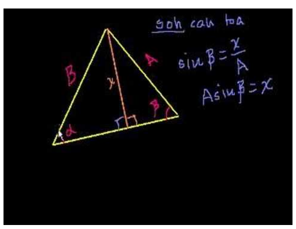
# 4.6 Law of Sines

Here you will further explore solving non-right triangles in cases where a corresponding side and angle are given using the Law of Sines.

When given a right triangle, you can use basic trigonometry to solve for missing information. When given SSS or SAS, you can use the Law of Cosines to solve for the missing information. But what happens when you are given two sides of a triangle and an angle that is not included? There are many ways to show two triangles are congruent, but SSA is not one of them. Why not?

### Watch This

http://www.youtube.com/watch?v=APNkWrD-U1k Khan Academy: Proof: Law of Sines

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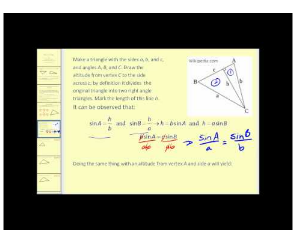
http://www.youtube.com/watch?v=dxYVBbSXYkA James Sousa: The Law of Sines: The Basics



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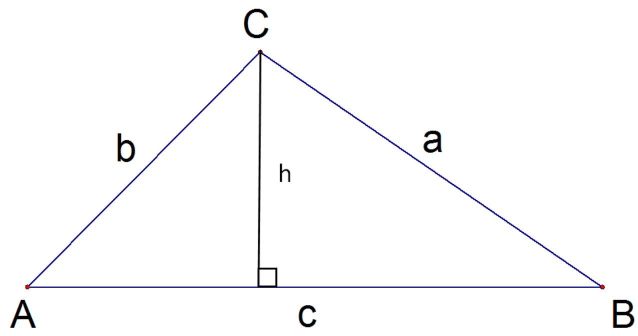
### Guidance

When given two sides and an angle that is not included between the two sides, you can use the Law of Sines. The Law of Sines states that in every triangle the ratio of each side to the sine of its corresponding angle is always the same. Essentially, it clarifies the general concept that opposite the largest angle is always the longest side.

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Here is a proof of the Law of Sines:




Looking at the right triangle formed on the left:

$$\sin A = \frac{h}{b}$$
$$h = b \sin A$$

Looking at the right triangle formed on the right:

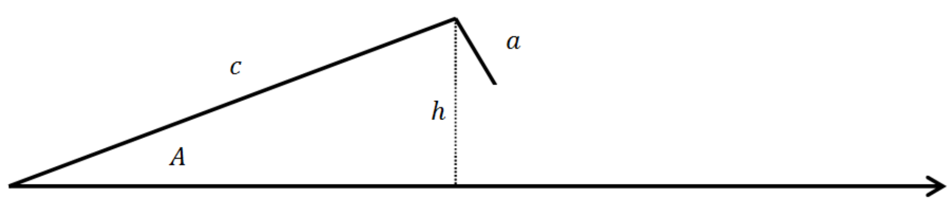
$$\sin B = \frac{h}{a}$$
$$h = a \sin B$$

Equating the heights which must be identical:

$$a \sin B = b \sin A$$
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

The best way to use the Law of Sines is to draw an extremely consistent picture each and every time even if that means redrawing and relabeling a picture. The reason why the consistency is important is because sometimes given SSA information defines zero, one or even two possible triangles.

Always draw the given angle in the bottom left with the two given sides above.




In this image side $a$ is deliberately too short, but in most problems you will not know this. You will need to compare $a$ to the height.

$$\sin A = \frac{h}{c}$$
$$h = c \ \sin A$$

**Case 1:** $a < h$

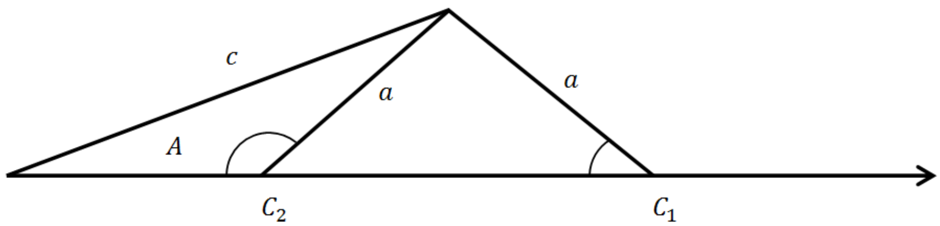
Simply put, side $a$ is not long enough to reach the opposite side and the triangle is impossible.

**Case 2:** $a = h$

Side $a$ just barely reaches the opposite side forming a $90^\circ$ angle.

**Case 3:** $h < a < c$

In this case side $a$ can swing toward the interior of the triangle or the exterior of the triangle- there are two possible triangles. This is called the ambiguous case because the given information does not uniquely identify one triangle. To solve for both triangles, use the Law of Sines to solve for angle $C_1$ first and then use the supplement to determine $C_2$.

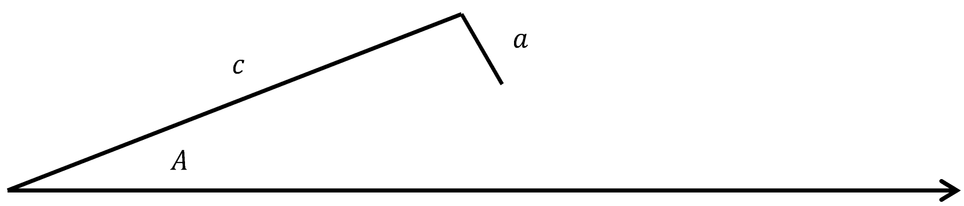



**Case 4:** $c \leq a$

In this case, side $a$ can only swing towards the exterior of the triangle, only producing $C_1$.

**Example A**

$\angle A = 40^\circ, c = 13$, and $a = 2$. If possible, find $\angle C$.



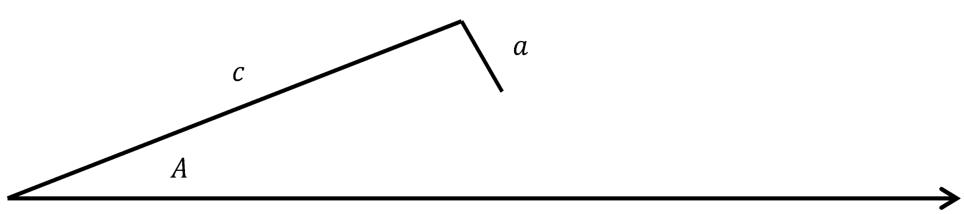

**Solution:**

$$\sin 40^\circ = \frac{h}{13}$$
$$h = 13 \sin 40^\circ \approx 8.356$$

Because $a < h$ $(2 < 8.356)$, this information does not form a proper triangle.

**Example B**

$\angle A = 17^\circ, c = 14$, and $a = 4.0932\ldots$ If possible, find $\angle C$.



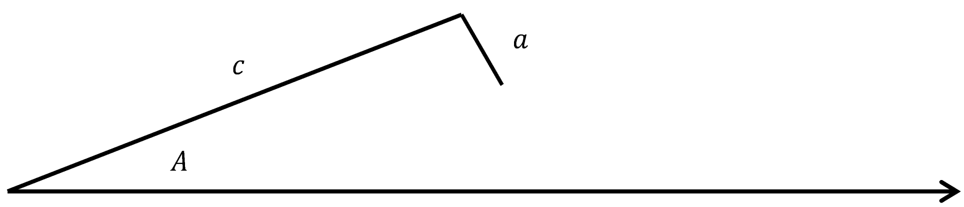
**Solution:**

$$\sin 17^\circ = \frac{h}{14}$$
$$h = 14\ \sin 17^\circ \approx 4.0932\ldots$$

Since $a = h$, this information forms exactly one triangle and angle $C$ must be $90^\circ$.

**Example C**

$\angle A = 22^\circ, c = 11$ and $a = 9$. If possible, find $\angle C$.




**Solution:**

$$\sin 22^\circ = \frac{h}{11}$$
$$h = 11\ \sin 22^\circ \approx 4.12\ldots$$

Since $h < a < c$, there must be two possible angles for angle $C$.

Apply the Law of Sines:

$$\frac{9}{\sin 22^\circ} = \frac{11}{\sin C_1}$$
$$9 \sin C_1 = 11 \sin 22^\circ$$
$$\sin C_1 = \frac{11 \sin 22^\circ}{9}$$
$$C_1 = \sin^{-1}\left(\frac{11 \sin 22^\circ}{9}\right) \approx 27.24\ldots^\circ$$
$$C_2 = 180 - C_1 = 152.75\ldots^\circ$$

**Concept Problem Revisited**

SSA is not a method from Geometry that shows two triangles are congruent because it does not always define a unique triangle.

## Vocabulary

***Ambiguous*** means that the given information may not uniquely identify one triangle.

## Guided Practice

1. Given $\Delta ABC$ where $A = 10^\circ, b = 10, a = 11$, find $\angle B$.

2. Given $\Delta ABC$ where $A = 12^\circ, B = 50^\circ, a = 14$ find $b$.

3. Given $\Delta ABC$ where $A = 70^\circ, b = 8, a = 3$, find $\angle B$ if possible.

**Answers:**

1. $\frac{10}{\sin B} = \frac{11}{\sin 10^\circ}$

$B = \sin^{-1}\left(\frac{10 \sin 10^\circ}{11}\right) \approx 9.08\ldots^\circ$

2. $\frac{14}{\sin 12^\circ} = \frac{b}{\sin 50^\circ}$

$b = \frac{14 \sin 50^\circ}{\sin 12^\circ} \approx 51.58\ldots$

3. $\sin 70^\circ = \frac{h}{8}$

$h = 8 \sin 70^\circ \approx 7.51\ldots$

Because $a < h$, this triangle is impossible.

## Practice

For 1-3, draw a picture of the triangle and state how many triangles could be formed with the given values.

1. $A = 30^\circ, a = 13, b = 15$

2. $A = 22^\circ, a = 21, b = 12$

3. $A = 42^\circ, a = 36, b = 37$

For 4-7, find all possible measures of $\angle B$ (if any exist) for each of the following triangle values.

4. $A = 86^\circ, a = 15, b = 11$

5. $A = 30^\circ, a = 24, b = 43$

6. $A = 48^\circ, a = 34, b = 39$

7. $A = 80^\circ, a = 22, b = 20$

For 8-12, find the length of $b$ for each of the following triangle values.

8. $A = 94^\circ, a = 31, B = 34^\circ$

9. $A = 112^\circ, a = 12, B = 15^\circ$

10. $A = 78^\circ, a = 20, B = 16^\circ$

11. $A = 54^\circ, a = 15, B = 112^\circ$

12. $A = 39^\circ, a = 9, B = 98^\circ$

13. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce two triangles?

14. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce no triangles?

15. In $\Delta ABC, b = 10$ and $\angle A = 39^\circ$. What's a possible value for $a$ that would produce one triangle?

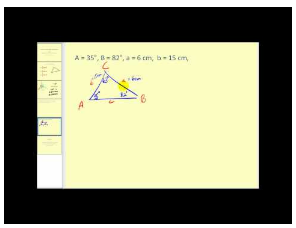
# 4.7 Area of a Triangle

Here you'll use the sine ratio to find the area of non-right triangles in which two sides and the included angle measure are known.

From geometry you already know that the area of a triangle is $\frac{1}{2} \cdot b \cdot h$.

What if you are given the sides of a triangle are 5 and 6 and the angle between the sides is $\frac{\pi}{3}$? You are not directly given the height, but can you still figure out the area of the triangle?

### Watch This



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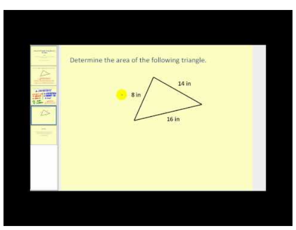
http://www.youtube.com/watch?v=mBFDq4bPXMs James Sousa: The Area of a Triangle Using Sine



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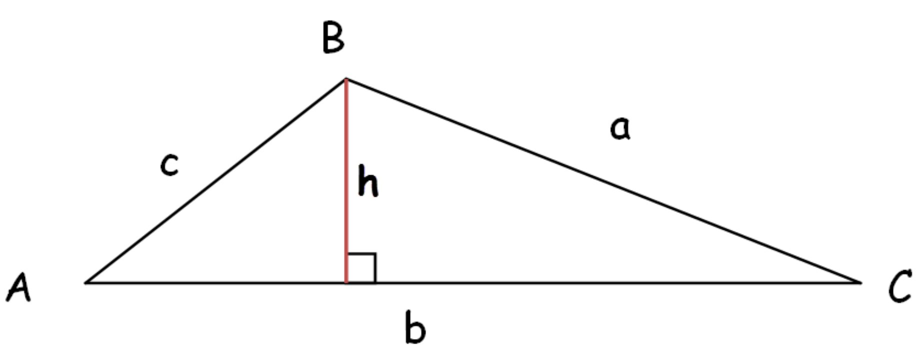
http://www.youtube.com/watch?v=Pi56pFy-8HU James Sousa: Heron's Formula

### Guidance

The sine function allows you to find the height of any triangle and substitute that value into the familiar triangle area formula.



Using the sine function, you can isolate $h$ for height:

$$\sin C = \frac{h}{a}$$
$$a \sin C = h$$

Substituting into the area formula:

$$Area = \frac{1}{2} b \cdot h$$
$$Area = \frac{1}{2} b \cdot a \cdot \sin C$$
$$Area = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$

**Example A**

Given $\Delta ABC$ with $A = 22^\circ, b = 6, c = 7$. What is the area?

**Solution:** The letters don't have to match exactly because the triangle or the formula can just be relabeled. The important part is that neither given side corresponds to the given angle.

$$Area = \frac{1}{2} bc \sin A$$
$$Area = \frac{1}{2} \cdot 6 \cdot 7 \cdot \sin 22^\circ \approx 7.86 \ldots units^2$$

**Example B**

Given $\Delta XYZ$ has area 28 square inches, what is the angle included between side length 8 and 9?

**Solution:**

$$Area = \frac{1}{2} \cdot a \cdot b \cdot \sin C$$
$$28 = \frac{1}{2} \cdot 8 \cdot 9 \cdot \sin C$$
$$\sin C = \frac{28 \cdot 2}{8 \cdot 9}$$
$$C = \sin^{-1}\left(\frac{28 \cdot 2}{8 \cdot 9}\right) \approx 51.05 \ldots^\circ$$

**Example C**

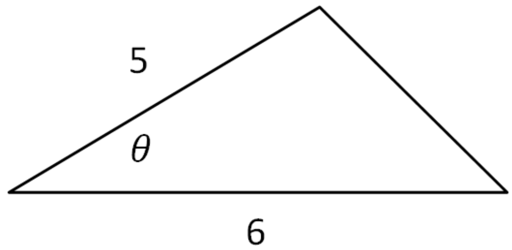
Given triangle $ABC$ with $A = 12^\circ, b = 4$ and $Area = 1.7\ un^2$, what is the length of side $c$?

**Solution:**

$$Area = \frac{1}{2} \cdot c \cdot b \cdot \sin A$$
$$1.7 = \frac{1}{2} \cdot c \cdot 4 \cdot \sin 12^\circ$$
$$c = \frac{1.7 \cdot 2}{4 \cdot \sin 12^\circ} \approx 4.08 \ldots$$

**Concept Problem Revisited**

What if you are given the sides of a triangle are 5 and 6 and the angle between the sides is $\theta = \frac{\pi}{3}$?




$Area = \frac{1}{2} \cdot 5 \cdot 6 \cdot \sin \frac{\pi}{3} \approx 12.99\ un^2$

## Vocabulary

The ***included angle*** between two sides of a triangle is the angle at the point where the two sides meet.

## Guided Practice

1. What is the area of $\Delta ABC$ with $A = 31°, b = 12, c = 14$?

2. What is the area of $\Delta XYZ$ with $x = 11, y = 12, z = 13$?

3. The area of a triangle is 3 square units. Two sides of the triangle are 4 units and 5 units. What is the measure of their included angle?

**Answers:**

1. $Area = \frac{1}{2} \cdot 12 \cdot 14 \cdot \sin 31° \approx 43.26 \ldots units^2$

2. Because none of the angles are given, there are two possible solution paths. You could use the Law of Cosines to find one angle. The angle opposite the side of length 11 is $52.02\ldots°$ therefore the area is:

$Area = \frac{1}{2} \cdot 12 \cdot 13 \cdot \sin 52.02 \ldots \approx 61.5\ un^2$

Another way to find the area is through the use of Heron's Formula which is:

$Area = \sqrt{s(s-a)(s-b)(s-c)}$

Where $s$ is the semi perimeter:

$s = \frac{a+b+c}{2}$

3. $3 = \frac{1}{2} \cdot 4 \cdot 5 \cdot \sin \theta$

$\theta = \sin^{-1}\left(\frac{3 \cdot 2}{4 \cdot 5}\right) \approx 17.45\ldots°$

## Practice

For 1-11, find the area of each triangle.

1. $\Delta ABC$ if $a = 13, b = 15$, and $\angle C = 70°$.
2. $\Delta ABC$ if $b = 8, c = 4$, and $\angle A = 58°$.
3. $\Delta ABC$ if $b = 34, c = 29$, and $\angle A = 125°$.
4. $\Delta ABC$ if $a = 3, b = 7$, and $\angle C = 81°$.

5. $\Delta ABC$ if $a = 4.8, c = 3.7$, and $\angle B = 54^\circ$.
6. $\Delta ABC$ if $a = 12, b = 5$, and $\angle C = 22^\circ$.
7. $\Delta ABC$ if $a = 3, b = 10$, and $\angle C = 65^\circ$.
8. $\Delta ABC$ if $a = 5, b = 9$, and $\angle C = 11^\circ$.
9. $\Delta ABC$ if $a = 5, b = 7$, and $c = 8$.
10. $\Delta ABC$ if $a = 7, b = 8$, and $c = 14$.
11. $\Delta ABC$ if $a = 12, b = 14$, and $c = 13$.
12. The area of a triangle is 12 square units. Two sides of the triangle are 8 units and 4 units. What is the measure of their included angle?
13. The area of a triangle is 23 square units. Two sides of the triangle are 14 units and 5 units. What is the measure of their included angle?
14. Given $\Delta DEF$ has area 32 square inches, what is the angle included between side length 9 and 10?
15. Given $\Delta GHI$ has area 15 square inches, what is the angle included between side length 7 and 11?

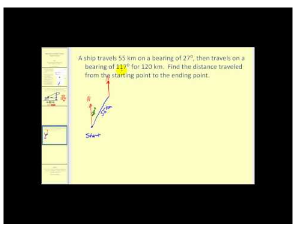
# 4.8 Applications of Basic Triangle Trigonometry

Here you will apply your knowledge of trigonometry and problem solving in context.

Deciding when to use SOH, CAH, TOA, Law of Cosines or the Law of Sines is not always obvious. Sometimes more than one approach will work and sometimes correct computations can still lead to incorrect results. This is because correct interpretation is still essential.

If you use both the Law of Cosines and the Law of Sines on a triangle with sides 4, 7, 10 you end up with conflicting answers. Why?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=-QOEcnuGQwo James Sousa: Solving Right Triangles-Part 2 Applications

### Guidance

When applying trigonometry, it is important to have a clear toolbox of mathematical techniques to use. Some of the techniques may be review like the fact that all three angles in a triangle sum to be 180°, other techniques may be new like the Law of Cosines. There also may be some properties that are true and make sense but have never been formally taught.

**Toolbox:**

- The three angles in a triangle sum to be 180°.
- There are 360° in a circle and this can help us interpret negative angles as positive angles.
- The Pythagorean Theorem states that for legs $a, b$ and hypotenuse $c$ in a right triangle, $a^2 + b^2 = c^2$.
- The Triangle Inequality Theorem states that for any triangle, the sum of any two of the sides must be greater than the third side.
- The Law of Cosines: $c^2 = a^2 + b^2 - 2ab\cos C$
- The Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B}$ or $\frac{\sin A}{a} = \frac{\sin B}{b}$ (Be careful for the ambiguous case)
- SOH CAH TOA is a mnemonic device to help you remember the three original trig functions:

$$\sin\theta = \frac{opp}{hyp} \quad \cos\theta = \frac{adj}{hyp} \quad \tan\theta = \frac{opp}{adj}$$

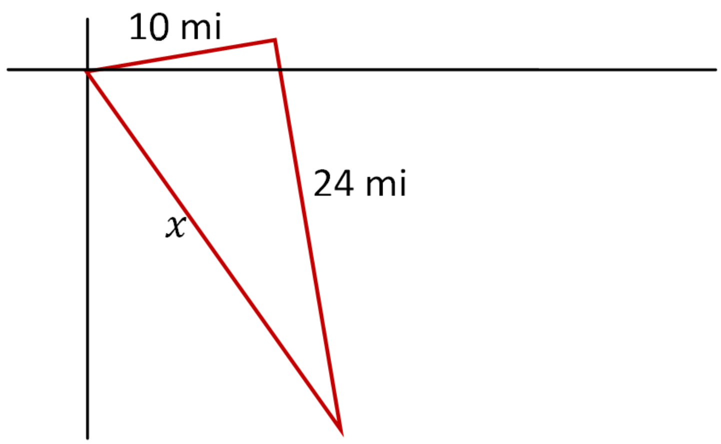
- 30-60-90 right triangles have side ratios $x, x\sqrt{3}, 2x$
- 45-45-90 right triangles have side ratios $x, x, x\sqrt{2}$

- Pythagorean number triples are exceedingly common and should always be recognized in right triangle problems. Examples of triples are 3, 4, 5 and 5, 12, 13.

**Example A**

Bearing is how direction is measured at sea. North is $0^\circ$, East is $90^\circ$, South is $180^\circ$ and West is $270^\circ$. A ship travels 10 miles at a bearing of $88^\circ$ and then turns $90^\circ$ to the right to avoid an iceberg for 24 miles. How far is the ship from its original position?

**Solution:** First draw a clear sketch.



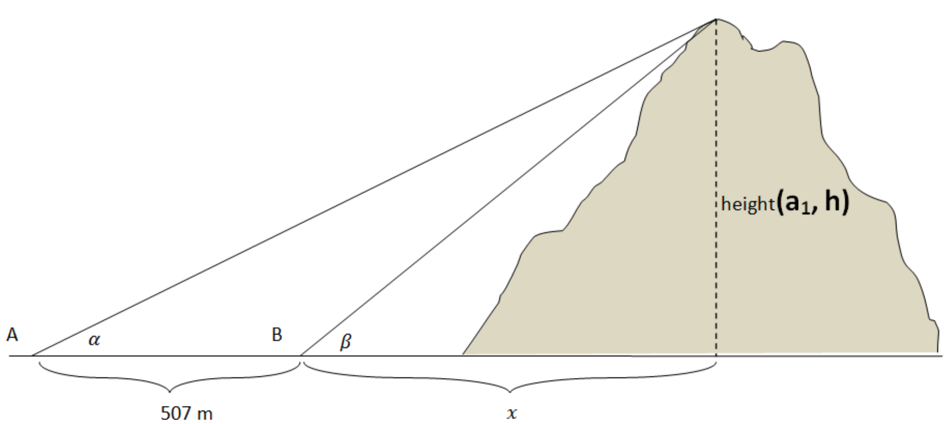

Next, recognize the right triangle with legs 10 and 24. This is a multiple of the 5, 12, 13 Pythagorean number triple and so the distance $x$ must be 26 miles.

**Example B**

A surveying crew is given the job of verifying the height of a cliff. From point $A$, they measure an angle of elevation to the top of the cliff to be $\alpha = 21.567^\circ$. They move 507 meters closer to the cliff and find that the angle to the top of the cliff is now $\beta = 25.683^\circ$. How tall is the cliff?

*Note that* $\alpha$ *is just the Greek letter alpha and in this case it stands for the number* $21.567^\circ$. $\beta$ *is the Greek letter beta and it stands for the number* $25.683^\circ$.

**Solution:** First, sketch the image and label what you know.




Next, because the height is measured at a right angle with the ground, set up two equations. Remember that $\alpha$ and $\beta$ are just numbers, not variables.

$$\tan\alpha = \frac{h}{507+x}$$
$$\tan\beta = \frac{h}{x}$$

Both of these equations can be solved for $h$ and then set equal to each other to find $x$.

$$h = \tan\alpha(507+x) = x\tan\beta$$
$$507\tan\alpha + x\tan\alpha = x\tan\beta$$
$$507\tan\alpha = x\tan\beta - x\tan\alpha$$
$$507\tan\alpha = x(\tan\beta - \tan\alpha)$$
$$x = \frac{507\tan\alpha}{\tan\beta-\tan\alpha} = \frac{507\tan 21.567^\circ}{\tan 25.683^\circ - \tan 21.567^\circ} \approx 228.7\ meters$$

Since the problem asked for the height, you need to substitute $x$ back and solve for $h$.

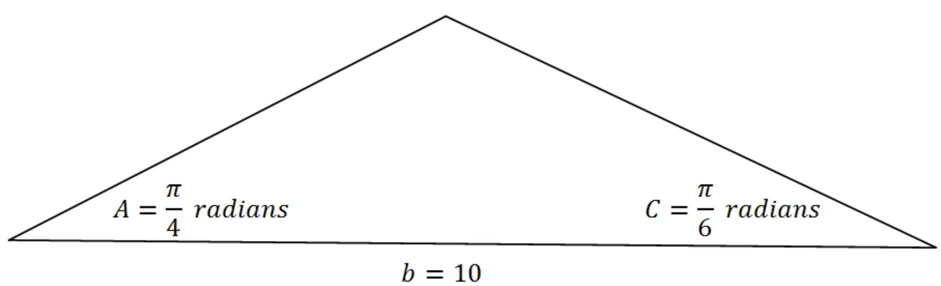
$h = x\tan\beta = 228.7\tan 25.683^\circ \approx 109.99\ meters$

**Example C**

Given a triangle with SSS or SAS you know to use the Law of Cosines. In triangles where there are corresponding angles and sides like AAS or SSA it makes sense to use the Law of Sines. What about ASA?

Given $\Delta ABC$ with $A = \frac{\pi}{4}\ radians, C = \frac{\pi}{6}\ radians$ and $b = 10\ in$ what is $a$?

**Solution:** First, draw a picture.




The sum of the angles in a triangle is 180°. Since this problem is in radians you either need to convert this rule to radians, or convert the picture to degrees.

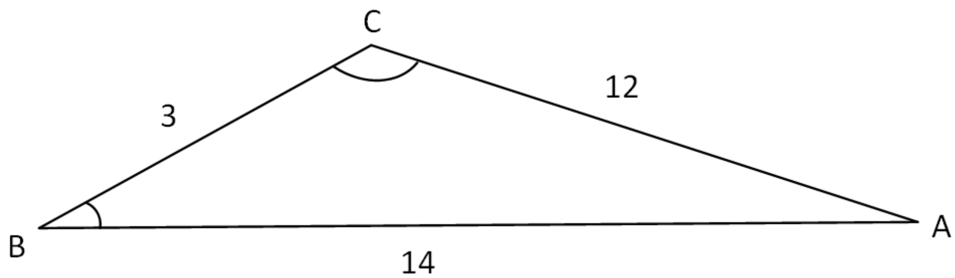
$$A = \frac{\pi}{4}\cdot\frac{180^\circ}{\pi} = 45^\circ$$
$$C = \frac{\pi}{6}\cdot\frac{180^\circ}{\pi} = 30^\circ$$

The missing angle must be $\angle B = 105^\circ$. Now you can use the Law of Sines to solve for $a$.

$$\frac{\sin 105^\circ}{10} = \frac{\sin 45^\circ}{a}$$
$$a = \frac{10\sin 45^\circ}{\sin 105^\circ} \approx 7.32\ in$$

**Concept Problem Revisited**

Sometimes when using the Law of Sines you can get answers that do not match the Law of Cosines. Both answers can be correct computationally, but the Law of Sines may involve interpretation when the triangle is obtuse. The Law of Cosines does not require this interpretation step.




First, use Law of Cosines to find $\angle B$:

$$12^2 = 3^2 + 14^2 - 2 \cdot 3 \cdot 14 \cdot \cos B$$
$$\angle B = \cos^{-1}\left(\frac{12^2 - 3^2 - 14^2}{-2 \cdot 3 \cdot 14}\right) \approx 43.43\ldots^\circ$$

Then, use Law of Sines to find $\angle C$. Use the unrounded value for $B$ even though a rounded value is shown.

$$\frac{\sin 43.43^\circ}{12} = \frac{\sin C}{14}$$
$$\frac{14 \sin 43.43^\circ}{12} = \sin C$$
$$\angle C = \sin^{-1}\left(\frac{14 \sin 43.43^\circ}{12}\right) \approx 53.3^\circ$$

Use the Law of Cosines to double check $\angle C$.

$$14^2 = 3^2 + 12^2 - 2 \cdot 3 \cdot 12 \cdot \cos C$$
$$C = \cos^{-1}\left(\frac{14^2 - 3^2 - 12^2}{-2 \cdot 3 \cdot 12}\right) \approx 126.7^\circ$$

Notice that the last two answers do not match, but they are supplementary. This is because this triangle is obtuse and the $\sin^{-1}\left(\frac{opp}{hyp}\right)$ function is restricted to only producing acute angles.

## Vocabulary

***Angle of elevation*** is the angle at which you view an object above the horizon.

***Angle of depression*** is the angle at which you view an object below the horizon. This can be thought of negative angles of elevation.

***Bearing*** is how direction is measured at sea. North is 0°, East is 90°, South is 180° and West is 270°.

Greek letters ***alpha*** and ***beta*** $(\alpha, \beta)$ are often used as placeholders for known angles. Unknown angles are often referred to as $\theta$ ***(theta).***

***ASA*** refers to the situation from geometry when there are two known angles in a triangle and one known side that is between the known angles.

### Guided Practice

1. The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{10}$. The lighthouse is 200 feet tall. Find the distance from the base of the lighthouse to the boat.

2. From the third story of a building (50 feet) David observes a car moving towards the building driving on the streets below. If the angle of depression of the car changes from 21° to 45° while he watches, how far did the car travel?

3. If a boat travels 4 miles SW and then 2 miles NNW, how far away is it from its starting point?

**Answers:**

1. When you draw a picture, you see that the given angle $\frac{\pi}{10}$ is not directly inside the triangle between the lighthouse, the boat and the base of the lighthouse. It is complementary to the angle you need.

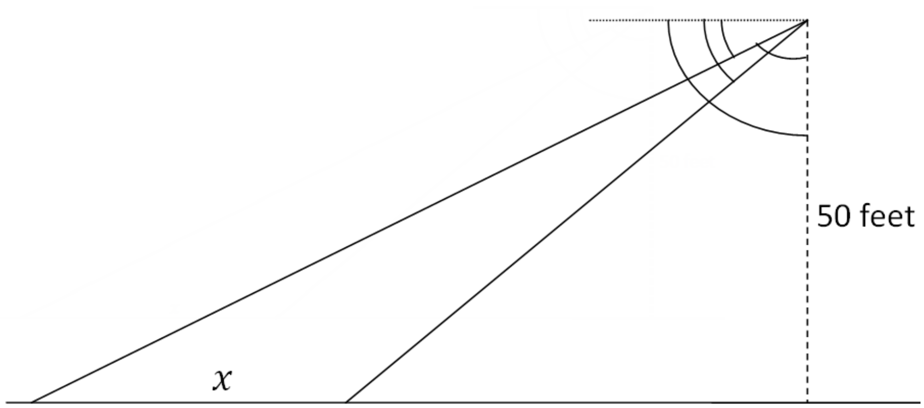
$$\begin{aligned}\frac{\pi}{10} + \theta &= \frac{\pi}{2} \\ \theta &= \frac{2\pi}{5}\end{aligned}$$

Now that you have the angle, use tangent to solve for $x$.

$$\begin{aligned}\tan\frac{2\pi}{5} &= \frac{x}{200} \\ x = 200\tan\frac{2\pi}{5} &\approx 615.5\ldots ft\end{aligned}$$

Alternatively, you could have noticed that $\frac{\pi}{10}$ is alternate interior angles with the angle of elevation of the lighthouse from the boat's perspective. This would yield the same distance for $x$.

2. Draw a very careful picture:



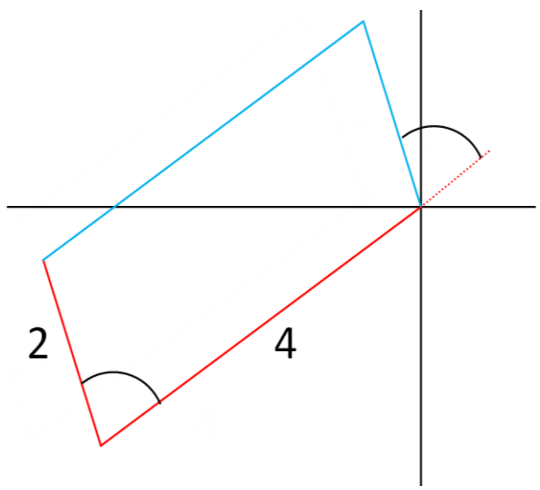
In the upper right corner of the picture there are four important angles that are marked with angles. The measures of these angles from the outside in are $90^\circ, 45^\circ, 21^\circ, 69^\circ$. There is a 45-45-90 right triangle on the right, so the base must also be 50. Therefore you can set up and solve an equation for $x$.

$$\tan 69^\circ = \frac{x+50}{50}$$
$$x = 50\tan 69^\circ - 50 \approx 80.25\ldots ft$$

The hardest part of this problem is drawing a picture and working with the angles.

3. 4 miles SW and then 2 miles NNW

Translate SW and NNW into degrees bearing. SW is a bearing of 225° and NNW is a bearing of 315°. Draw a picture in two steps. Draw the original 4 miles traveled and draw the second 2 miles traveled from the origin. Then translate the second leg of the trip so it follows the first leg. This way you end up with a parallelogram, which has interior angles that are easier to calculate.




The angle between the two red line segments is 67.5° which can be seen if the red line is extended past the origin.

The shorter diagonal of the parallelogram is the required unknown information.

$$x^2 = 4^2 + 2^2 - 2\cdot 4\cdot 2\cdot \cos 67.5^\circ$$
$$x \approx 3.7\ miles$$

### Practice

The angle of depression of a boat in the distance from the top of a lighthouse is $\frac{\pi}{6}$. The lighthouse is 150 feet tall. You want to find the distance from the base of the lighthouse to the boat.

1. Draw a picture of this situation.

2. What methods or techniques will you use?

3. Solve the problem.

From the third story of a building (60 feet) Jeff observes a car moving towards the building driving on the streets below. The angle of depression of the car changes from 34° to 62° while he watches. You want to know how far the car traveled.

4. Draw a picture of this situation.

5. What methods or techniques will you use?

6. Solve the problem.

A boat travels 6 miles NW and then 2 miles SW. You want to know how far away the boat is from its starting point.

7. Draw a picture of this situation.

8. What methods or techniques will you use?

9. Solve the problem.

You want to figure out the height of a building. From point $A$, you measure an angle of elevation to the top of the building to be $\alpha = 10^\circ$. You move 50 feet closer to the building to point $B$ and find that the angle to the top of the building is now $\beta = 60^\circ$.

10. Draw a picture of this situation.

11. What methods or techniques will you use?

12. Solve the problem.

13. Given $\Delta ABC$ with $A = 40^\circ, C = 65^\circ$ and $b = 8\ in$, what is $a$?

14. Given $\Delta ABC$ with $A = \frac{\pi}{3}\ radians, C = \frac{\pi}{8}\ radians$ and $b = 12\ in$ what is $a$?

15. Given $\Delta ABC$ with $A = \frac{\pi}{6}\ radians, C = \frac{\pi}{4}\ radians$ and $b = 20\ in$ what is $a$?

The relationship between the sides and angles of all kinds of triangles were explored though the use of the six basic trigonometric functions.

# 4.9 References


1. CK-12 Foundation. . CCSA
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# CHAPTER 5 Trigonometric Functions

## Chapter Outline



Trigonometry is the study of the relationship between the angles and sides of triangles. Here you will extend this idea to functions that have a cyclical and repeating nature. Tides, the height of a person on a Ferris wheel and seasonal temperature can all be modeled through the use of sine and cosine functions. You will apply the basic rules of transformations to create equations and graphs that fit data.

# 5.1 The Unit Circle

Here you will use your knowledge of basic triangle trigonometry to identify key points and angles around a circle of radius one centered at the origin.

The**unit circle** is a circle of radius one, centered at the origin, that summarizes all the 30-60-90 and 45-45-90 triangle relationships that exist. When memorized, it is extremely useful for evaluating expressions like$\cos(135^\circ)$ or$\sin\left(-\frac{5\pi}{3}\right)$. It also helps to produce the parent graphs of sine and cosine.

How can you use the unit circle to evaluate $\cos(135^\circ)$ and $\sin\left(-\frac{5\pi}{3}\right)$?

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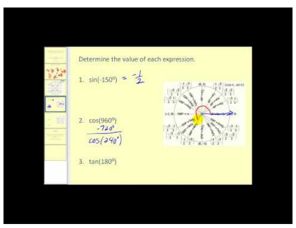

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http://www.youtube.com/watch?v=i56P6xzsB5Y James Sousa: Determine Trigonometric Function Values Using the Unit Circle

### Guidance

You already know how to translate between degrees and radians and the triangle ratios for 30-60-90 and 45-45-90 right triangles. In order to be ready to completely fill in and memorize a unit circle, two triangles need to be worked out. Start by finding the side lengths of a 30-60-90 triangle and a 45-45-90 triangle each with hypotenuse equal to 1.

**TABLE 5.1:**

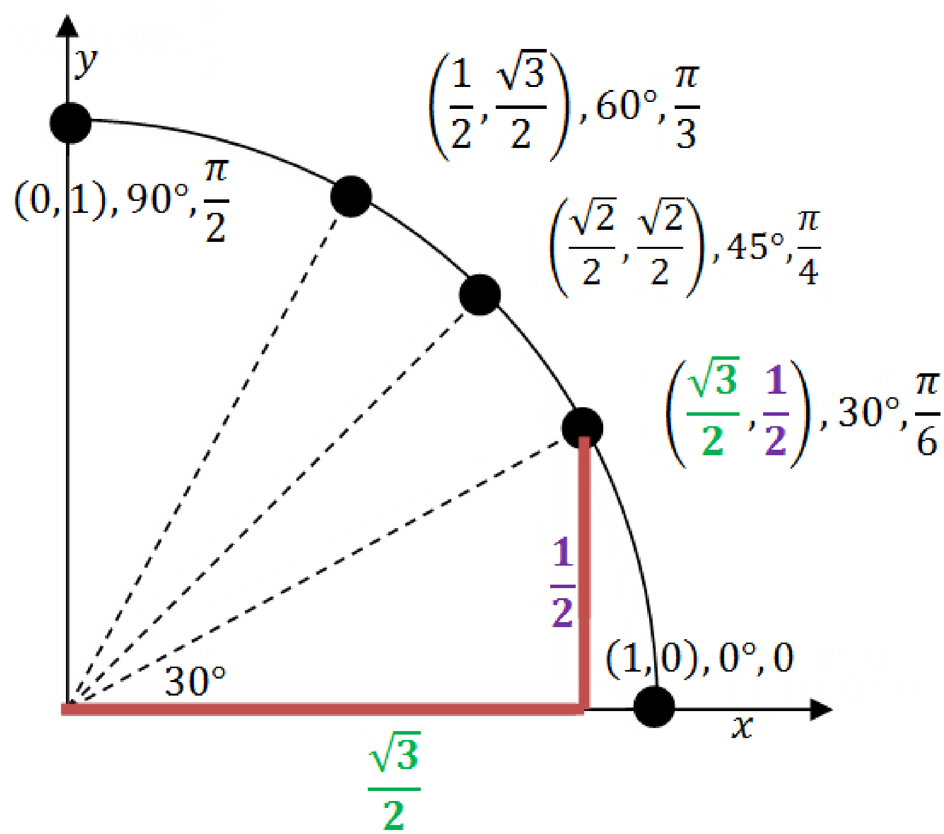
| | | |
|---|---|---|
| $30^\circ$ | $60^\circ$ | $90^\circ$ |
| $x$ | $x\sqrt{3}$ | $2x$ |
| $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |

**TABLE 5.2:**

| | | |
|---|---|---|
| $45^\circ$ | $45^\circ$ | $90^\circ$ |
| $x$ | $x$ | $x\sqrt{2}$ |
| $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 |

This is enough information to fill out the important points in the first quadrant of the unit circle. The values of the $x$ and $y$ coordinates for each of the key points are shown below. Remember that the $x$ and $y$ coordinates come from the lengths of the legs of the special right triangles, as shown specifically for the $30^\circ$ angle. ***Always remember to***

***measure the angle from the positive portion of the $x$-axis.***

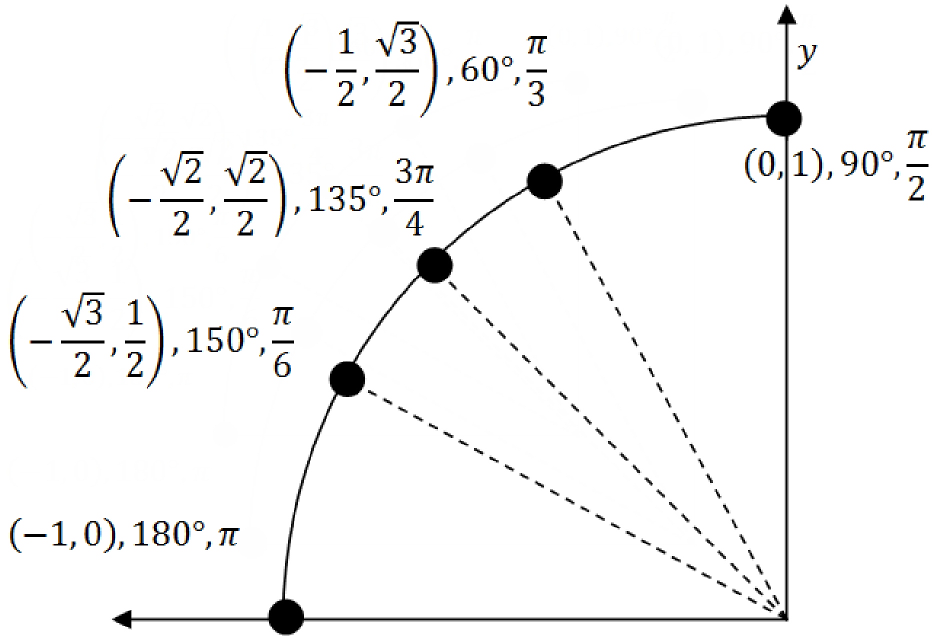



Knowing the first quadrant well is the key to knowing the entire unit circle. Every other point on the unit circle can be found using logic and this quadrant.

**Example A**

Use your knowledge of the first quadrant of the unit circle to identify the angles and important points of the second quadrant.

**Solution:** The heights are mirrored and equal which correspond to the $y$ values. The $x$ values are all negative.



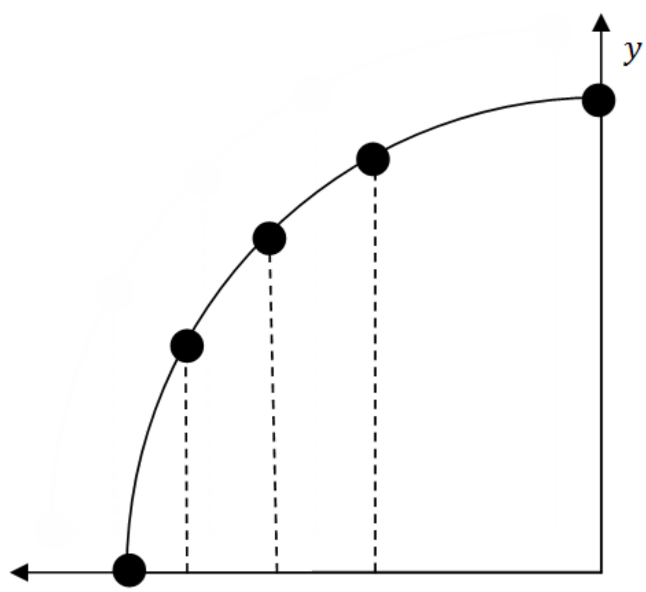

**Example B**

Identify a pattern in the heights of the points in the first quadrant to help you remember the points.

**Solution:** The heights of the points in the first quadrant are the $y$-coordinates which are: $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$

When rewritten, the pattern becomes clear: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

The three points in the middle are the most often mixed up. This pattern illustrates how they increase in size from small $\frac{1}{2}$, to medium $\frac{\sqrt{2}}{2}$, to large $\frac{\sqrt{3}}{2}$. When you fill in the unit circle, look for the heights that are small, medium and large and this will tell you were each value should go. Notice that the heights for these five points in the second quadrant are also $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$.

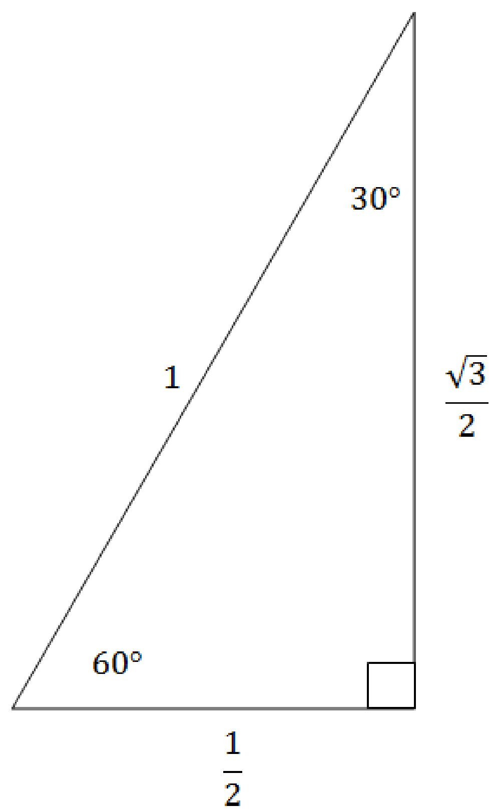



This technique also works for the widths. This can make memorizing the 16 points of the unit circle a matter of logic and the pattern: $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$.

**Example C**

Evaluate $\cos 60^\circ$ using the unit circle and right triangle trigonometry. What is the connection between the $x$ coordinate of the point and the cosine of the angle?

**Solution:** The point on the unit circle for $60^\circ$ is $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the point is one unit from the origin. This can be represented as a 30-60-90 triangle.




Since cosine is adjacent over hypotenuse, cosine turns out to be exactly the $x$ coordinate $\frac{1}{2}$.

**Concept Problem Revisited**

The $x$ value of a point along the unit circle corresponds to the cosine of the angle. The $y$ value of a point corresponds to the sine of the angle. When the angles and points are memorized, simply recall the $x$ or $y$ coordinate.

When evaluating $\cos(135^\circ)$ your thought process should be something like this:

You know $135^\circ$ goes with the point $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and cosine is the $x$ portion. So, $\cos(135^\circ) = -\frac{\sqrt{2}}{2}$.

When evaluating $\sin\left(-\frac{5\pi}{3}\right)$ your thought process should be something like this:

You know $-\frac{5\pi}{3}$ goes with the point $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and sine is the $y$ portion. So, $\sin\left(-\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

## Vocabulary

***Coterminal Angles*** are sets of angles such as $-10^\circ, 350^\circ, 710^\circ$ that start at the positive $x$-axis and end at the same terminal side. Since coterminal angles end at identical points along the unit circle, trigonometric expressions involving coterminal angles are equivalent: $\sin -10^\circ = \sin 350^\circ = \sin 710^\circ$.

### Guided Practice

1. Using knowledge of the first quadrant of the unit circle, identify the angles and important points of the third quadrant.

2. For each of the six trigonometric functions, identify the quadrants where they are positive and the quadrants where they are negative.

3. Evaluate the following trigonometric expressions using the unit circle.

a. $\sin\frac{\pi}{2}$

b. $\cos 210^\circ$

c. $\tan 315^\circ$

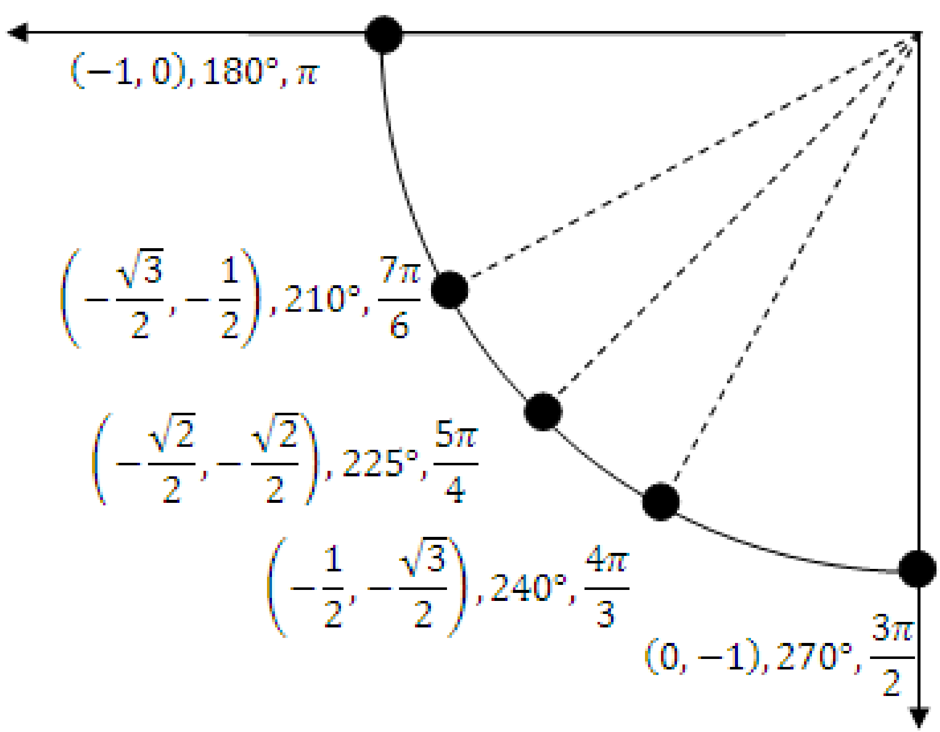
d. $\cot 270^\circ$

e. $\sec\frac{11\pi}{6}$

f. $\csc -\frac{5\pi}{4}$

**Answers:**

1. Both the $x$ values and $y$ values are negative and their respective coordinates correspond to those of the other quadrants.




2. In quadrant I, the hypotenuse, adjacent and opposite side are all positive. Thus all 6 trigonometric functions are positive.

In quadrant II the hypotenuse and opposite sides are positive and the adjacent side is negative. This means that every trigonometric expression involving an adjacent side is negative. Sine and its reciprocal cosecant are the only two trigonometric functions that do not refer to the adjacent side which makes them the only positive ones.

In quadrant III only the hypotenuse is positive. Thus the only trigonometric functions that are positive are tangent and its reciprocal cotangent because these functions refer to both adjacent and opposite sides which will both be negative.

In quadrant IV the hypotenuse and the adjacent sides are positive while the opposite side is negative. This means that only cosine and its reciprocal secant are positive.

A mnemonic device to remember which trigonometric functions are positive and which trigonometric functions are negative is "All Students Take Calculus." All refers to all the trigonometric functions are positive in quadrant I. The letter S refers to sine and its reciprocal cosecant that are positive in quadrant II. The letter T refers to tangent and its reciprocal cotangent that are positive in quadrant III. The letter C refers to cosine and its reciprocal secant that are positive in quadrant IV.

3. a. $\sin \frac{\pi}{2} = 1$

b. $\cos 210^\circ = -\frac{\sqrt{3}}{2}$

c. $\tan 315^\circ = -1$

d. $\cot 270^\circ = 0$

e. $\sec \frac{11\pi}{6} = \frac{1}{\cos \frac{11\pi}{6}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

f. $\csc -\frac{5\pi}{4} = \frac{1}{\sin -\frac{5\pi}{4}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$

## Practice

Name the angle between 0° and 360° that is coterminal with...

1. $-20^\circ$
2. $475^\circ$
3. $-220^\circ$
4. $690^\circ$
5. $-45^\circ$

Use your knowledge of the unit circle to help determine whether each of the following trigonometric expressions is positive or negative.

6. $\tan 143^\circ$
7. $\cos \frac{\pi}{3}$
8. $\sin 362^\circ$
9. $\csc \frac{3\pi}{4}$

Use your knowledge of the unit circle to evaluate each of the following trigonometric expressions.

10. $\cos 120^\circ$
11. $\sec \frac{\pi}{3}$
12. $\tan 225^\circ$
13. $\cot 120^\circ$
14. $\sin \frac{11\pi}{6}$
15. $\csc 240^\circ$
16. Find $\sin\theta$ and $\tan\theta$ if $\cos\theta = \frac{\sqrt{3}}{2}$ and $\cot\theta > 0$.
17. Find $\cos\theta$ and $\tan\theta$ if $\sin\theta = -\frac{1}{2}$ and $\sec\theta < 0$.

18. Draw the complete unit circle (all four quadrants) and label the important points.

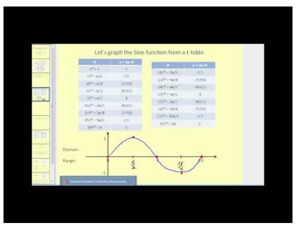
# 5.2 The Sinusoidal Function Family

Here you will see how the graphs of sine and cosine come from the unit circle.

The cosine function is the $x$coordinates of the unit circle and the sine function is the $y$coordinates. Since the unit circle has radius one and is centered at the origin, both sine and cosine oscillate between positive and negative one.

What happens when the circle is not centered at the origin and does not have a radius of 1?

### Watch This



MEDIA

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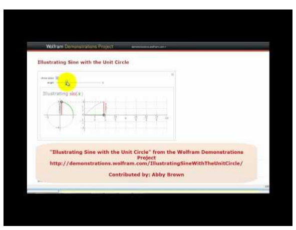
http://www.youtube.com/watch?v=nXx2PsgMjYA James Sousa: Graphing the Sine and Cosine Functions



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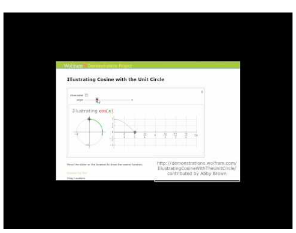
http://www.youtube.com/watch?v=QNQAkUUHNxo James Sousa: Animation: Graphing the Sine Function Using the Unit Circle



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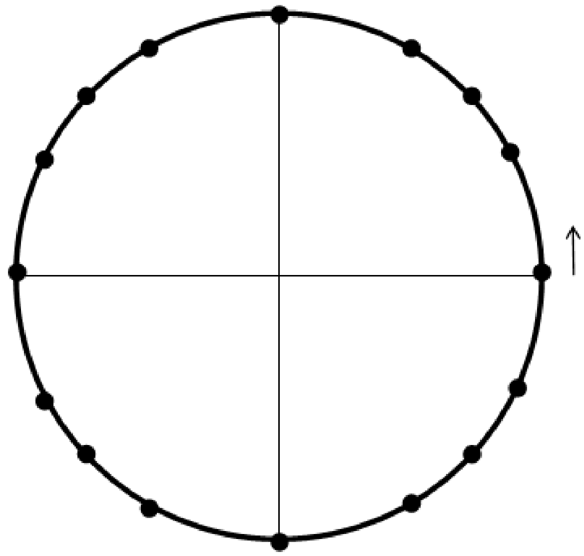
http://www.youtube.com/watch?v=tcjZOGaeoeo James Sousa: Animation: Graphing the Cosine Function Using the Unit Circle

### Guidance

Consider a Ferris wheel that spins evenly with a radius of 1 unit. It starts at (1, 0) or an angle of 0 radians and spins counterclockwise at a rate of one cycle per $2\pi$ minutes (so you can use time is equal to radians).



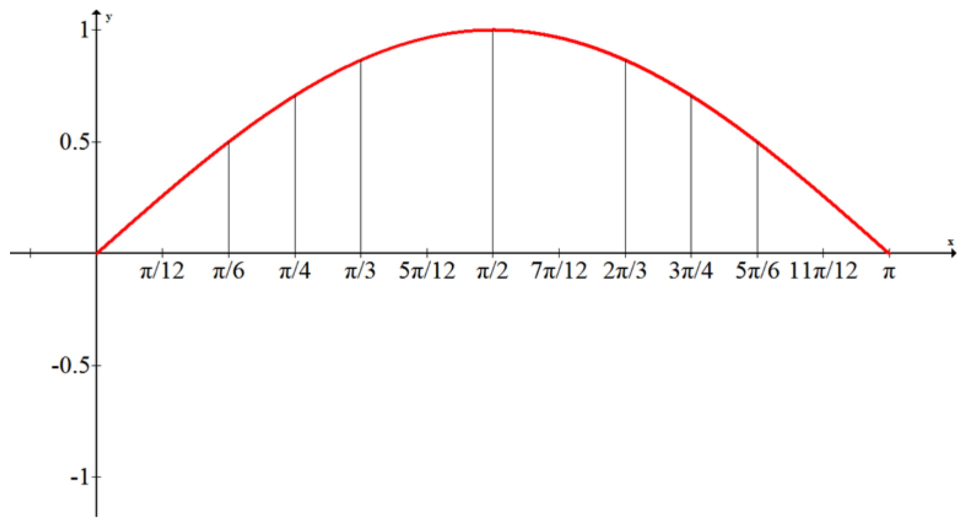
The 16 points around the circle are chosen because they correspond to the key points of the unit circle. Their heights ($y$-values) and widths ($x$-values) are already known and can be filled in.

First consider the height at each of the points as you travel around half of the circle from the starting location. Keep track of your work in a table.

### TABLE 5.3:

| **Angle (radians)** | **Height (units)** |
|---|---|
| 0 | 0 |
| $\frac{\pi}{6}$ | $\frac{1}{2}$ |
| $\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2} \approx 0.707$ |
| $\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2} \approx 0.866$ |
| $\frac{\pi}{2}$ | 1 |
| $\frac{2\pi}{3}$ | $\frac{\sqrt{3}}{2} \approx 0.866$ |
| $\frac{3\pi}{4}$ | $\frac{\sqrt{2}}{2} \approx 0.707$ |
| $\frac{5\pi}{6}$ | $\frac{1}{2}$ |
| $\pi$ | 0 |

Notice the symmetry of the height around $\frac{\pi}{2}$ and see the rest of the table in the examples. Once the table is finished, you can plot these points on a regular coordinate plane where the $x$ axis is the angle and the $y$ axis is the height. This is the first part of the graph of the sine function.




You will see a complete cycle of the sine function in Example A. In Example B, you will see how the $x$-coordinates produce the plot of the cosine curve.

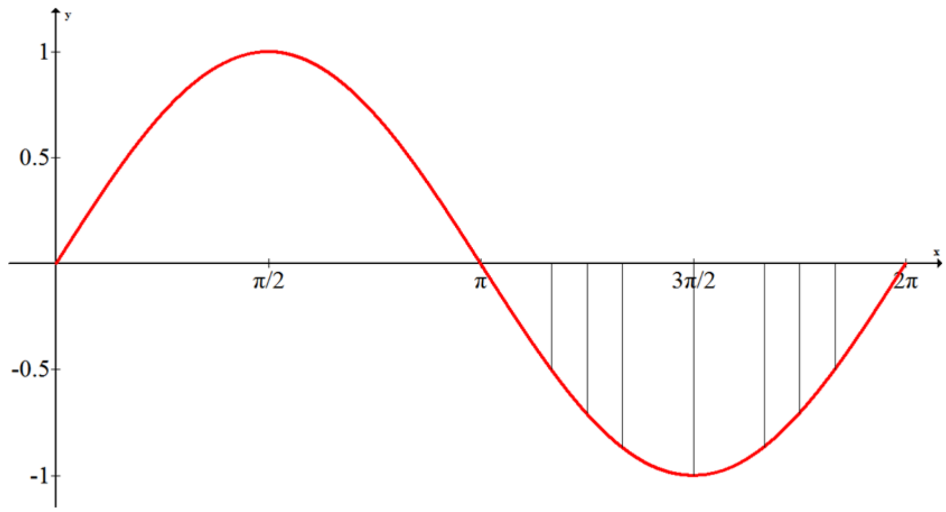
**Example A**

Finish the table for heights of the points in quadrants III and IV and draw an entire cycle (known as a period) of the sine function.

**Solution:**

TABLE 5.4:

| Angle (radians) | Height (units) |
|---|---|
| $\pi$ | 0 |
| $\frac{7\pi}{6}$ | $-\frac{1}{2}$ |
| $\frac{5\pi}{4}$ | $-\frac{\sqrt{2}}{2} \approx -0.707$ |
| $\frac{4\pi}{3}$ | $-\frac{\sqrt{3}}{2} \approx -0.866$ |
| $\frac{3\pi}{2}$ | -1 |
| $\frac{5\pi}{3}$ | $-\frac{\sqrt{3}}{2} \approx -0.866$ |
| $\frac{7\pi}{4}$ | $-\frac{\sqrt{2}}{2} \approx -0.707$ |
| $\frac{11\pi}{6}$ | $-\frac{1}{2}$ |
| $2\pi$ | 0 |




**Example B**

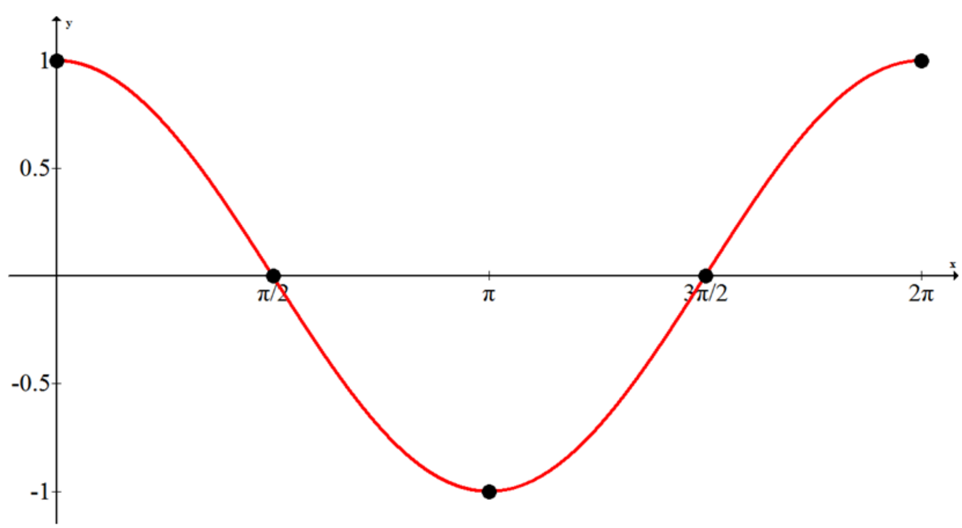
Use your knowledge of the angles $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ on the unit circle to get a complete cycle of the cosine graph.

**Solution:** The cosine function is the $x$-coordinates of the unit circle and measures width. By referring to a unit circle or your memory, you can fill out a much shorter table than before.

TABLE 5.5:

| Angle (radians) | Width (units) |
|---|---|
| 0 | 1 |
| $\frac{\pi}{2}$ | 0 |
| $\pi$ | -1 |
| $\frac{3\pi}{2}$ | 0 |
| $2\pi$ | 1 |

First plot these five points and then connect them with a smooth curve. This will produce the cosine graph.

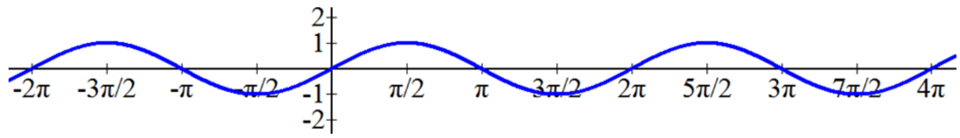


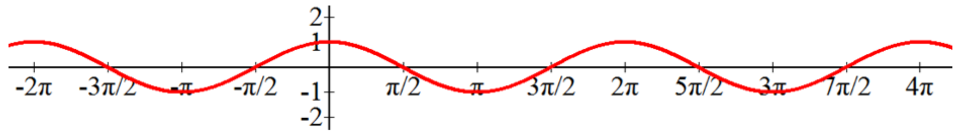

Determining these five main points is the key to graphing sine or cosine graphs even when the graph is shifted or stretched.

**Example C**

What happens on either side of the sine and cosine graphs? Can you explain why?

**Solution:** The graphs of the sine (blue) and cosine (red) functions repeat forever in both directions.




If you think about the example with the Ferris wheel, the ride will keep on spinning and has been spinning forever. This is why the same cycle of the graph repeats over and over.

**Concept Problem Revisited**

The unit circle produces the parent function sine and cosine graphs. When the unit circle is shifted up or down, made wider or narrower, or spun faster or slower in either direction, then the graphs of the sine and cosine functions will be transformed using basic function transformation rules.

### Vocabulary

The ***sinusoidal function family*** refers to either sine or cosine waves since they are the same except for a horizontal shift. This function family is also called the **periodic function family** because the function repeats after a given period of time.

### Guided Practice

1. How are the sine and cosine graphs the same and how are they different?

2. Where are two maximums and two minimums of the sine graph?

3. In the interval $[-2\pi, 4\pi)$ where does cosine have zeroes?

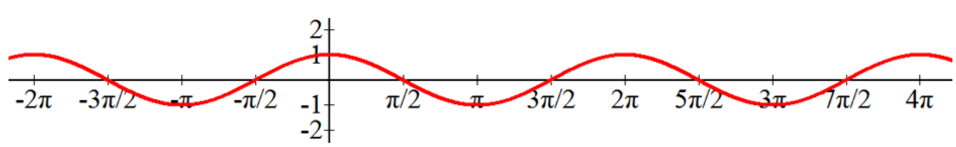
**Answers:**

1. The sine graph is the same as the cosine graph offset by $\frac{\pi}{2}$. Besides this shift, both curves are identical due to the perfect symmetry of circles.

2. One maximum of the sine graph occurs at $\left(\frac{\pi}{2}, 1\right)$. One minimum occurs at $\left(\frac{3\pi}{2}, -1\right)$. This is one cycle of the sine graph. Since it completes a cycle every $2\pi$, when you add $2\pi$ to an $x$-coordinate you will be on the same point of the cycle giving you another maximum or minimum.

$\left(\frac{5\pi}{2}, 1\right)$ is another maximum. $\left(\frac{7\pi}{2}, -1\right)$ is another minimum.

3. Observe where the cosine curve has $x$-coordinates equal to zero. Note that $4\pi$ is excluded from the interval. The values are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.



### Practice

1. Sketch $p(x) = \sin x$ from memory.

2. Sketch $j(x) = \cos x$ from memory.

3. Where do the maximums of the cosine graph occur?

4. Where do the minimums of the cosine graph occur?

5. Find all the zeroes of the sine function on the interval $\left[-\pi, \frac{5\pi}{2}\right]$.

6. Find all the zeroes of the cosine function on the interval $\left(-\frac{\pi}{2}, \frac{7\pi}{2}\right]$.

7. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $f(x) = 2\cos x$ will look like.

8. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph $g(x) = \cos x + 2$ of will look like.

9. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $h(x) = \cos(x - \pi)$ will look like.

10. Preview: Using your knowledge of function transformations and the cosine graph, predict what the graph of $k(x) = -\cos x$ will look like.

# 5.3 Amplitude of Sinusoidal Functions

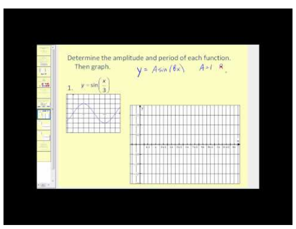
Here you will see how changing the radius of a circle affects the graph of the sine function through a vertical stretch.

The amplitude of the sine and cosine functions is the distance between the sinusoidal axis and the maximum or minimum value of the function. In relation to sound waves, amplitude is a measure of how loud something is.

What is the most common mistake made when graphing the amplitude of a sine wave?

### Watch This

Watch the portion of this video discussing amplitude:




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=qJ-oUV7xL3w James Sousa: Amplitude and Period of Sine and Cosine

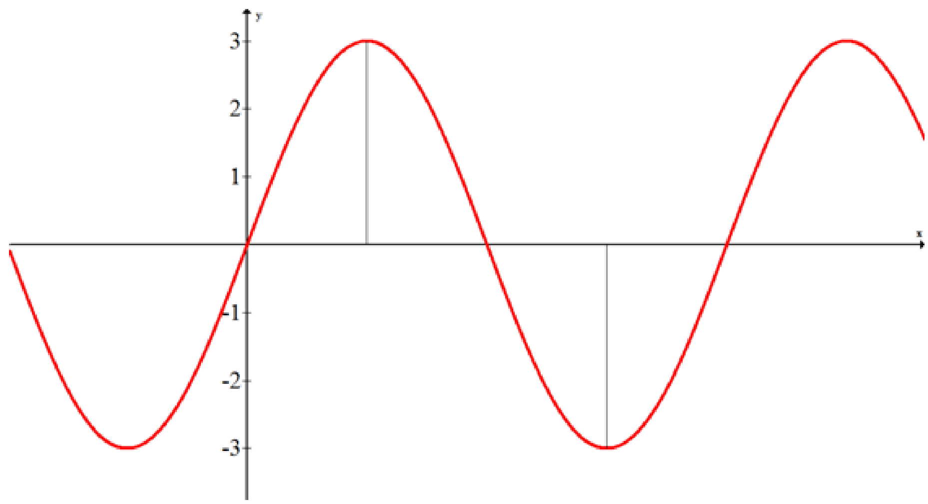
### Guidance

The general form a sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

The cosine function can just as easily be substituted and for many problems it will be easier to use a cosine equation. Since both the sine and cosine waves are identical except for a horizontal shift, it all depends on where you see the wave starting.

The coefficient $a$ is the amplitude (which fortunately also starts with the letter a). When there is no number present, then the amplitude is 1. The best way to define amplitude is through a picture. Below is the graph of the function $f(x) = 3 \cdot \sin x$, which has an amplitude of 3.



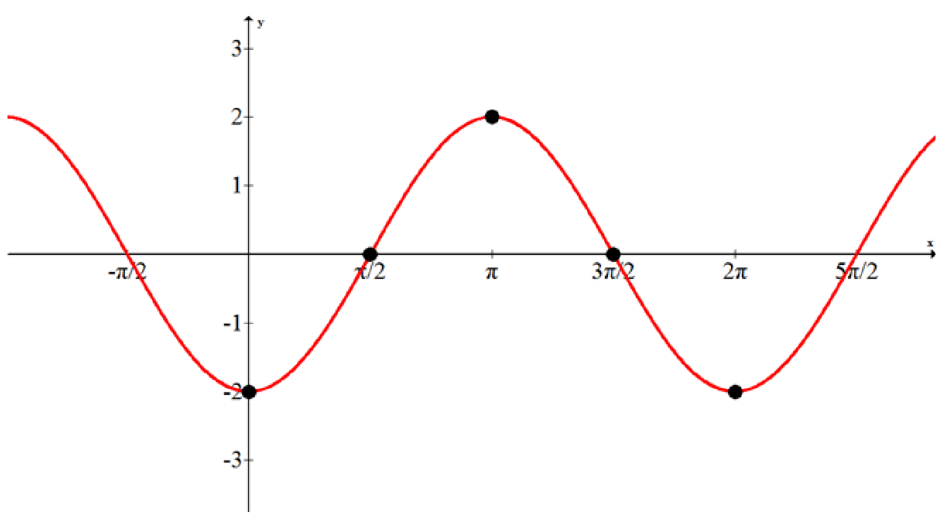

Notice that the amplitude is 3, not 6. This corresponds to the absolute value of the maximum and minimum values of the function. If the function had been $f(x) = -3 \cdot \sin x$, then the whole graph would be reflected across the $x$ axis.

Also notice that the $x$ axis on the graph above is unlabeled. This is to show that amplitude is a vertical distance. The sinusoidal axis is the neutral horizontal line that lies between the crests and the troughs (or peaks and valleys if you prefer). For this function, the sinusoidal axis was just the $x$ axis, but if the whole graph were shifted up, the sinusoidal axis would no longer be the $x$ axis. Instead, it would still be the horizontal line directly between the crests and troughs.

**Example A**

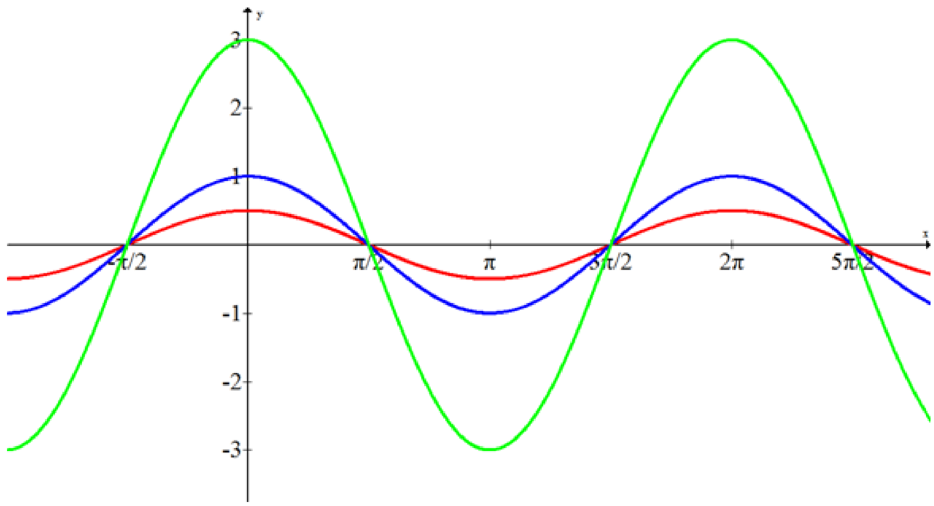
Graph the following function by first plotting main points: $f(x) = -2 \cdot \cos x$.

**Solution:** The amplitude is 2, which means the maximum values will be at 2 and the minimum values will be at -2. Normally with a basic cosine curve the points corresponding to $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ fall above, on or below the line in the following sequence: above, on, below, on, above. The negative sign switches above with below. The whole graph is reflected across the $x$-axis.




**Example B**

Write a cosine equation for each of the following functions.




**Solution:** The amplitudes of the three functions are 3, 1 and $\frac{1}{2}$ and none of them are reflected across the $x$-axis.

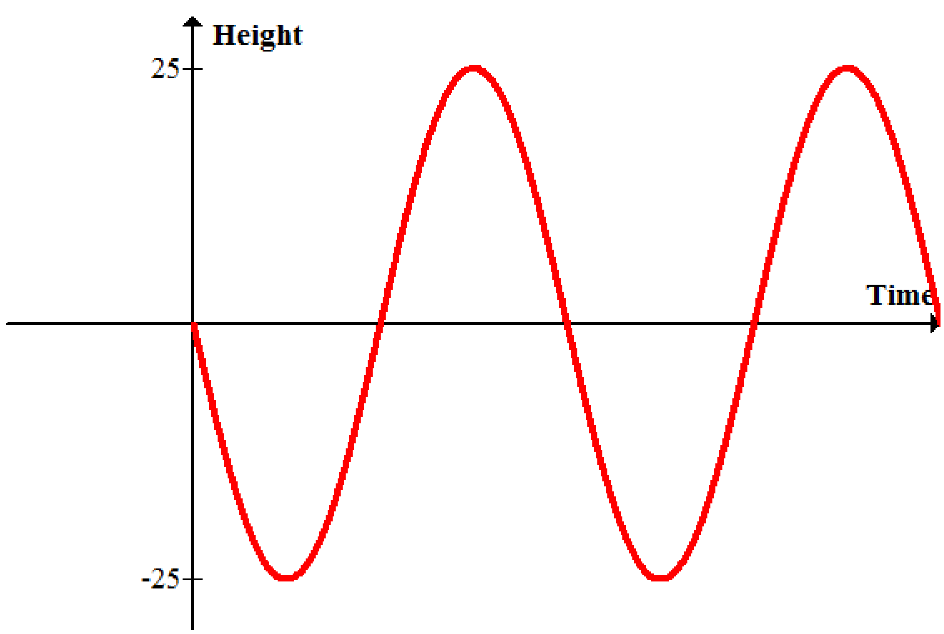
$$\begin{aligned} f(x) &= 3 \cdot \cos x \\ h(x) &= \cos x \\ g(x) &= \frac{1}{2} \cdot \cos x \end{aligned}$$

Note that amplitude itself is always positive.

**Example C**

A Ferris wheel with radius 25 feet sits next to a platform. The ride starts at the platform and travels down to start. Model the height versus time of the ride.

**Solution:** Since no information is given about the time, simply label the $x$ axis as time. At time zero the height is zero. Initially the height will decrease as the ride goes below the platform. Eventually, the wheel will find the minimum and start to increase again all the way until it reaches a maximum.



**Concept Problem Revisited**

The most common mistake is doubling or halving the amplitude unnecessarily. Many people forget that the number $a$ in the equation, like the 3 in $f(x) = 3\sin x$, is the distance from the $x$ axis to both the peak and the valley. It is not the total distance from the peak to the valley.

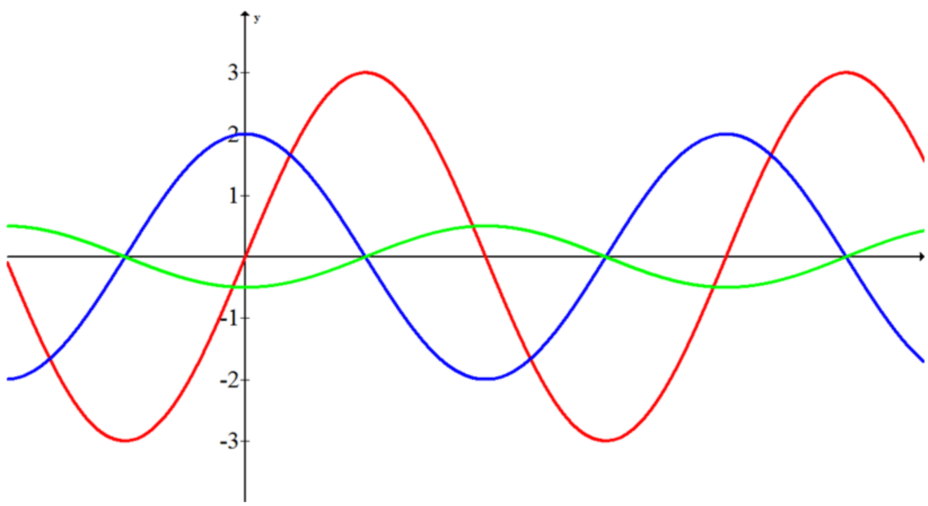
## Vocabulary

The ***amplitude*** of the sine or a cosine function is the shortest vertical distance between the sinusoidal axis and the maximum or minimum value.

The ***sinusoidal axis*** is the neutral horizontal line that lies between the crests and the troughs of the graph of the function.

## Guided Practice

1. Identify the amplitudes of the following four functions:




2. Graph the following function: $f(x) = -8\sin x$.

3. Find the amplitude of the function $f(x) = -3\cos x$ and use the language of transformations to describe how the graph is related to the parent function $y = \cos x$.

**Answers:**

1. The red function has amplitude 3. The blue function has amplitude 2. The green function has amplitude $\frac{1}{2}$.

2. First identify where the function starts and ends. In this case, one cycle (period) is from 0 to $2\pi$. Usually sine functions start at the sinusoidal axis and have one bump up and then one bump down, but the negative sign swaps directions. Lastly, instead of going up and down only one unit, this function has amplitude of 8. Thus the pattern is:

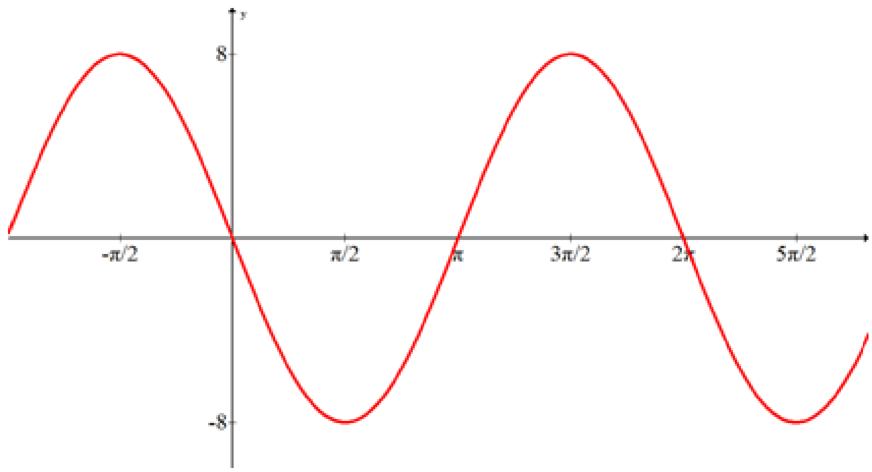
Starts at height 0

Then down to -8.

Then back to 0.

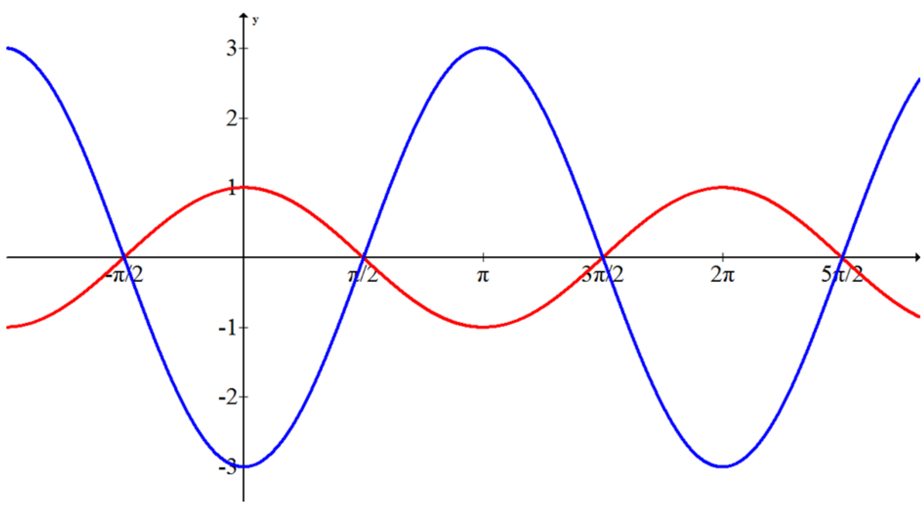
Then up to 8

Then back to 0.




Plotting these 5 points first is an essential step to sketching an approximate curve.

3. The new function is reflected across the $x$ axis and vertically stretched by a factor of 3.




## Practice

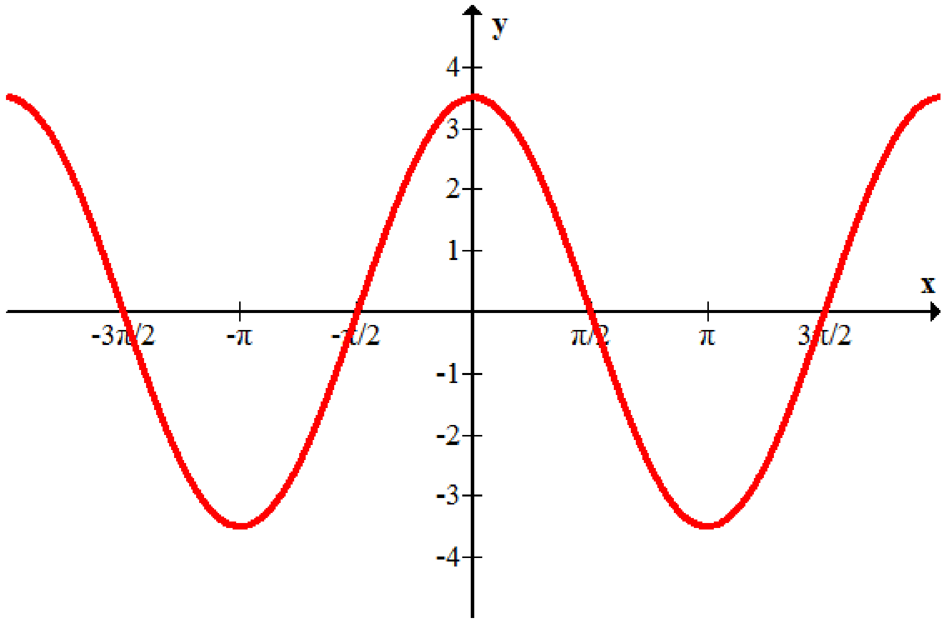
1. Explain how to find the amplitude of a sinusoidal function from its equation.

2. Explain how to find the amplitude of a sinusoidal function from its graph.

Find the amplitude of each of the following functions.

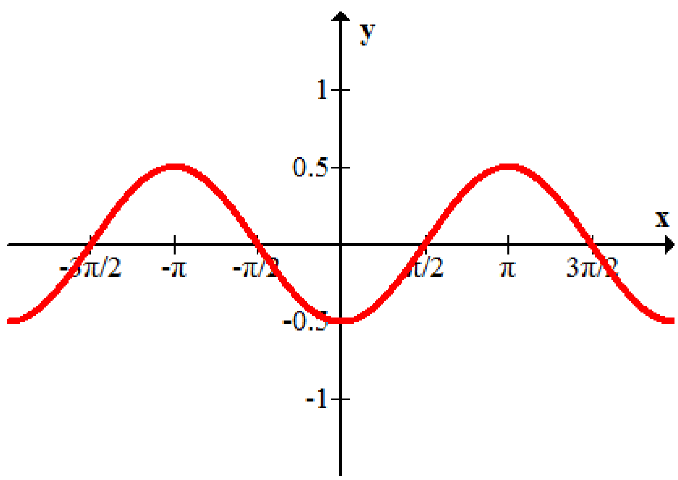
3. $g(x) = -5\cos x$

4.



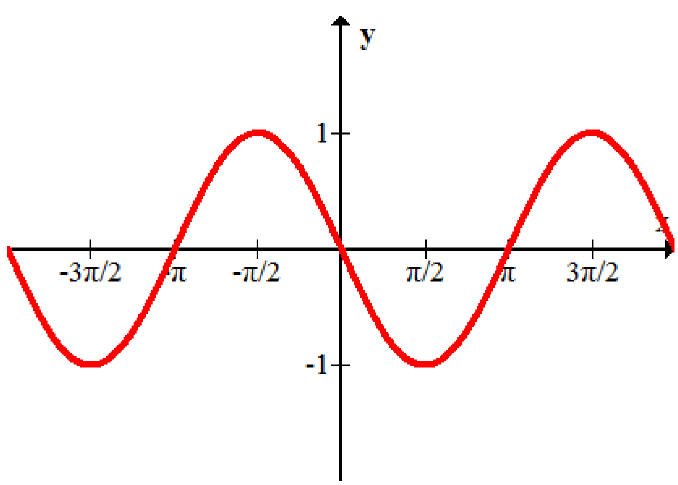

5. $f(x) = \frac{1}{2} \sin x$

6.




7. $j(x) = 3.12 \cos x$

8.



Sketch each of the following functions.

9. $f(x) = 3\sin x$

10. $g(x) = -4\cos x$

11. $h(x) = \pi \sin x$

12. $k(x) = -1.2\cos x$

13. $p(x) = \frac{2}{3}\cos x$

14. $m(x) = -\frac{1}{2}\sin x$

15. Preview: $r(x) = 3\sin x + 2$

# 5.4 Vertical Shift of Sinusoidal Functions

Here you will explore the how a vertical shift of a sinusoidal function is represented in an equation and in a graph.

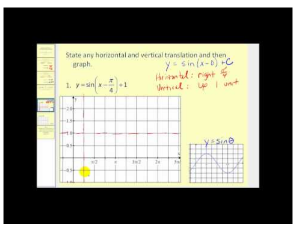
Your knowledge of transformations, specifically vertical shift, apply directly to sinusoidal functions. In practice, sketching shifted sine and cosine functions requires greater attention to detail and more careful labeling than other functions. Can you describe the following transformation in words?

$f(x) = \sin x \rightarrow g(x) = -3\sin x - 4$

In what order do the reflection, stretch and shift occur? Is there a difference?

### Watch This

Watch the portions of this video focusing on vertical translations:



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=DswBtrtvR5M James Sousa: Horizontal and Vertical Translations of Sine and Cosine

### Guidance

The general form of a sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

Recall that $a$ controls amplitude and the $\pm$ controls reflection. Here you will see how $d$ controls the vertical shift.

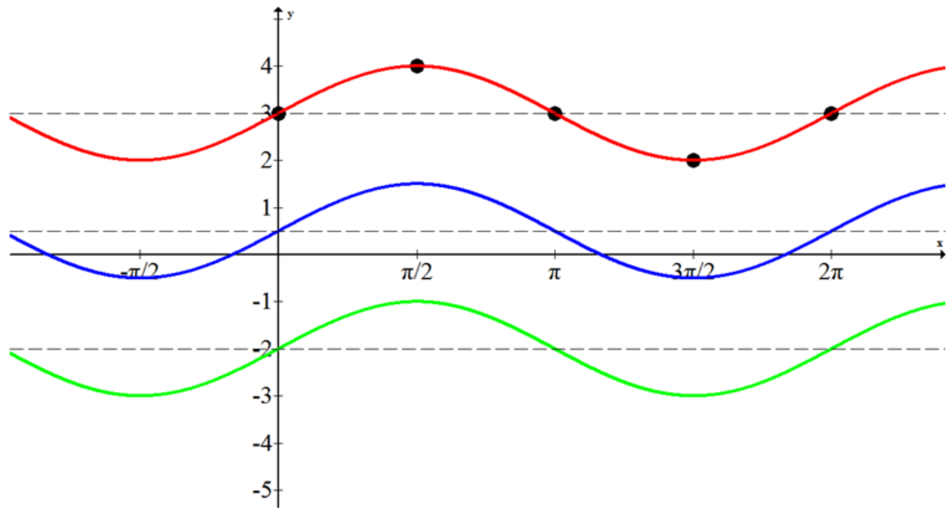
The most straightforward way to think about vertical shift of sinusoidal functions is to focus on the sinusoidal axis, the horizontal line running through the middle of the sine or cosine wave. At the start of the problem identify the vertical shift and immediately draw the new sinusoidal axis. Then proceed to graph amplitude and reflection about **that axis** as opposed to the $x$ axis.

**Example A**

Graph the following three functions.

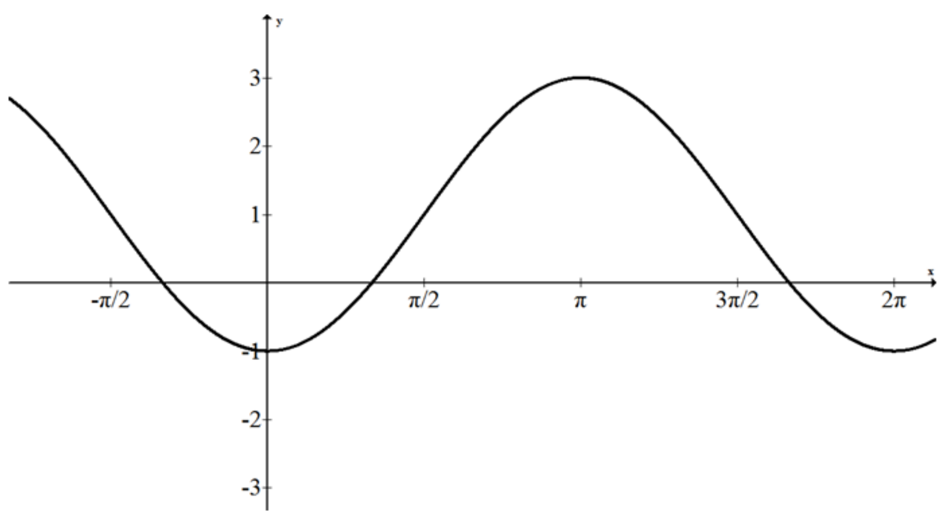
$$\begin{aligned} f(x) &= \sin x + 3 \\ g(x) &= \sin x - 2 \\ h(x) &= \sin x + \frac{1}{2} \end{aligned}$$

**Solution:** First draw the new sinusoidal axis for each graph. Then, draw a complete sine wave for each one. Remember to draw the five important points that separate each quadrant to get a clear sense of the graph. Right now every cycle starts at 0 and ends at $2\pi$ but this will not always be the case.



### Example B

Identify the equation of the following transformed cosine graph.



**Solution:** Since there is no sinusoidal axis given, you must determine the vertical shift, stretch and reflection. The peak occurs at $(\pi, 3)$ and the trough occurs at (0, -1) so the horizontal line directly between +3 and -1 is $y = 1$. Since the sinusoidal axis has been shifted up by one unit $d = 1$. From this height, the graph goes two above and two below which means that the amplitude is 2. Since this cosine graph starts its cycle at (0, -1) which is a lowest point, it is a negative cosine. The function is $f(x) = -2\cos x + 1$.

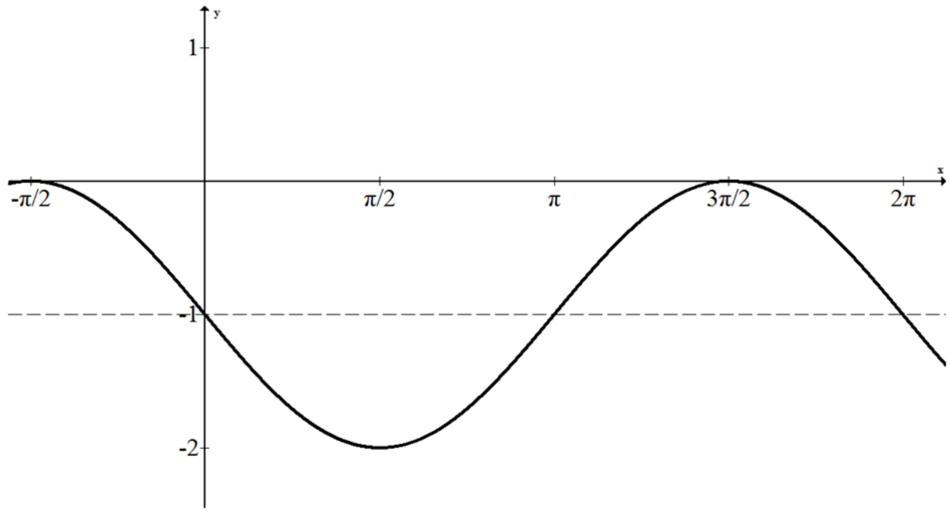
### Example C

Graph the following function: $f(x) = -\sin x - 1$.

**Solution:** Identify the important information. Then draw the sinusoidal axis.

- $a = 1$

- $d = -1$
- Reflection over $x$ axis.




Note that it is critical that you know the shape of a regular sine graph and a negative sine graph.

**Concept Problem Revisited**

The following transformation can be described in basically two ways.

$f(x) = \sin x \to g(x) = -3\sin x - 4$

The first is to describe the stretching and reflecting first and then the vertical shift. This is the most logical way to discuss the transformation verbally because then the numbers like 3 and -4 can be explicitly identified in the graph.

The second way to describe the transformation is to attempt to say the vertical shift first. In this case the vertical shift would initially be $-\frac{4}{3}$, and then the vertical stretch would magnify this distance from the $x$-axis. This is significantly less intuitive. If a description showed the vertical shift to be -4 initially followed by a stretch by a factor of 3, the sinusoidal axis would move to $y = 12$ which is incorrect.

The order in describing the transformation matters. When describing vertical transformations it is most intuitive to simply describe the transformations in the same order as the order of operations.

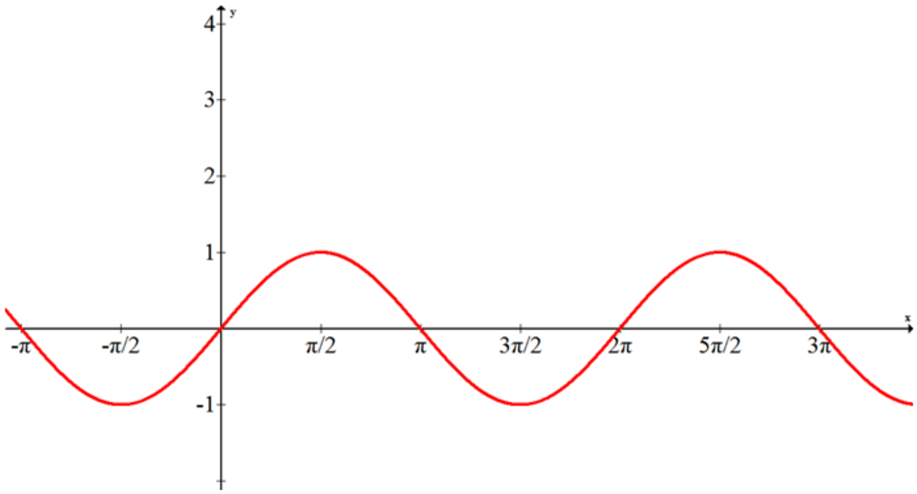
## Vocabulary

The ***sinusoidal axis*** is the horizontal line that runs through the middle of the sine or cosine wave.

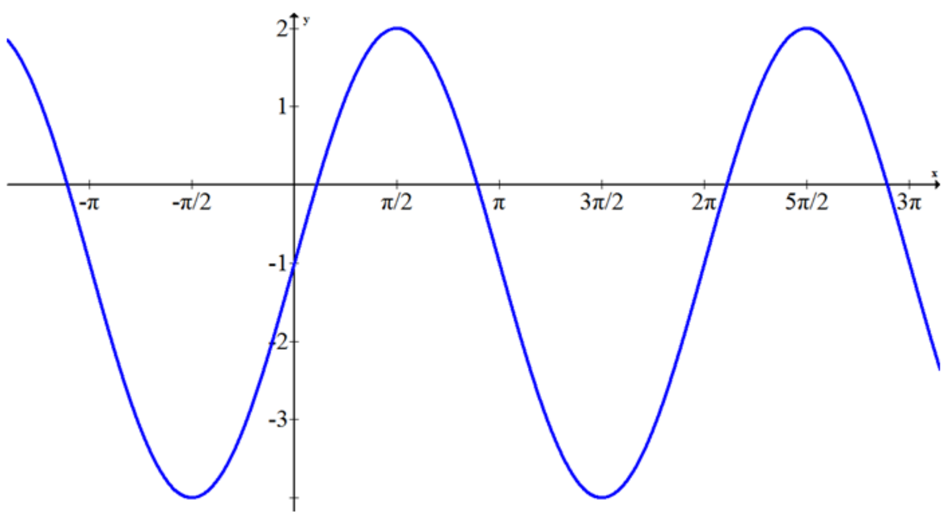
***Vertical shift*** is a rigid transformation that moves every point vertically by a set amount.

## Guided Practice

1. Transform the following sine graph in two ways. First, transform the sine graph by shifting it vertically up 1 unit and then stretching it vertically by a factor of 2 units. Second, transform the sine graph by stretching it vertically by a factor of 2 units and then shifting it vertically up 1 unit.



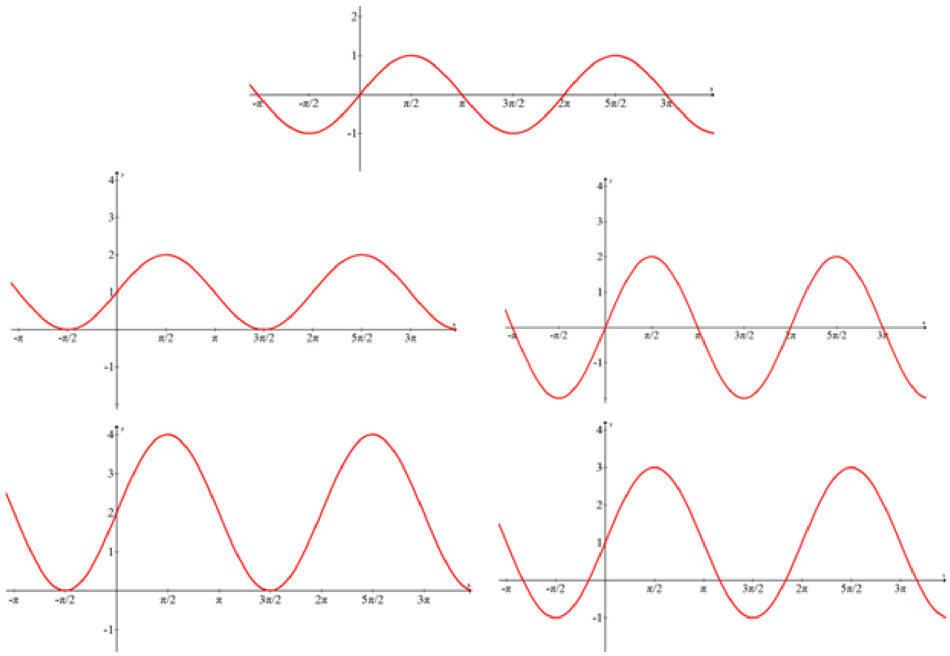

2. What equation models the following graph?




3. Graph the following function: $f(x) = -2 \cdot \cos x + 1$.

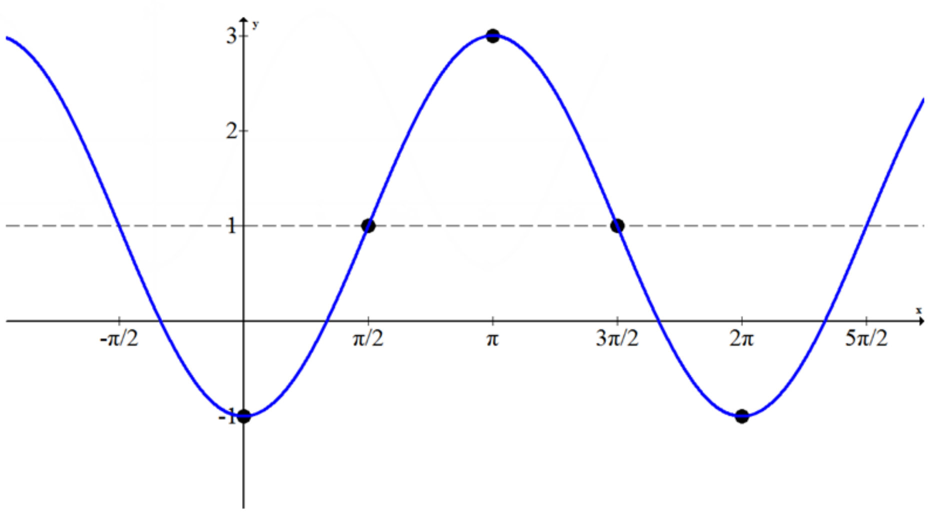
**Answers:**

1. When doing ordered transformations it is good to show where you start and where you end up so that you can effectively compare and contrast the outcomes. See how both transformations start with a regular sine wave. The two columns represent the sequence of transformations that produce different outcomes.



2. $f(x) = 3 \cdot \sin x - 1$

3. First draw the horizontal sinusoidal axis and identify the five main points for the cosine wave. Be careful to note that the amplitude is 2 and the cosine wave starts and ends at a low point because of the negative sign.




## Practice

Graph each of the following functions that have undergone a vertical stretch, reflection, and/or a vertical shift.

1. $f(x) = -2\sin x + 4$
2. $g(x) = \frac{1}{2}\cos x - 1$
3. $h(x) = 3\sin x + 2$
4. $j(x) = -1.5\cos x + \frac{1}{2}$
5. $k(x) = \frac{2}{3}\sin x - 3$

Find the minimum and maximum values of each of the following functions.

6. $f(x) = -3\sin x + 1$

7. $g(x) = 2\cos x - 4$

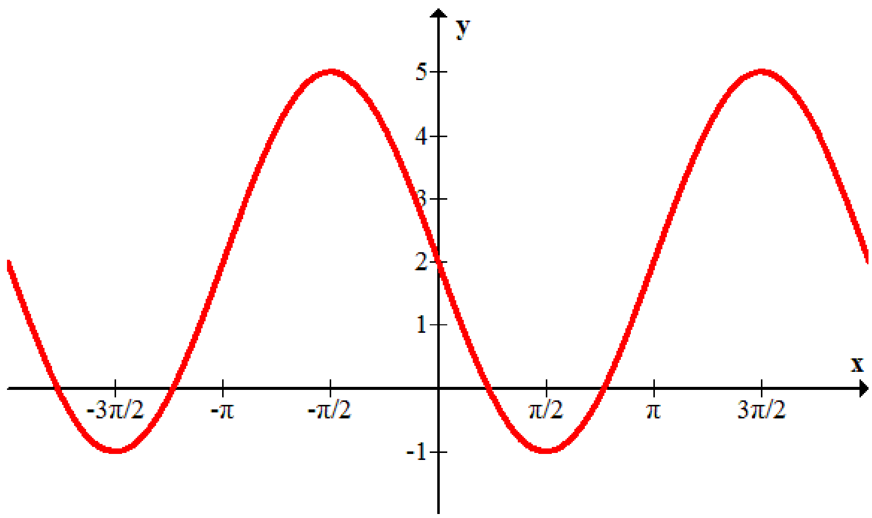
8. $h(x) = \frac{1}{2}\sin x + 1$

9. $j(x) = -\cos x + 5$

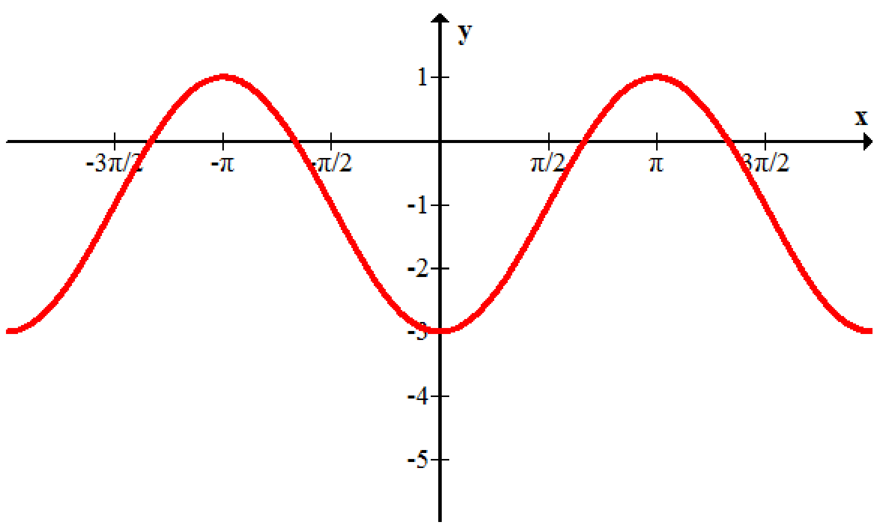
10. $k(x) = \sin(x) - 1$

Give the equation of each function graphed below.

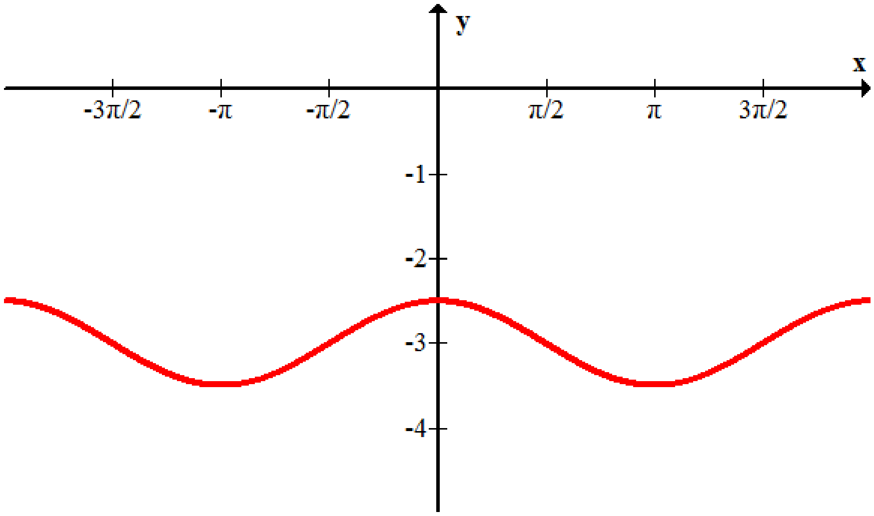
11.



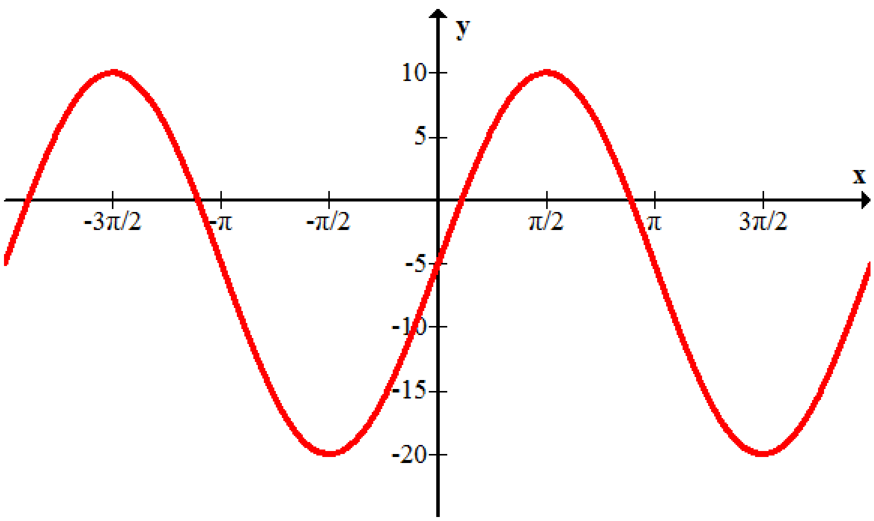

12,



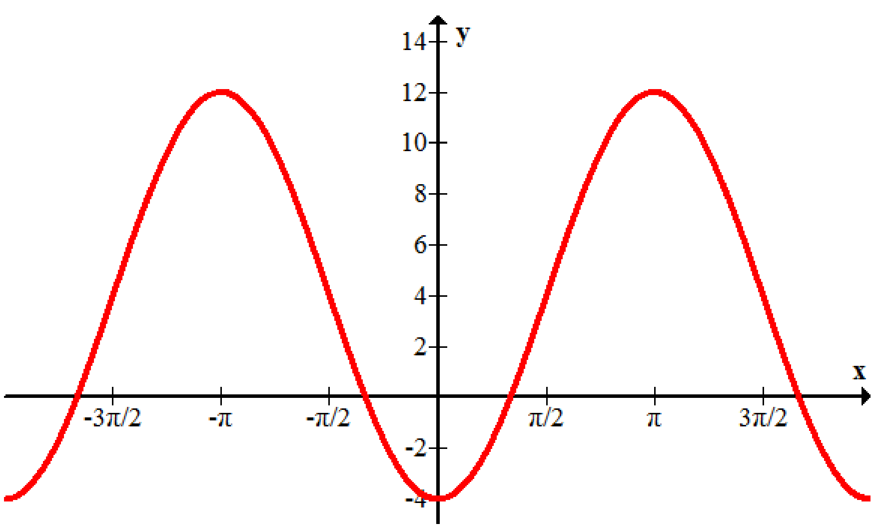

13.


y
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1
-2
-3
-4

14.


y
x
10
5
-3π/2
-π
-π/2
π/2
π
3π/2
-5
-10
-15
-20

15.


y
x
14
12
10
8
6
4
2
-3π/2
-π
-π/2
π/2
π
3π/2
-2
-4

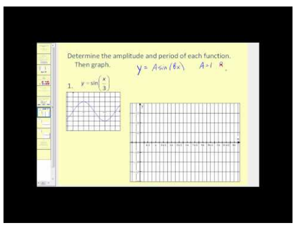
# 5.5 Frequency and Period of Sinusoidal Functions

Here you will apply your knowledge of horizontal stretching transformations to sine and cosine functions.

The transformation rules about horizontal stretching and shrinking directly apply to sine and cosine graphs. If a sine graph is horizontally stretched by a factor of $\frac{1}{2}$ that is the same as a horizontal compression by a factor of 2.

How does the equation change when a sine or cosine graph is stretched by a factor of 3?

### Watch This



MEDIA
Click image to the left for more content.

http://www.youtube.com/watch?v=qJ-oUV7xL3w James Sousa: Amplitude and Period of Sine and Cosine

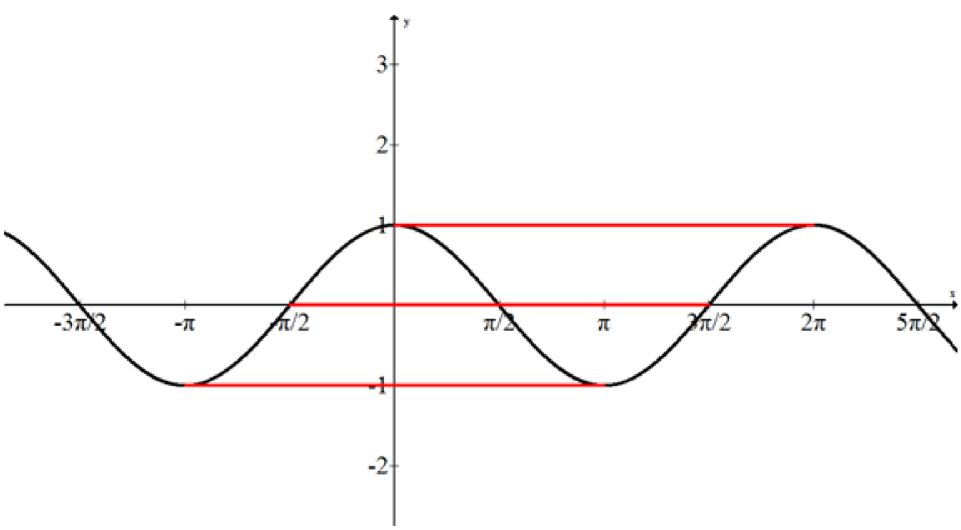
### Guidance

The general equation for a sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

The $\pm$ controls the reflection across the $x$-axis. The coefficient $a$ controls the amplitude. The constant $d$ controls the vertical shift. Here you will see that the coefficient $b$ controls the horizontal stretch.

Horizontal stretch is measured for sinusoidal functions as their periods. This is why this function family is also called the periodic function family. The period of a sinusoid is the length of a complete cycle. For basic sine and cosine functions, the period is $2\pi$. This length can be measured in multiple ways. In word problems and in other tricky circumstances, it may be most useful to measure from peak to peak.



The ability to measure the period of a function in multiple ways allows different equations to model an identical graph. In the image above, the top red line would represent a regular cosine wave. The center red line would represent a regular sine wave with a horizontal shift. The bottom red line would represent a negative cosine wave with a horizontal shift. This flexibility in perspective means that many of the examples, guided practice and practice problems may have multiple solutions. For now, try to always choose the function that has a period starting at $x = 0$.

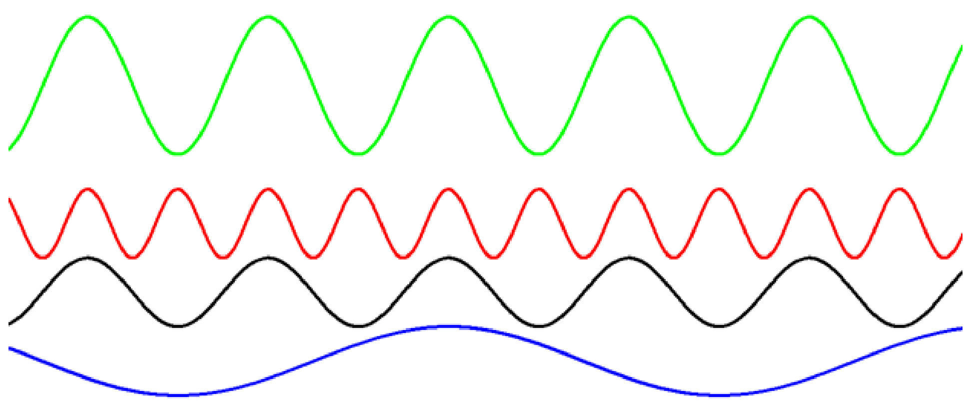
Frequency is a different way of measuring horizontal stretch. For sound, frequency is known as pitch. With sinusoidal functions, frequency is the number of cycles that occur in $2\pi$. A shorter period means more cycles can fit in $2\pi$ and thus a higher frequency. Period and frequency are inversely related by the equation:

$$\text{period} = \frac{2\pi}{\text{frequency}}$$

The equation of a basic sine function is $f(x) = \sin x$. In this case $b$, the frequency, is equal to 1 which means one cycle occurs in $2\pi$. This relationship is a common mistake in graphing sinusoidal functions. Students find $b = \frac{1}{2}$ and then mistakenly conclude that the period is $\frac{1}{2}$ when it is in fact stretched to $4\pi$.

**Example A**

Rank each wave by period from shortest to longest.



**Solution:**

The red wave has the shortest period.

The green and black waves have equal periods. A common mistake is to see that the green wave has greater amplitude and confuse that with greater periods.

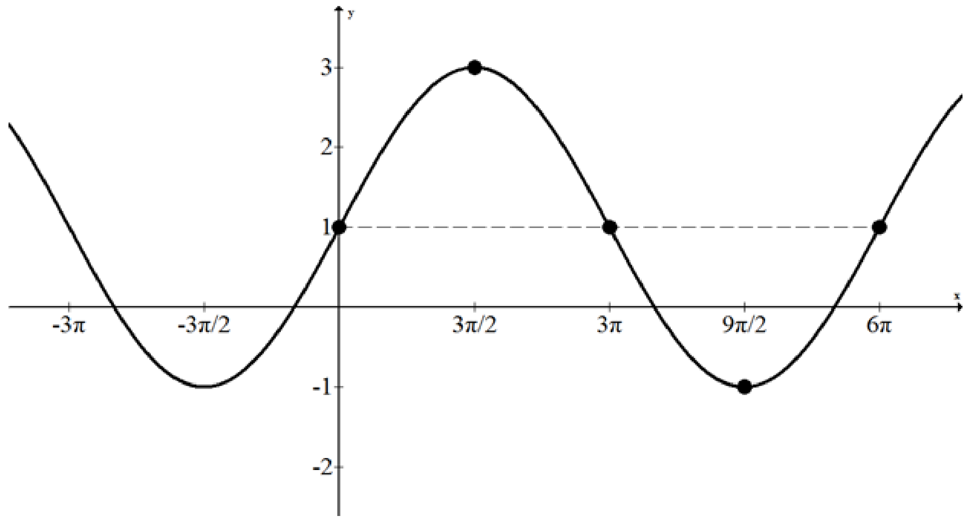
The blue wave has the longest period.

**Example B**

Identify the amplitude, vertical shift, period and frequency of the following function. Then graph the function.

$$f(x) = 2\sin\left(\frac{x}{3}\right) + 1$$

**Solution:** $a = 2, b = \frac{1}{3}, d = 1$. Since $b = \frac{1}{3}$ (frequency), then the period must be $6\pi$.




Often the most challenging part of graphing periodic functions is labeling the axes. Since the period is 6π, start by drawing the sinusoidal axis shifted appropriately. Then divide the 6π into four parts so that the 5 guiding points of the sine graph can be plotted with the amplitude and reflection in mind. The very last thing to do is to draw and extend the curve. Many students try to draw the curve too early and end up having to redo their work.

**Example C**

A measuring stick on a dock measures high tide to be 18 feet and low tide to be 6 feet. It takes about 6 hours for the tide to switch between low and high tides. Determine a graphical and algebraic model for the tides knowing that at $t = 0$ there is a high tide.

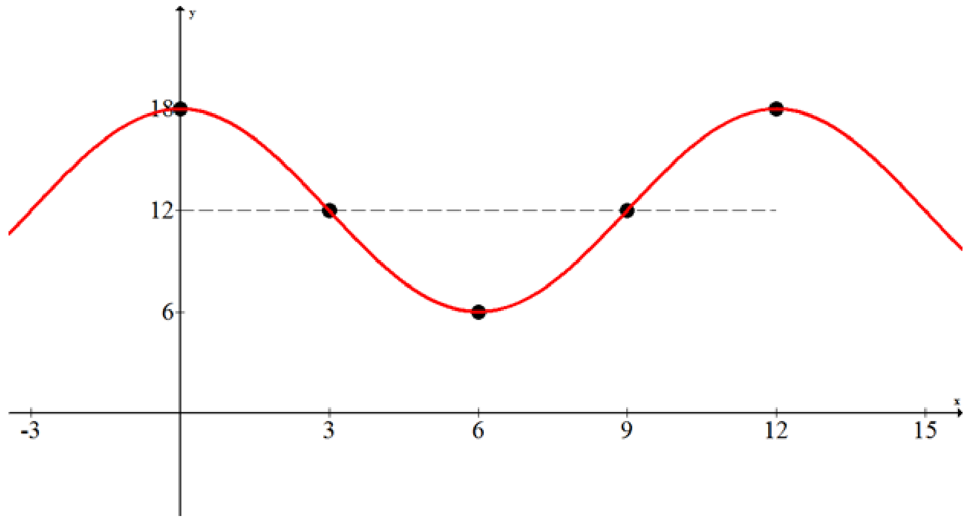
**Solution:** Usually the best course of action for word problems is to identify information, plot points, sketch and then finally come up with an equation.

From the given information you can deduce the following points. Notice how the sinusoidal axis can be assumed to be the average of the high and low tides.

### TABLE 5.6:

| Time (hours) | Water level (feet) |
|---|---|
| 0 | 18 |
| 6 | 6 |
| 12 | 18 |
| 3 | $\frac{18+6}{2} = 12$ |
| 9 | 12 |

By plotting those points and filling in the sinusoidal axis you can observe a cosine graph.




The amplitude is 6 so $a = 6$. There is no vertical reflection. Since the period is 12 you can determine the frequency $b$:

$12 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{6}$

The vertical shift is 12 so $d = 12$. Thus you have all the pieces to make an algebraic model:

$f(x) = +6 \cdot \cos\left(\frac{\pi}{6}x\right) + 12$

**Concept Problem Revisited**

If a sine graph is horizontally stretched by a factor of 3 then the general equation has $b = \frac{1}{3}$. This is because $b$ is the frequency and counts the number (or fraction) of a period that fits in a normal period of $2\pi$. Graphically, the sine wave will make a complete cycle in $6\pi$ units like Example B.

### Vocabulary

***Period*** is the distance it takes for a repeating function to make one complete cycle.

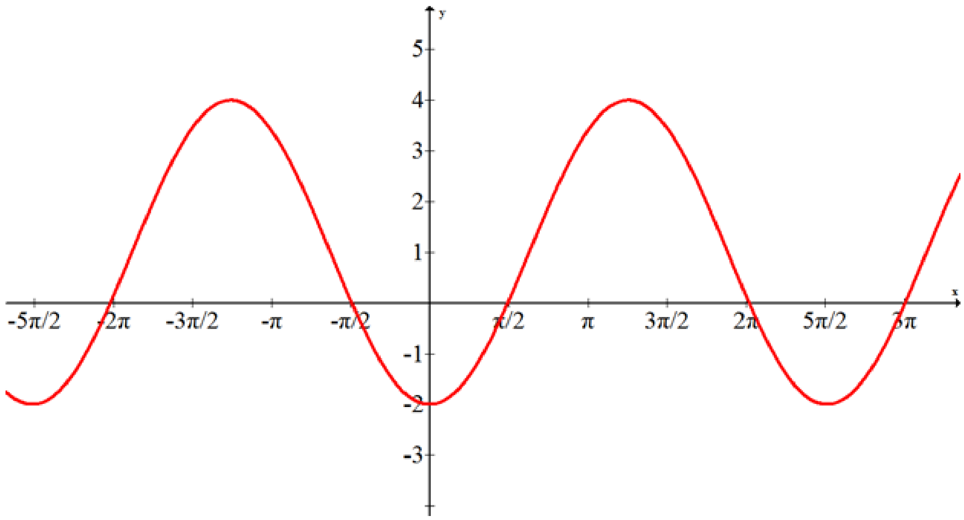
***Frequency*** is the number of cycles a function makes in a set amount of time or distance on the $x$ axis. For sine and cosine graphs this distance is $2\pi$.

### Guided Practice

1. A fish is caught in a water wheel by the side of a river. Initially the fish is 2 feet below the surface of the water. Twenty seconds later the fish is 14 feet in the air at the top of the water wheel. Model the fish's height in a graph and an equation.

2. Graph the following function: $g(x) = -\cos(8x) + 2$.

3. Given the following graph, identify the amplitude, period, and frequency and create an algebraic model.

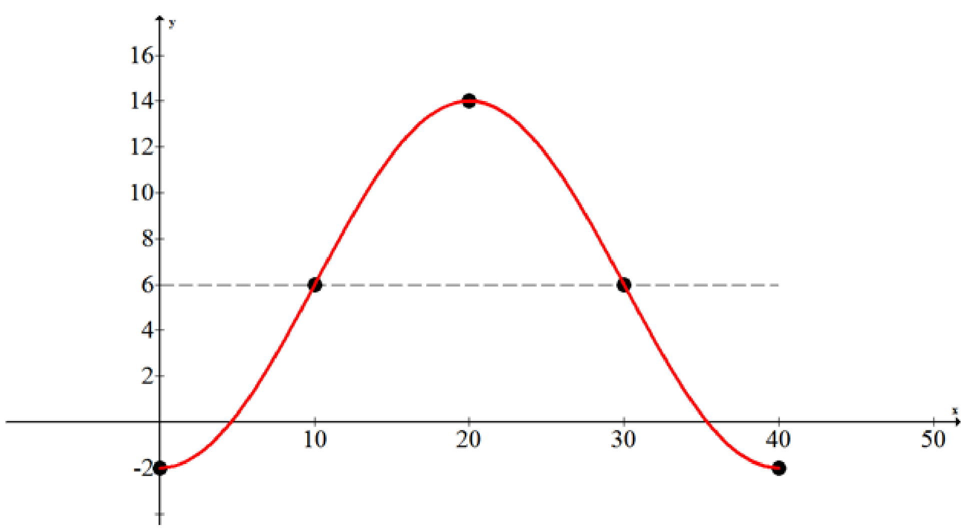



**Answers:**

1. Use logic to identify five key points. Use those key points to come up with a sketch. Use the sketch to identify information for the equation.

### TABLE 5.7:

| Time (seconds) | Fish height (feet) |
|---|---|
| 0 | -2 |
| 20 | 14 |
| 40 | -2 |
| 10 | $\frac{-2+14}{2} = 6$ |
| 30 | 6 |

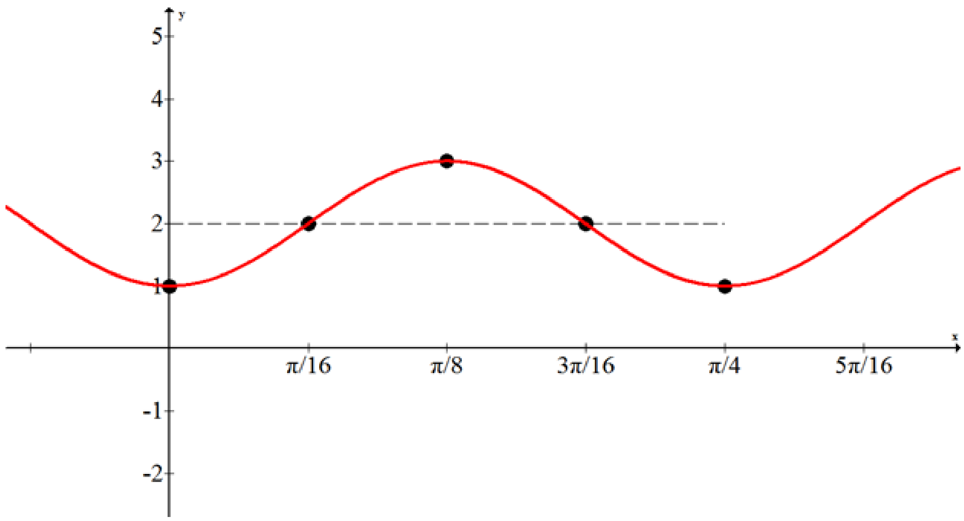



The amplitude is 8 so $a = 8$. The function looks like a negative cosine graph. The vertical shift is $d = 6$ and the period is 40.

$$40 = \frac{2\pi}{b} \to b = \frac{\pi}{20}$$
$$f(x) = -8 \cdot \cos\left(\frac{\pi}{20}x\right) + 6$$

Notice how the labeling on the graph is extremely deliberate. On both the $x$ and $y$ axes, only the most important intervals are labeled. This keeps the sketch accurate, evenly spaced on your paper and easy to read.

2. The labeling is the most important and challenging part of this problem. The amplitude is 1. The shape is a negative cosine. The vertical shift is 2. The period is $\frac{2\pi}{8} = \frac{\pi}{4}$. Working with this small period may be challenging at first, but remember that halving fractions is as simple as doubling the denominator.




3. The amplitude is 3. The shape is a negative cosine. The period is $\frac{5\pi}{2}$ which implies that $b = \frac{4}{5}$. The vertical shift is 1. $f(x) = -3 \cdot \cos\left(\frac{4}{5}x\right) + 1$.

## Practice

Find the frequency and period of each function below.

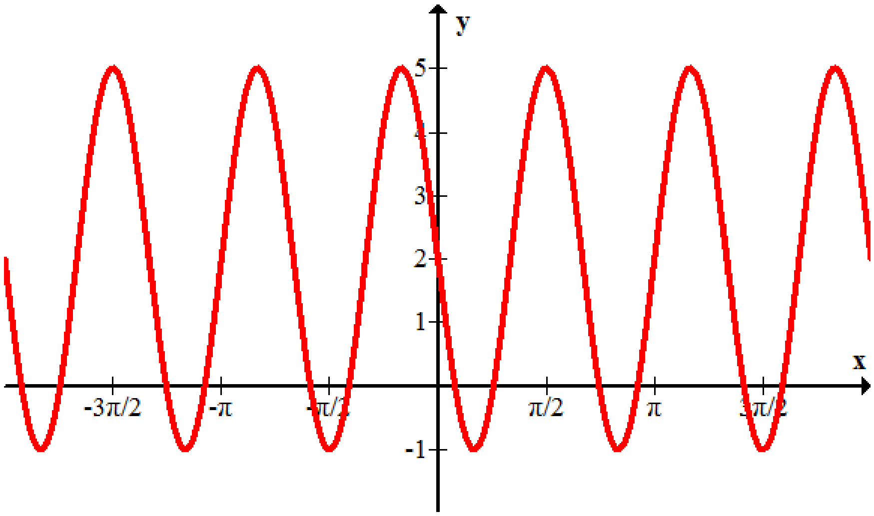
1. $f(x) = \sin(4x) + 1$
2. $g(x) = -3\cos(2x)$
3. $h(x) = \cos\left(\frac{1}{2}x\right) + 2$
4. $k(x) = -2\sin\left(\frac{3}{4}x\right) + 1$
5. $j(x) = 4\cos(3x) - 1$

Graph each of the following functions.

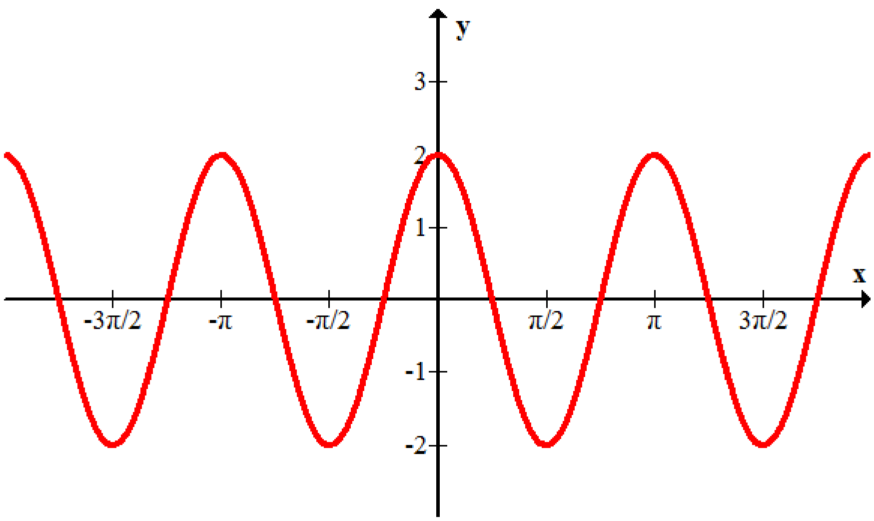
6. $f(x) = 3\sin(2x) + 1$
7. $g(x) = 2.5\cos(\pi x) - 4$
8. $h(x) = -\sin(4x) - 3$
9. $k(x) = \frac{1}{2}\cos(2x)$
10. $j(x) = -2\sin\left(\frac{3}{4}x\right) - 1$

Create an algebraic model for each of the following graphs.

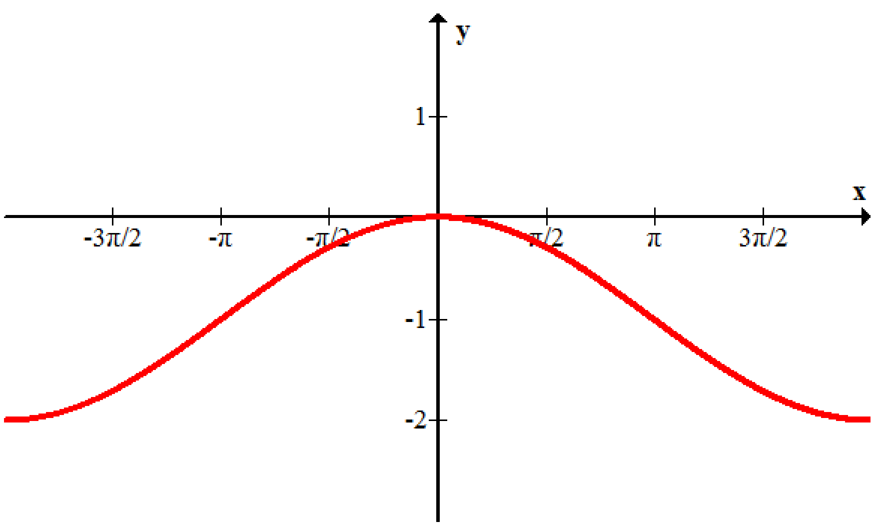
11.


y
5
4
3
2
1
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1

12.


y
3
2
1
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1
-2

13.


y
1
x
-3π/2
-π
-π/2
π/2
π
3π/2
-1
-2

14. At time 0 it is high tide and the water at a certain location is 10 feet high. At low tide 6 hours later, the water is 2 feet high. Given that tides can be modeled by sinusoidal functions, find a graph that models this scenario.

15. Find the equation that models the scenario in the previous problem.

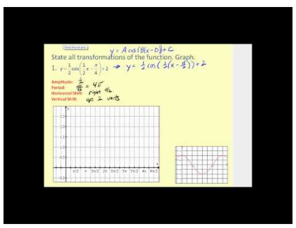
# 5.6 Phase Shift of Sinusoidal Functions

Here you will apply all the different transformations, including horizontal shifting, to sinusoidal functions.

A periodic function that does not start at the sinusoidal axis or at a maximum or a minimum has been shifted horizontally. This horizontal movement invites different people to see different starting points since a sine wave does not have a beginning or an end.

What are five other ways of writing the function $f(x) = 2 \cdot \sin x$?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=wUzARNIkH-g James Sousa: Graphing Sine and Cosine with Various Transformations

### Guidance

The general sinusoidal function is:

$f(x) = \pm a \cdot \sin(b(x+c)) + d$

The constant $c$ controls the horizontal shift. If $c = \frac{\pi}{2}$ then the sine wave is shifted left by $\frac{\pi}{2}$. If $c = -3$ then the sine wave is shifted right by 3. This is the opposite direction than you might expect, but it is consistent with the rules of transformations for all functions.

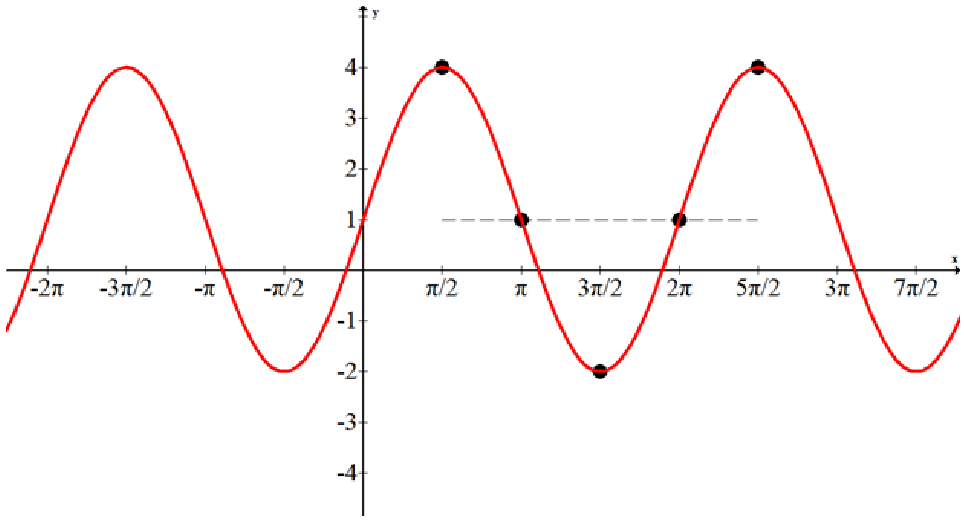
Generally $b$ is always written to be positive. If you run into a situation where $b$ is negative, use your knowledge of even and odd functions to rewrite the function.

$$\cos(-x) = \cos(x)$$
$$\sin(-x) = -\sin(x)$$

**Example A**

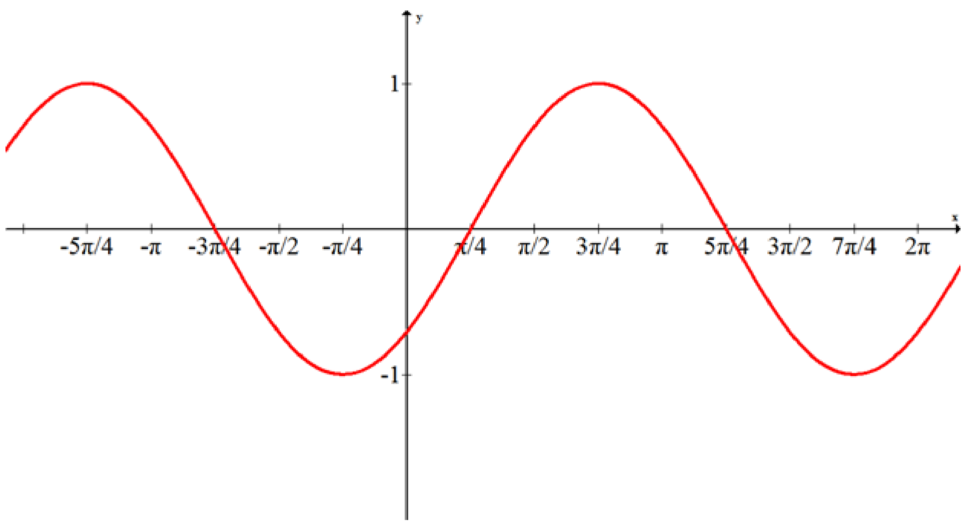
Graph the following function: $f(x) = 3 \cdot \cos\left(x - \frac{\pi}{2}\right) + 1$.

**Solution:** First find the start and end of one period and sketch only that portion of the sinusoidal axis. Then plot the 5 important points for a cosine graph while keeping the amplitude in mind.




**Example B**

Given the following graph, identify equivalent sine and cosine algebraic models.




**Solution:** Either this is a sine function shifted right by $\frac{\pi}{4}$ or a cosine graph shifted left $\frac{5\pi}{4}$.

$f(x) = \sin\left(x - \frac{\pi}{4}\right) = \cos\left(x + \frac{5\pi}{4}\right)$

**Example C**

At $t = 5$ minutes William steps up 2 feet to sit at the lowest point of the Ferris wheel that has a diameter of 80 feet. A full hour later he finally is let off the wheel after making only a single revolution. During that hour he wondered how to model his height over time in a graph and equation.

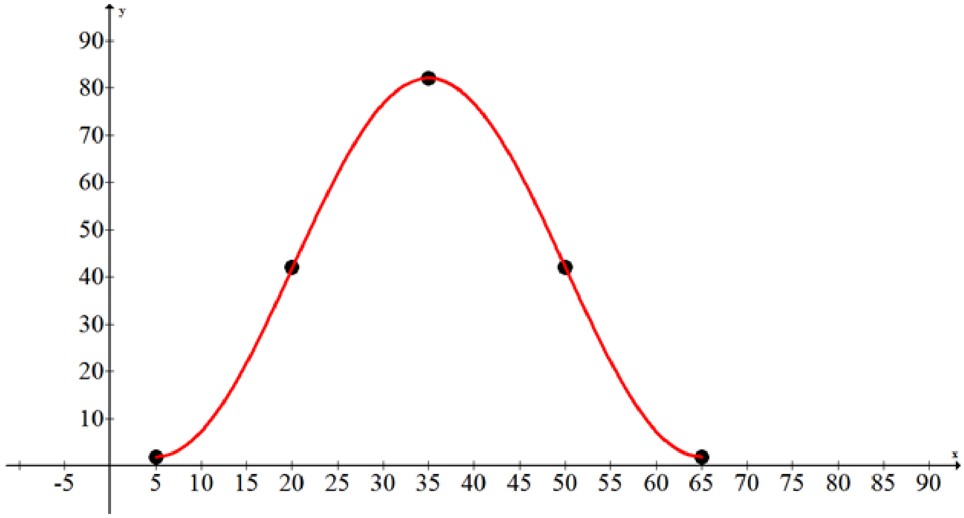
**Solution:** Since the period is 60 which works extremely well with the 360° in a circle, this problem will be shown in degrees.

TABLE 5.8:

| Time (minutes) | Height (feet) |
|---|---|
| 5 | 2 |
| 20 | 42 |
| 35 | 82 |

TABLE 5.8: (continued)

| | |
|---|---|
| 50 | 42 |
| 65 | 2 |




William chooses to see a negative cosine in the graph. He identifies the amplitude to be 40 feet. The vertical shift of the sinusoidal axis is 42 feet. The horizontal shift is 5 minutes to the right.

The period is 60 (not 65) minutes which implies $b = 6$ when graphed in degrees.

$60 = \frac{360}{b}$

Thus one equation would be:

$f(x) = -40 \cdot \cos(6(x-5)) + 42$

**Concept Problem Revisited**

The function $f(x) = 2 \cdot \sin x$ can be rewritten an infinite number of ways.

$$2 \cdot \sin x = -2 \cdot \cos\left(x + \frac{\pi}{2}\right) = 2 \cdot \cos\left(x - \frac{\pi}{2}\right) = -2 \cdot \sin(x - \pi) = 2 \cdot \sin(x - 8\pi)$$

It all depends on where you choose start and whether you see a positive or negative sine or cosine graph.

### Vocabulary

***Phase shift*** is a typical horizontal shift left or right that is used primarily with periodic functions.

### Guided Practice

1. Tide tables report the times and depths of low and high tides. Here is part of tide report from Salem, Massachusetts dated September 19, 2006.

TABLE 5.9:

| | | |
|---|---|---|
| 10:15 AM | 9 ft. | High Tide |
| 4:15 PM | 1 ft. | Low Tide |
| 10:15 PM | 9 ft. | High Tide |

Find an equation that predicts the height based on the time. Choose when $t = 0$ carefully.

2. Use the equation from Guided Practice #1 to predict the height of the tide at 6:05 AM.

3. Use the equation from Guided Practice #1 to find out when the tide will be at exactly 8 ft on September $19^{th}$.

**Answers:**

1. There are two logical places to set $t = 0$. The first is at midnight the night before and the second is at 10:15 AM. The first option illustrates a phase shift that is the focus of this concept, but the second option produces a simpler equation. Set $t = 0$ to be at midnight and choose units to be in minutes.

### TABLE 5.10:

| Time (hours : minutes) | Time (minutes) | Tide (feet) |
|---|---|---|
| 10:15 | 615 | 9 |
| 16:15 | 975 | 1 |
| 22:15 | 1335 | 9 |
| | $\frac{615+975}{2} = 795$ | 5 |
| | $\frac{1335+975}{2} = 1155$ | 5 |

These numbers seem to indicate a positive cosine curve. The amplitude is four and the vertical shift is 5. The horizontal shift is 615 and the period is 720.

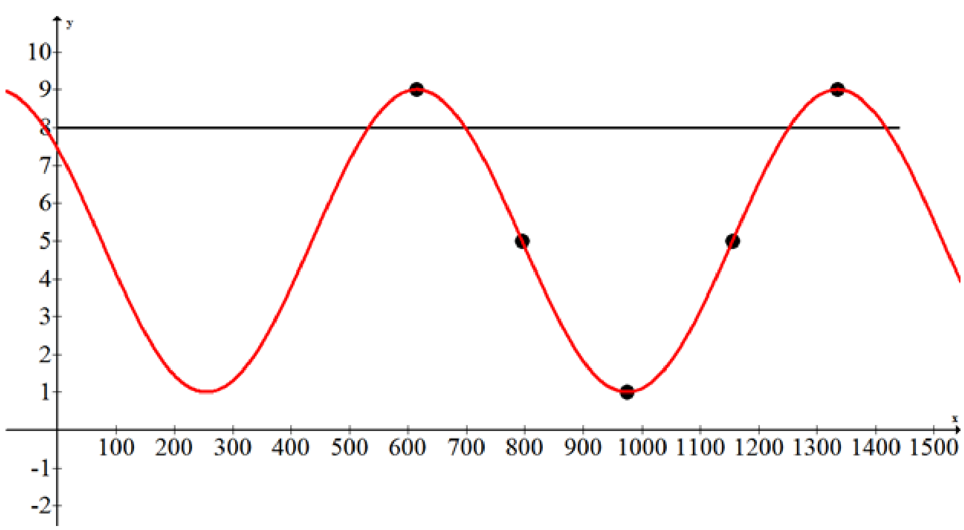
$720 = \frac{2\pi}{b} \rightarrow b = \frac{\pi}{360}$

Thus one equation is:

$f(x) = 4 \cdot \cos\left(\frac{\pi}{360}(x - 615)\right) + 5$

2. The height at 6:05 AM or 365 minutes is: $f(365) \approx 2.7057$ feet.

3. This problem is slightly different from question 2 because instead of giving $x$ and using the equation to find the $y$, this problem gives the $y$ and asks you to find the $x$. Later you will learn how to solve this algebraically, but for now use the power of the intersect button on your calculator to intersect the function with the line $y = 8$. Remember to find all the $x$ values between 0 and 1440 to account for the entire 24 hours.




There are four times within the 24 hours when the height is exactly 8 feet. You can convert these times to hours and minutes if you prefer.

$t \approx 532.18$ (8:52), 697.82 (11:34), 1252.18 (20 : 52), 1417.82 (23:38)

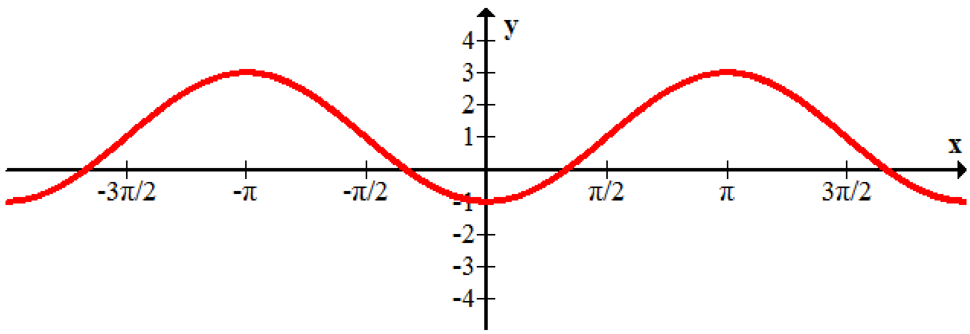
### Practice

Graph each of the following functions.

1. $f(x) = 2\cos\left(x - \frac{\pi}{2}\right) - 1$
2. $g(x) = -\sin(x - \pi) + 3$
3. $h(x) = 3\cos(2(x - \pi))$
4. $k(x) = -2\sin(2x - \pi) + 1$
5. $j(x) = -\cos\left(x + \frac{\pi}{2}\right)$

Give one possible sine equation for each of the graphs below.

6.

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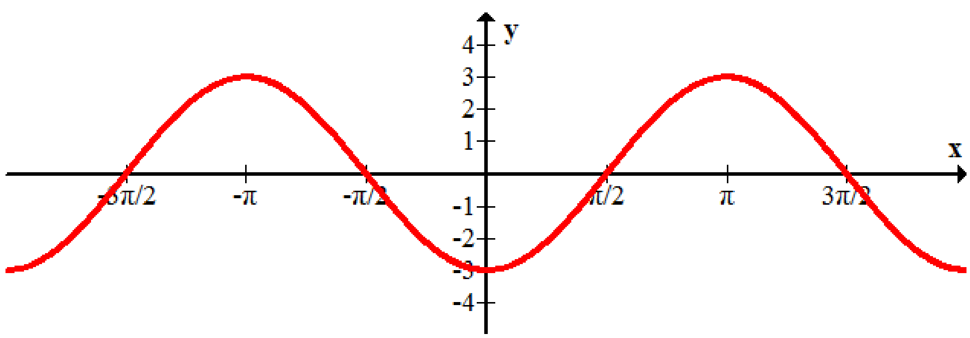


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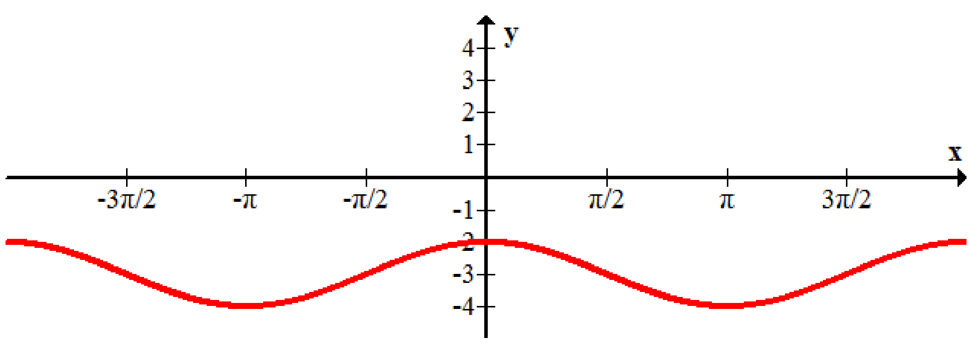
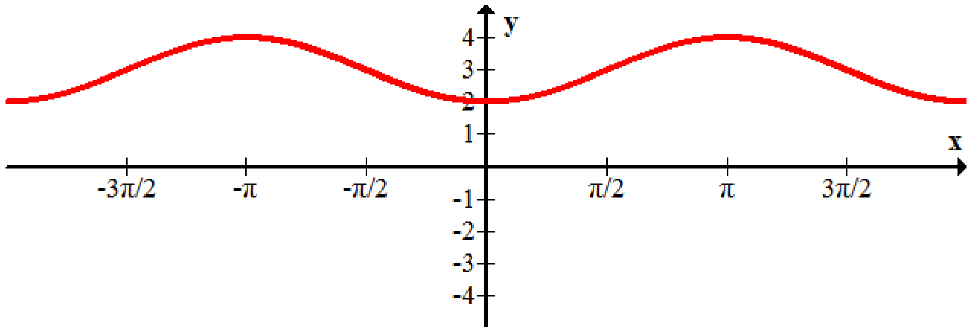



Give one possible cosine function for each of the graphs below.

9.

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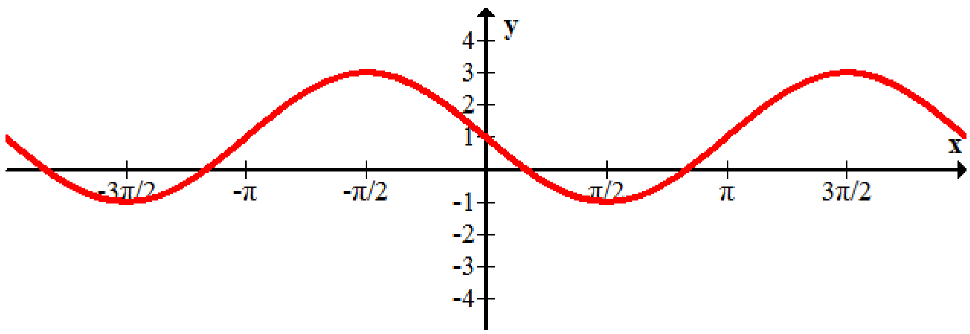


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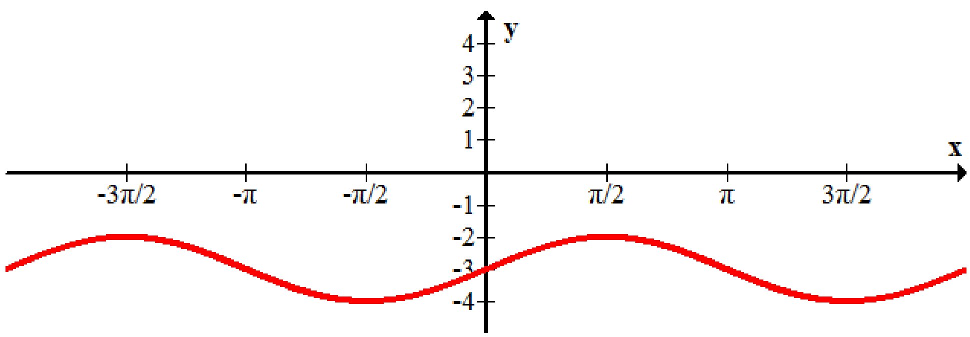


The temperature over a certain 24 hour period can be modeled with a sinusoidal function. At 3:00, the temperature for the period reaches a low of $22^\circ F$. At 15:00, the temperature for the period reaches a high of $40^\circ F$.

12. Find an equation that predicts the temperature based on the time in minutes. Choose $t = 0$ to be midnight.

13. Use the equation from #12 to predict the temperature at 4:00 PM.

14. Use the equation from #12 to predict the temperature at 8:00 AM.

15. Use the equation from #12 to predict the time(s) it will be $32^\circ F$.

# 5.7 Graphs of Other Trigonometric Functions

Here you will extend what you know about rational functions and sine and cosine functions to produce graphs of the four other trigonometric functions: tangent, secant, cosecant, and cotangent. You will also see how the four coefficients of a general sinusoidal function affect the new functions in similar ways.

If you already know the relationship between the equation and graph of sine and cosine functions then the other four functions can be found by identifying zeroes, asymptotes and key points. Are the four new functions transformations of the sine and cosine functions?

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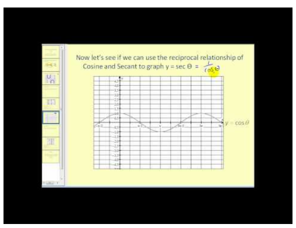

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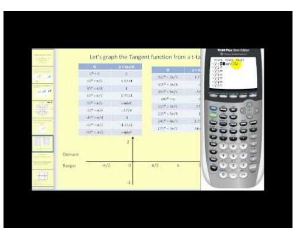
http://www.youtube.com/watch?v=5kOgBAVnrCI James Sousa: Graphing the Cosecant and Secant Functions



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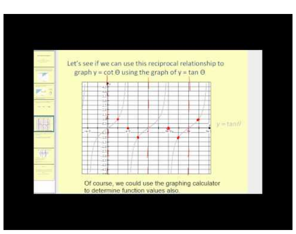
http://www.youtube.com/watch?v=6pc95eHUJhY James Sousa: Graphing the Tangent Function



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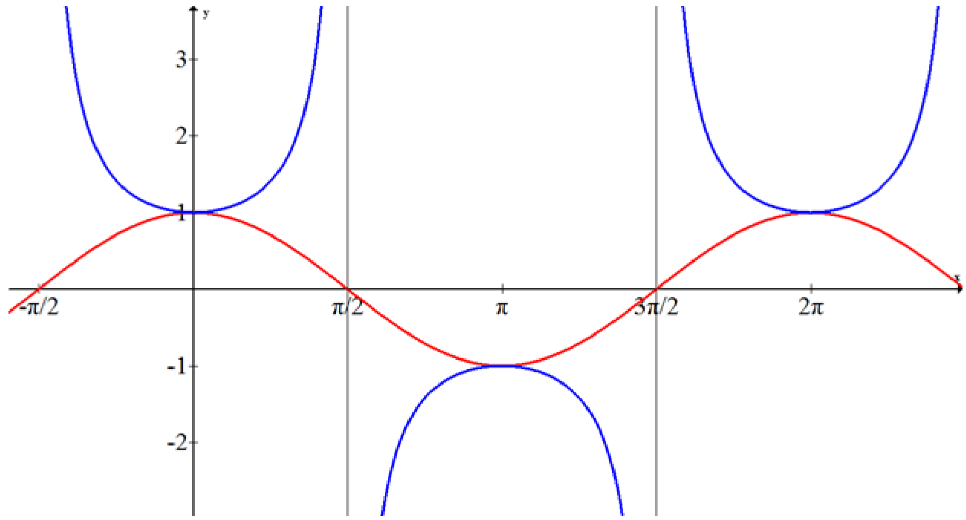
http://www.youtube.com/watch?v=5_65h_i8Yxg James Sousa: Graphing the Cotangent Function

### Guidance

Since secant is the inverse of cosine the graphs are very closely related.




Notice wherever cosine is zero, secant has a vertical asymptote and where $\cos x = 1$ then $\sec x = 1$ as well. These two logical pieces allow you to graph any secant function of the form:

$f(x) = \pm a \cdot \sec(b(x+c)) + d$

The method is to graph it as you would a cosine and then insert asymptotes and the secant curves so they touch the cosine curve at its maximum and minimum values. This technique is identical to graphing cosecant graphs. Simply use the sine graph to find the location and asymptotes.

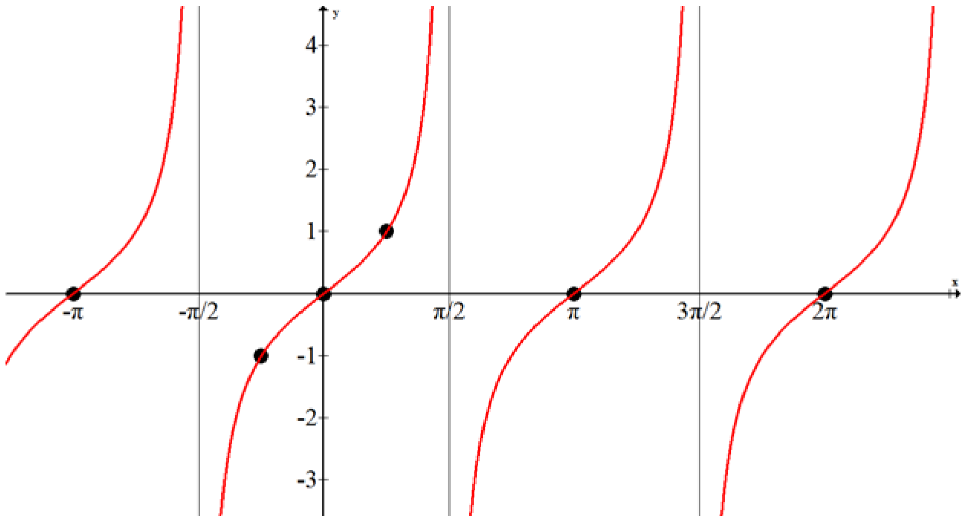
The tangent and cotangent graphs are more difficult because they are a ratio of the sine and cosine functions.

- $\tan x = \frac{\sin x}{\cos x}$
- $\cot x = \frac{\cos x}{\sin x}$

The way to think through the graph of $f(x) = \tan x$ is to first determine its asymptotes. The asymptotes occur when the denominator, cosine, is zero. This happens at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}\ldots$ The next thing to plot is the zeros which occur when the numerator, sine, is zero. This happens at $0, \pm\pi, \pm, 2\pi\ldots$ From the unit circle and basic right triangle trigonometry, you already know some values of $\tan x$:

- $\tan\frac{\pi}{4} = 1$
- $\tan\left(-\frac{\pi}{4}\right) = -1$

By plotting all this information, you get a very good sense as to what the graph of tangent looks like and you can fill in the rest.



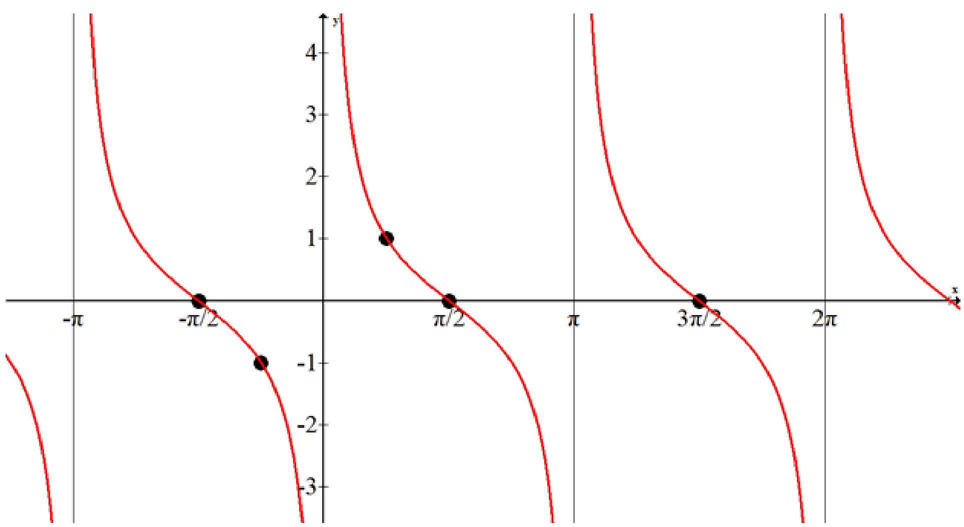

Notice that the period of tangent is π **not** 2π**,** because it has a shorter cycle.

The graph of cotangent can be found using identical logic as tangent. It is shown in Example A.

**Example A**

Graph $f(x) = \cot x$

**Solution:** You know $\cot x = \frac{1}{\tan x}$. This means that the graph of cotangent will have zeros wherever tangent has asymptotes and asymptotes wherever tangent has zeroes. You also know that where tangent is 1, cotangent is also 1.

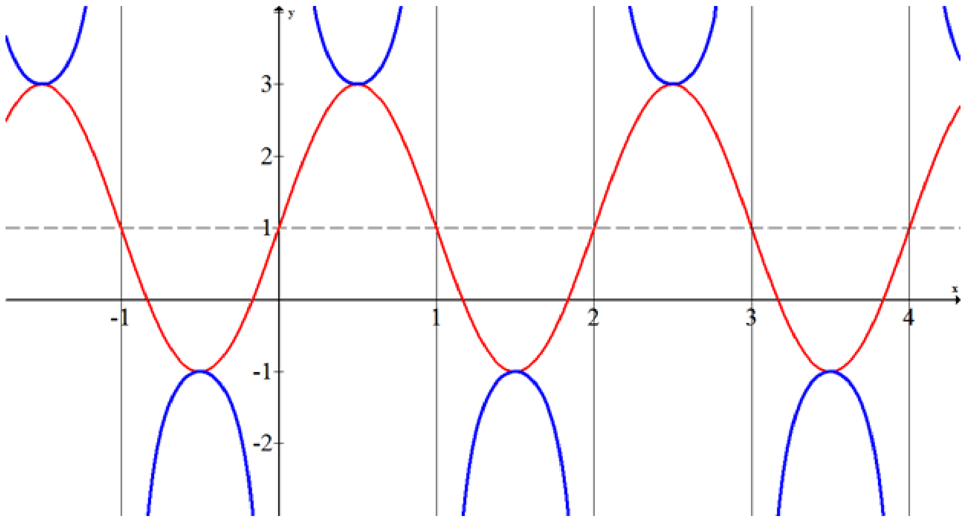



**Example B**

Graph the function $f(x) = -2 \cdot \csc(\pi(x-1)) + 1$.

**Solution:** Graph the function as if it were a sine function. Then insert asymptotes wherever the sine function crosses the sinusoidal axis. Lastly add in the cosecant curves.

The amplitude is 2. The shape is negative sine. The function is shifted up one unit and to the right one unit.




Note that only the blue portion of the graph represents the given function.

**Example C**

How do you write a tangent function as a cotangent function?

**Solution:** There are two main ways to go between a tangent function and a cotangent function. The first method was discussed in Example A: $f(x) = \tan x = \frac{1}{\cot x}$.

The second approach involves two transformations. Start by reflecting across the $x$ or the $y$ axis. Notice that this produces an identical result. Next shift the function to the right or left by $\frac{\pi}{2}$. Again this produces an identical result. $f(x) = \tan x = -\cot\left(x - \frac{\pi}{2}\right)$.

**Concept Problem Revisited**

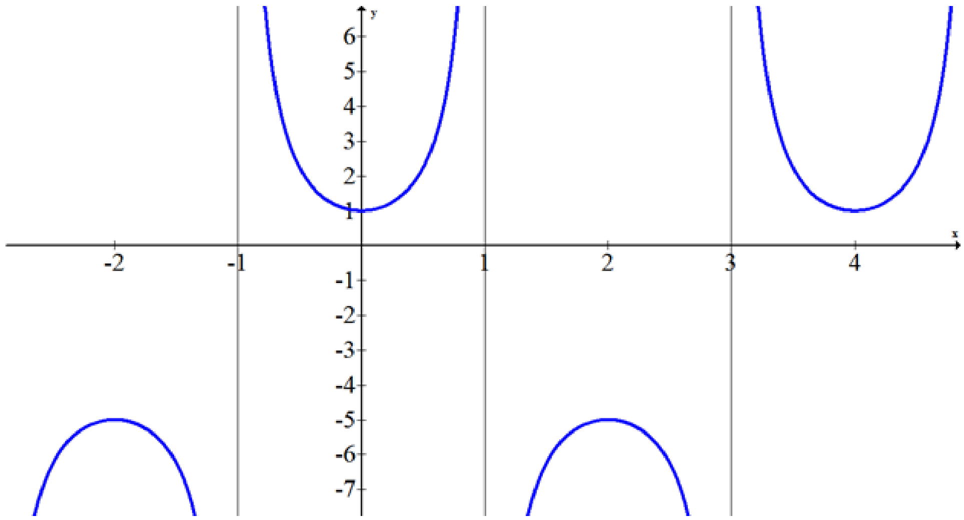
The four new functions are not purely transformations of the sine and cosine functions. However, secant and cosecant are transformations of each other as are tangent and cotangent.

## Vocabulary

A ***transformation*** is one that preserves the essential nature of the original graph. It does not introduce discontinuities or change the ordering of the points. Examples of transformations are stretching, reflecting and shifting.

## Guided Practice

1. Find the equation of the function in the following graph.

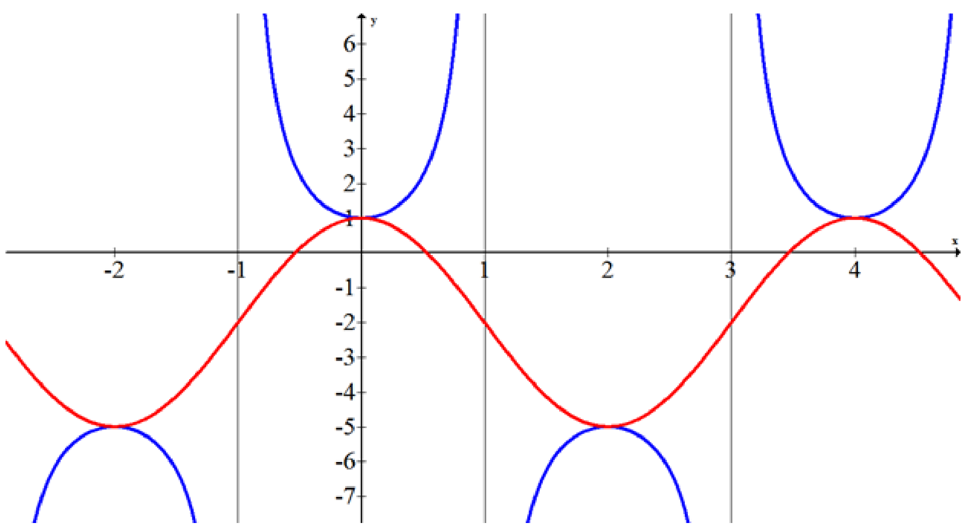


2. Graph the function $f(x) = -5 \cdot \sec\left(\frac{\pi}{3}(x-2)\right) - 4$.

3. Where are the asymptotes for tangent and why do they occur?

**Answers:**

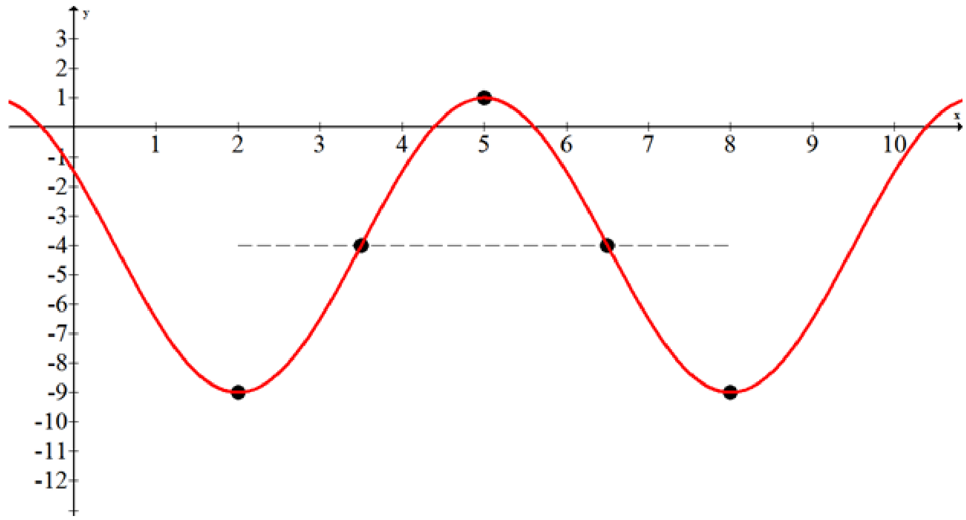
1. If you connect the relative maximums and minimums of the function, it produces a shifted cosine curve that is easier to work with.



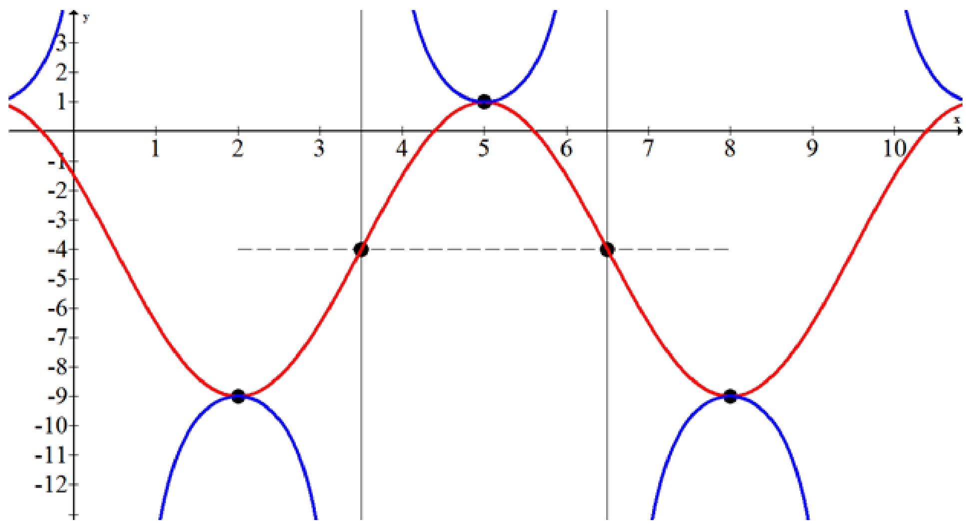
The amplitude is 3. The vertical shift is 2 down. The period is 4 which implies that $b = \frac{\pi}{2}$. The shape is positive cosine and if you choose to start at $x = 0$ there is no phase shift.

$f(x) = 3 \cdot \csc\left(\frac{\pi}{2}x\right) - 2$

2. First graph the function as if it were a cosine. The vertical shift is -4. The horizontal shift is to the right 2. This gives a starting point for the period. Since $b = \frac{\pi}{3}$ the period must be 6.




Then add in the asymptotes and secant curves. Note that the solution is only the blue portion of the curve.




3. Since $\tan x = \frac{\sin x}{\cos x}$ the asymptotes occur whenever $\cos x = 0$ which is $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

### Practice

1. What function can you use to help you make a sketch of $f(x) = \sec x$? Why?

2. What function can you use to help you make a sketch of $g(x) = \csc x$? Why?

Make a sketch of each of the following from memory.

3. $f(x) = \sec x$

4. $g(x) = \csc x$

5. $h(x) = \tan x$

6. $k(x) = \cot x$

Graph each of the following.

7. $f(x) = 2\csc(x) + 1$

8. $g(x) = 2\csc\left(\frac{\pi}{2}x\right) + 1$

9. $h(x) = 2\csc\left(\frac{\pi}{2}(x-3)\right)+1$
10. $j(x) = \cot\left(\frac{\pi}{2}x\right)+3$
11. $k(x) = -\sec\left(\frac{\pi}{3}(x+1)\right)-4$
12. $m(x) = -\tan(x)+1$
13. $p(x) = -2\tan\left(x-\frac{\pi}{2}\right)+1$
14. Find two ways to write $\sec x$ in terms of other trigonometric functions.
15. Find two ways to write $\csc x$ in terms of other trigonometric functions.

# 5.8 Graphs of Inverse Trigonometric Functions

Here you will graph the final form of trigonometric functions, the inverse trigonometric functions. You will learn why the entire inverses are not always included and you will apply basic transformation techniques.

In order for inverses of functions to be functions, the original function must pass the horizontal line test. Though none of the trigonometric functions pass the horizontal line test, you can restrict their domains so that they can pass. Then the inverses are produced just like with normal functions. Once you have the basic inverse functions, the normal transformation rules apply.

Why is $\sin^{-1}(\sin 370^\circ) \neq 370^\circ$? Don't the arcsin and sin just cancel out?

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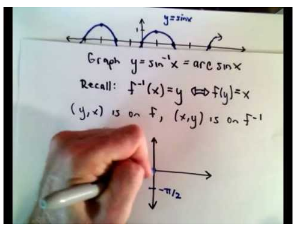

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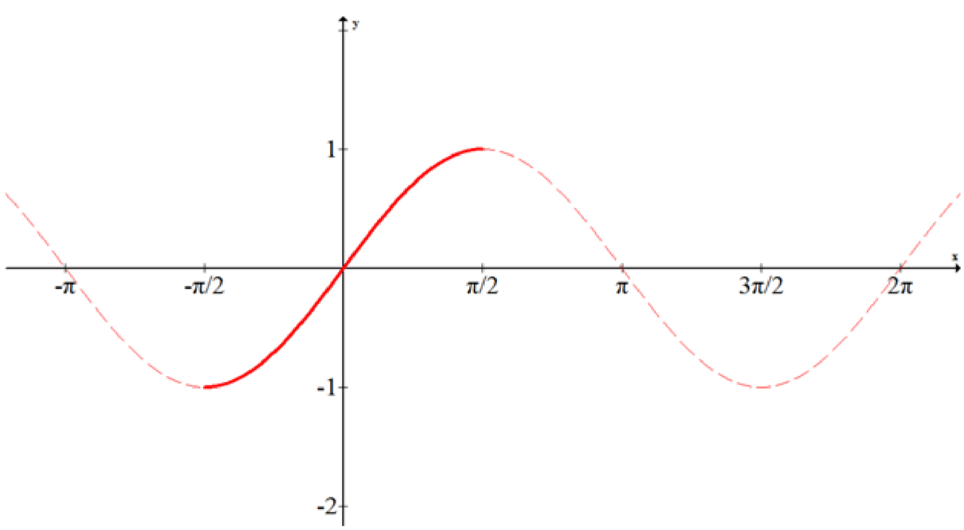
http://www.youtube.com/watch?v=bBBUMHe900U Inverse Trigonometric Functions

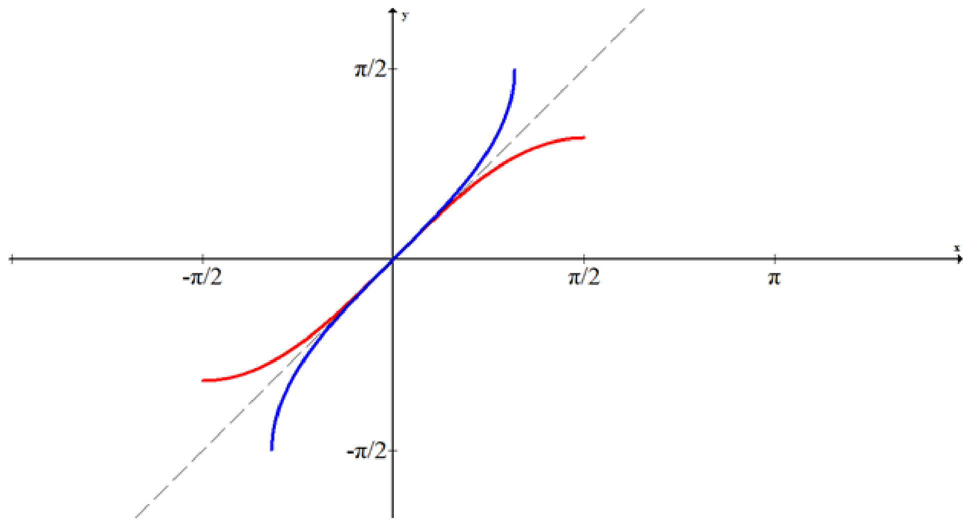
### Guidance

Since none of the six trigonometric functions pass the horizontal line test, you must restrict their domains before finding inverses of these functions. This is just like the way $y = \sqrt{x}$ is the inverse of $y = x^2$ when you restrict the domain to $x \geq 0$.

Consider the sine graph:



As a general rule, the restrictions to the domain are either the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ or $[0, \pi]$ to keep things simple. In this case sine is restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, as shown above. To find the inverse, reflect the bold portion across the line $y = x$. The blue curve below shows $f(x) = \sin^{-1} x$.

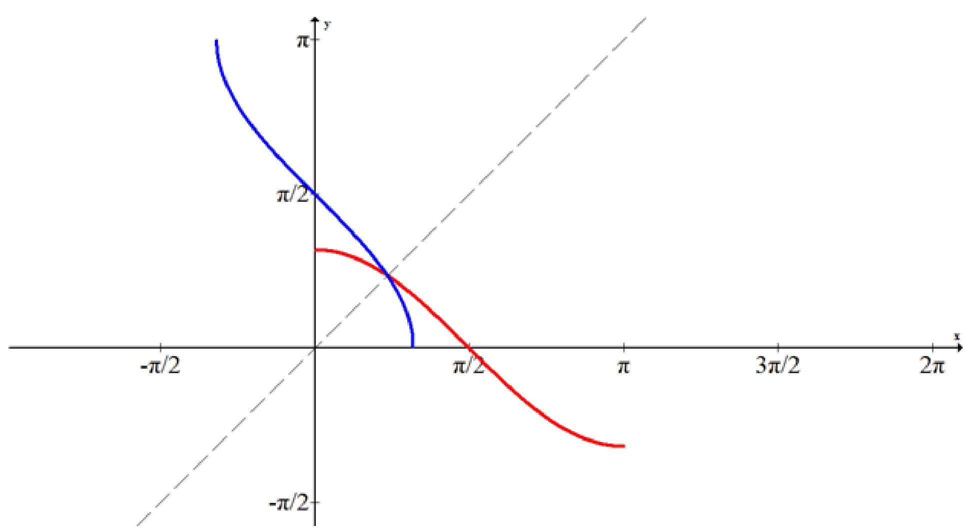



The result of this inversion is that arcsine will only ever produce angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. You must use logic and common sense to interpret these numbers in context.

**Example A**

What is the graph of $f(x) = \cos^{-1} x$?

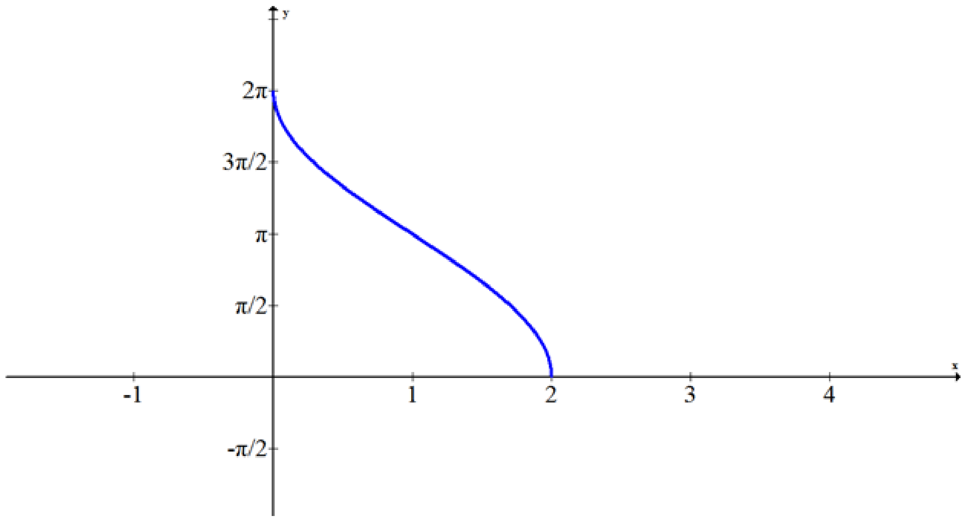
**Solution:** Graph the portion of cosine that fits the horizontal line test (the interval $[0, \pi]$) and reflect across the line $y = x$.




**Example B**

Graph the function $f(x) = 2\cos^{-1}(x - 1)$.

**Solution:** Since the graph of $f(x) = \cos^{-1} x$ was done in Example A, now you just need to shift it right one unit and stretch it vertically by a factor of 2. It intersected the $x$ axis at 1 before and now it will intersect at 2. It reached a height of $\pi$ before and now it will reach a height of $2\pi$.




**Example C**

Evaluate the following expression with and without a calculator using right triangles and your knowledge of inverse trigonometric functions.

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)$

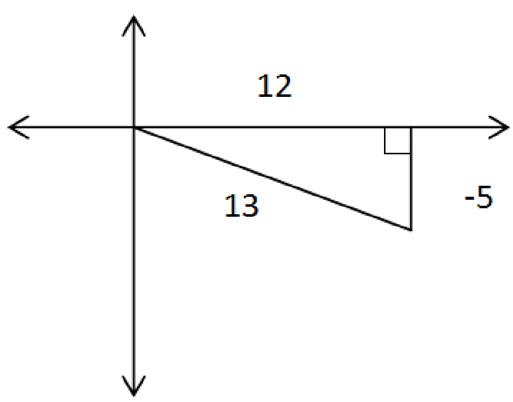
**Solution:** When using a calculator it can be extremely confusing trying to tell the difference between $\sin^{-1} x$ and $(\sin x)^{-1}$. In order to be able to effectively calculate this out it is best to write the expression explicitly only in terms of functions that your calculator does have.

The hardest part of this question is seeing the csc as a function (which produces an angle) on a ratio of a hypotenuse of 13 and an opposite side of -5. The sine of the inverse ratio must produce the same angle, so you can substitute it.

- $\csc^{-1}\left(-\frac{13}{5}\right) = \sin^{-1}\left(-\frac{5}{13}\right)$
- $\cot(\theta) = \frac{1}{\tan\theta}$

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right) = \frac{1}{\tan\left(\sin^{-1}\left(-\frac{5}{13}\right)\right)} = -\frac{12}{5}$

Not using a calculator is usually significantly easier. Start with your knowledge that $\csc^{-1}\left(-\frac{13}{5}\right)$ describes an angle in the fourth or the second quadrant because those are the two quadrants where cosecant is negative. Since $\csc^{-1}\theta$ has range $-\frac{\pi}{2}, \frac{\pi}{2}$, it only produces angles in quadrant I or quadrant IV (see Guided Practice 2). This triangle must then be in the fourth quadrant. All you need to do is draw the triangle and identify the cotangent ratio.




Cotangent is adjacent over opposite.

$\cot\left(\csc^{-1}\left(-\frac{13}{5}\right)\right)=-\frac{12}{5}$

**Concept Problem Revisited**

Since arcsine only produces angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ or $-90^\circ$ to $+90^\circ$ the result of $\sin^{-1}(\sin 370^\circ)$ is $10^\circ$ which is coterminal to $370^\circ$.

## Vocabulary

***Restricted domain*** refers to the fact that when creating an inverse you sometimes must cut off the domain of most of the function, saving the largest possible portion so that when the inverse is created it is also a function.

## Guided Practice

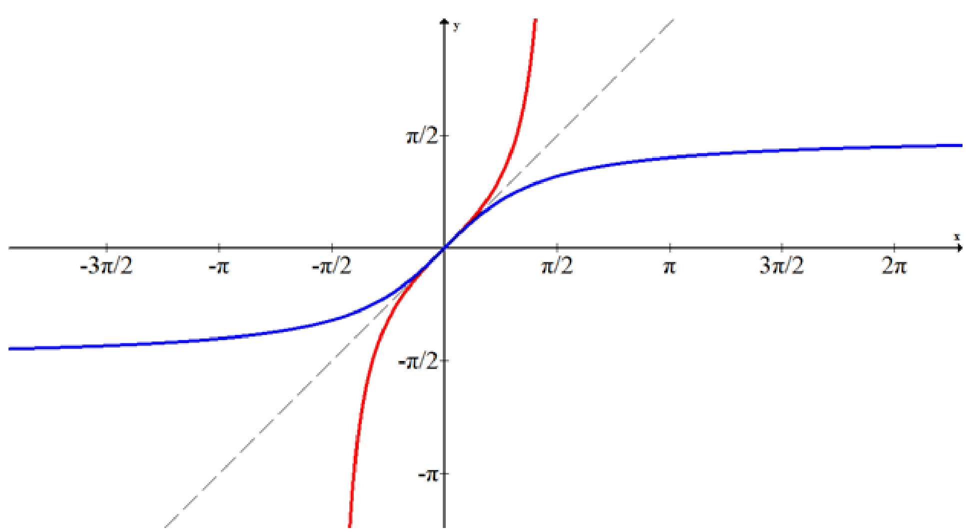
1. What is the graph of $y=\tan^{-1}x$?

2. What is the graph of $y=\csc^{-1}x$?

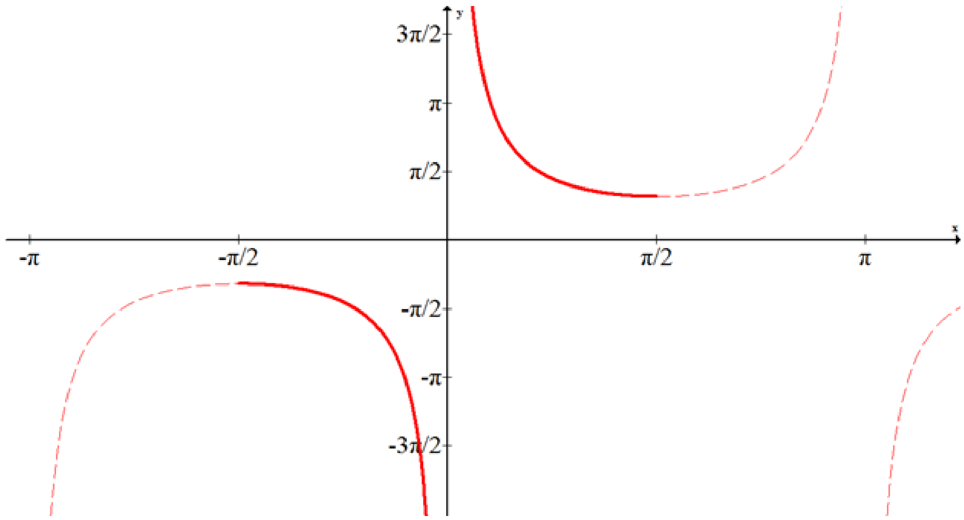
3. Evaluate the expression $\csc\left(\cot^{-1}\left[-\frac{8}{6}\right]\right)$.

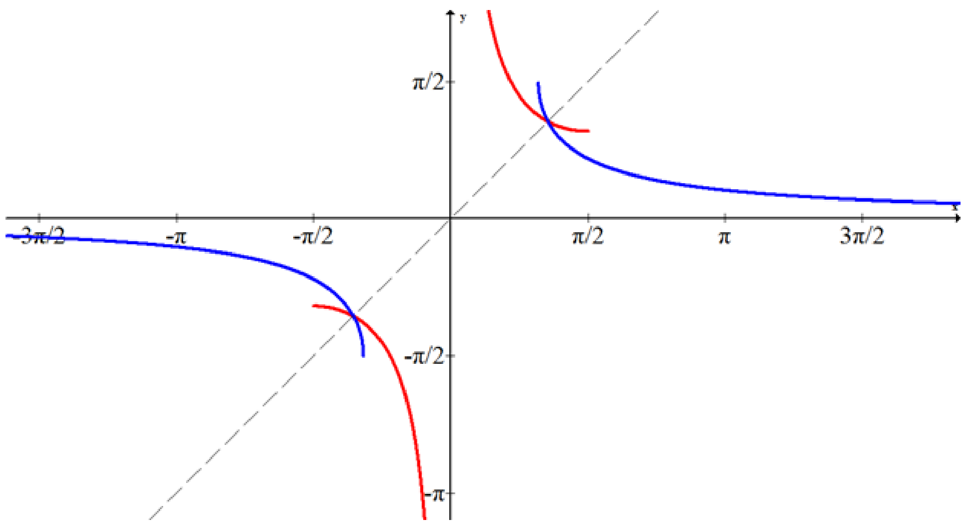
**Answers:**

1. Graph the portion of tangent that fits the horizontal line test and reflect across the line $y=x$. Note that the graph of arctan is in blue.




2. Graph the portion of cosecant that fits the horizontal line test and reflect across the line $y=x$.




Note that $f(x) = \csc^{-1} x$ is in blue. Also note that the word "arc-co-secant" is too cumbersome to use because of the train of prefixes.

3.

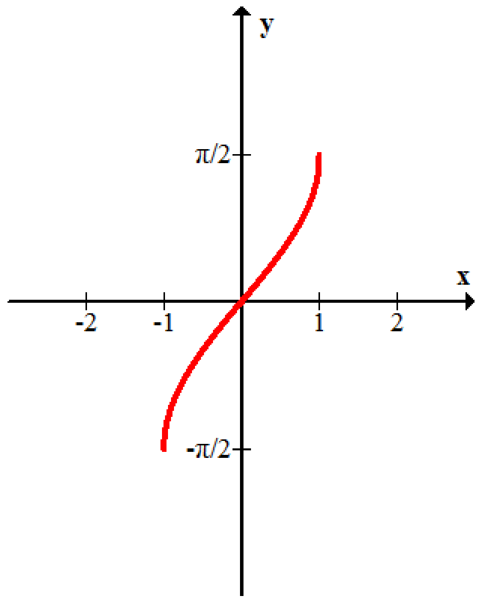
$$\begin{aligned}
\csc\left(\cot^{-1}\left[-\frac{8}{6}\right]\right) &= \frac{1}{\sin\left(\cot^{-1}\left[-\frac{8}{6}\right]\right)} \\
&= \frac{1}{\sin\left(\tan^{-1}\left(-\frac{6}{8}\right)\right)} \\
&= -\frac{10}{6} \\
&= -\frac{5}{3}
\end{aligned}$$

### Practice

1. Graph $f(x) = \cot^{-1} x$.
2. Graph $g(x) = \sec^{-1} x$.

Name each of the following graphs.

3.

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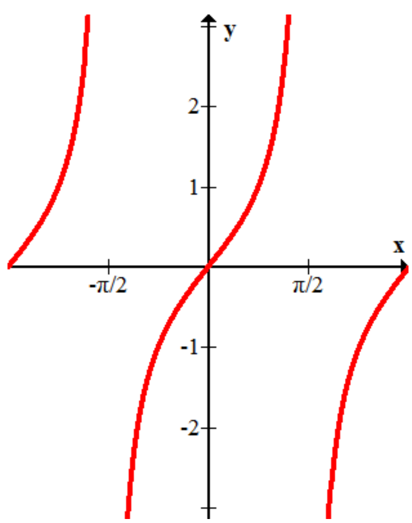


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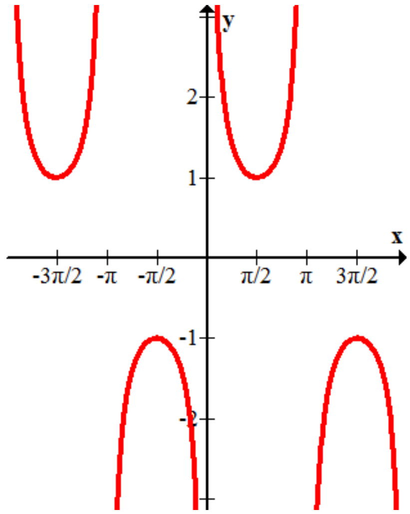


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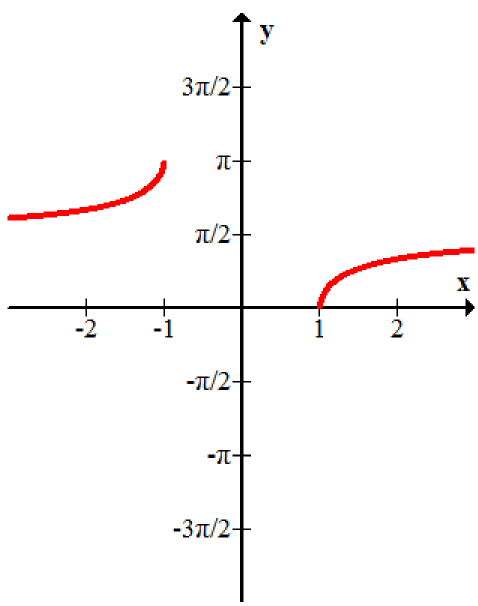


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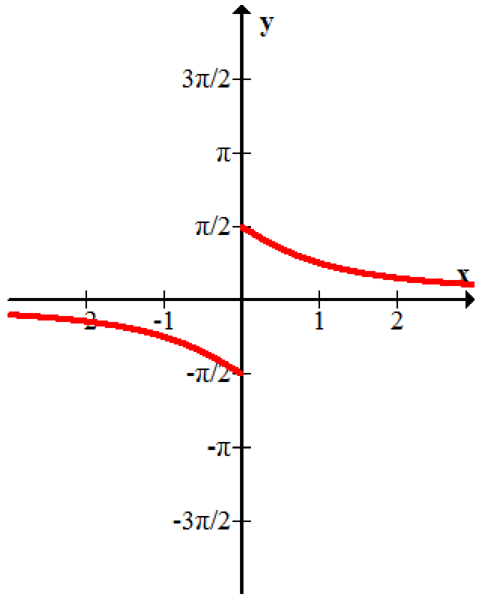


Graph each of the following functions using your knowledge of function transformations.

8. $h(x) = 3\sin^{-1}(x+1)$

9. $k(x) = 2\sin^{-1}(x) + \frac{\pi}{2}$

10. $m(x) = -\cos^{-1}(x-2)$

11. $j(x) = \cot^{-1}(x) + \pi$

12. $p(x) = -2\tan^{-1}(x-1)$

13. $q(x) = \csc^{-1}(x-2)$

14. $r(x) = -\sec^{-1}(x) + 4$

15. $t(x) = \csc^{-1}(x+1) - \frac{3\pi}{2}$

16. $v(x) = 2\sec^{-1}(x+2) + \frac{\pi}{2}$

17. $w(x) = -\cot^{-1}(x) - \frac{\pi}{2}$

Evaluate each expression.

18. $\sec\left(\tan^{-1}\left[\frac{3}{4}\right]\right)$

19. $\cot\left(\csc^{-1}\left[\frac{13}{12}\right]\right)$

20. $\csc\left(\tan^{-1}\left[\frac{4}{3}\right]\right)$

The height of a point on a circle as it spins over time creates a sine wave. You learned that there are many ways to change this function and each depends on manipulating the circle where the wave comes from. The circle can be shifted up or down, made larger or smaller, spun faster or slower or even started at different places. Each of these changes produces a transformation in the graph.

# 5.9 References


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# CHAPTER 6 Analytic Trigonometry

## Chapter Outline



Here you will focus on the algebraic manipulations of trigonometric functions. You will learn the trigonometric identities, which are a list of equivalent trigonometric statements. These will serve as a toolbox that, together with algebra, will help you to combine, reduce, and simplify trigonometric expressions.

# 6.1 Basic Trigonometric Identities

Here you will simplify trigonometric expressions using the reciprocal, quotient, odd-even and cofunction identities. You will also apply these simplification techniques in trigonometric proofs.

The basic trigonometric identities are ones that can be logically deduced from the definitions and graphs of the six trigonometric functions. Previously, some of these identities have been used in a casual way, but now they will be formalized and added to the toolbox of trigonometric identities.

How can you use the trigonometric identities to simplify the following expression?

$$\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1}$$

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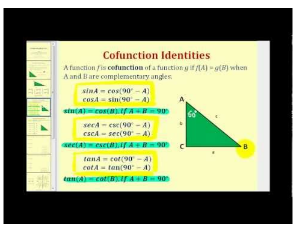


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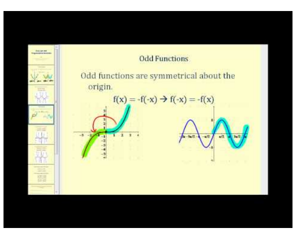
http://youtu.be/_gkuml–4_Q James Sousa: Cofunction Identities

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http://www.youtube.com/watch?v=YbU8Sq0quWE James Sousa: Even and Odd Trigonometric Identities

### Guidance

The reciprocal identities refer to the connections between the trigonometric functions like sine and cosecant. Sine is opposite over hypotenuse and cosecant is hypotenuse over opposite. This logic produces the following six identities.

- $\sin\theta = \frac{1}{\csc\theta}$
- $\cos\theta = \frac{1}{\sec\theta}$
- $\tan\theta = \frac{1}{\cot\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$

The quotient identities follow from the definition of sine, cosine and tangent.

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$

The odd-even identities follow from the fact that only cosine and its reciprocal secant are even and the rest of the trigonometric functions are odd.

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\tan(-\theta) = -\tan\theta$
- $\cot(-\theta) = -\cot\theta$
- $\sec(-\theta) = \sec\theta$
- $\csc(-\theta) = -\csc\theta$

The cofunction identities make the connection between trigonometric functions and their "co" counterparts like sine and cosine. Graphically, all of the cofunctions are reflections and horizontal shifts of each other.

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc\theta$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec\theta$

**Example A**

If $\sin\theta = 0.87$, find $\cos\left(\theta - \frac{\pi}{2}\right)$.

**Solution:** While it is possible to use a calculator to find $\theta$, using identities works very well too.

First you should factor out the negative from the argument. Next you should note that cosine is even and apply the odd-even identity to discard the negative in the argument. Lastly recognize the cofunction identity.

$\cos\left(\theta - \frac{\pi}{2}\right) = \cos\left(-\left(\frac{\pi}{2} - \theta\right)\right) = \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta = 0.87$

**Example B**

Use identities to simplify the following expression: $\tan x \cot x + \frac{\sin x \cot x (\sec x)^2}{\sec x}$.

**Solution:** Start by rewriting the expression and replacing one or two terms that you see will cancel. In this case, replace the $\cot x = \frac{1}{\tan x}$ and cancel the secant term.

$= \tan x \cdot \frac{1}{\tan x} + \sin x \cot x \sec x$

Cancel the tangents to make a one and then use the quotient and reciprocal identities to rewrite the right part of the expression in terms of just sines and cosines. Lastly you should cancel and simplify.

$$\begin{aligned}
&= 1 + \sin x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} \\
&= 1 + 1 \\
&= 2
\end{aligned}$$

**Example C**

Use identities to prove the following: $\cot(-\beta)\cot\left(\frac{\pi}{2} - \beta\right)\sin(-\beta) = \cos\left(\beta - \frac{\pi}{2}\right)$.

**Solution:** When doing trigonometric proofs, it is vital that you start on one side and only work with that side until you derive what is on the other side. Sometimes it may be helpful to work from both sides and find where the two sides meet, but this work is not considered a proof. You will have to rewrite your steps so they follow from only one side. In this case, work with the left side and keep rewriting it until you have $\cos\left(\beta-\frac{\pi}{2}\right)$.

$$\begin{aligned}
\cot(-\beta)\cot\left(\frac{\pi}{2}-\beta\right)\sin(-\beta) &= -\cot\beta\tan\beta\cdot-\sin\beta\\
&= -1\cdot-\sin\beta\\
&= \sin\beta\\
&= \cos\left(\frac{\pi}{2}-\beta\right)\\
&= \cos\left(-\left(\beta-\frac{\pi}{2}\right)\right)\\
&= \cos\left(\beta-\frac{\pi}{2}\right)
\end{aligned}$$

**Concept Problem Revisited**

The following trigonometric expression can be simplified to be equivalent to negative tangent.

$$\begin{aligned}
\left[\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\sin(-\theta)}\right]^{-1} &= \frac{\sin(-\theta)}{\sin\left(\frac{\pi}{2}-\theta\right)}\\
&= \frac{-\sin\theta}{\cos\theta}\\
&= -\tan\theta
\end{aligned}$$

### Vocabulary

An ***identity*** is a mathematical sentence involving the symbol "=" that is always true for variables within the domains of the expressions on either side. In the concept problem, the equivalent expressions are meaningless if the denominator of the rational expression ends up as zero. This is why identities only work within a valid domain.

***Cofunctions*** are functions that are identical except for a reflection and horizontal shift. Examples are sine and cosine, tangent and cotangent, secant and cosecant.

A ***proof*** is a derivation where two sides of an expression are shown to be equivalent through a sequence of logical steps.

### Guided Practice

1. Prove the quotient identity for tangent using the definition of sine, cosine and tangent.

2. If $\cos\left(\theta-\frac{\pi}{2}\right)=0.68$ then determine $\csc(-\theta)$.

3. Prove the following trigonometric identity by working with only one side.

$$\cos x\sin x\tan x\cot x\sec x\csc x=1$$

**Answers:**

1. When tangent, sine and cosine are replaced with the shorthand for side ratios the equivalence becomes a matter of algebra.

$$\begin{aligned}\tan\theta &= \frac{opp}{adj}\\ &= \frac{\left(\frac{opp}{hyp}\right)}{\left(\frac{adj}{hyp}\right)}\\ &= \frac{\sin\theta}{\cos\theta}\end{aligned}$$

2. As Example C and Example A show, $\cos\left(\theta-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-\theta\right)$.

$$\begin{aligned}0.68 &= \cos\left(\theta-\frac{\pi}{2}\right)\\ &= \cos\left(\frac{\pi}{2}-\theta\right)\\ &= \sin(\theta)\end{aligned}$$

Then, $\csc(-\theta)=-\csc\theta$

$$\begin{aligned}&= -\frac{1}{\sin\theta}\\ &= -(0.68)^{-1}\end{aligned}$$

3.

$$\begin{aligned}\cos x\sin x\tan x\cot x\sec x\csc x &= \cos x\sin x\tan x\cdot\frac{1}{\tan x}\cdot\frac{1}{\cos x}\cdot\frac{1}{\sin x}\\ &= 1\end{aligned}$$

### Practice

1. Prove the quotient identity for cotangent using sine and cosine.
2. Explain why $\cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$ using graphs and transformations.
3. Explain why $\sec\theta=\frac{1}{\cos\theta}$.
4. Prove that $\tan\theta\cdot\cot\theta=1$.
5. Prove that $\sin\theta\cdot\csc\theta=1$.
6. Prove that $\sin\theta\cdot\sec\theta=\tan\theta$.
7. Prove that $\cos\theta\cdot\csc\theta=\cot\theta$.
8. If $\sin\theta=0.81$, what is $\sin(-\theta)$?
9. If $\cos\theta=0.5$, what is $\cos(-\theta)$?
10. If $\cos\theta=0.25$, what is $\sec(-\theta)$?
11. If $\csc\theta=0.7$, what is $\sin(-\theta)$?
12. How can you tell from a graph if a function is even or odd?
13. Prove $\frac{\tan x\cdot\sec x}{\csc x}\cdot\cot x=\tan x$.

14. Prove $\frac{\sin^2 x \cdot \sec x}{\tan x} \cdot \csc x = 1$.

15. Prove $\cos x \cdot \tan x = \sin x$.

# 6.2 Sum and Difference Identities

Here you will add six identities to your toolbox: the sum and difference identities for sine, cosine and tangent. You will use these identities along with previous identities for proofs and simplifying expressions.

With your knowledge of special angles like the sine and cosine of $30^\circ$ and $45^\circ$, you can find the sine and cosine of $15^\circ$, the difference of $45^\circ$ and $30^\circ$, and $75^\circ$, the sum of $45^\circ$ and $30^\circ$. Using what you know about the unit circle and the sum and difference identities, how do you determine $\sin 15^\circ$ and $\sin 75^\circ$?

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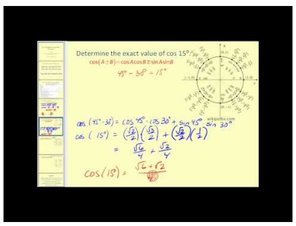

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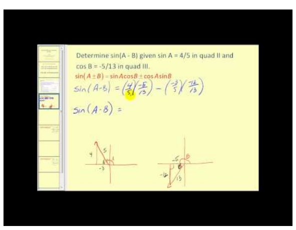
http://www.youtube.com/watch?v=H-0jQTzfkWQ James Sousa: Sum and Difference Identities for Cosine



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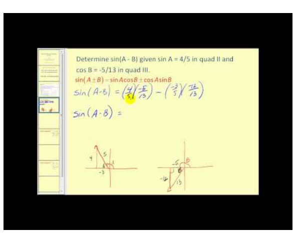

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http://www.youtube.com/watch?v=OQP78bwYcWw James Sousa: Sum and Difference Identities for Tangent

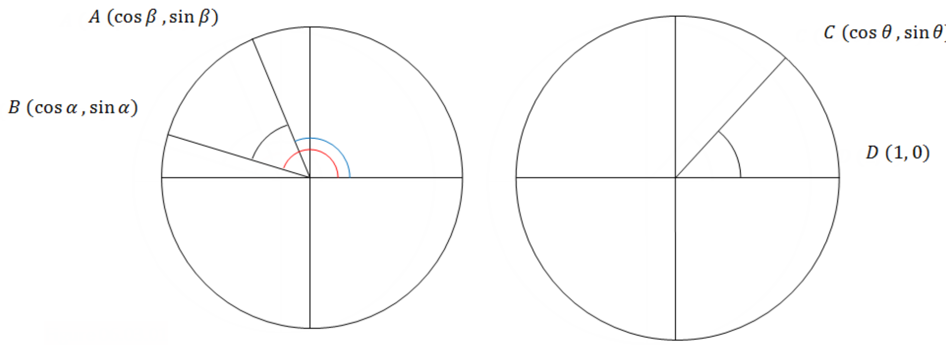
### Guidance

There are some intuitive but incorrect formulas for sums and differences with respect to trigonometric functions. The form below does not work for any trigonometric function and is one of the most common **incorrect** guesses as to the sum and difference identity.

$\sin(\theta+\beta) \neq \sin\theta + \sin\beta$

First look at the derivation of the cosine difference identity:

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$




Start by drawing two arbitrary angles $\alpha$ and $\beta$. In the image above $\alpha$ is the angle in red and $\beta$ is the angle in blue. The difference $\alpha-\beta$ is noted in black as $\theta$. The reason why there are two pictures is because the image on the right has the same angle $\theta$ in a rotated position. This will be useful to work with because the length of $\overline{AB}$ will be the same as the length of $\overline{CD}$.

$$\begin{aligned}
\overline{AB} &= \overline{CD} \\
\sqrt{(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2} &= \sqrt{(\cos\theta-1)^2+(\sin\theta-0)^2} \\
(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2 &= (\cos\theta-1)^2+(\sin\theta-0)^2
\end{aligned}$$

$$(\cos\alpha)^2 - 2\cos\alpha\cos\beta + (\cos\beta)^2 + (\sin\alpha)^2 - 2\sin\alpha\sin\beta + (\sin\beta)^2 = (\cos\theta-1)^2 + (\sin\theta)^2$$

$$\begin{aligned}
2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= (\cos\theta)^2-2\cos\theta+1+(\sin\theta)^2 \\
2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= 1-2\cos\theta+1 \\
-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta &= -2\cos\theta \\
\cos\alpha\cos\beta+\sin\alpha\sin\beta &= \cos\theta \\
&= \cos(\alpha-\beta)
\end{aligned}$$

The proofs for sine and tangent are left to examples and exercises. They are listed here for your reference. Cotangent, secant and cosecant are excluded because you can use reciprocal identities to get those once you have sine, cosine and tangent.

**Summary:**

- $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$
- $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
- $\tan(\alpha\pm\beta) = \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)} = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

**Example A**

Prove the cosine of a sum identity.

**Solution:** Start with the cosine of a difference and make a substitution. Then use the odd-even identity.

$\cos\alpha\cos\beta+\sin\alpha\sin\beta=\cos(\alpha-\beta)$

Let $\gamma=-\beta$

$$\begin{aligned}\cos\alpha\cos(-\gamma)+\sin\alpha\sin(-\gamma)&=\cos(\alpha+\gamma)\\ \cos\alpha\cos\gamma-\sin\alpha\sin\gamma&=\cos(\alpha+\gamma)\end{aligned}$$

**Example B**

Find the exact value of $\tan 15^\circ$ without using a calculator.

**Solution:** $\tan 15^\circ=\tan(45^\circ-30^\circ)=\frac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}=\frac{1-\frac{\sqrt{3}}{3}}{1+1\cdot\frac{\sqrt{3}}{3}}=\frac{3-\sqrt{3}}{3+\sqrt{3}}$

A final solution will not have a radical in the denominator. In this case multiplying through by the conjugate of the denominator will eliminate the radical. This technique is very common in PreCalculus and Calculus.

$$\begin{aligned}&=\frac{(3-\sqrt{3})\cdot(3-\sqrt{3})}{(3+\sqrt{3})\cdot(3-\sqrt{3})}\\ &=\frac{(3-\sqrt{3})^2}{9-3}\\ &=\frac{(3-\sqrt{3})^2}{6}\end{aligned}$$

**Example C**

Evaluate the expression exactly without using a calculator.

$\cos 50^\circ\cos 5^\circ+\sin 50^\circ\sin 5^\circ$

**Solution:** Once you know the general form of the sum and difference identities then you will recognize this as cosine of a difference.

$\cos 50^\circ\cos 5^\circ+\sin 50^\circ\sin 5^\circ=\cos(50^\circ-5^\circ)=\cos 45^\circ=\frac{\sqrt{2}}{2}$

**Concept Problem Revisited**

In order to evaluate $\sin 15^\circ$ and $\sin 75^\circ$ exactly without a calculator, you need to use the sine of a difference and sine of a sum.

$$\begin{aligned}\sin(45^\circ-30^\circ)&=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}-\sqrt{2}}{4}\\ \sin(45^\circ+30^\circ)&=\sin 45^\circ\cos 30^\circ+\cos 45^\circ\sin 30^\circ=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

### Vocabulary

The ***Greek letters*** used in this concept refer to unknown angles. They are $\alpha$-***alpha***, $\beta$- ***beta***, $\theta$- ***theta***, $\gamma$- ***gamma***.

The symbol $\pm$ is short hand for "***plus or minus***." The symbol $\mp$ is shorthand for "***minus or plus***." The order is important because for cosine of a sum, the negative sign is used on the other side of the identity. This is the opposite of sine of a sum, where a positive sign is used on the other side of the identity.

### Guided Practice

1. Prove the sine of a difference identity.
2. Use a sum or difference identity to find an exact value of $\cot\left(\frac{5\pi}{12}\right)$.
3. Prove the following identity:

$\frac{\sin(x-y)}{\sin(x+y)} = \frac{\tan x - \tan y}{\tan x + \tan y}$

**Answers:**

1. Start with the cofunction identity and then distribute and work out the cosine of a sum and cofunction identities.

$$\begin{aligned}\sin(\alpha-\beta) &= \cos\left(\frac{\pi}{2}-(\alpha-\beta)\right)\\ &= \cos\left(\left(\frac{\pi}{2}-\alpha\right)+\beta\right)\\ &= \cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\end{aligned}$$

2.Start with the definition of cotangent as the inverse of tangent.

$$\begin{aligned}\cot\left(\frac{5\pi}{12}\right) &= \frac{1}{\tan\left(\frac{5\pi}{12}\right)}\\ &= \frac{1}{\tan\left(\frac{9\pi}{12}-\frac{4\pi}{12}\right)}\\ &= \frac{1}{\tan(135^\circ-60^\circ)}\\ &= \frac{1+\tan 135^\circ \tan 60^\circ}{\tan 135^\circ - \tan 60^\circ}\\ &= \frac{1+(-1)\cdot\sqrt{3}}{(-1)-\sqrt{3}}\\ &= \frac{(1-\sqrt{3})}{(-1-\sqrt{3})}\\ &= \frac{(1-\sqrt{3})^2}{(-1+\sqrt{3})\cdot(1-\sqrt{3})}\\ &= \frac{(1-\sqrt{3})^2}{-(1-3)}\\ &= \frac{(1-\sqrt{3})^2}{2}\end{aligned}$$

3. Here are the steps:

$$\begin{aligned}
\frac{\sin(x-y)}{\sin(x+y)} &= \frac{\tan x - \tan y}{\tan x + \tan y} \\
\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} &= \\
\frac{\sin x \cos y - \cos x \sin y}{\sin x \cos y + \cos x \sin y} \cdot \frac{\left(\frac{1}{\cos x \cdot \cos y}\right)}{\left(\frac{1}{\cos x \cdot \cos y}\right)} &= \\
\frac{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) - \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)}{\left(\frac{\sin x \cancel{\cos y}}{\cos x \cdot \cancel{\cos y}}\right) + \left(\frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cdot \cos y}\right)} &= \\
\frac{\tan x - \tan y}{\tan x + \tan y} &=
\end{aligned}$$

### Practice

Find the exact value for each expression by using a sum or difference identity.

1. $\cos 75^\circ$

2. $\cos 105^\circ$

3. $\cos 165^\circ$

4. $\sin 105^\circ$

5. $\sec 105^\circ$

6. $\tan 75^\circ$

7. Prove the sine of a sum identity.

8. Prove the tangent of a sum identity.

9. Prove the tangent of a difference identity.

10. Evaluate without a calculator: $\cos 50^\circ \cos 10^\circ - \sin 50^\circ \sin 10^\circ$.

11. Evaluate without a calculator: $\sin 35^\circ \cos 5^\circ - \cos 35^\circ \sin 5^\circ$.

12. Evaluate without a calculator: $\sin 55^\circ \cos 5^\circ + \cos 55^\circ \sin 5^\circ$.

13. If $\cos\alpha\cos\beta = \sin\alpha\sin\beta$, then what does $\cos(\alpha+\beta)$ equal?

14. Prove that $\tan\left(x+\frac{\pi}{4}\right) = \frac{1+\tan x}{1-\tan x}$.

15. Prove that $\sin(x+\pi) = -\sin x$.

# 6.3 Pythagorean Identities - PreCalculus

Here you will prove and use the Pythagorean identities for the six trigonometric functions to simplify expressions and write proofs.

The Pythagorean Theorem works on right triangles. If you consider the $x$coordinate of a point along the unit circle to be the cosine and the $y$coordinate of the point to be the sine and the distance to the origin to be 1 then the Pythagorean Theorem immediately yields the identity:

$$y^2+x^2=1$$
$$\sin^2 x+\cos^2 x=1$$

An observant student may guess that other Pythagorean identities exist with the rest of the trigonometric functions. Is $\tan^2 x+\cot^2 x=1$ a legitimate identity?

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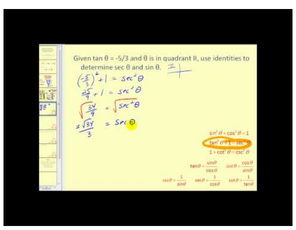

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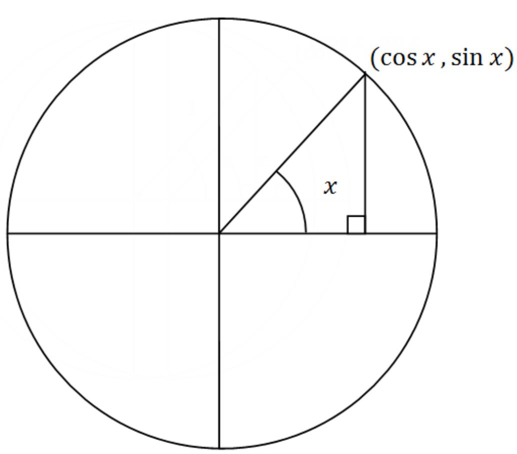
http://www.youtube.com/watch?v=OmJ5fxyXrfg James Sousa: Fundamental Identities: Reciprocal, Quotient, Pythagorean

### Guidance

The proof of the Pythagorean identity for sine and cosine is essentially just drawing a right triangle in a unit circle, identifying the cosine as the $x$ coordinate, the sine as the $y$ coordinate and 1 as the hypotenuse.



$\cos^2 x+\sin^2 x=1$

Most people rewrite the order of the sine and cosine so that the sine comes first.

$\sin^2 x+\cos^2 x=1$

The two other Pythagorean identities are:

- $1+\cot^2 x=\csc^2 x$
- $\tan^2 x+1=\sec^2 x$

To derive these two Pythagorean identities, divide the original Pythagorean identity by $\sin^2 x$ and $\cos^2 x$ respectively.

**Example A**

Derive the following Pythagorean identity: $1+\cot^2 x=\csc^2 x$.

**Solution:** First start with the original Pythagorean identity and then divide through by $\sin^2 x$ and simplify.

$$\begin{aligned}\frac{\sin^2 x}{\sin^2 x}+\frac{\cos^2 x}{\sin^2 x}&=\frac{1}{\sin^2 x}\\ 1+\cot^2 x&=\csc^2 x\end{aligned}$$

**Example B**

Simplify the following expression: $\frac{\sin x(\csc x-\sin x)}{1-\sin x}$.

**Solution:**

$$\begin{aligned}\frac{\sin x(\csc x-\sin x)}{1-\sin x}&=\frac{\sin x\cdot\csc x-\sin^2 x}{1-\sin x}\\ &=\frac{1-\sin^2 x}{1-\sin x}\\ &=\frac{(1-\sin x)(1+\sin x)}{1-\sin x}\\ &=1+\sin x\end{aligned}$$

Note that factoring the Pythagorean identity is one of the most powerful applications. This is very common and is a technique that you should feel comfortable using.

**Example C**

Prove the following trigonometric identity: $(\sec^2 x+\csc^2 x)-(\tan^2 x+\cot^2 x)=2$.

**Solution:** Group the terms and apply a different form of the second two Pythagorean identities which are $1+\cot^2 x=\csc^2 x$ and $\tan^2 x+1=\sec^2 x$.

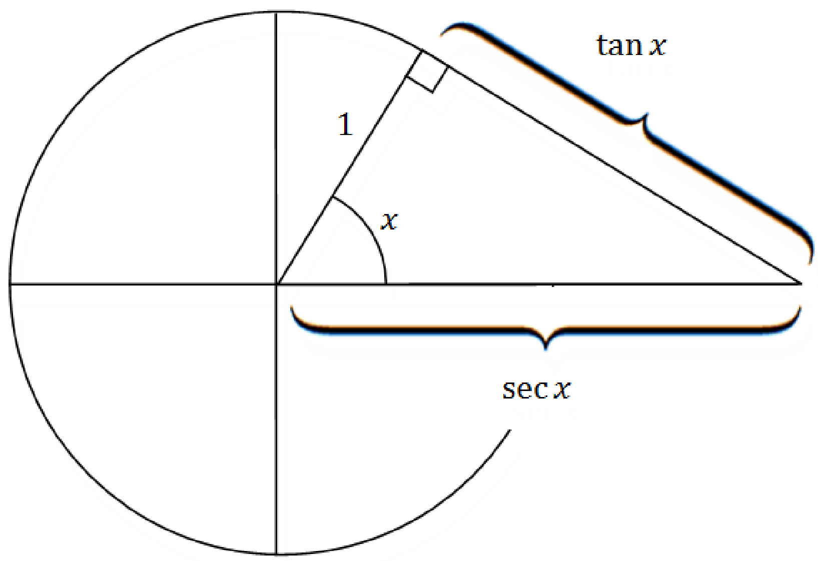
$$\begin{aligned}(\sec^2 x+\csc^2 x)-(\tan^2 x+\cot^2 x)&=\sec^2 x-\tan^2 x+\csc^2 x-\cot^2 x\\ &=1+1\\ &=2\end{aligned}$$

**Concept Problem Revisited**

Cofunctions are not always connected directly through a Pythagorean identity.

$\tan^2 x + \cot^2 x \neq 1$

Visually, the right triangle connecting tangent and secant can also be observed in the unit circle. Most people do not know that tangent is named "tangent" because it refers to the distance of the line tangent from the point on the unit circle to the $x$ axis. Look at the picture below and think about why it makes sense that $\tan x$ and $\sec x$ are as marked. $\tan x = \frac{opp}{adj}$. Since the adjacent side is equal to 1 (the radius of the circle), $\tan x$ simply equals the opposite side. Similar logic can explain the placement of $\sec x$.




### Vocabulary

The ***Pythagorean Theorem*** states that the sum of the squares of the two legs in a right triangle will always be the square of the hypotenuse.

The ***Pythagorean Identity*** states that since sine and cosine are equal to two legs in a right triangle with a hypotenuse of 1, then their relationship is that of the Pythagorean Theorem.

### Guided Practice

1. Derive the following Pythagorean identity:

$\tan^2 x + 1 = \sec^2 x$

2. Simplify the following expression.

$(\sec^2 x)(1 - \sin^2 x) - \left(\frac{\sin x}{\csc x} + \frac{\cos x}{\sec x}\right)$

3. Simplify the following expression.

$(\cos t - \sin t)^2 + (\cos t + \sin t)^2$

**Answers:**

1.

$$\frac{\sin^2 x}{\cos^2 x}+\frac{\cos^2 x}{\cos^2 x}=\frac{1}{\cos^2 x}$$
$$\tan^2 x+1=\sec^2 x$$

2.

$$(\sec^2 x)(1-\sin^2 x)-\left(\frac{\sin x}{\csc x}+\frac{\cos x}{\sec x}\right)$$
$$=\sec^2 x\cdot\cos^2 x-(\sin^2 x+\cos^2 x)$$
$$=1-1$$
$$=0$$

3. Note that initially, the expression is not the same as the Pythagorean identity.

$$(\cos t-\sin t)^2+(\cos t+\sin t)^2$$
$$=\cos^2 t-2\cos t\sin t+\sin^2 t+\cos^2 t+2\cos t\sin t+\sin^2 t$$
$$=1-2\cos t\sin t+1+2\cos t\sin t$$
$$=2$$

### Practice

Prove each of the following:

1. $(1-\cos^2 x)(1+\cot^2 x)=1$
2. $\cos x(1-\sin^2 x)=\cos^3 x$
3. $\sin^2 x=(1-\cos x)(1+\cos x)$
4. $\sin x=\frac{\sin^2 x+\cos^2 x}{\csc x}$
5. $\sin^4 x-\cos^4 x=\sin^2 x-\cos^2 x$
6. $\sin^2 x\cos^3 x=(\sin^2 x-\sin^4 x)(\cos x)$

Simplify each expression as much as possible.

7. $\tan^3 x\csc^3 x$
8. $\frac{\csc^2 x-1}{\sec^2 x}$
9. $\frac{1-\sin^2 x}{1+\sin x}$
10. $\sqrt{1-\cos^2 x}$
11. $\frac{\sin^2 x-\sin^4 x}{\cos^2 x}$
12. $(1+\tan^2 x)(\sec^2 x)$
13. $\frac{\sin^2 x+\tan^2 x+\cos^2 x}{\sec x}$
14. $\frac{1+\tan^2 x}{\csc^2 x}$
15. $\frac{1-\sin^2 x}{\cos x}$

# 6.4 Double, Half, and Power Reducing Identities

Here you will prove and use the double, half, and power reducing identities.

These identities are significantly more involved and less intuitive than previous identities. By practicing and working with these advanced identities, your toolbox and fluency substituting and proving on your own will increase. Each identity in this concept is named aptly. Double angles work on finding $\sin 80^\circ$ if you already know $\sin 40^\circ$. Half angles allow you to find $\sin 15^\circ$ if you already know $\sin 30^\circ$. Power reducing identities allow you to find $\sin^2 15^\circ$ if you know the sine and cosine of $30^\circ$.

What is $\sin^2 15^\circ$?

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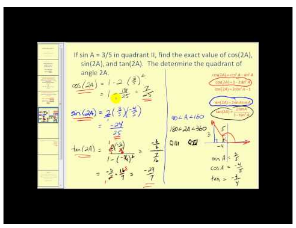

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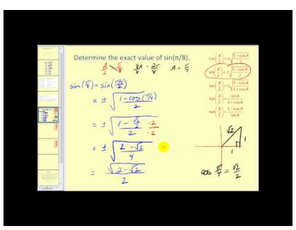
http://www.youtube.com/watch?v=-zhCYiHcVIE James Sousa: Double Angle Identities



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http://www.youtube.com/watch?v=Rp61qiglwfg James Sousa: Half Angle Identities

### Guidance

The double angle identities are proved by applying the sum and difference identities. They are left as practice problems. These are the double angle identities.

- $\sin 2x = 2\sin x \cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

The power reducing identities allow you to write a trigonometric function that is squared in terms of smaller powers. The proofs are left as guided practice and practice problems.

- $\sin^2 x = \frac{1-\cos 2x}{2}$

- $\cos^2 x = \frac{1+\cos 2x}{2}$
- $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

The half angle identities are a rewritten version of the power reducing identities. The proofs are left as practice problems.

- $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$
- $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$
- $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

**Example A**

Rewrite $\sin^4 x$ as an expression without powers greater than one.

**Solution:** While $\sin x \cdot \sin x \cdot \sin x \cdot \sin x$ does technically solve this question, try to get the terms to sum together not multiply together.

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 \\
&= \left(\frac{1-\cos 2x}{2}\right)^2 \\
&= \frac{1-2\cos 2x+\cos^2 2x}{4} \\
&= \frac{1}{4}\left(1-2\cos 2x+\frac{1+\cos 4x}{2}\right)
\end{aligned}$$

**Example B**

Write the following expression with only $\sin x$ and $\cos x$ : $\sin 2x + \cos 3x$.

**Solution:**

$$\begin{aligned}
\sin 2x+\cos 3x &= 2\sin x\cos x+\cos(2x+x) \\
&= 2\sin x\cos x+\cos 2x\cos x-\sin 2x\sin x \\
&= 2\sin x\cos x+(\cos^2 x-\sin^2 x)\cos x-(2\sin x\cos x)\sin x \\
&= 2\sin x\cos x+\cos^3 x-\sin^2 x\cos x-2\sin^2 x\cos x \\
&= 2\sin x\cos x+\cos^3 x-3\sin^2 x\cos x
\end{aligned}$$

**Example C**

Use half angles to find an exact value of $\tan 22.5^\circ$ without using a calculator.

**Solution:** $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

$\tan 22.5^\circ = \tan \frac{45^\circ}{2} = \pm\sqrt{\frac{1-\cos 45^\circ}{1+\cos 45^\circ}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\frac{\frac{2}{2}-\frac{\sqrt{2}}{2}}{\frac{2}{2}+\frac{\sqrt{2}}{2}}} = \pm\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$

Sometimes you may be requested to get all the radicals out of the denominator.

**Concept Problem Revisited**

In order to fully identify $\sin^2 15^\circ$ you need to use the power reducing formula.

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\sin^2 15^\circ = \frac{1-\cos 30^\circ}{2} = \frac{1}{2} - \frac{\sqrt{3}}{4}$$

## Vocabulary

An ***identity*** is a statement proved to be true once so that it can be used as a substitution in future simplifications and proofs.

## Guided Practice

1. Prove the power reducing identity for sine.

   $\sin^2 x = \frac{1-\cos 2x}{2}$
2. Simplify the following identity: $\sin^4 x - \cos^4 x$.
3. What is the period of the following function?

   $f(x) = \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$

**Answers:**

1. Start with the double angle identity for cosine.

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = (1-\sin^2 x) - \sin^2 x$$
$$\cos 2x = 1 - 2\sin^2 x$$

This expression is an equivalent expression to the double angle identity and is often considered an alternate form.

$$2\sin^2 x = 1 - \cos 2x$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$

2. Here are the steps:

$$\begin{aligned}\sin^4 x - \cos^4 x &= (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x)\\ &= -(\cos^2 x - \sin^2 x)\\ &= -\cos 2x\end{aligned}$$

3. $f(x) = \sin 2x \cdot \cos x + \cos 2x \cdot \sin x$ so $f(x) = \sin(2x+x) = \sin 3x$. Since $b = 3$ this implies the period is $\frac{2\pi}{3}$.

### Practice

Prove the following identities.

1. $\sin 2x = 2\sin x\cos x$

2. $\cos 2x = \cos^2 x - \sin^2 x$

3. $\tan 2x = \frac{2\tan x}{1-\tan^2 x}$

4. $\cos^2 x = \frac{1+\cos 2x}{2}$

5. $\tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$

6. $\sin \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}$

7. $\cos \frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$

8. $\tan \frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}$

9. $\csc 2x = \frac{1}{2}\csc x\sec x$

10. $\cot 2x = \frac{\cot^2 x-1}{2\cot x}$

Find the value of each expression using half angle identities.

11. $\tan 15^\circ$

12. $\tan 22.5^\circ$

13. $\sec 22.5^\circ$

14. Show that $\tan \frac{x}{2} = \frac{1-\cos x}{\sin x}$.

15. Show that $\tan \frac{x}{2} = \frac{\sin x}{1+\cos x}$.

# 6.5 Trigonometric Equations

Here you will solve equations that contain trigonometric functions. You will also learn to identify when an equation is an identity and when it is has no solutions.

Solving a trigonometric equation is just like solving a regular equation. You will use factoring and other algebraic techniques to get the variable on one side. The biggest difference with trigonometric equations is the opportunity for there to be an infinite number of solutions that must be described with a pattern. The equation$\cos x = 1$ has many solutions including 0 and$2\pi$. How would you describe all of them?

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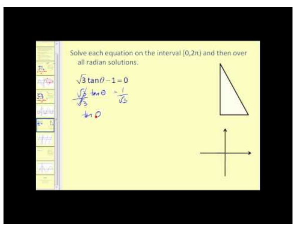

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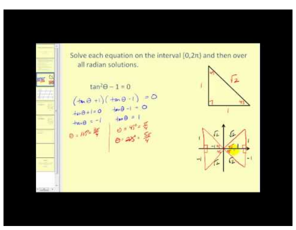
http://www.youtube.com/watch?v=26EWKD2Xha4 James Sousa: Solve Trigonometric Equations I



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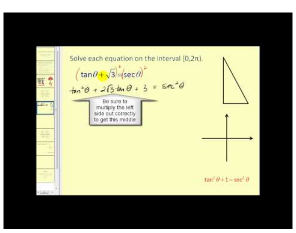
http://www.youtube.com/watch?v=ABKO3ta_Azw James Sousa: Solve Trigonometric Equations II



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http://www.youtube.com/watch?v=7thuFLqC7z0 James Sousa: Solve Trigonometric Equations III

### Guidance

The identities you have learned are helpful in solving trigonometric equations. The goal of solving an equation hasn't changed. Do whatever it takes to get the variable alone on one side of the equation. Factoring, especially with the Pythagorean identity, is critical.

When solving trigonometric equations, try to give exact (non-rounded) answers. If you are working with a calculator, keep in mind that while some newer calculators can provide exact answers like $\frac{\sqrt{3}}{2}$, most calculators will produce a decimal of 0.866... If you see a decimal like 0.866..., try squaring it. The result might be a nice fraction like $\frac{3}{4}$. Then you can logically conclude that the original decimal must be the square root of $\frac{3}{4}$ or $\frac{\sqrt{3}}{2}$.

When solving, if the two sides of the equation are always equal, then the equation is an identity. If the two sides of an equation are never equal, as with $\sin x = 3$, then the equation has no solution.

**Example A**

Solve the following equation algebraically and confirm graphically on the interval $[-2\pi, 2\pi]$.

$\cos 2x = \sin x$

**Solution:**

$$\begin{aligned}\cos 2x &= \sin x\\ 1-2\sin^2 x &= \sin x\\ 0 &= 2\sin^2 x + \sin x - 1\\ 0 &= (2\sin x - 1)(\sin x + 1)\end{aligned}$$

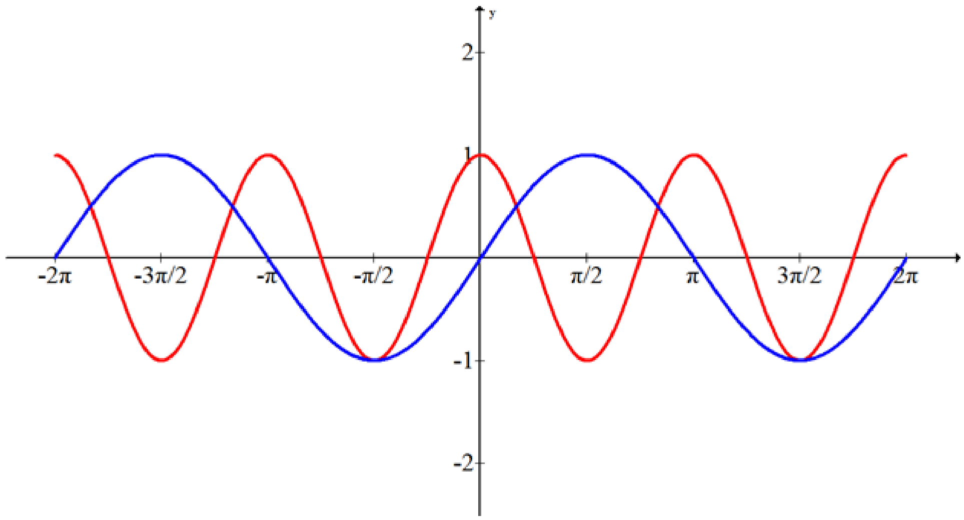
Solving the first part set equal to zero within the interval yields:

$$\begin{aligned}0 &= 2\sin x - 1\\ \frac{1}{2} &= \sin x\\ x &= \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{11\pi}{6}, -\frac{7\pi}{6}\end{aligned}$$

Solving the second part set equal to zero yields:

$$\begin{aligned}0 &= \sin x + 1\\ -1 &= \sin x\\ x &= -\frac{\pi}{2}, \frac{3\pi}{2}\end{aligned}$$

These are the six solutions that will appear as intersections of the two graphs $f(x) = \cos 2x$ and $g(x) = \sin x$.




**Example B**

Determine the general solution to the following equation.

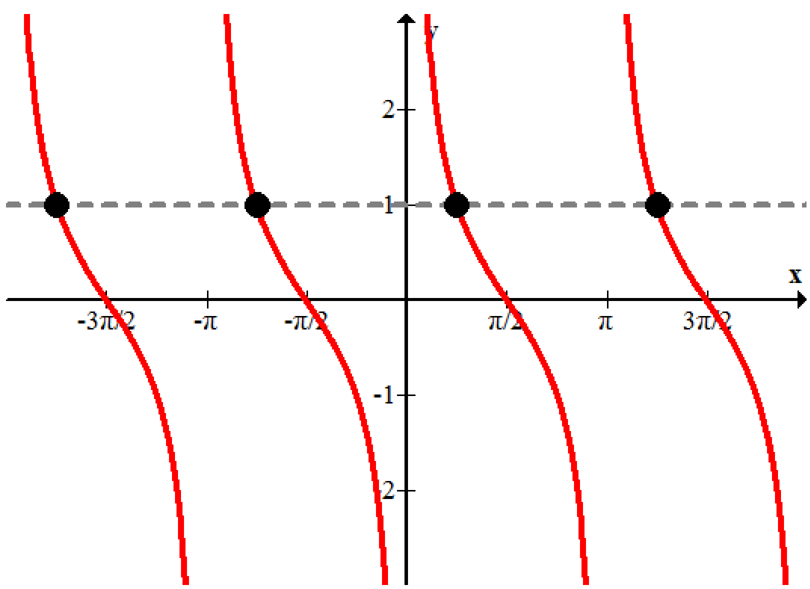
$\cot x - 1 = 0$

**Solution:**

$$\begin{aligned} \cot x - 1 &= 0 \\ \cot x &= 1 \end{aligned}$$

One solution is $x = \frac{\pi}{4}$. However, since this question asks for the general solution, you need to find every possible solution. You have to know that cotangent has a period of $\pi$ which means if you add or subtract $\pi$ from $\frac{\pi}{4}$ then it will also yield a height of 1. To capture all these other possible $x$ values you should use this notation.

$x = \frac{\pi}{4} \pm n \cdot \pi$ where $n$ is a integer




**Example C**

Solve the following equation.

$4\cos^2 x-1=3-4\sin^2 x$

**Solution:**

$$\begin{aligned}4\cos^2 x-1&=3-4\sin^2 x\\4\cos^2 x+4\sin^2 x&=3+1\\4(\cos^2 x+\sin^2 x)&=4\\4&=4\end{aligned}$$

This equation is always true which means the right side is always equal to the left side. This is an identity.

**Concept Problem Revisited**

The equation $\cos x=1$ has many solutions. When you type $\cos^{-1} 1$ on your calculator, it will yield only one solution which is 0. In order to describe all the solutions you must use logic and the graph to figure out that cosine also has a height of 1 at $-2\pi, 2\pi, -4\pi, 4\pi\ldots$ Luckily all these values are sequences in a clear pattern so you can describe them all in general with the following notation:

$x=0\pm n\cdot 2\pi$ where $n$ is an integer, or $x=\pm n\cdot 2\pi$ where $n$ is an integer

### Vocabulary

The terms ***"general solution," "completely solve"***, and ***"solve exactly"*** mean you must find solutions to an equation without the use of a calculator. In addition, trigonometric equations may have an infinite number of solutions that repeat in a certain pattern because they are periodic functions. When you see these directions remember to find all the solutions by using notation like in Example B.

### Guided Practice

1. Solve the following equation on the interval $(2\pi, 4\pi)$.

   $2\sin x+1=0$

2. Solve the following equation exactly.

   $2\cos^2 x+3\cos x-2=0$

3. Create an equation that has the solutions:

   $\frac{\pi}{4}\pm n\cdot 2\pi$ where $n$ is an integer

**Answers:**

1. First, solve for the solutions within one period and then use logic to find the solutions in the correct interval.

$$\begin{aligned}2\sin x+1&=0\\\sin x&=-\frac{1}{2}\\x&=\frac{7\pi}{6},\frac{11\pi}{6}\end{aligned}$$

You must add $2\pi$ to each of these solutions to get solutions that are in the interval.

$x = \frac{19\pi}{6}, \frac{23\pi}{6}$

2. Start by factoring:

$$\begin{aligned}2\cos^2 x + 3\cos x - 2 &= 0\\(2\cos x - 1)(\cos x + 2) &= 0\end{aligned}$$

Note that $\cos x \neq -2$ which means only one equation needs to be solved for solutions.

$$\begin{aligned}2\cos x - 1 &= 0\\\cos x &= \frac{1}{2}\\x &= \frac{\pi}{3}, -\frac{\pi}{3}\end{aligned}$$

These are the solutions within the interval $-\pi$ to $\pi$. Since this represents one full period of cosine, the rest of the solutions are just multiples of $2\pi$ added and subtracted to these two values.

$x = \pm\frac{\pi}{3} \pm n \cdot 2\pi$ where $n$ is an integer

3. There are an infinite number of possible equations that will work. When you see the $\frac{\pi}{4}$ you should think either of where tangent is equal to one or where sine/cosine is equal to $\frac{\sqrt{2}}{2}$. The problem with both of these initial guesses is that tangent repeats every $\pi$ not every $2\pi$, and sine/cosine have a second place where they reach a height of 1. An option that works is:

$\tan\frac{x}{2} = 1$

This equation works because the period of $\tan\frac{x}{2}$ is $2\pi$.

### Practice

Solve each equation on the interval $[0, 2\pi)$.

1. $3\cos^2\frac{x}{2} = 3$
2. $4\sin^2 x = 8\sin^2\frac{x}{2}$

Find approximate solutions to each equation on the interval $[0, 2\pi)$.

3. $3\cos^2 x + 10\cos x + 2 = 0$
4. $\sin^2 x + 3\sin x = 5$
5. $\tan^2 x + \tan x = 3$
6. $\cot^2 x + 5\tan x + 14 = 0$
7. $\sin^2 x + \cos^2 x = 1$

Solve each equation on the interval $[0, 360^\circ)$.

8. $2\sin\left(x - \frac{\pi}{2}\right) = 1$
9. $4\cos(x - \pi) = 4$

Solve each equation on the interval $[2\pi, 4\pi)$.

10. $\cos^2 x + 2\cos x + 1 = 0$

11. $3\sin x = 2\cos^2 x$
12. $\tan x \sin^2 x = \tan x$
13. $\sin^2 x + 1 = 2\sin x$
14. $\sec^2 x = 4$
15. $\sin^2 x - 4 = \cos^2 x - \cos 2x - 4$

The various trigonometric identities were introduced, proved and added to your toolbox. You learned what proofs look like in trigonometry and used the identities to derive the identities themselves. You solved trigonometric equations and bridged the connection between analytic trigonometry, which is very algebraic, with graphical representations.

# 6.6 References

1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
4. CK-12 Foundation. . CCSA
5. CK-12 Foundation. . CCSA

# CHAPTER 7 Vectors

## Chapter Outline



Graphically, vectors are arrows in the coordinate plane. They have length and direction. Algebraically, they allow a whole new way to think about points, lines, angles. Most importantly, they provide an opportunity for you to apply your knowledge of trigonometry in context.

# 7.1 Basic Properties of Vectors

Here you will find out what a vector is algebraically and graphically.

An airplane being pushed off course by wind and a swimmer's movement across a moving river are both examples of vectors in action. Points in the coordinate plane describe location. Vectors, on the other hand, have no location and indicate only direction and magnitude. Vectors can describe the strength of forces like gravity or speed and direction of a ship at sea. Vectors are extremely useful in modeling complex situations in the real world.

What are other differences between vectors and points?

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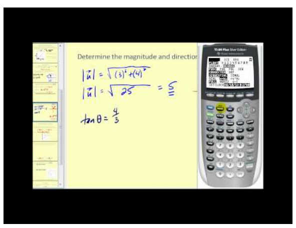

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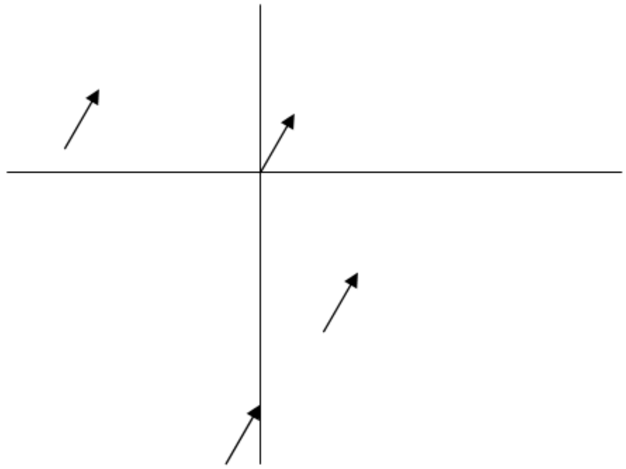
http://www.youtube.com/watch?v=IKzR0Odurm0 James Sousa: Introduction to Vectors

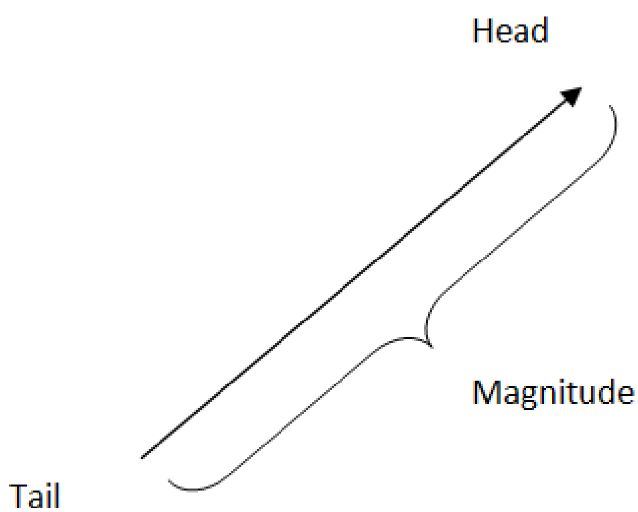
### Guidance

A two dimensional vector is represented graphically as an arrow with a tail and a head. The head is the arrow and is also called the terminal point. When finding the vector between two points start with the terminal point and subtract the initial point (the tail).

The two defining characteristics of a vector are its magnitude and its direction. The magnitude is shown graphically by the length of the arrow and the direction is indicated by the angle that the arrow is pointing. Notice how the following vector is shown multiple times on the same coordinate plane. This emphasizes that the location on the coordinate plane does not matter and is not unique. Each representation of the vector has identical direction and magnitude.



One way to define a vector is as a line segment with a direction. Vectors are said to be equal if they have the ***same magnitude*** and the ***same direction.*** The **absolute value of a vector** is the same as the **length of the line segment** or the **magnitude of the vector**. Magnitude can be found by using the Pythagorean Theorem or the distance formula.




There are a few different ways to write a vector $v$.

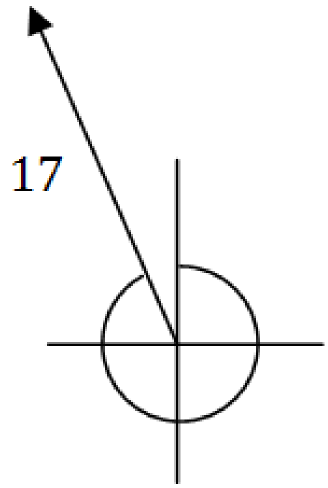
$$v, \overrightarrow{v}, \vec{v}, \text{ or } v \text{ with a } \sim \text{ underneath}$$

When you write about vectors algebraically there are a few ways to describe a specific vector. First, you could describe its angle and magnitude as $r, \theta$. Second, you could describe it as an ordered pair: $< x, y >$. Notice that when discussing vectors you should use the brackets $<>$ instead of parentheses because it helps avoid confusion between a vector and a point. Vectors can be multidimensional.

**Example A**

A ship is traveling NNW at 17 knots (nautical mph). Describe this ship's movement in a vector.

**Solution:** NNW is halfway between NW and N. When describing ships at sea, it is best to use bearing which has $0^\circ$ as due North and $270^\circ$ as due West. This makes NW equal to $315^\circ$ and NNW equal to $337.5^\circ$.



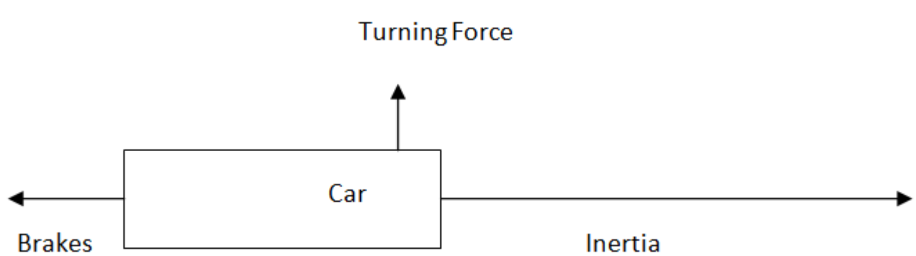

When you see this picture, it turns into a basic trig question to find the $x$ and $y$ components of the vector. Note that the reference angle that the vector makes with the negative portion of the $x$ axis is $67.5^\circ$.

$$\sin 67.5^\circ = \frac{y}{17}, \cos 67.5^\circ = \frac{x}{17}$$
$$< x, y > \approx < -6.5, 15.7 >$$

**Example B**

A car driving 40 mph brakes and turns around a corner. Draw the approximate force vectors acting on the car as if you were looking down on a map.

**Solution:** The primary force is the car's inertia. This is the force acting on the car to keep it moving forward in a straight line. There are three other forces that affect the car. First is gravity, but since this vector is perpendicular to the other vectors it would require a third dimension which will not be considered at this time. The second force is the brakes that act to slow the car down. This force is not as strong as the inertia force. The third force is the act of turning which nudges the front portion of the car to one side. The picture below shows a sketch of the forces acting on the car.




**Example C**

Consider the points: $A(1,3), B(-4,-6), C(5,-13)$. Find the vectors in component form of $\overrightarrow{AB}, \overrightarrow{BA}, \overrightarrow{AC}, \overrightarrow{CB}$.

**Solution:** Remember that when finding the vector between two points, start with the terminal point and subtract the initial point.

$$\overrightarrow{AB} = < -5, -9 >$$
$$\overrightarrow{BA} = < 5, 9 >$$
$$\overrightarrow{AC} = < 4, -16 >$$
$$\overrightarrow{CB} = < -9, 7 >$$

**Concept Problem Revisited**

There are many differences between points and vectors. Points are locations and vectors are made up of distance and angles. Parentheses are used for points and <> are used for vectors. One relationship between vectors and points is that a point plus a vector will yield a new point. It is as if there is a starting place and then a vector tells you where to go from that point. Without the starting point, the vector could start from anywhere.

### Vocabulary

A ***vector*** is a set of instructions indicating direction and magnitude.

The ***tail of a vector*** is the ***initial point*** where the vector starts.

The ***head of a vector*** is the ***terminal point*** where it ends.

A ***force diagram*** is a collection of vectors that each represent a force like gravity or wind acting on an object.

***Magnitude*** refers to the length of the vector and is associated with the strength of the force or the speed of the object.

***Bearing*** is measured with $0^\circ$ as due North, $90^\circ$ as East, $180^\circ$ as South and $270^\circ$ as West.

### Guided Practice

1. A father is pulling his daughter up a hill. The hill has a $20^\circ$ incline. The daughter is on a sled which sits on the ground and has a rope that the father pulls as he walks. The rope makes a $39^\circ$ angle with the slope. Draw a force diagram showing how these forces act on the daughter's center of gravity:

a. The force of gravity.

b. The force holding the daughter in the sled to the ground.

c. The force pulling the daughter backwards down the slope.

d. The force of the father pulling the daughter up the slope.

2. Center the force diagram from the previous question into the origin and identify the angle between each consecutive force vector.

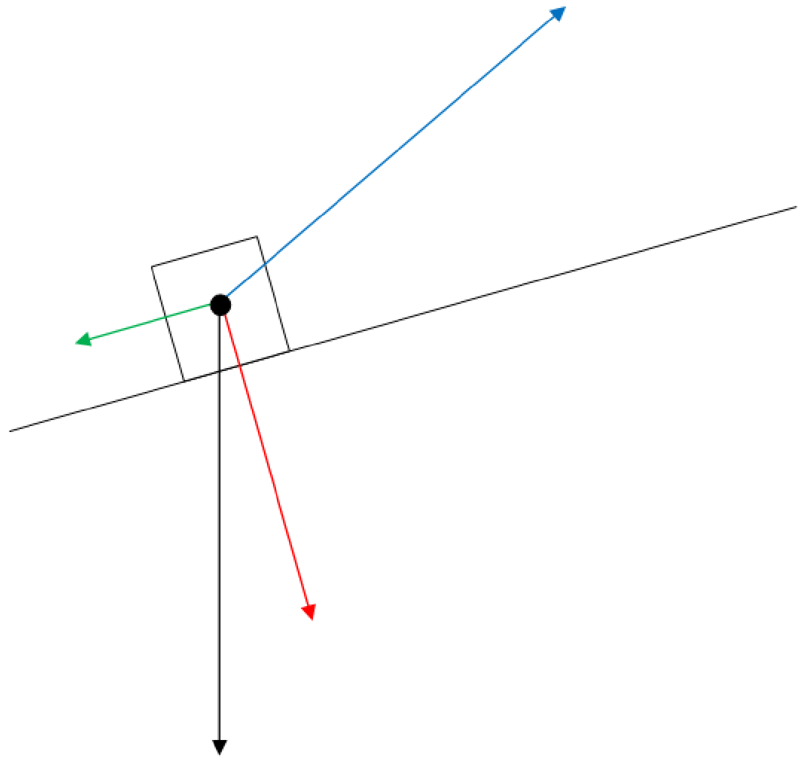
3. Given the following vectors and point, compute the sum.

$A = (1,3)$, $\overrightarrow{v} = < 4, 8 >$, $\overrightarrow{u} = < -1, -5 >$

$A + \overrightarrow{v} + \overrightarrow{u} = ?$

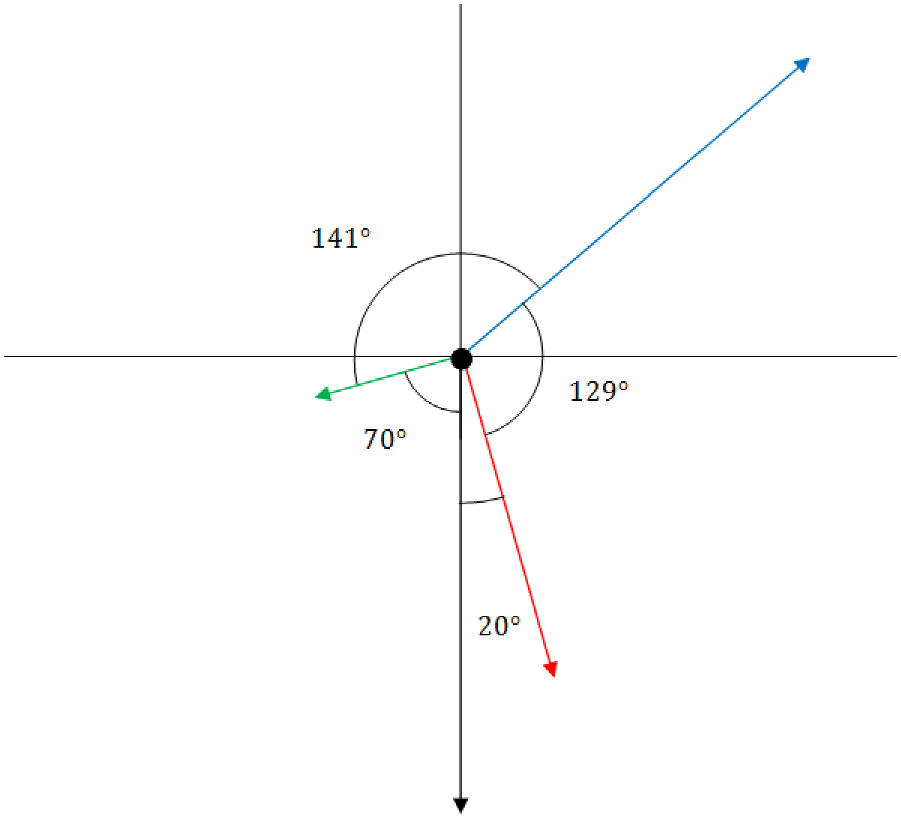
**Answers:**

1. The girl's center of gravity is represented by the black dot. The force of gravity is the black arrow straight down. The green arrow is gravity's effect pulling the girl down the slope. The red arrow is gravity's effect pulling the girl straight into the slope. The blue arrow represents the force that the father is exerting as he pulls the girl up the hill.



Notice that the father's force vector (blue) is longer than the force pulling the girl down the hill. This means that over time they will make progress and ascend the hill. Also note that the father is wasting some of his energy lifting rather than just pulling. If he could pull at an angle directly opposing the force pulling the girl down the hill, then he would be using all of his energy efficiently.

2. The $x$ and $y$ axis are included as reference and note that the gravity vector overlaps with the negative $y$ axis. In order to find each angle, you must use your knowledge of supplementary, complementary and vertical angles and all the clues from the question. To check, see if all the angles sum to be 360°.




3. $A = (1,3), \overrightarrow{v} =< 4,8 >, \overrightarrow{u} =< -1,-5 >$. $A + \overrightarrow{v} + \overrightarrow{u} = (4,6)$.

### Practice

1. Describe what a vector is and give a real-life example of something that a vector could model.

Consider the points: $A(3,5), B(-2,-4), C(1,-12), D(-5,7)$. Find the vectors in component form of:

2. $\overrightarrow{AB}$
3. $\overrightarrow{BA}$
4. $\overrightarrow{AC}$
5. $\overrightarrow{CB}$
6. $\overrightarrow{AD}$
7. $\overrightarrow{DA}$
8. What is $C + \overrightarrow{CB}$? Compute this algebraically and describe why the answer makes sense.
9. Use your answer to the previous problem to help you determine $D + \overrightarrow{DA}$ without doing any algebra.
10. A ship is traveling SSW at 13 knots. Describe this ship's movement in a vector.
11. A vector that describes a ships movement is $< 5\sqrt{2}, 5\sqrt{2} >$. What direction is the ship traveling in and what is its speed in knots?

For each of the following vectors, draw the vector on a coordinate plane starting at the origin and find its magnitude.

12. $< 3,7 >$
13. $< -3,4 >$

14. $< -5, 10 >$

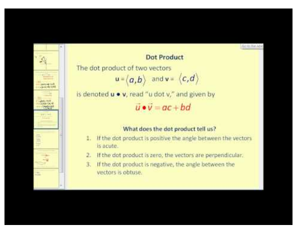
15. $< 6, -8 >$

# 7.2 Operations with Vectors

Here you will add and subtract vectors with vectors and vectors with points.

When two or more forces are acting on the same object, they combine to create a new force. A bird flying due south at 10 miles an hour in a headwind of 2 miles an hour only makes headway at a rate of 8 miles per hour. These forces directly oppose each other. In real life, most forces are not parallel. What will happen when the headwind has a slight crosswind as well, blowing NE at 2 miles per hour. How far does the bird get in one hour?

### Watch This



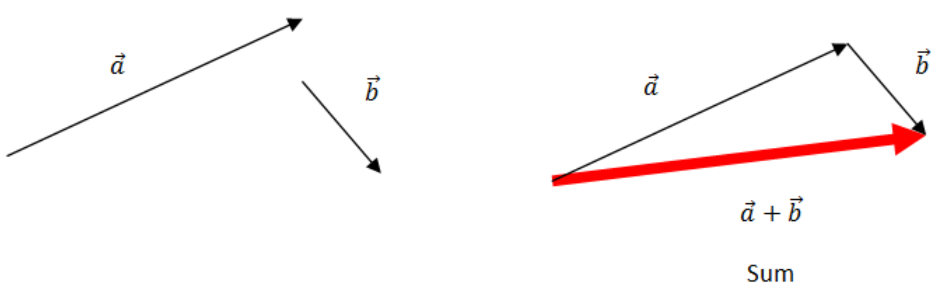
MEDIA

Click image to the left for more content.

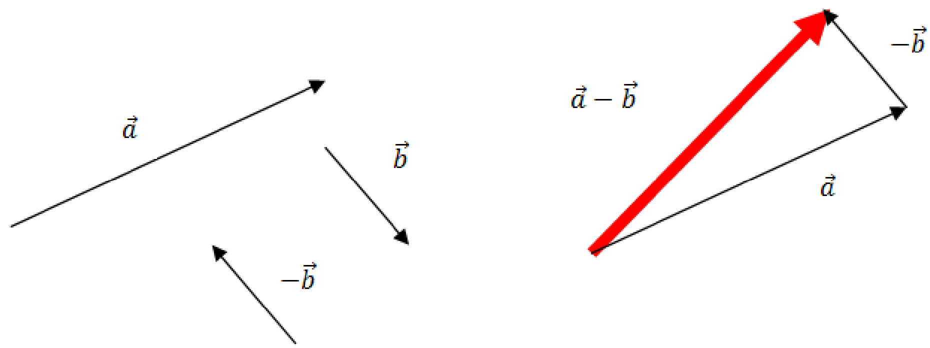
http://www.youtube.com/watch?v=EYIxFJXoUvA James Sousa: Vector Operations

### Guidance

When adding vectors, place the tail of one vector at the head of the other. This is called the **tail-to-head rule.** The vector that is formed by joining the tail of the first vector with the head of the second is called the **resultant vector**.




Vector subtraction reverses the direction of the second vector. $\overrightarrow{a}-\overrightarrow{b}=\overrightarrow{a}+(-\overrightarrow{b})$:

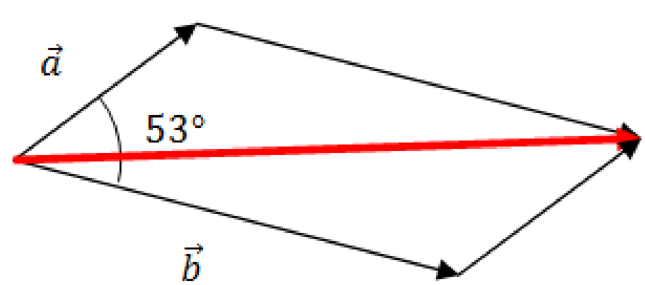



**Scalar multiplication** means to multiply a vector by a number. This changes the magnitude of the vector, but not its direction. If $\overrightarrow{v} =< 3,4 >$, then $2\overrightarrow{v} =< 6,8 >$.

**Example A**

Two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, have magnitudes of 5 and 9 respectively. The angle between the vectors is 53°. Find $|\overrightarrow{a}+\overrightarrow{b}|$.

**Solution:** Adding vectors can be done in either order (just like with regular numbers). Subtracting vectors must be done in a specific order or else the vector will be negative (just like with regular numbers). In either case, use geometric reasoning and the law of cosines with the parallelogram that is formed to find the magnitude of the resultant vector.



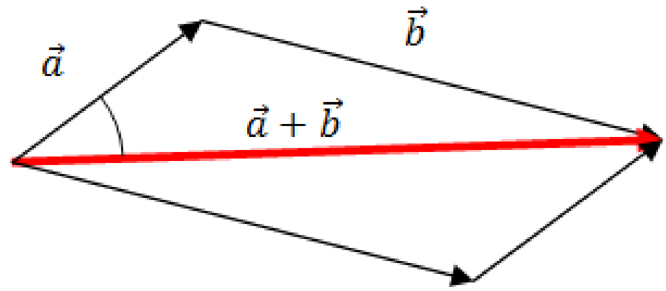

In order to fine the magnitude of the resulting vector $(x)$, note the triangle on the bottom that has sides 9 and 5 with included angle 127°.

$$x^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 127^\circ$$
$$x \approx 12.66$$

**Example B**

Using the picture from Example A, what is the angle that the sum $\overrightarrow{a}+\overrightarrow{b}$ makes with $\overrightarrow{a}$?

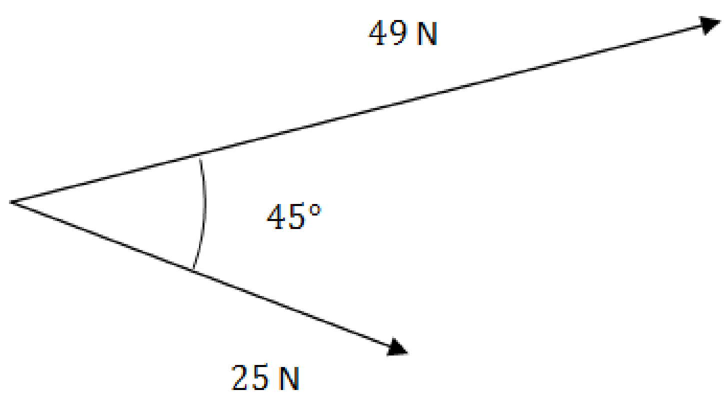
**Solution:** Start by drawing a good picture and labeling what you know. $|\overrightarrow{a}| = 5, |\overrightarrow{b}| = 9, |\overrightarrow{a}+\overrightarrow{b}| \approx 12.66$ . Since you know three sides of the triangle and you need to find one angle, this is the SSS application of the Law of Cosines.



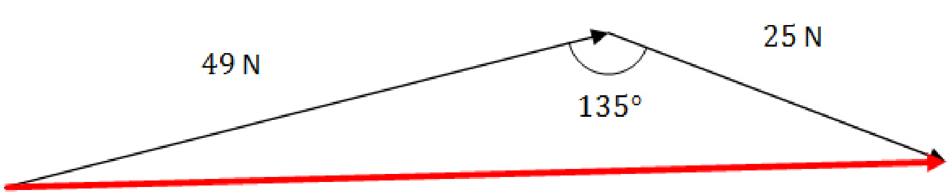

$$9^2 = 12.66^2 + 5^2 - 2 \cdot 12.66 \cdot 5 \cdot \cos\theta$$
$$\theta = 34.6^\circ$$

**Example C**

Elaine started a dog walking business. She walks two dogs at a time named Elvis and Ruby. They each pull her in different directions at a 45° angle with different forces. Elvis pulls at a force of 25 *N* and Ruby pulls at a force of 49 *N*. How hard does Elaine need to pull so that she can stay balanced? *Note: N stands for Newtons which is the standard unit of force.*




**Solution:** Even though the two vectors are centered at Elaine, the forces are added which means that you need to use the tail-to-head rule to add the vectors together. Finding the angle between each component vector requires logical use of supplement angles.

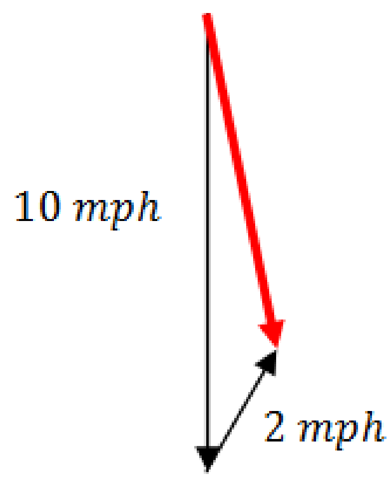


$$x^2 = 49^2 + 25^2 - 2 \cdot 49 \cdot 25 \cdot \cos 135^\circ$$
$$x \approx 68.98\ N$$

In order for Elaine to stay balanced, she will need to counteract this force with an equivalent force of her own in the exact opposite direction.

**Concept Problem Revisited**

A bird flying due south at 10 miles an hour with a cross headwind of 2 mph heading NE would have a force diagram that looks like this:




The angle between the bird's vector and the wind vector is 45° which means this is a perfect situation for the Law of Cosines. Let $x =$ the red vector.

$$x^2 = 10^2 + 2^2 - 2 \cdot 10 \cdot 2 \cdot \cos 45^\circ$$
$$x \approx 8.7$$

The bird is blown slightly off track and travels only about 8.7 miles in one hour.

## Vocabulary

A ***resultant vector*** is the vector that is produced when two or more vectors are summed or subtracted. It is also what is produced when a single vector is scaled by a constant.

***Scaling a vector*** means that the components are each multiplied by a common scale factor. For example: $4 \cdot < 3, 2 > = < 12, 8 >$

## Guided Practice

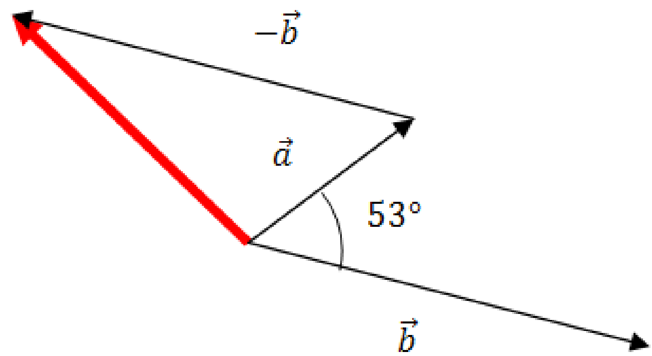
1. Find the magnitude of $|\vec{a} - \vec{b}|$ from Example A.

2. Consider vector $\overrightarrow{v} = < 2, 5 >$ and vector $\overrightarrow{u} = < -1, 9 >$. Determine the component form of the following: $3\overrightarrow{v} - 2\overrightarrow{u}$.

3. An airplane is flying at a bearing of 270° at 400 mph. A wind is blowing due south at 30 mph. Does this cross wind affect the plane's speed?

**Answers:**

1.




The angle between $-\overrightarrow{b}$ and $\overrightarrow{a}$ is 53° because in the diagram $\overrightarrow{b}$ is parallel to $-\overrightarrow{b}$ so you can use the fact that alternate interior angles are congruent. Since the magnitudes of vectors $\overrightarrow{a}$ and $-\overrightarrow{b}$ are known to be 5 and 9, this becomes an application of the Law of Cosines.

$$y^2 = 9^2 + 5^2 - 2 \cdot 9 \cdot 5 \cdot \cos 53^\circ$$
$$y \approx 7.2$$

2. Do multiplication first for each term, followed by vector subtraction.

$$\begin{aligned} 3 \cdot \overrightarrow{v} - 2 \cdot \overrightarrow{u} &= 3 \cdot < 2,5 > -2 \cdot < -1,9 > \\ &= < 6,15 > - < -2,18 > \\ &= < 8,-3 > \end{aligned}$$

3. Since the cross wind is perpendicular to the plane, it pushes the plane south as the plane tries to go directly east. As a result the plane still has an airspeed of 400 mph but the groundspeed (true speed) needs to be calculated.

$$400^2 + 30^2 = x^2$$
$$x \approx 401$$

## Practice

Consider vector $\overrightarrow{v} = < 1,3 >$ and vector $\overrightarrow{u} = < -2,4 >$.

1. Determine the component form of $5\overrightarrow{v} - 2\overrightarrow{u}$.
2. Determine the component form of $-2\overrightarrow{v} + 4\overrightarrow{u}$.
3. Determine the component form of $6\overrightarrow{v} + \overrightarrow{u}$.
4. Determine the component form of $3\overrightarrow{v} - 6\overrightarrow{u}$.
5. Find the magnitude of the resultant vector from #1.
6. Find the magnitude of the resultant vector from #2.
7. Find the magnitude of the resultant vector from #3.
8. Find the magnitude of the resultant vector from #4.
9. The vector $< 3,4 >$ starts at the origin. What is the direction of the vector?
10. The vector $< -1,2 >$ starts at the origin. What is the direction of the vector?
11. The vector $< 3,-4 >$ starts at the origin. What is the direction of the vector?
12. A bird flies due south at 8 miles an hour with a cross headwind blowing due east at 15 miles per hour. How far does the bird get in one hour?
13. What direction is the bird in the previous problem actually moving?
14. A football is thrown at 50 miles per hour due north. There is a wind blowing due east at 8 miles per hour. What is the actual speed of the football?
15. What direction is the football in the previous problem actually moving?

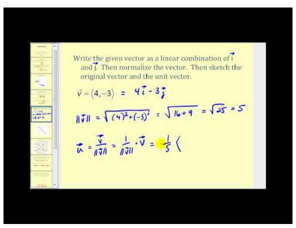
# 7.3 Resolution of Vectors into Components

Here you will find unit vectors and you will convert vectors into linear combinations of standard unit vectors and component vectors.

Sometimes working with horizontal and vertical components of a vector can be significantly easier than working with just an angle and a magnitude. This is especially true when combining several forces together.

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of 41°. Connie pulls with 10 lb of force at an angle of 100°. Cynthia pulls with 12 lb of force at an angle of 200°. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?

### Watch This



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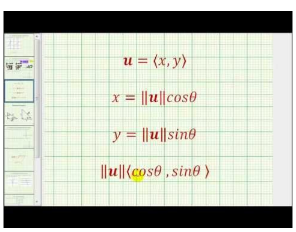
http://www.youtube.com/watch?v=Ect0fBnBlLc James Sousa: The Unit Vector



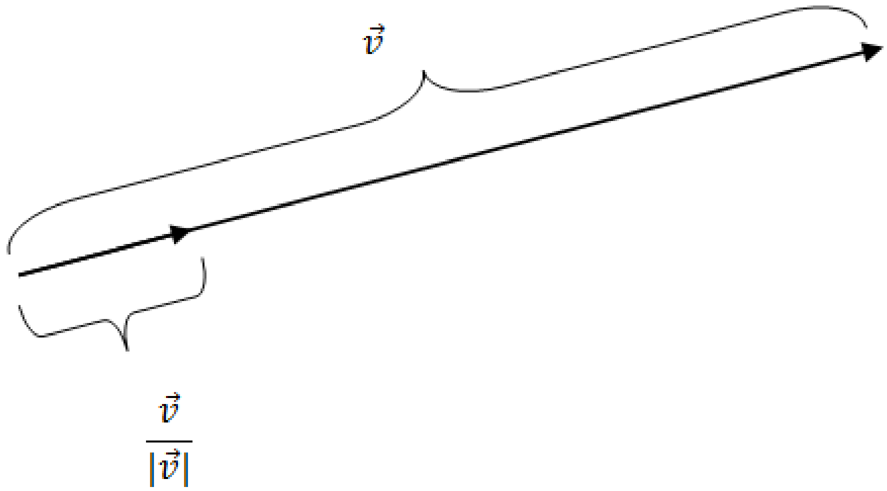

MEDIA

Click image to the left for more content.

http://youtu.be/WZ3xzVHT0mc James Sousa: Find the Component Form of a Vector Given Magnitude and Direction

### Guidance

A unit vector is a vector of length one. Sometimes you might wish to scale a vector you already have so that it has a length of one. If the length was five, you would scale the vector by a factor of $\frac{1}{5}$ so that the resulting vector has magnitude of 1. Another way of saying this is that a unit vector in the direction of vector $\vec{v}$ is $\frac{\vec{v}}{|\vec{v}|}$.




There are two standard unit vectors that make up all other vectors in the coordinate plane. They are $\overrightarrow{i}$ which is the vector $< 1,0 >$ and $\overrightarrow{j}$ which is the vector $< 0,1 >$. These two unit vectors are perpendicular to each other. A linear combination of $\overrightarrow{i}$ and $\overrightarrow{j}$ will allow you to uniquely describe any other vector in the coordinate plane. For instance the vector $< 5,3 >$ is the same as $5\overrightarrow{i}+3\overrightarrow{j}$.

Often vectors are initially described as an angle and a magnitude rather than in component form. Working with vectors written as an angle and magnitude requires extremely precise geometric reasoning and excellent pictures. One advantage of rewriting the vectors in component form is that much of this work is simplified.

**Example A**

A plane has a bearing of 60° and is going 350 mph. Find the component form of the velocity of the airplane.

**Solution:** A bearing of 60° is the same as a 30° on the unit circle which corresponds to the point $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. When written as a vector $< \frac{\sqrt{3}}{2},\frac{1}{2} >$ is a unit vector because it has magnitude 1. Now you just need to scale by a factor of 350 and you get your answer of $< 175\sqrt{3},175 >$.

**Example B**

Consider the plane flying in Example A. If there is wind blowing with the bearing of 300° at 45 mph, what is the component form of the total velocity of the airplane?

**Solution:** A bearing of 300° is the same as 150° on the unit circle which corresponds to the point $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$. You can now write and then scale the wind vector.

$45\cdot < -\frac{\sqrt{3}}{2},\frac{1}{2} > = < -\frac{45\sqrt{3}}{2},\frac{45}{2} >$

Since both the wind vector and the velocity vector of the airplane are written in component form, you can simply sum them to find the component vector of the total velocity of the airplane.

$< 175\sqrt{3},175 > + < -\frac{45\sqrt{3}}{2},\frac{45}{2} > = < \frac{305\sqrt{3}}{2},\frac{395}{2} >$

**Example C**

Consider the plane and wind in Example A and Example B. Find the actual ground speed and direction of the plane (as a bearing).

**Solution:** Since you already know the component vector of the total velocity of the airplane, you should remember that these components represent an $x$ distance and a $y$ distance and the question asks for the hypotenuse.

$$\left(\frac{305\sqrt{3}}{2}\right)^2+\left(\frac{395}{2}\right)^2=c^2$$
$$329.8\approx c$$

The airplane is traveling at about 329.8 mph.

Since you know the $x$ and $y$ components, you can use tangent to find the angle. Then convert this angle into bearing.

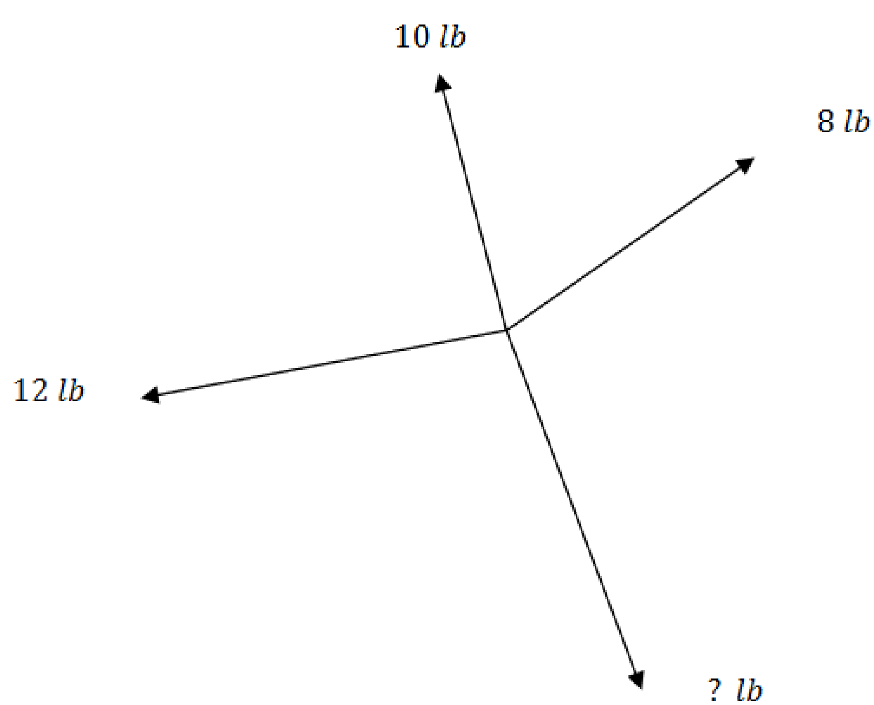
$$\tan\theta=\frac{\left(\frac{395}{2}\right)}{\left(\frac{305\sqrt{3}}{2}\right)}$$
$$\theta\approx 36.8^\circ$$

An angle of $36.8^\circ$ on the unit circle is equivalent to a bearing of $53.2^\circ$.

Note that you can do the entire problem in bearing by just switching sine and cosine, but it is best to truly understand what you are doing every step of the way and this will probably involve always going back to the unit circle.

**Concept Problem Revisited**

Consider four siblings fighting over a candy in a four way tug of war. Lanie pulls with 8 lb of force at an angle of $41^\circ$. Connie pulls with 10 lb of force at an angle of $100^\circ$. Cynthia pulls with 12 lb of force at an angle of $200^\circ$. How much force and in what direction does poor little Terry have to pull the candy so it doesn't move?




To add the three vectors together would require several iterations of the Law of Cosines. Instead, write each vector in component form and set equal to a zero vector indicating that the candy does not move.

$$\overrightarrow{L}+\overrightarrow{CON}+\overrightarrow{CYN}+\overrightarrow{T}=<0,0>$$

$$<8\cdot\cos 41^\circ, 8\cdot\sin 41^\circ> + <10\cdot\cos 100^\circ, 10\cdot\sin 100^\circ>$$
$$+<12\cdot\cos 200^\circ, 12\cdot\sin 200^\circ> + \overrightarrow{T} = <0,0>$$

Use a calculator to add all the $x$ components and bring them to the far side and the $y$ components and then subtract from the far side to get:

$$\overrightarrow{T}\approx<6.98,-10.99>$$

Turning this component vector into an angle and magnitude yields how hard and in what direction he would have to pull. Terry will have to pull with about 13 lb of force at an angle of 302.4°.

## Vocabulary

A ***unit vector*** is a vector of magnitude one.

***Component form*** means in the form $<x,y>$. To translate from magnitude $r$ and direction $\theta$, use the relationship $<r\cdot\cos\theta, r\cdot\sin\theta>=<x,y>$.

The ***standard unit vectors*** are $\overrightarrow{i}$ which is the vector $<1,0>$ and $\overrightarrow{j}$ which is the vector $<0,1>$.

A ***linear combination*** of vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ means a multiple of one plus a multiple of the other.

## Guided Practice

$\overrightarrow{v}=<2,-5>, \overrightarrow{u}=<-3,2>, \overrightarrow{t}=<-4,-3>, \overrightarrow{r}=<5,y>$

$B=(4,-5), P=(-3,8)$

1. Solve for $y$ in vector $\overrightarrow{r}$ to make $\overrightarrow{r}$ perpendicular to $\overrightarrow{t}$.
2. Find the unit vectors in the same direction as $\overrightarrow{u}$ and $\overrightarrow{t}$.
3. Find the point 10 units away from $B$ in the direction of $P$.

**Answers:**

1. $\overrightarrow{t}$ has slope $\frac{3}{4}$ which means that $\overrightarrow{r}$ must have slope $-\frac{4}{3}$. A vector's slope is found by putting the $y$ component over the $x$ component just like with $\frac{rise}{run}$.

$$\frac{y}{5}=-\frac{4}{3}$$
$$y=-\frac{20}{3}$$

2. To find a unit vector, divide each vector by its magnitude.

$\frac{\overrightarrow{u}}{|\overrightarrow{u}|}=<\frac{-3}{\sqrt{13}},\frac{2}{\sqrt{13}}>, \frac{\overrightarrow{t}}{|\overrightarrow{t}|}=<\frac{-4}{5},\frac{-3}{5}>$

3. The vector $\overrightarrow{BP}$ is $< -7, 13 >$. First take the unit vector and then scale it so it has a magnitude of 10.

$$\frac{BP}{|BP|} =< \frac{-7}{\sqrt{218}}, \frac{13}{\sqrt{218}} >$$
$$10 \cdot \frac{BP}{|BP|} =< \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} >$$

You end up with a vector that is ten units long in the right direction. The question asked for a point from $B$ which means you need to add this vector to point $B$.

$(4, -5) + < \frac{-70}{\sqrt{218}}, \frac{130}{\sqrt{218}} > \approx (-0.74, 3.8)$

### Practice

$\overrightarrow{v} =< 1, -3 >, \overrightarrow{u} =< 2, 5 >, \overrightarrow{t} =< 9, -1 >, \overrightarrow{r} =< 2, y >$

$A = (-3, 2), B = (5, -2)$

1. Solve for $y$ in vector $\overrightarrow{r}$ to make $\overrightarrow{r}$ perpendicular to $\overrightarrow{t}$.

2. Find the unit vector in the same direction as $\overrightarrow{u}$.

3. Find the unit vector in the same direction as $\overrightarrow{t}$.

4. Find the unit vector in the same direction as $\overrightarrow{v}$.

5. Find the unit vector in the same direction as $\overrightarrow{r}$.

6. Find the point exactly 3 units away from $A$ in the direction of $B$.

7. Find the point exactly 6 units away from $B$ in the direction of $A$.

8. Find the point exactly 5 units away from $A$ in the direction of $B$.

9. Jack and Jill went up a hill to fetch a pail of water. When they got to the top of the hill, they were very thirsty so they each pulled on the bucket. Jill pulled at 30° with 20 lbs of force. Jack pulled at 45° with 28 lbs of force. What is the resulting vector for the bucket?

10. A plane is flying on a bearing of 60° at 400 mph. Find the component form of the velocity of the plane. What does the component form tell you?

11. A baseball is thrown at a 70° angle with the horizontal with an initial speed of 30 mph. Find the component form of the initial velocity.

12. A plane is flying on a bearing of 200° at 450 mph. Find the component form of the velocity of the plane.

13. A plane is flying on a bearing of 260° at 430 mph. At the same time, there is a wind blowing at a bearing of 30° at 60 mph. What is the component form of the velocity of the plane?

14. Use the information from the previous problem to find the actual ground speed and direction of the plane.

15. Wind is blowing at a magnitude of 40 mph with an angle of 25° with respect to the east. What is the velocity of the wind blowing to the north? What is the velocity of the wind blowing to the east?

# 7.4 Dot Product and Angle Between Two Vectors

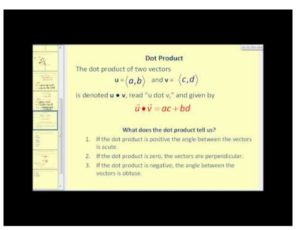
Here you will compute the dot product between two vectors and interpret its meaning.

While two vectors cannot be strictly multiplied like numbers can, there are two different ways to find the product between two vectors. The cross product between two vectors results in a new vector perpendicular to the other two vectors. You can study more about the cross product between two vectors when you take Linear Algebra. The second type of product is the dot product between two vectors which results in a regular number. This number represents***how much of one vector goes in the direction of the other***. In one sense, it indicates how much the two vectors agree with each other. This concept will focus on the dot product between two vectors.

What is the dot product between $< -1, 1 >$ and $< 4, 4 >$? What does the result mean?

### Watch This

Watch the portion of this video focusing on the dot product:

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Click image to the left for more content.

http://www.youtube.com/watch?v=EYIxFJXoUvA James Sousa: Vector Operations

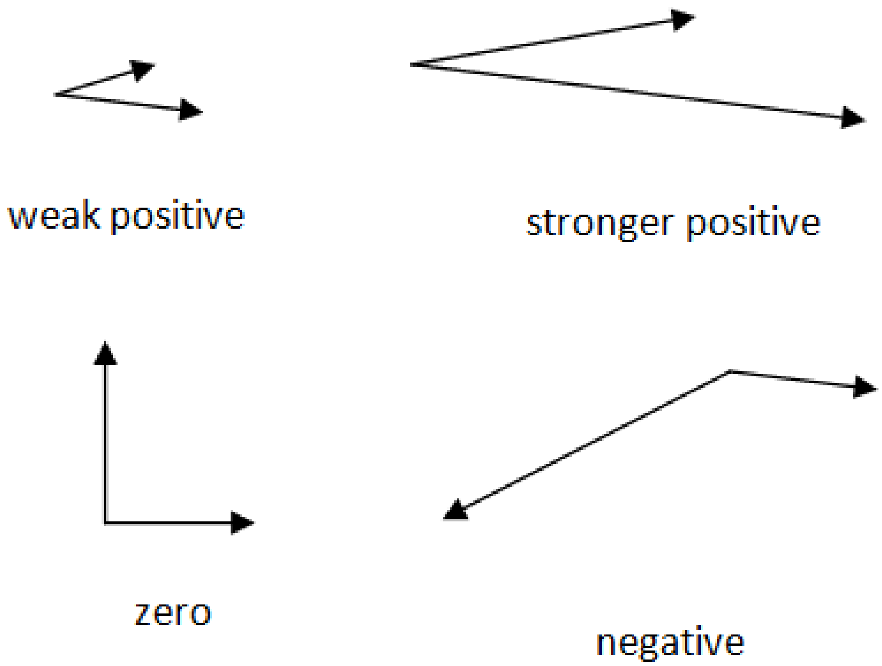
### Guidance

The dot product is defined as:

$u \cdot v = < u_1, u_2 > \cdot < v_1, v_2 > = u_1 v_1 + u_2 v_2$

This procedure states that you multiply the corresponding values and then sum the resulting products. It can work with vectors that are more than two dimensions in the same way.

Before trying this procedure with specific numbers, look at the following pairs of vectors and relative estimates of their dot product.




Notice how vectors going in generally the same direction have a positive dot product. Think of two forces acting on a single object. A positive dot product implies that these forces are working together at least a little bit. Another way of saying this is the angle between the vectors is less than $90^\circ$.

There are a many important properties related to the dot product that you will prove in the examples, guided practice and practice problems. The two most important are 1) what happens when a vector has a dot product with itself and 2) what is the dot product of two vectors that are perpendicular to each other.

- $v \cdot v = |v|^2$
- $v$ and $u$ are perpendicular if and only if $v \cdot u = 0$

The dot product can help you determine the angle between two vectors using the following formula. Notice that in the numerator the dot product is required because each term is a vector. In the denominator only regular multiplication is required because the magnitude of a vector is just a regular number indicating length.

$$\cos\theta = \frac{u \cdot v}{|u||v|}$$

**Example A**

Show the commutative property holds for the dot product between two vectors. In other words, show that $u \cdot v = v \cdot u$.

**Solution:** This proof is for two dimensional vectors although it holds for any dimensional vectors.

Start with the vectors in component form.

$$u =< u_1, u_2 >$$
$$v =< v_1, v_2 >$$

Then apply the definition of dot product and rearrange the terms. The commutative property is already known for regular numbers so we can use that.

$$u \cdot v =< u_1, u_2 > \cdot < v_1, v_2 >$$
$$= u_1 v_1 + u_2 v_2$$
$$= v_1 u_1 + v_2 u_2$$
$$=< v_1, v_2 > \cdot < u_1, u_2 >$$
$$\& = v \cdot u$$

**Example B**

Find the dot product between the following vectors: $< 3,1 > \cdot < 5,-4 >$

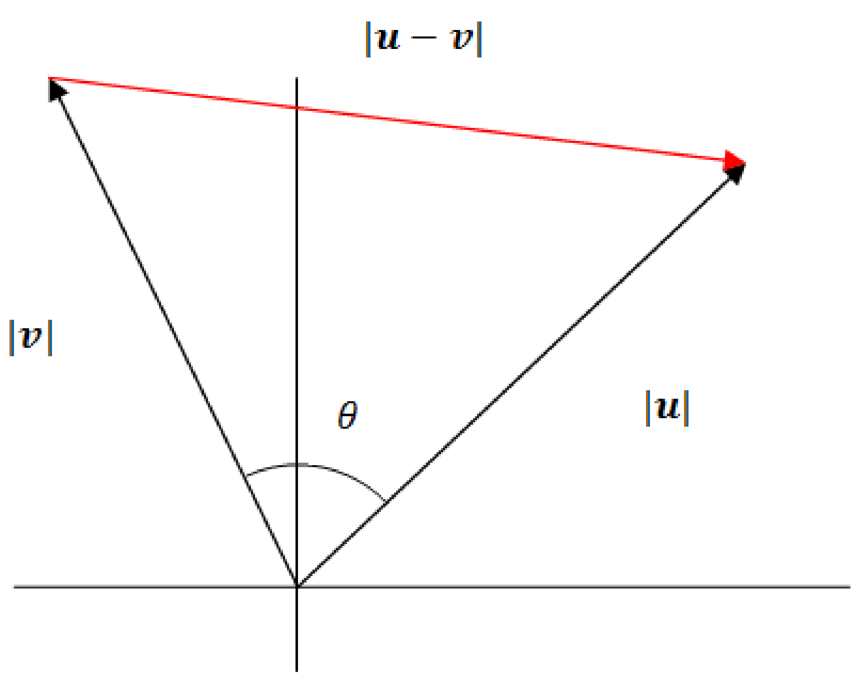
**Solution:** $< 3,1 > \cdot < 5,-4 >= 3 \cdot 5 + 1 \cdot (-4) = 15 - 4 = 11$

**Example C**

Prove the angle between two vectors formula:

$\cos\theta = \frac{u \cdot v}{|u||v|}$

**Solution:** Start with the law of cosines.




$$|u-v|^2 = |v|^2 + |u|^2 - 2|v||u|\cos\theta$$
$$(u-v)\cdot(u-v) =$$
$$u \cdot u - 2u \cdot v + v \cdot v =$$
$$|u|^2 - 2u \cdot v + |v|^2 =$$
$$-2u \cdot v = -2|v||u|\cos\theta$$
$$\frac{u \cdot v}{|u||v|} = \cos\theta$$

**Concept Problem Revisited**

The dot product between the two vectors $<-1,1>$ and $<4,4>$ can be computed as:

$(-1)(4)+1(4)=-4+4=0$

The result of zero makes sense because these two vectors are perpendicular to each other.

### Vocabulary

The ***dot product*** is also known as **inner product** and **scalar product**. It is one of two kinds of products taken between vectors. It produces a number that can be interpreted to tell how much one vector goes in the direction of the other.

### Guided Practice

1. Show the distributive property holds under the dot product**.**

$u\cdot(v+w)=uv+uw$

2. Find the dot product between the following vectors.

$(4i-2j)\cdot(3i-8j)$

3. What is the angle between $v=<3,5>$ and $u=<2,8>$?

**Answers:**

1. This proof will work with two dimensional vectors although the property does hold in general.

$u=<u_1,u_2>,v=<v_1,v_2>,w=<w_1,w_2>$

$$\begin{aligned}
u\cdot(v+w)&=u\cdot(<v_1,v_2>+<w_1,w_2>)\\
&=u\cdot<v_1+w_1,v_2+w_2>\\
&=<u_1,u_2>\cdot<v_1+w_1,v_2+w_2>\\
&=u_1(v_1+w_1)+u_2(v_2+w_2)\\
&=u_1v_1+u_1w_1+u_2v_2+u_2w_2\\
&=u_1v_1+u_2v_2+u_1w_1+u_2w_2\\
&=u\cdot v+v\cdot w
\end{aligned}$$

2. The standard unit vectors can be written as component vectors.

$<4,-2>\cdot<3,-8>=12+(-2)(-8)=12+16=28$

3. Use the angle between two vectors formula.

$v=<3,5>$ and $u=<2,8>$

$$\frac{u \cdot v}{|u||v|} = \cos\theta$$
$$\frac{<3,5> \cdot <2,8>}{\sqrt{34} \cdot \sqrt{68}} = \cos\theta$$
$$\frac{6+35}{\sqrt{34} \cdot \sqrt{68}} = \cos\theta$$
$$\cos^{-1}\left(\frac{41}{\sqrt{34} \cdot \sqrt{68}}\right) = \theta$$
$$31.49 \approx \theta$$

### Practice

Find the dot product for each of the following pairs of vectors.

1. $<2,6> \cdot <-3,5>$
2. $<5,-1> \cdot <4,4>$
3. $<-3,-4> \cdot <2,2>$
4. $<3,1> \cdot <6,3>$
5. $<-1,4> \cdot <2,9>$

Find the angle between each pair of vectors below.

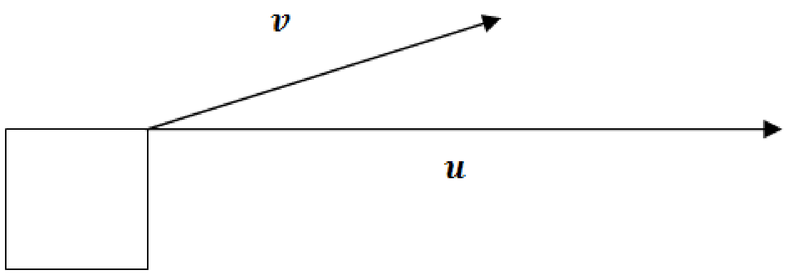
6. $<2,6> \cdot <-3,5>$
7. $<5,-1> \cdot <4,4>$
8. $<-3,-4> \cdot <2,2>$
9. $<3,1> \cdot <6,3>$
10. $<-1,4> \cdot <2,9>$
11. What is $v \cdot v$?
12. How can you use the dot product to find the magnitude of a vector?
13. What is $0 \cdot v$?
14. Show that $(cu) \cdot v = u \cdot (cv)$ where $c$ is a constant.
15. Show that $<2,3>$ is perpendicular to $<1.5,-1>$.

# 7.5 Vector Projection

Here you will project one vector onto another and apply this technique as it relates to force.

Projecting one vector onto another explicitly answers the question: "how much of one vector goes in the direction of the other vector?" The dot product is useful because it produces a scalar quantity that helps to answer this question. In this concept, you will produce an actual vector not just a scalar.

Why is vector projection useful when considering pulling a box in the direction of $v$ instead of horizontally in the direction of $u$?

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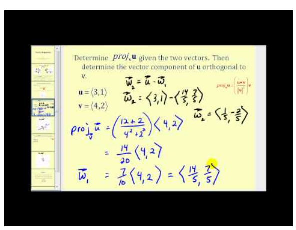

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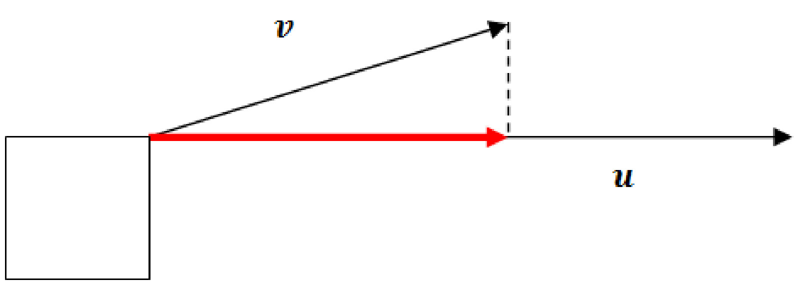
Click image to the left for more content.

http://www.youtube.com/watch?v=3VlxQPeNJFs James Sousa: Vector Projection

### Guidance

Consider the question from the concept.



The definition of vector projection for the indicated red vector is the called $proj_u v$. When you read $proj_u v$ you should say "the vector projection of $v$ onto $u$." This implies that the new vector is going in the direction of $u$. Conceptually, this means that if someone was pulling the box at an angle and strength of vector $v$ then some of their energy would be wasted pulling the box up and some of the energy would actually contribute to pulling the box horizontally.

The definition of scalar projection is simply the length of the vector projection. When the scalar projection is positive it means that the angle between the two vectors is less than 90°. When the scalar projection is negative it means that the two vectors are heading in opposite directions.

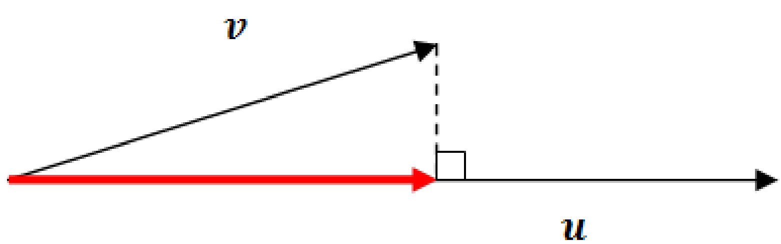
The vector projection formula can be written two ways. The version on the left is most simplified, but the version on the right makes the most sense conceptually. The proof is demonstrated in Example A.

$proj_u v = \left(\frac{v \cdot u}{|u|^2}\right) u = \left(\frac{v \cdot u}{|u|}\right) \frac{u}{|u|}$

**Example A**

Prove the vector projection formula.

**Solution:** Given two vectors $u, v$, what is $proj_u v$?




First note that the projected vector in red will go in the direction of $u$. This means that it will be a product of the unit vector $\frac{u}{|u|}$ and the length of the red vector (the scalar projection). In order to find the scalar projection, note the right triangle, the unknown angle $\theta$ between the two vectors and the cosine ratio.

$\cos\theta = \frac{scalar\ projection}{|v|}$

Recall that $\cos\theta = \frac{u \cdot v}{|u||v|}$. Now just substitute and simplify to find the length of the scalar projection.

$$\begin{aligned} \cos\theta &= \frac{scalar\ projection}{|v|} \\ \frac{u \cdot v}{|u||v|} &= \frac{scalar\ projection}{|v|} \\ \frac{u \cdot v}{|u|} &= scalar\ projection \end{aligned}$$

Now you have the length of the vector projection and the direction you want it to go:

$proj_u v = \left(\frac{u \cdot v}{|u|}\right) \frac{u}{|u|}$

**Example B**

Find the scalar projection of vector $v = <3, 4>$ onto vector $u = <5, -12>$.

**Solution:** As noted in Example A, the scalar projection is the magnitude of the vector projection. This was shown to be $\left(\frac{u \cdot v}{|u|}\right)$ where $u$ is the vector being projected onto.

$\frac{u \cdot v}{|u|} = \frac{<5,-12> \cdot <3,4>}{13} = \frac{15-48}{13} = -\frac{33}{13}$

**Example C**

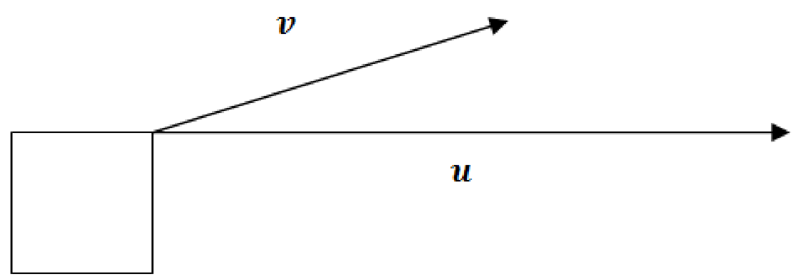
Find the vector projection of vector $v =< 3,4 >$ onto vector $u =< 5,-12 >$

**Solution:** Since the scalar projection has already been found in Example B, you should multiply the scalar by the "onto" unit vector.

$-\frac{33}{13} < \frac{5}{13}, -\frac{12}{13} > = < -\frac{165}{169}, \frac{396}{169} >$

**Concept Problem Revisited**

Vector projection is useful in physics applications involving force and work.




When the box is pulled by vector $v$ some of the force is wasted pulling up against gravity. In real life this may be useful because of friction, but for now this energy is inefficiently wasted in the horizontal movement of the box.

## Vocabulary

The ***vector projection*** is the vector produced when one vector is resolved into two component vectors, one that is parallel to the second vector and one that is perpendicular to the second vector. The parallel vector is the vector projection.

The ***scalar projection*** is the length of the vector projection. When the scalar projection is negative it means that the two vectors are heading in opposite directions and the angle between the vectors is greater than $90^\circ$.

## Guided Practice

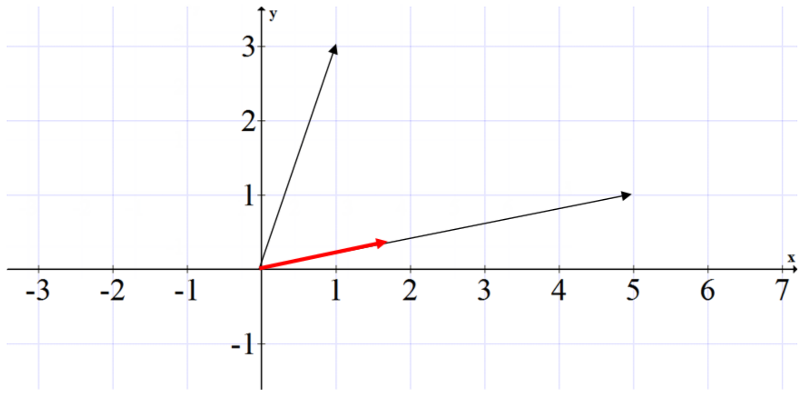
1. Sketch vectors $< 1,3 >$ and $< 5,2 >$. What is the vector projection of $< 1,3 >$ onto $< 5,2 >$? Sketch the projection.

2. Sketch the vector $< -2,-2 >$ and $< 4,-2 >$. Explain using a sketch why a negative scalar projection of $< -2,-2 >$ onto $< 4,-2 >$ makes sense.

3. A father is pulling his daughter up a hill. The hill has a $20^\circ$ incline. The daughter is on a sled that sits on the ground and has a rope that the father pulls with a force of 100 lb as he walks. The rope makes a $39^\circ$ angle with the slope. What is the effective force that the father exerts moving his daughter and the sled up the hill?

**Answers:**

1.

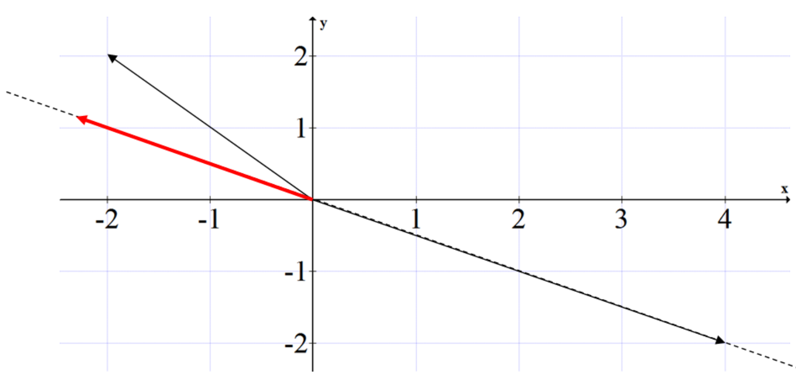



The formula for vector projection where $u$ is the onto vector is:

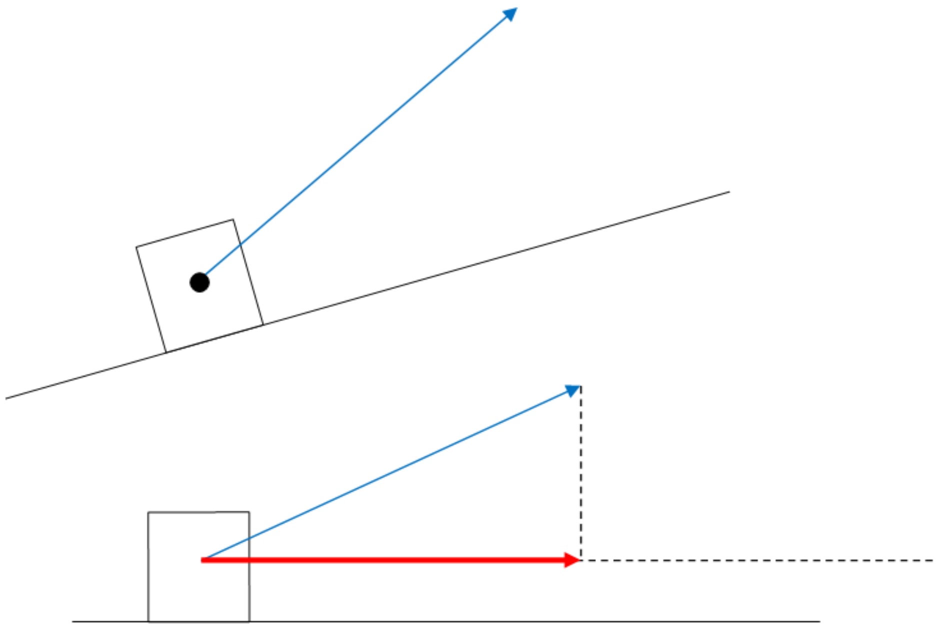
$$
\begin{aligned}
proj_u v = \left(\frac{u \cdot v}{|u|}\right)\frac{u}{|u|} &= \left(\frac{<1,3> \cdot <5,1>}{\sqrt{26}}\right)\frac{<5,1>}{\sqrt{26}} \\
&= \left(\frac{5+3}{26}\right) <5,1> \\
&= \frac{4}{13} <5,1> \\
&= <\frac{20}{13}, \frac{4}{13}>
\end{aligned}
$$

The graph confirms the result of the vector projection.

2. First plot the two vectors and extend the "onto" vector. When the vector projection occurs, the vector $<-2,2>$ goes in the opposite direction of the vector $<4,-2>$. This will create a vector projection going in the opposite direction of $<4,-2>$.




3. The box represents the girl and the sled. The blue arrow indicates the father's 100 lb force. Notice that the question asks for simply the amount of force which means scalar projection. Since this is not dependent on the slope of this hill, we can rotate our perspective and still get the same scalar projection.



The components of the father's force vector is $100 < \cos 39^\circ, \sin 39^\circ >$ and the "onto" vector is any vector horizontally to the right. Since we are only looking for the length of the horizontal component and you already have the angle between the two vectors, the scalar projection is:

$100 \cdot \cos 39^\circ \approx 77.1\ lb$

### Practice

1. Sketch vectors $< 2,4 >$ and $< 2,1 >$.

2. What is the vector projection of $< 2,4 >$ onto $< 2,1 >$? Sketch the projection.

3. Sketch vectors $< -2,1 >$ and $< -1,3 >$.

4. What is the vector projection of $< -1,3 >$ onto $< -2,1 >$? Sketch the projection.

5. Sketch vectors $< 6,2 >$ and $< 8,1 >$.

6. What is the vector projection of $< 6,2 >$ onto $< 8,1 >$? Sketch the projection.

7. Sketch vectors $< 1,7 >$ and $< 6,3 >$.

8. What is the vector projection of $< 1,7 >$ onto $< 6,3 >$? Sketch the projection.

9. A box is on the side of a hill inclined at $30^\circ$. The weight of the box is 40 pounds. What is the magnitude of the force required to keep the box from sliding down the hill?

10. Sarah is on a sled on the side of a hill inclined at $60^\circ$. The weight of Sarah and the sled is 125 pounds. What is the magnitude of the force required for Sam to keep Sarah from sliding down the hill.

11. A 1780 pound car is parked on a street that makes an angle of $15^\circ$ with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

12. A 1900 pound car is parked on a street that makes an angle of $10^\circ$ with the horizontal. Find the magnitude of the force required to keep the car from rolling down the hill.

13. A 30 pound force that makes an angle of $32^\circ$ with an inclined plane is pulling a box up the plane. The inclined plane makes a $20^\circ$ angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

14. A 22 pound force that makes an angle of $12^\circ$ with an inclined plane is pulling a box up the plane. The inclined plane makes a $25^\circ$ angle with the horizontal. What is the magnitude of the effective force pulling the box up the plane?

15. Anne pulls a wagon on a horizontal surface with a force of 50 pounds. The handle of the wagon makes an angle of $30^\circ$ with the ground. What is the magnitude of the effective force pulling the wagon?

You learned that you can add and subtract vectors just like you can numbers. You saw that vectors also have other algebraic properties and essentially create an entirely new algebraic structure. There are two ways the product of vectors can be taken and you studied one way called the dot product. Lastly, you learned to calculate how much of one vector goes into the direction of another vector.

# 7.6 References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# CHAPTER 8 Systems and Matrices

## Chapter Outline



You have solved systems of equations in Algebra 1 and Algebra 2 using methods such as substitution and elimination. These ideas can be extended and improved upon using matrices, which are an array of numbers.

# 8.1 Systems of Two Equations and Two Unknowns

Here you will review how to solve a system of two equations and two unknowns using the elimination method.

The cost of two cell phone plans can be written as a system of equations based on the number of minutes used and the base monthly rate. As a consumer, it would be useful to know when the two plans cost the same and when is one plan cheaper.

Plan A costs \$40 per month plus \$0.10 for each minute of talk time.

Plan B costs \$25 per month plus \$0.50 for each minute of talk time.

Plan B has a lower starting cost, but since it costs more per minute, it may not be the right plan for someone who likes to spend a lot of time on the phone. When do the two plans cost the same amount?

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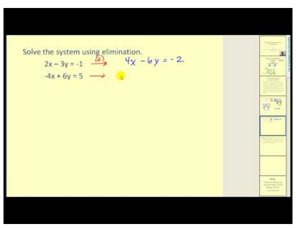


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http://www.youtube.com/watch?v=ova8GSmPV4o James Sousa: Solving Systems of Equations by Elimination

### Guidance

There are many ways to solve a system that you have learned in the past including substitution and graphical intersection. Here you will focus on solving using elimination because the knowledge and skills used will transfer directly into using matrices.

When solving a system, the first thing to do is to count the number of variables that are missing and the number of equations. The number of variables needs to be the same or fewer than the number of equations. Two equations and two variables can be solved, but one equation with two variables cannot.

Get into the habit of always writing systems in standard form: $Ax+By=C$. This will help variables line up, avoid +/- errors and lay the groundwork for using matrices. Once two equations with two variables are in standard form, decide which variable you want to eliminate, scale each equation as necessary by multiplying through by constants and then add the equations together. This procedure should reduce both the number of equations and the number of variables leaving one equation and one variable. Solve and substitute to determine the value of the second variable.

**Example A**

Solve the following system of equations: $5x+12y=72$ and $3x-2y=18$.

**Solution:** Here is a system of two equations and two variables in standard form. Notice that there is an $x$ column and a $y$ column on the left hand side and a constant column on the right hand side when you rewrite the equations as shown. Also notice that if you add the system as written no variable will be eliminated.

Equation 1: $5x+12y=72$

Equation 2: $3x-2y=18$

Strategically choose to eliminate $y$ by scaling the second equation by 6 so that the coefficient of $y$ will match at 12 and -12.

$$\begin{aligned} 5x+12y &= 72 \\ 18x-12y &= 108 \end{aligned}$$

Add the two equations:

$$\begin{aligned} 23x &= 180 \\ x &= \frac{180}{23} \end{aligned}$$

The value for $x$ could be substituted into either of the original equations and the result could be solved for $y$; however, since the value is a fraction it will be easier to repeat the elimination process in order to solve for $x$. This time you will take the first two equations and eliminate $x$ by making the coefficients of $x$ to be 15 and -15. Scale the first equation by a factor of 3 and scale the second equation by a factor of -5.

Equation 1: $15x+36y=216$

Equation 2: $-15x+10y=-90$

Adding the two equations:

$$\begin{aligned} 0x+46y &= 126 \\ y &= \frac{126}{46}=\frac{63}{23} \end{aligned}$$

The point $\left(\frac{180}{23},\frac{63}{23}\right)$ is where these two lines intersect.

**Example B**

Solve the following system of equations:

$$\begin{aligned} 6x-7y &= 8 \\ 15x-14y &= 21 \end{aligned}$$

**Solution:** Scaling the first equation by -2 will allow the $y$ term to be eliminated when the equations are summed.

$$\begin{aligned} -12x+14y &= -16 \\ 15x-14y &= 21 \end{aligned}$$

The sum is:

$$\begin{aligned} 3x &= 5 \\ x &= \frac{5}{3} \end{aligned}$$

You can substitute $x$ into the first equation to solve for $y$.

$$\begin{aligned}6 \cdot \frac{5}{3} - 7y &= 8\\ 10 - 7y &= 8\\ -7y &= -2\\ y &= \frac{2}{7}\end{aligned}$$

The point $\left(\frac{5}{3}, \frac{2}{7}\right)$ is where these two lines intersect.

**Example C**

Solve the following system of equations:

$$\begin{aligned}5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11\\ \frac{1}{x} + \frac{1}{y} &= 4\end{aligned}$$

**Solution:** The strategy of elimination still applies. You can eliminate the $\frac{1}{y}$ term if the second equation is scaled by a factor of -2.

$$\begin{aligned}5 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} &= 11\\ -2 \cdot \frac{1}{x} - 2 \cdot \frac{1}{y} &= -8\end{aligned}$$

Add the equations together and solve for $x$.

$$\begin{aligned}-3 \cdot \frac{1}{x} + 0 \cdot \frac{1}{y} &= 3\\ -3 \cdot \frac{1}{x} &= 3\\ \frac{1}{x} &= -1\\ x &= -1\end{aligned}$$

Substitute into the second equation and solve for $y$.

$$\begin{aligned}\frac{1}{-1} + \frac{1}{y} &= 4\\ -1 + \frac{1}{y} &= 4\\ \frac{1}{y} &= 5\\ y &= \frac{1}{5}\end{aligned}$$

The point $\left(-1, \frac{1}{5}\right)$ is the point of intersection between these two curves.

**Concept Problem Revisited**

Plan A costs $40 per month plus $0.10 for each minute of talk time.

Plan B costs $25 per month plus $0.50 for each minute of talk time.

If you want to find out when the two plans cost the same, you can represent each plan with an equation and solve the system of equations. Let $y$ represent cost and $x$ represent number of minutes.

$$y = 0.10x + 40$$
$$y = 0.50x + 25$$

First you put these equations in standard form.

$$x - 10y = -400$$
$$x - 2y = -50$$

Then you scale the second equation by -1 and add the equations together and solve for $y$.

$$-8y = -350$$
$$y = 43.75$$

To solve for $x$, you can scale the second equation by -5, add the equations together and solve for $x$.

$$-4x = -150$$
$$x = 37.5$$

The equivalent costs of plan A and plan B will occur at 37.5 minutes of talk time with a cost of $43.75.

## Vocabulary

A ***system of equations*** is two or more equations.

***Standard form*** for the equation of a line is $Ax + By = C$.

To ***scale an equation*** means to multiply every term by a constant.

## Guided Practice

1. Solve the following system using elimination:

$$20x + 6y = -32$$
$$18x - 14y = 10$$

2. Solve the following system using elimination:

$$5x - y = 22$$
$$-2x + 7y = 19$$

3. Solve the following system using elimination:

$$11 \cdot \frac{1}{x} - 5 \cdot \frac{1}{y} = -38$$
$$9 \cdot \frac{1}{x} + 2 \cdot \frac{1}{y} = -25$$

**Answers:**

1. Start by scaling both of the equations by $\frac{1}{2}$. Then notice that you have $3y$ and $-7y$. Rescale the first equation by 7 and the second equation by 3 to make the coefficients of $y$ at 21 and -21. There are a number of possible ways to eliminate $y$.

$$70x + 21y = -112$$
$$27x - 21y = 15$$

Add, solve for $x = -1$, substitute and solve for $y$.

Final Answer: (-1, -2)

2. Start by scaling the first equation by 7 and notice that the $y$ coefficient will immediately be eliminated when the equations are summed.

$$35x - 7y = 154$$
$$-2x + 7y = 19$$

Add, solve for $x = \frac{173}{33}$. Instead of substituting, practice eliminating $x$ by scaling the first equation by 2 and the second equation by 5.

$$10x - 2y = 44$$
$$-10x + 35y = 95$$

Add, solve for $y$.

Final Answer: $\left(\frac{173}{33}, \frac{139}{33}\right)$

3. To eliminate $\frac{1}{y}$, scale the first equation by 2 and the second equation by 5.

To eliminate $\frac{1}{x}$, scale the first equation by -9 and the second equation by 11.

Final Answer: $\left(-\frac{1}{3}, 1\right)$

### Practice

Solve each system of equations using the elimination method.

1. $x+y=-4; -x+2y=13$
2. $\frac{3}{2}x-\frac{1}{2}y=\frac{1}{2}; -4x+2y=4$
3. $6x+15y=1; 2x-y=19$
4. $x-\frac{2y}{3}=\frac{-2}{3}; 5x-2y=10$
5. $-9x-24y=-243; \frac{1}{2}x+y=\frac{21}{2}$
6. $5x+\frac{28}{3}y=\frac{176}{3}; y+x=10$
7. $2x-3y=50; 7x+8y=-10$
8. $2x+3y=1; 2y=-3x+14$
9. $2x+\frac{3}{5}y=3; \frac{3}{2}x-y=-5$
10. $5x=9-2y; 3y=2x-3$
11. How do you know if a system of equations has no solution?
12. If a system of equations has no solution, what does this imply about the relationship of the curves on the graph?
13. Give an example of a system of two equations with two unknowns with an infinite number of solutions. Explain how you know the system has an infinite number of solutions.
14. Solve:

$$12\cdot\frac{1}{x}-18\cdot\frac{1}{y}=4$$
$$8\cdot\frac{1}{x}+9\cdot\frac{1}{y}=5$$

15. Solve:

$$14\cdot\frac{1}{x}-5\cdot\frac{1}{y}=-3$$
$$7\cdot\frac{1}{x}+2\cdot\frac{1}{y}=3$$

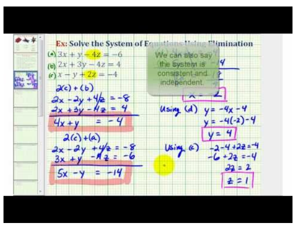
# 8.2 Systems of Three Equations and Three Unknowns

Here you will extend your knolwedge of systems of equations to three equations and three unknowns.

Later, you will learn about matrices and how to row reduce which will allow you to solve systems of equations in a new way. In order to set you up so that using matrices is logical and helpful, it is important to first solve a few systems of three equations using a very specific type of variable elimination.

When solving systems, what are you allowed to do to each equation?

### Watch This



MEDIA

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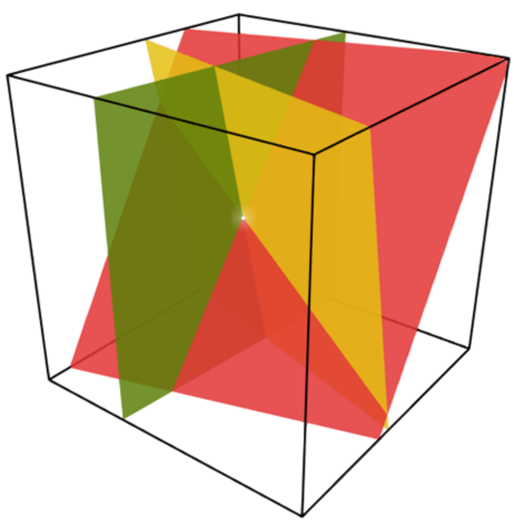
http://youtu.be/3RbVSvvRyeI James Sousa: Ex 1: System of Three Equations with Three Unknowns Using Elimination

### Guidance

A system of three equations with three unknowns represents three planes in three dimensional space. When solving the system, you are figuring out how the planes intersect. It is possible that the three planes could intersect in a point:



It is also possible for two or more planes to be parallel or each pair of planes to intersect in a line. In either of these cases the three planes do not intersect at a single point and the system is said to have no solution. If the three planes intersect at a line or a plane, there are an infinite number of solutions.

The following system of equations has the solution (1, 3, 7). You can verify this by substituting 1 for $x$, 3 for $y$, and 7 for $z$ into each equation.

$$\begin{aligned} x+2y-z&=0\\ 7x-0y+z&=14\\ 0x+y+z&=10 \end{aligned}$$

One thing to be mindful of when given a system of equations is whether or not the equations are linearly independent. Three equations are linearly independent if each equation cannot be produced by a linear combination of the other two.

When solving a system of three equations and three variables, there are a few general guidelines that can be helpful:

- Start by trying to eliminate the first variable in the second row.
- Next eliminate the first and second variables in the third row. This will create zero coefficients in the lower right hand corner.
- Repeat this process for the upper right hand corner and you should end up with a very nice diagonal indicating what $x, y$ and $z$ equal.

**Example A**

Even though you know the solution, solve the system of equations below:

$$\begin{aligned} x+2y-z&=0\\ 7x-0y+z&=14\\ 0x+y+z&=10 \end{aligned}$$

**Solution:** There are a number of ways to solve this system. Common techniques involve swapping rows, dividing and multiplying a row by a constant and adding or subtracting a multiple of one row to another.

*Step 1: Swap rows 2 and 3. Change* -0 to +0.

$$\begin{aligned} x+2y-z&=0\\ 0x+y+z&=10\\ 7x+0y+z&=14 \end{aligned}$$

*Step 2: Subtract 7 times row 1 to row 3.*

$$\begin{aligned} x+2y-z&=0\\ 0x+y+z&=10\\ 0x-14y+8z&=14 \end{aligned}$$

*Step 3: Add 14 times row 2 to row 3*

$$
\begin{aligned}
x+2y-z&=0\\
0x+y+z&=10\\
0x+0y+22z&=154
\end{aligned}
$$

*Step 4: Divide row 3 by 22.*

$$
\begin{aligned}
x+2y-z&=0\\
0x+y+z&=10\\
0x+0y+z&=7
\end{aligned}
$$

*Step 5: Subtract row 3 from row 2*

$$
\begin{aligned}
x+2y-z&=0\\
0x+y+0z&=3\\
0x+0y+z&=7
\end{aligned}
$$

*Step 6: Add row 3 to row 1*

$$
\begin{aligned}
x+2y+0z&=7\\
0x+y+0z&=3\\
0x+0y+z&=7
\end{aligned}
$$

*Step 7: Subtract 2 times row 2 to row 1.*

$$
\begin{aligned}
x+0y+0z&=1\\
0x+y+0z&=3\\
0x+0y+z&=7
\end{aligned}
$$

The solution to the system is (1, 3, 7) exactly as it should be.

**Example B**

Is the following system linearly independent or dependent? How do you know?

$$
\begin{aligned}
3x+2y+z&=8\\
x+y+z&=3\\
5x+4y+3z&=14\\
6x+6y+6z&=18
\end{aligned}
$$

**Solution:** With four equations and three unknowns there must be at least one dependent equation. The simplest method of seeing linearly dependence is to notice that one equation is just a multiple of the other. In this case the fourth equation is just six times the second equation and so it is dependent.

Most people will not notice that the third equation is also dependent. It is common to start doing a problem and notice somewhere along the way that all the variables in a row disappear. This means that the original equations were dependent. In this case, the third equation is the first equation plus two times the second equation. This means they are dependent.

**Example C**

Reduce the following system to a system of two equations and two unknowns.

$$\begin{aligned} 3x+2y+z&=7\\ 4x+0y+z&=6\\ 6x-y+0z&=5 \end{aligned}$$

**Solution:** Strategically swapping rows so that the zero coefficients do not live on the diagonal is a clever starting move.

*Step 1: Swap rows 2 and 3.*

$$\begin{aligned} 3x+2y+z&=7\\ 6x-y+0z&=5\\ 4x+0y+z&=6 \end{aligned}$$

*Step 2: Scale row 3 by a factor of 3. Subtract 2 times row 1 from for 2.*

$$\begin{aligned} 3x+2y+z&=7\\ 0x-5y-2z&=-9\\ 12x+0y+3z&=18 \end{aligned}$$

*Step 3: Subtract 4 times row 1 from row 3.*

$$\begin{aligned} 3x+2y+z&=7\\ 0x-5y-2z&=-9\\ 0x-8y-z&=-10 \end{aligned}$$

*Step 4: Scale the second row by 8 and the third row by 5.*

$$\begin{aligned} 3x+2y+z&=7\\ 0x-40y-16z&=-72\\ 0x-40y-5z&=-50 \end{aligned}$$

*Step 5: Subtract row 2 from row 3.*

$$\begin{aligned} 3x+2y+z&=7\\ 0x-40y-16z&=-72\\ 0x+0y+11z&=+22 \end{aligned}$$

*Step 6: Scale row 3 to find z.*

$$\begin{aligned} 3x+2y+z&=7\\ 0x-40y-16z&=-72\\ 0x+0y+z&=2 \end{aligned}$$

Now that $z=2$, rewrite the system so it becomes a system of three equations and three unknowns. Any iteration of the first two rows will work. This iteration is from step 6.

$$\begin{aligned} 3x+2y+2&=7\\ 0x-40y-32&=-72 \end{aligned}$$

**Concept Problem Revisited**

When solving a system of three equations with three unknowns, you are allowed to add and subtract rows, swap rows and scale rows. These three operations should allow you to eliminate the coefficients of the variables in a systematic way.

## Vocabulary

***Swapping rows*** means rewriting the system so that two rows change places. This is purely a visual reorganization and should help you problem solve and see what to do next.

***Eliminating the variable*** means making the coefficient equal to zero thereby removing that variable from that particular equation and reducing the number of unknown quantities.

***Scaling a row*** means multiplying every coefficient in the row by any number you choose (besides zero). This can be helpful for getting coefficients to match so that they can be eliminated.

***Consistent systems*** are systems that have at least one solution. If the equations are ***linearly independent***, then the system will have just one solution.

***Inconsistent systems*** are systems that have no solution. An example is $x=2, x=3$.

## Guided Practice

1. When Kaitlyn went to the store with ten dollars she saw that she had some choices about what to buy. She could get one apple, one onion and one basket of blueberries for 9 dollars. She could get two apples and two onions for

10 dollars. She could also get two onions and one basket of blueberries for 10 dollars. Write a system of equations with variables $a, o$ and $b$ representing each of the three things she can buy.

2. Solve the system described in # 1.

3. Show that the following system is dependent.

$$\begin{aligned} x+y+z&=9\\ x+2y+3z&=22\\ 2x+3y+4z&=31 \end{aligned}$$

**Answers:**

1. Here is the system of equations:

$$\begin{aligned} a+o+b&=9\\ 2a+2o&=10\\ 2o+b&=10 \end{aligned}$$

2. Rewrite the system using $x, y$ and $z$ so that $o$ and 0 do not get mixed up. Include coefficients of 0 so that each column represents one variable.

*Step 1: Rewrite*

$$\begin{aligned} 1x+1y+1z&=9\\ 2x+2y+0z&=10\\ 0x+2y+1z&=10 \end{aligned}$$

*Step 2: Subtract 2 times row 1 from row 2.*

$$\begin{aligned} 1x+1y+1z&=9\\ 0x+0y-2z&=-8\\ 0x+2y+1z&=10 \end{aligned}$$

*Step 3: Swap row 2 and row 3. Then scale row 3.*

$$\begin{aligned} 1x+1y+1z&=9\\ 0x+2y+1z&=10\\ 0x+0y+z&=4 \end{aligned}$$

At this point you can see from the third equation that $z=4$. From the second equation, $2y+4=10$, so $y=3$. Finally you can see from the first equation that $x+3+4=9$ so $x=2$. Apples cost 2 dollars each, onions cost 3 dollars each and blueberries cost 4 dollars each.

3. You could notice that the third equation is simply the sum of the other two. What happens when you do not notice and try to solve the system as if it were independent?

*Step 1: Rewrite the system.*

$$\begin{aligned} x+y+z &= 9 \\ x+2y+3z &= 22 \\ 2x+3y+4z &= 31 \end{aligned}$$

*Step 2: Subtract 2 times row 1 from row 3.*

$$\begin{aligned} x+y+z &= 9 \\ x+2y+3z &= 22 \\ 0x+1y+2z &= 13 \end{aligned}$$

*Step 3: Subtract row 1 from row 2.*

$$\begin{aligned} x+y+z &= 9 \\ 0x+1y+2z &= 13 \\ 0x+1y+2z &= 13 \end{aligned}$$

At this point when you subtract row 2 from row 3, all the coefficients in row 3 disappear. This means that you will end up with the following system of only two equations and three unknowns. Since the unknowns outnumber the equations, the system does not have a solution of one point.

$$\begin{aligned} x+y+z &= 9 \\ 0x+1y+2z &= 13 \end{aligned}$$

### Practice

1. An equation with three variables represents a plane in space. Describe all the ways that three planes could interact in space.

2. What does it mean for equations to be linearly dependent?

3. How can you tell that a system is linearly dependent?

4. If you have linearly independent equations with four unknowns, how many of these equations would you need in order to get one solution?

5. Solve the following system of equations:

$$\begin{aligned} 3x-4y+z &= -17 \\ 6x+y-3z &= 4 \\ -x-y+5z &= 16 \end{aligned}$$

6. Show that the following system is dependent:

$$\begin{aligned} 2x-2y+z&=5\\ 6x+y-3z&=2\\ 4x+3y-4z&=-3 \end{aligned}$$

7. Solve the following system of equations:

$$\begin{aligned} 4x+y+z&=15\\ -2x+3y+4z&=38\\ -x-y+3z&=16 \end{aligned}$$

8. Solve the following system of equations:

$$\begin{aligned} 3x-2y+3z&=6\\ x+3y-3z&=-14\\ -x+y+5z&=22 \end{aligned}$$

9. Solve the following system of equations:

$$\begin{aligned} 3x-y+z&=-10\\ 6x-2y+2z&=-20\\ -x-y+4z&=12 \end{aligned}$$

10. Solve the following system of equations:

$$\begin{aligned} x-3y+6z&=-30\\ 4x+2y-3z&=18\\ -8x-3y+2z&=-22 \end{aligned}$$

11. Solve the following system of equations.

$$\begin{aligned} x+2y+2z+w&=5\\ 2x+y+2z-0w&=5\\ 3x+3y+3z+2w&=12\\ x+0y+z+w&=1 \end{aligned}$$

A parabola goes through (3, -9.5), (6, -32), and (-4, 8).

12. Write a system of equations that you could use to solve to find the equation of the parabola. *Hint: Use the general equation* $Ax^2+Bx+C=y$.

13. Solve the system of equations from #12.

A parabola goes through (-2, 3), (2, 19), and (1, 6).

14. Write a system of equations that you could use to solve to find the equation of the parabola. *Hint: Use the general equation* $Ax^2 + Bx + C = y$.

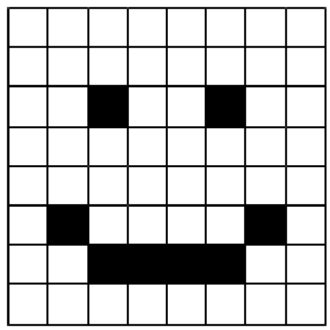
15. Solve the system of equations from #14.

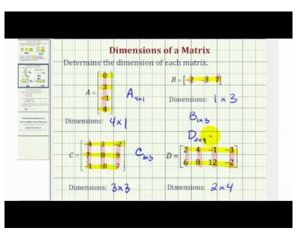
# 8.3 Matrices to Represent Data

Here you will learn what a matrix is and how to use one to represent data.

A matrix is a rectangular array of numbers representing data in a variety of forms. Computers work very heavily with matrices because operations with matrices are efficient with memory. Matrices can represent statistical data with numbers, but also graphical data with pictures.

How might you use a matrix to write the following image as something a computer could recognize and work with?






MEDIA

Click image to the left for more content.

http://youtu.be/ilFJYjfKYjk James Sousa: Dimensions of a Matrix

### Guidance

A matrix is a means of storing information effectively and efficiently. The rows and columns each mean something very specific and the location of a number is just as important as its value. The following are all examples of matrices:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

The entries in a matrix can be written out using brackets like [ ], but they can also be described individually using a set of 2 subscript indices $i$ and $j$ that stand for the row number and the column number. Alternatively, the matrix can be named with just a capital letter like $A$.

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Square matrices have the same number of rows as columns. The **order** of a matrix describes the number of rows and the number of columns in the matrix. The following matrix is said to have order $2 \times 3$ because it has two rows and three columns. A $1 \times 1$ matrix is just a regular number.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

The identity matrix of order $n \times n$ has zeros everywhere except along the main diagonal where it has ones. Just like the number 1 has an important property with numbers, the identity matrix of any order has special properties as well.

$$[1], \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When you turn the rows of a matrix into the columns of a new matrix, the two matrices are transpositions of one another. The superscript $T$ stands for transpose. Sometimes using the transpose of a matrix is more useful than using the matrix itself.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

A triangular matrix is not a matrix in the shape of a triangle. Rather, a lower triangular matrix is a square matrix where every entry below the diagonal is zero. An upper triangular matrix is a square matrix where every entry above the diagonal is zero. The following is a lower triangular matrix. When you work with solving matrices, look for triangular matrices because they are much easier to solve.

$$\begin{bmatrix} 2 & 3 & 4 & 5 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 2 & -9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

A diagonal matrix is both upper and lower triangular which means all the entries except those along the diagonal are zero. The identity matrix is a special case of a diagonal matrix.

**Example A**

Organize the driving distances between Sacramento $(A)$, Dallas $(B)$, Albany $(C)$ and Las Vegas $(D)$ in a matrix.

**Solution:**

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1727 & 2847 & 560 \\ 1727 & 0 & 1648 & 1219 \\ 2847 & 1648 & 0 & 2552 \\ 560 & 1219 & 2552 & 0 \end{bmatrix} \end{array}$$

Note that this matrix is symmetric across the main diagonal. Symmetric matrices are important, just like triangular matrices.

**Example B**

Kate runs three bakeries and each bakery sells bagels and muffins. The rows represent the bakeries and the columns represent bagels (left) and muffins (right) sold. Answer the following questions about Kate's sales.

$$K = \begin{bmatrix} 144 & 192 \\ 115 & 127 \\ 27 & 34 \end{bmatrix}$$

a. What does 127 represent?
b. How many muffins did Kate sell in total?
c. How many bagels did Kate sell in her first location?
d. Which location is doing poorly?

**Solution:**

a. 127 represents the number of muffins that Kate sold in her second location. You know this because it is in the muffin column and the second row.
b. The total muffins sold is equal to the sum of the right hand column. $192+127+34=353$
c. Kate sold 144 bagels at her first location.
d. The third location is doing much worse than the other two locations.

**Example C**

Identify the order of the following matrices

$$A = \begin{bmatrix} 1 & 3 & 4 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 21 & 45 & 1 \\ 34 & 1 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 25 & 235 \\ 562 & 562 \\ 4 & 413 \\ 454 & 33 \\ 1 & 141 \end{bmatrix}$$

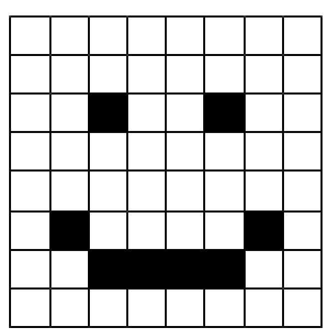
**Solution:** $A$ is $1 \times 4$, $B$ is $2 \times 3$, $C$ is $5 \times 2$. Note that $4 \times 1, 3 \times 2, 2 \times 5$ are not the same orders and would be incorrect.

**Concept Problem Revisited**

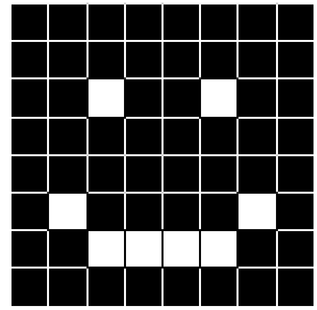
By writing every hollow square as a 0 and a blank square as a 1 a computer could read the picture:

When you use computers to manipulate images, the computer just manipulates the numbers. In this case, if you swap zeros and ones, you get the negative image.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$



Real photos and computer images have matrices that are much larger and include more numbers than just zero and one to account for more colors.

### Vocabulary

A ***matrix*** is a rectangular array of numbers used to represent data in a compact and efficient way.

A ***square matrix*** is a matrix with the same number of rows and columns.

An ***identity matrix*** is a matrix with zeros everywhere except along the diagonal where there are ones.

A ***diagonal matrix*** is a matrix with zeros everywhere except along the diagonal where the numbers can be anything.

The ***transpose of a matrix*** is a new matrix whose columns and rows have been switched. This changes the order of the matrix from, for example, $3 \times 2$ to $2 \times 3$.

A ***triangular matrix*** is described as either ***upper or lower triangular.*** That portion of the matrix is entirely zeros. These kinds of matrices are easier to solve.

A ***symmetric matrix*** is a square matrix with reflection symmetry across the main diagonal.

### Guided Practice

1. Geetha took the SAT three times. The first time she scored 460 on math and 540 on verbal. The second time she scored 540 on math and 620 on verbal. The third time she scored 650 on math and 670 on verbal. Use a matrix to represent Geetha's scores.

2. Write out the $5 \times 4$ matrix whose entries are $a_{ij} = \frac{i+j}{j}$.

3. Create a $3 \times 3$ matrix for each of the following:

a. Diagonal Matrix

b. Lower Triangular

c. Symmetric

d. Identity

**Answers:**

1. Let the rows represent each sitting and the columns represent math and verbal.

$$G = \begin{bmatrix} 460 & 540 \\ 540 & 620 \\ 650 & 670 \end{bmatrix}$$

2.

$$\begin{bmatrix} 2 & \frac{3}{2} & \frac{4}{3} & \frac{5}{4} & \frac{6}{5} \\ 3 & 2 & \frac{5}{3} & \frac{3}{2} & \frac{7}{5} \\ 4 & \frac{5}{2} & 2 & \frac{7}{4} & \frac{8}{5} \\ 5 & 3 & \frac{7}{3} & 2 & \frac{9}{5} \end{bmatrix}$$

3. While the identity matrix does technically work for all the parts of this problem, it does not highlight the differences between each definition. Here are possible answers.

a. Diagonal Matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

b. Lower Triangular

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 3 & 14 \\ 0 & 0 & 5 \end{bmatrix}$$

c. Symmetric

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 14 \\ 1 & 14 & 5 \end{bmatrix}$$

d. Identity

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Practice

State the order of each of the following matrices:

1. $A = \begin{bmatrix} 4 & 2 & 4 & 7 \\ 5 & 2 & 1 & 0 \end{bmatrix}$

2. $B = \begin{bmatrix} 0 & 1 \\ 34 & 1 \end{bmatrix}$

3. $C = \begin{bmatrix} 2 & 62 \\ 14 & 3 \\ 4 & 3 \\ 1 & 11 \end{bmatrix}$

4. $D = \begin{bmatrix} 12 & 0 & 2 \\ 0 & 3 & 3 \\ 4 & 0 & 1 \\ 1 & 4 & 0 \end{bmatrix}$

5. $E = \begin{bmatrix} 1 & 11 \end{bmatrix}$

6. Give an example of a $1 \times 1$ matrix.

7. Give an example of a $3 \times 2$ matrix.

8. If a symmetric matrix is also lower triangular, what type of matrix is it?

9. Write out the $2 \times 3$ matrix whose entries are $a_{ij} = i - j$.

Morgan worked for three weeks during the summer earning money on Mondays, Tuesdays, Wednesdays, Thursdays, and Fridays. The following matrix represents his earnings.

$$\begin{bmatrix} 24 & 22 & 32 \\ 25 & 28 & 30 \\ 30 & 28 & 32 \\ 10 & 15 & 19 \\ 35 & 32 & 30 \end{bmatrix}$$

10. What do the rows and columns represent?

11. How much money did Morgan make in the first week?

12. How much money did Morgan make on Tuesdays?

13. What day of the week was most profitable?

14. What day of the week was least profitable?

15. Is the following a matrix? Explain.

$$\begin{bmatrix} dogs & 0 \\ cats & 3 \\ sheep & 0 \\ ducks & 4 \end{bmatrix}$$

# 8.4 Matrix Algebra

Here you will add, subtract and multiply matrices. As a result you will discover the algebraic properties of matrices.

Algebra refers to your ability to manipulate variables and unknowns based on rules and properties. Matrix algebra is extremely similar to the algebra you already know for numbers with a few important differences. What are these differences?

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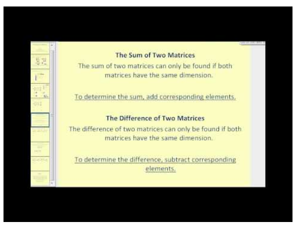

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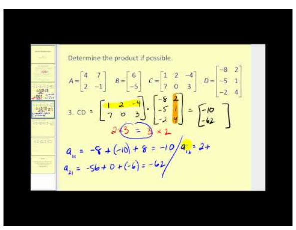
http://www.youtube.com/watch?v=iNty4CSFIpU James Sousa: Matrix Addition, Subtraction, and Scalar Multiplication



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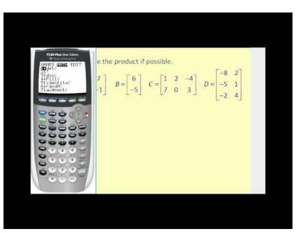
http://www.youtube.com/watch?v=6Hmzu-WKCjc James Sousa: Matrix Multiplication



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http://www.youtube.com/watch?v=C7Dc414qmlk James Sousa: Matrix Multiplication on the Calculator

### Guidance

Two matrices of the same order can be added by summing the entries in the corresponding positions.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 7 \end{bmatrix}$$

Two matrices of the same order can be subtracted by subtracting the entries in the corresponding positions.

$$\begin{bmatrix} 10 & 9 & 8 \\ 7 & 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \end{bmatrix}$$

You can find the product of matrix $A$ and matrix $B$ if the number of columns in matrix $A$ matches the number of rows in matrix $B$. Another way to remember this is when you write the orders of matrix $A$ and matrix $B$ next to each other they must be connected by the same number. The resulting matrix has the number of rows from the first matrix and the number of columns from the second matrix.

$(2 \times 3) \cdot (3 \times 5) = (2 \times 5)$

To compute the first entry of the resulting $2 \times 5$ matrix you should match the first row from the first matrix and the first column of the second matrix. The arithmetic operation to combine these numbers is identical to taking the dot product between two vectors.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 11 & ? & ? & ? & ? \\ 21 & ? & ? & ? & ? \end{bmatrix}$$

- The entry in the first row first column of the new matrix is computed as $1 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$.
- The entry in the second row first column of the new matrix is computed as $5 \cdot 0 + 6 \cdot 2 + 9 \cdot 1 = 21$.
- The rest of the entries of this product are left to Example A.

## Properties of Matrix Algebra

- Commutatively holds for matrix addition. This means that when matrices $A$ and $B$ can be added (when they have matching orders), then: $A + B = B + A$
- Commutatively **does not** hold in general for matrix multiplication.
- Associativity holds for both multiplication and addition. $(AB)C = A(BC), (A+B)+C = A+(B+C)$
- Distribution over addition and subtraction holds. $A(B \pm C) = AB \pm AC$

### Example A

Complete the entries of the matrix multiplication introduced in the guidance section.

$$\begin{bmatrix} 1 & 4 & 3 \\ 5 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 0 & 2 & 1 \\ 1 & 1 & 3 & 0 & 1 \end{bmatrix}$$

**Solution:** Two of the arithmetic operations are shown.

$$C = \begin{bmatrix} 11 & 4 & 12 & 9 & 7 \\ 21 & 14 & 42 & 17 & 15 \end{bmatrix}$$

$$c_{12} = 1 \cdot 1 + 4 \cdot 0 + 3 \cdot 1 = 4$$

$$c_{22} = 5 \cdot 1 + 6 \cdot 0 + 9 \cdot 1 = 14$$

### Example B

Show the commutative property does not hold by demonstrating $AB \neq BA$

$$A=\begin{bmatrix}0 & -1 & 8\\ 1 & 2 & 0\\ 4 & 3 & 12\end{bmatrix}, \quad B=\begin{bmatrix}1 & 5 & 1\\ 2 & 2 & 1\\ 4 & 3 & 0\end{bmatrix}$$

**Solution:**

$$AB=\begin{bmatrix}30 & 22 & -1\\ 5 & 9 & 3\\ 58 & 62 & 7\end{bmatrix}$$
$$BA=\begin{bmatrix}9 & 12 & 20\\ 6 & 5 & 28\\ 3 & 2 & 32\end{bmatrix}$$

**Example C**

Compute the following matrix arithmetic: $10\cdot(2A-3C)\cdot B$.

$A=\begin{bmatrix}1 & 2\\ 4 & 5\end{bmatrix}, B=\begin{bmatrix}0 & 1 & 2\\ 4 & 3 & 2\end{bmatrix}, C=\begin{bmatrix}12 & 0\\ 1 & 3\end{bmatrix}$

**Solution:** When a matrix is multiplied by a scalar (such as with $2A$), multiply each entry in the matrix by the scalar.

$$2A=\begin{bmatrix}2 & 4\\ 8 & 10\end{bmatrix}$$
$$-3C=\begin{bmatrix}-36 & 0\\ -3 & -9\end{bmatrix}$$
$$2A-3C=\begin{bmatrix}-34 & 4\\ 5 & 1\end{bmatrix}$$

Since the associative property holds, you can either distribute the ten or multiply by matrix $B$ next.

$$(2A-3C)\cdot B=\begin{bmatrix}16 & -22 & -60\\ 4 & 8 & 12\end{bmatrix}$$
$$10\cdot(2A-3C)\cdot B=\begin{bmatrix}160 & -220 & -600\\ 40 & 80 & 120\end{bmatrix}$$

**Concept Problem Revisited**

The main difference between matrix algebra and regular algebra with numbers is that matrices do not have the commutative property for multiplication. There are other complexities that matrices have, but many of them stem from the fact that for most matrices $AB \neq BA$.

## Vocabulary

***Matrix operations*** are addition, subtraction and multiplication. Division involves a multiplicative inverse that is not discussed at this point.

### Guided Practice

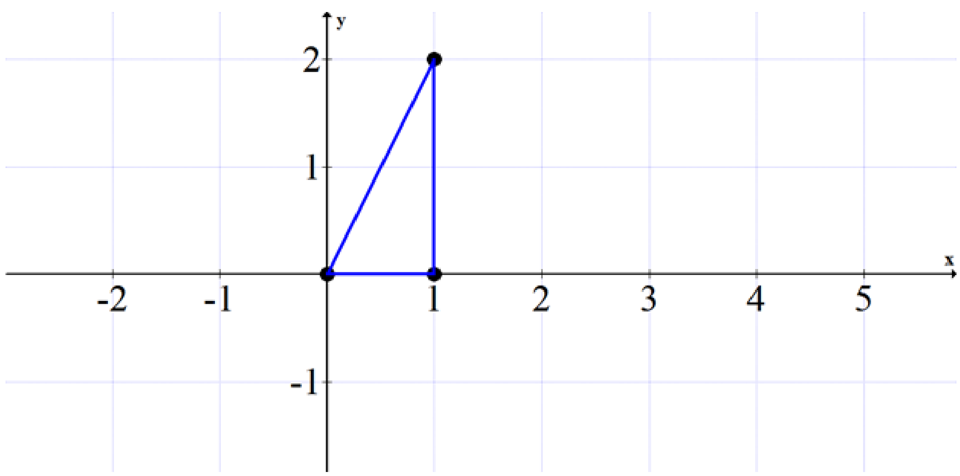
1. Show that a $3 \times 3$ identity matrix works as the multiplicative identity.

2. Use your calculator to input and compute the following matrix operations.

$$A = \begin{bmatrix} 54 & 65 & 12 \\ 235 & 322 & 167 \\ 413 & 512 & 123 \end{bmatrix}, \quad B = \begin{bmatrix} 163 & 212 & 466 \\ 91 & 221 & 184 \\ 42 & 55 & 42 \end{bmatrix}$$
$$A^T \cdot B \cdot A - 100A$$

3. Matrix multiplication can be used as a transformation in the coordinate system. Consider the triangle with coordinates (0, 0) (1, 2) and (1, 0) the following matrix:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix}$$



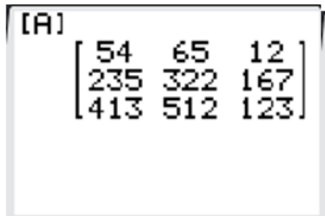

What does the new picture look like?

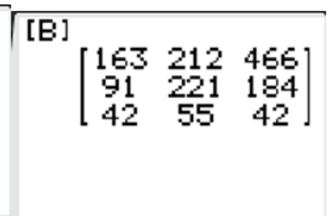
**Answers:**

1. A $3 \times 3$ matrix multiplied by the identity should yield the original matrix.

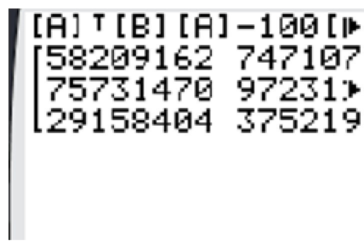
$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
$$a_{11} = a \cdot 1 + b \cdot 0 + c \cdot 0 = a$$
$$a_{12} = a \cdot 0 + b \cdot 1 + c \cdot 0 = b$$
$$\vdots$$

2. Most graphing calculators like the TI-84 can do operations on matrices. Find where you can enter matrices and enter the two matrices.




Then type in the appropriate operation and see the result. The TI-84 has a built in Transpose button.




The actual numbers on this guided practice are less important than the knowledge that your calculator can perform all of the matrix algebra demonstrated in this concept. It is useful to fully know the capabilities of the tools at your disposal, but it should not replace knowing why the calculator does what it does.

3. The matrix simplifies to become:

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

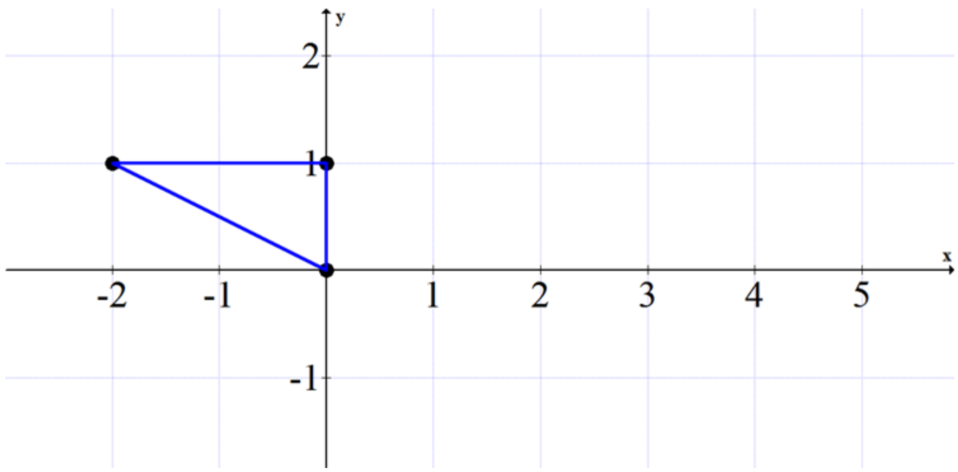
When applied to each point as a transformation, a new point is produced. Note that $\begin{bmatrix} x & y \end{bmatrix}$ is a matrix representing each original point and $\begin{bmatrix} x' & y' \end{bmatrix}$ is the new point. The $x'$ is read as "$x$ prime" and is a common way to refer to a result after a transformation.

$$\begin{bmatrix} x & y \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x' & y' \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$




Notice how the matrix transformation rotates graphs in a counterclockwise direction 90°.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} = \begin{bmatrix} -y & x \end{bmatrix}$$

The matrix transformation applied in the following order will rotate a graph clockwise 90°.

$$\begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90 & \cos 90 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$$

## Practice

Do #1-#11 without your calculator.

$$A = \begin{bmatrix} 2 & 7 \\ 3 & 8 \end{bmatrix}, B = \begin{bmatrix} 0 & 5 & 1 \\ 3 & 4 & 6 \end{bmatrix}, C = \begin{bmatrix} 14 & 6 \\ 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 5 & 0 \\ 1 & 2 \end{bmatrix}$$

1. Find $AC$. If not possible, explain.
2. Find $BA$. If not possible, explain.
3. Find $CA$. If not possible, explain.
4. Find $4B^T$. If not possible, explain.
5. Find $A+C$. If not possible, explain.
6. Find $D-A$. If not possible, explain.
7. Find $2(A+C-D)$. If not possible, explain.
8. Find $(A+C)B$. If not possible, explain.
9. Find $B(A+C)$. If not possible, explain.
10. Show that $A(C+D) = AC + AD$.
11. Show that $A(C-D) = AC - AD$.

Practice using your calculator for #12-#15.

$$E = \begin{bmatrix} 312 & 59 & 34 \\ 342 & 156 & 189 \\ 783 & 23 & 133 \end{bmatrix}, F = \begin{bmatrix} 33 & 72 & 21 \\ 93 & 41 & 94 \\ 62 & 75 & 72 \end{bmatrix}, G = \begin{bmatrix} 11 & 735 & 67 \\ 93 & 456 & 2 \\ 94 & 34 & 0 \end{bmatrix}$$

12. Find $E+F+G$.
13. Find $2E$.

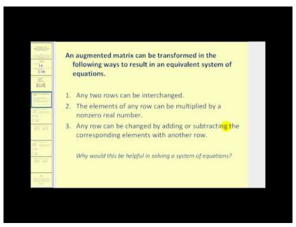
14. Find $4F$.
15. Find $(E+F)G$.

# 8.5 Row Operations and Row Echelon Forms

Here you will manipulate matrices using row operations into row echelon form and reduced row echelon form.

Applying row operations to reduce a matrix is a procedural skill that takes lots of writing, rewriting and careful arithmetic. The payoff for being able to transform a matrix into a simplified form will become clear later. For now, what does the simplified form mean for a matrix?

### Watch This

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Click image to the left for more content.

http://www.youtube.com/watch?v=LsnOlNjWWug James Sousa: Introduction to Augmented Matrices

### Guidance

There are only three operations that are permitted to act on matrices. They are the exact same operations that are permitted when solving a system of equations.

1. Add a multiple of one row to another row.
2. Scale a row by multiplying through by a non-zero constant.
3. Swap two rows.

Using these three operations, your job is to simplify matrices into **row echelon form.** Row echelon form must meet three requirements.

1. The leading coefficient of each row must be a one.
2. All entries in a column below a leading one must be zero.
3. All rows that just contain zeros are at the bottom of the matrix.

Here are some examples of matrices in row echelon form:

$$\begin{bmatrix} 1 & 14 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 5 & 6 \\ 0 & 0 & 1 & 4 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Reduced row echelon form** also has one extra stipulation compared with row echelon form.

4. Every leading coefficient of 1 must be the only non-zero element in that column.

Here are some examples of matrices in reduced row echelon form:

$$\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}1&0&3\\0&1&4\end{bmatrix},\begin{bmatrix}1&2&0&0&6\\0&0&1&0&7\\0&0&0&1&-2\\0&0&0&0&0\end{bmatrix}$$

Putting a matrix into reduced row echelon form is a result of performing **Gauss-Jordan elimination.** The process illustrated in this concept is named after those mathematicians.

### Example A

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix}3&7\\2&5\end{bmatrix}$$

**Solution:** In each step of the solution, only one of the three row operations will be used. Specific shorthand will be introduced.

$$\begin{bmatrix}3&7\\2&5\end{bmatrix}\begin{array}{ccc}&\rightarrow&\\ \rightarrow&\cdot 3&\rightarrow\end{array}\begin{bmatrix}3&7\\6&15\end{bmatrix}\begin{array}{ccc}&\rightarrow&\\ \rightarrow&-2\cdot I&\rightarrow\end{array}\begin{bmatrix}3&7\\0&1\end{bmatrix}$$

Note that the $\cdot 3$ in between the first two matrices indicates that the second row is scaled by a factor of 3. The $-2\cdot I$ between the next two matrices indicates that the second row has two times the first row subtracted from it. The $I$ is a roman numeral referring to the row number.

$$\begin{bmatrix}3&7\\0&1\end{bmatrix}\begin{array}{ccc}\rightarrow&-7II&\rightarrow\\ &\rightarrow&\end{array}\begin{bmatrix}3&0\\0&1\end{bmatrix}\begin{array}{ccc}\rightarrow&\cdot\frac{1}{3}&\rightarrow\\ &\rightarrow&\end{array}\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Row reducing a $2\times 2$ matrix to become the identity matrix is an exercise that illustrates the fact that the rows were linearly independent.

### Example B

Put the following matrix into reduced row echelon form.

$$\begin{bmatrix}2&4&0\\0&3&1\\1&2&4\end{bmatrix}$$

**Solution:**

$$\begin{bmatrix}2&4&0\\0&3&1\\1&2&4\end{bmatrix}\begin{array}{ccc}&\rightarrow&\\&\rightarrow&\\ \rightarrow&-\frac{I}{2}&\rightarrow\end{array}\begin{bmatrix}2&4&0\\0&3&1\\0&0&4\end{bmatrix}\begin{array}{ccc}&\rightarrow&\\&\rightarrow&\\ \rightarrow&\div 4&\rightarrow\end{array}\begin{bmatrix}2&4&0\\0&3&1\\0&0&1\end{bmatrix}\begin{array}{ccc}\rightarrow&\div 2&\rightarrow\\ \rightarrow&\div 3&\rightarrow\\&\rightarrow&\end{array}\begin{bmatrix}1&2&0\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix}$$

Note that in the preceding step, two operations were used. This is acceptable when the operations do not interfere or interact with each other.

$$\begin{bmatrix}1&2&0\\0&1&\frac{1}{3}\\0&0&1\end{bmatrix}\begin{array}{ccc}&\rightarrow&\\ \rightarrow&-\frac{III}{3}&\rightarrow\\&\rightarrow&\end{array}\begin{bmatrix}1&2&0\\0&1&0\\0&0&1\end{bmatrix}\begin{array}{ccc}\rightarrow&-2II&\rightarrow\\&\rightarrow&\\&\rightarrow&\end{array}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Again, row reducing a $3\times 3$ matrix to become the identity matrix is just an exercise that illustrates the fact that the rows were linearly independent.

### Example C

In a single $3\times 3$ matrix, describe the general approach of Gauss-Jordan elimination. In other words, which locations would you try to focus on first?

**Solution:** One approach is to try to get a one in the A position. Then get a zero in position B and position C by multiplying by a multiple of row 1. Then try to get a zero in position D.

$$\begin{bmatrix}A & I & G\\ B & H & F\\ C & D & E\end{bmatrix}$$

Every matrix may have a different strategy and as long as you use the three row operations, you will be on the right track. One thing to be very careful of is to try to avoid fractions within your matrix. Scale the row to eliminate the fraction.

**Concept Problem Revisited**

There are two forms of a matrix that are most simplified. The most important is reduced row echelon form that follows the four stipulations from the guidance section. An example of a matrix in reduced row echelon form is:

$$\begin{bmatrix}1 & 0 & 0 & 2 & 43\\ 0 & 1 & 0 & 2 & 3\\ 0 & 0 & 1 & 98 & 5\end{bmatrix}$$

### Vocabulary

***Row operations*** are swapping rows, adding a multiple of one row to another or scaling a row by multiplying through by a scalar.

***Row echelon form*** is a matrix that has a leading one at the start of every non-zero row, zeros below every leading one and all rows containing only zeros at the bottom of the matrix.

***Reduced row echelon form*** is the same as row echelon form with one additional stipulation: that every other entry in a column with a leading one must be zero.

### Guided Practice

1. Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix}0 & 4 & 5\\ 2 & 6 & 8\end{bmatrix}$$

2. Reduce the following matrix to row echelon form.

$$\begin{bmatrix}3 & 6\\ 2 & 4\\ 5 & 17\end{bmatrix}$$

3. Reduce the following matrix to reduced row echelon form.

$$\begin{bmatrix}3 & 4 & 1 & 0\\ 5 & -1 & 0 & 1\end{bmatrix}$$

**Answers:**

1.

$$\begin{bmatrix}0 & 4 & 5\\ 2 & 6 & 8\end{bmatrix}\begin{matrix}\rightarrow & II & \rightarrow\\ \rightarrow & I & \rightarrow\end{matrix}\begin{bmatrix}2 & 6 & 8\\ 0 & 4 & 5\end{bmatrix}\begin{matrix}\rightarrow & \div 2 & \rightarrow\\ & \rightarrow & \end{matrix}\begin{bmatrix}1 & 3 & 4\\ 0 & 4 & 5\end{bmatrix}$$

$$\begin{matrix} & \rightarrow & \\ \rightarrow & -I & \rightarrow\end{matrix}\begin{bmatrix}1 & 3 & 4\\ 0 & 1 & 1\end{bmatrix}\begin{matrix}\rightarrow & -3II & \rightarrow\\ & \rightarrow & \end{matrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}$$

2. $$\begin{bmatrix}3 & 6\\ 2 & 4\\ 5 & 17\end{bmatrix}\begin{matrix}\rightarrow & \div 3 & \rightarrow\\ \rightarrow & \div 2 & \rightarrow\\ & \rightarrow & \end{matrix}\begin{bmatrix}1 & 2\\ 1 & 2\\ 5 & 17\end{bmatrix}\begin{matrix} & \rightarrow & \\ \rightarrow & -I & \rightarrow\\ \rightarrow & -5I & \rightarrow\end{matrix}\begin{bmatrix}1 & 2\\ 0 & 0\\ 0 & 7\end{bmatrix}\begin{matrix} & \rightarrow & \\ \rightarrow & III & \rightarrow\\ \rightarrow & II & \rightarrow\end{matrix}\begin{bmatrix}1 & 2\\ 0 & 7\\ 0 & 0\end{bmatrix}\begin{matrix} & \rightarrow & \\ \rightarrow & \div 7 & \rightarrow\\ & \rightarrow & \end{matrix}\begin{bmatrix}1 & 2\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

3.

$$\begin{bmatrix} 3 & 4 & 1 & 0 \\ 5 & -1 & 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow & \cdot 5 & \rightarrow \\ \rightarrow & \cdot 3 & \rightarrow \end{matrix} \begin{bmatrix} 15 & 20 & 5 & 0 & \\ & 15 & -3 & 0 & 3 \end{bmatrix} \begin{matrix} & \rightarrow & \\ \rightarrow & -I & \rightarrow \end{matrix} \begin{bmatrix} 15 & 20 & 5 & 0 \\ 0 & -23 & -5 & 3 \end{bmatrix}$$

$$\begin{matrix} \rightarrow & \cdot 23 & \rightarrow \\ \rightarrow & \cdot 20 & \rightarrow \end{matrix} \begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix}$$

$$\begin{bmatrix} 345 & 460 & 115 & 0 \\ 0 & -460 & -115 & 60 \end{bmatrix} \begin{matrix} \rightarrow & +II & \rightarrow \\ & \rightarrow & \end{matrix} \begin{bmatrix} 345 & 0 & 0 & 60 \\ 0 & -460 & -115 & 60 \end{bmatrix}$$

$$\begin{matrix} \rightarrow & \div 345 & \rightarrow \\ \rightarrow & \div -460 & \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \frac{60}{345} \\ 0 & 1 & \frac{115}{460} & -\frac{60}{460} \end{bmatrix}$$

Notice how fractions were avoided until the final step. Adding and subtracting large numbers in a matrix is easier to handle than adding and subtracting small numbers because then you don't need to find a common denominator.

### Practice

1. Give an example of a matrix in row echelon form.
2. Give an example of a matrix in reduced row echelon form.
3. What are the three row operations you are allowed to perform when reducing a matrix?
4. If a square matrix reduces to the identity matrix, what does that mean about the rows of the original matrix?

Use the following matrix for 5-6.

$$A = \begin{bmatrix} -3 & -4 & -12 \\ 4 & 4 & 12 \\ -11 & -12 & -35 \end{bmatrix}$$

5. Reduce matrix $A$ to row echelon form.
6. Reduce matrix $A$ to reduced row echelon form. Are the rows of matrix $A$ linearly independent?

Use the following matrix for 7-8.

$$B = \begin{bmatrix} 3 & -4 & 8 \\ 9 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

7. Reduce matrix $B$ to row echelon form.
8. Reduce matrix $B$ to reduced row echelon form. Are the rows of matrix $B$ linearly independent?

Use the following matrix for 9-10.

$$C = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 3 & 6 & -3 & 1 \\ 6 & 12 & -7 & 0 \end{bmatrix}$$

9. Reduce matrix $C$ to row echelon form.
10. Reduce matrix $C$ to reduced row echelon form. Are the rows of matrix $C$ linearly independent?

Use the following matrix for 11-12.

$$D = \begin{bmatrix} 1 & 1 \\ 3 & 4 \\ 2 & 3 \end{bmatrix}$$

11. Reduce matrix $D$ to row echelon form.

12. Reduce matrix $D$ to reduced row echelon form. Are the rows of matrix $D$ linearly independent?

Use the following matrix for 13-14.

$$E = \begin{bmatrix} -5 & -6 & -12 \\ -1 & -1 & -2 \\ 2 & 2 & 4 \end{bmatrix}$$

13. Reduce matrix $E$ to row echelon form.

14. Reduce matrix $E$ to reduced row echelon form. Are the rows of matrix $E$ linearly independent?

Use the following matrix for 15-16.

$$F = \begin{bmatrix} -23 & 6 & 3 \\ 2 & -\frac{1}{2} & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

15. Reduce matrix $F$ to row echelon form.

16. Reduce matrix $F$ to reduced row echelon form. Are the rows of matrix $F$ linearly independent?

# 8.6 Augmented Matrices

Here you will solve systems of equations using augmented matrices.

The reason why the rules for row reducing matrices are the same as the rules for eliminating coefficients when solving a system of equations is because you are essentially doing the same thing in each case. When you write and rewrite the equation every time you end up writing down lots of extra information. Matrices take care of this information by embedding it in the location of each entry. How would you use matrices to write the following system of equations?

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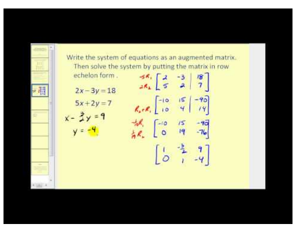

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http://www.youtube.com/watch?v=BWBckWPjfpw James Sousa: Augmented Matrices: Row Echelon Form

### Guidance

In order to represent a system as a matrix equation, first write all the equations in standard form so that the coefficients of the variables line up in columns. Then copy down just the coefficients in a matrix array. Next copy the variables in a variable matrix and the constants into a constant matrix.

$$\begin{aligned} x+y+z&=9\\ x+2y+3z&=22\\ 2x+3y+4z&=31 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 9\\ 22\\ 31 \end{bmatrix}$$

The reason why this works is because of the way matrix multiplication is defined.

$$\begin{bmatrix} 1 & 1 & 1\\ 1 & 2 & 3\\ 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} x\\ y\\ z \end{bmatrix} = \begin{bmatrix} 1x+1y+1z\\ 1x+2y+3z\\ 2z+3y+4z \end{bmatrix} = \begin{bmatrix} 9\\ 22\\ 31 \end{bmatrix}$$

Notice how putting brackets around the two matrices on the right does very little to hide the fact that this is just a regular system of 3 equations and 3 variables.

Once you have your system represented as a matrix you can solve it using an augmented matrix. An augmented matrix is two matrices that are joined together and operated on as if they were a single matrix. In the case of solving a system, you need to augment the coefficient matrix and the constant matrix. The vertical line indicates the separation between the coefficient matrix and the constant matrix.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right]$$

Reduce the matrix to reduced row echelon form and you will find the solution to the system, if one exists.

**Example A**

Attempt to solve the system from the guidance section.

**Solution:**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 1 & 2 & 3 & 22 \\ 2 & 3 & 4 & 31 \end{array}\right] \begin{array}{ccc} & \rightarrow & \\ \rightarrow & -I & \rightarrow \\ \rightarrow & -2I & \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 1 & 2 & 13 \end{array}\right] \begin{array}{ccc} & \rightarrow & \\ & \rightarrow & \\ \rightarrow & -II & \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 13 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is dependent which means there are an infinite number of solutions.

**Example B**

Solve the following system using an augmented matrix.

$$\begin{aligned} x+y+z &= 6 \\ x-y-z &= -4 \\ x+2y+3z &= 14 \end{aligned}$$

**Solution:**

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & -4 \\ 1 & 2 & 3 & 14 \end{array}\right] \begin{array}{ccc} & \rightarrow & \\ \rightarrow & -I & \\ \rightarrow & -I & \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -2 & -10 \\ 0 & 1 & 2 & 8 \end{array}\right] \begin{array}{ccc} \rightarrow & -III & \rightarrow \\ \rightarrow & +3III & \rightarrow \\ & \rightarrow & \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 1 & 2 & 8 \end{array}\right] \begin{array}{ccc} & \rightarrow & \\ & \rightarrow & \\ \rightarrow & -II & \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & -2 & -6 \end{array}\right] \begin{array}{ccc} & \rightarrow & \\ & \rightarrow & \\ \rightarrow & \div -2 & \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{cc} \rightarrow & +III \\ \rightarrow & -4III \\ & \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Every matrix can be interpreted as its own linear system. The final augmented matrix can be interpreted as:

$$\begin{aligned} 1x+0y+0z &= 1 \\ 0x+1y+0z &= 2 \\ 0x+0y+1z &= 3 \end{aligned}$$

Which means $x=1, y=2, z=3$.

**Example C**

Solve the following system using augmented Matrices.

$$\begin{aligned} w+x+z&=11\\ w+x&=9\\ x+y&=7\\ y+z&=5 \end{aligned}$$

**Solution:** While substitution would work in this problem, the idea is to demonstrate how augmented matrices will work even with larger matrices.

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11\\ 1 & 1 & 0 & 0 & 9\\ 0 & 1 & 1 & 0 & 7\\ 0 & 0 & 1 & 1 & 5 \end{array}\right] \begin{array}{c} \rightarrow\\ \rightarrow IV\\ \rightarrow II \rightarrow\\ \rightarrow III \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11\\ 0 & 1 & 1 & 0 & 7\\ 0 & 0 & 1 & 1 & 5\\ 1 & 1 & 0 & 0 & 9 \end{array}\right] \begin{array}{c} \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow -I \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11\\ 0 & 1 & 1 & 0 & 7\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & -1 & -2 \end{array}\right] \begin{array}{c} \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow \cdot(-1) \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 11\\ 0 & 1 & 1 & 0 & 7\\ 0 & 0 & 1 & 1 & 5\\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \rightarrow -IV \rightarrow\\ \rightarrow\\ \rightarrow -IV \rightarrow\\ \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9\\ 0 & 1 & 1 & 0 & 7\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \rightarrow\\ \rightarrow -III \rightarrow\\ \rightarrow\\ \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 9\\ 0 & 1 & 0 & 0 & 4\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{c} \rightarrow -II \rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5\\ 0 & 1 & 0 & 0 & 4\\ 0 & 0 & 1 & 0 & 3\\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$w=5, x=4, y=3, z=2$

**Concept Problem Revisited**

If you were to write the system as a matrix system you could write:

$$\begin{aligned} 5x+y&=6\\ x+y&=10 \end{aligned}$$

$$\begin{bmatrix} 5 & 1\\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x\\ y \end{bmatrix} = \begin{bmatrix} 6\\ 10 \end{bmatrix}$$

### Vocabulary

An ***augmented matrix*** is a matrix formed when two matrices are joined together and operated on as if they were a single matrix.

### Guided Practice

1. Use an augmented matrix to solve the following system.

$$\begin{aligned} 2x+y+z&=16\\ 2y+6z&=0\\ x+y&=10 \end{aligned}$$

2. Use an augmented matrix to solve the following system.

$$3x + y = -15$$
$$x + 2y = 15$$

3. Use an augmented matrix to solve the following system.

$$-a + b - c = 0$$
$$2a - 2b - 3c = 25$$
$$3a - 4b + 3c = 2$$

**Answers:**

1. The row reduction steps are not shown. Only the initial and final augmented matrices are shown.

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 16 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & 0 & 10 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

2. $\left[\begin{array}{cc|c} 3 & 1 & -15 \\ 1 & 2 & 15 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -9 \\ 0 & 1 & 12 \end{array}\right]$

3. $\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 2 & -2 & -3 & 25 \\ 3 & -4 & 3 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right]$

## Practice

Solve the following systems of equations using augmented matrices. If one solution does not exist, explain why not.

1.

$$4x - 2y = -20$$
$$x - 3y = -15$$

2.

$$3x + 5y = 33$$
$$-x - 2y = -13$$

3.

$$x + 4y = 11$$
$$3x + 12y = 33$$

4.

$$-3x+y=-7$$
$$-x+4y=5$$

5.

$$3x+y=6$$
$$-6x-2y=10$$

6.

$$2x-y+z=4$$
$$4x+7y-z=38$$
$$-x+3y+2z=23$$

7.

$$4x+y-z=-16$$
$$-3x+4y+z=18$$
$$x+y-3z=-17$$

8.

$$3x+2y-3z=7$$
$$-x+5y+2z=29$$
$$x+2y+z=15$$

9.

$$2x+y-2z=4$$
$$-4x-2y+4z=-8$$
$$3x+y-z=5$$

10.

$$-x+3y+z=11$$
$$3x+y+2z=27$$
$$5x-y-z=5$$

11.

$$\begin{aligned} 3x+2y+4z &= 21 \\ -2x+3y+z &= -11 \\ x+2y-3z &= -3 \end{aligned}$$

12.

$$\begin{aligned} -x+2y-6z &= 4 \\ 8x+5y+3z &= -8 \\ 2x-4y+12z &= 5 \end{aligned}$$

13.

$$\begin{aligned} 3x+5y+8z &= 37 \\ -6x+3y+z &= 42 \\ x+3y-2z &= 5 \end{aligned}$$

14.

$$\begin{aligned} 4x+y-6z &= -38 \\ 2x+7y+8z &= 108 \\ -3x+2y-3z &= -15 \end{aligned}$$

15.

$$\begin{aligned} 6x+3y-2z &= -22 \\ -4x-2y+4z &= 28 \\ 3x+3y+2z &= 7 \end{aligned}$$

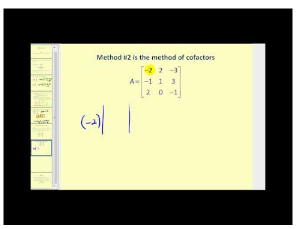
# 8.7 Determinant of Matrices

Here you will find the determinants of $2 \times 2$ and higher order matrices.

A**determinant** is a number computed from the entries in a square matrix. It has many properties and interpretations that you will explore in linear algebra. This concept is focused on the procedure of calculating determinants. Once you know how to calculate the determinant of a$2 \times 2$ matrix, then you will be able to calculate the determinant of a$3 \times 3$matrix. Once you know how to calculate the determinant of a$3 \times 3$ matrix you can calculate the determinant of a$4 \times 4$ and so on.

A logical question about determinants is where does the procedure come from? Why are determinants defined in the way that they are?

### Watch This

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http://www.youtube.com/watch?v=OI07C1HsOuc James Sousa: Determinants

### Guidance

The determinant of a matrix $A$ is written as $|A|$. For a $2 \times 2$ matrix $A$, the value is calculated as:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Notice how the diagonals are multiplied and then subtracted.

The determinant of a $3 \times 3$ matrix is more involved.

$$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Usually you will start by looking at the top row, although any row or column will work. Then use the checkerboard pattern for signs (shown below) and create smaller $2 \times 2$ matrices.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The smaller $2\times 2$ matrices are the entries that remain when the row and column of the coefficient you are working with are ignored.

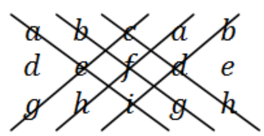
$\det B = |B| = +a\cdot\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\cdot\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\cdot\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Next take the determinant of the smaller $2\times 2$ matrices and you get a long string of computations.

$$\begin{aligned} &= +a(ei-fh)-b(di-fg)+c(dh-eg) \\ &= aei-afh-bdi+bfg+cdh-ceg \\ &= aei+bfg+cdh-ceg-afh-bdi \end{aligned}$$

Most people do not remember this sequence. A French mathematician named Sarrus demonstrated a great device to memorize the computation of the determinant for $3\times 3$ matrices. The first step is simply to copy the first two columns to the right of the matrix. Then draw three diagonal lines going down and to the right.

$B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$




*Notice that they correspond exactly to the three positive terms of the determinant demonstrated above.* Next draw three diagonals going up and to the right. *These diagonals correspond exactly to the three negative terms.*

$\det B = aei+bfg+cdh-ceg-afh-bdi$

Sarrus's rule does not work for the determinants of matrices that are not of order $3\times 3$.

**Example A**

Find $\det A$ for $A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$

**Solution:** $\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 3\cdot 5 - 2\cdot 1 = 15-2 = 13$

**Example B**

Find $\det B$ for $B = \begin{bmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{bmatrix}$

**Solution:**

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 1 \\ 5 & 0 & 2 \\ 2 & 1 & 5 \end{vmatrix} &= 3\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} 5 & 2 \\ 2 & 5 \end{vmatrix} + 1\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix} \\ &= 3(0\cdot 5-2\cdot 1)-2(5\cdot 5-2\cdot 2)+1(5\cdot 1-2\cdot 0) \\ &= -6-42+5 = -43 \end{aligned}$$

**Example C**

Find the determinant of $B$ from example B using Sarrus's Rule.

**Solution:**

$$\begin{matrix} 3 & 2 & 1 & 3 & 2 \\ 5 & 0 & 2 & 5 & 0 \\ 2 & 1 & 5 & 2 & 1 \end{matrix}$$

**det** $B = 0+8+5-0-6-50 = -43$

As you can see, Sarrus's Rule is efficient and much of the calculations can be done mentally. Additionally, zero values make much of the multiplication easier.

**Concept Problem Revisited**

Determinants for $2 \times 2$ matrices are defined the way they are because of the general solution to a system of 2 variables and 2 equations.

$$\begin{aligned} ax+by &= e \\ cx+dy &= f \end{aligned}$$

To eliminate the $x$, scale the first equation by $c$ and the second equation by a.

$$\begin{aligned} acx+bcy &= ec \\ acx+ady &= af \end{aligned}$$

Subtract the second equation from the first and solve for $y$.

$$\begin{aligned} ady-bcy &= af-ec \\ y(ad-bc) &= af-ec \\ y &= \frac{af-ec}{ad-bc} \end{aligned}$$

When you solve for $x$ you also get $ad-bc$ in the denominator of the general solution. This pattern led people to start using this strategy in solving systems of equations. The determinant is defined in this way so it will always be the denominator of the general solution of either variable.

## Vocabulary

The ***determinant*** of a matrix is a number calculated from the entries in a matrix. The procedure is derived from solving linear systems.

***Sarrus's rule*** is a memorization technique that enables you to compute the determinant of $3 \times 3$ matrices efficiently.

## Guided Practice

1. Find the determinant of the following matrix.

$$C = \begin{bmatrix} -4 & 12 \\ 1 & -3 \end{bmatrix}$$

2. Find the determinant of the following matrix.

$$D = \begin{bmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{bmatrix}$$

3. Find the determinant of the following $4 \times 4$ matrix by carefully choosing the row or column to work with.

$$E = \begin{bmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{bmatrix}$$

**Answers:**

1. $\det C = \begin{vmatrix} -4 & 12 \\ 1 & -3 \end{vmatrix} = 12 - 12 = 0$

2. $\det D = \begin{vmatrix} 4 & 8 & 3 \\ 0 & 1 & 7 \\ 12 & 5 & 13 \end{vmatrix} = 4 \cdot 13 + 8 \cdot 7 \cdot 12 + 0 - 36 - 5 \cdot 7 \cdot 4 - 0 = 548$

3. Notice that the third column is made up with zeros and a one. Choose this column to make up the coefficients because then instead of having to evaluate the determinant of four individual $3 \times 3$ matrices, you only need to do one.

$$\begin{vmatrix} 4 & 5 & 0 & 2 \\ -1 & -3 & 0 & 3 \\ 4 & 8 & 1 & 5 \\ -3 & 2 & 0 & 9 \end{vmatrix} = 0 \cdot \begin{vmatrix} -1 & -3 & 3 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ 4 & 8 & 5 \\ -3 & 2 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix} - 0 \cdot \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ 4 & 8 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 5 & 2 \\ -1 & -3 & 3 \\ -3 & 2 & 9 \end{vmatrix}$$

$$= 4 \cdot (-3) \cdot 9 + 5 \cdot 3 \cdot (-3) + 2 \cdot (-1) \cdot 2 - 18 - 24 - (-45)$$

$$= -154$$

## Practice

Find the determinants of each of the following matrices.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. $\begin{bmatrix} 2 & 6 & 4 & 6 \\ 0 & 1 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ -6 & 2 & 3 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 5 & 0 & 0 & 1 \\ 2 & 1 & 8 & 3 \\ 9 & 3 & 2 & 6 \\ -4 & 2 & 5 & 1 \end{bmatrix}$

15. Can you find the determinant for any matrix? Explain.

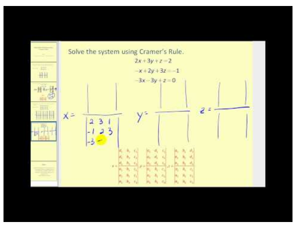
16. The following matrix has a determinant of zero: $\begin{bmatrix} 6 & 4 \\ 3 & 2 \end{bmatrix}$. If the determinant of a matrix is zero, what does that say about the rows of the matrix?

# 8.8 Cramer's Rule

Here you will solve systems of equations using Cramer's Rule.

A system of equations can be represented and solved in general using matrices and determinants. This method can be significantly more efficient than eliminating variables in equations. What does it mean for a solution method to be more efficient? Is Cramer's Rule the most efficient means of solving a system of equations?

## Watch This



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Click image to the left for more content.

http://www.youtube.com/watch?v=ItxF3IjC_uw James Sousa: Cramer's Rule

## Guidance

The determinant is defined in a seemingly arbitrary way; however, when you look at the general solution for a $2 \times 2$ matrix, the reasoning why it is defined this way is apparent.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

When you solve the system above for $y$ and $x$, you get the following:

$$\begin{aligned} y &= \frac{af - ce}{ad - bc} \\ x &= \frac{bf - de}{ad - bc} \end{aligned}$$

Note that the system can be represented by the matrix and the solutions can be written as ratios of two determinants. The determinant in the denominator is of the coefficient matrix. The numerator of the $x$ solution is the determinant of the new matrix whose columns are made up of the $y$ coefficients and the solution coefficients. The numerator of the $y$ solution is the determinant of the new matrix made up of the $x$ coefficients and the solution coefficients.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

This is a fantastic improvement over solving systems using substitution or elimination. Cramer's Rule also works with larger order matrices. For a system of 3 variables and 3 equations the reasoning is identical.

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = l$$

The system can be represented as a matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

The three solutions can be represented as a ratio of determinants.

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}}$$

Remember that evaluating the determinants of $3 \times 3$ matrices using Sarrus's rule is very efficient.

**Example A**

Represent the following system of equations as a matrix equation.

$$\begin{aligned} y-13 &= -3x \\ x &= 19-4y \end{aligned}$$

**Solution:** First write each equation in standard form.

$$\begin{aligned} 3x+y &= 13 \\ x+4y &= 19 \end{aligned}$$

Then write as a coefficient matrix times a variable matrix equal to a solution matrix.

$\begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 19 \end{bmatrix}$

**Example B**

Solve the system from Example A using Cramer's Rule.

**Solution:**

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{\begin{vmatrix} 13 & 1 \\ 19 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{13 \cdot 4 - 19 \cdot 1}{3 \cdot 4 - 1 \cdot 1} = \frac{33}{11} = 3$$

$$y = \frac{\begin{vmatrix} 3 & 13 \\ 1 & 19 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix}} = \frac{3 \cdot 19 - 13}{11} = \frac{44}{11} = 4$$

**Example C**

What is $y$ equal to in the following system?

$$\begin{aligned} x+2y-z &= 0 \\ 7x-0y+z &= 14 \\ 0x+y+z &= 10 \end{aligned}$$

**Solution:** If you attempted to solve this using elimination, it would take over a page of writing and rewriting to solve. Cramer's Rule speeds up the solving process.

$\begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 14 \\ 10 \end{bmatrix}$

$$y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & -1 \\ 7 & 14 & 1 \\ 0 & 10 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -1 \\ 7 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{14+0+(-70)-0-10-0}{0+0+(-7)-0-1-14} = \frac{-66}{-22} = 3$$

**Concept Problem Revisited**

Example C reminds you of the fact that a problem done with traditional coefficient elimination can take over a page of writing and rewriting. Efficiency partly means requiring less time and space. If this was all that efficiency meant then it would not make sense to solve systems of two equations with two unknowns using matrices because the solution could be found more quickly using substitution. However, the other part of efficiency is minimizing the number of decisions that have to be made. A computer is very good at adding, subtracting and multiplying numbers, but not very good at deciding whether eliminating $x$ or eliminating $y$ would be better. This is why a definite algorithm using matrices and Cramer's Rule is more efficient.

### Vocabulary

A ***matrix equation*** represents a system of equations by multiplying a coefficient matrix and a variable matrix to get a solution matrix.

### Guided Practice

1. Solve the following system using Cramer's Rule.

$$\begin{aligned} 5x + 12y &= 72 \\ 18x - 12y &= 108 \end{aligned}$$

2. Solve the following system using Cramer's Rule and your calculator.

$$\begin{aligned} 70x + 21y &= -112 \\ 27x - 21y &= 15 \end{aligned}$$

3. What is the value of $z$ in the following system?

$$\begin{aligned} 3x + 2y + z &= 7 \\ 4x + 0y + z &= 6 \\ 6x - y + 0z &= 5 \end{aligned}$$

**Answers:**

1.

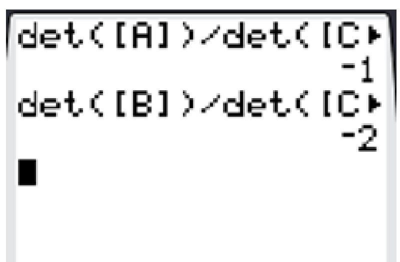
$$x = \frac{\begin{vmatrix} 72 & 12 \\ 108 & -12 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{72 \cdot (-12) - 12 \cdot 108}{5 \cdot (-12) - 12 \cdot 18} = \frac{-2160}{276} = \frac{180}{23}$$

$$y = \frac{\begin{vmatrix} 5 & 72 \\ 18 & 108 \end{vmatrix}}{\begin{vmatrix} 5 & 12 \\ 18 & -12 \end{vmatrix}} = \frac{5 \cdot 108 - 72 \cdot 18}{276} = \frac{-756}{276} = -\frac{63}{23}$$

2. Input the following three matrices into your calculator. Matrix $A$ has columns that are the constants and the $y$ coefficients. Matrix $B$ has columns that are $x$ coefficients and the constants. Matrix $C$ is just the coefficient matrix.

$$A = \begin{bmatrix} -112 & 21 \\ 15 & -21 \end{bmatrix}$$
$$B = \begin{bmatrix} 70 & -112 \\ 27 & 15 \end{bmatrix}$$
$$C = \begin{bmatrix} 70 & 21 \\ 27 & -21 \end{bmatrix}$$

Then compute $x = \frac{\det A}{\det C}$ and $y = \frac{\det B}{\det C}$




The solution is $x = -1, y = -2$

3.

$$3x + 2y + z = 7$$
$$4x + 0y + z = 6$$
$$6x - y + 0z = 5$$

$$z = \frac{\begin{vmatrix} 3 & 2 & 7 \\ 4 & 0 & 6 \\ 6 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & 1 \\ 4 & 0 & 1 \\ 6 & -1 & 0 \end{vmatrix}} = \frac{0 + 2 \cdot 6 \cdot 6 + 7 \cdot 4 \cdot (-1) - 0 - (-1) \cdot 6 \cdot 3 - 5 \cdot 4 \cdot 2}{0 + 2 \cdot 1 \cdot 6 + 1 \cdot 4 \cdot (-1) - 0 - (-1) \cdot 1 \cdot 3 - 0} = \frac{22}{11} = 2$$

### Practice

Solve the following systems of equations using Cramer's Rule. If one solution does not exist, explain.

1.

$$\begin{aligned} 4x - 2y &= -20 \\ x - 3y &= -15 \end{aligned}$$

2.

$$\begin{aligned} 3x + 5y &= 33 \\ -x - 2y &= -13 \end{aligned}$$

3.

$$\begin{aligned} x + 4y &= 11 \\ 3x + 12y &= 33 \end{aligned}$$

4.

$$\begin{aligned} -3x + y &= -7 \\ -x + 4y &= 5 \end{aligned}$$

5.

$$\begin{aligned} 3x + y &= 6 \\ -6x - 2y &= 10 \end{aligned}$$

6. Use Cramer's Rule to solve for $x$ in the following system:

$$\begin{aligned} 2x - y + z &= 4 \\ 4x + 7y - z &= 38 \\ -x + 3y + 2z &= 23 \end{aligned}$$

7. Use Cramer's Rule to solve for $y$ in the following system:

$$\begin{aligned} 4x + y - z &= -16 \\ -3x + 4y + z &= 18 \\ x + y - 3z &= -17 \end{aligned}$$

8. Use Cramer's Rule to solve for $z$ in the following system:

$$\begin{aligned} 3x+2y-3z &= 7 \\ -x+5y+2z &= 29 \\ x+2y+z &= 15 \end{aligned}$$

9. Use Cramer's Rule to solve for $x$ in the following system:

$$\begin{aligned} 2x+y-2z &= -5 \\ -4x-2y+3z &= 2 \\ 3x+y-z &= 3 \end{aligned}$$

10. Use Cramer's Rule to solve for $y$ in the following system:

$$\begin{aligned} -x+3y+z &= 11 \\ 3x+y+2z &= 27 \\ 5x-y-z &= 5 \end{aligned}$$

11. Use Cramer's Rule to solve for $z$ in the following system:

$$\begin{aligned} 3x+2y+4z &= 21 \\ -2x+3y+z &= -11 \\ x+2y-3z &= -3 \end{aligned}$$

Solve the following systems of equations using Cramer's Rule. Practice using your calculator to help with at least one problem. If one solution does not exist, explain.

12.

$$\begin{aligned} -x+2y-6z &= 4 \\ 8x+5y+3z &= -8 \\ 2x-4y+12z &= 5 \end{aligned}$$

13.

$$\begin{aligned} 3x+5y+8z &= 37 \\ -6x+3y+z &= 42 \\ x+3y-2z &= 5 \end{aligned}$$

14.

$$\begin{aligned} 4x+y-6z &= -38 \\ 2x+7y+8z &= 108 \\ -3x+2y-3z &= -15 \end{aligned}$$

15.

$$
\begin{aligned}
6x+3y-2z&=-22\\
-4x-2y+4z&=28\\
3x+3y+2z&=7
\end{aligned}
$$

16. When using Cramer's Rule to solve a system of equations you will occasionally find that the determinant of the coefficient matrix is zero. When this happens, how can you tell whether your system has no solution or infinite solutions?

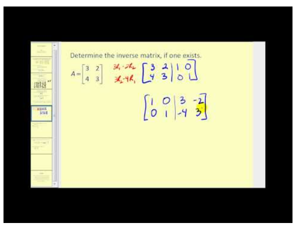
# 8.9 Inverse Matrices

Here you will learn how to find the inverse of a matrix and how to solve a system of equations using an inverse matrix.

Two numbers are multiplicative inverses if their product is 1. Every number besides the number 0 has a multiplicative inverse. For matrices, two matrices are inverses of each other if they multiply to be the identity matrix.

What kinds of matrices do not have inverses?

### Watch This




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=KBYvP6YG58g James Sousa: Inverse Matrices Using Augmented Matrices

### Guidance

Consider a matrix $A$ that has inverse $A^{-1}$. How do you find matrix $A^{-1}$ if you just have matrix $A$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}, A^{-1} = ?$$

The answer is that you augment matrix $A$ with the identity matrix and row reduce.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

The row reducing is demonstrated in Example A. The right part of the augmented Matrix is the inverse matrix $A^{-1}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & 0 \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

Fractions are usually unavoidable when computing inverses.

One reason why inverses are so powerful is because they allow you to solve systems of equations with the same logic as you would solve a single linear equation. Consider the following system based on the coefficients of matrix $A$ from above.

$$\begin{aligned} x+2y+3z &= 96 \\ x+0y+z &= 36 \\ 0x+2y-z &= -12 \end{aligned}$$

By writing this system as a matrix equation you get:

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \end{aligned}$$

If this were a normal linear equation where you had a constant times the variable equals a constant, you would multiply both sides by the multiplicative inverse of the coefficient. Do the same in this case.

$$\begin{aligned} A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\ \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \end{aligned}$$

All that is left is for you to perform the matrix multiplication to get the solution. See Example B.

**Example A**

Show the steps for finding the inverse matrix $A$ from the guidance section.

**Solution:**

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \; -I \; \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \; +II \; \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & -2 & -1 & 1 & 0 \\ 0 & 0 & -3 & -1 & 1 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \; \div(-2) \; \rightarrow \\ \rightarrow \; \div(-3) \; \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{c} \rightarrow \; -3III \; \rightarrow \\ \rightarrow \; -III \; \rightarrow \\ \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right] \begin{array}{c} \rightarrow \; -2II \; \rightarrow \\ \rightarrow \\ \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

The matrix on the right is the inverse matrix $A^{-1}$.

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

**Example B**

Solve the following system of equations using inverse matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$$

**Solution:**

$$\begin{aligned}
A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\
A^{-1} \cdot A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{1}{3} & \frac{4}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{1}{3} \cdot 96 + \frac{4}{3} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{6} \cdot 96 - \frac{1}{6} \cdot 36 + \frac{1}{3} \cdot (-12) \\ \frac{1}{3} \cdot 96 - \frac{1}{3} \cdot 36 - \frac{1}{3} \cdot (-12) \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 12 \\ 6 \\ 24 \end{bmatrix}
\end{aligned}$$

**Example C**

Find the inverse of the following matrix.

$$\begin{bmatrix} 1 & 6 \\ 4 & 24 \end{bmatrix}$$

**Solution:**

$$\left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 4 & 24 & 0 & 1 \end{array}\right] \rightarrow \begin{matrix} \rightarrow \\ -4I \end{matrix} \rightarrow \left[\begin{array}{cc|cc} 1 & 6 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{array}\right]$$

This matrix is not invertible because its rows are not linearly independent. To test to see if a square matrix is invertible, check whether or not the determinant is zero. If the determinant is zero then the matrix is not invertible because the rows are not linearly independent.

**Concept Problem Revisited**

Non-square matrices do not generally have inverses. Square matrices that have determinants equal to zero do not have inverses.

## Vocabulary

***Multiplicative inverses*** are two numbers or matrices whose product is one or the identity matrix.

## Guided Practice

1. Confirm matrix $A$ and $A^{-1}$ are inverses by computing $A^{-1} \cdot A$ and $A \cdot A^{-1}$.

$$A=\begin{bmatrix}1&2&3\\1&0&1\\0&2&-1\end{bmatrix}, A^{-1}=\begin{bmatrix}-\frac{1}{3}&\frac{4}{3}&\frac{1}{3}\\\frac{1}{6}&-\frac{1}{6}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}$$

2. Use a calculator to compute $A^{-1}$, compute $A^{-1}\cdot A$, compute $A\cdot A^{-1}$ and compute $A^{-1}\cdot\begin{bmatrix}96\\36\\-12\end{bmatrix}$.

3. The identity matrix happens to be its own inverse. Find another matrix that is its own inverse.

**Answers:**

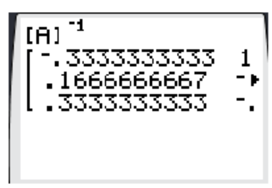
1.

$$A^{-1}\cdot A=\begin{bmatrix}-\frac{1}{3}&\frac{4}{3}&\frac{1}{3}\\\frac{1}{6}&-\frac{1}{6}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}&-\frac{1}{3}\end{bmatrix}\cdot\begin{bmatrix}1&2&3\\1&0&1\\0&2&-1\end{bmatrix}=$$
$$a_{11}=-\frac{1}{3}\cdot 1+\frac{4}{3}\cdot 1+\frac{1}{3}\cdot 0=1$$
$$a_{22}=\frac{1}{6}\cdot 2-\frac{1}{6}\cdot 0+\frac{1}{3}\cdot 2=1$$
$$a_{33}=\frac{1}{3}\cdot 3-\frac{1}{3}\cdot 1-\frac{1}{3}(-1)=1$$

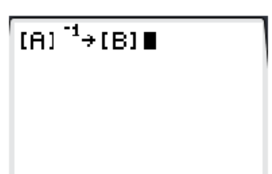
Note that the rest of the entries turn out to be zero. This is left for you to confirm.

2. Start by entering just matrix $A$ into the calculator.

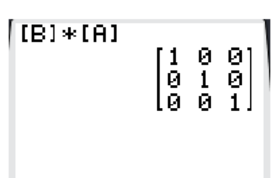
To compute matrix $A^{-1}$ use the inverse button programmed into the calculator. Do not try to raise the matrix to the negative one exponent. This will not work.



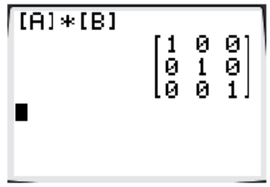

Note that the calculator may return decimal versions of the fractions and will not show the entire matrix on its limited display. You will have to scroll to the right to confirm that $A^{-1}$ matches what you have already found. Once you have found $A^{-1}$ go ahead and store it as matrix $B$ so you do not need to type in the entries.




$A^{-1}\cdot A=B\cdot A$

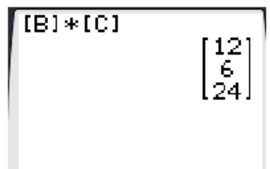


$A \cdot A^{-1} = A \cdot B$




$A^{-1} \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix} = B \cdot C$

You need to create matrix $C = \begin{bmatrix} 96 \\ 36 \\ -12 \end{bmatrix}$




Being able to effectively use a calculator should improve your understanding of matrices and allow you to check all the work you do by hand.

3. Helmert came up with a very clever matrix that happens to be its own inverse. Here are the $2 \times 2$ and the $3 \times 3$ versions.

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

### Practice

Find the inverse of each of the following matrices, if possible. Make sure to do some by hand and some with your calculator.

1. $\begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

2. $\begin{bmatrix} -3 & 6 \\ 2 & 5 \end{bmatrix}$

3. $\begin{bmatrix} -1 & 2 \\ 2 & 0 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

5. $\begin{bmatrix} 6 & 5 \\ 2 & -2 \end{bmatrix}$

6. $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

7. $\begin{bmatrix} -1 & 3 & -4 \\ 4 & 2 & 1 \\ 1 & 2 & 5 \end{bmatrix}$

8. $\begin{bmatrix} 4 & 5 & 8 \\ 9 & 0 & 1 \\ 0 & 3 & -2 \end{bmatrix}$

9. $\begin{bmatrix} 0 & 7 & -1 \\ 2 & -3 & 1 \\ 6 & 8 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 4 & 2 & -3 \\ 2 & 4 & 5 \\ 1 & 8 & 0 \end{bmatrix}$

11. $\begin{bmatrix} -2 & -6 & -12 \\ -1 & -5 & -2 \\ 2 & 3 & 4 \end{bmatrix}$

12. $\begin{bmatrix} -2 & 6 & 3 \\ 2 & 4 & 0 \\ -8 & 2 & 1 \end{bmatrix}$

13. Show that Helmert's $2 \times 2$ matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$

14. Show that Helmert's $3 \times 3$ matrix is its own inverse: $\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$.

15. Non-square matrices sometimes have left inverses, where $A^{-1} \cdot A = I$, or right inverses, where $A \cdot A^{-1} = I$. Why can't non-square matrices have "regular" inverses?

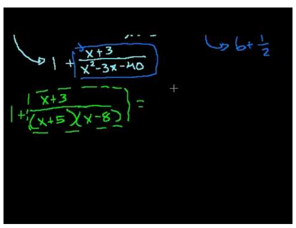
# 8.10 Partial Fractions

Here you will apply what you know about systems and matrices to decompose rational expressions into the sum of several partial fractions.

When given a rational expression like $\frac{4x-9}{x^2-3x}$ it is very helpful in calculus to be able to write it as the sum of two simpler fractions like $\frac{3}{x}+\frac{1}{x-3}$. The challenging part is trying to get from the initial rational expression to the simpler fractions.

You may know how to add fractions and go from two or more separate fractions to a single fraction, but how do you go the other way around?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=S-XKGBesRzk Khan Academy: Partial Fraction Expansion 1

### Guidance

**Partial fraction decomposition** is a procedure that reverses adding fractions with unlike denominators. The most challenging part is coming up with the denominators of each individual partial fraction. See if you can spot the pattern.

$\frac{6x-1}{x^2(x-1)(x^2+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}+\frac{Dx+E}{x^2+2}$

In this example each individual factor must be represented. Linear factors that are raised to a power greater than one must have each successive power included as a separate denominator. Quadratic terms that do not factor to be linear terms are included with a numerator that is a linear function of $x$.

**Example A**

Use partial fractions to decompose the following rational expression.

$\frac{7x^2+x+6}{x^3+3x}$

**Solution:** First factor the denominator and identify the denominators of the partial fractions.

$\frac{7x^2+x+6}{x(x^2+3)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$

When the fractions are eliminated by multiplying through by the LCD the equation becomes:

$$7x^2+x+6=A(x^2+3)+x(Bx+C)$$
$$7x^2+x+6=Ax^2+3A+Bx^2+Cx$$

Notice the squared term, linear term and constant term form a system of three equations with three variables.

$$\begin{aligned} A+B&=7\\ C&=1\\ 3A&=6 \end{aligned}$$

In this case it is easy to see that $A=2, B=5, C=1$. Often, the resulting system of equations is more complex and would benefit from your knowledge of solving systems using matrices.

$\frac{7x^2+x+6}{x(x^2+3)}=\frac{2}{x}+\frac{5x+1}{x^2+3}$

**Example B**

Decompose the following rational expression.

$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}$

**Solution:** First identify the denominators of the partial fractions.

$$\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}+\frac{D}{x}+\frac{E}{x^2}$$

When the entire fraction is multiplied through by $(x-1)^3x^2$ the equation results to:

$$\begin{aligned} &5x^4-3x^3-x^2+4x-1\\ &=A(x-1)^2x^2+B(x-1)x^2+Cx^2+D(x-1)^3x+E(x-1)^3 \end{aligned}$$

Multiplication of each term can be done separately to be extra careful.

$$\begin{aligned} &Ax^4-2Ax^3+Ax^2\\ &Bx^3-Bx^2\\ &Cx^2\\ &Dx^4-3Dx^3+3Dx^2-Dx\\ &Ex^3-3Ex^2+3Ex-E \end{aligned}$$

Group terms with the same power of $x$ and set equal to the corresponding term.

$$\begin{aligned} 5x^4&=Ax^4+Dx^4\\ -3x^3&=-2Ax^3+Bx^3-3D^3+Ex^3\\ -x^2&=Ax^2-Bx^2+Cx^2+3Dx^2-3Ex^2\\ 4x&=-Dx+3Ex\\ -1&=-E \end{aligned}$$

From these 5 equations, every $x$ can be divided out. Assume that $x\neq 0$ because if it were, then the original expression would be undefined.

$$
\begin{aligned}
5 &= A+D \\
-3 &= -2A+B-3D+E \\
-1 &= A-B+C+3D-3E \\
4 &= -D+3E \\
-1 &= E
\end{aligned}
$$

This is a system of equations of five variables and 5 equations. Some of the equations can be solved using logic and substitution like $E=-1,\ D=-7,\ A=12$. You can use any method involving determinants or matrices. In this case it is easiest to substitute known values into equations with one unknown value to get more known values and repeat.

$$
\begin{aligned}
B &= 1 \\
C &= 6 \\
\frac{5x^4-3x^3-x^2+4x-1}{(x-1)^3x^2} &= \frac{12}{x-1}+\frac{1}{(x-1)^2}+\frac{6}{(x-1)^3}+\frac{-7}{x}+\frac{-1}{x^2}
\end{aligned}
$$

**Example C**

Use matrices to complete the partial fraction decomposition of the following rational expression.

$$\frac{2x+4}{(x-1)(x+3)}$$

**Solution:**

$$
\begin{aligned}
\frac{2x+4}{(x-1)(x+3)} &= \frac{A}{x+1}+\frac{B}{x+3} \\
2x+4 &= Ax+3A+Bx+B
\end{aligned}
$$

$$
\begin{aligned}
2 &= A+B \\
4 &= 3A+B
\end{aligned}
$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 1 & 4 \end{array}\right] \to \begin{array}{c} \to \\ -3\cdot I \end{array} \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right] \to \begin{array}{c} \to \\ \div -2 \end{array} \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \to \begin{array}{c} -II \\ \to \end{array} \to \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

$$A=1, B=1$$

$$\frac{2x+4}{(x-1)(x+3)} = \frac{1}{x+1} + \frac{1}{x+3}$$

**Concept Problem Revisited**

To decompose the rational expression into the sum of two simpler fractions you need to use partial fraction decomposition.

$$\frac{4x-9}{x^2-3x} = \frac{A}{x} + \frac{B}{x-3}$$
$$4x-9 = A(x-3)+Bx$$
$$4x-9 = Ax-3A+Bx$$

Notice that the constant term -9 must be equal to the constant term $-3A$ and that the terms with $x$ must be equal as well.

$$-9 = -3A$$
$$4 = A+B$$

Solving this system yields:

$A = 3, \quad B = 1$

Therefore,

$\frac{4x-9}{x^2-3x} = \frac{3}{x} + \frac{1}{x-3}$

## Vocabulary

***Partial fraction decomposition*** is a procedure that undoes the operation of adding fractions with unlike denominators. It separates a rational expression into the sum of rational expressions with unlike denominators.

## Guided Practice

1. Use matrices to help you decompose the following rational expression.

$\frac{5x-2}{(2x-1)(3x+4)}$

2. Confirm Example C by adding the partial fractions.

$\frac{2x+4}{(x-1)(x+3)} = \frac{1}{x+1} + \frac{1}{x+3}$

3. Confirm Guided Practice #1 by adding the partial fractions.

$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{-\frac{26}{11}}{3x+4}$

**Answers:**

1.

$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{A}{2x-1} + \frac{B}{3x+4}$$
$$5x-2 = A(3x+4)+B(2x-1)$$
$$5x-2 = 3Ax+4A+2Bx-B$$
$$5 = 3A+2B$$
$$-2 = 4A-B$$

$$\left[\begin{array}{cc|c}3&2&5\\4&-1&-2\end{array}\right] \begin{array}{c}\rightarrow\ \cdot 4\ \rightarrow\\ \rightarrow\ \cdot 3\ \rightarrow\end{array} \left[\begin{array}{cc|c}12&8&20\\12&-3&-6\end{array}\right] \begin{array}{c}\rightarrow\\ \rightarrow\ -I\ \rightarrow\end{array} \left[\begin{array}{cc|c}12&8&20\\0&-11&-26\end{array}\right] \begin{array}{c}\rightarrow\ \cdot 11\ \rightarrow\\ \rightarrow\ \cdot 8\ \rightarrow\end{array} \left[\begin{array}{cc|c}132&88&220\\0&-88&-208\end{array}\right]$$
$$\begin{array}{c}\rightarrow\ +II\ \rightarrow\\ \rightarrow\end{array} \left[\begin{array}{cc|c}132&0&12\\0&-88&-208\end{array}\right] \begin{array}{c}\rightarrow\ \div 132\ \rightarrow\\ \rightarrow\ \div -88\ \rightarrow\end{array} \left[\begin{array}{cc|c}1&0&\frac{1}{11}\\0&1&\frac{26}{11}\end{array}\right]$$

$$A = \frac{1}{11}, B = -\frac{26}{11}$$
$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$$

2. $\frac{1}{x+1} + \frac{1}{x+3} = \frac{x+3}{(x+1)(x+3)} + \frac{x+1}{(x+1)(x+3)} = \frac{2x+4}{(x+1)(x+3)}$

3.

$$\frac{5x-2}{(2x-1)(3x+4)} = \frac{\frac{1}{11}}{2x-1} + \frac{\frac{26}{11}}{3x+4}$$
$$5x-2 = \frac{1}{11}(3x+4) + \frac{26}{11}(2x-1)$$
$$55x-22 = 3x+4+26(2x-1)$$
$$55x-22 = 3x+4+52x-26$$
$$55x-22 = 55x-22$$

## Practice

Decompose the following rational expressions. Practice using matrices with at least one of the problems.

1. $\frac{3x-4}{(x-1)(x+4)}$
2. $\frac{2x+1}{x^2(x-3)}$
3. $\frac{x+1}{x(x-5)}$
4. $\frac{x^2+3x+1}{x(x-3)(x+6)}$
5. $\frac{3x^2+2x-1}{x^2(x+2)}$
6. $\frac{x^2+1}{x(x-1)(x+1)}$
7. $\frac{4x^2-9}{x^2(x-4)}$
8. $\frac{2x-4}{(x+7)(x-3)}$
9. $\frac{3x-4}{x^2(x^2+1)}$
10. $\frac{2x+5}{(x-3)(x^2+4)}$
11. $\frac{3x^2+2x-5}{x^2(x-3)(x^2+1)}$

12. Confirm your answer to #1 by adding the partial fractions.

13. Confirm your answer to #3 by adding the partial fractions.

14. Confirm your answer to #6 by adding the partial fractions.

15. Confirm your answer to #9 by adding the partial fractions.

A system of equations is a collection of two or more equations with multiple unknown variables. You learned that solving a system means solving for these variables. Graphically this means finding where the lines, curves or planes intersect. You learned that a matrix is a rectangular array of numbers that corresponds to the coefficients in a system. Using a matrix to represent the system is incredibly powerful because it makes it easier to solve for the variables. In this chapter on systems and matrices, you reviewed techniques for solving systems in two and three dimensions, learned the basic idea of matrices, exercised different properties of matrices and finally used matrices to solve systems.

# 8.11 References


1. CK-12 Foundation. . CCSA
2. CK-12 Foundation. . CCSA
3. CK-12 Foundation. . CCSA
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# CHAPTER 9 Conics

## Chapter Outline



Conics are an application of analytic geometry. Here you will get a chance to work with shapes like circles that you have worked with before. You will also get to see the equations and definitions that turn circles into curved ellipses and the rest of the conic sections.

# 9.1 General Form of a Conic

Here you will see how each conic section is the intersection of a plane and a cone, review completing the square and start working with the general equation of a conic.

Conics are a family of graphs that include parabolas, circles, ellipses and hyperbolas. All of these graphs come from the same general equation and by looking and manipulating a specific equation you can learn to tell which conic it is and how it can be graphed.

What is the one essential skill that enables you to manipulate the equation of a conic in order to sketch its graph?

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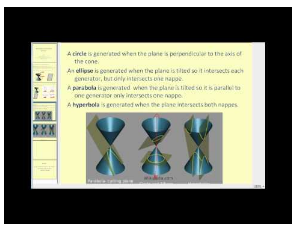

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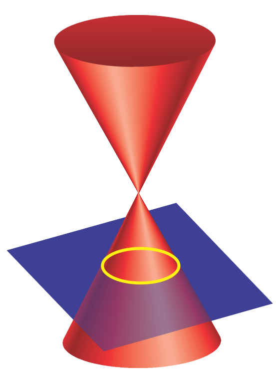
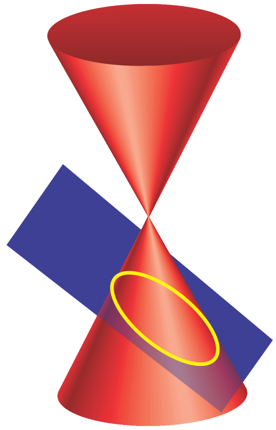
http://www.youtube.com/watch?v=iJOcn9C9y4w James Sousa: Introduction to Conic Sections

### Guidance

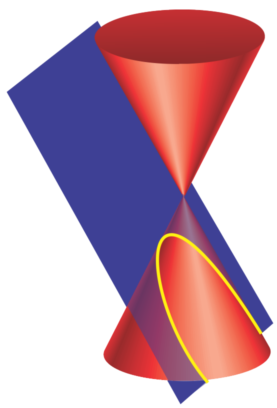
The word conic comes from the word cone which is where the shapes of parabolas, circles, ellipses and hyperbolas originate. Consider two cones that open up in opposite directions and a plane that intersects it horizontally. A flat intersection would produce a perfect circle.



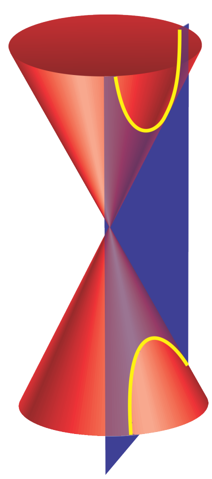
To produce an ellipse, tilt the plane so that the circle becomes elongated and oval shaped. Notice that the angle that the plane is tilted is still less steep than the slope of the side of the cone.



As you tilt the plane even further and the slope of the plane equals the slope of the cone edge you produce a parabola. Since the slopes are equal, a parabola only intersects one of the cones.



Lastly, if you make the plane steeper still, the plane ends up intersecting both the lower cone and the upper cone creating the two parts of a hyperbola.



The intersection of three dimensional objects in three dimensional space to produce two dimensional graphs is quite challenging. In practice, the knowledge of where conics come from is not widely used. It will be more important for you to be able to manipulate an equation into standard form and graph it in a regular coordinate plane. The regular form of a conic is:

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

Before you start manipulating the general form of a conic equation you should be able to recognize whether it is a circle, ellipse, parabola or hyperbola. In standard form, the two coefficients to examine are $A$ and $C$.

- For **circles,** the coefficients of $x^2$ and $y^2$ are the same sign and the same value: $A = C$
- For **ellipses,** the coefficients of $x^2$ and $y^2$ are the same sign and different values: $A, C > 0$, $A \neq C$
- For **hyperbolas,** the coefficients of $x^2$ and $y^2$ are opposite signs: $C < 0 < A$ or $A < 0 < C$
- For **parabolas,** either the coefficient of $x^2$ or $y^2$ must be zero: $A = 0$ or $C = 0$

Each specific type of conic has its own graphing form, but in all cases the technique of completing the square is essential. The examples review completing the square and recognizing conics.

**Example A**

Complete the square in the expression $x^2 + 6x$. Demonstrate graphically what completing the square represents.

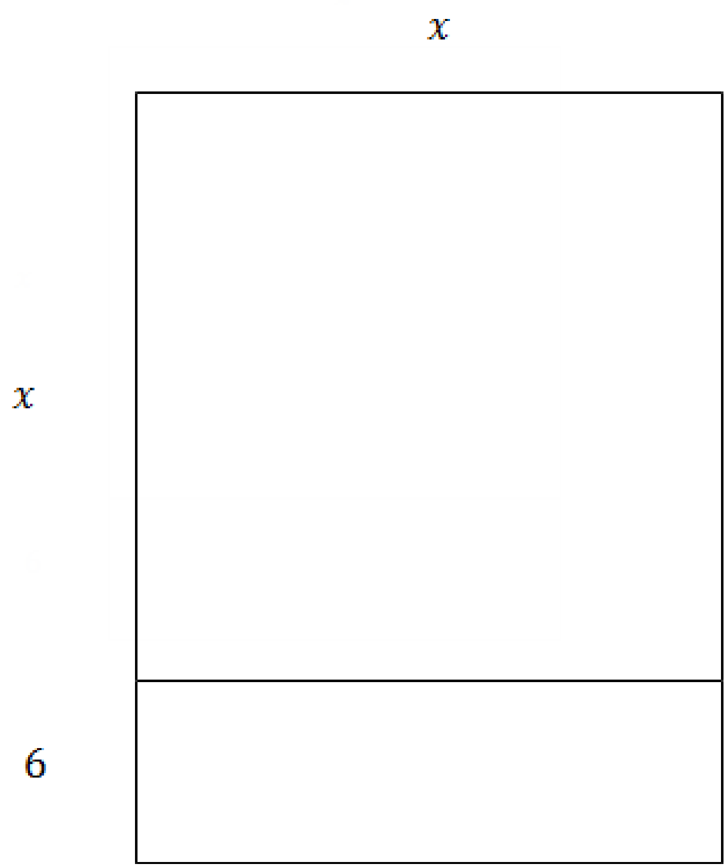
**Solution:** Algebraically, completing the square just requires you to divide the coefficient of $x$ by 2 and square the result. In this case $\left(\frac{6}{2}\right)^2 = 3^2 = 9$. Since you cannot add nine to an expression without changing its value, you must simultaneously add nine and subtract nine so the net change will be zero.

$x^2 + 6x + 9 - 9$

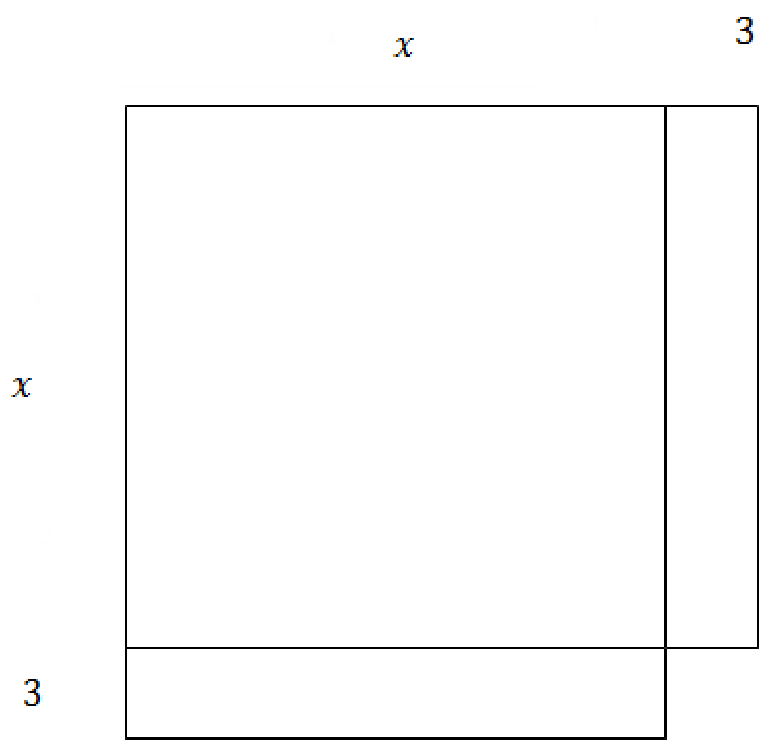
Now you can factor by recognizing a perfect square.

$(x+3)^2 - 9$

Graphically the original expression $x^2 + 6x$ can be represented by the area of a rectangle with sides $x$ and $(x+6)$.




The term "complete the square" has visual meaning as well algebraic meaning. The rectangle can be rearranged to be more square-like so that instead of small rectangle of area $6x$ at the bottom, there is a rectangle of area $3x$ on two sides of the $x^2$ square.




Notice what is missing to make this shape a perfect square? A little corner square of 9 is missing which is why the 9 should be added to make the perfect square of $(x+3)(x+3)$.

**Example B**

What type of conic is each of the following relations?

a. $5y^2-2x^2=-25$
b. $x=-\frac{1}{2}y^2-3$
c. $4x^2+6y^2=36$
d. $x^2-\frac{1}{4}y=1$
e. $-\frac{x^2}{8}+\frac{y^2}{4}=1$
f. $-x^2+99y^2=12$

**Solution:**

a. Hyperbola because the $x^2$ and $y^2$ coefficients are different signs.
b. Parabola (sideways) because the $x^2$ term is missing.
c. Ellipse because the $x^2$ and $y^2$ coefficients are different values but the same sign.
d. Parabola (upright) because the $y^2$ term is missing.
e. Hyperbola because the $x^2$ and $y^2$ coefficients are different signs.
f. Hyperbola because the $x^2$ and $y^2$ coefficients are different signs.

**Example C**

Complete the square for both the $x$ and $y$ terms in the following equation.

$x^2+6x+2y^2+16y=0$

**Solution:** First write out the equation with space so that there is room for the terms to be added to both sides. Since this is an equation, it is appropriate to add the values to both sides instead of adding and subtracting the same value simultaneously. As you rewrite with spaces, factor out any coefficient of the $x^2$ or $y^2$ terms since your algorithm for completing the square only works when this coefficient is one.

$x^2+6x+\_\_+2(y^2+8y+\_\_)=0$

Next complete the square by adding a nine and what looks like a 16 on the left (it is actually a 32 since it is inside the parentheses).

$x^2+6x+9+2(y^2+8y+16)=9+32$

Factor.

$(x+3)^2+2(y+4)^2=41$

**Concept Problem Revisited**

The one essential skill that you need for conics is completing the square. If you can do problems like Example C then you will be able to graph every type of conic.

## Vocabulary

***Completing the square*** is a procedure that enables you to combine squared and linear terms of the same variable into a perfect square of a binomial.

***Conics*** are a family of graphs (not functions) that come from the same general equation. This family is the intersection of a two sided cone and a plane in three dimensional space.

## Guided Practice

1. Identify the type of conic in each of the following relations.

a. $3x^2=3y^2+18$
b. $y=4(x-3)^2+2$
c. $x^2+y^2=4$
d. $y^2+2y+x^2-6x=12$
e. $\frac{x^2}{6}+\frac{y^2}{12}=1$
f. $x^2-y^2+4=0$

2. Complete the square in the following expression.

$6y^2-36y+4$

3. Complete the square for both $x$ and $y$ in the following equation.

$-3x^2-24x+4y^2-32y=8$

**Answers:**

1. a. The relation is a hyperbola because when you move the $3y^2$ to the left hand side of the equation, it becomes negative and then the coefficients of $x^2$ and $y^2$ have opposite signs.

b. Parabola

c. Circle

d. Circle

e. Ellipse

f. Hyperbola

2.

$$\begin{aligned}
&6y^2-36y+4\\
&6(y^2-6y+__)+4\\
&6(y^2-6y+9)+4-54\\
&6(y-3)^2-50
\end{aligned}$$

3.

$$\begin{aligned}
-3x^2-24x+4y^2-32y&=8\\
-3(x^2+8x+__)+4(y^2-8y+__)&=8\\
-3(x^2+8x+16)+4(y^2-8y+16)&=8-48+64\\
-3(x+4)^2+4(y-4)^2&=24
\end{aligned}$$

## Practice

Identify the type of conic in each of the following relations.

1. $3x^2+4y^2=12$
2. $x^2+y^2=9$
3. $\frac{x^2}{4}+\frac{y^2}{9}=1$
4. $y^2+x=11$
5. $x^2+2x-y^2+6y=15$
6. $x^2=y-1$

Complete the square for $x$ and/or $y$ in each of the following expressions.

7. $x^2+4x$
8. $y^2-8y$
9. $3x^2+6x+4$
10. $3y^2+9y+15$
11. $2x^2-12x+1$

Complete the square for $x$ and/or $y$ in each of the following equations.

12. $4x^2-16x+y^2+2y=-1$
13. $9x^2-54x+y^2-2y=-81$
14. $3x^2-6x-4y^2=9$
15. $y=x^2+4x+1$

# 9.2 Parabolas

Here you will define a parabola in terms of its directrix and focus, graph parabolas vertically and horizontally, and use a new graphing form of the parabola equation.

When working with parabolas in the past you probably used vertex form and analyzed the graph by finding its roots and intercepts. There is another way of defining a parabola that turns out to be more useful in the real world. One of the many uses of parabolic shapes in the real world is satellite dishes. In these shapes it is vital to know where the receptor point should be placed so that it can absorb all the signals being reflected from the dish.

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

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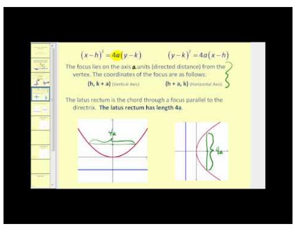

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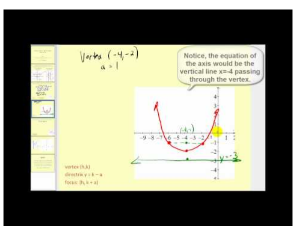

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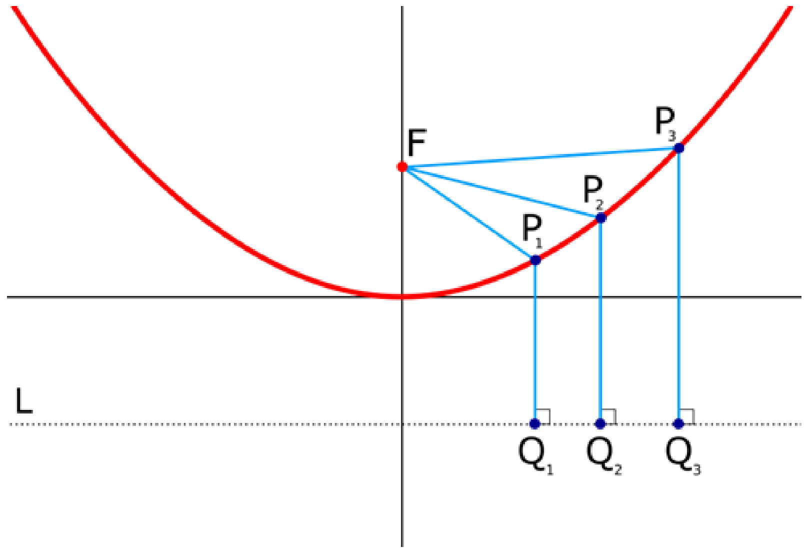
### Guidance

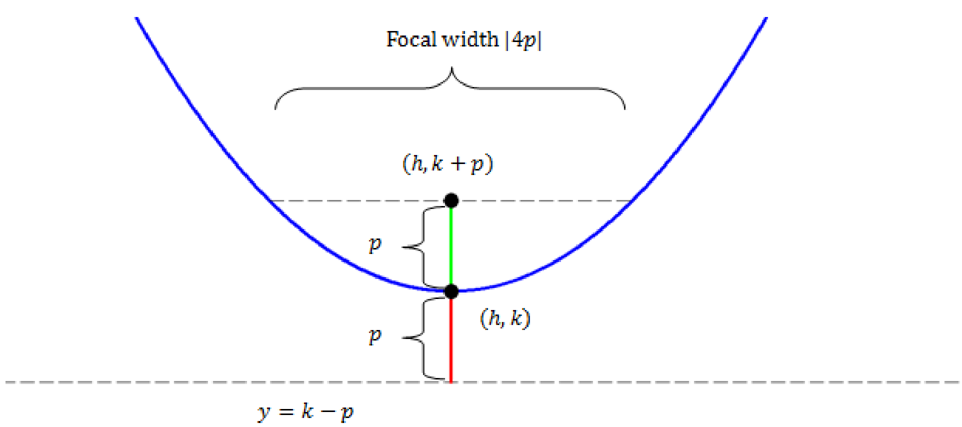
The definition of a parabola is the collection of points equidistant from a point called the focus and a line called the directrix.




Notice how the three points $P_1, P_2, P_3$ are each connected by a blue line to the focus point $F$ and the directrix line $L$.

$$\overline{FP_1} = \overline{P_1Q_1}$$
$$\overline{FP_2} = \overline{P_2Q_2}$$
$$\overline{FP_3} = \overline{P_3Q_3}$$

There are two graphing equations for parabolas that will be used in this concept. The only difference is one equation graphs parabolas opening vertically and one equation graphs parabolas opening horizontally. You can recognize the parabolas opening vertically because they have an $x^2$ term. Likewise, parabolas opening horizontally have a $y^2$ term. The general equation for a parabola opening vertically is $(x-h)^2 = \pm 4p(y-k)$. The general equation for a parabola opening horizontally is $(y-k)^2 = \pm 4p(x-h)$.




Note that the vertex is still $(h,k)$. The parabola opens upwards or to the right if the $4p$ is positive. The parabola opens down or to the left if the $4p$ is negative. The focus is just a point that is distance $p$ away from the vertex. The directrix is just a line that is distance $p$ away from the vertex in the opposite direction. You can sketch how wide the parabola is by noting the focal width is $|4p|$.

Once you put the parabola into this graphing form you can sketch the parabola by plotting the vertex, identifying $p$ and plotting the focus and directrix and lastly determining the focal width and sketching the curve.

**Example A**

Identify the following conic, put it into graphing form and identify its vertex, focal length $(p)$, focus, directrix and focal width.

$2x^2+16x+y=0$

**Solution:** This is a parabola because the $y^2$ coefficient is zero.

$$\begin{aligned}x^2+8x &= -\frac{1}{2}y\\ x^2+8x+16 &= -\frac{1}{2}y+16\\ (x+4)^2 &= -\frac{1}{2}(y-32)\\ (x+4)^2 &= -4\cdot\frac{1}{8}(y-32)\end{aligned}$$

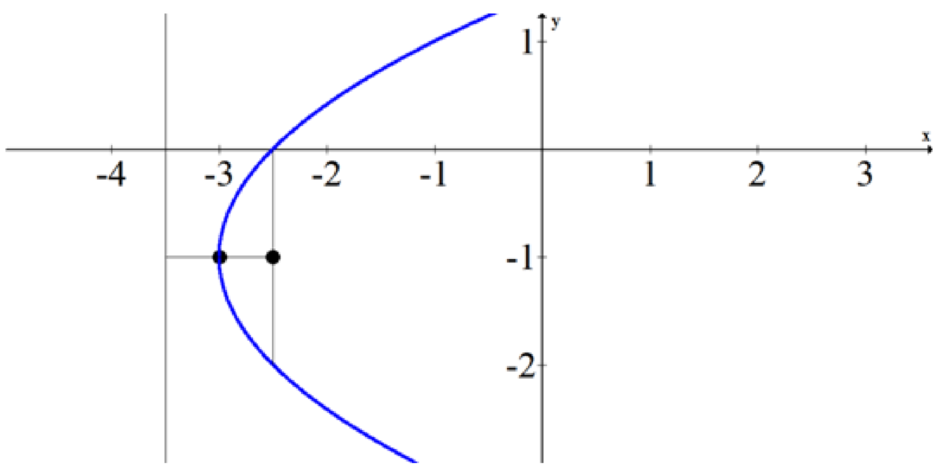
The vertex is (-4, 32). The focal length is $p=\frac{1}{8}$. This parabola opens down which means that the focus is at $\left(-4,32-\frac{1}{8}\right)$ and the directrix is horizontal at $y=32+\frac{1}{8}$. The focal width is $\frac{1}{2}$.

**Example B**

Sketch the following parabola and identify the important pieces of information.

$(y+1)^2=4\cdot\frac{1}{2}\cdot(x+3)$

**Solution:**

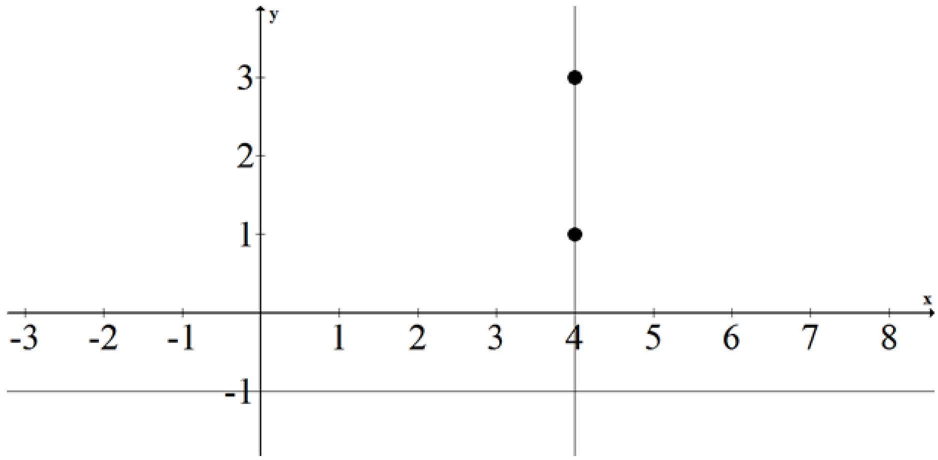



The vertex is at (-3, -1). The parabola is sideways because there is a $y^2$ term. The parabola opens to the right because the $4p$ is positive. The focal length is $p=\frac{1}{2}$ which means the focus is $\frac{1}{2}$ to the right of the vertex at (-2.5, -1) and the directrix is $\frac{1}{2}$ to the left of the vertex at $x=-3.5$. The focal width is 2 which is why the width of the parabola stretches from (-2.5, 0) to (-2.5, -2).

**Example C**

What is the equation of a parabola that has a focus at (4, 3) and a directrix of $y=-1$?

**Solution:** It would probably be useful to graph the information that you have in order to reason about where the vertex is.




The vertex must be halfway between the focus and the directrix. This places it at (4, 1). The focal length is 2. The parabola opens upwards. This is all the information you need to create the equation.

$(x-4)^2 = 4 \cdot 2 \cdot (y-1)$

OR $(x-4)^2 = 8(y-1)$

**Concept Problem Revisited**

Where should the receptor be located on a satellite dish that is four feet wide and nine inches deep?

Since real world problems do not come with a predetermined coordinate system, you can choose to make the vertex of the parabola at (0, 0). Then, if everything is done in inches, another point on the parabola will be (24, 9). *(Many people might mistakenly believe the point* (48, 9) *is on the parabola but remember that half this width stretches to* (-24, 9) *as well.)* Using these two points, the focal width can be found.

$$\begin{aligned}(x-0)^2 &= 4p(y-0)\\(24-0)^2 &= 4p(9-0)\\\frac{24^2}{4 \cdot 9} &= p\\16 &= p\end{aligned}$$

The receptor should be sixteen inches away from the vertex of the parabolic dish.

### Vocabulary

The ***focus*** of a parabola is the point that the parabola seems to curve around.

The ***directrix*** of a parabola is the line that the parabola seems to curve away from.

A ***parabola*** is the collection of points that are equidistant from a fixed focus and directrix.

### Guided Practice

1. What is the equation of a parabola with focus at (2, 3) and directrix at $y = 5$?
2. What is the equation of a parabola that opens to the right with focal width from (6, -7) to (6, 12)?
3. Sketch the following conic by putting it into graphing form and identifying important information.

$y^2 - 4y + 12x - 32 = 0$

**Answers:**

1. The vertex must lie directly between the focus and the directrix, so it must be at (2, 4). The focal length is therefore equal to 1. The parabola opens downwards.

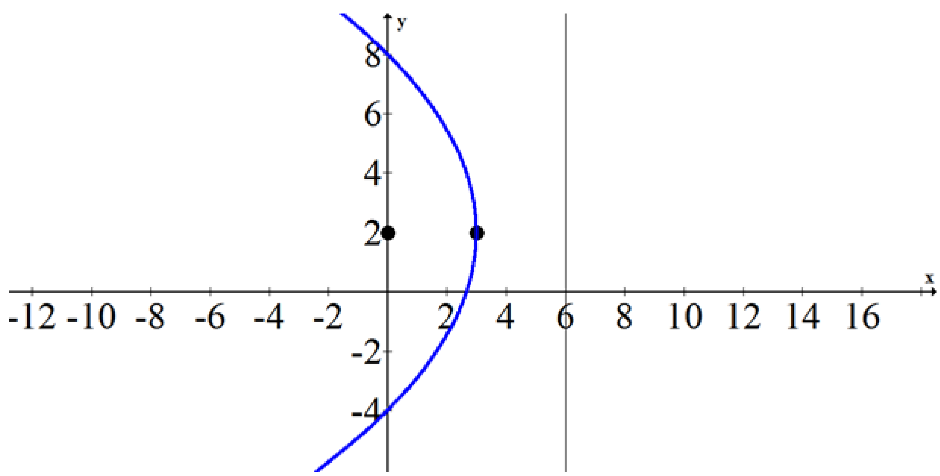
$(x-2)^2 = -4 \cdot 1 \cdot (y-4)$

2. The focus is in the middle of the focal width. The focus is $\left(6, \frac{5}{2}\right)$. The focal width is 19 which is four times the focal length so the focal length must be $\frac{19}{4}$. The vertex must be a focal length to the left of the focus, so the vertex is at $\left(6 - \frac{19}{4}, \frac{5}{2}\right)$. This is enough information to write the equation of the parabola.

$\left(y - \frac{5}{2}\right)^2 = 4 \cdot \frac{19}{4} \cdot \left(x - 6 + \frac{19}{4}\right)$

3. $y^2 - 4y + 12x - 32 = 0$

$$\begin{aligned} y^2 - 4y &= -12x + 32 \\ y^2 - 4y + 4 &= -12x + 32 + 4 \\ (y-2)^2 &= -12(x-3) \\ (y-2)^2 &= -4 \cdot 3 \cdot (x-3) \end{aligned}$$




The vertex is at (3, 2). The focus is at (0, 2). The directrix is at $x = 6$.

## Practice

1. What is the equation of a parabola with focus at (1, 4) and directrix at $y = -2$?
2. What is the equation of a parabola that opens to the left with focal width from (-2, 5) to (-2, -7)?
3. What is the equation of a parabola that opens to the right with vertex at (5, 4) and focal width of 12?
4. What is the equation of a parabola with vertex at (1, 8) and directrix at $y = 12$?
5. What is the equation of a parabola with focus at (-2, 4) and directrix at $x = 4$?
6. What is the equation of a parabola that opens downward with a focal width from (-4, 9) to (16, 9)?
7. What is the equation of a parabola that opens upward with vertex at (1, 11) and focal width of 4?

Sketch the following parabolas by putting them into graphing form and identifying important information.

8. $y^2+2y-8x+33=0$

9. $x^2-8x+20y+36=0$

10. $x^2+6x-12y-15=0$

11. $y^2-12y+8x+4=0$

12. $x^2+6x-4y+21=0$

13. $y^2+14y-2x+59=0$

14. $x^2+12x-\frac{8}{3}y+\frac{92}{3}=0$

15. $x^2+2x-\frac{4}{5}y+1=0$

# 9.3 Circles

Here you will formalize the definition of a circle, translate a conic from standard form into graphing form, and graph circles.

A circle is the collection of points that are the same distance from a single point. What is the connection between the Pythagorean Theorem and a circle?

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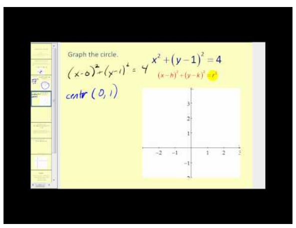

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http://www.youtube.com/watch?v=g1xa7PvYV3I Conic Sections: The Circle

## Guidance

A circle is the collection of points that are equidistant from a single point. This single point is called the center of the circle. A circle does not have a focus or a directrix, instead it simply has a center. Circles can be recognized immediately from the general equation of a conic when the coefficients of $x^2$ and $y^2$ are the same sign and the same value. Circles are not functions because they do not pass the vertical line test. The distance from the center of a circle to the edge of the circle is called the radius of the circle. The distance from one end of the circle through the center to the other end of the circle is called the diameter. The diameter is equal to twice the radius.

The graphing form of a circle is:

$(x-h)^2+(y-k)^2=r^2$

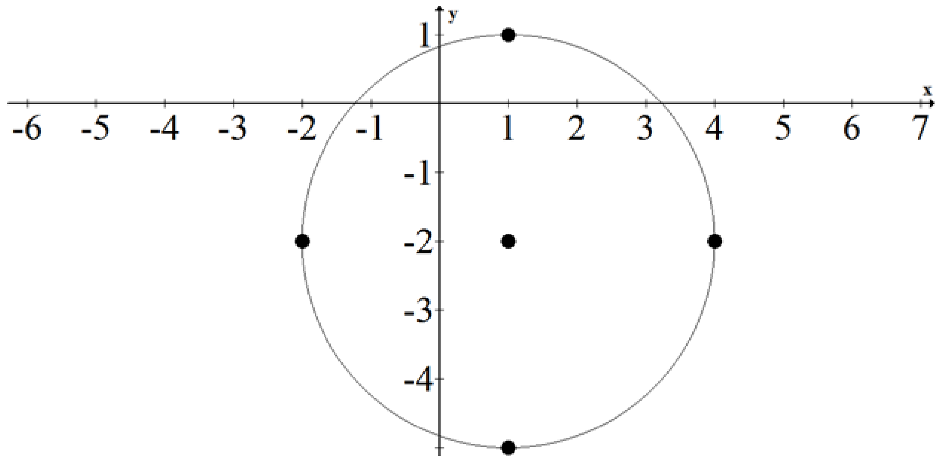
The center of the circle is at $(h,k)$ and the radius of the circle is $r$. Note that this looks remarkably like the Pythagorean Theorem.

**Example A**

Graph the following circle.

$(x-1)^2+(y+2)^2=9$

**Solution:** Plot the center and the four points that are exactly 3 units from the center.




**Example B**

Turn the following equation into graphing form for a circle. Identify the center and the radius.

$36x^2+36y^2-24x+36y-275=0$

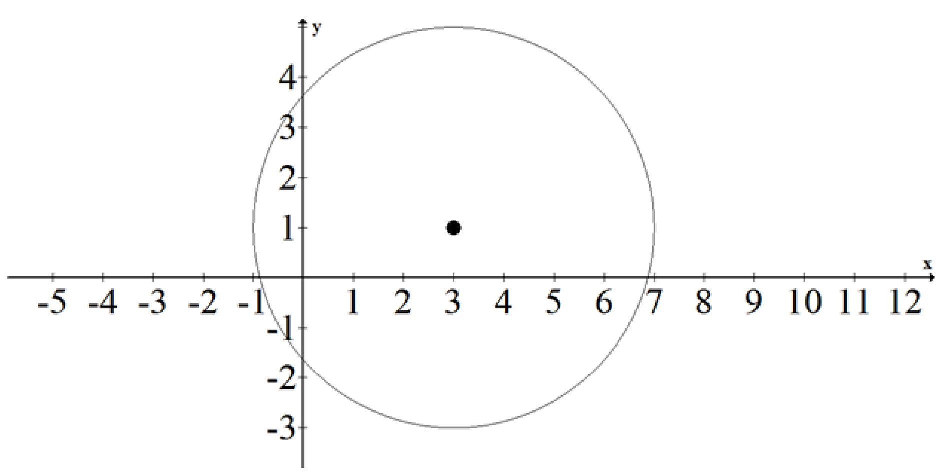
**Solution:** Complete the square and then divide by the coefficient of $x^2$ and $y^2$

$$
\begin{aligned}
&36x^2-24x+36y^2+36y=275\\
&36\left(x^2-\frac{2}{3}x+\_\_\right)+36(y^2+y+\_\_)=275\\
&36\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+36\left(y^2+y+\frac{1}{4}\right)=275+4+9\\
&36\left(x-\frac{1}{3}\right)^2+36\left(y+\frac{1}{2}\right)^2=288\\
&\left(x-\frac{1}{3}\right)^2+\left(y+\frac{1}{2}\right)^2=8
\end{aligned}
$$

The center is $\left(\frac{1}{3},-\frac{1}{2}\right)$. The radius is $\sqrt{8}=2\sqrt{2}$.

**Example C**

Write the equation of the following circle.



**Solution:**

The center of the circle is at (3, 1) and the radius of the circle is $r = 4$. The equation is $(x-3)^2 + (y-1)^2 = 16$.

**Concept Problem Revisited**

The reason why the graphing form of a circle looks like the Pythagorean Theorem is because each $x$ and $y$ coordinate along the outside of the circle forms a perfect right triangle with the radius as the hypotenuse.

## Vocabulary

The ***radius*** of a circle is the distance from the center of the circle to the outside edge.

The ***center*** of a circle is the point that defines the location of the circle.

A ***circle*** is the collection of points that are equidistant from a given point.

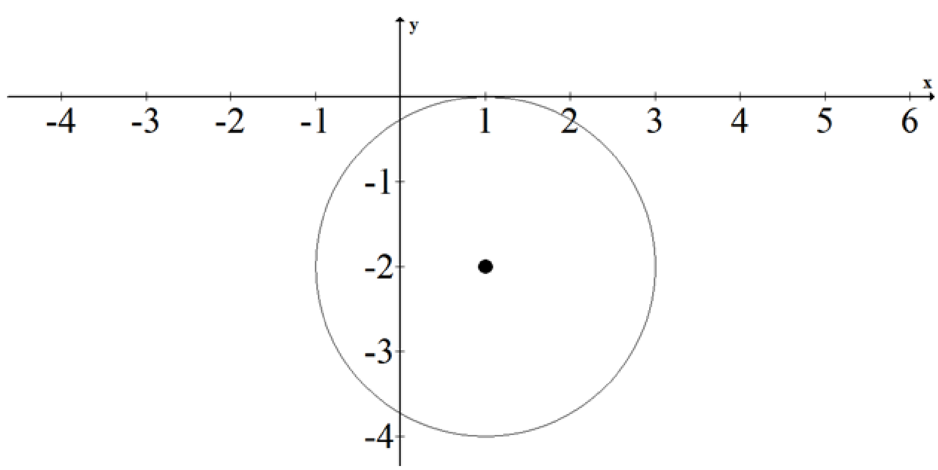
## Guided Practice

1. Graph the following conic: $(x+2)^2 + (y-1)^2 = 1$

2. Translate the following conic from standard form to graphing form.

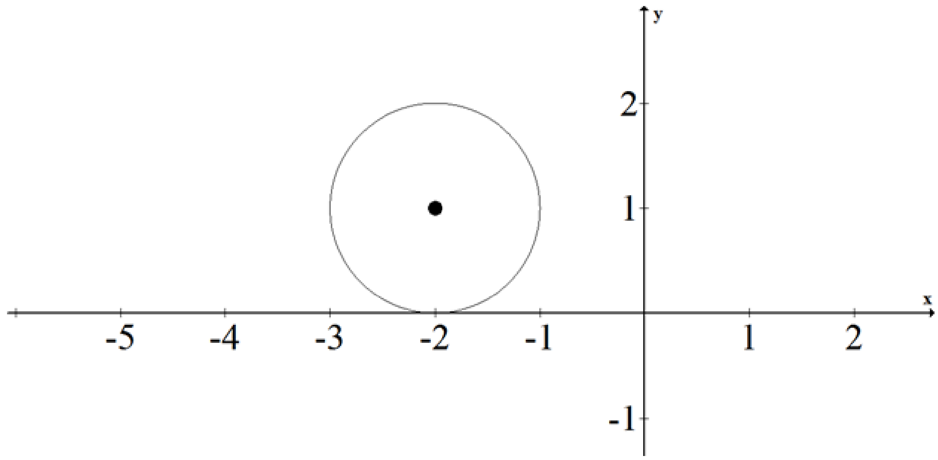
$x^2 - 34x + y^2 + 24y + \frac{749}{2} = 0$

3. Write the equation for the following circle.




**Answers:**

1.




2.

$$\begin{aligned}
x^2-34x+y^2+24y+\frac{749}{2}&=0\\
x^2-34x+y^2+24y&=-\frac{749}{2}\\
x^2-34x+289+y^2+24y+144&=-\frac{749}{2}+289+144\\
(x-17)^2+(y+12)^2&=\frac{117}{2}
\end{aligned}$$

3.

$(x-1)^2+(y+2)^2=4$

### Practice

Graph the following conics:

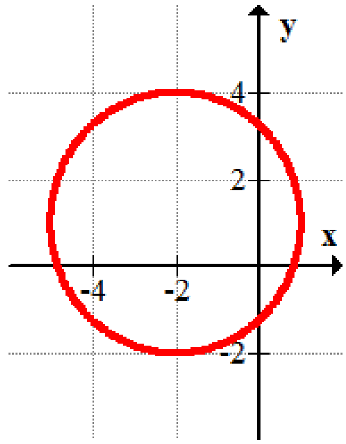
1. $(x+4)^2+(y-3)^2=1$
2. $(x-7)^2+(y+1)^2=4$
3. $(y+2)^2+(x-1)^2=9$
4. $x^2+(y-5)^2=8$
5. $(x-2)^2+y^2=16$

Translate the following conics from standard form to graphing form.

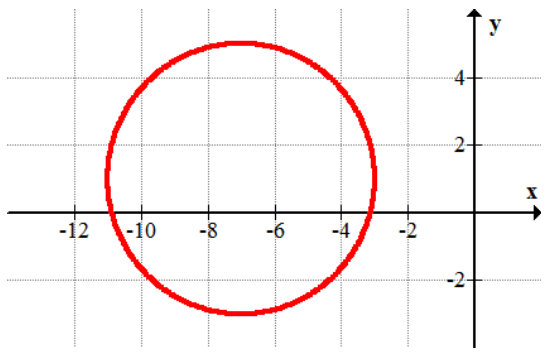
6. $x^2-4x+y^2+10y+18=0$
7. $x^2+2x+y^2-8y+1=0$
8. $x^2-6x+y^2-4y+12=0$
9. $x^2+2x+y^2+14y+25=0$
10. $x^2-2x+y^2-2y=0$

Write the equations for the following circles.

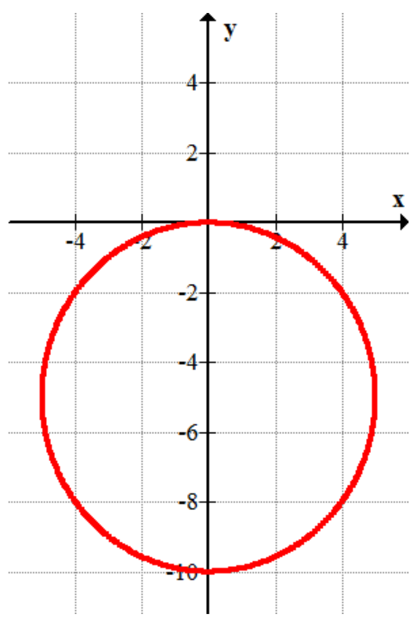
11.



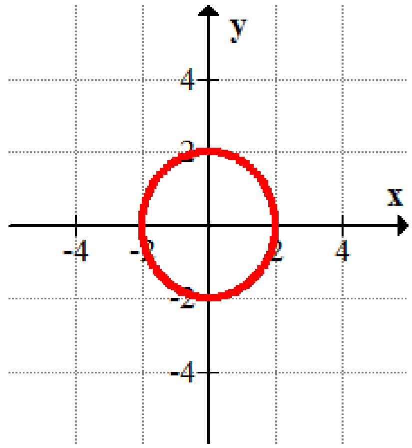

12.



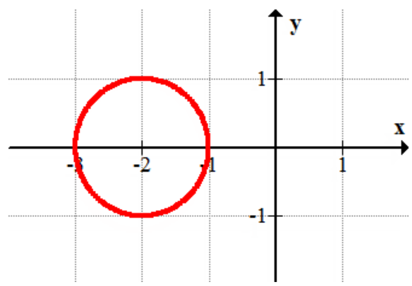

13.




14.




15.



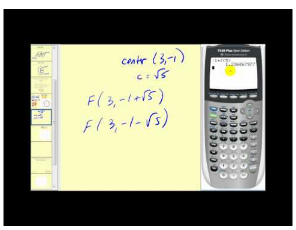
# 9.4 Ellipses

Here you will translate ellipse equations from standard conic form to graphing form, graph ellipses and identify the different axes. You will also identify eccentricity and solve word problems involving ellipses.

An ellipse is commonly known as an oval. Ellipses are just as common as parabolas in the real world with their own uses. Rooms that have elliptical shaped ceilings are called whisper rooms because if you stand at one focus point and whisper, someone standing at the other focus point will be able to hear you.

Ellipses look similar to circles, but there are a few key differences between these shapes. Ellipses have both an $x$-radius and a $y$-radius while circles have only one radius. Another difference between circles and ellipses is that an ellipse is defined as the collection of points that are a set distance from two focus points while circles are defined as the collection of points that are a set distance from one center point. A third difference between ellipses and circles is that not all ellipses are similar to each other while all circles are similar to each other. Some ellipses are narrow and some are almost circular. How do you measure how strangely shaped an ellipse is?

### Watch This



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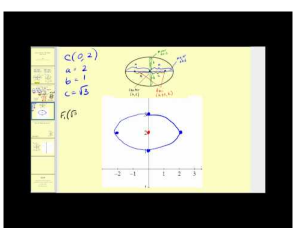
http://www.youtube.com/watch?v=LVumLCx3fQo James Sousa: Conic Sections: The Ellipse part 1



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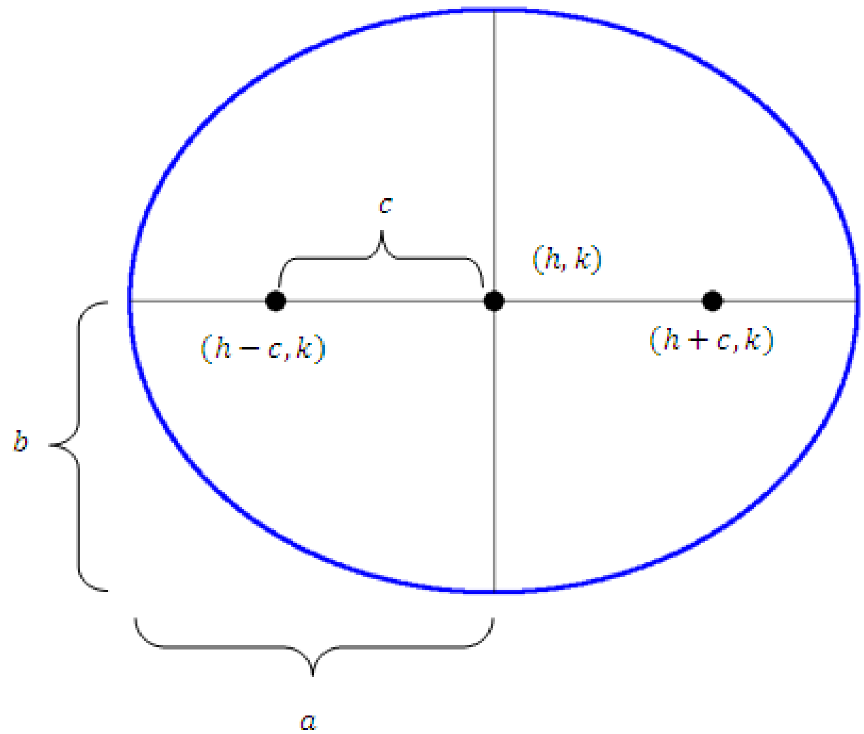
http://www.youtube.com/watch?v=oZB69DY0q9A James Sousa: Conic Sections: The Ellipse part 2

### Guidance

An ellipse has two foci. For every point on the ellipse, the sum of the distances to each foci is constant. This is what defines an ellipse. Another way of thinking about the definition of an ellipse is to allocate a set amount of string and fix the two ends of the string so that there is some slack between them. Then use a pencil to pull the string taught and trace the curve all the way around both fixed points. You will trace an ellipse and the fixed end points of the string will be the foci. Foci is the plural form of focus. In the picture below, $(h,k)$ is the center of the ellipse and the other two marked points are the foci.




The general equation for an ellipse is:

$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

In this case the major axis is horizontal because $a$, the $x$-radius, is larger. If the $y$-radius were larger, then $a$ and $b$ would reverse. In other words, the coefficient $a$ always comes from the length of the semi major axis (the longer axis) and the coefficient $b$ always comes from the length of the semi minor axis (the shorter axis).

In order to find the locations of the two foci, you will need to find the focal radius represented as $c$ using the following relationship:

$a^2-b^2=c^2$

Once you have the focal radius, measure from the center along the major axis to locate the foci. The general shape of an ellipse is measured using eccentricity. Eccentricity is a measure of how oval or how circular the shape is. Ellipses can have an eccentricity between 0 and 1 where a number close to 0 is extremely circular and a number close to 1 is less circular. Eccentricity is calculated by:

$e=\frac{c}{a}$

Ellipses also have two directrix lines that correspond to each focus but on the outside of the ellipse. The distance from the center of the ellipse to each directrix line is $\frac{a^2}{c}$.

**Example A**

Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{x^2}{25}+\frac{y^2}{16}=1$

**Solution:** The center of this ellipse is at (0, 0). The semi major axis is $a=5$ and travels horizontally. This means that the vertices are at (5, 0) and (-5, 0). The semi-minor axis is $b=4$ and travels vertically.

$$\begin{aligned}25-16&=c^2\\3&=c\end{aligned}$$

The focal radius is 3. This means that the foci are at (3, 0) and (-3, 0).

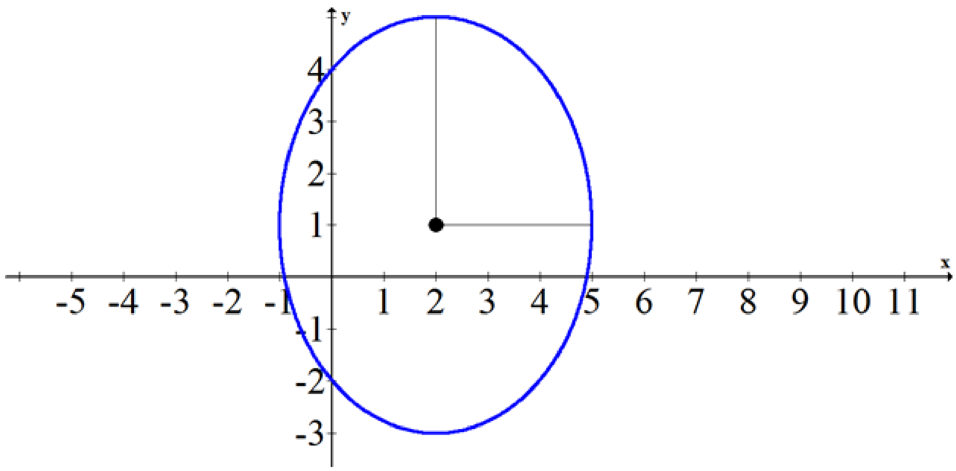
The eccentricity is $e = \frac{3}{5}$.

**Example B**

Sketch the following ellipse.

$\frac{(y-1)^2}{16} + \frac{(x-2)^2}{9} = 1$

**Solution:** Plotting the foci are usually important, but in this case the question simply asks you to sketch the ellipse. All you need is the center, $x$-radius and $y$-radius.




**Example C**

Put the following conic into graphing form.

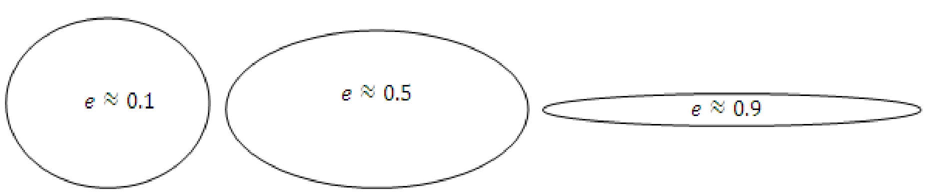
$25x^2 - 150x + 36y^2 + 72y - 639 = 0$

**Solution:**

$$\begin{aligned}
25x^2 - 150x + 36y^2 + 72y - 639 &= 0 \\
25(x^2 - 6x) + 36(y^2 + 2y) &= 639 \\
25(x^2 - 6x + 9) + 36(y^2 + 2y + 1) &= 639 + 225 + 36 \\
25(x-3)^2 + 36(y+1)^2 &= 900 \\
\frac{25(x-3)^2}{900} + \frac{36(y+1)^2}{900} &= \frac{900}{900} \\
\frac{(x-3)^2}{36} + \frac{(y+1)^2}{25} &= 1
\end{aligned}$$

**Concept Problem Revisited**

Ellipses are measured using their eccentricity. Here are three ellipses with estimated eccentricity for you to compare.



Eccentricity is the ratio of the focal radius to the semi major axis: $e = \frac{c}{a}$.

### Vocabulary

The ***semi-major axis*** is the distance from the center of the ellipse to the furthest point on the ellipse. The letter $a$ represents the length of the semi-major axis.

The ***major axis*** is the longest distance from end to end of an ellipse. This distance is twice that of the semi-major axis.

The ***semi-minor axis*** is the distance from the center to the edge of the ellipse on the axis that is perpendicular to the semi-major axis. The letter $b$ represents the length of the semi-minor axis.

An ***ellipse*** is the collection of points whose sum of distances from two foci is constant.

The ***foci*** in an ellipse are the two points that the ellipse curves around.

***Eccentricity*** is a measure of how oval or how circular the shape is. It is the ratio of the focal radius to the semi major axis: $e = \frac{c}{a}$.

### Guided Practice

1. Find the vertices (endpoints of the major axis), foci and eccentricity of the following ellipse.

$\frac{(x-2)^2}{4} + \frac{(y+1)^2}{16} = 1$

2. Sketch the following ellipse.

$(x-3)^2 + \frac{(y-1)^2}{9} = 1$

3. Put the following conic into graphing form.

$9x^2 - 9x + 4y^2 + 12y + \frac{9}{4} = -8$

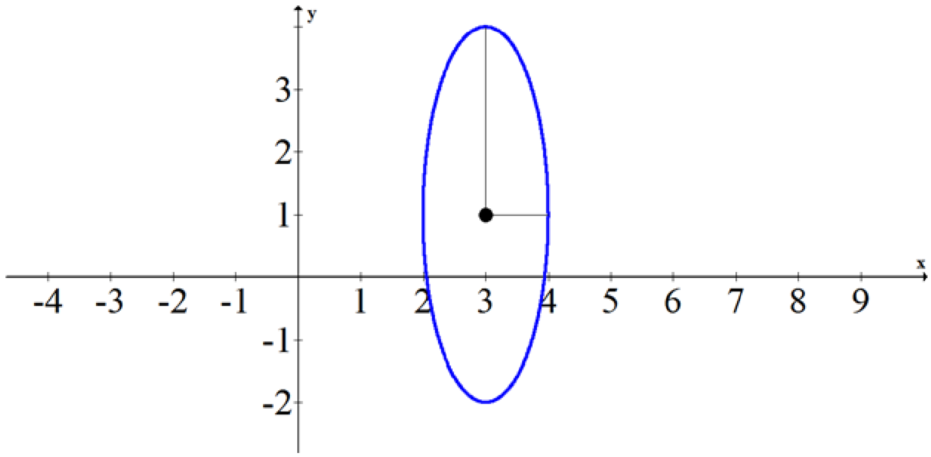
**Answers:**

1. The center of the ellipse is at (2, -1). The major axis is vertical which means the semi major axis is $a = 4$. The vertices are (2, 3) and (2, -5).

$$16^2 - 4^2 = c^2$$
$$4\sqrt{15} = \sqrt{240} = c$$

Thus the foci are $(2, -1 + 4\sqrt{15})$ and $(2, -1 - 4\sqrt{15})$

2.




3.

$$\begin{aligned}
9x^2 - 9x + 4y^2 + 12y + \frac{9}{4} &= -8 \\
9x^2 - 9x + \frac{9}{4} + 4y^2 + 12y &= -8 \\
9\left(x^2 - x - \frac{1}{4}\right) + 4(y^2 + 3y) &= -8 \\
9\left(x - \frac{1}{2}\right)^2 + 4\left(y^2 + 3y + \frac{9}{4}\right) &= -8 + 4 \cdot \frac{9}{4} \\
9\left(x - \frac{1}{2}\right)^2 + 4\left(y + \frac{3}{2}\right)^2 &= 1 \\
\frac{(x - \frac{1}{2})^2}{\frac{1}{9}} + \frac{(y + \frac{3}{2})^2}{\frac{1}{4}} &= 1
\end{aligned}$$

### Practice

Find the vertices, foci, and eccentricity for each of the following ellipses.

1. $\frac{(x-1)^2}{4} + \frac{(y+5)^2}{16} = 1$
2. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$
3. $(x-2)^2 + \frac{(y-1)^2}{4} = 1$

Now sketch each of the following ellipses *(note that they are the same as the ellipses in #1 - #3)*.

4. $\frac{(x-1)^2}{4} + \frac{(y+5)^2}{16} = 1$
5. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{4} = 1$
6. $(x-2)^2 + \frac{(y-1)^2}{4} = 1$

Put each of the following equations into graphing form.

7. $x^2 + 2x + 4y^2 + 56y + 197 = 16$
8. $x^2 - 8x + 9y^2 + 18y + 25 = 9$

9. $9x^2 - 36x + 4y^2 + 16y + 52 = 36$

Find the equation for each ellipse based on the description.

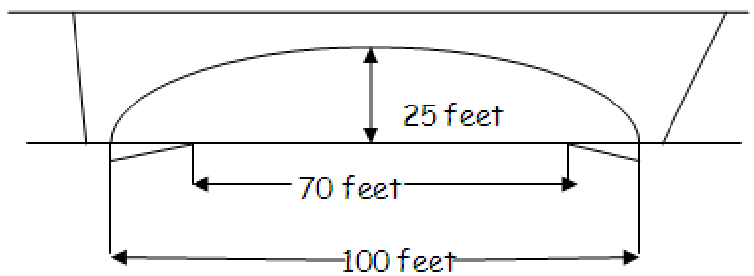
10. An ellipse with vertices (4, -2) and (4, 8) and minor axis of length 6.

11. An ellipse with minor axis from (4, -1) to (4, 3) and major axis of length 12.

12. An ellipse with minor axis from (-2, 1) to (-2, 7) and one focus at (2, 4).

13. An ellipse with one vertex at (6, -15), and foci at (6, 10) and (6, -14).

A bridge over a roadway is to be built with its bottom the shape of a semi-ellipse 100 feet wide and 25 feet high at the center. The roadway is to be 70 feet wide.




14. Find one possible equation of the ellipse that models the bottom of the bridge.

15. What is the clearance between the roadway and the overpass at the edge of the roadway?

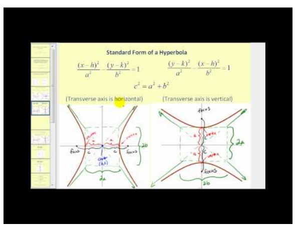
# 9.5 Hyperbolas

Here you will translate conic equations into graphing form and graph hyperbolas. You will also learn how to measure the eccentricity of a hyperbola and solve word problems.

Hyperbolas are relations that have asymptotes. When graphing rational functions you often produce a hyperbola. In this concept, hyperbolas will not be oriented in the same way as with rational functions, but the basic shape of a hyperbola will still be there.

Hyperbolas can be oriented so that they open side to side or up and down. One of the most common mistakes that you can make is to forget which way a given hyperbola should open. What are some strategies to help?

### Watch This



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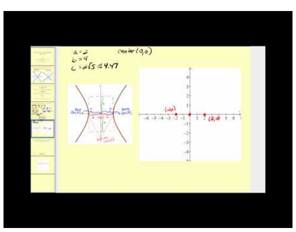
http://www.youtube.com/watch?v=i6vM82SNAUk James Sousa: Conic Sections: The Hyperbola part 1



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http://www.youtube.com/watch?v=6Xahrwp6LkI James Sousa: Conic Sections: The Hyperbola part 2

### Guidance

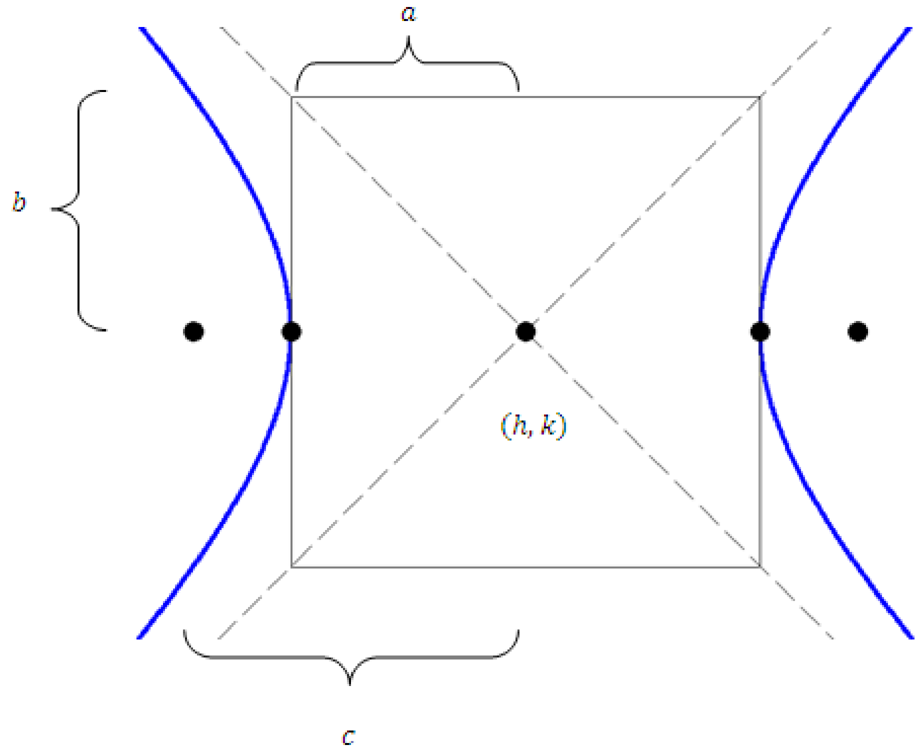
A hyperbola has two foci. For every point on the hyperbola, the difference of the distances to each foci is constant. This is what defines a hyperbola. The graphing form of a hyperbola that opens side to side is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

A hyperbola that opens up and down is:

$$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$$

Notice that for hyperbolas, $a$ goes with the positive term and $b$ goes with the negative term. It does not matter which constant is larger.




When graphing, the constants $a$ and $b$ enable you to draw a rectangle around the center. The transverse axis travels from vertex to vertex and has length $2a$. The conjugate axis travels perpendicular to the transverse axis through the center and has length $2b$. The foci lie beyond the vertices so the eccentricity, which is measured as $e = \frac{c}{a}$, is larger than 1 for all hyperbolas. Hyperbolas also have two directrix lines that are $\frac{a^2}{c}$ away from the center (not shown on the image).

The focal radius is $a^2 + b^2 = c^2$.

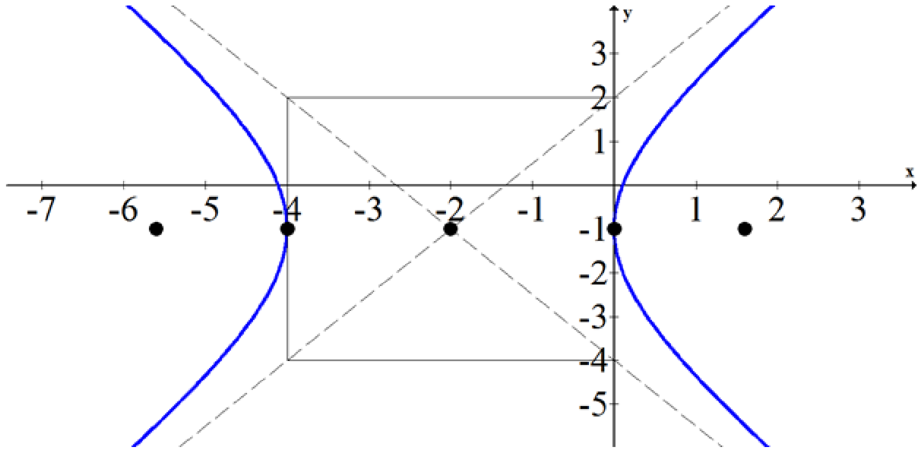
**Example A**

Put the following hyperbola into graphing form and sketch it.

$9x^2 - 4y^2 + 36x - 8y - 4 = 0$

**Solution:**

$$\begin{aligned} 9(x^2+4x) - 4(y^2+2y) &= 4 \\ 9(x^2+4x+4) - 4(y^2+2y+1) &= 4+36-4 \\ 9(x+2)^2 - 4(y+1)^2 &= 36 \\ \frac{(x+2)^2}{4} - \frac{(y+1)^2}{9} &= 1 \end{aligned}$$




**Example B**

Find the equation of the hyperbola with foci at (-3, 5) and (9, 5) and asymptotes with slopes of $\pm\frac{4}{3}$.

**Solution:** The center is between the foci at (3, 5). The focal radius is $c = 6$. The slope of the asymptotes is always the rise over run inside the box. In this case since the hyperbola is horizontal and $a$ is in the $x$ direction the slope is $\frac{b}{a}$. This makes a system of equations.

$$\frac{b}{a} = \pm\frac{4}{3}$$
$$a^2 + b^2 = 6^2$$

When you solve, you get $a = \sqrt{13}$, $b = \frac{4}{3}\sqrt{13}$.

$\frac{(x-3)^2}{13} - \frac{(y-5)^2}{\frac{16}{9}\cdot 13} = 1$

**Example C**

Find the equation of the conic that has a focus point at $(1, 2)$, a directrix at $x = 5$, and an eccentricity equal to $\frac{3}{2}$. Use the property that the distance from a point on the hyperbola to the focus is equal to the eccentricity times the distance from that same point to the directrix:

$\overline{PF} = e\overline{PD}$

**Solution:** This relationship bridges the gap between ellipses which have eccentricity less than one and hyperbolas which have eccentricity greater than one. When eccentricity is equal to one, the shape is a parabola.

$\sqrt{(x-1)^2 + (y-2)^2} = \frac{3}{2}\sqrt{(x-5)^2}$

Square both sides and rearrange terms so that it is becomes a hyperbola in graphing form.

$$\begin{aligned}
x^2-2x+1+(y-2)^2 &= \frac{9}{4}(x^2-10x+25)\\
x^2-2x+1-\frac{9}{4}x^2+\frac{90}{4}x-\frac{225}{4}+(y-2)^2 &= 0\\
-\frac{5}{4}x^2+\frac{92}{4}x+(y-2)^2 &= \frac{221}{4}\\
-5x^2+92x+4(y-2)^2 &= 221\\
-5\left(x^2-\frac{92}{5}x\right)+4(y-2)^2 &= 221
\end{aligned}$$

$$\begin{aligned}
-5\left(x^2-\frac{92}{5}x+\frac{92^2}{10^2}\right)+4(y-2)^2 &= 221-\frac{2116}{5}\\
-5\left(x-\frac{92}{10}\right)^2+4(y-2)^2 &= -\frac{1011}{5}\\
\left(x-\frac{92}{10}\right)^2-(y-2)^2 &= \frac{1011}{100}\\
\frac{\left(x-\frac{92}{10}\right)^2}{\left(\frac{1011}{100}\right)}-\frac{(y-2)^2}{\left(\frac{1011}{100}\right)} &= 1
\end{aligned}$$

**Concept Problem Revisited**

The best strategy to remember which direction the hyperbola opens is often the simplest. Consider the hyperbola $x^2-y^2=1$. This hyperbola opens side to side because $x$ can clearly never be equal to zero. This is a basic case that shows that when the negative is with the $y$ value then the hyperbola opens up side to side.

### Vocabulary

***Eccentricity*** is the ratio between the length of the focal radius and the length of the semi transverse axis. For hyperbolas, the eccentricity is greater than one.

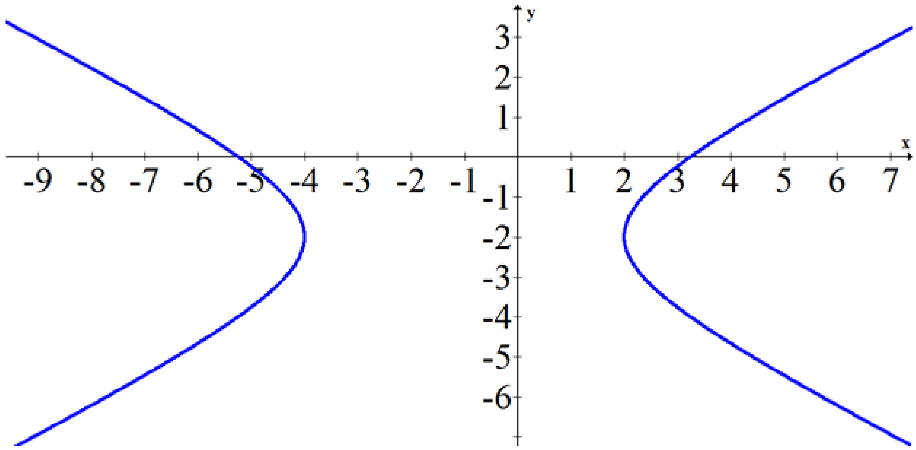
A ***hyperbola*** is the collection of points that share a constant difference between the distances between two focus points.

### Guided Practice

1. Completely identify the components of the following conic.

$$9x^2-16y^2-18x+96y+9=0$$

2. Given the following graph, estimate the equation of the conic.



3. Find the equation of the hyperbola that has foci at (13, 5) and (-17, 5) with asymptote slopes of $\pm\frac{3}{4}$.

**Answers:**

1. $9x^2-16y^2-18x+96y+9=0$

$$9(x^2-2x)-16(y^2-6y)=-9$$
$$9(x^2-2x+1)-16(y^2-6y+9)=-9+9-144$$
$$9(x-1)^2-16(y-3)^2=144$$
$$-\frac{(x-1)^2}{16}+\frac{(y-3)^2}{9}=1$$

Shape: Hyperbola that opens vertically.

Center: (1, 3)

$a=3$

$b=4$

$c=5$

$e=\frac{c}{a}=\frac{5}{3}$

$d=\frac{a^2}{c}=\frac{9}{5}$

Foci: (1, 8), (1, -2)

Vertices: (1, 6), (1, 0)

Equations of asymptotes: $(x-1)=\pm\frac{3}{4}(y-3)$

Note that it is easiest to write the equations of the asymptotes in point-slope form using the center and the slope.

Equations of directrices: $y=3\pm\frac{9}{5}$

2. Since exact points are not marked, you will need to estimate the slope of asymptotes to get an approximation for $a$ and $b$. The slope seems to be about $\pm\frac{2}{3}$. The center seems to be at (-1, -2). The transverse axis is 6 which means $a=3$.

$\frac{(x+1)^2}{9}-\frac{(y+2)^2}{4}=1$

3. The center of the conic must be at (-2, 5). The focal radius is $c=15$. The slopes of the asymptotes are $\pm\frac{3}{4}=\frac{b}{a}$.

$a^2+b^2=c^2$

Since 3, 4, 5 is a well known Pythagorean number triple it should be clear to you that $a = 12, b = 9$.

$\frac{(x+2)^2}{12^2} - \frac{(y-5)^2}{9^2} = 1$

## Practice

Use the following equation for #1 - #5: $x^2 + 2x - 4y^2 - 24y - 51 = 0$

1. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
2. Identify whether the hyperbola opens side to side or up and down.
3. Find the location of the vertices.
4. Find the equations of the asymptotes.
5. Sketch the hyperbola.

Use the following equation for #6 - #10: $-9x^2 - 36x + 16y^2 - 32y - 164 = 0$

6. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
7. Identify whether the hyperbola opens side to side or up and down.
8. Find the location of the vertices.
9. Find the equations of the asymptotes.
10. Sketch the hyperbola.

Use the following equation for #11 - #15: $x^2 - 6x - 9y^2 - 54y - 81 = 0$

11. Put the hyperbola into graphing form. Explain how you know it is a hyperbola.
12. Identify whether the hyperbola opens side to side or up and down.
13. Find the location of the vertices.
14. Find the equations of the asymptotes.
15. Sketch the hyperbola.

# 9.6 Degenerate Conics

Here you will discover what happens when a conic equation can't be put into graphing form.

The general equation of a conic is$Ax^2+Bxy+Cy^2+Dx+Ey+F=0$. This form is so general that it encompasses all regular lines, singular points and degenerate hyperbolas that look like an X. This is because there are a few special cases of how a plane can intersect a two sided cone. How are these degenerate shapes formed?

### Guidance

Degenerate conic equations simply cannot be written in graphing form. There are three types of degenerate conics:

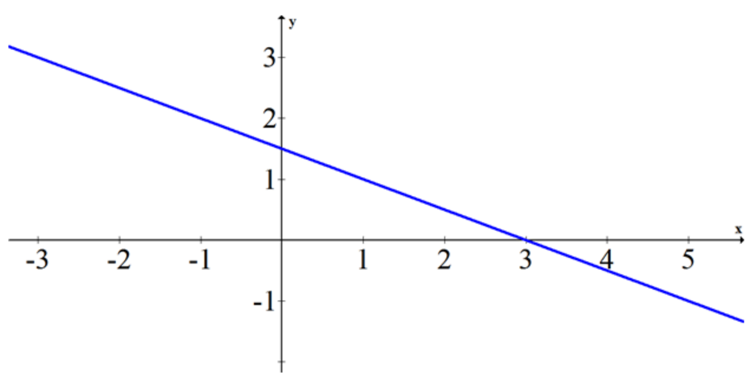
1. ***A singular point***, which is of the form: $\frac{(x-h)^2}{a}+\frac{(y-k)^2}{b}=0$. You can think of a singular point as a circle or an ellipse with an infinitely small radius.
2. ***A line***, which has coefficients $A=B=C=0$ in the general equation of a conic. The remaining portion of the equation is $Dx+Ey+F=0$, which is a line.
3. ***A degenerate hyperbola***, which is of the form: $\frac{(x-h)^2}{a}-\frac{(y-k)^2}{b}=0$. The result is two intersecting lines that make an "X" shape. The slopes of the intersecting lines forming the X are $\pm\frac{b}{a}$. This is because $b$ goes with the $y$ portion of the equation and is the rise, while $a$ goes with the $x$ portion of the equation and is the run.

**Example A**

Transform the conic equation into standard form and sketch.

$0x^2+0xy+0y^2+2x+4y-6=0$

**Solution:** This is the line $y=-\frac{1}{2}x+\frac{3}{2}$.




**Example B**

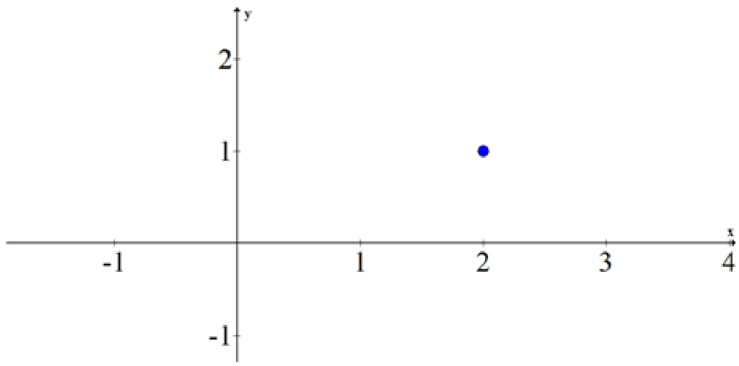
Transform the conic equation into standard form and sketch.

$3x^2-12x+4y^2-8y+16=0$

**Solution:** $3x^2-12x+4y^2-8y+16=0$

$$\begin{aligned}
3(x^2-4x)+4(y^2-2y) &= -16\\
3(x^2-4x+4)+4(y^2-2y+1) &= -16+12+4\\
3(x-2)^2+4(y-1)^2 &= 0\\
\frac{(x-2)^2}{4}+\frac{(y-1)^2}{3} &= 0
\end{aligned}$$

The point (2, 1) is the result of this degenerate conic.




**Example C**

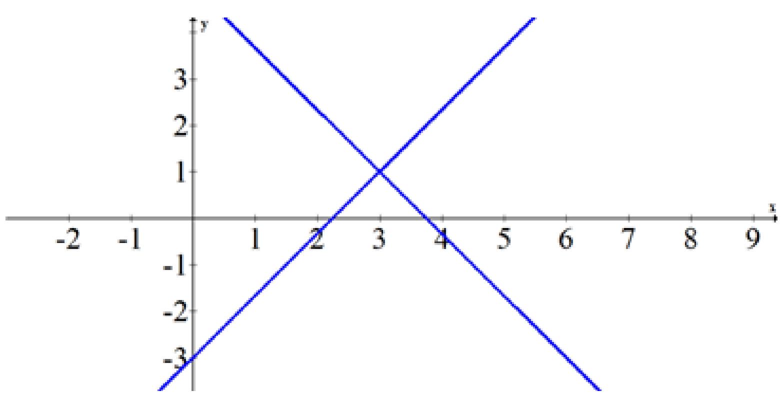
Transform the conic equation into standard form and sketch.

$16x^2-96x-9y^2+18y+135=0$

**Solution:** $16x^2-96x-9y^2+18y+135=0$

$$\begin{aligned}
16(x^2-6x)-9(y^2-2y) &= -135\\
16(x^2-6x+9)-9(y^2-2y+1) &= -135+144-9\\
16(x-3)^2-9(y-1)^2 &= 0\\
\frac{(x-3)^2}{9}-\frac{(y-1)^2}{16} &= 0
\end{aligned}$$

This is a degenerate hyperbola.



**Concept Problem Revisited**

When you intersect a plane with a two sided cone where the two cones touch, the intersection is a **single point**. When you intersect a plane with a two sided cone so that the plane touches the edge of one cone, passes through the central point and continues touching the edge of the other conic, this produces a **line**. When you intersect a plane with a two sided cone so that the plane passes vertically through the central point of the two cones, it produces a **degenerate hyperbola**.

## Vocabulary

A ***degenerate conic*** is a conic that does not have the usual properties of a conic. Since some of the coefficients of the general equation are zero, the basic shape of the conic is merely a point, a line or a pair of lines. The connotation of the word degenerate means that the new graph is less complex than the rest of conics.

## Guided Practice

1. Create a conic that describes just the point (4, 7).

2. Transform the conic equation into standard form and sketch.

$-4x^2+8x+y^2+4y=0$

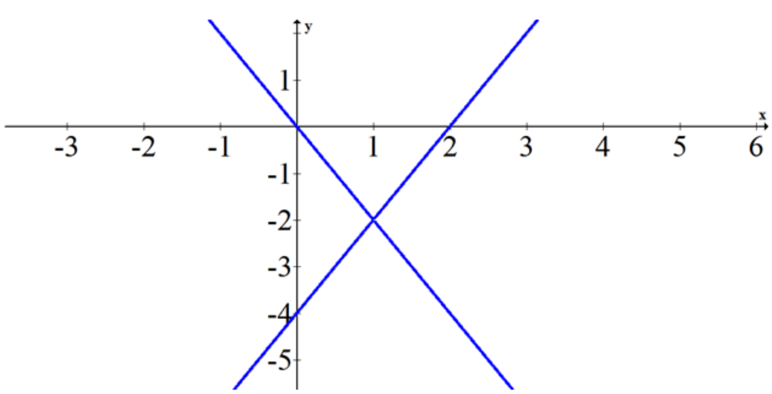
3. Can you tell just by looking at a conic in general form if it is a degenerate conic?

**Answers:**

1. $(x-4)^2+(y-7)^2=0$

2.

$$\begin{aligned}
-4x^2+8x+y^2+4y&=0\\
-4(x^2-2x)+(y^2+4y)&=0\\
-4(x^2-2x+1)+(y^2+4y+4)&=-4+4\\
-4(x-1)^2+(y+2)^2&=0\\
\frac{(x-1)^2}{1}-\frac{(y+2)^2}{4}&=0
\end{aligned}$$




3. In general you cannot tell if a conic is degenerate from the general form of the equation. You can tell that the degenerate conic is a line if there are no $x^2$ or $y^2$ terms, but other than that you must always try to put the conic equation into graphing form and see whether it equals zero because that is the best way to identify degenerate conics.

### Practice

1. What are the three degenerate conics?

Change each equation into graphing form and state what type of conic or degenerate conic it is.

2. $x^2-6x-9y^2-54y-72=0$

3. $4x^2+16x-9y^2+18y-29=0$

4. $9x^2+36x+4y^2-24y+72=0$

5. $9x^2+36x+4y^2-24y+36=0$

6. $0x^2+5x+0y^2-2y+1=0$

7. $x^2+4x-y+8=0$

8. $x^2-2x+y^2-6y+6=0$

9. $x^2-2x-4y^2+24y-35=0$

10. $x^2-2x+4y^2-24y+33=0$

Sketch each conic or degenerate conic.

11. $\frac{(x+2)^2}{4}+\frac{(y-3)^2}{9}=0$

12. $\frac{(x-3)^2}{9}+\frac{(y+3)^2}{16}=1$

13. $\frac{(x+2)^2}{9}-\frac{(y-1)^2}{4}=1$

14. $\frac{(x-3)^2}{9}-\frac{(y+3)^2}{4}=0$

15. $3x+4y=12$

You learned that a conic section is the family of shapes that are formed by the different ways a flat plane intersects a two sided cone in three dimensional space. Parabolas, circles, ellipses and hyperbolas each have their own graphing form of equations that helped you identify information about them like the focus and the directrix.

# 9.7 References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# CHAPTER 10 Polar and Parametric Equations

## Chapter Outline

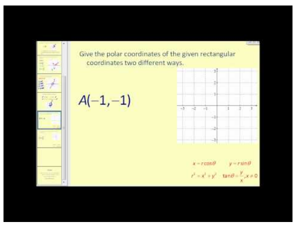


The Cartesian coordinate system with $(x, y)$ coordinates is great for representing some functions and relations, but is very limiting in some ways. The polar coordinate system allows you to easily graph spirals and other shapes that are not functions. Parametric equations allow you to have both $x$ and $y$ depend on a third variable $t$. Here you will explore both polar and parametric equations.

# 10.1 Polar and Rectangular Coordinates

In the rectangular coordinate system, points are identified by their distances from the $x$and $y$axes. In the**polar coordinate system,** points are identified by their angle on the unit circle and their distance from the origin. You can use basic right triangle trigonometry to translate back and forth between the two representations of the same point. How are lines and other functions affected by this new coordinate system?

## Watch This

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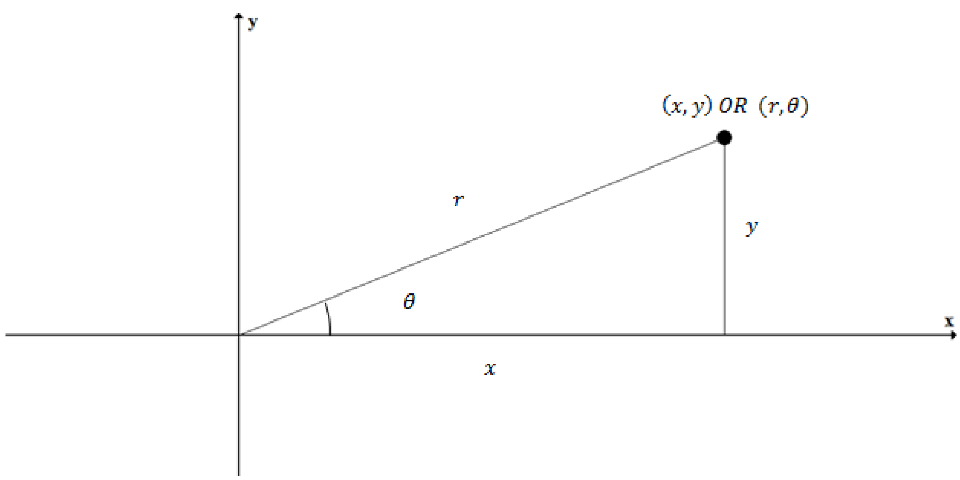
http://www.youtube.com/watch?v=-tZR3ggdoIU James Sousa: Polar Coordinates

## Guidance

Rectangular coordinates are the ordinary $(x,y)$ coordinates that you are used to.




Polar coordinates represent the same point, but describe the point by its distance from the origin $(r)$ and its angle on the unit circle $(\theta)$. To translate back and forth between polar and rectangular coordinates you should use the basic trig relationships:

$$\sin\theta = \frac{y}{r} \rightarrow r\cdot\sin\theta = y$$
$$\cos\theta = \frac{x}{r} \rightarrow r\cdot\cos\theta = x$$

You can also express the relationship between $x, y$ and $r$ using the Pythagorean Theorem.

$x^2 + y^2 = r^2$

Note that coordinates in polar form are not unique. This is because there are an infinite number of coterminal angles that point towards any given $(x, y)$ coordinate.

Once you can translate back and forth between points, use the same substitutions to change equations too. A polar equation is written with the radius as a function of the angle. This means an equation in polar form should be written in the form $r = \_\_$.

**Example A**

Convert the point (3, 4) to polar coordinates in three different ways.

**Solution:** $\tan\theta = \frac{4}{3}$

$$\begin{aligned}
&\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1^\circ \\
&r^2 = 3^2 + 4^2 \\
&r = 5
\end{aligned}$$

Three equivalent polar coordinates for the point (3, 4) are:

$(5, 53.1^\circ)$, $(5, 413.1^\circ)$, $(-5, 233.1^\circ)$

Notice how the third coordinate points in the opposite direction and has a seemingly negative radius. This means go in the opposite direction of the angle.

**Example B**

Write the equation of the line in polar form: $y = -x + 1$.

**Solution:** Make substitutions for $y$ and $x$. Then, solve for $r$.

$$\begin{aligned}
r \cdot \sin\theta &= -r \cdot \cos\theta + 1 \\
r \cdot \sin\theta + r \cdot \cos\theta &= 1 \\
r(\sin\theta + \cos\theta) &= 1 \\
r &= \frac{1}{\sin\theta + \cos\theta}
\end{aligned}$$

**Example C**

Express the following equation using rectangular coordinates: $r = \frac{8}{1 + 2\cos\theta}$.

**Solution:** Use the fact that $r = \pm\sqrt{x^2 + y^2}$ and $r\cos\theta = x$.

$$\begin{aligned}
r + 2r \cdot \cos\theta &= 8 \\
\pm\sqrt{x^2 + y^2} + 2x &= 8 \\
\pm\sqrt{x^2 + y^2} &= 8 - 2x \\
x^2 + y^2 &= 64 - 32x + 4x^2 \\
-3x^2 + 32x + y^2 - 64 &= 0
\end{aligned}$$

This is the equation of a hyperbola.

**Concept Problem Revisited**

The general way to express a line $y = mx + b$ in polar form is $r = \frac{b}{\sin\theta - m\cdot\cos\theta}$.

## Vocabulary

The ***polar coordinate system*** defines each point by its angle on the unit circle $(\theta)$ and its distance from the origin $(r)$. Points in the polar coordinate system are written as $(r,\theta)$.

The ***rectangular coordinate system*** defines each point by its distance from the $x$ and $y$ axes. Points in the rectangular coordinate system are written as $(x,y)$.

## Guided Practice

1. Sketch the following polar equation: $r = 3$.

2. Sketch the following polar equation: $r = \theta$ with $\theta : 0 \le \theta \le 2\pi$.

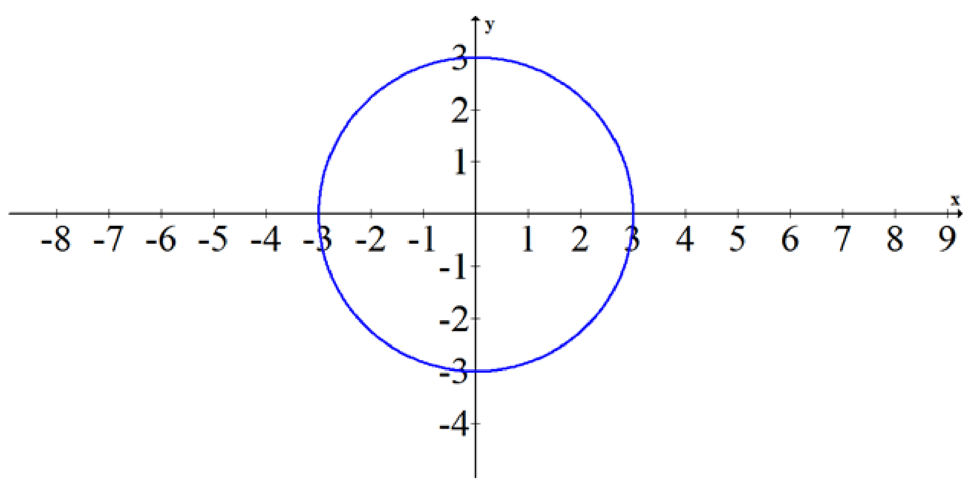
3. Translate the following polar expression into rectangular coordinates and then graph.

$$r = 2\cdot\sec\left(\theta - \frac{\pi}{2}\right)$$

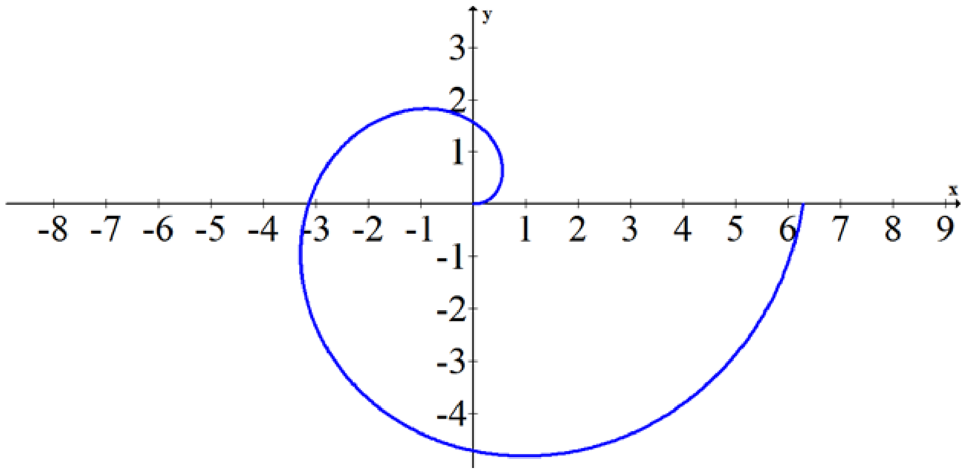
**Answers:**

1. Since theta is not in the equation, it can vary freely. This simple equation produces a perfect circle of radius 3 centered at the origin.

You can show this equation is equivalent to $x^2 + y^2 = 9$

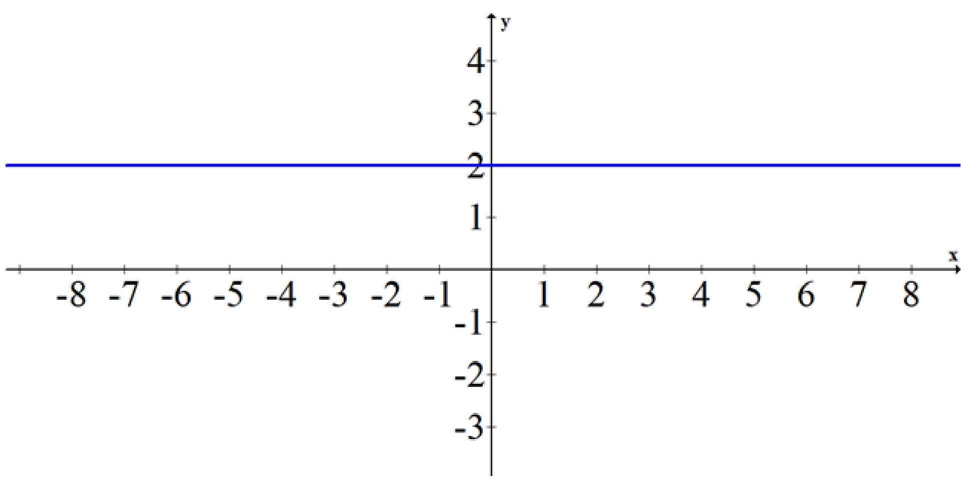



2. The equation $r = \theta$ is an example of a polar equation that cannot be easily expressed in rectangular form. In order to sketch the graph, identify a few key points: $(0,0), \left(\frac{\pi}{2},\frac{\pi}{2}\right), (\pi,\pi), \left(\frac{3\pi}{2},\frac{3\pi}{2}\right), (2\pi,2\pi)$. You should see that the shape is very recognizable as a spiral.




3. Simplify the polar equation first before converting to rectangular coordinates.

$$\begin{aligned} r &= 2 \cdot \sec\left(\theta - \frac{\pi}{2}\right) \\ r \cdot \cos\left(\theta - \frac{\pi}{2}\right) &= 2 \\ r \cdot \cos\left(\frac{\pi}{2} - \theta\right) &= 2 \\ r \cdot \sin\theta &= 2 \\ y &= 2 \end{aligned}$$




## Practice

Plot the following polar coordinates.

1. $\left(3, \frac{5\pi}{6}\right)$
2. $\left(2, \frac{\pi}{2}\right)$
3. $\left(4, -\frac{7\pi}{6}\right)$
4. $\left(-2, \frac{5\pi}{3}\right)$

Give two alternate sets of coordinates for each point.

5. $(2,60^\circ)$

6. $(5,330^\circ)$

7. $(2,210^\circ)$

Graph each equation.

8. $r=4$

9. $\theta=\frac{\pi}{4}$

10. $r=2\theta$ with $\theta:0\leq\theta\leq 2\pi$.

Convert each point to rectangular form.

11. $\left(4,\frac{2\pi}{3}\right)$

12. $\left(3,\frac{\pi}{4}\right)$

13. $\left(5,\frac{\pi}{3}\right)$

Convert each point to polar form using radians where $0\leq\theta<2\pi$.

14. (1, 3)

15. (1, -4)

16. (2, 6)

Convert each equation to polar form.

17. $x=3$

18. $2x+4y=2$

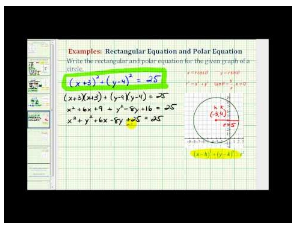
Here you will use polar and rectangular coordinates to identify a point and you will use polar and rectangular equations to identify lines and curves.

# 10.2 Polar Equations of Conics

Here you will find equations and graphs for various conics, including those whose major axis is at a slant.

Polar coordinates allow you to extend your knowledge of conics in a new context. Calculators are an excellent tool for graphing polar conics. What settings do you need to know in order to properly use your calculator?

### Watch This

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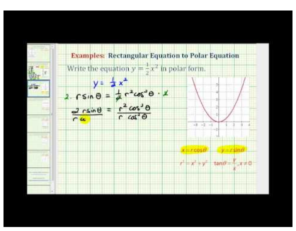
http://youtu.be/ZFNSMfWvVNY James Sousa: Find the Rectangular and Polar Equation of a Circle from a Graph

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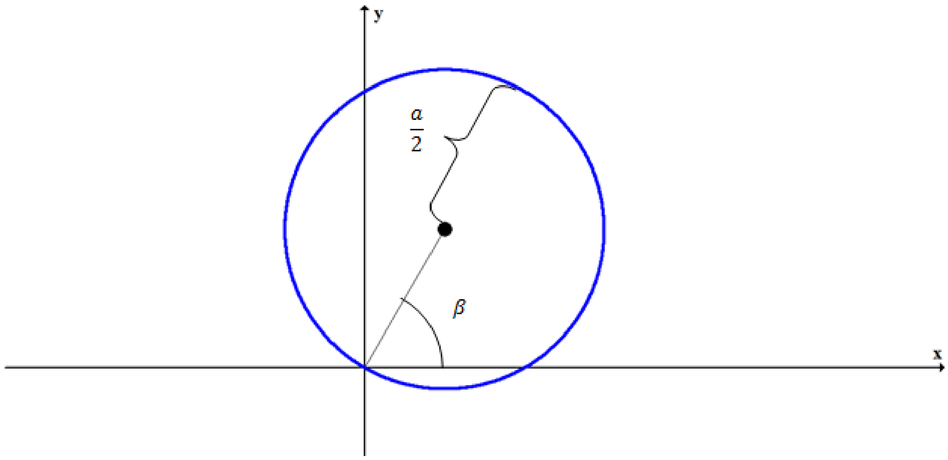
http://youtu.be/XkZAtV8jNb4 James Sousa: Find the Polar Equation for a Parabola

### Guidance

Polar equations refer to the radius $r$ as a function of the angle $\theta$. There are a few typical polar equations you should be able to recognize and graph directly from their polar form.

The following polar function is a circle of radius $\frac{a}{2}$ passing through the origin with a center at angle $\beta$.

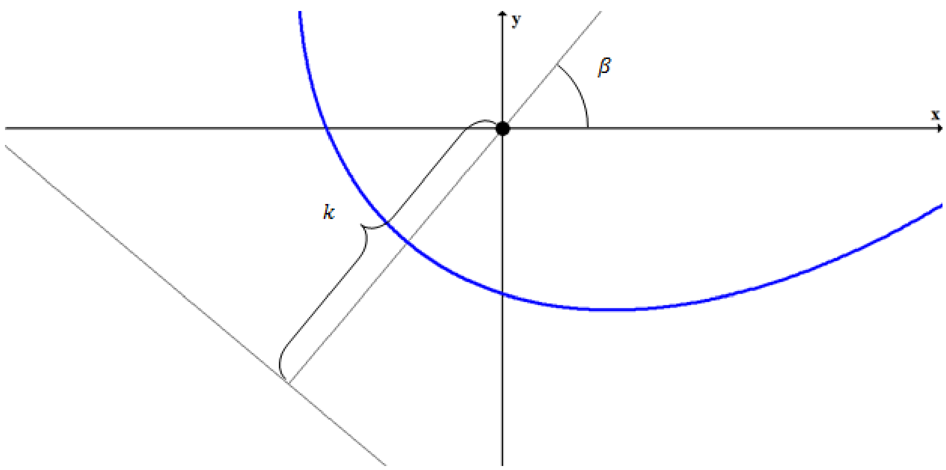
$r = a \cdot \cos(\theta - \beta)$




There are other ways of representing a circle like this using cofunction identities and coterminal angles.

Ellipses, parabolas and hyperbolas have a common general polar equation. Just like with the circle, there are other ways of representing these relations using cofunction and coterminal angles; however, this general form is easiest to use because each parameter can be immediately interpreted in a graph.

$r = \frac{k \cdot e}{1 - e \cdot \cos(\theta - \beta)}$



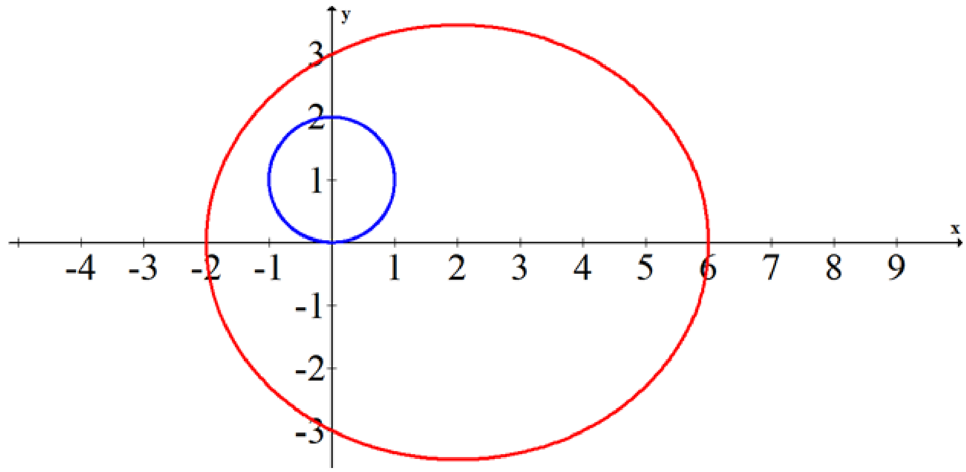

One of the focus points of a conic written in this way is always at the pole (the origin). The angle $\beta$ indicates the angle towards the center if the conic is an ellipse, the opening direction if the conic is a parabola and the angle away from the center if the conic is a hyperbola. The eccentricity $e$ should tell you what conic it is. The constant $k$ is the distance from the focus at the pole to the nearest directrix. This directrix lies in the opposite direction indicated by $\beta$.

There are many opportunities for questions involving partial information with polar conics. A few relationships that are often useful for solving these questions are:

- $e = \frac{c}{a} = \frac{\overline{PF}}{\overline{PD}} \rightarrow \overline{PF} = e \cdot \overline{PD}$
- Ellipses: $k = \frac{a^2}{c} - c$
- Hyperbolas: $k = c - \frac{a^2}{c}$

### Example A

A great way to discover new types of graphs in polar coordinates is to experiment on your own with your calculator. Try to come up with equations and graphs that look similar to the following two polar functions.




**Solution:** The circle in blue has a center at 90° and has a diameter of 2. Its equation is $r = 2\cos(\theta - 90°)$.

The red ellipse appears to have center at (2, 0) with $a = 4$ and $c = 2$. This means the eccentricity is $e = \frac{1}{2}$. In order to write the equation in polar form you still need to find $k$.

$k = \frac{a^2}{c} - c = \frac{4^2}{2} - 2 = 8 - 2 = 6$

Thus the equation for the ellipse is:

$r = \frac{6 \cdot \frac{1}{2}}{1 - \frac{1}{2} \cdot \cos(\theta)}$

**Example B**

Identify the center, foci, vertices and equations of the directrix lines for the following conic:

$r = \frac{20}{4 - 5 \cdot \cos\left(\theta - \frac{3\pi}{4}\right)}$

**Solution:** First the polar equation needs to be in graphing form. This means that the denominator needs to look like $1 - e \cdot \cos(\theta - \beta)$.

$$r = \frac{20}{4 - 5 \cdot \cos\left(\theta - \frac{3\pi}{4}\right)} \cdot \frac{\frac{1}{4}}{\frac{1}{4}} = \frac{5}{1 - \frac{5}{4} \cdot \cos\left(\theta - \frac{3\pi}{4}\right)} = \frac{4 \cdot \frac{5}{4}}{1 - \frac{5}{4} \cdot \cos\left(\theta - \frac{3\pi}{4}\right)}$$

$$e = \frac{5}{4}, \quad k = 4, \quad \beta = \frac{3\pi}{4} = 135°$$

Using this information and the relationships you were reminded of in the guidance, you can set up a system and solve for $a$ and $c$.

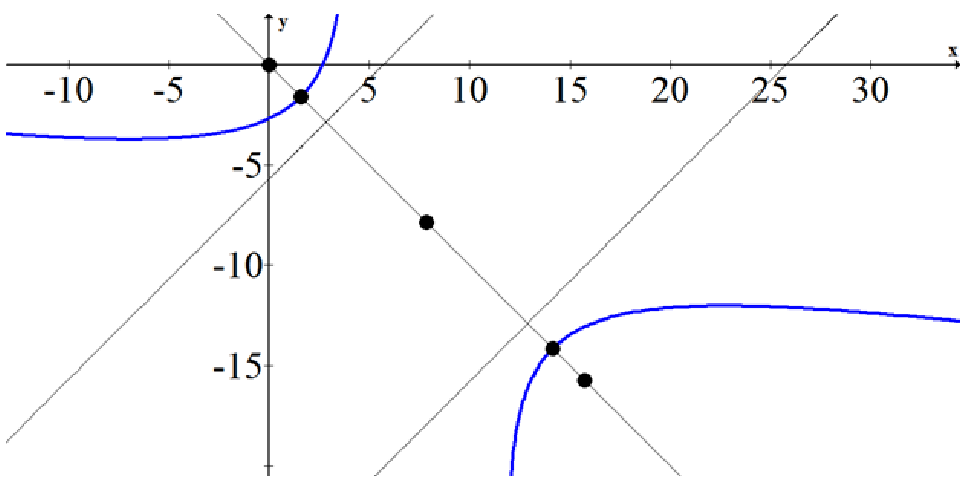
$$4 = c - \frac{a^2}{c}$$
$$\frac{5}{4} = \frac{c}{a} \rightarrow \frac{4}{5} = \frac{a}{c} \rightarrow \frac{4c}{5} = a$$
$$4 = c - \left(\frac{4c}{5}\right)^2 \cdot \frac{1}{c}$$
$$4 = c - \frac{16c^2}{25c}$$
$$4 = \frac{9c}{25}$$
$$\frac{100}{9} = c$$
$$\frac{80}{9} = a$$

The center is the point $\left(\frac{100}{9}, \frac{7\pi}{4}\right)$ which is much more convenient to write in polar coordinates. The closest directrix is the line $r = 4 \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. The other directrix is the line $r = \left(2 \cdot \frac{100}{9} - 4\right) \cdot \sec\left(\theta - \frac{7\pi}{4}\right)$. One focus is at the pole, the other focus is the point $\left(\frac{200}{9}, \frac{7\pi}{4}\right)$. The vertices are at the center plus or minus $a$ in the same angle: $\left(\frac{100}{9} \pm \frac{80}{9}, \frac{7\pi}{4}\right)$

**Example C**

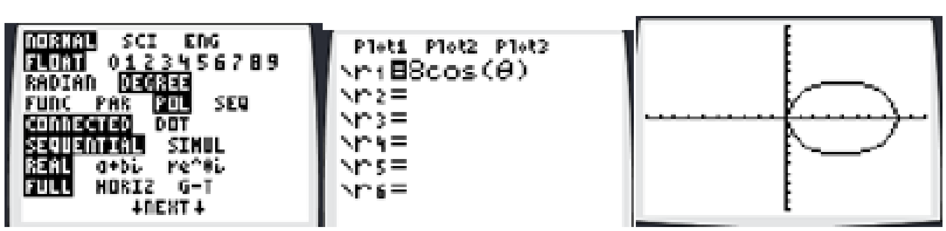
Graph the conic from Example B.

**Solution:**




**Concept Problem Revisited**

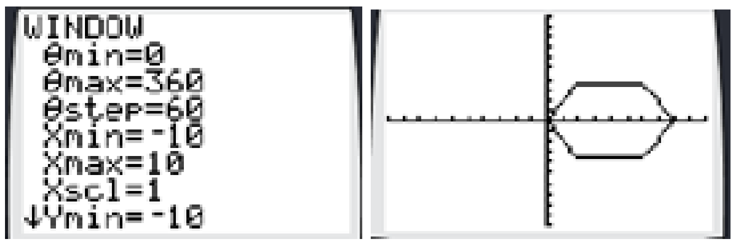
Most calculators have a polar coordinate mode. On the TI-84, the mode can be switched to polar in the mode menu. This changes the graphing features. You can choose to be in radians or degrees and graphs will look the same. When you graph a circle of the form $r = 8 \cdot \cos\theta$. you should see the following on your calculator.



When you go to the window setting you should notice that in addition to $X_{min}, X_{max}$ there are new settings called $\theta_{min}, \theta_{max}$ and $\theta_{step}$.

If $\theta_{min}$ and $\theta_{max}$ do not span an entire period, you may end up missing part of your polar graph.

The $\theta_{step}$ controls how accurate the graph should be. If you put $\theta_{step}$ at a low number like 0.1 the graph will plot extremely slowly because the calculator is doing 3600 cosine calculations. On the other hand if $\theta_{step} = 30$ then the calculator will do fewer calculations producing a rough circle, but probably not accurate enough for your purposes.




### Vocabulary

A ***parameter*** is a constant in a general equation that takes on a specific value in a specific equation. In this concept, the parameters are $e, k, \beta, d$. The variables $x, y, r, \theta$ are not parameters.

The ***pole*** is how the origin is described in polar form.

***Eccentricity*** is a parameter associated with conic sections. It is defined by the formulas $e = \frac{c}{a}$ or more generally, $e = \frac{\overline{PF}}{\overline{PD}}$.

### Guided Practice

1. Convert the following conic from polar form to rectangular form.

$r = \frac{3}{2-\cos\theta}$

2. Graph the following conic.

$r = \frac{3}{2-\cos(\theta-30^\circ)}$

3. Translate the following conic to polar form.

$(x-3)^2 + (y+4)^2 = 25$

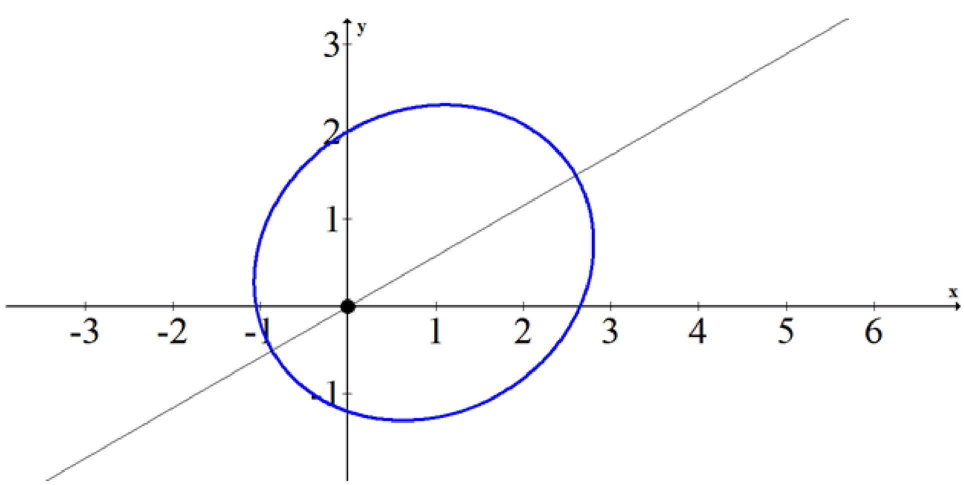
**Answers:**

1. There are many ways to convert from polar form to rectangular form. You should become comfortable with the algebra.

$$
\begin{aligned}
r &= \frac{3}{2-\cos\theta} \\
r(2-\cos\theta) &= 3 \\
2r - r\cdot\cos\theta &= 3 \\
2r &= 3 + r\cdot\cos\theta = 3 + y \\
4r^2 &= 9 + 6y + y^2 \\
4(x^2+y^2) &= 9 + 6y + y^2 \\
4x^2 + 4y^2 &= 9 + 6y + y^2 \\
4x^2 + 3y^3 + 6y &= 9 \\
4x^2 + 3(y^2+2y+1) &= 9+3 \\
4x^2 + 3(y+1)^2 &= 12 \\
\frac{x^2}{3} + \frac{(y+1)^2}{4} &= 1
\end{aligned}
$$

2. Convert to the standard conic form.

$$
\begin{aligned}
&r = \frac{3}{2-\cos(\theta-30^\circ)} \\
&r = \frac{3}{2-\cos(\theta-30^\circ)}\cdot\frac{\frac{1}{2}}{\frac{1}{2}} = \frac{\frac{3}{2}}{1-\frac{1}{2}\cdot\cos(\theta-30^\circ)} = \frac{3\cdot\frac{1}{2}}{1-\frac{1}{2}\cdot\cos(\theta-30^\circ)} \\
&k = 3, \quad e = \frac{1}{2}, \quad \beta = 30^\circ
\end{aligned}
$$




3. Expand the original equation and then translate to polar coordinates:

$$(x-3)^2+(y+4)^2=25$$
$$x^2-6x+9+y^2+8y+16=25$$
$$r^2-6x+8y=0$$
$$r^2-6r\cdot\cos\theta+8r\cdot\sin\theta=0$$
$$r-6\cos\theta+8\sin\theta=0$$
$$r=6\cos\theta-8\sin\theta$$

### Practice

Convert the following conics from polar form to rectangular form. Then, identify the conic.

1. $r=\frac{5}{3-\cos\theta}$
2. $r=\frac{4}{2-\cos\theta}$
3. $r=\frac{2}{2-\cos\theta}$
4. $r=\frac{3}{2-4\cos\theta}$
5. $r=5\cos(\theta)$

Graph the following conics.

6. $r=\frac{5}{4-2\cos(\theta-90^\circ)}$
7. $r=\frac{5}{3-7\cos(\theta-60^\circ)}$
8. $r=\frac{3}{3-3\cos(\theta-30^\circ)}$
9. $r=\frac{1}{2-\cos(\theta-60^\circ)}$
10. $r=\frac{3}{6-3\cos(\theta-45^\circ)}$

Translate the following conics to polar form.

11. $\frac{(x-1)^2}{4}+\frac{y^2}{3}=1$
12. $(x-5)^2+(y+12)^2=169$
13. $x^2+(y+1)^2=1$
14. $(x-1)^2+y^2=1$
15. $-3x^2-4x+y^2-1=0$

# 10.3 Parameters and Parameter Elimination

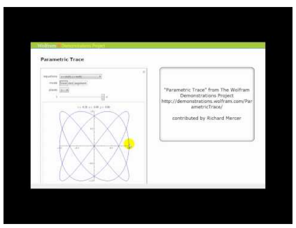
Here you will represent equations and graphs in a new way with parametric equations.

In a parametric equation, the variables $x$and $y$are not dependent on one another. Instead, both variables are dependent on a third variable,$t$. Usually $t$will stand for time. A real world example of the relationship between $x, y$and $t$is the height, weight and age of a baby.

Both the height and the weight of a baby depend on time, but there is also clearly a positive relationship between just the height and weight of the baby. By focusing on the relationship between the height and the weight and letting time hide in the background, you create a parametric relationship between the three variables.

What other types of real world situations are modeled with parametric equations?

### Watch This



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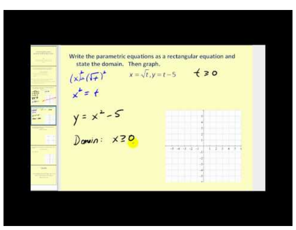
http://www.youtube.com/watch?v=Fz6p4aC9e2Q James Sousa: Introduction to Parametric Equations



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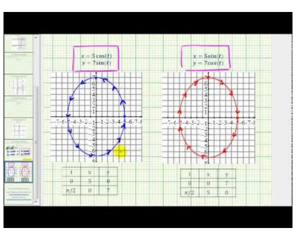
http://www.youtube.com/watch?v=tW6N7DFTvrM James Sousa: Converting Parametric Equation to Rectangular Form



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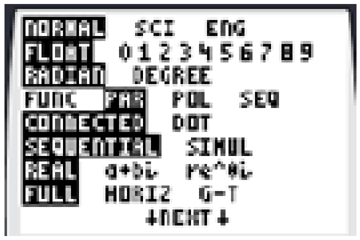
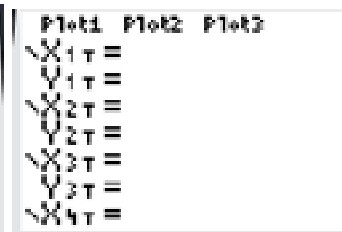
http://youtu.be/zs0Nw0tb4y8 James Sousa: Parametric Equations for an Ellipse in Cartesian Form

### Guidance

In your graphing calculator there is a parametric mode. Once you put your calculator into parametric mode, on the graphing screen you will no longer see $y =$ __, instead, you will see:






Notice how for plot one, the calculator is asking for two equations based on variable $T$:

$$x_{1T} = f(t)$$
$$y_{1T} = g(t)$$

In order to transform a parametric equation into a normal one, you need to do a process called "eliminating the parameter." To do this, you must solve the $x = f(t)$ equation for $t = f^{-1}(x)$ and substitute this value of $t$ into the $y$ equation. This will produce a normal function of $y$ based on $x$.

There are two major benefits of graphing in parametric form. First, it is straightforward to graph a portion of a regular function using the $T_{min}, T_{max}$ and $T_{step}$ in the window setting. Second, parametric form enables you to graph projectiles in motion and see the effects of time.

**Example A**

Eliminate the parameter in the following equations.

$$x = 6t - 2$$
$$y = 5t^2 - 6t$$

**Solution:** $x = 6t - 2$ So $\frac{x+2}{6} = t$. Now, substitute this value for $t$ into the second equation:

$y = 5\left(\frac{x+2}{6}\right)^2 - 6\left(\frac{x+2}{6}\right)$

**Example B**

For the given parametric equation, graph over each interval of $t$.

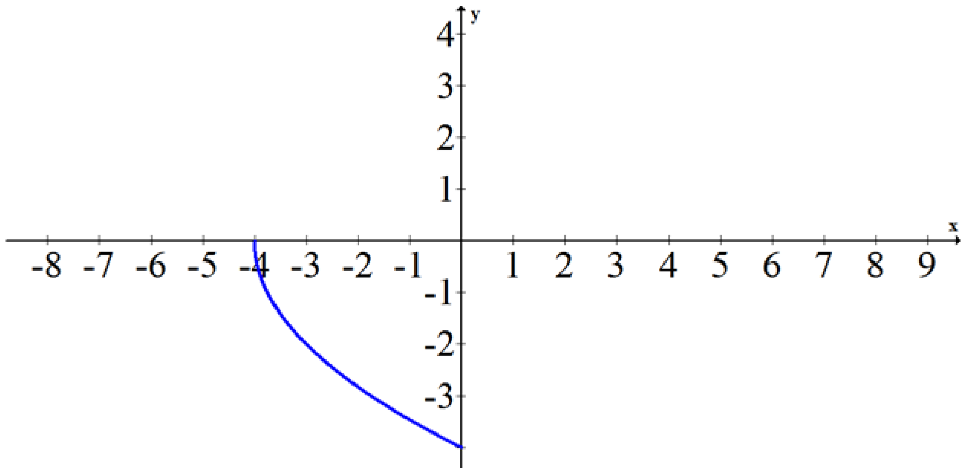
$x = t^2 - 4$

$y = 2t$

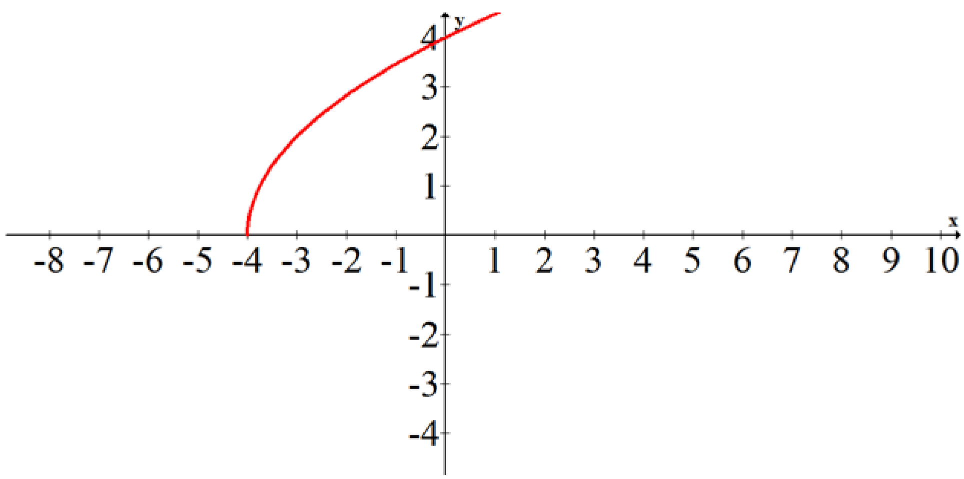
a. $-2 \le t \le 0$
b. $0 \le t \le 5$
c. $-3 \le t \le 2$

**Solution:**

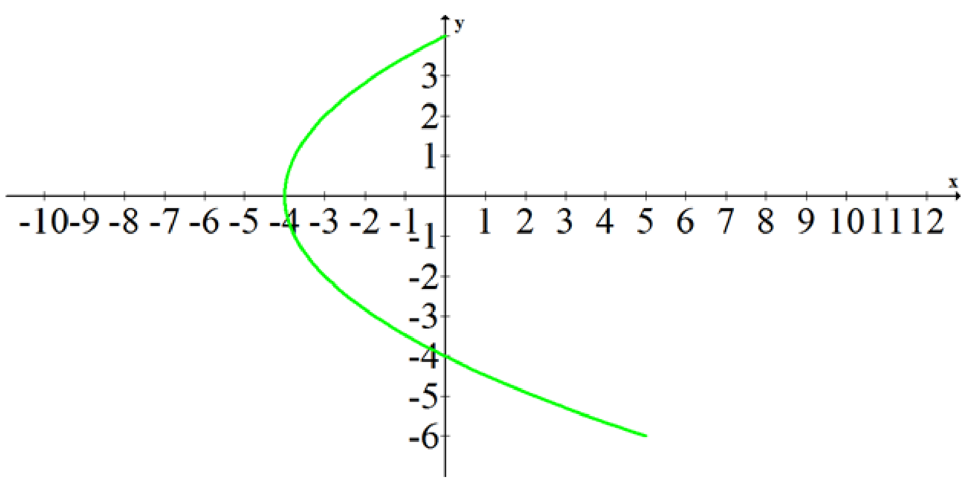
a. A good place to start is to find the coordinates where $t$ indicates the graph will start and end. For $-2 \le t \le 0$, $t = -2$ and $t = 0$ indicate that the points (0, -4) and (-4, 0) are the endpoints of the graph.




b. $0 \leq t \leq 5$




c. $-3 \leq t \leq 2$




### Example C

Eliminate the parameter and graph the following parametric curve.

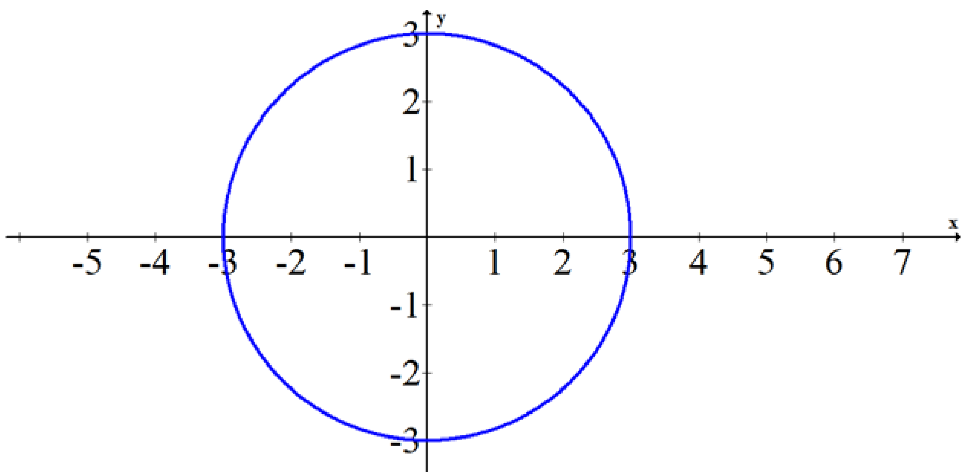
$x = 3 \cdot \sin t$

$y = 3 \cdot \cos t$

**Solution:** When parametric equations involve trigonometric functions you can use the Pythagorean Identity, $\sin^2 t + \cos^2 t = 1$. In this problem, $\sin t = \frac{x}{3}$ (from the first equation) and $\cos t = \frac{y}{3}$ (from the second equation). Substitute these values into the Pythagorean Identity and you have:

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$
$$x^2 + y^2 = 9$$

This is a circle centered at the origin with radius 3.




**Concept Problem Revisited**

Parametric equations are often used when only a portion of a graph is useful. By limiting the domain of $t$, you can graph the precise interval of the function you want. Parametric equations are also useful when two different variables jointly depend on a third variable and you wish to look at the relationship between the two dependent variables. This is very common in statistics where an underlying variable may actually be the cause of a problem and the observer can only examine the relationship between the outcomes that they see. In the physical world, parametric equations are exceptional at graphing position over time because the horizontal and vertical vectors of objects in free motion are each dependent on time, yet independent of one another.

### Vocabulary

***"Eliminating the parameter"*** is a phrase that means to turn a parametric equation that has $x = f(t)$ and $y = g(t)$ into just a relationship between $y$ and $x$.

***Parametric form*** refers to a relationship that includes $x = f(t)$ and $y = g(t)$. ***Parameterization*** also means parametric form.

### Guided Practice

1. Find the parameterization for the line segment connecting the points (1, 3) and (4, 8).

2. A tortoise and a hare start 202 feet apart and then race to a flag halfway between them. The hare decides to take a nap and give the tortoise a 21 second head start. The hare runs at 9.8 feet per second and the tortoise hustles along at 3.2 feet per second. Who wins this epic race and by how much?

3. Use your calculator to model the race in #2.

**Answers:**

1. Use the fact that a point plus a vector yields another point. A vector between these points is $< 4-1, 8-3 >=< 3,5 >$

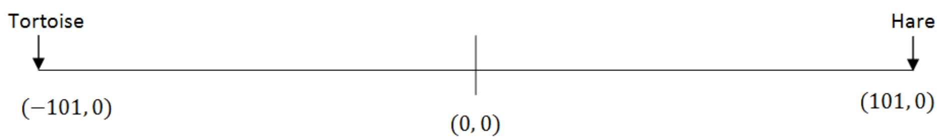
Thus the point (1, 3) plus $t$ times the vector $< 3,5 >$ will produce the point (4, 8) when $t = 1$ and the point (1, 3) when $t = 0$.

$(x,y) = (1,3) + t\cdot < 3,5 >, \text{ for } 0 \le t \le 1$

You than then break up this vector equation into parametric form.

$$x = 1 + 3t$$
$$y = 3 + 5t$$
$$0 \le t \le 1$$

2. First draw a picture and then represent each character with a set of parametric equations.




The tortoise's position is (-101, 0) at $t = 0$ and (-97.8, 0) at $t = 1$. You can deduce that the equation modeling the tortoise's position is:

$$x_1 = -101 + 3.2 \cdot t$$
$$y_1 = 0$$

The hare's position is (101, 0) at $t = 21$ and (91.2, 0) at $t = 22$. Note that it does not make sense to make equations modeling the hare's position before 21 seconds have elapsed because the Hare is napping and not moving. You can set up an equation to solve for the hare's theoretical starting position had he been running the whole time.

$$x_2 = b - 9.8t$$
$$101 = b - 9.8 \cdot 21$$
$$305.8 = b$$

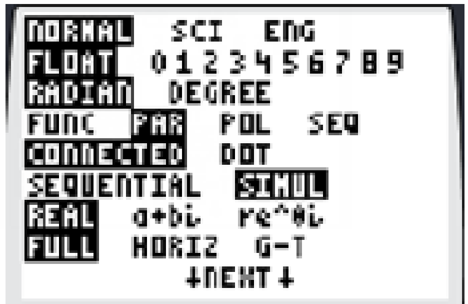
The hare's position equation after $t = 21$ can be modeled by:

$$x_2 = 305.8 - 9.8 \cdot t$$
$$y_2 = 0$$

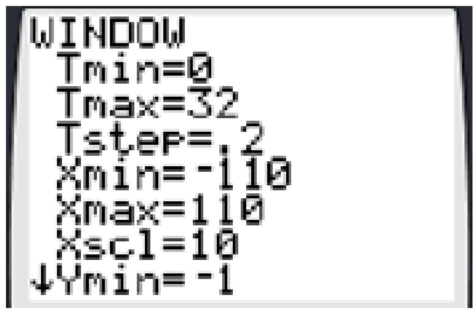
The tortoise crosses $x = 0$ when $t \approx 31.5$. The hare crosses $x = 0$ when $t \approx 31.2$. The hare wins by about 1.15 feet.

3. There are many settings you should know for parametric equations that bring questions like this to life. The TI-84 has features that allow you to see the race happen.

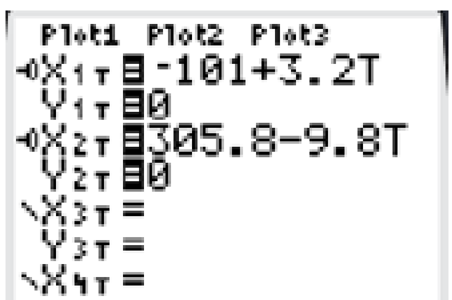
First, set the mode to simultaneous graphing. This will show both the tortoise and hare's position at the same time.



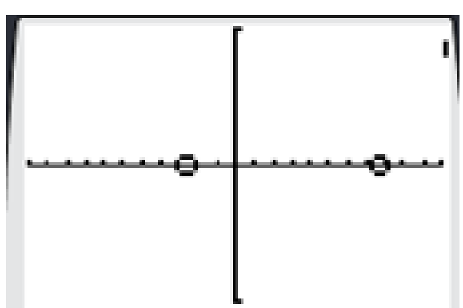

Next, change the graphing window so that $t$ varies between 0 and 32 seconds. The $T_{step}$ determines how often the calculator will calculate points. The larger the $T_{step}$, the faster and less accurately the graph will plot. Also change the $x$ to vary between -110 and 110 so you can see the positions of both characters.




Input the parametric equations. Toggle to the left of the $x$ and change the cursor from a line to a line with a bubble at the end. This shows their position more clearly.




Now when you graph you should watch the race unfold as the two position graphs race towards each other.



### Practice

Eliminate the parameter in the following sets of parametric equations.

1. $x = 3t - 1;\ y = 4t^2 - 2t$

2. $x = 3t^2 + 6t;\ y = 2t - 1$

3. $x = t + 2;\ y = t^2 + 4t + 4$

4. $x = t - 5;\ y = t^3 + 1$

5. $x = t + 4;\ y = t^2 - 5$

For the parametric equation $x = t, y = t^2 + 1$, graph over each interval of $t$.

6. $-2 \leq t \leq -1$

7. $-1 \leq t \leq 0$

8. $-1 \leq t \leq 1$

9. $-2 \leq t \leq 2$

10. $-5 \leq t \leq 5$

11. Eliminate the parameter and graph the following parametric curve: $x = \sin t,\ y = -4 + 3\cos t$.

12. Eliminate the parameter and graph the following parametric curve: $x = 1 + 2\cos t,\ y = 1 + 2\sin t$.

13. Using the previous problem as a model, find a parameterization for the circle with center (2, 4) and radius 3.

14. Find the parameterization for the line segment connecting the points (2, 7) and (1, 4).

15. Find a parameterization for the ellipse $\frac{x^2}{4} + \frac{y^2}{25} = 1$. Use the fact that $\cos^2 t + \sin^2 t = 1$. Check your answer with your calculator.

16. Find a parameterization for the ellipse $\frac{(x-4)^2}{9} + \frac{(y+1)^2}{36} = 1$. Check your answer with your calculator.

# 10.4 Parametric Inverses

Here you will use your knowledge of both inverses and parametric equations to solve problems.

You have learned that a graph and its inverse are reflections of each other across the line $y = x$. You have also learned that in order to find an inverse algebraically, you can switch the $x$ and $y$ variables and solve for $y$. Parametric equations actually make finding inverses easier because both the $x$ and $y$ variables are based on a third variable $t$. All you need to do to find the inverse of a set of parametric equations and switch the functions for $x$ and $y$.

Is the inverse of a function always a function?

### Watch This

MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=4Y14XhPD7Os James Sousa: Graphing Parametric Equations in the TI84

### Guidance

To find the inverse of a parametric equation you must switch the function of $x$ with the function of $y$. This will switch all the points from $(x, y)$ to $(y, x)$ and also has the effect of visually reflecting the graph over the line $y = x$.

**Example A**

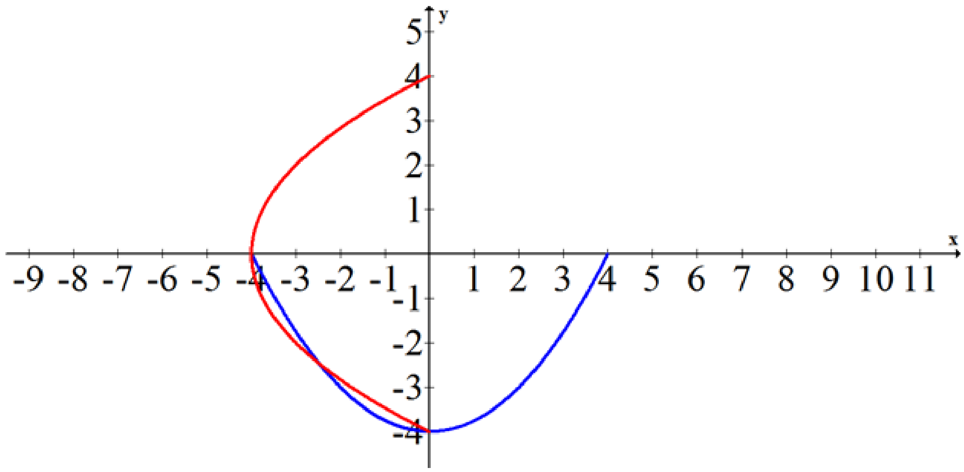
Find and graph the inverse of the parametric function on the domain $-2 < t < 2$.

$$x = 2t$$
$$y = t^2 - 4$$

**Solution:** Switch the $x$ and $y$ functions and graph.

$$x = t^2 - 4$$
$$y = 2t$$

The original function is shown in blue and the inverse is shown in red.




**Example B**

Is the point (4, 8) in the following function or its inverse?

$$x = 2t^2 - 2$$
$$y = t^2 - 1$$

**Solution:** Try to solve for a matching $t$ in the original function.

### TABLE 10.1:

| | |
|---|---|
| $x = 2t^2 - 2$<br>$4 = 2t^2 - 2$<br>$6 = 2t^2$<br>$3 = t^2$<br>$\pm\sqrt{3} = t$ | $y = t^2 - 1$<br>$8 = t^2 - 1$<br>$9 = t^2$<br>$\pm 3 = t$ |

The point does not satisfy the original function. Check to see if it satisfies the inverse.

### TABLE 10.2:

| | |
|---|---|
| $x = t^2 - 1$<br>$4 = t^2 - 1$<br>$\pm\sqrt{5} = t$ | $y = 2t^2 - 2$<br>$8 = 2t^2 - 2$<br>$10 = 2t^2$<br>$\pm\sqrt{5} = t$ |

The point does satisfy the inverse of the function.

**Example C**

Parameterize the following function and then graph the function and its inverse.

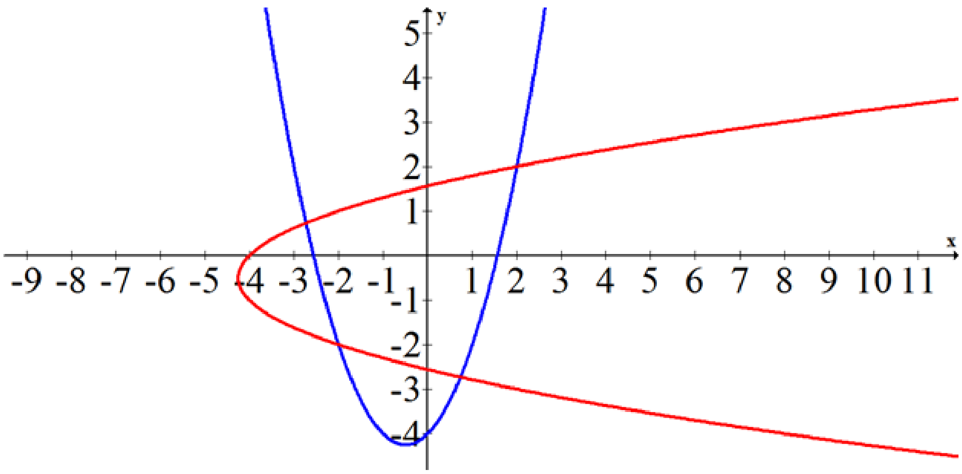
$f(x) = x^2 + x - 4$

**Solution:** For the original function, the parameterization is:

$$x = t$$
$$y = t^2 + t - 4$$

The inverse is:

$$x = t^2 + t - 4$$
$$y = t$$




**Concept Problem Revisited**

The inverse of a function is not always a function. In order to see whether the inverse of a function will be a function, you must perform the horizontal line test on the original function. If the function passes the horizontal line test then the inverse will be a function. If the function does not pass the horizontal line test then the inverse produces a relation rather than a function.

## Vocabulary

Two functions are ***inverses*** if for every point $(a, b)$ on the first function there exists a point $(b, a)$ on the second function.

An ***intersection*** for two sets of parametric equations happens when the points exist at the same $x, y$ and $t$.

## Guided Practice

1. Find the points of intersection of the function and its inverse from Example C.

2. Does the point (-2, 6) live on the following function or its inverse?

$x = t^2 - 10$

$y = \frac{t}{2} - 4$

3. Identify where the following parametric function intersects with its inverse.

$x = 4t$

$y = t^2 - 16$

**Answers:**

1. The parameterized function is:

$$x_1 = t$$
$$y_1 = t^2 + t - 4$$

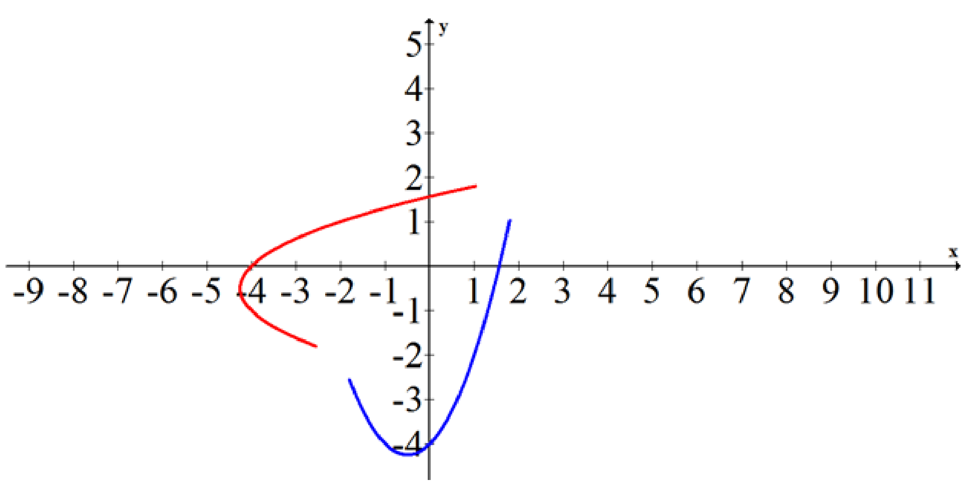
The inverse is:

$$x_2 = t^2 + t - 4$$
$$y_2 = t$$

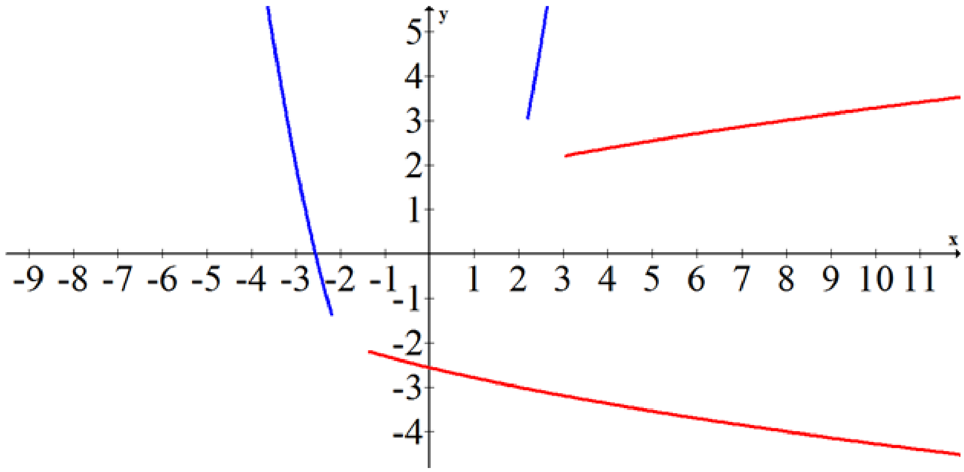
To find where these intersect, set $x_1 = x_2$ and $y_1 = y_2$ and solve.

$$t = t^2 + t - 4$$
$$t^2 = 4$$
$$t = \pm 2$$

You still need to actually calculate the points of intersection on the graph. You can tell from the graph in Example C that there seem to be four points of intersection. Since $t$ can mean time, the question of intersection is more complicated than simply overlapping. It means that the points are at the same $x$ and $y$ coordinate at the same time. Note what the graphs look like when $-1.8 < t < 1.8$.




Note what the graphs look like $t > 2.2$ or $t < -2.2$




Notice how when these partial graphs are examined there is no intersection at anything besides $t = \pm 2$ and the points (2, 2) and (-2, -2) While the paths of the graphs intersect in four places, they intersect at the same time only twice.

2. First check to see if the point (-2, 6) produces a matching time for the original function.

### TABLE 10.3:

| | |
|---|---|
| $-2 = t^2 - 10$<br>$8 = t^2$<br>$\pm 2\sqrt{2} = t$ | $6 = \frac{t}{2} - 4$<br>$20 = t$ |

The point does not live on the original function. Now, you must check to see if it lives on the inverse.

### TABLE 10.4:

| | |
|---|---|
| $6 = t^2 - 10$<br>$\pm 4 = t$ | $-2 = \frac{t}{2} - 6$<br>$4 = \frac{t}{2}$<br>$8 = t$ |

The point does not live on the inverse either.

3. $x_1 = 4t;\ y_1 = t^2 - 16$ The inverse is:

$$x_2 = t^2 - 16$$
$$y_2 = 4t$$

Solve for $t$ when $x_1 = x_2$ and $y_1 = y_2$.

$$4t = t^2 - 16$$
$$0 = t^2 - 4t - 16$$
$$t = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot (-16)}}{2} = \frac{4 \pm 4\sqrt{5}}{2} = 2 \pm 2\sqrt{5}$$

The points that correspond to these two times are:

$$x = 4(2+2\sqrt{5}),\ y = (2+2\sqrt{5})^2 - 16$$
$$x = 4(2-2\sqrt{5}),\ y = (2-2\sqrt{5})^2 - 16$$

### Practice

Use the function $x = t-4;\ y = t^2+2$ for #1 - #3.

1. Find the inverse of the function.
2. Does the point (-2, 6) live on the function or its inverse?
3. Does the point (0, 1) live on the function or its inverse?

Use the relation $x = t^2;\ y = 4-t$ for #4 - #6.

4. Find the inverse of the relation.
5. Does the point (4, 0) live on the relation or its inverse?
6. Does the point (0, 4) live on the relation or its inverse?

Use the function $x = 2t+1;\ y = t^2-3$ for #7 - #9.

7. Find the inverse of the function.
8. Does the point (1, 5) live on the function or its inverse?
9. Does the point (9, 13) live on the function or its inverse?

Use the function $x = 3t+14;\ y = t^2-2t$ for #10 - #11.

10. Find the inverse of the function.
11. Identify where the parametric function intersects with its inverse.

Use the relation $x = t^2;\ y = 4t-4$ for #12 - #13.

12. Find the inverse of the relation.
13. Identify where the relation intersects with its inverse.
14. Parameterize $f(x) = x^2+x-6$ and then graph the function and its inverse.
15. Parameterize $f(x) = x^2+3x+2$ and then graph the function and its inverse.

# 10.5 Applications of Parametric Equations

Here you will use parametric equations to represent the vertical and horizontal motion of objects over time.

A regular function has the ability to graph the height of an object over time. Parametric equations allow you to actually graph the complete position of an object over time. For example, parametric equations allow you to make a graph that represents the position of a point on a Ferris wheel. All the details like height off the ground, direction, and speed of spin can be modeled using the parametric equations.

What is the position equation and graph of a point on a Ferris wheel that starts at a low point of 6 feet off the ground, spins counterclockwise to a height of 46 feet off the ground, then goes back down to 6 feet in 60 seconds?

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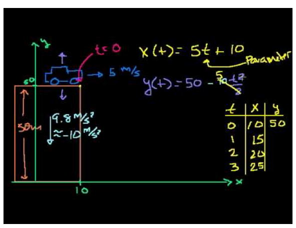

MEDIA
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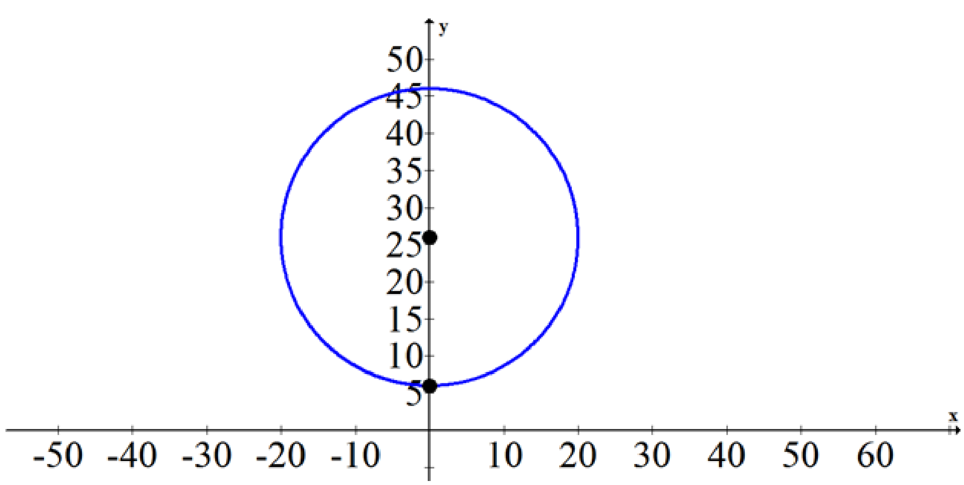
http://www.youtube.com/watch?v=m6c6dlmUT1c Academy: Parametric Equations 1

### Guidance

There are two types of parametric equations that are typical in real life situations. The first is circular motion as was described in the concept problem. The second is projectile motion.

Parametric equations that describe circular motion will have $x$ and $y$ as periodic functions of sine and cosine. Either $x$ will be a sine function and $y$ will be a cosine function or the other way around. The best way to come up with parametric equations is to first draw a picture of the circle you are trying to represent.



Next, it is important to note the starting point, center point and direction. You should already have the graphs of sine and cosine memorized so that when you see a pattern in words or as a graph, you can identify what you see as $+\sin, -\sin, +\cos, -\cos$. In this example, the vertical component starts at a low point of 6, travels to a middle point of 26 and then a height of 46 and back down. This is a $-\cos$ pattern. The amplitude of the $-\cos$ is 20 and the vertical shift is 26. Lastly, the period is 60. You can use the period to help you find *b*.

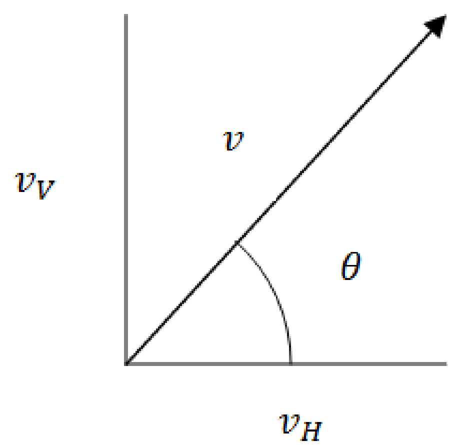
$$60 = \frac{2\pi}{b}$$
$$b = \frac{\pi}{30}$$

Thus the vertical parameterization is:

$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$

Try to find the horizontal parameterization on your own. The solution will be discussed in the "Concept Problem Revisited" section.

Projectile motion has a vertical component that is quadratic and a horizontal component that is linear. This is because there are 3 parameters that influence the position of an object in flight: starting height, initial velocity, and the force of gravity. The horizontal component is independent of the vertical component. This means that the starting horizontal velocity will remain the horizontal velocity for the entire flight of the object.




Note that gravity, $g$, has a force of about $-32\ ft/s^2$ or $-9.81\ m/s^2$. The examples and practice questions in this concept will use feet.

If an object is launched from the origin at a velocity of $v$ then it has horizontal and vertical components that can be found using basic trigonometry.

$$\sin\theta = \frac{v_V}{v} \rightarrow v \cdot \sin\theta = v_V$$
$$\cos\theta = \frac{v_H}{v} \rightarrow v \cdot \cos\theta = v_H$$

The horizontal component is basically finished. The only adjustments that would have to be made are if the starting location is not at the origin, wind is added or if the projectile travels to the left instead of the right. See Example A.

$x = t \cdot v \cdot \cos\theta$

The vertical component also needs to include gravity and the starting height. The general equation for the vertical component is:

$y = \frac{1}{2} \cdot g \cdot t^2 + t \cdot v \cdot \sin\theta + k$

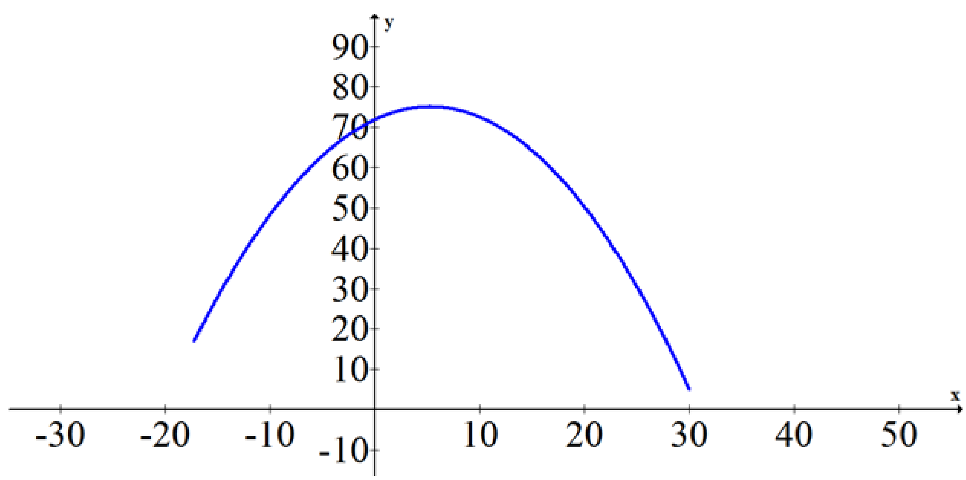
The constant $g$ represents gravity, $t$ represents time, $v$ represents initial velocity and $k$ represents starting height. You will explore this equation further in calculus and physics. Note that in this concept, most answers will be found and confirmed using technology such as your graphing calculator.

**Example A**

A ball is thrown from the point (30, 5) at an angle of $\frac{4\pi}{9}$ to the left at an initial velocity of 68 $ft/s$. Model the position of the ball over time using parametric equations. Use your graphing calculator to graph your equations for the first four seconds while the ball is in the air.

**Solution:** The horizontal component is $x = -t \cdot 68 \cdot \cos(\frac{4\pi}{9}) + 30$. Note the negative sign because the object is traveling to the left and the +30 because the object starts at (30, 5).

The vertical component is $y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 68 \cdot \sin(\frac{4\pi}{9}) + 5$. Note that $g = -32$ because gravity has a force of $-32$ $ft/s^2$ and the $+5$ because the object starts at (30, 5).

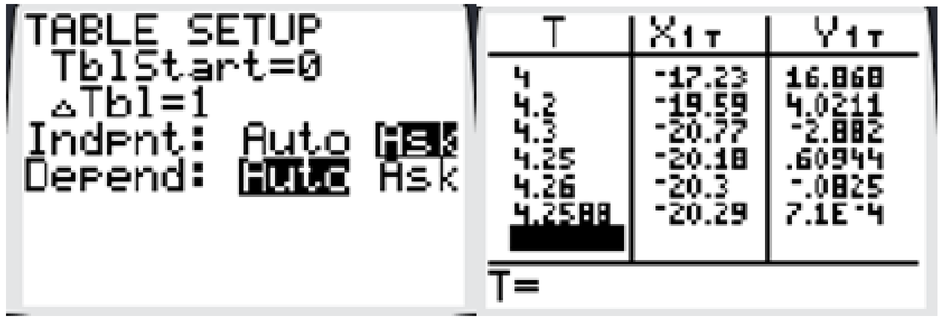



**Example B**

When does the ball reach its maximum and when does the ball hit the ground? How far did the person throw the ball?

**Solution:** To find when the function reaches its maximum, you can find the vertex of the parabola. Analytically this is messy because of the decimal coefficients in the quadratic. Use your calculator to approximate the maximum after you have graphed it. Depending on how small you make your $T_{step}$ should find the maximum height to be about 75 feet.

To find out when the ball hits the ground, you can set the vertical component equal to zero and solve the quadratic equation. You can also use the table feature on your calculator to determine when the graph goes from having a positive vertical value to a negative vertical value. The benefit for using the table is that it simultaneously tells you the $x$ value of the zero.




After about 4.2588 seconds the ball hits the ground at (-20.29, 0). This means the person threw the ball from (30, 5) to (-20.29, 0), a horizontal distance of just over 50 feet.

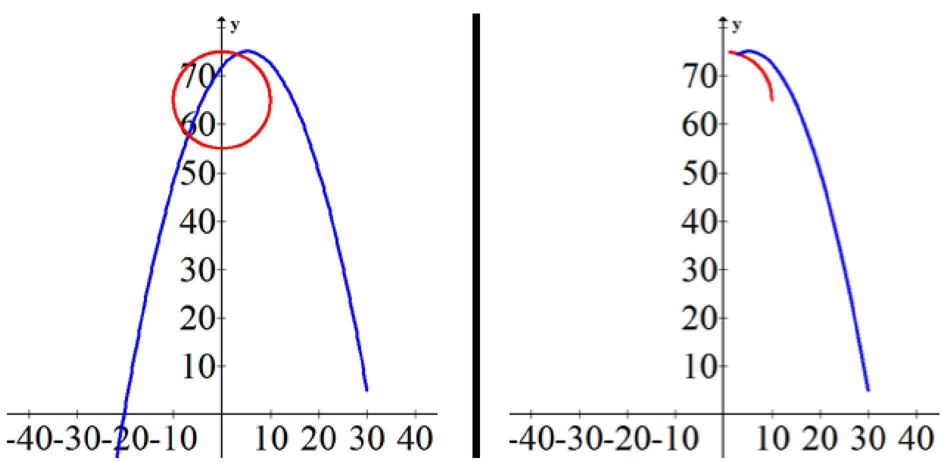
**Example C**

Kieran is on a Ferris wheel and his position is modeled by the parametric equations:

$$x_K = 10 \cdot \cos\left(\frac{\pi}{5}t\right)$$
$$y_K = 10 \cdot \sin\left(\frac{\pi}{5}t\right) + 65$$

Jason throws the ball modeled by the equation in Example A towards Kieran who can catch the ball if it gets within three feet. Does Kieran catch the ball?

**Solution:** This question is designed to demonstrate the power of your calculator. If you simply model the two equations simultaneously and ignore time you will see several points of intersection. This graph is shown below on the left. These intersection points are not interesting because they represent where Kieran and the ball are at the same place but at different moments in time.




When the $T_{max}$ is adjusted to 2.3 so that each graph represents the time from 0 to 2.3, you get a better sense that at about 2.3 seconds the two points are close. This graph is shown above on the right.

You can now use your calculator to help you determine if the distance between Kieran and the ball actually does go below 3 feet. Start by plotting the ball's position in your calculator as $x_1$ and $y_1$ and Kieran's position as $x_2$ and $y_2$. Then, plot a new parametric equation that compares the distance between these two points over time. You can put this under $x_3$ and $y_3$. Note that you can find the $x_1, x_2, y_1, y_2$ entries in the vars and parametric menu.

$$x_3 = t$$
$$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Now when you graph, you should change your window settings and let $t$ vary between 0 and 4, the $x$ window show between 0 and 4 and the $y$ window show between 0 and 5. This way it should be clear if the distance truly does get below 3 feet.


Depending on how accurate your $T_{step}$ is, you should find that the distance is below 3 feet. Kieran does indeed catch the ball.

**Concept Problem Revisited**

The parametric equations for the point on the wheel are:

$$x = 20\sin\left(\frac{\pi}{30}t\right)$$
$$y = -20\cos\left(\frac{\pi}{30}t\right) + 26$$

The horizontal parameterization is found by noticing that the $x$ values start at 0, go up to 20, go back to 0, then down to -20, and finally back to 0. This is a $+\sin$ pattern with amplitude 20. The period is the same as with the vertical component.

### Vocabulary

A calculator can reference ***internal variables*** like $x_1, y_1$ that have already been set in the calculator's memory to form new variables like $x_3, y_3$.

The ***horizontal and vertical components*** of parametric equations are the $x =$ and $y =$ functions respectively.

### Guided Practice

1. At what velocity does a football need to be thrown at a 45° angle in order to make it all the way across a football field?

2. Suppose Danny stands at the point (300, 0) and launches a football at 67 mph at an angle of 45° towards Johnny who is at the origin. Suppose Johnny also throws a football towards Danny at 60 mph at an angle of 50° at the exact same moment. There is a 4 mph breeze in Johnny's favor. Do the balls collide in midair?

3. Nikki got on a Ferris wheel ten seconds ago. She started 2 feet off the ground at the lowest point of the wheel and will make a complete cycle in four minutes. The ride reaches a maximum height of 98 feet and spins

clockwise. Write parametric equations that model Nikki's position over time. Where will Nikki be three minutes from now?

**Answers:**

1. A football field is 100 yards or 300 feet. The parametric equations for a football thrown from (300, 0) back to the origin at speed $v$ are:

$$x = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$y = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot sin\left(\frac{\pi}{4}\right)$$

Substituting the point (0, 0) in for $(x, y)$ produces a system of two equations with two variables $v, t$.

$$0 = -t \cdot v \cdot \cos\left(\frac{\pi}{4}\right) + 300$$
$$0 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot v \cdot \sin\left(\frac{\pi}{4}\right)$$

You can solve this system many different ways.

$t = \frac{5\sqrt{3}}{2} \approx 4.3\ seconds, v = 40\sqrt{6} \approx 97.98\ ft/s$

In order for someone to throw a football at a 45° angle all the way across a football field, they would need to throw at about 98 $ft/s$ which is about 66.8 mph.

$\frac{98\ feet}{1\ sec} \cdot \frac{3600\ sec}{1\ hour} \cdot \frac{1\ mile}{5280\ feet} \approx \frac{66.8\ miles}{1\ hour}$

2. Calculate the velocity of each person and of the wind in feet per second:

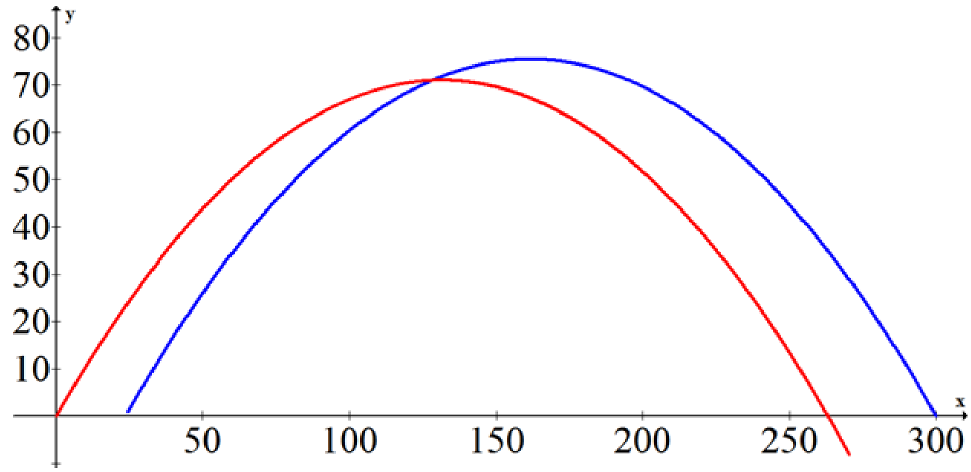
$$\frac{67\ miles}{1\ hour} \cdot \frac{1\ hour}{3600\ sec} \cdot \frac{5280\ feet}{1\ mile} \approx 98.27\ ft/sec$$
$$\frac{60\ miles}{1\ hour} \cdot \frac{1\ hour}{3600\ sec} \cdot \frac{5280\ feet}{1\ mile} = 88\ ft/sec$$
$$\frac{4\ miles}{1\ hour} \cdot \frac{1\ hour}{3600\ sec} \cdot \frac{5280\ feet}{1\ mile} \approx 5.87\ ft/sec$$

The location of Danny's ball can be described with the following parametric equations (in radians). Note that the wind simply adds a linear term.

$$x_1 = -t \cdot 98.27 \cdot \cos\left(\frac{\pi}{4}\right) + 300 + 5.87t$$
$$y_1 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 98.27 \cdot \sin\left(\frac{\pi}{4}\right)$$

The location of Johnny's ball can be described with the following parametric equations.

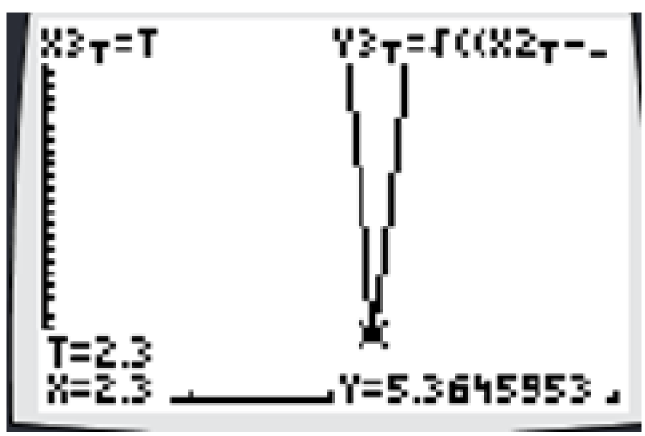
$$x_2 = t \cdot 88 \cdot \cos\left(\frac{5\pi}{18}\right) + 5.87t$$
$$y_2 = \frac{1}{2} \cdot (-32) \cdot t^2 + t \cdot 88 \cdot \sin\left(\frac{5\pi}{18}\right)$$




You might observe that the two graphs overlap at about 125 feet, but this is unimportant because they are probably at that location at different times. Plot the distance over time and see how close the footballs actually get in midair.

$$x_3 = t$$
$$y_3 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

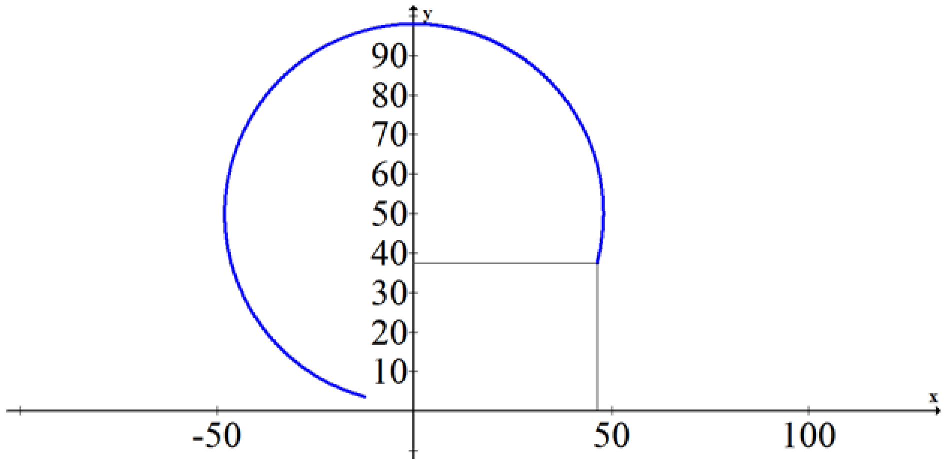
The TI-83 may run slower than the TI-84 when you do this calculation depending on how small your $T_{step}$ is. The resulting graph is not a parabola and the calculator cannot find a minimum in the way it can in other circumstances. Still, you can trace and determine that the footballs get 5.36 feet or closer in midair.




3. Don't let the 10 second difference confuse you. In order to deal with the time difference, use $\left(t + \frac{1}{6}\right)$ instead of $t$ in each equation. When $t = 0$, ten seconds $\left(\frac{1}{6}\text{ of a minute}\right)$ have already elapsed.

$$x = -48 \cdot \sin\left(\frac{\pi}{2}\left(t + \frac{1}{6}\right)\right)$$
$$y = -48 \cdot \cos\left(\frac{\pi}{2}\left(t + \frac{1}{6}\right)\right) + 50$$

At $t = 3, x \approx 46.36$ and $y \approx 37.58$




### Practice

Candice gets on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 103 feet, making a complete cycle in 5 minutes.

1. Write a set of parametric equations to model Candice's position.
2. Where will Candice be in two minutes?
3. Where will Candice be in four minutes?

One minute ago Guillermo got on a Ferris wheel at its lowest point, 3 feet off the ground. The Ferris wheel spins clockwise to a maximum height of 83 feet, making a complete cycle in 6 minutes.

4. Write a set of parametric equations to model Guillermo's position.
5. Where will Guillermo be in two minutes?
6. Where will Guillermo be in four minutes?

Kim throws a ball from (0, 5) to the right at 50 mph at a 45° angle.

7. Write a set of parametric equations to model the position of the ball.
8. Where will the ball be in 2 seconds?
9. How far does the ball get before it lands?

David throws a ball from (0, 7) to the right at 70 mph at a 60° angle. There is a 6 mph wind in David's favor.

10. Write a set of parametric equations to model the position of the ball.
11. Where will the ball be in 2 seconds?
12. How far does the ball get before it lands?

Suppose Riley stands at the point (250, 0) and launches a football at 72 mph at an angle of 60° towards Kristy who is at the origin. Suppose Kristy also throws a football towards Riley at 65 mph at an angle of 45° at the exact same moment. There is a 6 mph breeze in Kristy's favor.

13. Write a set of parametric equations to model the position of Riley's ball.
14. Write a set of parametric equations to model the position of Kristy's ball.
15. Graph both functions and explain how you know that the footballs don't collide even though the two graphs intersect.

You learned that the polar coordinate system identifies points by their angle $(\theta)$ and distance to the origin $(r)$. Polar equations allow you to more easily represent circles and spirals with equations. Parametric equations are a set of two functions, each dependent on the same third variable. You saw that parametric equations are especially useful when working with an object in flight where its vertical and horizontal components don't theoretically interact.

# 10.6 References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# CHAPTER 11 Complex Numbers

## Chapter Outline



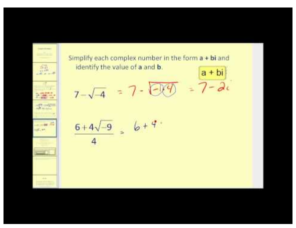
Complex numbers are based on the fact that $\sqrt{-1}$ is defined to be the imaginary number $i$, and that normal algebraic and graphical calculations are still valid even with imaginary numbers. In order to understand the reasoning behind $i$, you will look at how the Fundamental Theorem of Algebra requires the number of roots of a polynomial to match the degree of the polynomial. You will then discover how the complex number system interacts with the real number system.

# 11.1 Fundamental Theorem of Algebra

Here you will state the connection between zeroes of a polynomial and the Fundamental Theorem of Algebra and start to use complex numbers.

You have learned that a quadratic has at most two real zeroes and a cubic has at most three real zeros. You may have noticed that the number of real zeros is always less than or equal to the degree of the polynomial. By looking at a graph you can see when a parabola crosses the$x$ axis 0, 1 or 2 times, but what does this have to do with complex numbers?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=NeTRNpBI17I James Sousa: Complex Numbers

### Guidance

A real number is any rational or irrational number. When a real number is squared, it will always produce a non-negative value. Complex numbers include real numbers and another type of number called imaginary numbers. Unlike real numbers, imaginary numbers produce a negative value when squared. The square root of negative one is defined to be the imaginary number $i$.

$i = \sqrt{-1}$ and $i^2 = -1$

Complex numbers are written with a real component and an imaginary component. All complex numbers can be written in the form $a+bi$. When the imaginary component is zero, the number is simply a real number. This means that real numbers are a subset of complex numbers.

The **Fundamental Theorem of Algebra** states that an $n^{th}$ degree polynomial with real or complex coefficients has, with multiplicity, exactly $n$ complex roots. This means a cubic will have exactly 3 roots, some of which may be complex.

Multiplicity refers to when a root counts more than once. For example, in the following function the only root occurs at $x = 3$.

$f(x) = (x-3)^2$

The Fundamental Theorem of Algebra states that this $2^{nd}$ degree polynomial must have exactly 2 roots with multiplicity. This means that the root $x = 3$ has multiplicity 2. One way to determine the multiplicity is to simply look at the degree of each of the factors in the factorized polynomial.

$g(x) = (x-1)(x-3)^4(x+2)$

This function has 6 roots. The first two roots $x = 1$ and $x = -2$ have multiplicities of 1 because the power of each of their binomial factors is 1. The third root $x = 3$ has a multiplicity of 4 because the power of its binomial factor is

4. Keep in mind that all polynomials can be written in factorized form like the above polynomial, due to a theorem called the Linear Factorization Theorem.

**Example A**

Identify the zeroes of the following complex polynomial.

$f(x) = x^2 + 9$

**Solution:** Set $y = 0$ and solve for $x$. This will give you the two zeros.

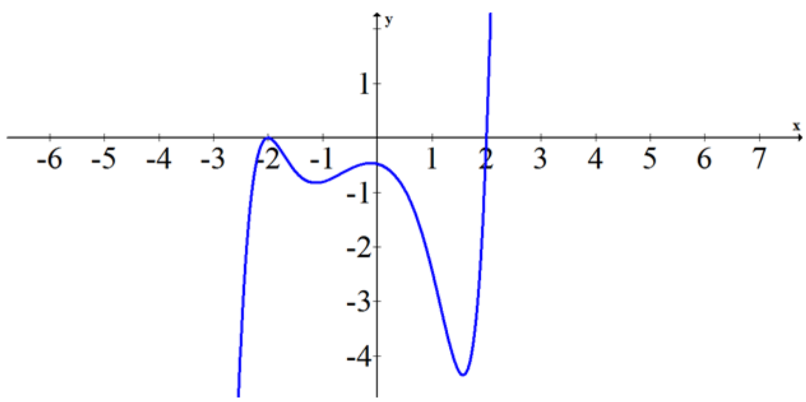
$$\begin{aligned} 0 &= x^2 + 9 \\ -9 &= x^2 \\ \pm 3i &= x \end{aligned}$$

Thus the linear factorization of the function is:

$f(x) = (x - 3i)(x + 3i)$

**Example B**

Examine the following graph and make conclusions about the number and type of zeros of this $7^{th}$ degree polynomial.




**Solution:** A $7^{th}$ degree polynomial has 7 roots. Three real roots are visible. The root at $x = -2$ has multiplicity 2 and the root at $x = 2$ has multiplicity 1. The other four roots appear to be imaginary and the clues are the relative maximums and minimums that do not cross the $x$ axis.

**Example C**

Identify the polynomial that has the following five roots. $x = 0, 2, 3, \pm\sqrt{5}i$

**Solution:** Write the function in factorized form.

$f(x) = (x - 0)(x - 2)(x - 3)(x - \sqrt{5}i)(x + \sqrt{5}i)$

When you multiply through, it will be helpful to do the complex conjugates first. The complex conjugates are $(x - \sqrt{5}i)(x + \sqrt{5}i)$.

$$\begin{aligned} f(x) &= x(x^2 - 5x + 6)(x^2 - 5 \cdot (-1)) \\ f(x) &= (x^3 - 5x^2 + 6x)(x^2 + 5) \\ f(x) &= x^5 - 5x^4 + 6x^3 + 5x^3 - 25x^2 + 30x \\ f(x) &= x^5 - 5x^4 + 11x^3 - 25x^2 + 30x \end{aligned}$$

**Concept Problem Revisited**

When a parabola fails to cross the $x$ axis it still has 2 roots. These two roots happen to be imaginary numbers. The function $f(x) = x^2 + 4$ does not cross the $x$ axis, but its roots are $x = \pm 2i$.

## Vocabulary

A ***complex number*** is a number written in the form $a + bi$ where both $a$ and $b$ are real numbers. When $b = 0$, the result is a real number and when $a = 0$ the result is an imaginary number.

An ***imaginary number*** is the square root of a negative number. $\sqrt{-1}$ is defined to be the imaginary number $i$.

***Complex conjugates*** are pairs of complex numbers with real parts that are identical and imaginary parts that are of equal magnitude but opposite signs. $1 + 3i$ and $1 - 3i$ or $5i$ and $-5i$ are examples of complex conjugates.

## Guided Practice

1. Write the polynomial that has the following roots: 4 (with multiplicity 3), 2 (with multiplicity 2) and 0.

2. Factor the following polynomial into its linear factorization and state all of its roots.

$$f(x) = x^4 - 5x^3 + 7x^2 - 5x + 6$$

3. Can you create a polynomial with real coefficients that has one imaginary root? Why or why not?

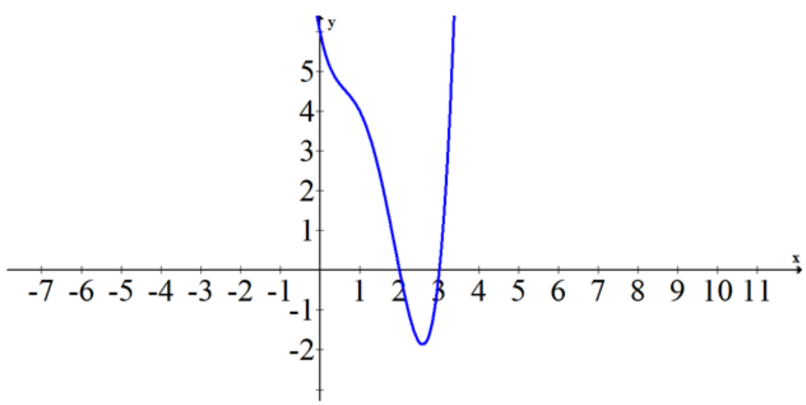
**Answers:**

1. $f(x) = (x-4)^3 \cdot (x-2)^2 \cdot x$

2. You can use polynomial long division to obtain the following factorization.

$$f(x) = (x-3)(x-2)(x-i)(x+i)$$

If you need a place to start, it is helpful to look at the graph of the polynomial and notice that the graph shows you exactly where the real roots appear.




3. No, if a polynomial has real coefficients then either it has no imaginary roots, or the imaginary roots come in pairs of complex conjugates (so that the imaginary portions cancel out when the factors are multiplied).

## Practice

For 1 - 4, find the polynomial with the given roots.

1. 2 (with multiplicity 2), 4 (with multiplicity 3), 1, $\sqrt{2}i, -\sqrt{2}i$.

2. 1, -3 (with multiplicity 3), $-1$, $\sqrt{3}i, -\sqrt{3}i$

3. 5 (with multiplicity 2), -1 (with multiplicity 2), $2i, -2i$

4. $i, -i$, $\sqrt{2}i, -\sqrt{2}i$

For each polynomial, factor into its linear factorization and state all of its roots.

5. $f(x) = x^5 + 4x^4 - 2x^3 - 14x^2 - 3x - 18$

6. $g(x) = x^4 - 1$

7. $h(x) = x^6 - 12x^5 + 61x^4 - 204x^3 + 532x^2 - 864x + 576$

8. $j(x) = x^7 - 11x^6 + 49x^5 - 123x^4 + 219x^3 - 297x^2 + 243x - 81$

9. $k(x) = x^5 + 3x^4 - 11x^3 - 15x^2 + 46x - 24$

10. $m(x) = x^6 - 12x^4 + 23x^2 + 36$

11. $n(x) = x^6 - 3x^5 - 10x^4 - 32x^3 - 81x^2 - 85x - 30$

12. $p(x) = x^6 + 4x^5 + 7x^4 + 12x^3 - 16x^2 - 112x - 112$

13. How can you tell the number of roots that a polynomial has from its equation?

14. Explain the meaning of the term “multiplicity”.

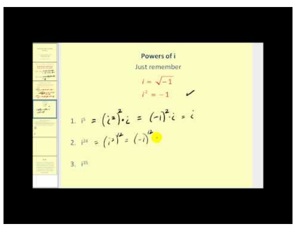
15. A polynomial with real coefficients has one root that is $\sqrt{3}i$. What other root(s) must the polynomial have?

# 11.2 Arithmetic with Complex Numbers

Here you will add, subtract, multiply and divide complex numbers. You will also find the absolute value of complex numbers and plot complex numbers in the complex plane.

The idea of a complex number can be hard to comprehend, especially when you start thinking about absolute value. In the past you may have thought of the absolute value of a number as just the number itself or its positive version. How should you think about the absolute value of a complex number?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=htiloYIILqs James Sousa: Complex Number Operations

### Guidance

Complex numbers follow all the same rules as real numbers for the operations of adding, subtracting, multiplying and dividing. There are a few important ideas to remember when working with complex numbers:

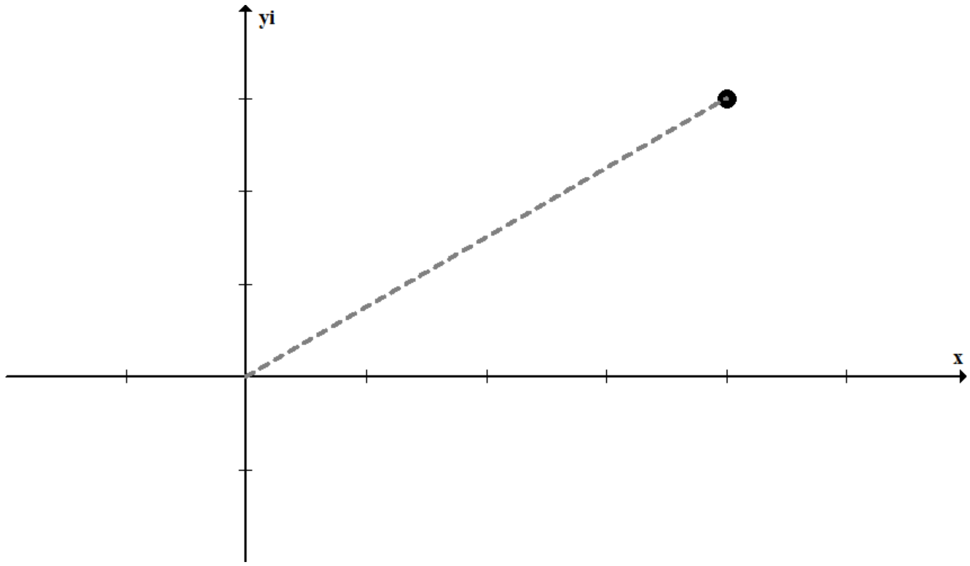
1. When simplifying, you must remember to combine imaginary parts with imaginary parts and real parts with real parts. For example, $4+5i+2-3i=6+2i$.

2. If you end up with a complex number in the denominator of a fraction, eliminate it by multiplying both the numerator and denominator by the complex conjugate of the denominator.

3. The powers of $i$ are:

- $i=\sqrt{-1}$
- $i^2=-1$
- $i^3=-\sqrt{-1}=-i$
- $i^4=1$
- $i^5=i$
- . . . and the pattern repeats

The complex plane is set up in the same way as the regular $x,y$ plane, except that real numbers are counted horizontally and complex numbers are counted vertically. The following is the number $4+3i$ plotted in the complex number plane. Notice how the point is four units over and three units up.




The absolute value of a complex number like $|4+3i|$ is defined as the distance from the complex number to the origin. You can use the Pythagorean Theorem to get the absolute value. In this case, $|4+3i| = \sqrt{4^2+3^2} = \sqrt{25} = 5$.

**Example A**

Multiply and simplify the following complex expression.

$(2+3i)(1-5i)-3i+8$

**Solution:** $(2+3i)(1-5i)-3i+8$

$$\begin{aligned}&= 2-10i+3i-15i^2-3i+8\\&= 10-10i+15\\&= 25-10i\end{aligned}$$

**Example B**

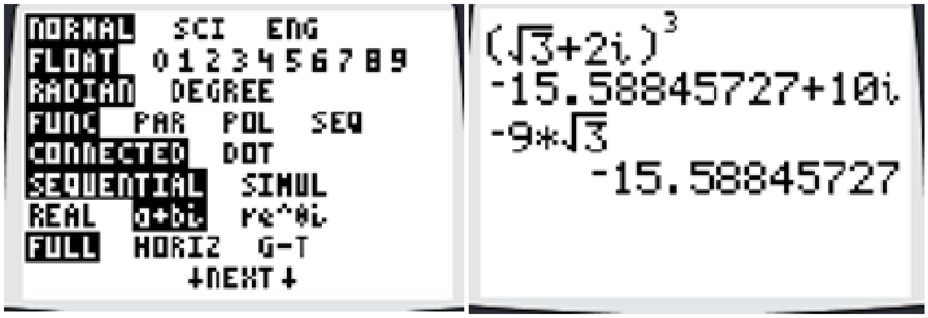
Compute the following power by hand and use your calculator to support your work.

$\left(\sqrt{3}+2i\right)^3$

**Solution:** $\left(\sqrt{3}+2i\right)\cdot\left(\sqrt{3}+2i\right)\cdot\left(\sqrt{3}+2i\right)$

$$\begin{aligned}&= \left(3+4i\sqrt{3}-4\right)\left(\sqrt{3}+2i\right)\\&= \left(-1+4i\sqrt{3}\right)\left(\sqrt{3}+2i\right)\\&= -\sqrt{3}-2i+12i-8\sqrt{3}\\&= -9\sqrt{3}+10i\end{aligned}$$

A TI-84 can be switched to imaginary mode and then compute exactly what you just did. Note that the calculator will give a decimal approximation for $-9\sqrt{3}$.




**Example C**

Simplify the following complex expression.

$\frac{7-9i}{4-3i}+\frac{3-5i}{2i}$

**Solution:** To add fractions you need to find a common denominator.

$$\begin{aligned}
&\frac{(7-9i)\cdot 2i}{(4-3i)\cdot 2i}+\frac{(3-5i)\cdot(4-3i)}{2i\cdot(4-3i)}\\
&=\frac{14i+18}{8i+6}+\frac{12-20i-9i-15}{8i+6}\\
&=\frac{15-15i}{8i+6}
\end{aligned}$$

Lastly, eliminate the imaginary component from the denominator by using the conjugate.

$$\begin{aligned}
&=\frac{(15-15i)\cdot(8i-6)}{(8i+6)\cdot(8i-6)}\\
&=\frac{120i-90+120+90i}{100}\\
&=\frac{30i+30}{100}\\
&=\frac{3i+3}{10}
\end{aligned}$$

**Concept Problem Revisited**

A better way to think about the absolute value is to define it as the distance from a number to zero. In the case of complex numbers where an individual number is actually a coordinate on a plane, zero is the origin.

## Vocabulary

The ***absolute value of a complex number*** is the distance from the complex number to the origin.

The ***complex number plane*** is just like the regular $x,y$ coordinate system except that the horizontal component is the real portion of the complex number $(a)$ and the vertical component is the complex portion of the number $(b)$.

A ***complex number*** is a number written in the form $a+bi$ where both $a$ and $b$ are real numbers. When $b=0$, the result is a real number and when $a=0$ the result is an imaginary number.

An ***imaginary number*** is the square root of a negative number. $\sqrt{-1}$ is defined to be the imaginary number $i$.

***Complex conjugates*** are pairs of complex numbers with real parts that are identical and imaginary parts that are of equal magnitude but opposite signs. $1+3i$ and $1-3i$ or $5i$ and $-5i$ are examples of complex conjugates.

### Guided Practice

1. Simplify the following complex number.

$i^{2013}$

2. Plot the following complex number on the complex coordinate plane and determine its absolute value.

$-12+5i$

3. For $a=3+4i, b=1-2i$ compute the sum, difference and product of $a$ and $b$.

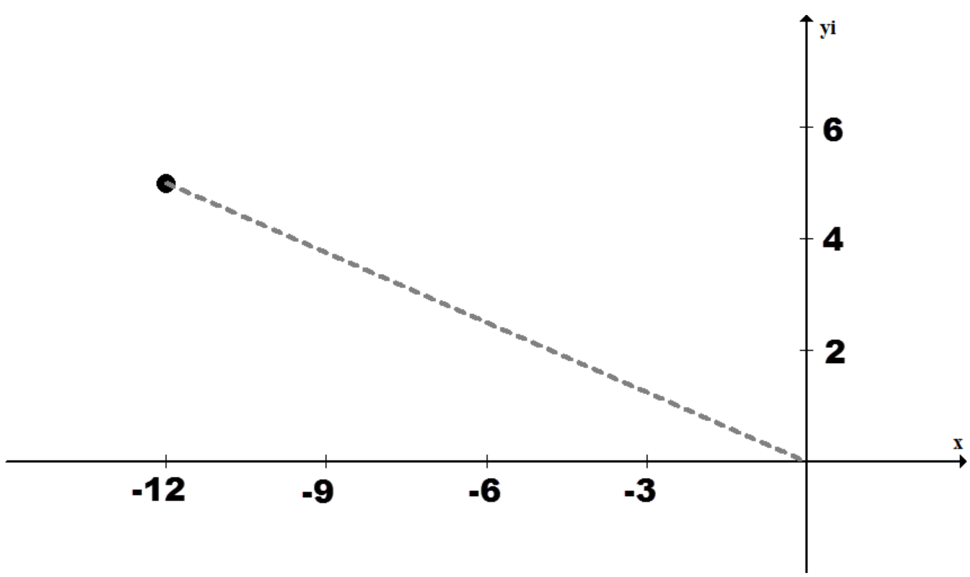
**Answers:**

1. When simplifying complex numbers, $i$ should not have a power greater than 1. The powers of $i$ repeat in a four part cycle:

$$
\begin{aligned}
i^5 &= i = \sqrt{-1} \\
i^6 &= i^2 = -1 \\
i^7 &= i^3 = -\sqrt{-1} = -i \\
i^8 &= i^4 = 1
\end{aligned}
$$

Therefore, you just need to determine where 2013 is in the cycle. To do this, determine the remainder when you divide 2013 by 4. The remainder is 1 so $i^{2013}=i$.

2.




The sides of the right triangle are 5 and 12, which you should recognize as a Pythagorean triple with a hypotenuse of 13. $|-12+5i|=13$.

3.

$$
\begin{aligned}
a+b &= (3+4i)+(1-2i) = 4-2i \\
a-b &= (3+4i)-(1-2i) = 2+6i \\
a\cdot b &= (3+4i)\cdot(1-2i) = 3-6i+4i+8 = 11-2i
\end{aligned}
$$

### Practice

Simplify the following complex numbers.

1. $i^{252}$
2. $i^{312}$
3. $i^{411}$
4. $i^{2345}$

For each of the following, plot the complex number on the complex coordinate plane and determine its absolute value.

5. $6-8i$
6. $2+i$
7. $4-2i$
8. $-5i+1$

Let $c=2+7i$ and $d=3-5i$.

9. What is $c+d$ ?
10. What is $c-d$ ?
11. What is $c \cdot d$ ?
12. What is $2c-4d$ ?
13. What is $2c \cdot 4d$ ?
14. What is $\frac{c}{d}$ ?
15. What is $c^2-d^2$ ?

# 11.3 Trigonometric Polar Form of Complex Numbers

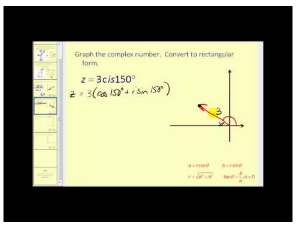
Here you will use basic right triangle trigonometry to represent complex points in the polar plane. You will also use the trigonometric form of complex numbers to multiply and divide complex numbers.

You already know how to represent complex numbers in the complex plane using rectangular coordinates and you already know how to multiply and divide complex numbers. Representing these points and performing these operations using trigonometric polar form will make your computations more efficient.

What are the two ways to multiply the following complex numbers?

$\left(1+\sqrt{3}i\right)\left(\sqrt{2}-\sqrt{2}i\right)$

### Watch This



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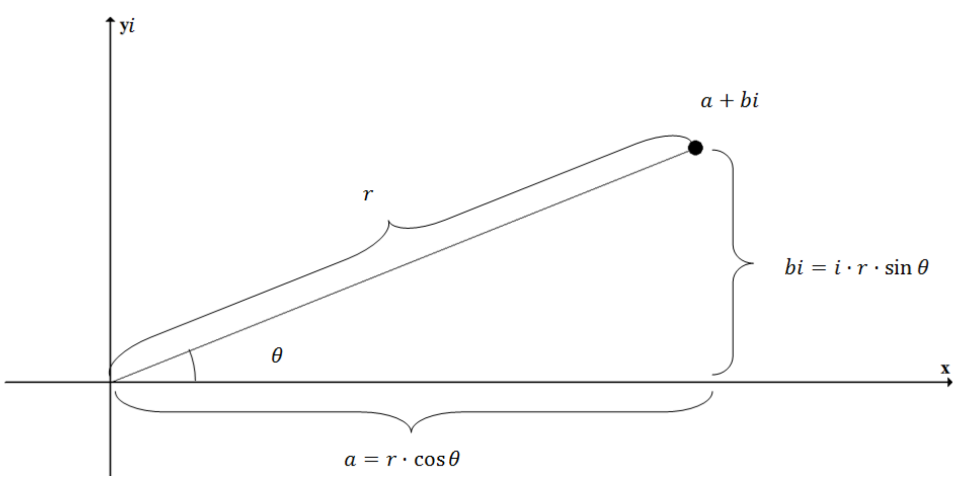
Click image to the left for more content.

http://www.youtube.com/watch?v=Zha7ZF8aVhU James Sousa: Trigonometric Form of Complex Numbers

### Guidance

Any point represented in the complex plane as $a+bi$ can be represented in polar form just like any point in the rectangular coordinate system. You will use the distance from the point to the origin as $r$ and the angle that the point makes as $\theta$.




As you can see, the point $a+bi$ can also be represented as $r\cdot\cos\theta+i\cdot r\cdot\sin\theta$. The trigonometric polar form can be abbreviated by factoring out the $r$ and noting the first letters:

$r(\cos\theta + i\cdot\sin\theta) \to r\cdot \text{cis } \theta$

The abbreviation $r\cdot \text{cis } \theta$ is read as "$r$ kiss theta." It allows you to represent a point as a radius and an angle. One great benefit of this form is that it makes multiplying and dividing complex numbers extremely easy. For example:

Let: $z_1 = r_1\cdot \text{cis } \theta_1, z_2 = r_2\cdot \text{cis } \theta_2$ with $r_2 \neq 0$.

Then:

$$z_1\cdot z_2 = r_1\cdot r_2\cdot \text{cis } (\theta_1+\theta_2)$$
$$z_1 \div z_2 = \frac{r_1}{r_2}\cdot \text{cis}(\theta_1-\theta_2)$$

For basic problems, the amount of work required to compute products and quotients for complex numbers given in either form is roughly equivalent. For more challenging questions, trigonometric polar form becomes significantly advantageous.

**Example A**

Convert the following complex number from rectangular form to trigonometric polar form.

$1-\sqrt{3}i$

**Solution:** The radius is the absolute value of the number.

$r^2 = 1^2 + \left(-\sqrt{3}\right)^2 \to r = 2$

The angle can be found with basic trig and the knowledge that the opposite side is always the imaginary component and the adjacent side is always the real component.

$\tan\theta = -\frac{\sqrt{3}}{1} \to \theta = 60^\circ$

Thus the trigonometric form is 2 cis $60^\circ$.

**Example B**

Convert the following complex number from trigonometric polar form to rectangular form.

$4 \text{ cis}\left(\frac{3\pi}{4}\right)$

**Solution:** $4 \text{ cis } \left(\frac{3\pi}{4}\right) = 4\left(\cos\left(\frac{3\pi}{4}\right) + i\cdot\sin\left(\frac{3\pi}{4}\right)\right) = 4\left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right) = -2\sqrt{2}+2\sqrt{2}i$

**Example C**

Divide the following complex numbers.

$\frac{4 \text{ cis } 32^\circ}{2 \text{ cis } 2^\circ}$

**Solution:** $\frac{4 \text{ cis } 32^\circ}{2 \text{ cis } 2^\circ} = \frac{4}{2} \text{ cis } (32^\circ - 2^\circ) = 2 \text{ cis } (30^\circ)$

**Concept Problem Revisited**

In rectangular coordinates:

$\left(1+\sqrt{3}i\right)\left(\sqrt{2}-\sqrt{2}i\right) = \sqrt{2}-\sqrt{2}i+\sqrt{6}i+\sqrt{6}$

In trigonometric polar coordinates, $1+\sqrt{3}i = 2 \text{ cis } 60^\circ$ and $\sqrt{2}-\sqrt{2}i = 2 \text{ cis } -45^\circ$. Therefore:

$\left(1+\sqrt{3}i\right)\left(\sqrt{2}-\sqrt{2}i\right) = 2 \text{ cis } 60^\circ \cdot 2 \text{ cis } -45^\circ = 4 \text{ cis } 105^\circ$

## Vocabulary

***Trigonometric polar form*** of a complex number describes the location of a point on the complex plane using the angle and the radius of the point.

The abbreviation $r \cdot \text{cis}\ \theta$ stands for $r \cdot (\cos\theta + i \cdot \sin\theta)$ and is how trigonometric polar form is typically observed.

## Guided Practice

1. Translate the following complex number from trigonometric polar form to rectangular form.

   $5\ \text{cis}\ 270^\circ$

2. Translate the following complex number from rectangular form into trigonometric polar form.

   $8$

3. Multiply the following complex numbers in trigonometric polar form.

   $4\ \text{cis}\ 34^\circ \cdot 5\ \text{cis}\ 16^\circ \cdot \frac{1}{2}\ \text{cis}\ 100^\circ$

**Answers:**

1. $5\ \text{cis}\ 270^\circ = 5(\cos 270^\circ + i \cdot \sin 270^\circ) = 5(0 - i) = -5i$

2. $8 = 8\ \text{cis}\ 0^\circ$

3.

$$\begin{aligned} &4\ \text{cis}\ 34^\circ \cdot 5\ \text{cis}\ 16^\circ \cdot \frac{1}{2}\ \text{cis}\ 100^\circ \\ &= 4 \cdot 5 \cdot \frac{1}{2} \cdot \text{cis}\ (34^\circ + 16^\circ + 100^\circ) = 10\ \text{cis}\ 150 \end{aligned}$$

Note how much easier it is to do products and quotients in trigonometric polar form.

## Practice

Translate the following complex numbers from trigonometric polar form to rectangular form.

1. $5\ \text{cis}\ 270^\circ$
2. $2\ \text{cis}\ 30^\circ$
3. $-4\ \text{cis}\ \frac{\pi}{4}$
4. $6\ \text{cis}\ \frac{\pi}{3}$
5. $2\ \text{cis}\ \frac{5\pi}{2}$

Translate the following complex numbers from rectangular form into trigonometric polar form.

6. $2 - i$
7. $5 + 12i$
8. $6i + 8$
9. $i$

Complete the following calculations and simplify.

10. $2\ \text{cis}\ 22^\circ \cdot \frac{1}{5}\ \text{cis}\ 15^\circ \cdot 3\ \text{cis}\ 95^\circ$
11. $9\ \text{cis}\ 98^\circ \div 3\ \text{cis}\ 12^\circ$

12. $15 \text{ cis } \frac{\pi}{4} \cdot 2 \text{ cis } \frac{\pi}{6}$
13. $-2 \text{ cis } \frac{2\pi}{3} \div 15 \text{ cis } \frac{7\pi}{6}$

Let $z_1 = r_1 \cdot \text{cis } \theta_1$ and $z_2 = r_2 \cdot \text{cis } \theta_2$ with $r_2 \neq 0$.

14. Use the trigonometric sum and difference identities to prove that $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \text{cis } (\theta_1 + \theta_2)$.
15. Use the trigonometric sum and difference identities to prove that $z_1 \div z_2 = \frac{r_1}{r_2} \cdot \text{cis } (\theta_1 - \theta_2)$.

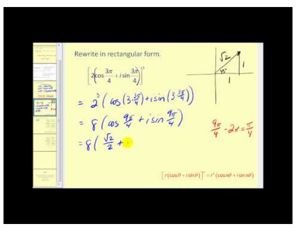
# 11.4 De Moivre's Theorem and nth Roots

Here you will learn about De Moivre's Theorem, which will help you to raise complex numbers to powers and find roots of complex numbers.

You know how to multiply two complex numbers together and you've seen the advantages of using trigonometric polar form, especially when multiplying more than two complex numbers at the same time. Because raising a number to a whole number power is repeated multiplication, you also know how to raise a complex number to a whole number power.

What is a geometric interpretation of squaring a complex number?

### Watch This

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Click image to the left for more content.

http://www.youtube.com/watch?v=Sf9gEzcVZkU James Sousa: De Moivre's Theorem: Powers of Complex Numbers in Trig Form

### Guidance

Recall that if $z_1 = r_1 \cdot \text{cis } \theta_1$ and $z_2 = r_2 \cdot \text{cis } \theta_2$ with $r_2 \neq 0$, then $z_1 \cdot z_2 = r_1 \cdot r_2 \cdot \text{cis } (\theta_1 + \theta_2)$.

If $z_1 = z_2 = z = r \text{ cis } \theta$ then you can determine $z^2$ and $z^3$:

$$z^2 = r \cdot r \cdot \text{cis } (\theta + \theta) = r^2 \text{ cis } (2 \cdot \theta)$$
$$z^3 = r^3 \text{ cis } (3 \cdot \theta)$$

De Moivre's Theorem simply generalizes this pattern to the power of any positive integer.

$z^n = r^n \cdot \text{cis } (n \cdot \theta)$

In addition to raising a complex number to a power, you can also take square roots, cube roots and $n^{th}$ roots of complex numbers. Suppose you have complex number $z = r \text{ cis } \theta$ and you want to take the $n^{th}$ root of $z$. In other words, you want to find a number $v = s \cdot \text{cis } \beta$ such that $v^n = z$. Do some substitution and manipulation:

$$v^n = z$$
$$(s \cdot \text{cis } \beta)^n = r \cdot \text{cis } \theta$$
$$s^n \cdot \text{cis } (n \cdot \beta) = r \cdot \text{cis } \theta$$

You can see at this point that to find $s$ you need to take the $n^{th}$ root of $r$. The trickier part is to find the angles, because $n \cdot \beta$ could be any angle coterminal with $\theta$. This means that there are $n$ different $n^{th}$ roots of $z$.

$$\begin{aligned} n \cdot \beta &= \theta + 2\pi k \\ \beta &= \frac{\theta + 2\pi k}{n} \end{aligned}$$

The number $k$ can be all of the counting numbers including zeros up to $n-1$. So if you are taking the $4^{th}$ root, then $k = 0, 1, 2, 3$.

Thus the $n^{th}$ root of a complex number requires $n$ different calculations, one for each root:

$v = \sqrt[n]{r} \cdot \text{cis} \left(\frac{\theta+2\pi k}{n}\right) \ for \ \{k \ \varepsilon \ I | 0 \leq k \leq n-1\}$

**Example A**

Find the cube root of the number 8.

**Solution:** Most students know that $2^3 = 8$ and so know that 2 is the cube root of 8. However, they don't realize that there are two other cube roots that they are missing. Remember to write out $k = 0, 1, 2$ and use the unit circle whenever possible to help you to find all three cube roots.

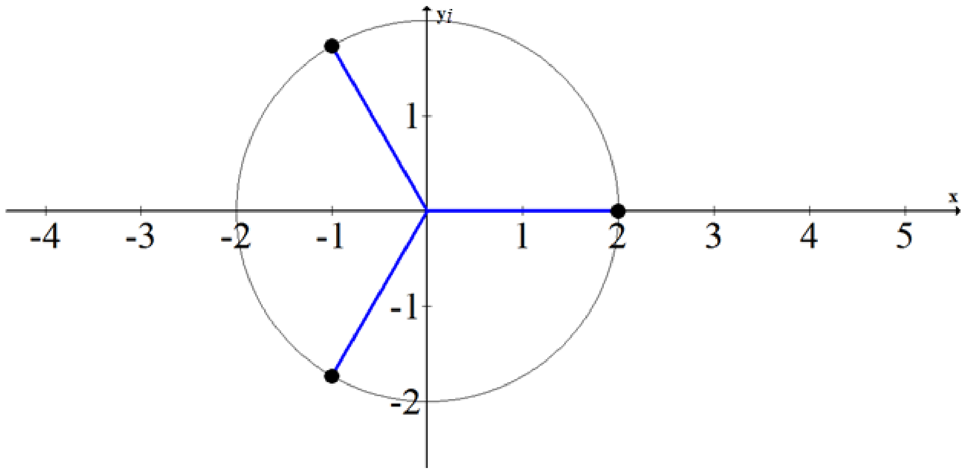
$$\begin{aligned} 8 &= 8 \text{ cis } 0 = (s \cdot \text{cis } \beta)^3 \\ z_1 &= 2 \cdot \text{cis } \left(\frac{0+2\pi \cdot 0}{3}\right) = 2 \text{ cis } 0 = 2(\cos 0 + i \cdot \sin 0) = 2(1+0) = 2 \\ z_2 &= 2 \cdot \text{cis } \left(\frac{0+2\pi \cdot 1}{3}\right) = 2 \text{ cis} \left(\frac{2\pi}{3}\right) \\ &= 2\left(\cos\left(\frac{2\pi}{3}\right) + i \cdot \sin\left(\frac{2\pi}{3}\right)\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -1 + i\sqrt{3} \\ z_3 &= 2 \cdot \text{cis} \left(\frac{0+2\pi \cdot 2}{3}\right) = 2 \text{ cis} \left(\frac{4\pi}{3}\right) \\ &= 2\left(\cos\left(\frac{4\pi}{3}\right) + i \cdot \sin\left(\frac{4\pi}{3}\right)\right) = 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - i\sqrt{3} \end{aligned}$$

The cube roots of 8 are 2, $-1 + i\sqrt{3}, -1 - i\sqrt{3}$.

**Example B**

Plot the roots of 8 graphically and discuss any patterns you notice.

**Solution:**




The three points are equally spaced around a circle of radius 2. Only one of the points, $2+0i$, is made up of only real numbers. The other two points have both a real and an imaginary component which is why they are off of the $x$ axis.

As you become more comfortable with roots, you can just determine the number of points that need to be evenly spaced around a certain radius circle and find the first point. The rest is just logic.

**Example C**

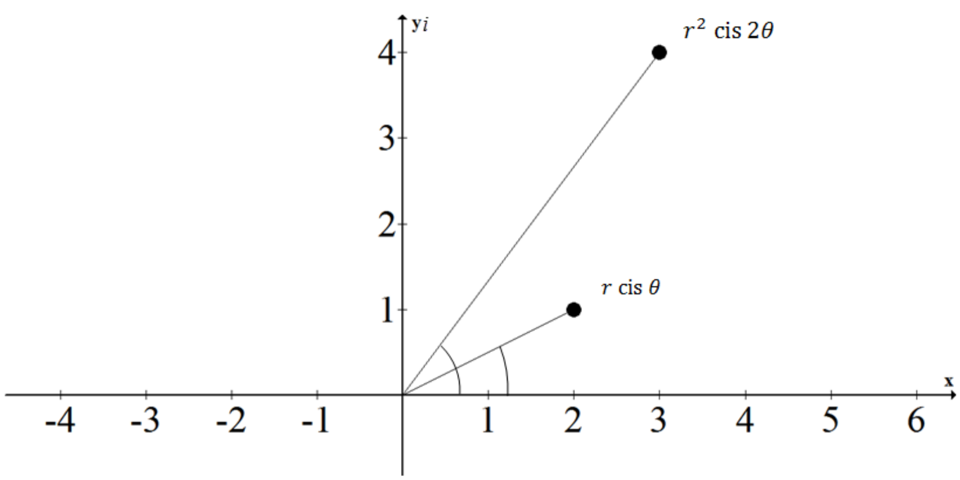
What are the fourth roots of $16 \text{ cis } 48^\circ$?

**Solution:** There will be 4 points, each $90^\circ$ apart with the first point at $2 \text{ cis } (12^\circ)$.

$2 \text{ cis } (12^\circ),\ 2 \text{ cis } (102^\circ),\ 2 \text{ cis } (192^\circ),\ 2 \text{ cis } (282^\circ)$

**Concept Problem Revisited**

Squaring a complex number produces a new complex number. The angle gets doubled and the magnitude gets squared, so geometrically you see a rotation.




## Vocabulary

The $n^{th}$ ***roots of unity*** refer to the $n^{th}$ roots of the number 1.

The ***real axis*** is the $x$ axis and the ***imaginary axis*** is the $y$ axis. Together they make the complex coordinate plane.

### Guided Practice

1. Check the three cube roots of 8 to make sure they are truly cube roots.
2. Solve for $z$ by finding the nth root of the complex number.

   $z^3 = 64 - 64\sqrt{3}i$

3. Use De Moivre's Theorem to evaluate the following power.

   $\left(\sqrt{2} - \sqrt{2}i\right)^6$

**Answers:**

1.

$$\begin{aligned}
z_1^3 &= 2^3 = 8 \\
z_2^3 &= \left(-1 + i\sqrt{3}\right)^3 \\
&= \left(-1 + i\sqrt{3}\right) \cdot \left(-1 + i\sqrt{3}\right) \cdot (-1 + i\sqrt{3}) \\
&= \left(1 - 2i\sqrt{3} - 3\right) \cdot \left(-1 + i\sqrt{3}\right) \\
&= \left(-2 - 2i\sqrt{3}\right) \cdot \left(-1 + i\sqrt{3}\right) \\
&= 2 - 2i\sqrt{3} + 2i\sqrt{3} + 6 \\
&= 8
\end{aligned}$$

Note how many steps and opportunities there are for making a mistake when multiplying multiple terms in rectangular form. When you check $z_3$, use trigonometric polar form.

$$\begin{aligned}
z_3^3 &= 2^3 \text{ cis } \left(3 \cdot \frac{4\pi}{3}\right) \\
&= 8(\cos 4\pi + i \cdot \sin 4\pi) \\
&= 8(1 + 0) \\
&= 8
\end{aligned}$$

2. First write the complex number in cis form. Remember to identify $k = 0, 1, 2$. This means the roots will appear every $\frac{360^\circ}{3} = 120^\circ$.

$$\begin{aligned}
z^3 &= 64 - 64\sqrt{3}i = 128 \cdot \text{cis } 300^\circ \\
z_1 &= 128^{\frac{1}{3}} \cdot \text{cis } \left(\frac{300}{3}\circ\right) = 128^{\frac{1}{3}} \cdot \text{cis}(100^\circ) \\
z_2 &= 128^{\frac{1}{3}} \cdot \text{cis } (220^\circ) \\
z_3 &= 128^{\frac{1}{3}} \cdot \text{cis } (340^\circ)
\end{aligned}$$

3. First write the number in trigonometric polar form, then apply De Moivre's Theorem and simplify.

$$\begin{aligned}\left(\sqrt{2}-\sqrt{2}i\right)^6 &= (2 \text{ cis } 315^\circ)^6 \\ &= 2^6 \cdot \text{cis } (6 \cdot 315^\circ) \\ &= 64 \cdot \text{cis } (1890^\circ) \\ &= 64 \cdot \text{cis } (1890^\circ) \\ &= 64 \cdot \text{cis } (90^\circ) \\ &= 64(\cos 90^\circ + i \cdot \sin 90^\circ) \\ &= 64(0+i) \\ &= 64i\end{aligned}$$

### Practice

Use De Moivre's Theorem to evaluate each expression. Write your answers in rectangular form.

1. $(1+i)^5$
2. $\left(1-\sqrt{3}i\right)^3$
3. $(1+2i)^6$
4. $\left(\sqrt{3}-i\right)^5$
5. $\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^4$
6. Find the cube roots of $3+4i$.
7. Find the $5^{th}$ roots of $32i$.
8. Find the $5^{th}$ roots of $1+\sqrt{5}i$.
9. Find the $6^{th}$ roots of - 64 and plot them on the complex plane.
10. Use your answers to #9 to help you solve $x^6+64=0$.

For each equation: a) state the number of roots, b) calculate the roots, and c) represent the roots graphically.

11. $x^3=1$
12. $x^8=1$
13. $x^{12}=1$
14. $x^4=16$
15. $x^3=27$

You learned the motivation for complex numbers by studying the Fundamental Theorem of Algebra. Then you learned how complex numbers are used in common operations. You learned different ways of representing complex numbers. Finally, you took powers of roots of complex numbers using De Moivre's Theorem.

# 11.5 References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# Chapter 12 Discrete Math

## Chapter Outline



Discrete math is all about patterns, sequences, summing numbers, counting and probability. Many of these topics you will revisit in classes after Calculus. The goal here is to familiarize you with the important notation and the habits of thinking that accompany a mature way of looking at problems.

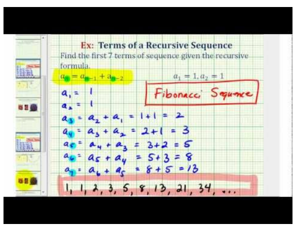
# 12.1 Recursion

Here you will define patterns recursively and use recursion to solve problems.

When you look at a pattern, there are many ways to describe it. You can describe patterns explicitly by stating how each term $a_k$is obtained from the term number$k$. You can also describe patterns recursively by stating how each new term $a_k$is obtained from the previous term$a_{k-1}$. Recursion defines an entire sequence based on the first term and the pattern between consecutive terms. The Fibonacci sequence is a famous recursive sequence, but how is it represented using recursion?

$0, 1, 1, 3, 5, 8, 13, 21, 34, \ldots$

### Watch This

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http://youtu.be/RjsyEWDEQe0 James Sousa: Finding Terms in a Sequence Given the Recursive Formula

### Guidance

When most people see a pattern they see how consecutive terms are related to one another. You might describe patterns with phrases like the ones below:

TABLE 12.1:

| Pattern | Recursive Description |
|---|---|
| $3, 6, 12, 24, \ldots$ | "Each term is twice as big as the previous term" |
| $3, 6, 9, 12, \ldots$ | "Each term is three more than the previous term" |

Each phrase is a sign of recursive thinking that defines each term as a function of the previous term.

$$a_k = f(a_{k-1})$$

In some cases, a recursive formula can be a function of the previous two or three terms. Keep in mind that the downside of a recursively defined sequence is that it is impossible to immediately know the $100^{th}$ term without knowing the $99^{th}$ term.

**Example A**

For the Fibonacci sequence, determine the first eleven terms and the sum of these terms.

**Solution:** $0+1+1+2+3+5+8+13+21+34+55 = 143$

**Example B**

Write a recursive definition that fits the following sequence.

$3, 7, 11, 15, 18, \ldots$

**Solution:** In order to write a recursive definition for a sequence you must define the pattern and state the first term. With this information, others would be able to replicate your sequence without having seen it for themselves.

$a_1 = 3$

$a_k = a_{k-1} + 4$

**Example C**

What are the first nine terms of the sequence defined by:

$a_1 = 1$

$a_k = \frac{1}{k} + 1$?

**Solution:** $1, 2, \frac{3}{2}, \frac{5}{3}, \frac{8}{3}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}$

**Concept Problem Revisited**

The Fibonacci sequence is represented by the recursive definition:

$a_1 = 0$

$a_2 = 1$

$a_k = a_{k-2} + a_{k-1}$

### Vocabulary

A ***recursively defined pattern or sequence*** is a sequence with terms that are defined based on the prior term(s) in the sequence.

An ***explicit pattern or sequence*** is a sequence with terms that are defined based on the term number.

### Guided Practice

1. The Lucas sequence is like the Fibonacci sequence except that the starting numbers are 2 and 1 instead of 1 and 0. What are the first ten terms of the Lucas sequence?

2. Zeckendorf's Theorem states that every positive integer can be represented uniquely as a sum of nonconsecutive Fibonacci numbers. What is the Zeckendorf representation of the number 50 and the number 100?

3. Consider the following pattern generating rule:

> *If the last number is odd, multiply it by 3 and add 1.*
>
> *If the last number is even, divide the number by 2.*
>
> *Repeat.*

Try a few different starting numbers and see if you can state what you think always happens.

**Answers:**

1. $2, 1, 3, 4, 7, 11, 18, 29, 47, 76$

2. $50 = 34 + 13 + 3;\ 100 = 89 + 8 + 3$

3. You can choose any starting positive integer you like. Here are the sequences that start with 7 and 15.

$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 \dots$

$15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 \dots$

You could make the conjecture that any starting number will eventually lead to the repeating sequence 4, 2, 1.

This problem is called the Collatz Conjecture and is an unproven statement in mathematics. People have used computers to try all the numbers up to $5 \times 2^{60}$ and many mathematicians believe it to be true, but since all natural numbers are infinite in number, this test does not constitute a proof.

### Practice

Write a recursive definition for each of the following sequences.

1. $3, 7, 11, 15, 19, \dots$
2. $3, 9, 27, 81, \dots$
3. $3, 6, 9, 12, 15, \dots$
4. $3, 6, 12, 24, 48, \dots$
5. $1, 4, 16, 64, \dots$
6. Find the first 6 terms of the following sequence:

$b_1 = 2$

$b_2 = 8$

$b_k = 6b_{k-1} - 4b_{k-2}$

7. Find the first 6 terms of the following sequence:

$c_1 = 4$

$c_2 = 18$

$c_k = 2c_{k-1} + 5c_{k-2}$

Suppose the Fibonacci sequence started with 2 and 5.

8. List the first 10 terms of the new sequence.
9. Find the sum of the first 10 terms of the new sequence.

Write a recursive definition for each of the following sequences. *These are trickier!*

10. $1, 4, 13, 40, \dots$
11. $1, 5, 17, 53, \dots$
12. $2, 11, 56, 281, \dots$
13. $2, 3, 6, 18, 108, \dots$
14. $4, 6, 11, 18, 30, \dots$
15. $7, 13, 40, 106, 292, \dots$

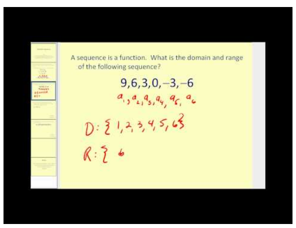
# 12.2 Arithmetic and Geometric Sequences

Here you will identify different types of sequences and use sequences to make predictions.

A sequence is a list of numbers with a common pattern. The common pattern in an arithmetic sequence is that the same number is added or subtracted to each number to produce the next number. The common pattern in a geometric sequence is that the same number is multiplied or divided to each number to produce the next number.

Are all sequences arithmetic or geometric?

### Watch This



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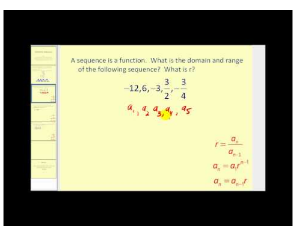
http://www.youtube.com/watch?v=jExpsJTu9o8 James Sousa: Arithmetic Sequences



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http://www.youtube.com/watch?v=XHyeLKZYb2w James Sousa: Geometric Sequences

### Guidance

A sequence is just a list of numbers separated by commas. A sequence can be finite or infinite. If the sequence is infinite, the first few terms are followed by an ellipsis $(\ldots)$ indicating that the pattern continues forever.

**An infinite sequence**: $1, 2, 3, 4, 5, \ldots$

**A finite sequence:** $2, 4, 6, 8$

In general, you describe a sequence with subscripts that are used to index the terms. The $k^{th}$ term in the sequence is $a_k$.

$a_1, a_2, a_3, a_4, \ldots, a_k, \ldots$

Arithmetic sequences are defined by an initial value $a_1$ and a common difference $d$.

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_1 + d \\
a_3 &= a_1 + 2d \\
a_4 &= a_1 + 3d \\
&\vdots \\
a_n &= a_1 + (n-1)d
\end{aligned}
$$

Geometric sequences are defined by an initial value $a_1$ and a common ratio $r$.

$$
\begin{aligned}
a_1 &= a_1 \\
a_2 &= a_1 \cdot r \\
a_3 &= a_1 \cdot r^2 \\
a_4 &= a_1 \cdot r^3 \\
&\vdots \\
a_n &= a_1 \cdot r^{n-1}
\end{aligned}
$$

If a sequence does not have a common ratio or a common difference, it is neither an arithmetic or a geometric sequence. You should still try to figure out the pattern and come up with a formula that describes it.

**Example A**

For each of the following three sequences, determine if it is arithmetic, geometric or neither.

a. $0.135, 0.189, 0.243, 0.297, \ldots$
b. $\frac{2}{9}, \frac{1}{6}, \frac{1}{8}, \ldots$
c. $0.54, 1.08, 3.24, \ldots$

**Solution:**

a. The sequence is arithmetic because the common difference is 0.054.
b. The sequence is geometric because the common ratio is $\frac{3}{4}$.
c. The sequence is not arithmetic because the differences between consecutive terms are 0.54 and 2.16 which are not common. The sequence is not geometric because the ratios between consecutive terms are 2 and 3 which are not common.

**Example B**

For the following sequence, determine the common ratio or difference, the next three terms, and the $2013^{th}$ term.

$\frac{2}{3}, \frac{5}{3}, \frac{8}{3}, \frac{11}{3}, \ldots$

**Solution:** The sequence is arithmetic because the difference is exactly 1 between consecutive terms. The next three terms are $\frac{14}{3}, \frac{17}{3}, \frac{20}{3}$. An equation for this sequence would is:

$a_n = \frac{2}{3} + (n-1) \cdot 1$

Therefore, the $2013^{th}$ term requires 2012 times the common difference added to the first term.

$a_{2013} = \frac{2}{3} + 2012 \cdot 1 = \frac{2}{3} + \frac{6036}{3} = \frac{6038}{3}$

**Example C**

For the following sequence, determine the common ratio or difference and the next three terms.

$\frac{2}{3}, \frac{4}{9}, \frac{6}{27}, \frac{8}{81}, \frac{10}{243}, \ldots$

**Solution:** This sequence is neither arithmetic nor geometric. The differences between the first few terms are $-\frac{2}{9}, -\frac{2}{9}, -\frac{10}{81}, -\frac{14}{243}$. While there was a common difference at first, this difference did not hold through the sequence. **Always check the sequence in multiple places to make sure that the common difference holds up throughout.**

The sequence is also not geometric because the ratios between the first few terms are $\frac{2}{3}, \frac{1}{2}, \frac{4}{9}$. These ratios are not common.

Even though you cannot get a common ratio or a common difference, it is still possible to produce the next three terms in the sequence by noticing the numerator is an arithmetic sequence with starting term of 2 and a common difference of 2. The denominators are a geometric sequence with an initial term of 3 and a common ratio of 3. The next three terms are:

$\frac{12}{3^6}, \frac{14}{3^7}, \frac{16}{3^8}$

**Concept Problem Revisited**

Example C shows that some patterns that use elements from both arithmetic and geometric series are neither arithmetic nor geometric. Two famous sequences that are neither arithmetic nor geometric are the Fibonacci sequence and the sequence of prime numbers.

**Fibonacci Sequence:** $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

**Prime Numbers:** $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

## Vocabulary

A ***sequence*** is a list of numbers separated by commas.

The common pattern in an ***arithmetic sequence*** is that the same number is added or subtracted to each number to produce the next number. This is called the ***common difference.***

The common pattern in a ***geometric sequence*** is that the same number is multiplied or divided to each number to produce the next number. This is called the ***common ratio.***

## Guided Practice

1. What is the tenth term in the following sequence?

$-12, 6, -3, \frac{3}{2}, \ldots$

2. What is the tenth term in the following sequence?

$-1, \frac{2}{3}, \frac{7}{3}, 4, \frac{17}{3}, \ldots$

3. Find an equation that defines the $a_k$ term for the following sequence.

$0, 3, 8, 15, 24, 35, \ldots$

**Answers:**

1. The sequence is geometric and the common ratio is $-\frac{1}{2}$. The equation is $a_n = -12 \cdot \left(-\frac{1}{2}\right)^{n-1}$. The tenth term is:

$-12 \cdot \left(-\frac{1}{2}\right)^9 = \frac{3}{128}$

2. The pattern might not be immediately recognizable, but try ignoring the $\frac{1}{3}$ in each number to see the pattern a

different way.

$-3, 2, 7, 12, 17, \ldots$

You should see the common difference of 5. This means the common difference from the original sequence is $\frac{5}{3}$. The equation is $a_n = -1 + (n-1)\left(\frac{5}{3}\right)$. The $10^{th}$ term is:

$-1 + 9 \cdot \left(\frac{5}{3}\right) = -1 + 3 \cdot 5 = -1 + 15 = 14$

3. The sequence is not arithmetic nor geometric. It will help to find the pattern by examining the common differences and then the common differences of the common differences. This numerical process is connected to ideas in calculus.

0, 3, 8, 15, 24, 35

3, 5, 7, 9, 11

2, 2, 2, 2

Notice when you examine the common difference of the common differences the pattern becomes increasingly clear. Since it took *two* layers to find a constant function, this pattern is *quadratic* and very similar to the perfect squares.

$1, 4, 9, 16, 25, 36, \ldots$

The $a_k$ term can be described as $a_k = k^2 - 1$

### Practice

Use the sequence $1, 5, 9, 13, \ldots$ for questions 1-3.

1. Find the next three terms in the sequence.

2. Find an equation that defines the $a_k$ term of the sequence.

3. Find the $150^{th}$ term of the sequence.

Use the sequence $12, 4, \frac{4}{3}, \frac{4}{9}, \ldots$ for questions 4-6.

4. Find the next three terms in the sequence.

5. Find an equation that defines the $a_k$ term of the sequence.

6. Find the $17^{th}$ term of the sequence.

Use the sequence $10, -2, \frac{2}{5}, -\frac{2}{25}, \ldots$ for questions 7-9.

7. Find the next three terms in the sequence.

8. Find an equation that defines the $a_k$ term of the sequence.

9. Find the $12^{th}$ term of the sequence.

Use the sequence $\frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}, \ldots$ for questions 10-12.

10. Find the next three terms in the sequence.

11. Find an equation that defines the $a_k$ term of the sequence.

12. Find the $314^{th}$ term of the sequence.

13. Find an equation that defines the $a_k$ term for the sequence $4, 11, 30, 67, \ldots$

14. Explain the connections between arithmetic sequences and linear functions.

15. Explain the connections between geometric sequences and exponential functions.

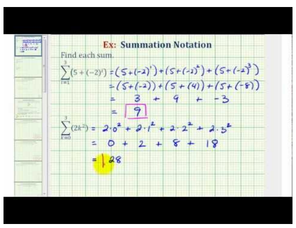
# 12.3 Sigma Notation

Here you will learn how to represent the sum of sequences of numbers using sigma notation.

Writing the sum of long lists of numbers that have a specific pattern is not very efficient. Summation notation allows you to use the pattern and the number of terms to represent the same sum in a much more concise way. How can you use sigma notation to represent the following sum?

$1+4+9+16+25+\cdots+144$

### Watch This

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http://www.youtube.com/watch?v=0L0rU17hHuM&feature=youtu.be James Sousa: Find a Sum Written in Summation/Sigma Notation

### Guidance

A **series** is a sum of a sequence. The Greek capital letter sigma is used for summation notation because it stands for the letter $S$ as in sum.

Consider the following general sequence and note that the subscript for each term is an index telling you the term number.

$a_1, a_2, a_3, a_4, a_5$

When you write the sum of this sequence in a series, it can be represented as a sum of each individual term or abbreviated using a capital sigma.

$$a_1+a_2+a_3+a_4+a_5=\sum_{i=1}^{5} a_i$$

The three parts of sigma notation that you need to be able to read are the argument, the lower index and the upper index. The argument, $a_i$, tells you what terms are added together. The lower index, $i=1$, tells you where to start and the upper index, 5, tells you where to end. You should practice reading and understanding sigma notation because it is used heavily in Calculus.

**Example A**

Write out all the terms of the series.

$$\sum_{k=4}^{8} 2k$$

**Solution:**

$$\sum_{k=4}^{8} 2k=2\cdot 4+2\cdot 5+2\cdot 6+2\cdot 7+2\cdot 8$$

**Example B**

Write the sum in sigma notation: $2+3+4+5+6+7+8+9+10$

**Solution:**

$2+3+4+5+6+7+8+9+10=\sum_{i=2}^{10} i$

**Example C**

Write the sum in sigma notation.

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}$

**Solution:**

$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}=\sum_{i=1}^{7} \frac{1}{i^2}$

**Concept Problem Revisited**

The hardest part when first using sigma representation is determining how each pattern generalizes to the $k^{th}$ term. Once you know the $k^{th}$ term, you know the argument of the sigma. For the sequence creating the series below, $a_k = k^2$. Therefore, the argument of the sigma is $i^2$.

$1+4+9+16+25+\cdots+144=1^2+2^2+3^2+4^2+\cdots 12^2=\sum_{i=1}^{12} i^2$

## Vocabulary

***Sigma notation*** is also known as ***summation notation*** and is a way to represent a sum of numbers. It is especially useful when the numbers have a specific pattern or would take too long to write out without abbreviation.

## Guided Practice

1. Write out all the terms of the sigma notation and then calculate the sum.

$\sum_{k=0}^{4} 3k-1$

2. Represent the following infinite series in summation notation.

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots$

3. Is there a way to represent an infinite product? How would you represent the following product?

$1\cdot\sin\left(\frac{360}{3}\right)\cdot\sin\left(\frac{360}{4}\right)\cdot\sin\left(\frac{360}{5}\right)\cdot\sin\left(\frac{360}{6}\right)\cdot\sin\left(\frac{360}{7}\right)\cdot\ldots$

**Answers:**

1.

$$\begin{aligned}\sum_{k=0}^{4} 3k-1 &= (3\cdot 0-1)+(3\cdot 1-1)+(3\cdot 2-1)+(3\cdot 3-1)+(3\cdot 4-1)\\ &= -1+2+5+8+11\end{aligned}$$

2. There are an infinite number of terms in the series so using an infinity symbol in the upper limit of the sigma is appropriate.

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=\sum_{i=1}^{\infty} \frac{1}{2^i}$

3. Just like summation uses a capital Greek letter for $S$, product uses a capital Greek letter for $P$ which is the capital form of $\pi$.

$$1 \cdot \sin\left(\frac{360}{2 \cdot 3}\right) \cdot \sin\left(\frac{360}{2 \cdot 4}\right) \cdot \sin\left(\frac{360}{2 \cdot 5}\right) \cdot \sin\left(\frac{360}{2 \cdot 6}\right) \cdot \sin\left(\frac{360}{2 \cdot 7}\right) \cdot \ldots = \prod_{i=3}^{\infty} \sin\left(\frac{360}{2 \cdot i}\right)$$

This infinite product is the result of starting with a circle of radius 1 and inscribing a regular triangle inside the circle. Then you inscribe a circle inside the triangle and a square inside the new circle. The shapes alternate being inscribed within each other as they are nested inwards: circle, triangle, circle, square, circle, pentagon, ... The question that this calculation starts to answer is whether this process reduces to a number or to zero.

### Practice

For 1-5, write out all the terms of the sigma notation and then calculate the sum.

1. $\sum_{k=1}^{5} 2k-3$
2. $\sum_{k=0}^{8} 2^k$
3. $\sum_{i=1}^{4} 2 \cdot 3^i$
4. $\sum_{i=1}^{10} 4i-1$
5. $\sum_{i=0}^{4} 2 \cdot \left(\frac{1}{3}\right)^i$

Represent the following series in summation notation with a lower index of 0.

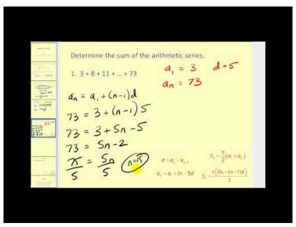
6. $1+4+7+10+13+16+19+22$
7. $3+5+7+9+11$
8. $8+7+6+5+4+3+2+1$
9. $5+6+7+8$
10. $3+6+12+24+48+\cdots$
11. $10+5+\frac{5}{2}+\frac{5}{4}$
12. $4-8+16-32+64\cdots$
13. $2+4+6+8+\cdots$
14. $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots$
15. $\frac{2}{3}+\frac{2}{9}+\frac{2}{27}+\frac{2}{81}+\cdots$

# 12.4 Arithmetic Series

Here you will learn to compute finite arithmetic series more efficiently than just adding the terms together one at a time.

While it is possible to add arithmetic series one term at a time, it is not feasible or efficient when there are a large number of terms. What is a clever way to add up all the whole numbers between 1 and 100?

### Watch This



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http://www.youtube.com/watch?v=Dj1JZIdIwwo James Sousa: Arithmetic Series

### Guidance

The key to adding up a finite arithmetic series is to pair up the first term with the last term, the second term with the second to last term and so on. The sum of each pair will be equal. Consider a generic series:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots a_n$$

When you pair the first and the last terms and note that $a_n = a_1 + (n-1)k$ the sum is:

$$a_1 + a_n = a_1 + a_1 + (n-1)k = 2a_1 + (n-1)k$$

When you pair up the second and the second to last terms you get the same sum:

$$a_2 + a_{n-1} = (a_1 + k) + (a_1 + (n-2)k) = 2a_1 + (n-1)k$$

The next logical question to ask is: how many pairs are there? If there are $n$ terms total then there are exactly $\frac{n}{2}$ pairs. If $n$ happens to be even then every term will have a partner and $\frac{n}{2}$ will be a whole number. If $n$ happens to be odd then every term but the middle one will have a partner and $\frac{n}{2}$ will include a $\frac{1}{2}$ pair that represents the middle term with no partner. Here is the general formula for arithmetic series:

$\sum_{i=1}^{n} a_i = \frac{n}{2}(2a_1 + (n-1)k)$ where $k$ is the common difference for the terms in the series.

**Example A**

Add up the numbers between one and ten (inclusive) in two ways.

**Solution:** One way to add up lists of numbers is to pair them up for easier mental arithmetic.

$$\begin{aligned}1+2+3+4+5+6+7+8+9+10 &= 3+7+11+15+19\\ &= 10+26+19\\ &= 36+19\\ &= 55\end{aligned}$$

Another way is to note that $1+10=2+9=3+8=4+7=5+6=11$. There are 5 pairs of 11 which total 55.

**Example B**

Evaluate the following sum.

$\sum_{k=0}^{5} 5k-2$

**Solution:** The first term is -2, the last term is 23 and there are 6 terms making 3 pairs. A common mistake is to forget to count the 0 index.

$\sum_{k=0}^{5} 5k-2 = \frac{6}{2}\cdot(-2+23) = 3\cdot 21 = 63$

**Example C**

Try to evaluate the sum of the following geometric series using the same technique as you would for an arithmetic series.

$\frac{1}{8}+\frac{1}{2}+2+8+32$

**Solution:**

The real sum is: $\frac{341}{8}$

When you try to use the technique used for arithmetic sequences you get: $3\left(\frac{1}{8}+32\right)=\frac{771}{8}$

It is important to know that geometric series have their own method for summing. The method learned in this concept only works for arithmetic series.

**Concept Problem Revisited**

Gauss was a mathematician who lived hundreds of years ago and there is an anecdote told about him when he was a young boy in school. When misbehaving, his teacher asked him to add up all the numbers between 1 and 100 and he stated 5050 within a few seconds.

You should notice that $1+100=2+99=\cdots=101$ and that there are exactly 50 pairs that sum to be 101. $50\cdot 101=5050$.

## Vocabulary

An ***arithmetic series*** is a sum of numbers whose consecutive terms form an arithmetic sequence.

## Guided Practice

1. Sum the first 15 terms of the following arithmetic sequence.

$-1, \frac{2}{3}, \frac{7}{3}, 4, \frac{17}{3}\dots$

2. Sum the first 100 terms of the following arithmetic sequence.

$-7,-4,-1,2,5,8,\dots$

3. Evaluate the following sum.

$\sum_{i=0}^{500} 2i-312$

**Answers:**

1. The initial term is -1 and the common difference is $\frac{5}{3}$.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{15}{2}\left(2(-1)+(15-1)\frac{5}{3}\right)\\ &= \frac{15}{2}\left(-2+14\cdot\frac{5}{3}\right)\\ &= 160\end{aligned}$$

2. The initial term is -7 and the common difference is 3.

$$\begin{aligned}\sum_{i=1}^{n} a_i &= \frac{n}{2}(2a_1+(n-1)k)\\ &= \frac{100}{2}(2(-7)+(100-1)3)\\ &= 14150\end{aligned}$$

3. The initial term is -312 and the common difference is 2.

$$\begin{aligned}\sum_{i=0}^{500} 2i-312 &= \frac{501}{2}(2(-312)+(501-1)2)\\ &= 94188\end{aligned}$$

## Practice

1. Sum the first 24 terms of the sequence $1,5,9,13,\ldots$
2. Sum the first 102 terms of the sequence $7,9,11,13,\ldots$
3. Sum the first 85 terms of the sequence $-3,-1,1,3,\ldots$
4. Sum the first 97 terms of the sequence $\frac{1}{3},\frac{2}{3},1,\frac{4}{3},\ldots$
5. Sum the first 56 terms of the sequence $-\frac{2}{3},\frac{1}{3},\frac{4}{3},\ldots$
6. Sum the first 91 terms of the sequence $-8,-4,0,4,\ldots$

Evaluate the following sums.

7. $\sum_{i=0}^{300} 3i+18$
8. $\sum_{i=0}^{215} 5i+1$

9. $\sum_{i=0}^{100} i-15$

10. $\sum_{i=0}^{85} -13i+1$

11. $\sum_{i=0}^{212} -2i+6$

12. $\sum_{i=0}^{54} 6i-9$

13. $\sum_{i=0}^{167} -5i+3$

14. $\sum_{i=0}^{341} 6i+102$

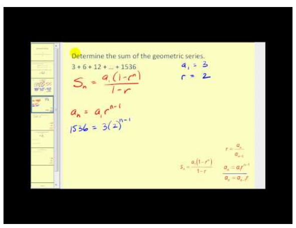
15. $\sum_{i=0}^{452} -7i-\frac{5}{2}$

# 12.5 Geometric Series

Here you will sum infinite and finite geometric series and categorize geometric series as convergent or divergent.

An advanced factoring technique allows you to rewrite the sum of a finite geometric series in a compact formula. An infinite geometric series is more difficult because sometimes it sums to be a number and sometimes the sum keeps on growing to infinity. When does an infinite geometric series sum to be just a number and when does it sum to be infinity?

### Watch This

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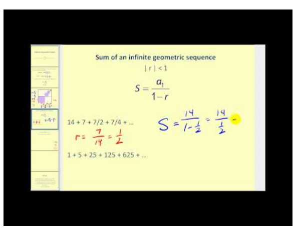
http://www.youtube.com/watch?v=mYg5gKlJjHc James Sousa: Geometric Series

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http://www.youtube.com/watch?v=RLZXFhvdlV8 James Sousa: Infinite Geometric Series

### Guidance

Recall the advanced factoring technique for the difference of two squares and, more generally, two terms of any power (5 in this case).

$$a^2-b^2=(a-b)(a+b)$$
$$a^5-b^5=(a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$$
$$a^n-b^n=(a-b)(a^{n-1}+\cdots+b^{n-1})$$

If the first term is one then $a=1$. If you replace $b$ with the letter $r$, you end up with:

$1-r^n=(1-r)(1+r+r^2+\cdots r^{n-1})$

You can divide both sides by $(1-r)$ because $r \neq 1$.

$1+r+r^2+\cdots r^{n-1}=\frac{1-r^n}{1-r}$

The left side of this equation is a geometric series with starting term 1 and common ratio of $r$. Note that even though the ending exponent of $r$ is $n-1$, there are a total of $n$ terms on the left. To make the starting term not one, just scale both sides of the equation by the first term you want, $a_1$.

$a_1 + a_1 r + a_1 r^2 + \cdots a_1 r^{n-1} = a_1 \left(\frac{1-r^n}{1-r}\right)$

This is the sum of a finite geometric series.

To sum an infinite geometric series, you should start by looking carefully at the previous formula for a finite geometric series. As the number of terms get infinitely large ($n \to \infty$) one of two things will happen.

$a_1 \left(\frac{1-r^n}{1-r}\right)$

**Option 1:** The term $r^n$ will go to infinity or negative infinity. This will happen when $|r| \geq 1$. When this happens, the sum of the infinite geometric series does not go to a specific number and the series is said to be **divergent.**

**Option 2:** The term $r^n$ will go to zero. This will happen when $|r| < 1$. When this happens, the sum of the infinite geometric series goes to a certain number and the series is said to be **convergent**.

One way to think about these options is think about what happens when you take $0.9^{100}$ and $1.1^{100}$.

$$0.9^{100} \approx 0.00002656$$
$$1.1^{100} \approx 13780$$

As you can see, even numbers close to one either get very small quickly or very large quickly.

The formula for calculating the sum of an infinite geometric series that converges is:

$\sum_{i=1}^{\infty} a_1 \cdot r^{i-1} = a_1 \left(\frac{1}{1-r}\right)$

Notice how this formula is the same as the finite version but with $r^n = 0$, just as you reasoned.

**Example A**

Compute the sum of the following infinite geometric series.

$0.2 + 0.02 + 0.002 + 0.0002 + \cdots$

**Solution:** You can tell just by looking at the sum that the infinite sum will be the repeating decimal $0.\overline{2}$. You may recognize this as the fraction $\frac{2}{9}$, but if you don't, this is how you turn a repeating decimal into a fraction.

Let $x = 0.\overline{2}$

Then $10x = 2.\overline{2}$

Subtract the two equations and solve for $x$.

$$10x - x = 2.\overline{2} - 0.\overline{2}$$
$$9x = 2$$
$$x = \frac{2}{9}$$

**Example B**

Why does an infinite series with $r = 1$ diverge?

**Solution:** If $r = 1$ this means that the common ratio between the terms in the sequence is 1. This means that each number in the sequence is the same. When you add up an infinite number of any finite numbers (even fractions close to zero) you will always get infinity or negative infinity. The only exception is 0. This case is trivial because a geometric series with an initial value of 0 is simply the following series, which clearly sums to 0:

$0+0+0+0+\cdots$

**Example C**

What is the sum of the first 8 terms in the following geometric series?

$4+2+1+\frac{1}{2}+\cdots$

**Solution:** The first term is 4 and the common ratio is $\frac{1}{2}$.

$SUM = a_1\left(\frac{1-r^n}{1-r}\right) = 4\left(\frac{1-\left(\frac{1}{2}\right)^8}{1-\frac{1}{2}}\right) = 4\left(\frac{\frac{255}{256}}{\frac{1}{2}}\right) = \frac{255}{32}$

**Concept Problem Revisited**

An infinite geometric series converges if and only if $|r| < 1$. Infinite arithmetic series never converge.

## Vocabulary

To ***converge*** means the sum approaches a specific number.

To ***diverge*** means the sum does not converge, and so usually goes to positive or negative infinity. It could also mean that the series oscillates infinitely.

A ***partial sum*** of an infinite sum is the sum of all the terms up to a certain point. Considering partial sums can be useful when analyzing infinite sums.

## Guided Practice

1. Compute the sum from Example A using the infinite summation formula and confirm that the sum truly does converge.

2. Does the following geometric series converge or diverge? Does the sum go to positive or negative infinity?

$-2+2-2+2-2+\cdots$

3. You put $100 in a bank account at the end of every year for 10 years. The account earns 6% interest. How much do you have total at the end of 10 years?

**Answers:**

1. The first term of the sequence is $a_1 = 0.2$. The common ratio is 0.1. Since $|0.1| < 1$, the series does converge.

$0.2\left(\frac{1}{1-0.1}\right) = \frac{0.2}{0.9} = \frac{2}{9}$

2. The initial term is -2 and the common ratio is -1. Since the $|-1| \geq 1$, the series is said to diverge. Even though the series diverges, it does not approach negative or positive infinity. When you look at the partial sums (the sums up to certain points) they alternate between two values:

$-2, 0, -2, 0, \dots$

This pattern does not go to a specific number. Just like a sine or cosine wave, it does not have a limit as it approaches infinity.

3. The first deposit gains 9 years of interest: $100 \cdot 1.06^9$

The second deposit gains 8 years of interest: $100 \cdot 1.06^8$. This pattern continues, creating a geometric series. The last term receives no interest at all.

$100 \cdot 1.06^9 + 100 \cdot 1.06^8 + \cdots 100 \cdot 1.06 + 100$

Note that normally geometric series are written in the opposite order so you can identify the starting term and the common ratio more easily.

$a_1 = 100, r = 1.06$

The sum of the 10 years of deposits is:

$a_1\left(\frac{1-r^n}{1-r}\right) = 100\left(\frac{1-1.06^{10}}{1-1.06}\right) \approx \$1318.08$

### Practice

Find the sum of the first 15 terms for each geometric sequence below.

1. $5, 10, 20, \ldots$
2. $2, 8, 32, \ldots$
3. $5, \frac{5}{2}, \frac{5}{4}, \ldots$
4. $12, 4, \frac{4}{3}, \ldots$
5. $\frac{1}{3}, 1, 3, \ldots$

For each infinite geometric series, identify whether the series is convergent or divergent. If convergent, find the number where the sum converges.

6. $5 + 10 + 20 + \cdots$
7. $2 + 8 + 32 + \cdots$
8. $5 + \frac{5}{2} + \frac{5}{4} + \cdots$
9. $12 + 4 + \frac{4}{3} + \cdots$
10. $\frac{1}{3} + 1 + 3 + \cdots$
11. $6 + 2 + \frac{2}{3} + \cdots$
12. You put \$5000 in a bank account at the end of every year for 30 years. The account earns 2% interest. How much do you have total at the end of 30 years?
13. You put \$300 in a bank account at the end of every year for 15 years. The account earns 4% interest. How much do you have total at the end of 10 years?
14. You put \$10,000 in a bank account at the end of every year for 12 years. The account earns 3.5% interest. How much do you have total at the end of 12 years?
15. Why don't infinite arithmetic series converge?

# 12.6 Counting with Permutations and Combinations

Here you will review counting using decision charts, permutations and combinations.

Sometimes it makes sense to count the number of ways for an event to occur by looking at each possible outcome. However, when there are a large number of outcomes this method quickly becomes inefficient. If someone asked you how many possible regular license plates there are for the state of California, it would not be feasible to count each and every one. Instead, you would need to use the fact that on the typical California license plate there are four numbers and three letters. Using this information, about how many license plates could there be?

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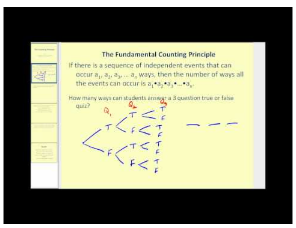


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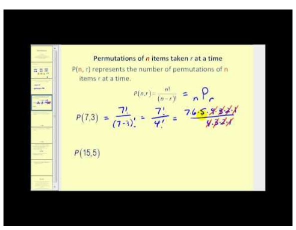
http://www.youtube.com/watch?v=qJ7AYDmHVRE James Sousa: The Counting Principle

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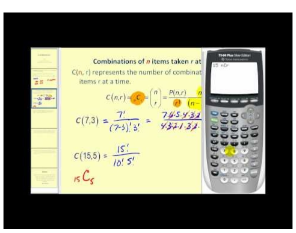
http://www.youtube.com/watch?v=JyRKTesp6fQ James Sousa: Permutations



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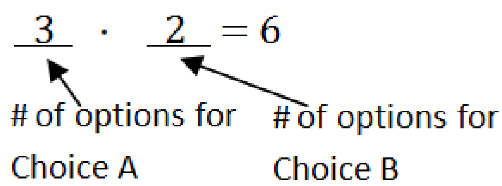
http://www.youtube.com/watch?v=SGn1913lOYM James Sousa: Combinations

### Guidance

Consider choice $A$ with three options $(A_1, A_2, A_3)$ and choice $B$ with two options $(B_1, B_2)$. If you had to choose an option from $A$ and then an option from $B$, the overall total number of options would be $3 \cdot 2 = 6$. The options are $A_1B_1, A_1B_2, A_2B_1, A_2B_2, A_3B_1, A_3B_2$.

You can see where the six comes from by making a decision chart and using the Fundamental Counting Principle. First, determine how many decisions you are making. Here, there are only two decisions to make (1: choose an option from $A$; 2: choose an option from $B$), so you will have two "slots" in your decision chart. Next, think about how many possibilities there are for the first choice (in this case there are 3) and how many possibilities there are for the second choice (in this case there are 2). The Fundamental Counting Principle says that you can multiply those numbers together to get the total number of outcomes.




Another type of counting question is when you have a given number of objects, you want to choose some (or all) of them, and you want to know how many ways there are to do this. For example, a teacher has a classroom of 30 students, she wants 5 of them to do a presentation, and she wants to know how many ways this could happen. These types of questions have to do with **combinations** and **permutations.** The difference between combinations and permutations has to do with whether or not the order that you are choosing the objects matters.

- A teacher choosing a group to make a presentation would be a **combination** problem, because **order does not matter.**
- A teacher choosing $1^{st}$, $2^{nd}$, and $3^{rd}$ place winners in a science fair would be a **permutation** problem, because the **order matters** (a student getting $1^{st}$ place vs. $2^{nd}$ place are different outcomes).

Recall that the factorial symbol, !, means to multiply every whole number up to and including that whole number together. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$. The factorial symbol is used in the formulas for permutations and combinations.

**Combination Formula:** The number of ways to choose $k$ objects from a group of $n$ objects is –

$${}_nC_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

**Permutation Formula:** The number of ways to choose **and arrange** $k$ objects from a group of $n$ objects is –

$${}_nP_k = k!\binom{n}{k} = k! \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!}$$

Notice that in both permutation and combination problems you are not allowed to repeat your choices. Any time you are allowed to repeat and order does not matter, you can use a decision chart. (Problems with repetition where order does not matter are more complex and are not discussed in this text.)

Whenever you are doing a counting problem, the first thing you should decide is if the problem is a decision chart problem, a permutation problem, or a combination problem. You will find that permutation problems can also be solved with decision charts. The opposite is not true. There are many decision chart problems (ones where you are allowed to repeat choices) that could not be solved with the permutation formula.

*Note: Here you have only begun to explore counting problems. For more information about combinations, permutations, and other types of counting problems, consult a Probability text.*

**Example A**

You are going on a road trip with 4 friends in a car that fits 5 people. How many different ways can everyone sit if you have to drive the whole way?

**Solution:** A decision chart is a great way of thinking about this problem. You have to sit in the driver's seat. There are four options for the first passenger seat. Once that person is seated there are three options for the next passenger seat. This goes on until there is one person left with one seat.

$1 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 24$

**Example B**

How many different ways can the gold, silver and bronze medals be awarded in an Olympic event with 12 athletes competing?

**Solution:** Since the order does matter with the three medals, this is a permutation problem. You will start with 12 athletes and then choose and arrange 3 different winners.

${}_{12}P_3 = \frac{12!}{(12-3)!} = \frac{12!}{9!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot \ldots}{9 \cdot \ldots} = 12 \cdot 11 \cdot 10 = 1320$

Note that you could also use a decision chart to decide how many possibilities are there for gold (12) how many possibilities are there for silver (11 since one already has gold) and how many possibilities are there for bronze (10). You can use a decision chart for any permutation problem.

$12 \cdot 11 \cdot 10 = 1320$

**Example C**

You are deciding which awards you are going to display in your room. You have 8 awards, but you only have room to display 4 awards. Right now you are not worrying about how to arrange the awards so the order does not matter. In how many ways could you choose the 4 awards to display?

**Solution:** Since order does not matter, this is a combination problem. You start with 8 awards and then choose 4.

${}_8C_4 = \binom{8}{4} = \frac{8!}{4!(8-4)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 7 \cdot 2 \cdot 5 = 70$

Note that if you try to use a decision chart with this question, you will need to do an extra step of reasoning. There are 8 options I could choose first, then 7 left, then 6 and lastly 5.

$8 \cdot 7 \cdot 6 \cdot 5 = 1680$

This number is so big because it takes into account order, which you don't care about. It is the same result you would get if you used the permutation formula instead of the combination formula. To get the right answer, you need to divide this number by the number of ways 4 objects can be arranged, which is $4! = 24$. This has to do with the connection between the combination formula and the permutation formula.

**Concept Problem Revisited**

A license plate that has 3 letters and 4 numbers can be represented by a decision chart with seven spaces. You can use a decision chart because order definitely does matter with license plates. The first spot is a number, the next three spots are letters and the last three spots are numbers. Note that when choosing a license plate, repetition is allowed.

$$10 \cdot 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 26^3 \cdot 10^4 = 175,760,000$$

This number is only approximate because in reality there are certain letter and number combinations that are not allowed, some license plates have extra symbols, and some commercial and government license plates have more numbers, fewer letters or blank spaces.

## Vocabulary

A ***combination*** is the number of ways of choosing $k$ objects from a total of $n$ objects (order does not matter).

A ***permutation*** is the number of ways of choosing and arranging $k$ objects from a total of $n$ objects (order does matter)

A ***decision chart*** is a sequence of numbers that multiply together where each number represents the number of possible options for that slot.

## Guided Practice

1. There are 20 hockey players on a pro NHL team, two of which are goalies. In how many different ways can 5 skaters and 1 goalie be on the ice at the same time?

2. In how many different ways could you score a 70% on a 10 question test where each question is weighted equally and is either right or wrong?

3. How many different 4 digit ATM passwords are there? Assume you can repeat digits.

**Answers:**

1. The question asks for how many on the ice, implying that order does not matter. This is combination problem with two combinations. You need to choose 1 goalie out of a possible of 2 and choose 5 skaters out of a possible 18.

$\binom{2}{1}\binom{18}{5} = 2 \cdot \frac{18!}{5! \cdot 13!} = 17136$

2. The order of the questions you got right does not matter, so this is a combination problem.

$\binom{10}{7} = \frac{10!}{7!3!} = 120$

3. Order does matter. There are 10 digits and repetition is allowed. You can use a decision chart for each of the four options.

$10 \cdot 10 \cdot 10 \cdot 10 = 10,000$

## Practice

Simplify each of the following expressions so that they do not have a factorial symbol.

1. $\frac{7!}{3!}$

2. $\frac{110!}{105!5!}$

3. $\frac{52!}{49!}$

4. In how many ways can you choose 3 objects from a set of 9 objects?

5. In how many ways can you choose and arrange 4 objects from a set of 15 objects?

First, state whether each problem is a **permutation/decision chart** problem or a **combination** problem. Then, solve.

6. Suppose you need to choose a new combination for your combination lock. You have to choose 3 numbers, each different and between 0 and 40. How many combinations are there?

7. You just won a contest where you can choose 2 friends to go with you to a concert. You have five friends who are available and want to go. In how many ways can you choose the friends?

8. You want to construct a 3 digit number from the digits 4, 6, 8, 9. How many possible numbers are there?

9. There are 12 workshops at a conference and Sam has to choose 3 to attend. In how many ways can he choose the 3 to attend?

10. 9 girls and 5 boys are finalists in a contest. In how many ways can $1^{st}$, $2^{nd}$, and $3^{rd}$ place winners be chosen?

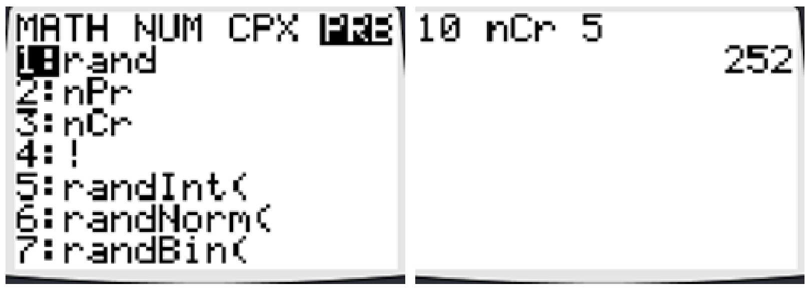
11. For the special at a restaurant you can choose 3 different items from the 10 item menu. How many different combinations of meals could you get?

12. You visit 12 colleges and want to apply to 4 of them. In how many ways could you choose the four to apply to?

13. For the 12 colleges you visited, you want to rank your top five. In how many ways could you rank your top 5?

14. ***Explain why the following problem is not strictly a permutation or combination problem***: The local ice cream shop has 12 flavors. You decide to buy 2 scoops in a dish. In how many ways could you do this if you are allowed to get two of the same scoop?

15. Your graphing calculator has the combination and permutation formulas built in. Push the MATH button and scroll to the right to the PRB list. You should see ${}_nP_r$ and ${}_nC_r$ as options. In order to use these: 1) On your home screen type the value for $n$; 2) Select ${}_nP_r$ or ${}_nC_r$; 3) Type the value for $k$ ($r$ on the calculator). Use your calculator to verify that ${}_{10}C_5 = 252$.



# 12.7 Basic Probability

Here you will calculate the probability of simple and compound events.

Most are familiar with how flipping a coin or rolling dice works and yet probability remains one of the most counterintuitive branches of mathematics for many people. The idea that flipping a coin and getting 10 heads in a row is just as unlikely as getting the following sequence of heads and tails is hard to comprehend.

$HHTHTTTTHTH$

Assume a plane crashes on average once every 100 days (extremely inaccurate). Given a plane crashed today, what day in the next 100 days is the plane most likely to crash next?

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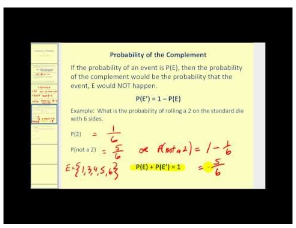

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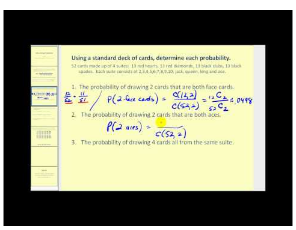
http://www.youtube.com/watch?v=YWt_u5l_jHs James Sousa: Introduction to Probability



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http://www.youtube.com/watch?v=IZAMLgS5x6w James Sousa: Determining Probability

### Guidance

Probability is the chance of an event occurring. Simple probability is defined as the number of outcomes you are looking for (also called successes) divided by the total number of outcomes. The notation $P(E)$ is read "the probability of event $E$".

$P(E) = \frac{successes}{possible\ outcomes}$

Probabilities can be represented with fractions, decimals, or percents. Since the number of possible outcomes is in the denominator, the probability is always between zero and one. A probability of 0 means the event will definitely not happen, while a probability of 1 means the event will definitely happen.

$0 \leq P(E) \leq 1$

The probability of something not happening is called the **complement** and is found by subtracting the probability from one.

$P(E^C) = 1 - P(E)$

You will often be looking at probabilities of two or more independent experiments. Experiments are independent when the outcome of one experiment has no effect on the outcome of the other experiment. If there are two experiments, one with outcome $A$ and the other with outcome $B$, then the probability of $A$ and $B$ is:

$P(A \text{ and } B) = P(A) \cdot P(B)$

The probability of $A$ or $B$ is:

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

**Example A**

If you are dealt one card from a 52 card deck, what is the probability that you are dealt a heart? What is the probability that you are dealt a 3? What is the probability that you are dealt the three of hearts?

**Solution:** There are 13 hearts in a deck of 52 cards. $P(heart) = \frac{13}{52} = \frac{1}{4}$

There are 4 threes in the deck of 52. $P(three) = \frac{4}{52} = \frac{1}{13}$

There is only one three of hearts. $P(three \text{ and } heart) = \frac{1}{52}$

**Example B**

Dean and his friend Randy like to play a special poker game with their friends. Dean goes home a winner 60% of the time and Randy goes home a winner 75% of the time.

a. What is the probability that they both win in the same night?
b. What is the probability that Randy wins and Dean loses?
c. What is the probability that they both lose?

**Solution:** First represent the information with probability symbols.

Let $D$ be the event that Dean wins. Let $R$ be the event that Randy wins. The complement of each probability is when Dean or Randy loses instead.

$P(D) = 0.60, \quad P(D^C) = 0.40$

$P(R) = 0.75, \quad P(R^C) = 0.25$

a. $P(D \text{ and } R) = P(D) \cdot P(R) = 0.60 \cdot 0.75 = 0.45$
b. $P(R \text{ and } D^C) = P(R) \cdot P(D^C) = 0.75 \cdot 0.40 = 0.30$
c. $P(D^C \text{ and } R^C) = P(D^C) \cdot P(R^C) = 0.40 \cdot 0.25 = 0.10$

**Example C**

If a plane crashes on average once every hundred days, what is the probability that the plane will crash in the next 100 days?

**Solution:** The naïve and incorrect approach would be to interpret the question as "what is the sum of the probabilities for each of the days?" Since there are 100 days and each day has a probability of 0.01 for a plane crash, then by this logic, there is a 100% chance that a plane crashes. This isn't true because if on average the plane crashes once every hundred days, some stretches of 100 days there will be more crashes and some stretches there will be no crashes. The 100% solution does not hold.

In order to solve this question, you need to rephrase the question and ask a slightly different one that will help as an intermediate step. What is the probability that a plane does not crash in the next 100 days?

In order for this to happen it must not crash on day 1 and not crash on day 2 and not crash on day 3 etc.

The probability of the plane not crashing on any day is $P(no\ crash) = 1 - P(crash) = 1 - 0.01 = 0.99$.

The product of each of these probabilities for the 100 days is:

$0.99^{100} \approx 0.366$

Therefore, the probability that a plane does not crash in the next 100 days is about 36.6%. To answer the original question, the probability that a plane does crash in the next 100 days is $1-0.366 = 0.634$ or about 63.4%.

**Concept Problem Revisited**

Whether or not a plane crashes today does not matter. The probability that a plane crashes tomorrow is $p = 0.01$. The probability that it crashes any day in the next 100 days is equally $p = 0.01$. The key part of the question is the word "next".

The probability that a plane does not crash on the first day and does crash on the second day is a compound probability, which means you multiply the probability of each event.

$P(Day\ 1\ no\ crash\ AND\ Day\ 2\ crash) = 0.99 \cdot 0.01 = 0.0099$

Notice that this probability is slightly smaller than 0.01. Each successive day has a slightly smaller probability of being the next day that a plane crashes. Therefore, the day with the highest probability of a plane crashing next is tomorrow.

## Vocabulary

The ***probability*** of an event is the number of outcomes you are looking for (called successes) divided by the total number of outcomes.

The **complement of an event** is the event not happening.

***Independent events*** are events where the occurrence of the first event does not impact the probability of the second event.

## Guided Practice

1. Jack is a basketball player with a free throw average of 0.77. What is the probability that in a game where he has 8 shots that he makes all 8? What is the probability that he only makes 1?

2. If it has a 20% chance of raining on Tuesday, your phone has 30% chance of running out of batteries, and there is a 10% chance that you forget your wallet, what is the probability that you are in the rain without money or a phone?

3. Consider the previous question with the rain, wallet and phone. What is the probability that at least one of the three events does occur?

**Answers:**

1. Let $J$ represent the event that Jack makes the free throw shot and $J^C$ represent the event that Jack misses the shot.

$P(J) = 0.77,\ P(J^C) = 0.23$

The probability that Jack makes all 8 shots is the same as Jack making one shot and making the second shot and making the third shot etc.

$P(J)^8 = 0.77^8 \approx 12.36\%$

There are 8 ways that Jack could make 1 shot and miss the rest. The probability of each of these cases occurring is:

$P(J^C)^7 \cdot P(J) = 0.23^7 \cdot 0.77$

Therefore, the overall probability of Jack making 1 shot and missing the rest is:

$0.23^7 \cdot 0.77 \cdot 8 = 0.0002097 = 0.02097\%$

2. While a pessimist may believe that all the improbable negative events will occur at the same time, the actual probability of this happening is less than one percent:

$0.20 \cdot 0.30 \cdot 0.1 = 0.006 = 0.6\%$

3. The naïve approach would be to simply add the three probabilities together. This is incorrect. The better way to approach the problem is to ask the question: what is the probability that none of the events occur?

$0.8 \cdot 0.7 \cdot 0.9 = 0.504$

The probability that at least one occurs is the complement of none occurring.

$1 - 0.504 = 0.496 = 49.6\%$

## Practice

A card is chosen from a standard deck.

1. What's the probability that the card is a queen?

2. What's the probability that the card is a queen or a spade?

You toss a nickel, a penny, and a dime.

3. List all the possible outcomes (the elements in the sample space).

4. What is the probability that the nickel comes up heads?

5. What is the probability that none of the coins comes up heads?

6. What is the probability that at least one of the coins comes up heads?

A bag contains 7 red marbles, 9 blue marbles, and 10 green marbles. You reach in the bag and choose 4 marbles, one after the other, without replacement.

7. What is the probability that all 4 marbles are red?

8. What is the probability that you get a red marble, then a blue marble, then 2 green marbles?

You take a 40 question multiple choice test and believe that for each question you have a 55% chance of getting it right.

9. What is the probability that you get all the questions right?

10. What is the probability that you get all of the questions wrong?

A player rolls a pair of standard dice. Find each probability.

11. $P(\textit{sum is even})$

12. $P(\textit{sum is } 7)$

13. $P(\textit{sum is at least } 3)$

14. You want to construct a 3 digit number at random from the digits 4, 6, 8, 9 without repeating digits. What is the probability that you construct the number 684?

15. In poker, a straight is 5 cards in a row (ex. 3, 4, 5, 6, 7), NOT all the same suit (if they are all the same suit it is considered a straight flush or a royal flush). A straight can start or end with an Ace. What's the probability of a straight? *For an even bigger challenge, see if you can calculate the probabilities for all of the poker hands.*

# 12.8 Binomial Theorem

Here you will apply the Binomial Theorem to expand binomials that are raised to a power. In order to do this you will use your knowledge of sigma notation and combinations.

The Binomial Theorem tells you how to expand a binomial such as $(2x-3)^5$ without having to compute the repeated distribution. What is the expanded version of$(2x-3)^5$?

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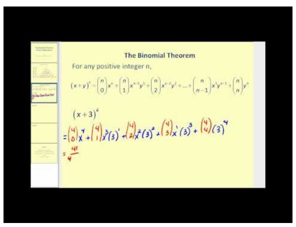


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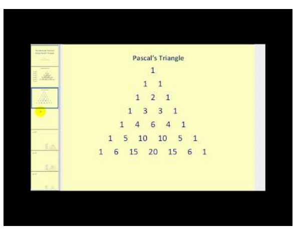
http://www.youtube.com/watch?v=YxysKtqpbVI James Sousa: The Binomial Theorem Using Combinations

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http://www.youtube.com/watch?v=NLQmQGA4a3M James Sousa: The Binominal Theorem Using Pascal's Triangle

### Guidance

The Binomial Theorem states:

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

Writing out a few terms of the summation symbol helps you to understand how this theorem works:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n$$

Going from one term to the next in the expansion, you should notice that the exponents of $a$ decrease while the exponents of $b$ increase. You should also notice that the coefficients of each term are combinations. Recall that $\binom{n}{0}$ is the number of ways to choose 0 objects from a set of $n$ objects.

Another way to think about the coefficients in the Binomial Theorem is that they are the numbers from Pascal's Triangle. Look at the expansions of $(a+b)^n$ below and notice how the coefficients of the terms are the numbers in Pascal's Triangle.

$$(a+b)^0 = 1$$

$$(a+b)^1 = 1a + 1b$$

$$(a+b)^2 = 1a^2 + 2ab + 1b^2$$

$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

$$\vdots$$

Be extremely careful when working with binomials of the form $(a-b)^n$. You need to remember to capture the negative with the second term as you write out the expansion: $(a-b)^n = (a+(-b))^n$.

**Example A**

Expand the following binomial using the Binomial Theorem.

$(m-n)^6$

**Solution:**

$$\begin{aligned}(m-n)^6 &= \binom{6}{0}m^6 + \binom{6}{1}m^5(-n)^1 + \binom{6}{2}m^4(-n)^2 + \binom{6}{3}m^3(-n)^3 \\ &+ \binom{6}{4}m^2(-n)^4 + \binom{6}{5}m^1(-n)^5 + \binom{6}{6}(-n)^6 \\ &= 1m^6 - 6m^5n + 15m^4n^2 - 20m^3n^3 + 15m^2n^4 - 6m^1n^5 + 1n^6\end{aligned}$$

**Example B**

What is the coefficient of the term $x^7y^9$ in the expansion of the binomial $(x+y)^{16}$?

**Solution:** The Binomial Theorem allows you to calculate just the coefficient you need.

$\binom{16}{9} = \frac{16!}{9!7!} = \frac{16\cdot15\cdot14\cdot13\cdot12\cdot11\cdot10}{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = 11,440$

**Example C**

What is the coefficient of $x^6$ in the expansion of $(4-3x)^7$?

**Solution:** For this problem you should calculate the whole term, since the 3 and the 4 in $(3-4x)$ will impact the coefficient of $x^6$ as well. $\binom{7}{6}4^1(-3x)^6 = 7\cdot4\cdot729x^6 = 20,412x^6$. The coefficient is 20,412.

**Concept Problem Revisited**

The expanded version of $(2x-3)^5$ is:

$$\begin{aligned}(2x-3)^5 &= \binom{5}{0}(2x)^5 + \binom{5}{1}(2x)^4(-3)^1 + \binom{5}{2}(2x)^3(-3)^2 \\ &+ \binom{5}{3}(2x)^2(-3)^3 + \binom{5}{4}(2x)^1(-3)^4 + \binom{5}{5}(-3)^6 \\ &= (2x)^5 + 5(2x)^4(-3)^1 + 10(2x)^3(-3)^2 \\ &+ 10(2x)^2(-3)^3 + 5(2x)^1(-3)^4 + (-3)^5 \\ &= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 810x - 243\end{aligned}$$

### Vocabulary

The ***Binomial Theorem*** is a theorem that states how to expand binomials that are raised to a power using combinations. The ***Binomial Theorem*** is:

$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$

### Guided Practice

1. What is the coefficient of $x^3$ in the expansion of $(x-4)^5$?

2. Compute the following summation.

$\sum_{i=0}^{4} \binom{4}{i}$

3. Collapse the following polynomial using the Binomial Theorem.

$32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$

**Answers:**

1. $\binom{5}{2} \cdot 1^3(-4)^2 = 160$

2. This is asking for $\binom{4}{0} + \binom{4}{1} + \cdots + \binom{4}{4}$, which are the sum of all the coefficients of $(a+b)^4$.

$1+4+6+4+1 = 16$

3. Since the last term is -1 and the power on the first term is a 5 you can conclude that the second half of the binomial is $(?-1)^5$. The first term is positive and $(2x)^5 = 32x^5$, so the first term in the binomial must be $2x$. The binomial is $(2x-1)^5$.

### Practice

Expand each of the following binomials using the Binomial Theorem.

1. $(x-y)^4$
2. $(x-3y)^5$
3. $(2x+4y)^7$
4. What is the coefficient of $x^4$ in $(x-2)^7$?
5. What is the coefficient of $x^3y^5$ in $(x+y)^8$?
6. What is the coefficient of $x^5$ in $(2x-5)^6$?
7. What is the coefficient of $y^2$ in $(4y-5)^4$?
8. What is the coefficient of $x^2y^6$ in $(2x+y)^8$?
9. What is the coefficient of $x^3y^4$ in $(5x+2y)^7$?

Compute the following summations.

10. $\sum_{i=0}^{9} \binom{9}{i}$
11. $\sum_{i=0}^{12} \binom{12}{i}$
12. $\sum_{i=0}^{8} \binom{8}{i}$

Collapse the following polynomials using the Binomial Theorem.

13. $243x^5 - 405x^4 + 270x^3 - 90x^2 = 15x - 1$

14. $x^7 - 7x^6y + 21x^5y^2 - 35x^4y^3 + 35x^3y^4 - 21x^2y^5 + 7xy^6 - y^7$

15. $128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

# 12.9 Induction Proofs

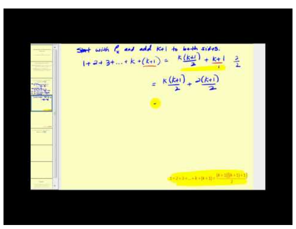
Here you will learn how to prove statements about numbers using induction.

Induction is one of many methods for proving mathematical statements about numbers. The basic idea is that you prove a statement is true for a small number like 1. This is called the base case. Then, you show that if the statement is true for some random number$k$, then it must also be true for$k+1$.

An induction proof is like dominoes set up in a line, where the base case starts the falling cascade of truth. Once you have shown that in general if the statement is true for $k$ then it must also be true for $k+1$, it means that once you show the statement is true for 1, then it must also be true for 2, and then it must also be true for 3, and then it must also be true for 4 and so on.

What happens when you forget the base case?

### Watch This



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Click image to the left for more content.

http://www.youtube.com/watch?v=QHkG0d5kZvE James Sousa: Mathematical Induction

### Guidance

Induction is a method of proof usually used to prove statements about positive whole numbers (the natural numbers). Induction has three steps:

1. The base case is where the statement is shown to be true for a specific number. Usually this is a small number like 1.
2. The inductive hypothesis is where the statement is assumed to be true for $k$.
3. The inductive step/proof is where you show that then the statement must be true for $k+1$.

These three logical pieces will show that the statement is true for every number greater than the base case.

Suppose you wanted to use induction to prove: $n \geq 1, 2+2^2+2^3+\cdots+2^n = 2^{n-1}-2$.

Start with the **Base Case**. Show that the statement works when $n=1$:

$2^1 = 2$ and $2^{1+1}-2 = 4-2 = 2$. Therefore, $2^1 = 2^{1+1}-2$. (Both sides are equal to 2)

Next, state your **Inductive Hypothesis**. Assume that the statement works for some random number $k$:

$2+2^2+\cdots+2^k = 2^{k+1}-2$ (You are assuming that this is a true statement)

Next, you will want to use algebra to manipulate the previous statement to **prove** that the statement is also true for $k+1$. So, you will be trying to show that $2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-2$. Start with the inductive hypothesis and multiply both sides of the equation by 2. Then, do some algebra to get the equation looking like you want.

*Inductive Hypothesis (starting equation):* $2+2^2+\cdots 2^k = 2^{k+1}-2$

*Multiply by 2:* $2(2+2^2+\cdots+2^k) = 2(2^{k+1}-2)$

*Rewrite:* $2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4$

*Add 2 to both sides:* $2+2^2+2^3+\cdots+2^{k+1} = 2^{k+1+1}-4+2$

*Simplify:* $2+2^2+\cdots+2^{k+1} = 2^{k+1+1}-2$

This is exactly what you were trying to prove! So, first you showed that the statement worked for $n=1$. Then, you showed that the if the statement works for one number than it must work for the next number. This means, the statement must be true for all numbers greater than or equal to 1.

The idea of induction can be hard to understand at first and it definitely takes practice. One thing that makes induction tricky is that there is not a clear procedure for the "proof" part. With practice, you will start to see some common algebra techniques for manipulating equations to prove what you are trying to prove.

**Example A**

There is something wrong with this proof. Can you explain what the mistake is?

*For* $n \geq 1 : 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

**Base Case:** $1 = 1^2 = \frac{1(1+1)(2\cdot 1+1)}{6} = \frac{1\cdot 2\cdot 3}{6} = \frac{6}{6} = 1$

**Inductive Hypothesis:** Assume the following statement is true:

$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$

**Proof:** You want to show the statement is true for $k+1$.

*"Since the statement is assumed true for k, which is any number, then it must be true for* $k+1$. *You can just substitute* $k+1$ *in."*

$1^2+2^2+3^2+\cdots+(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

**Solution:** This is the most common fallacy when doing induction proofs. The fact that the statement is assumed to be true for $k$ does not immediately imply that it is true for $k+1$ and you cannot just substitute in $k+1$ to produce what you are trying to show. his is equivalent to assuming true for all numbers and then concluding true for all numbers which is circular and illogical.

**Example B**

Write the base case, inductive hypothesis and what you are trying to show for the following statement. Do not actually prove it.

$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$

**Solution:**

**Base Case:** $1^3 = \frac{1^2(1+1)^2}{4}$ (Both sides are equal to 1)

**Inductive Hypothesis:** Assume the following statement is true:

$1^3+2^3+3^3+\cdots+k^3 = \frac{k^2(k+1)^2}{4}$

Next, you would want to prove that the following is true:

$1^3+2^3+3^3+\cdots+k^3+(k+1)^3 = \frac{(k+1)^2((k+1)+1)^2}{4}$

**Example C**

Prove the following statement: *For* $n \geq 1, 1^3+2^3+3^3+\cdots n^3 = (1+2+3+\cdots n)^2$.

**Solution:**

**Base Case(s):** Two base cases are shown however only one is actually necessary.

$$1^3 = 1^2$$
$$1^3 + 2^3 = 1 + 8 = 9 = 3^2 = (1+2)^2$$

**Inductive Hypothesis:** Assume the statement is true for some number $k$ . In other words, assume the following is true:

$1^3 + 2^3 + 3^3 + \cdots k^3 = (1+2+3+\cdots k)^2$

**Proof:** You want to show the statement is true for $k+1$. It is a good idea to restate what your goal is at this point. Your goal is to show that:

$1^3 + 2^3 + 3^3 + \cdots k^3 + (k+1)^3 = (1+2+3+\cdots k + (k+1))^2$

You need to start with the assumed case and do algebraic manipulations until you have created what you are trying to show (the equation above):

$1^3 + 2^3 + 3^3 + \cdots k^3 = (1+2+3+\cdots k)^2$

From the work you have done with arithmetic series you should notice:

$1+2+3+4+\cdots+k = \frac{k}{2}(2+(k-1)) = \frac{k(k+1)}{2}$

Substitute into the right side of the equation and add $(k+1)^3$ to both sides:

$1^3 + 2^3 + 3^3 + \cdots k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$

When you combine the right hand side algebraically you get the result of another arithmetic series.

$1^3 + 2^3 + 3^3 + \cdots k^3 + (k+1)^3 = \left(\frac{(k+1)(k+2)}{2}\right)^2 = (1+2+3+\cdots k + (k+1))^2$

$\therefore$

The symbol $\therefore$ is one of many indicators like QED that follow a proof to tell the reader that the proof is complete.

**Concept Problem Revisited**

If you forget the base case in an induction proof, then you haven't really proved anything. You can get silly results like this "proof" of the statement: "$1 = 3$"

**Base Case:** Missing

**Inductive Hypothesis:** $k = k+1$ where $k$ is a counting number.

**Proof:** Start with the assumption step and add one to both sides.

$$k = k+1$$
$$k+1 = k+2$$

Thus by transitivity of equality:

$$k = k+1 = k+2$$
$$k = k+2$$

Since $k$ is a counting number, $k$ could equal 1. Therefore:

$1 = 3$

### Vocabulary

The **base case** is the anchor step. It is the first domino to fall, creating a cascade and thus proving the statement true for every number greater than the base case.

The **inductive hypothesis** is the step where you assume the statement is true for $k$.

The **inductive step** is the **proof**. It is when you show the statement is true for $k+1$ using only the inductive hypothesis and algebra.

### Guided Practice

1. Write the base case, inductive hypothesis, and what you are trying to show for the following statement. Do not actually prove it.

$\quad$ $For\ n \geq 1,\ n^3+2n$ is divisible by 3 for any positive integer $n$

2. Complete the proof for the previous problem.

3. Prove the following statement using induction:

$\quad$ $For\ n \geq 1,\ 1+2+3+4+\cdots+n = \frac{n(n+1)}{2}$

**Answers:**

1. **Base Case:** $1^3+2\cdot 1 = 3$ which is divisible by 3.

**Inductive Step:** Assume the following is true for $k$:

$k^3+2k$ divisible by 3.

Next, you will want to show the following is true for $k+1$:

$(k+1)^3+2(k+1)$ is divisible by 3.

2. The goal is to show that $(k+1)^3+2(k+1)$ is divisible by 3 if you already know $k^3+2k$ is divisible by 3. Expand $(k+1)^3+2(k+1)$ to see what you get:

$$\begin{aligned}(k+1)^3+2(k+1) &= k^3+3k^2+3k+1+2k+2\\ &= (k^3+2k)+3(k^2+k+1)\end{aligned}$$

$k^3+2k$ is divisible by 3 by assumption (the inductive step) and $3(k^2+k+1)$ is clearly a multiple of 3 so is divisible by 3. The sum of two numbers that are divisible by is also divisible by 3.

$\therefore$

3. **Base Case:** $1 = \frac{1(1+1)}{2} = 1\cdot\frac{2}{2} = 1$

**Inductive Hypothesis:** $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

**Proof:** Start with what you know and work to showing it true for $k+1$.

Inductive Hypothesis: $1+2+3+4+\cdots+k = \frac{k(k+1)}{2}$

Add $k+1$ to both sides: $1+2+3+4+\cdots+k+(k+1) = \frac{k(k+1)}{2}+(k+1)$

Find a common denominator for the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{k^2+k}{2}+\frac{2k+2}{2}$

Simplify the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{k^2+3k+2}{2}$

Factor the numerator of the right side: $1+2+3+4+\cdots+k+(k+1) = \frac{(k+1)(k+2)}{2}$

Rewrite the right side: $1+2+3+4+\cdots+k+(k+1)=\frac{(k+1)((k+1)+1)}{2}$

$\therefore$

## Practice

For each of the following statements: a) show the base case is true; b) state the inductive hypothesis; c) state what you are trying to prove in the inductive step/proof. *Do not prove yet.*

1. For $n \geq 5,\ 4n < 2^n$.

2. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

3. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

4. For $n \geq 2,\ n^2 \geq 2n$.

5. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

6. For $n \geq 1,\ 0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Now, prove each of the following statements. Use your answers to problems 1-6 to help you get started.

7. For $n \geq 5,\ 4n < 2^n$.

8. For $n \geq 1,\ 8^n - 3^n$ is divisible by 5.

9. For $n \geq 1,\ 7^n - 1$ is divisible by 6.

10. For $n \geq 2,\ n^2 \geq 2n$.

11. For $n \geq 1,\ 4^n + 5$ is divisible by 3.

12. For $n \geq 1,\ 0^2 + 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

13. You should believe that the following statement is clearly false. What happens when you try to prove it true by induction?

For $n \geq 2,\ n^2 < n$

14. Explain why the base case is necessary for proving by induction.

15. The principles of inductive proof can be used for other proofs besides proofs about numbers. Can you prove the following statement from geometry using induction?

*The sum of the interior angles of any* $n-gon$ *is* $180(n-2)$ *for* $n \geq 3$.

You learned that recursion, how most people intuitively see patterns, is where each term in a sequence is defined by the term that came before. You saw that terms in a pattern can also be represented as a function of their term number. You learned about two special types of patterns called arithmetic sequences and geometric sequences that have a wide variety of applications in the real world. You saw that series are when terms in a sequence are added together. A strong understanding of patterns helped you to count efficiently, which in turn allowed you to compute both basic and compound probabilities. Finally, you learned that induction is a method of proof that allows you to prove your own mathematical statements.

# 12.10 References


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# CHAPTER 13 Finance

## Chapter Outline



Here you will review concepts of exponential growth and geometric series with a focus on the relationship between time and money.

# 13.1 Simple Interest

Here you'll learn to calculate the effect of time on the balance of a savings account growing by simple interest.

The basic concept of interest is that a dollar today is worth more than a dollar next year. If a person deposits \$100 into a bank account today at 6% simple interest, then in one year the bank owes the person that \$100 plus a few dollars more. If the person decides to leave it in the account and keep earning the interest, then after two years the bank would owe the person even more money. How much interest will the person earn each year? How much money will the person have after two years?

### Watch This


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http://www.khanacademy.org/math/precalculus/v/introduction-to-interest

### Guidance

Simple interest is defined as interest that only accumulates on the initial money deposited in the account. This initial money is called the principal. Another type of interest is compound interest where the interest also compounds on itself. In the real world, most companies do not use simple interest because it is considered too simple. You will practice with it here because it introduces the concept of the time value of money and that a dollar today is worth slightly more than a dollar in one year.

The formula for simple interest has 4 variables and all the problems and examples will give 3 and your job will be to find the unknown quantity using rules of Algebra.

$FV$ means *future value* and it stands for the amount in the account at some future time $t$.

$PV$ means *present value* and it stands for the amount in the account at time 0.

$t$ means time (usually years) that has elapsed between the present value and the future value. The value of $t$ indicates how long the money has been accumulating interest.

$i$ means the simple interest rate. If the interest rate is 6%, in the formula you will use the decimal version of 0.06. Here is the formula that shows the relationship between $FV$ and $PV$.

$FV = PV(1+t \cdot i)$

**Example A**

Linda invested \$1,000 for her child's college education. She saved it for 18 years at a bank which offered 5% simple interest. How much did she have at the end of 18 years?

**Solution:** First identify known and unknown quantities.

$$\begin{aligned} PV &= \$1,000 \\ t &= 18 \text{ years} \\ i &= 0.05 \\ FV &= \text{unknown so you will use } x \end{aligned}$$

Then substitute the values into the formula and solve to find the future value.

$$\begin{aligned} FV &= PV(1+t \cdot i) \\ x &= 1,000(1+18 \cdot 0.05) \\ x &= 1,000(1+0.90) \\ x &= 1,000(1.9) \\ x &= 1,900 \end{aligned}$$

Linda initially had \$1,000, but 18 years later with the effect of 5% simple interest, that money grew to \$1,900.

**Example B**

Tory put \$200 into a bank account that earns 8% simple interest. How much interest does Tory earn each year and how much does she have at the end of 4 years?

**Solution:** First you will focus on the first year and identify known and unknown quantities.

$$\begin{aligned} PV &= \$200 \\ t &= 1 \text{ year} \\ i &= 0.08 \\ FV &= \text{unknown so we will use } x \end{aligned}$$

Second, you will substitute the values into the formula and solve to find the future value.

$$\begin{aligned} FV &= PV(1+t \cdot i) \\ FV &= 200(1+1 \cdot 0.08) \\ FV &= 200 \cdot 1.08 \\ FV &= 216 \end{aligned}$$

The third thing you need to do is interpret and organize the information. Tory had \$200 to start with and then at the end of one year she had \$216. The additional \$16 is interest she has earned that year. Since the account is simple interest, she will keep earning \$16 dollars every year because her principal remains at \$200. The \$16 of interest earned that first year just sits there earning no interest of its own for the following three years.

**TABLE 13.1:**

| Year | Principal at Beginning of Year | Interest Earned that Year | Total Interest Earned |
|---|---|---|---|
| 1 | 200 | $200 \times .08 = 16$ | 16 |
| 2 | 200 | 16 | 32 |
| 3 | 200 | 16 | 48 |
| 4 | 200 | 16 | 64 |

At the end of 4 years, Tory will have \$264 on her account. \$64 will be interest. She earned \$16 in interest each year.

**Example C**

Amy has \$5000 to save and she wants to buy a car for \$10,000. For how many years will she need to save if she earns 10% simple interest? On the other hand, what will the simple interest rate need to be if she wants to save enough money in 15 years?

**Solution:** Notice that there are two separate problems. Let's start with the first problem and identify known and unknown quantities.

$$\begin{aligned} PV &= 5,000 \\ FV &= 10,000 \\ i &= 0.10 \\ t &= ? \end{aligned}$$

Now substitute and solve for $t$.

$$\begin{aligned} FV &= PV(1+t \cdot i) \\ 10,000 &= 5,000(1+t \cdot 0.10) \\ 2 &= 1+t \cdot 0.10 \\ 1 &= t \cdot 0.10 \\ t &= \frac{1}{0.10} = 10 \ years \end{aligned}$$

Now let's focus on the second problem and go through the process of identifying known and unknown quantities, substituting and solving.

$$\begin{aligned} PV = 5,000; \ FV &= 10,000; \ i = ?; \ t = 15 \ years \\ FV &= PV(1+t \cdot i) \\ 10,000 &= 5,000(1+15 \cdot i) \\ 2 &= 1+15i \\ 1 &= 15i \\ i &= \frac{1}{15} \approx 0.06667 = 6.667\% \end{aligned}$$

To answer the first question, Amy would need to save for 10 years getting a simple interest rate of 10%. For the second question, she would need to save for 15 years at a simple interest rate of about 6.667%.

**Concept Problem Revisited**

The person who deposits \$100 today at 6% simple interest will have \$106 in one year and \$112 in two years.

$$\begin{aligned} FV &= PV(1+t \cdot i) = 100(1+1 \cdot 0.06) = 100 \cdot 1.06 = 106 \\ FV &= PV(1+t \cdot i) = 100(1+2 \cdot 0.06) = 100 \cdot 1.12 = 112 \end{aligned}$$

### Vocabulary

***Principal*** is the amount initially deposited into the account. *Notice the spelling is principal, not principle*.

***Interest*** is the conversion of time into money.

### Guided Practice

1. How much will a person have at the end of 5 years if they invest $400 at 6% simple interest?
2. How long will it take $3,000 to grow to $4,000 at 4% simple interest?
3. What starting balance grows to $5,000 in 5 years with 10% simple interest?

**Answers:**

1. $PV = 400,\ t = 5,\ i = 0.06,\ FV = ?$

$$\begin{aligned} FV &= PV(1+t \cdot i) \\ &= 400(1+5 \cdot 0.06) \\ &= 400 \cdot 1.30 \\ &= \$520 \end{aligned}$$

2. $PV = 3,000,\ t = ?,\ i = 0.04,\ FV = 4,000$

$$\begin{aligned} 4,000 &= 3,000(1+t \cdot 0.04) \\ \frac{4}{3} &= 1+0.04t \\ \frac{1}{3} &= 0.04t \\ t &= \frac{1}{3 \cdot 0.04} \approx 8.333 \textit{ years} \end{aligned}$$

3. $PV = ?,\ FV = 5,000,\ t = 5,\ i = 0.10$

$$\begin{aligned} FV &= PV(1+t \cdot i) \\ 5,000 &= PV(1+5 \cdot 0.10) \\ PV &= \frac{5,000}{1+.50} \approx \$3,333.33 \end{aligned}$$

### Practice

1. How much will a person have at the end of 8 years if they invest $3,000 at 4.5% simple interest?
2. How much will a person have at the end of 6 years if they invest $2,000 at 3.75% simple interest?
3. How much will a person have at the end of 12 years if they invest $1,500 at 7% simple interest?
4. How much interest will a person earn if they invest $10,000 for 10 years at 5% simple interest?

5. How much interest will a person earn if they invest $2,300 for 49 years at 3% simple interest?
6. How long will it take $2,000 to grow to $5,000 at 3% simple interest?
7. What starting balance grows to $12,000 in 8 years with 10% simple interest?
8. Suppose you have $3,000 and want to have $35,000 in 25 years. What simple interest rate will you need?
9. How long will it take $1,000 to grow to $4,000 at 8% simple interest?
10. What starting balance grows to $9,500 in 4 years with 6.5% simple interest?
11. Suppose you have $1,500 and want to have $8,000 in 15 years. What simple interest rate will you need?
12. Suppose you have $800 and want to have $6,000 in 45 years. What simple interest rate will you need?
13. What starting balance grows to $2,500 in 2 years with 1.5% simple interest?
14. Suppose you invest $4,000 which earns 5% simple interest for the first 12 years and then 8% simple interest for the next 8 years. How much money will you have after 20 years?
15. Suppose you invest $10,000 which earns 2% simple interest for the first 8 years and then 5% simple interest for the next 7 years. How much money will you have after 15 years?

***Principal*** is the amount initially deposited into the account. *Notice the spelling is principal, not principle*.

***Interest*** is the conversion of time into money.

# 13.2 Compound Interest per Year

Here you'll explore how to compute an investment's growth given time and a compound interest rate.

If a person invests \$100 in a bank with 6% simple interest, they earn \$6 in the first year and \$6 again in the second year totaling \$112. If this was really how interest operated with most banks, then someone clever may think to withdraw the \$106 after the first year and immediately reinvest it. That way they earn 6% on \$106. At the end of the second year, the clever person would have earned \$6 like normal, plus an extra .36 cents totaling \$112.36. Thirty six cents may seem like not very much, but how much more would a person earn if they saved their \$100 for 50 years at 6% compound interest rather than at just 6% simple interest?

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http://www.khanacademy.org/finance-economics/core-finance/v/introduction-to-compound-interest

### Guidance

Compound interest allows interest to grow on interest. As with simple interest, $PV$ is defined as present value, $FV$ is defined as future value, $i$ is the interest rate, and $t$ is time. The formulas for simple and compound interest look similar, so be careful when reading problems in determining whether the interest rate is simple or compound. The following table shows the amount of money in an account earning compound interest over time:

**TABLE 13.2:**

| Year | Amount Ending in Account |
|---|---|
| 1 | $FV = PV(1+i)$ |
| 2 | $FV = PV(1+i)^2$ |
| 3 | $FV = PV(1+i)^3$ |
| 4 | $FV = PV(1+i)^4$ |
| ... | |
| $t$ | $FV = PV(1+i)^t$ |

An account with a present value of $PV$ that earns compound interest at $i$ percent annually for $t$ years has a future value of $FV$ shown below:

$$FV = PV(1+i)^t$$

**Example A**

Compute the amount ending in an account for years 1, 2, 3 and 4 for an initial deposit of \$100 at 3% compound interest.

**Solution:** $PV = 100,\ i = 0.03,\ t = 1,2,3\ and\ 4,\ FV = ?$

### TABLE 13.3:

| Year | Amount ending in Account |
|---|---|
| 1 | $FV = 100(1+0.03) = 103.00$ |
| 2 | $FV = 100(1+0.03)^2 = 106.09$ |
| 3 | $FV = 100(1+0.03)^3 \approx 109.27$ |
| 4 | $FV = 100(1+0.03)^4 \approx 112.55$ |

*Calculator shortcut: When doing repeated calculations that are just 1.03 times the result of the previous calculation, use the <ANS>button to create an entry that looks like <Ans*1.03>. Then, pressing enter repeatedly will rerun the previous entry producing the values on the right.*

**Example B**

How much will Kyle have in a savings account if he saves \$3,000 at 4% compound interest for 10 years?

**Solution:**

$PV = 3,000,\ i = 0.04,\ t = 10\ years,\ FV = ?$

$$FV = PV(1+i)^t$$
$$FV = 3000(1+0.04)^{10} \approx \$4,440.73$$

**Example C**

How long will it take money to double if it is in an account earning 8% compound interest?

**Estimation Solution:** The rule of 72 is an informal means of estimating how long it takes money to double. It is useful because it is a calculation that can be done mentally that can yield surprisingly accurate results. This can be very helpful when doing complex problems to check and see if answers are reasonable. The rule simply states $\frac{72}{i} \approx t$ where $i$ is written as an integer (i.e. 8% would just be 8).

In this case $\frac{72}{8} = 9 \approx t$, so it will take about 9 years.

**Exact Solution:** Since there is no initial value you are just looking for some amount to double. You can choose any amount for the present value and double it to get the future value even though specific numbers are not stated in the problem. Here you should choose 100 for $PV$ and 200 for $FV$.

$PV = 100,\ FV = 200,\ i = 0.08,\ t = ?$

$$FV = PV(1+i)^t$$
$$200 = 100(1+0.08)^t$$
$$2 = 1.08^t$$
$$\ln 2 = \ln 1.08^t$$
$$\ln 2 = t \cdot \ln 1.08$$
$$t = \frac{\ln 2}{\ln 1.08} = 9.00646$$

It will take just over 9 years for money (any amount) to double at 8%. This is extraordinarily close to your estimation and demonstrates how powerful the Rule of 72 can be in estimation.

**Concept Problem Revisited**

Earlier you were introduced to a concept problem contrasting \$100 for 50 years at 6% compound interest versus 6% simple. Now you can calculate how much more powerful compound interest is.

$PV = 100,\ t = 50,\ i = 6\%,\ FV = ?$

***Simple interest:***

$FV = PV(1+t \cdot i) = 100(1+50 \cdot 0.06) = 400$

***Compound interest:***

$FV = PV(1+i)^t = 100(1+0.06)^{50} \approx 1,842.02$

It is remarkable that simple interest grows the balance of the account to \$400 while compound interest grows it to about \$1,842.02. The additional money comes from interest growing on interest repeatedly.

### Vocabulary

***Compound interest*** is interest that grows not only on principal, but also on previous interest earned.

The ***Rule of 72*** states that the approximate amount of time that it will take an account earning simple interest to double is $t \approx \frac{72}{i}$, where $i$ is written as an integer.

### Guided Practice

1. How much will Phyllis have after 40 years if she invests \$20,000 in a savings account that earns 1% compound interest?

2. How long will it take money to double at 6% compound interest? Estimate using the rule of 72 and also find the exact answer.

3. What compound interest rate is needed to grow \$100 to \$120 in three years?

**Answers:**

1. $t = 40,\ PV = 20,000,\ i = 0.01$

$$\begin{aligned} FV &= PV(1+i)^t \\ &= 20,000(1+0.01)^{40} \\ &= \$29,777.27 \end{aligned}$$

2. Estimate: $\frac{72}{6} = 12 \approx$ years it will take to double

$PV = 100,\ FV = 200,\ i = 0.06,\ t = ?$

$$\begin{aligned} 200 &= 100(1+0.06)^t \\ 2 &= (1.06)^t \\ \ln 2 &= \ln 1.06^t = t\ \ln 1.06 \\ t &= \frac{\ln 2}{\ln 1.06} \approx 11.89\ \textit{years} \end{aligned}$$

3. $PV = 100, \ FV = 120, \ t = 3, \ i = ?$

$$\begin{aligned} FV &= PV(1+i)^t \\ 120 &= 100(1+i)^3 \\ \tfrac{1}{3} &= [(1+i)^3]^{\frac{1}{3}} \\ \tfrac{1}{3} &= 1+i \\ i &= 1.2^{\frac{1}{3}} - 1 \approx 0.06266 \end{aligned}$$

## Practice

For problems 1-10, find the missing value in each row using the compound interest formula.

### TABLE 13.4:

| Problem Number | $PV$ | $FV$ | $t$ | $i$ |
|---|---|---|---|---|
| 1. | \$1,000 | | 7 | 1.5% |
| 2. | \$1,575 | \$2,250 | 5 | |
| 3. | \$4,500 | \$5,534.43 | | 3% |
| 4. | | \$10,000 | 12 | 2% |
| 5. | \$1,670 | \$3,490 | 10 | |
| 6. | \$17,000 | \$40,000 | 25 | |
| 7. | \$10,000 | \$17,958.56 | | 5% |
| 8. | | \$50,000 | 30 | 8% |
| 9. | | \$1,000,000 | 40 | 6% |
| 10. | \$10,000 | | 50 | 7% |

11. How long will it take money to double at 4% compound interest? Estimate using the rule of 72 and also find the exact answer.

12. How long will it take money to double at 3% compound interest? Estimate using the rule of 72 and also find the exact answer.

13. Suppose you have \$5,000 to invest for 10 years. How much money would you have in 10 years if you earned 4% simple interest? How much money would you have in 10 years if you earned 4% compound interest?

14. Suppose you invest \$4,000 which earns 5% compound interest for the first 12 years and then 8% compound interest for the next 8 years. How much money will you have after 20 years?

15. Suppose you invest \$10,000 which earns 2% compound interest for the first 8 years and then 5% compound interest for the next 7 years. How much money will you have after 15 years?

# 13.3 Compound Interest per Period

Here you'll learn to compute future values with interest that accumulates semi-annually, monthly, daily, etc.

Clever Carol went to her bank which was offering 12% interest on its savings account. She asked very nicely if instead of having 12% at the end of the year, if she could have 6% after the first 6 months and then another 6% at the end of the year. Carol and the bank talked it over and they realized that while the account would still seem like it was getting 12%, Carol would actually be earning a higher percentage. How much more will Carol earn this way?

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### Guidance

Consider a bank that compounds and adds interest to accounts $k$ times per year. If the original percent offered is 12% then in one year that interest can be compounded:

- Once, with 12% at the end of the year $(k = 1)$
- Twice (semi-annually), with 6% after the first 6 months and 6% after the last six months $(k = 2)$
- Four times (quarterly), with 3% at the end of each 3 months $(k = 4)$
- Twelve times (monthly), with 1% at the end of each month $(k = 12)$

The intervals could even be days, hours or minutes. When intervals become small so does the amount of interest earned in that period, but since the intervals are small there are more of them. This effect means that there is a much greater opportunity for interest to compound.

The formula for interest compounding $k$ times per year for $t$ years at a nominal interest rate $i$ with present value $PV$ and future value $FV$ is:

$FV = PV\left(1+\frac{i}{k}\right)^{kt}$

Note: Just like simple interest and compound interest use the symbol $i$ to represent interest but they compound in very different ways, so does a nominal rate. As you will see in the examples, a nominal rate of 12% may actually yield more than 12%.

**Example A**

How much will Felix have in 4 years if he invests \$300 in a bank that offers 12% compounded monthly?

**Solution:** $FV = ?,\ PV = 300,\ t = 4,\ k = 12,\ i = 0.12$

$FV = PV\left(1+\frac{i}{k}\right)^{kt} = 300\left(1+\frac{0.12}{12}\right)^{12\cdot 4} \approx 483.67$

*Note: A very common mistake when typing the values into a calculator is using an exponent of 12 and then multiplying the whole quantity by 4 instead of using an exponent of* $(12 \cdot 4) = 48$ .

**Example B**

How many years will Matt need to invest his money at 6% compounded daily $(k = 365)$ if he wants his $3,000 to grow to $5,000?

**Solution:** $FV = 5,000,\ PV = 3,000,\ k = 365,\ i = 0.06,\ t = ?$

$$\begin{aligned}
FV &= PV\left(1+\frac{i}{k}\right)^{kt}\\
5,000 &= 3,000\left(1+\frac{0.06}{365}\right)^{365t}\\
\frac{5}{3} &= \left(1+\frac{0.06}{365}\right)^{365t}\\
\ln\frac{5}{3} &= \ln\left(1+\frac{0.06}{365}\right)^{365t}\\
\ln\frac{5}{3} &= 365t\cdot\ln\left(1+\frac{0.06}{365}\right)\\
t &= \frac{\ln\frac{5}{3}}{365\cdot\left(1+\frac{0.06}{365}\right)} = 8.514\ \textit{years}
\end{aligned}$$

**Example C**

What nominal interest rate compounded quarterly doubles money in 5 years?

**Solution:** $FV = 200,\ PV = 100,\ k = 4,\ i = ?,\ t = 5$

$$\begin{aligned}
FV &= PV\left(1+\frac{i}{k}\right)^{kt}\\
200 &= 100\left(1+\frac{i}{4}\right)^{4\cdot 5}\\
\tfrac{1}{20} &= \left[\left(1+\frac{i}{4}\right)^{20}\right]^{\frac{1}{20}}\\
2^{\frac{1}{20}} &= 1+\frac{i}{4}\\
i &= \left(2^{\frac{1}{20}}-1\right)4 \approx 0.1411 = 14.11\%
\end{aligned}$$

**Concept Problem Revisited**

If Clever Carol earned the 12% at the end of the year she would earn $12 in interest in the first year. If she compounds it $k = 2$ times per year then she will end up earning:

$FV = PV\left(1+\frac{i}{k}\right)^{kt} = 100\left(1+\frac{.12}{2}\right)^{2\cdot 1} = \$112.36$

### Vocabulary

**Nominal interest** is a number that resembles an interest rate, but it really is a sum of compound interest rates. A nominal rate of 12% compounded monthly is really 1% compounded 12 times.

$\left(1+\frac{.12}{12}\right)^{12}=(1+0.01)^{12}=1.1286$

The ***number of compounding periods*** is how often the interest will be accrued and added to the account balance, and is represented by the variable $k$.

### Guided Practice

1. How much will Steve have in 8 years if he invests \$500 in a bank that offers 8% compounded quarterly?

2. How many years will Mark need to invest his money at 3% compounded weekly $(k=52)$ if he wants his \$100 to grow to \$400?

3. What nominal interest rate compounded semi-annually doubles money in 18 years?

**Answers:**

1. $PV=500,\ t=8,\ i=8\%,\ FV=?,\ k=4$

$FV=PV\left(1+\frac{i}{k}\right)^{kt}=500\left(1+\frac{0.08}{4}\right)^{4\cdot 8}=\$942.27$

2. $FV=400,\ PV=100,\ k=52,\ i=0.03,\ t=?$

$$FV=PV\left(1+\frac{i}{k}\right)^{kt}$$
$$400=100\left(1+\frac{0.03}{52}\right)^{52\cdot t}$$
$$t=\frac{\ln 4}{52\cdot\ln\left(1+\frac{0.03}{52}\right)}=46.22\ years$$

3. $PV=100,\ FV=200,\ t=18,\ k=2,\ i=?$

$$FV=PV\left(1+\frac{i}{k}\right)^{kt}$$
$$200=100\left(1+\frac{i}{2}\right)^{2\cdot 18}$$
$$i=2\left(2^{\frac{1}{36}}-1\right)\approx 0.03888=3.888\%$$

### Practice

1. What is the length of a compounding period if $k=12$?
2. What is the length of a compounding period if $k=365$?
3. What would the value of $k$ be if interest was compounded every hour?
4. What would the value of $k$ be if interest was compounded every minute?
5. What would the value of $k$ be if interest was compounded every second?

For problems 6-15, find the missing value in each row using the compound interest formula.

### TABLE 13.5:

| Problem Number | $PV$ | $FV$ | $t$ | $i$ | $k$ |
|---|---|---|---|---|---|
| 6. | $1,000 | | 7 | 1.5% | 12 |
| 7. | $1,575 | $2,250 | 5 | | 2 |
| 8. | $4,000 | $5,375.67 | | 3% | 1 |
| 9. | | $10,000 | 12 | 2% | 365 |
| 10. | $10,000 | | 50 | 7% | 52 |
| 11. | $1,670 | $3,490 | 10 | | 4 |
| 12. | $17,000 | $40,000 | 25 | | 12 |
| 13. | $12,000 | | 3 | 5% | 365 |
| 14. | | $50,000 | 30 | 8% | 4 |
| 15. | | $1,000,000 | 40 | 6% | 2 |

# 13.4 Continuous Interest

Here you'll learn to use the force of interest to compute future values when interest is being compounded continuously.

Clever Carol realized that she makes more money when she convinces the bank to give her 12% in two chunks of 6% than only one time at 12%. Carol knew she could convince them to give her 1% at the end of each month for a total of 12% which would be even more than the two chunks of 6%. As Carol makes the intervals smaller and smaller, does she earn more and more money from the bank? Does this extra amount ever stop or does it keep growing forever?

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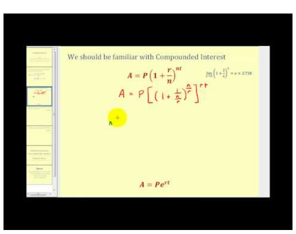

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### Guidance

Calculus deals with adding up an infinite number of infinitely small amounts. A result of calculus that is used in finance is the number $e$ as $k$ the number of compounding periods, approaches infinity.

$e \approx \left(1+\frac{1}{k}\right)^k \approx 2.71828\ldots$ as $k$ approaches infinity

This means that even when there are an infinite number of infinitely small compounding periods, there will be a limit on the interest earned in a year. The term for infinitely small compounding periods is continuous compounding.

The formula for finding the future value of a present value invested at a continuously compounding interest rate $r$ for $t$ years is:

$FV = PV \cdot e^{rt}$

**Example A**

What is the future value of $360 invested for 6 years at a continuously compounding rate of 5%?

**Solution:** $FV = ?,\ PV = 360,\ r = 0.05,\ t = 6$

$FV = PV \cdot e^{rt} = 360e^{0.05\cdot 6} = 360e^{0.30} \approx 485.95$

**Example B**

What is the continuously compounding rate that will grow $100 into $250 in just 2 years?

**Solution:** $PV = 100,\ FV = 250,\ r = ?,\ t = 2$

$$\begin{aligned}
FV &= PV \cdot e^{rt} \\
250 &= 100 \cdot e^{r2} \\
2.5 &= e^{r2} \\
\ln 2 &= 2r \\
r &= \frac{\ln 2}{2} \approx 0.3466 = 34.66\%
\end{aligned}$$

**Example C**

What amount invested at 7% continuously compounding yields $1,500 after 8 years?

**Solution:** $PV = ?\ FV = 1,500,\ t = 8,\ r = 0.07$

$$\begin{aligned}
FV &= PV \cdot e^{rt} \\
1,500 &= PV \cdot e^{0.07\cdot 8} \\
PV &= \frac{1,500}{e^{0.07\cdot 8}} \approx \$856.81
\end{aligned}$$

**Concept Problem Revisited**

Clever Carol could calculate the returns on each of the possible compounding periods for one year.

**For once per year,** $k = 1$**:**

$FV = PV(1+i)^t = 100(1+0.12)^1 = 112$

**For twice per year,** $k = 2$**:**

$FV = PV(1+i)^t = 100\left(1+\frac{0.12}{2}\right)^2 = 112.36$

**For twelve times per year, $k = 12$:**

$FV = PV(1+i)^t = 100\left(1+\frac{0.12}{12}\right)^{12} \approx 112.68$

At this point Carol might notice that while she more than doubled the number of compounding periods, she did not more than double the extra pennies. The growth slows down and approaches the continuously compounded growth result.

**For continuously compounding interest:**

$FV = PV \cdot e^{rt} = 100 \cdot e^{0.12 \cdot 1} \approx 112.75$

No matter how small Clever Carol might convince her bank to compound the 12%, the most she can earn is around 12.75 in interest.

### Vocabulary

A **continuously compounding interest rate** is the rate of growth proportional to the amount of money in the account at every instantaneous moment in time. It is equivalent to infinitely many but infinitely small compounding periods.

### Guided Practice

1. What is the future value of $500 invested for 8 years at a continuously compounding rate of 9%?

2. What is the continuously compounding rate which grows $27 into $99 in just 4 years?

3. What amount invested at 3% continuously compounding yields $9,000,000 after 200 years?

**Answers:**

1. $FV = 500e^{8 \cdot 0.09} \approx 1027.22$

2. $99 = 27e^{4r}$

Solving for $r$ yields: $r = 0.3248 = 32.48\%$

3. $9,000,000 = PVe^{200 \cdot 0.03}$

Solving for $PV$ yields: $PV = \$22,308.77$

### Practice

For problems 1-10, find the missing value in each row using the continuously compounding interest formula.

**TABLE 13.6:**

| Problem Number | $PV$ | $FV$ | $t$ | $r$ |
|---|---|---|---|---|
| 1. | $1,000 | | 7 | 1.5% |
| 2. | $1,575 | $2,250 | 5 | |
| 3. | $4,500 | $5,500 | | 3% |
| 4. | | $10,000 | 12 | 2% |
| 5. | $1,670 | $3,490 | 10 | |
| 6. | $17,000 | $40,000 | 25 | |
| 7. | $10,000 | $18,000 | | 5% |
| 8. | | $50,000 | 30 | 8% |
| 9. | | $1,000,000 | 40 | 6% |

**TABLE 13.6:** (continued)

| | | | | |
|---|---|---|---|---|
| 10. | $10,000 | | 50 | 7% |

11. How long will it take money to double at 4% continuously compounding interest?

12. How long will it take money to double at 3% continuously compounding interest?

13. Suppose you have $6,000 to invest for 12 years. How much money would you have in 10 years if you earned 3% simple interest? How much money would you have in 10 years if you earned 3% continuously compounding interest?

14. Suppose you invest $2,000 which earns 5% continuously compounding interest for the first 12 years and then 8% continuously compounding interest for the next 8 years. How much money will you have after 20 years?

15. Suppose you invest $7,000 which earns 1.5% continuously compounding interest for the first 8 years and then 6% continuously compounding interest for the next 7 years. How much money will you have after 15 years?

# 13.5 APR and APY (Nominal and Effective Rates)

Here you'll learn how to compare rates for loans and savings accounts to find more favorable deals.

In looking at an advertisement for a car you might see 2.5% APR financing on a $20,000 car. What does APR mean? What rate are they really charging you for the loan? Different banks may offer 8.1% annually, 8% compounded monthly or 7.9% compounded continuously. How much would you really be making if you put $100 in each bank? Which bank has the best deal?

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### Guidance

A nominal rate is an interest rate in name only. Banks, car dealerships and all companies will often advertise the interest rate that is most appealing to consumers who don't know the difference between APR and APY. In places like loans where the interest rate is working against you, they advertise a nominal rate that is lower than the effective rate. On the other hand, banks want to advertise the highest rates possible on their savings accounts so that people believe they are earning more interest.

In order to calculate what you are truly being charged, or how much money an account is truly making, it is necessary to use what you have learned about compounding interest and continuous interest. Then, you can make an informed decision about what is best.

**APR** stands for **Annual Percentage Rate**. It is a nominal rate and must be compounded according to the terms. The terms are usually monthly, so $k = 12$.

**APY** stands for **Annual Percentage Yield**. It is a true rate that states exactly how much money will be earned as interest.

**Example A**

If a credit card advertises 19.9% APR (annual rate compounded monthly) and you left $1000 unpaid, how much would you owe in a year?

**Solution:** First recognize that 19.9% APR is a nominal rate compounded monthly.

$FV = ?\ PV = 1000,\ i = .199,\ k = 12,\ t = 1$

$$FV = 1000\left(1 + \frac{0.199}{12}\right)^{12} \approx \$1,218.19$$

Notice that \$1,218.19 is an increase of about 21.82% on the original \$1,000. Many consumers expect to pay only \$199 in interest because they misunderstood the term APR. The effective interest on this account is about 21.82%, which is more than advertised.

Another interesting note is that just like there are rounding conventions in this math text (4 significant digits or dollars and cents), there are legal conventions for rounding interest rate decimals. Many companies include an additional 0.0049% because it rounds down for advertising purposes, but adds additional cost when it is time to pay up. For the purposes of these example problems and exercises, ignore this addition.

**Example B**

Three banks offer three slightly different savings accounts. Calculate the Annual Percentage Yield for each bank and choose which bank would be best to invest in.

*Bank A* offers 7.1% annual interest.

*Bank B* offers 7.0% annual interest compounded monthly.

*Bank C* offers 6.98% annual interest compounded continuously.

**Solution**: Since no initial amount is given, choose a $PV$ that is easy to work with like \$1 or \$100 and test just one year so $t = 1$. Once you have the future value for 1 year, you can look at the percentage increase from the present value to determine the APY.

### TABLE 13.7:

| Bank A | Bank B | Bank C |
|---|---|---|
| $FV = PV(1+i)^t$<br>$FV = 100(1+0.071)$<br>$FV = \$107.1$<br><br>$APY = 7.1\%$ | $FV = PV\left(1+\frac{i}{k}\right)^{kt}$<br>$FV = 100\left(1+\frac{0.07}{12}\right)^{12}$<br>$FV \approx 107.229$<br><br>$APY \approx 7.2290\%$ | $FV = PV \cdot e^{rt}$<br>$FV = 100e^{.0698}$<br>$FV \approx 107.2294$<br><br>$APY = 7.2294\%$ |

Bank A compounded only once per year so the APY was exactly the starting interest rate. However, for both Bank B and Bank C, the APY was higher than the original interest rates. While the APY's are very close, Bank C offers a slightly more favorable interest rate to an investor.

**Example C**

The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 4% compounding continuously?

**Solution:** Solve for APY for the bank where all information is given, the continuously compounding bank.

$FV = PV \cdot e^{rt} = 100 \cdot e^{0.04} \approx 104.08$

The APY is about 4.08%. Now you will set up an equation where you use the 104.08 you just calculated, but with the other banks interest rate.

$$FV = PV\left(1+\frac{i}{k}\right)^{kt}$$
$$104.08 = 100\left(1+\frac{i}{12}\right)^{12}$$
$$i = 12\left[\left(\frac{104.08}{100}\right)^{\frac{1}{12}} - 1\right] = 0.0400667$$

The second bank will need to offer slightly more than 4% to match the first bank.

**Concept Problem Revisited**

A loan that offers 2.5% APR that compounds monthly is really charging slightly more.

$\left(1+\frac{0.025}{12}\right)^{12} \approx 1.025288$

They are really charging about 2.529%.

The table below shows the APY calculations for three different banks offering 8.1% annually, 8% compounded monthly and 7.9% compounded continuously.

### TABLE 13.8:

| Bank A | Bank B | Bank C |
|---|---|---|
| $FV = PV(1+i)^t$<br>$FV = 100(1+0.081)$<br>$FV = \$108.1$<br>$APY = 8.1\ \%$ | $FV = PV\left(1+\frac{i}{k}\right)^{kt}$<br>$FV = 100\left(1+\frac{0.08}{12}\right)^{12}$<br>$FV \approx 108.299$<br>$APY \approx 8.300\%$ | $FV = PV \cdot e^{rt}$<br>$FV = 100e^{0.079}$<br>$FV\& \approx 108.22$<br>$APY \approx 8.22\%$ |

Even though Bank B does not seem to offer the best interest rate, or the most advantageous compounding strategy, it still offers the highest yield to the consumer.

## Vocabulary

**Nominal Interest Rate** is an interest rate in name only since a method of compounding needs to be associated with it in order to get a true effective interest rate. **APR** rates are nominal.

***Annual Effective Interest Rate*** is the true interest that is being charged or earned. **APY** rates are effective.

## Guided Practice

1. Which bank offers the best deal to someone wishing to deposit money?

- Bank A, offering 4.5% annually compounded
- Bank B, offering 4.4% compounded quarterly
- Bank C, offering 4.3% compounding continuously

2. What is the effective rate of a credit card interest charge of 34.99% APR compounded monthly?

3. Which bank offers the best deal to someone wishing to deposit money?

- Bank A, offering 10% annually compounded
- Bank B, offering 11% compounded quarterly
- Bank C, offering 12% compounding continuously

**Answers:**

1. The following table shows the APY calculations for the three banks.

### TABLE 13.9:

| Bank A | Bank B | Bank C |
|---|---|---|
| $FV = PV(1+i)^t$<br>$FV = 100(1+0.045)$<br>$APY = 4.5\%$ | $FV = PV\left(1+\frac{i}{k}\right)^{kt}$<br>$FV = 100\left(1+\frac{0.044}{4}\right)^4$<br>$APY \approx 4.473\%$ | $FV = PV \cdot e^{rt}$<br>$FV = 100e^{0.043}$<br>$APY = 4.394\%$ |

Bank B offers the best interest rate.

2. $\left(1+\frac{.3499}{12}\right)^{12} = 1.4118$ or a 41.18% effective interest rate.

3. The APY for Bank A remains at 10%. The APY for Bank C will be higher than Bank A because not only does it compound more often, it also has a higher compound rate.

## Practice

For problems 1-4, find the APY for each of the following bank accounts.

1. Bank A, offering 3.5% annually compounded.

2. Bank B, offering 3.4% compounded quarterly.

3. Bank C, offering 3.3% compounded monthly.

4. Bank D, offering 3.3% compounding continuously.

5. What is the effective rate of a credit card interest charge of 21.99% APR compounded monthly?

6. What is the effective rate of a credit card interest charge of 16.89% APR compounded monthly?

7. What is the effective rate of a credit card interest charge of 18.49% APR compounded monthly?

8. The APY for two banks are the same. What nominal interest rate would a monthly compounding bank need to offer to match another bank offering 3% compounding continuously?

9. The APY for two banks are the same. What nominal interest rate would a quarterly compounding bank need to offer to match another bank offering 1.5% compounding continuously?

10. The APY for two banks are the same. What nominal interest rate would a daily compounding bank need to offer to match another bank offering 2% compounding monthly?

11. Explain the difference between APR and APY.

12. Give an example of a situation where the APY is higher than the APR. Explain why the APY is higher.

13. Give an example of a situation where the APY is the same as the APR. Explain why the APY is the same.
14. Give an example of a situation where you would be looking for the highest possible APY.
15. Give an example of a situation where you would be looking for the lowest possible APY.

# 13.6 Annuities

Here you'll learn how to compute future values of periodic payments.

Sally knows she can earn a nominal rate of 6% convertible monthly in a retirement account, and she decides she can afford to save \$1,500 from her paycheck every month. How can you use geometric series to simplify the calculation of finding the future value of all these payments? How much money will Sally have saved in 30 years?

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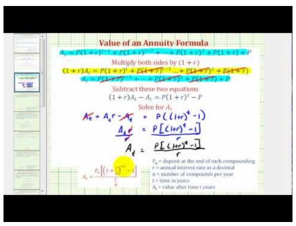


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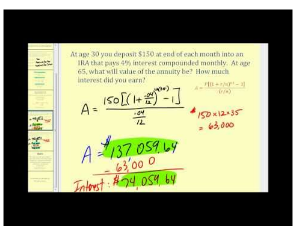

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### Guidance

An annuity is a series of equal payments that occur periodically. The word annuity comes from annual which means yearly. You will start by working with payments that occur once at the end of each year and then delve deeper to payments that occur monthly or any period.

Assume an investor saves $R$ dollars at the end of each year for $t$ years in an account that earns $i$ interest per period.

- The first payment $R$ will be in the bank account for $t-1$ years and grow to be: $R(1+i)^{t-1}$
- The second payment $R$ will be in the bank account for $t-2$ years and grow to be: $R(1+i)^{t-2}$
- This pattern continues until the last payment of $R$ that is deposited in the account right at $t$ years, so it doesn't earn any interest at all.

The account balance at this point in the future (Future Value, $FV$) is the sum of each individual $FV$ of all the payments:

$FV = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{t-2} + R(1+i)^{t-1}$

Recall that a geometric series with initial value $a$ and common ratio $r$ with $n$ terms has sum:

$a+ar+ar^2+\cdots+ar^{n-1}=a\cdot\frac{1-r^n}{1-r}$

So, a geometric series with starting value $R$ and common ratio $(1+i)$ has sum:

$$\begin{aligned} FV &= R\cdot\frac{1-(1+i)^n}{1-(1+i)} \\ &= R\cdot\frac{1-(1+i)^n}{-i} \\ &= R\cdot\frac{(1+i)^n-1}{i} \end{aligned}$$

This formula describes the relationship between $FV$ (the account balance in the future), $R$ (the annual payment), $n$ (the number of years) and $i$ (the interest per year).

It is extraordinarily flexible and will work even when payments occur monthly instead of yearly by rethinking what, $R$, $i$ and $n$ mean. The resulting Future Value will still be correct. If $R$ is monthly payments, then $i$ is the interest rate per month and $n$ is the number of months.

**Example A**

What will the future value of his IRA (special type of savings account) be if Lenny saves $5,000 a year at the end of each year for 35 years at an interest rate of 4%?

**Solution:** $R=5,000,\ i=0.04,\ n=35,\ FV=?$

$$\begin{aligned} FV &= R\cdot\frac{(1+i)^n-1}{i} \\ FV &= 5,000\cdot\frac{(1+0.04)^{35}-1}{0.04} \\ FV &= \$368,281.12 \end{aligned}$$

**Example B**

How long does Mariah need to save if she wants to retire with a million dollars and saves $10,000 a year at 5% interest?

**Solution:** $FV=1,000,000,\ R=10,000,\ i=0.05,\ n=?$

$$\begin{aligned} FV &= R\cdot\frac{(1+i)^n-1}{i} \\ 1,000,000 &= 10,000\cdot\frac{(1+0.05)^n-1}{0.05} \\ 100 &= \frac{(1+0.05)^n-1}{0.05} \\ 5 &= (1+0.05)^n-1 \\ 6 &= (1+0.05)^n \\ n &= \frac{\ln\ 6}{\ln\ 1.05}\approx 36.7\ years \end{aligned}$$

**Example C**

How much will Peter need to save each month if he wants to buy an $8,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

**Solution:** In this situation you will do all calculations in months instead of years. An adjustment in the interest rate and the time is required and the answer needs to be clearly interpreted at the end.

$FV = 8,000,\ R = ?,\ i = \frac{0.12}{12} = 0.01,\ n = 5 \cdot 12 = 60$

$$\begin{aligned} FV &= R \cdot \frac{(1+i)^n - 1}{i} \\ 8,000 &= R \cdot \frac{(1+0.01)^{60} - 1}{0.01} \\ R &= \frac{8,000 \cdot 0.01}{(1+0.01)^{60} - 1} \approx 97.96 \end{aligned}$$

Peter will need to save about $97.96 every month.

**Concept Problem Revisited**

Sally wanted to know how much she will have if she can earn 6% in a retirement account and she decides to save $1,500 from her paycheck every month.

$FV = ?,\ i = \frac{0.06}{12} = 0.005,\ n = 30 \cdot 12 = 360,\ R = 1,500$

$$\begin{aligned} FV &= R \cdot \frac{(1+i)^n - 1}{i} \\ FV &= 1,500 \cdot \frac{(1+0.005)^{360} - 1}{0.005} \\ FV &\approx 1,506,772.56 \end{aligned}$$

## Vocabulary

An ***annuity*** is a series of equal payments that occur periodically.

## Guided Practice

1. At the end of each quarter, Fermin makes a $200 deposit into a mutual fund. If his investment earns 8.1% interest compounded quarterly, what will his annuity be worth in 15 years?

2. What interest rate compounded semi-annually is required to grow a $25 semi-annual payment to $500 in 8 years?

3. How many years will it take to save $1,000 if Maria saves $20 every month at a 5% monthly interest rate?

**Answers:**

1. Quarterly means 4 times per year.

$FV = ?,\ R = 200,\ i = \frac{0.081}{4},\ n = 60$

$FV = 200 \cdot \frac{\left(1+\frac{0.081}{4}\right)^{60} - 1}{\frac{0.081}{4}} \approx \$23,008.71$

2. $FV = 500,\ R = 25,\ n = 16,\ i = ?$

$500 = 25 \cdot \frac{(1+i)^{16} - 1}{i}$ *or* $0 = 500 - 25 \cdot \frac{(1+i)^{16} - 1}{i}$

At this point it is not easy to solve using regular algebra. Systematic guess and check will work. This is an opportunity to use the solver function on the TI-83/84 calculator. Go to <MATH>, scroll down to find the <Solver>option and press <ENTER>. Enter the equation with $x$ instead of $i$ enter a guess where it says "$X =$", and let the calculator approximate the solution by pressing <ALPHA>then <ENTER>. You should get the answer:

$i \approx 0.029009$

Remember that the calculation is in semi-annual time periods, so the yearly nominal interest rate is $2 \cdot 0.029009 \approx 0.05802 = 5.802\%$

3. $FV = 1{,}000,\ R = 20,\ i = 0.05,\ n = ?$. Note that the calculation will be done in months. At the end you will convert your answer to years.

$$\begin{aligned}
FV &= R \cdot \frac{(1+i)^n - 1}{i} \\
1000 &= 20 \cdot \frac{(1+0.05)^n - 1}{0.05} \\
2.5 &= (1+0.05)^n - 1 \\
3.5 &= (1.05)^n \\
n &= \frac{\ln\ 3.5}{\ln\ 1.05} \approx 25.68\ months
\end{aligned}$$

It will take about 2.140 years.

## Practice

1. At the end of each month, Rose makes a \$400 deposit into a mutual fund. If her investment earns 6.1% interest compounded monthly, what will her annuity be worth in 30 years?

2. What interest rate compounded quarterly is required to grow a \$40 quarterly payment to \$1000 in 5 years?

3. How many years will it take to save \$10,000 if Sal saves \$50 every month at a 2% monthly interest rate?

4. How much will Bob need to save each month if he wants to buy a \$33,000 car with cash in 5 years? He can earn a nominal interest rate of 12% compounded monthly.

5. What will the future value of his IRA be if Cal saves \$5,000 a year at the end of each year for 35 years at an interest rate of 8%?

6. How long does Kathy need to save if she wants to retire with four million dollars and saves \$10,000 a year at 8% interest?

7. What interest rate compounded monthly is required to grow a \$416 monthly payment to \$80,000 in 10 years?

8. Every six months, Shanice makes a \$1000 deposit into a mutual fund. If her investment earns 5% interest compounded semi-annually, what will her annuity be worth in 25 years?

9. How much will Jen need to save each month if she wants to put \$60,000 down on a house in 5 years? She can earn a nominal interest rate of 8% compounded monthly.

10. How long does Adrian need to save if she wants to retire with three million dollars and saves \$5,000 a year at 10% interest?

11. What will the future value of her IRA be if Vanessa saves \$3,000 a year at the end of each year for 40 years at an interest rate of 6.7%?

12. At the end of each quarter, Justin makes a \$1,500 deposit into a mutual fund. If his investment earns 4.5%

interest compounded quarterly, what will her annuity be worth in 35 years?

13. What will the future value of his IRA be if Ted saves $3,500 a year at the end of each year for 25 years at an interest rate of 5.8%?

14. What interest rate compounded monthly is required to grow a $300 monthly payment to $1,000,000 in 35 years?

15. How much will Katie need to save each month if she wants to put $55,000 down in cash on a house in 2 years? She can earn a nominal interest rate of 6% compounded monthly.

# 13.7 Annuities for Loans

Here you'll learn how to compute present values of equal periodic payments.

Many people buy houses they cannot afford. This causes major problems for both the banks and the people who have their homes taken. In order to make wise choices when you buy a house, it is important to know how much you can afford to pay each period and calculate a maximum loan amount.

Joanna knows she can afford to pay \$12,000 a year for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What will she end up paying after 30 years?

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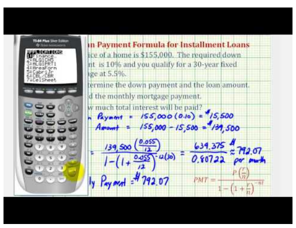

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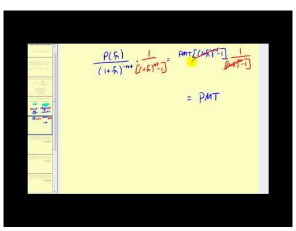

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### Guidance

The present value can be found from the future value using the regular compound growth formula:

$$PV(1+i)^n = FV$$
$$PV = \frac{FV}{(1+i)^n}$$

You also know the future value of an annuity:

$FV = R \cdot \frac{(1+i)^n - 1}{i}$

So by substitution, the formula for the present value of an annuity is:

$PV = R \cdot \frac{(1+i)^n - 1}{i} \cdot \frac{1}{(1+i)^n} = R \cdot \frac{(1+i)^n - 1}{i(1+i)^n} = R \cdot \frac{1-(1+i)^{-n}}{i}$

The present value of a series of equal payments $R$ with interest rate $i$ per period for $n$ periods is:

$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$

**Example A**

What is the monthly payment of a $1,000,000 house loan over 30 years with a nominal interest rate of 6% convertible monthly?

**Solution:** $PV = \$1,000,000,\ R = ?,\ i = 0.005,\ n = 360$

$$\begin{aligned} PV &= R \cdot \frac{1-(1+i)^{-n}}{i} \\ 1,000,000 &= R \cdot \frac{1-(1+0.005)^{-360}}{0.005} \\ R &= \frac{1,000,000 \cdot 0.005}{1-(1+0.005)^{-360}} \approx 5995.51 \end{aligned}$$

It is remarkable that in order to pay off a $1,000,000 loan you will have to pay $5,995.51 a month, every month, for thirty years. After 30 years, you will have made 360 payments of $5995.51, and therefore will have paid the bank more than $2.1 million, more than twice the original loan amount. It is no wonder that people can get into trouble taking on more debt than they can afford.

**Example B**

How long will it take to pay off a $20,000 car loan with a 6% annual interest rate convertible monthly if you pay it off in monthly installments of $500? What about if you tried to pay it off in monthly installments of $100?

**Solution:** $PV = \$20,000,\ R = \$500,\ i = \frac{0.06}{12} = 0.005,\ n = ?$

$$\begin{aligned} PV &= R \cdot \frac{1-(1+i)^{-n}}{i} \\ 20,000 &= 500 \cdot \frac{1-(1+0.005)^{-n}}{0.005} \\ 0.2 &= 1-(1+0.005)^{-n} \\ (1+0.005)^{-n} &= 0.8 \\ n &= -\frac{\ln 0.8}{\ln 1.005} \approx 44.74\ months \end{aligned}$$

For the $100 case, if you try to set up an equation and solve, there will be an error. This is because the interest on $20,000 is exactly $100 and so every month the payment will go to only paying off the interest. If someone tries to pay off less than $100, then the debt will grow.

**Example C**

It saves money to pay off debt faster in order to save money on interest. As shown in Example A, interest can more than double the cost of a 30 year mortgage. This example shows how much money can be saved by paying off more than the minimum.

Suppose a $300,000 loan has 6% interest convertible monthly with monthly payments over 30 years. What are the monthly payments? How much time and money would be saved if the monthly payments were larger by a fraction of $\frac{13}{12}$? This is like making 13 payments a year instead of just 12. First you will calculate the monthly payments if 12 payments a year are made.

$$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$$
$$300,000 = R \cdot \frac{1-(1+0.005)^{-360}}{0.005}$$
$$R = \$1,798.65$$

After 30 years, you will have paid $647,514.57, more than twice the original loan amount.

If instead the monthly payment was $\frac{13}{12} \cdot 1798.65 = 1948.54$, you would pay off the loan faster. In order to find out how much faster, you will make your unknown.

$$PV = R \cdot \frac{1-(1+i)^{-n}}{i}$$
$$300,000 = 1948.54 \cdot \frac{1-(1+0.005)^{-n}}{0.005}$$
$$0.7698 = 1-(1+0.005)^{-n}$$
$$(1+0.005)^{-n} = 0.23019$$
$$n = -\frac{\ln 0.23019}{\ln 1.005} \approx 294.5\ months$$

294.5 months is about 24.5 years. Paying fractionally more each month saved more than 5 years of payments.

$294.5\ months \cdot \$1,948.54 = \$573,847.99$

The loan ends up costing $573,847.99, which saves you more than $73,000 over the total cost if you had paid over 30 years.

**Concept Problem Revisited**

Joanna knows she can afford to pay $12,000 a year to pay for a house loan. Interest rates are 4.2% annually and most house loans go for 30 years. What is the maximum loan she can afford? What does she end up paying after 30 years? You can use the present value formula to calculate the maximum loan:

$PV = 12,000 \cdot \frac{1-(1+0.042)^{-30}}{0.042} \approx \$202,556.98$

For 30 years she will pay $12,000 a year. At the end of the 30 years she will have paid $\$12,000 \cdot 30 = \$360,000$ total

## Vocabulary

A ***loan*** is money borrowed that has to be paid back with interest.

A **mortgage** is a specific loan for a house.

## Guided Practice

1. Mackenzie obtains a 15 year student loan for $160,000 with 6.8% interest. What will her yearly payments be?

2. How long will it take Francisco to pay off a $16,000 credit card bill with 19.9% APR if he pays $800 per month? *Note: APR in this case means nominal rate convertible monthly.*

3. What will the monthly payments be on a credit card debt of $8,000 with 34.99% APR if it is paid off over 3 years?

**Answers:**

1. $PV = \$160,000,\ R = ?,\ n = 15,\ i = 0.068$

$$160,000 = R \cdot \frac{1-(1+0.068)^{-15}}{0.068}$$
$$R \approx \$17,345.88$$

2. $PV = \$16,000,\ R = \$600,\ n = ?,\ i = \frac{0.199}{12}$

$$16,000 = 600 \cdot \frac{1-\left(1+\frac{0.199}{12}\right)^{-n}}{\frac{0.199}{12}}$$
$$n = 24.50\ months$$

3. $PV = \$8,000,\ R = ?,\ n = 36,\ i = \frac{0.3499}{12}$

$$8,000 = R \cdot \frac{1-\left(1+\frac{0.3499}{12}\right)^{-36}}{\frac{0.3499}{12}}$$
$$R = \$361.84$$

### Practice

For problems 1-10, find the missing value in each row using the present value for annuities formula.

**TABLE 13.10:**

| Problem Number | *PV* | *R* | *n* (years) | *i* (annual) | Periods per year |
|---|---|---|---|---|---|
| 1. | | $4,000 | 7 | 1.5% | 1 |
| 2. | $15,575 | | 5 | 5% | 4 |
| 3. | $4,500 | $300 | | 3% | 12 |
| 4. | | $1,000 | 12 | 2% | 1 |
| 5. | $16,670 | | 10 | 10% | 4 |
| 6. | | $400 | 4 | 2% | 12 |
| 7. | $315,000 | $1,800 | | 5% | 12 |
| 8. | | $500 | 30 | 8% | 12 |
| 9. | | $1,000 | 40 | 6% | 4 |
| 10. | $10,000 | | 6 | 7% | 12 |

11. Charese obtains a 15 year student loan for $200,000 with 6.8% interest. What will her yearly payments be?

12. How long will it take Tyler to pay off a $5,000 credit card bill with 21.9% APR if he pays $300 per month? *Note: APR in this case means nominal rate convertible monthly*.

13. What will the monthly payments be on a credit card debt of $5,000 with 24.99% APR if it is paid off over 3 years?

14. What is the monthly payment of a $300,000 house loan over 30 years with a nominal interest rate of 2% convertible monthly?

15. What is the monthly payment of a $270,000 house loan over 30 years with a nominal interest rate of 3% convertible monthly?

The effects of interest on lump sum deposits and periodic deposits were explored. The key idea was that a dollar today is worth more than a dollar in a year.

# CHAPTER 14 Concepts of Calculus

## Chapter Outline



Newton and Leibniz invented Calculus a few hundred years ago to account for subtle paradoxes in their representations of the physical world. What happens when you add an infinite number of infinitely small numbers together? What happens when you multiply an infinitely small number by another infinitely small number? Calculus is a system of tools and a way of thinking about infinity that helps to make sense of these perceived paradoxes.

# 14.1 Limit Notation

Here you will write and read limit notation and use limit notation to describe the behavior of a function at a point and at infinity.

When learning about the end behavior of a rational function you described the function as either having a horizontal asymptote at zero or another number, or going to infinity. Limit notation is a way of describing this end behavior mathematically.

You already know that as $x$ gets extremely large then the function $f(x)=\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ goes to $\frac{8}{3}$ because the greatest powers are equal and $\frac{8}{3}$ is the ratio of the leading coefficients. How is this statement represented using limit notation?

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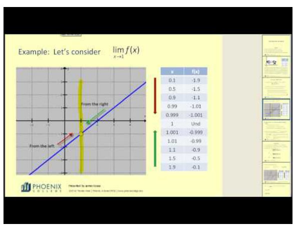

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http://www.youtube.com/watch?v=ahZ8LLtgu_w James Sousa: Introduction to Limits

### Guidance

Limit notation is a way of stating an idea that is a little more subtle than simply saying $x=5$ or $y=3$.

$\lim_{x\to a} f(x)=b$

"The limit of $f$ of $x$ as $x$ approaches $a$ is $b$"

The letter $a$ can be any number or infinity. The function $f(x)$ is any function of $x$. The letter $b$ can be any number. If the function goes to infinity, then instead of writing "$=\infty$" you should write that the limit does not exist or " *DNE*". This is because infinity is not a number. If a function goes to infinity then it has no limit.

While a function may never actually reach a height of $b$ it will get arbitrarily close to $b$. One way to think about the concept of a limit is to use a physical example. Stand some distance from a wall and then take a big step to get halfway to the wall. Take another step to go halfway to the wall again. If you keep taking steps that take you halfway to the wall then two things will happen. First, you will get extremely close to the wall but never actually reach the wall regardless of how many steps you take. Second, an observer who wishes to describe your situation would notice that the wall acts as a limit to how far you can go.

**Example A**

Translate the following statement into limit notation.

The limit of $y=4x^2$ as $x$ approaches 2 is 16.

**Solution:** $\lim_{x\to 2} 4x^2=16$

**Example B**

Translate the following mathematical statement into words.

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{2}\right)^i = 1$$

**Solution:** The limit of the sum of $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ as the number of terms approaches infinity is 1.

**Example C**

Use limit notation to represent the following mathematical statement.

$$\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots = \frac{1}{2}$$

**Solution:**

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{3}\right)^i = \frac{1}{2}$$

**Concept Problem Revisited**

The limit of $\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}$ as $x$ approaches infinity is $\frac{8}{3}$. This can be written using limit notation as:

$$\lim_{x\to\infty}\left(\frac{8x^4+4x^3+3x^2-10}{3x^4+6x^2+9x}\right) = \frac{8}{3}$$

### Vocabulary

***Limit notation*** is a way of expressing the fact that the function gets arbitrarily close to a value. In calculus or analysis you may define a limit in terms of the Greek letter epsilon $\varepsilon$ and delta $\delta$.

### Guided Practice

1. Describe the end behavior of the following rational function at infinity and negative infinity using limits.

$$f(x) = \frac{-5x^3+4x^2-10}{10x^3+3x^2+98}$$

2. Translate the following limit expression into words.

$$\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = x$$

3. What do you notice about the limit expression in # 2?

**Answers:**

1. Since the function has equal powers of $x$ in the numerator and in the denominator, the end behavior is $-\frac{1}{2}$ as $x$ goes to both positive and negative infinity.

$$\lim_{x\to\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right) = \lim_{x\to-\infty}\left(\frac{-5x^3+4x^2-10}{10x^3+3x^2+98}\right) = -\frac{1}{2}$$

2. The limit of the ratio of the difference between $f$ of quantity $x$ plus $h$ and $f$ of $x$ and $h$ as $h$ approaches 0 is $x$.

3. You should notice that $h \to 0$ does not mean $h = 0$ because if it did then you could not have a 0 in the denominator. You should also note that in the numerator, $f(x+h)$ and $f(x)$ are going to be super close together as $h$ approaches zero. Calculus will enable you to deal with problems that seem to look like $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

### Practice

Describe the end behavior of the following rational functions at infinity and negative infinity using limits.

1. $f(x) = \frac{2x^4+4x^2-1}{5x^4+3x+9}$

2. $g(x) = \frac{8x^3+4x^2-1}{2x^3+4x+7}$

3. $f(x) = \frac{x^2+2x^3-3}{5x^3+x+4}$

4. $f(x) = \frac{4x+4x^2-5}{2x^2+3x+3}$

5. $f(x) = \frac{3x^2+4x^3+4}{6x^3+3x^2+6}$

Translate the following statements into limit notation.

6. The limit of $y = 2x^2 + 1$ as $x$ approaches 3 is 19.

7. The limit of $y = e^x$ as $x$ approaches negative infinity is 0.

8. The limit of $y = \frac{1}{x}$ as $x$ approaches infinity is 0.

Use limit notation to represent the following mathematical statements.

9. $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \frac{1}{3}$

10. The series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges.

11. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$

12. $\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1$

Translate the following mathematical statements into words.

13. $\lim_{x \to 0} \frac{5x^2-4}{x+1} = -4$

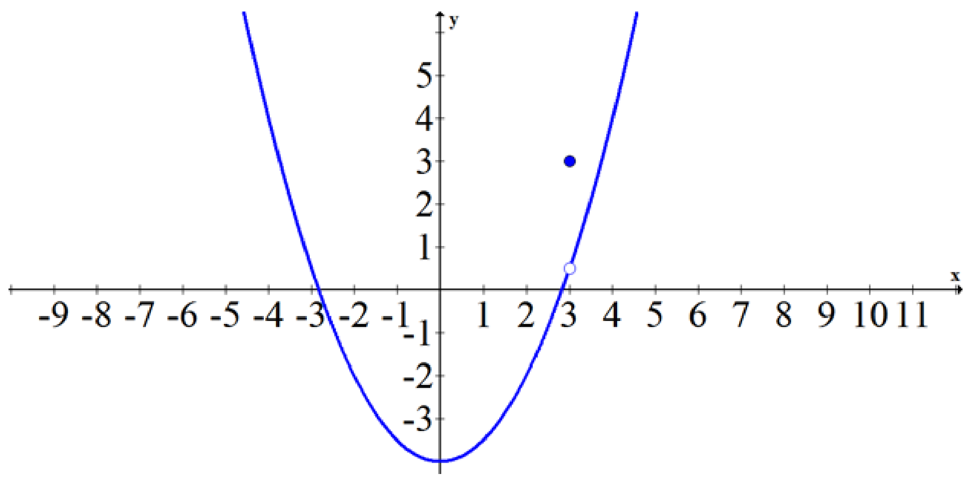
14. $\lim_{x \to 1} \frac{x^3-1}{x-1} = 3$

15. If $\lim_{x \to a} f(x) = b$, is it possible that $f(a) = b$? Explain.

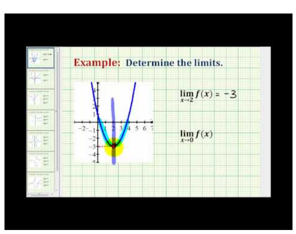
# 14.2 Graphs to Find Limits

Here you will use graphs to help you evaluate limits and refine your understanding of what a limit represents.

A limit can describe the end behavior of a function. This is called a limit at infinity or negative infinity. A limit can also describe the limit at any normal $x$ value. Sometimes this is simply the height of the function at that point. Other times this is what you would expect the height of the function to be at that point even if the height does not exist or is at some other point. In the following graph, what are $f(3)$, $\lim_{x \to 3} f(x)$, $\lim_{x \to \infty} f(x)$?




### Watch This



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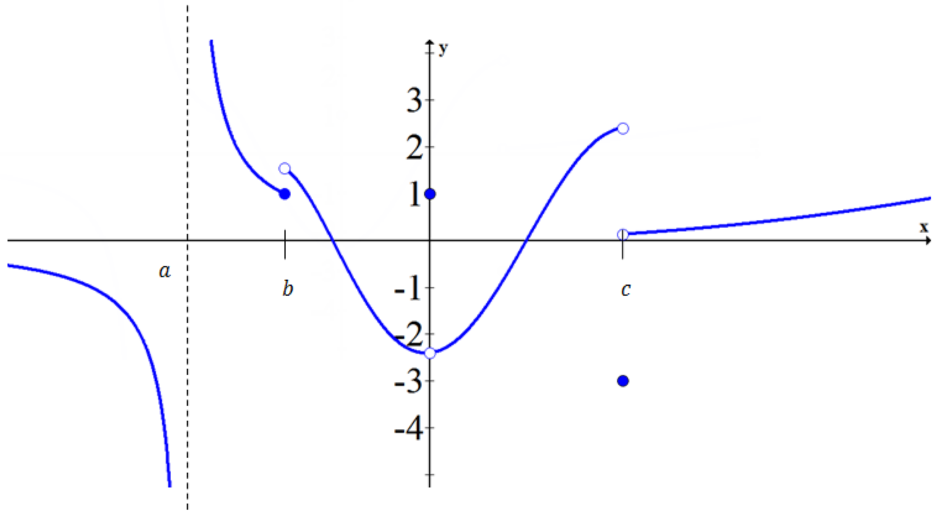
http://www.youtube.com/watch?v=LdewtuWi7fM James Sousa: Determining Basic Limits Graphically

### Guidance

When evaluating the limit of a function from its graph, you need to distinguish between the function evaluated at the point and the limit around the point.




Functions like the one above with discontinuities, asymptotes and holes require you to have a very solid understanding of how to evaluate and interpret limits.

At $x = a$, the function is undefined because there is a vertical asymptote. You would write:

$f(a) = DNE, \lim_{x \to a} f(x) = DNE$

At $x = b$, the function is defined because the filled in circle represents that it is the height of the function. This appears to be at about 1. However, since the two sides do not agree, the limit does not exist here either.

$f(b) = 1, \lim_{x \to b} f(x) = DNE$

At $x = 0$, the function has a discontinuity in the form of a hole. It is as if the point $(0, -2.4)$ has been lifted up and placed at $(0, 1)$. You can evaluate both the function and the limit at this point, however these quantities will not match. When you evaluate the function you have to give the actual height of the function, which is 1 in this case. When you evaluate the limit, you have to give what the height of the function is supposed to be based solely on the neighborhood around 0. Since the function appears to reach a height of -2.4 from both the left and the right, the limit does exist.

$f(0) = 1, \lim_{x \to 0} f(x) = -2.4$

At $x = c$, the limit does not exist because the left and right hand neighborhoods do not agree on a height. On the other hand, the filled in circle represents that the function is defined at $x = c$ to be -3.

$f(c) = -3, \lim\limits_{x \to c} f(x) = DNE$

At $x \to \infty$ you may only discuss the limit of the function since it is not appropriate to evaluate a function at infinity (you cannot find $f(\infty)$). Since the function appears to increase without bound, the limit does not exist.

$\lim\limits_{x \to \infty} f(x) = DNE$

At $x \to -\infty$ the graph appears to flatten as it moves to the left. There is a horizontal asymptote at $y = 0$ that this function approaches as $x \to -\infty$.

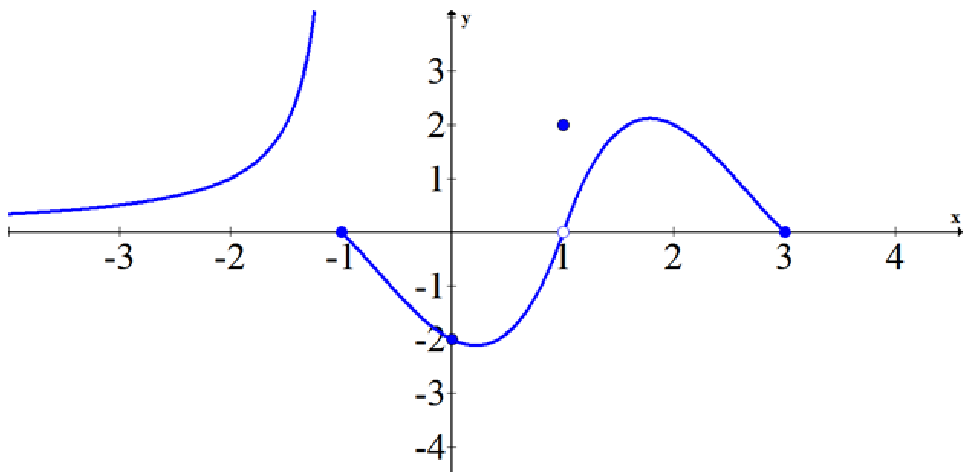
$\lim\limits_{x \to -\infty} f(x) = 0$

When evaluating limits graphically, your main goal is to determine whether the limit exists. The limit only exists when the left and right sides of the functions meet at a specific height. Whatever the function is doing at that point does not matter for the sake of limits. The function could be defined at that point, could be undefined at that point, or the point could be defined at some other height. Regardless of what is happening at that point, when you evaluate limits graphically, you only look at the neighborhood to the left and right of the function at the point.

**Example A**

Evaluate the following expressions using the graph of the function $f(x)$.

a. $\lim_{x \to -\infty} f(x)$
b. $\lim_{x \to -1} f(x)$
c. $\lim_{x \to 0} f(x)$
d. $\lim_{x \to 1} f(x)$
e. $\lim_{x \to 3} f(x)$
f. $f(-1)$
g. $f(2)$
h. $f(1)$
i. $f(3)$



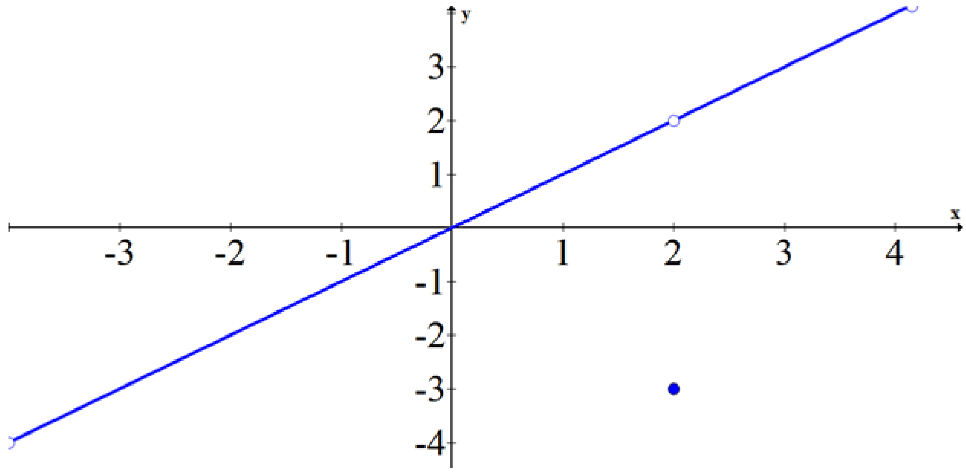

**Solution:**

a. $\lim_{x \to -\infty} f(x) = 0$
b. $\lim_{x \to -1} f(x) = DNE$
c. $\lim_{x \to 0} f(x) = -2$
d. $\lim_{x \to 1} f(x) = 0$
e. $\lim_{x \to 3} f(x) = DNE$ (This is because only one side exists and a regular limit requires both left and right sides to agree)
f. $f(-1) = 0$
g. $f(0) = -2$
h. $f(1) = 2$
i. $f(3) = 0$

**Example B**

Sketch a graph that has a limit at $x = 2$, but that limit does not match the height of the function.

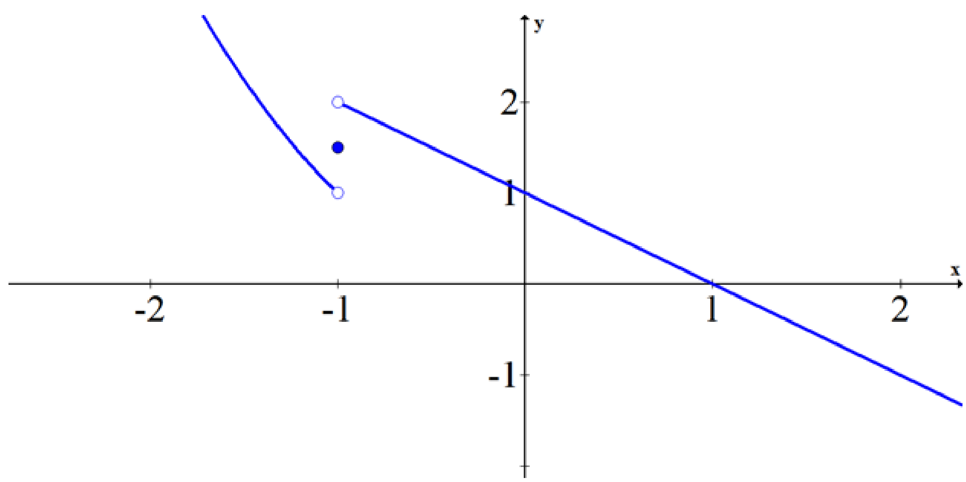
**Solution:** While there are an infinite number of graphs that fit this criteria, you should make sure your graph has a removable discontinuity at $x = 2$.




**Example C**

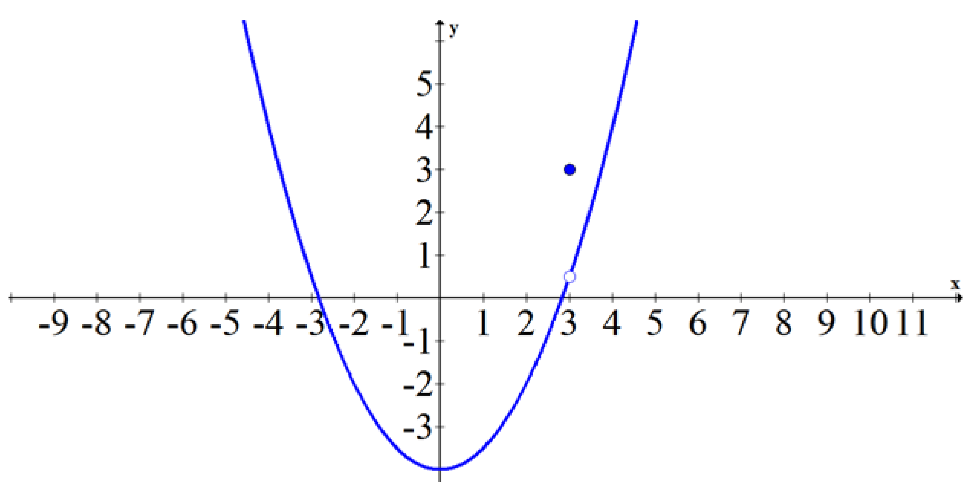
Sketch a graph that is defined at $x = -1$ but $\lim_{x \to -1} f(x)$ does not exist.

**Solution:** The graph must have either a jump or an infinite discontinuity at $x = -1$ and also have a solid hole filled in somewhere on that vertical line.




**Concept Problem Revisited**

Given the graph of the function $f(x)$ to be:



$\lim_{x \to 3} f(x) = \frac{1}{2}$

$f(3) = 3$

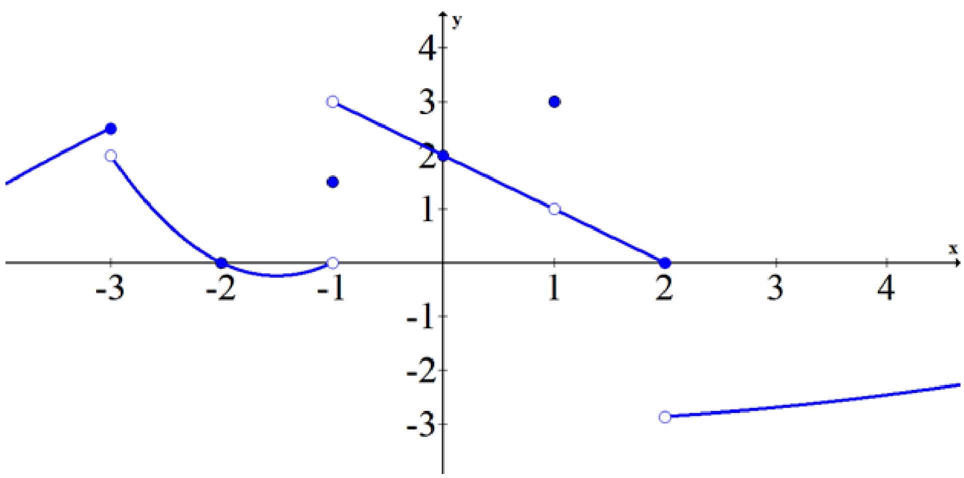
$\lim_{x \to \infty} f(x) = DNE$

### Vocabulary

The phrase ***"does not exist"*** or ***"DNE"*** is used with limits to imply that the limit does not approach a particular numerical value. Sometimes this means that the limit continues to grow bigger or smaller to infinity. Sometimes it means that the limit can't decide between two disagreeing values. There are times when instead of indicating the limit does not exist, you might choose to write the limit goes to infinity. This notation is technically not correct, but it does provide more useful information than writing DNE and therefore can be acceptable.

### Guided Practice

1. Identify everywhere where the limit does not exist and where the limit does exist in the following function.




2. Evaluate and explain how to find the limits as $x$ approaches 0 and 1 in the previous question.

3. Evaluate the limits of the following piecewise function at -2, 0 and 1.

$$f(x) = \begin{cases} 2 & x < -2 \\ -1 & x = -2 \\ -x-2 & -2 < x \leq 0 \\ x^2 & 0 < x < 1 \\ -2 & x = 1 \\ x^2 & 1 < x \end{cases}$$

**Answers:**

1. The limit does not exist at $x = -3, -1, 2, +\infty, -\infty$. At every other point (including $x = 1$), the limit does exist.

2. $\lim_{x \to 0} f(x) = 2,\ \lim_{x \to 1} f(x) = 1$

Both of these limits exist because the left hand and right hand neighborhoods of these points seem to approach the same height. In the case of the point $(0,2)$ the function happened to be defined there. In the case of the point $(1,1)$ the function happened to be defined elsewhere, but that does not matter. You only need to consider what the function does right around the point.

3. Since you already know how to graph piecewise functions (graph each function in the $x$ interval indicated) you can then observe graphically the limits at -2, 0 and 1.

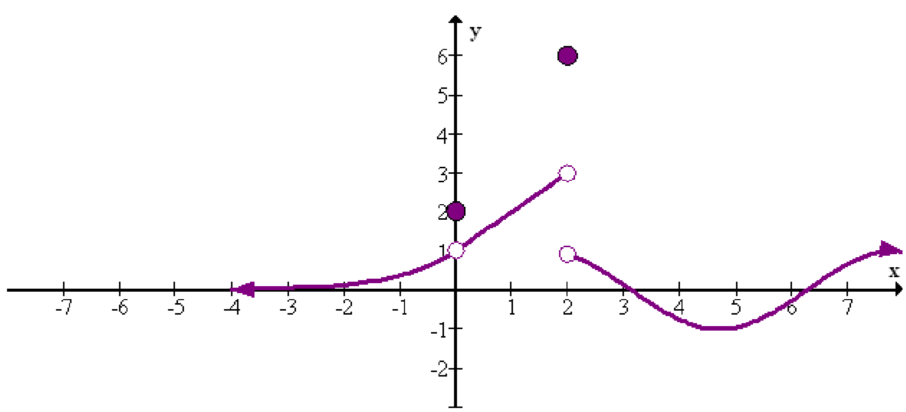
$\lim_{x \to -2} f(x) = 2$

$\lim_{x \to 0} f(x) = DNE$

$\lim_{x \to 1} f(x) = 1$

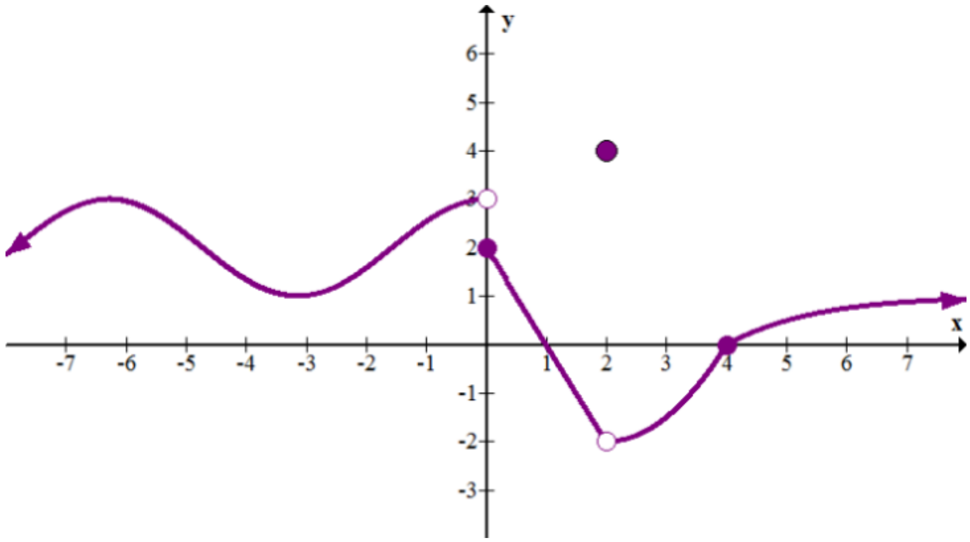
## Practice

Use the graph of $f(x)$ below to evaluate the expressions in 1-6.




1. $\lim_{x \to -\infty} f(x)$
2. $\lim_{x \to \infty} f(x)$
3. $\lim_{x \to 2} f(x)$
4. $\lim_{x \to 0} f(x)$
5. $f(0)$
6. $f(2)$

Use the graph of $g(x)$ below to evaluate the expressions in 7-13.




7. $\lim_{x\to-\infty} g(x)$
8. $\lim_{x\to\infty} g(x)$
9. $\lim_{x\to 2} g(x)$
10. $\lim_{x\to 0} g(x)$
11. $\lim_{x\to 4} g(x)$
12. $g(0)$
13. $g(2)$
14. Sketch a function $h(x)$ such that $h(2)=4$, but $\lim_{x\to 2} h(x) = DNE$.
15. Sketch a function $j(x)$ such that $j(2)=4$, but $\lim_{x\to 2} j(x) = 3$.

# 14.3 Tables to Find Limits

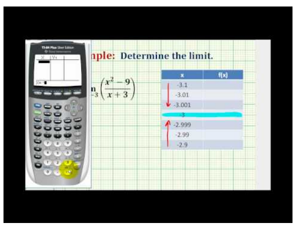
Here you will estimate limits using tables.

Calculators such as the TI-84 have a table view that allows you to make extremely educated guesses as to what the limit of a function will be at a specific point, even if the function is not actually defined at that point.

How could you use a table to calculate the following limit?

$\lim_{x \to -1} \frac{x^2+3x+2}{x+1}$

### Watch This

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http://www.youtube.com/watch?v=l7Tcay720vw James Sousa: Determine a Limit Numerically

### Guidance

If you were given the following information organized in a table, how would you fill in the center column?

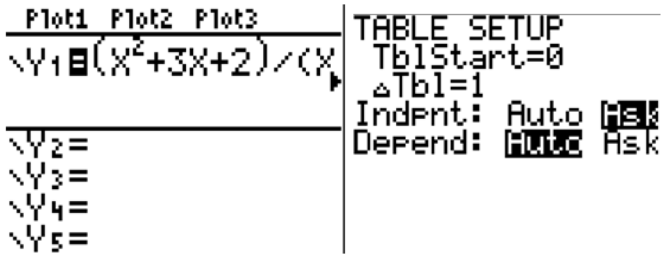
**TABLE 14.1:**

| | | | | | | |
|---|---|---|---|---|---|---|
| 3.9 | 3.99 | 3.999 | | 4.001 | 4.01 | 4.1 |
| 12.25 | 12.01 | 12.00001 | | 11.99999 | 11.99 | 11.75 |

It would be logical to see the symmetry and notice how the top row approaches the number 4 from the left and the right. It would also be logical to notice how the bottom row approaches the number 12 from the left and the right. This would lead you to the conclusion that the limit of the function represented by this table is 12 as the top row approaches 4. It would not matter if the value at 4 was undefined or defined to be another number like 17, the pattern tells you that the limit at 4 is 12.

Using tables to help evaluate limits requires this type of logic. To use a table on your calculator to evaluate a limit:

1. Enter the function on the $y =$ screen
2. Go to table set up and highlight "ask" for the independent variable
3. Go to the table and enter values close to the number that $x$ approaches




**Example A**

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-x-2}$

### TABLE 14.2:

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

**Solution:** While it is not necessary to use the table feature in the calculator, it is very efficient. Another option is to substitute the given $x$ values into the expression $\frac{x-2}{x^2-x-2}$ and record your results.

### TABLE 14.3:

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.34483 | 0.33445 | 0.33344 | 0.33322 | 0.33223 | 0.32258 |

The evidence suggests that the limit is $\frac{1}{3}$.

**Example B**

Complete the table and use the result to estimate the limit.

$\lim_{x \to 2} \frac{x-2}{x^2-4}$

### TABLE 14.4:

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

**Solution:** You can trick the calculator into giving a very exact answer by typing in 1.999999999999 because then the calculator rounds instead of producing an error.

### TABLE 14.5:

| $x$ | 1.9 | 1.99 | 1.999 | 2.001 | 2.01 | 2.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.25641 | 0.25063 | 0.25006 | 0.24994 | 0.24938 | 0.2439 |

The evidence suggests that the limit is $\frac{1}{4}$.

**Example C**

Complete the table and use the result to estimate the limit.

$\lim_{x \to 0} \frac{\sqrt{x+3} - \sqrt{3}}{x}$

**TABLE 14.6:**

| $x$ | -0.1 | -0.01 | -0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

**Solution:**

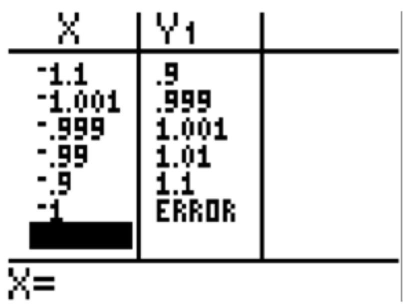
**TABLE 14.7:**

| $x$ | -0.1 | -0.01 | -0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.29112 | 0.28892 | 0.2887 | 0.28865 | 0.28843 | 0.28631 |

The evidence suggests that the limit is a number between 0.2887 and 0.28865. When you learn to find the limit analytically, you will know that the exact limit is $\frac{1}{2} \cdot 3\frac{1}{2} \approx 0.2886751346$.

**Concept Problem Revisited**

When you enter values close to -1 in the table you get $y$ values that are increasingly close to the number 1. This implies that the limit as $x$ approaches -1 is 1. Notice that when you evaluate the function at -1, the calculator produces an error. This should lead you to the conclusion that while the function is not defined at $x = -1$ , the limit does exist.




## Vocabulary

***Numerically*** is a term used to describe one of several different representations in mathematics. It refers to tables where the actual numbers are visible.

## Guided Practice

1. Graph the following function and the use a table to verify the limit as $x$ approaches 1.

$f(x) = \frac{x^3-1}{x-1}, x \neq 1$

2. Estimate the limit numerically.

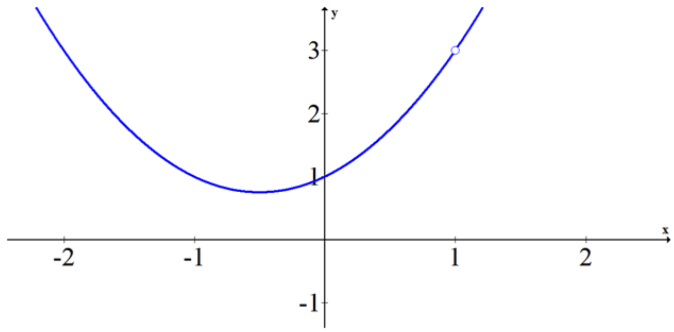
$\lim_{x \to 2} \frac{x^2-3x+2}{x-2}$

3. Estimate the limit numerically.

$\lim_{x \to 0} \frac{\left[\frac{4}{x+2}\right]-2}{x}$

**Answers:**

1. $\lim_{x \to 1} f(x) = 3$. This is because when you factor the numerator and cancel common factors, the function becomes a quadratic with a hole at the point $(1,3)$.




You can verify the limit in the table.

### TABLE 14.8:

| $x$ | $f(x)$ |
|---|---|
| .75 | 2.3125 |
| .9 | 2.71 |
| .99 | 2.9701 |
| .999 | 2.997 |
| 1 | Error |
| 1.001 | 3.003 |
| 1.01 | 3.0301 |
| 1.1 | 3.31 |
| 1.25 | 3.8125 |

2. $\lim_{x \to 2} f(x) = 1$

### TABLE 14.9:

| $x$ | 1.75 | 1.9 | 1.99 | 1.999 | 2 | 2.001 | 2.01 | 2.1 | 2.25 |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 0.75 | 0.9 | 0.99 | 0.999 | Error | 1.001 | 1.01 | 1.1 | 1.25 |

3. $\lim_{x \to 0} f(x) = 0$

### TABLE 14.10:

| $x$ | -0.1 | -0.01 | -0.001 | 0.001 | 0.01 | 0.1 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.20526 | 0.02005 | 0.002 | -0.002 | -0.02 | -0.1952 |

## Practice

Estimate the following limits numerically.

1. $\lim_{x \to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x \to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x \to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+2}$
4. $\lim_{x \to 0} \frac{\sqrt{x+2}-\sqrt{2}}{x}$
5. $\lim_{x \to 3} \left(\frac{1}{x-3} - \frac{9}{x^2-9}\right)$

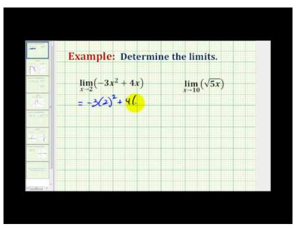
6. $\lim_{x \to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x \to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x \to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$
9. $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$
10. $\lim_{x \to 0} \frac{x^2+5x}{x}$
11. $\lim_{x \to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x \to 4} \frac{\sqrt{x}-2}{x-4}$
13. $\lim_{x \to 2} \frac{\sqrt{x+3}-2}{x-1}$
14. $\lim_{x \to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x \to -1} \frac{x-2}{x+1}$

# 14.4 Substitution to Find Limits

Here you will start to find limits analytically using substitution.

Finding limits for the vast majority of points for a given function is as simple as substituting the number that $x$ approaches into the function. Since this turns evaluating limits into an algebra-level substitution, most questions involving limits focus on the cases where substituting does not work. How can you decide if substitution is an appropriate analytical tool for finding a limit?

### Watch This

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http://www.youtube.com/watch?v=VLiMfJHZIpk James Sousa: Determine a Limit Analytically

### Guidance

Finding a limit analytically means finding the limit using algebraic means. In order to evaluate many limits, you can substitute the value that $x$ approaches into the function and evaluate the result. This works perfectly when there are no holes or asymptotes at that particular $x$ value. You can be confident this method works as long as you don't end up dividing by zero when you substitute.

If the function $f(x)$ has no holes or asymptote at $x = a$ then: $\lim_{x \to a} f(x) = f(a)$

Occasionally there will be a hole at $x = a$. The limit in this case is the height of the function if the hole did not exist. In other words, if the function is a rational expression with factors that can be canceled, cancel the term algebraically and then substitute into the resulting expression. If no factors can be canceled, it could be that the limit does not exist at that point due to asymptotes.

**Example A**

Which of the following limits can you determine using direct substitution? Find that limit.

$\lim_{x \to 2} \frac{x^2-4}{x-2}$, $\lim_{x \to 3} \frac{x^2-4}{x-2}$

**Solution:** The limit on the right can be evaluated using direct substitution because the hole exists at $x = 2$ not $x = 3$.

$\lim_{x \to 3} \frac{x^2-4}{x-2} = \frac{3^2-4}{3-2} = \frac{9-4}{1} = 5$

**Example B**

Evaluate the following limit by canceling first and then using substitution.

$\lim\limits_{x \to 2} \frac{x^2-4}{x-2}$

**Solution:**

$$\begin{aligned}\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}\\ &= \lim_{x\to 2}(x+2)\\ &= 2+2\\ &= 4\end{aligned}$$

**Example C**

Evaluate the following limit analytically: $\lim_{x\to 4}\frac{x^2-x-12}{x-4}$

**Solution:**

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-x-12}{x-4} &= \lim_{x\to 4}\frac{(x-4)(x+3)}{(x-4)}\\ &= \lim_{x\to 4}(x+3)\\ &= 4+3\\ &= 7\end{aligned}$$

**Concept Problem Revisited**

In order to decide whether substitution is an appropriate first step you can always just try it. You'll know it won't work if you end up trying to evaluate an expression with a denominator equal to zero. If this happens, go back and try to factor and cancel, and then try substituting again.

### Vocabulary

***Substitution*** is a method of determining limits where the value that $x$ is approaching is substituted into the function and the result is evaluated. This is one way to evaluate a limit ***analytically.***

### Guided Practice

1. Evaluate the following limit analytically.

$\lim\limits_{x\to 3}\frac{x^2-9}{x-3}$

2. Evaluate the following limit analytically.

$\lim\limits_{t\to 4}\sqrt{t+32}$

3. Evaluate the following limit analytically.

$\lim\limits_{y\to 4}\frac{3|y-1|}{y+4}$

**Answers:**

1. $\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{(t-3)}=\lim_{x\to 3}(x+3)=6$

2. $\lim_{t\to 4}\sqrt{t+32}=\sqrt{4+32}=\sqrt{36}=6$

3. $\lim_{y\to 4}\frac{3|y-1|}{y+4}=\frac{3|4-1|}{4+4}=\frac{3\cdot 3}{8}=\frac{9}{8}$

### Practice

Evaluate the following limits analytically.

1. $\lim_{x\to 5} \frac{x^2-25}{x-5}$
2. $\lim_{x\to -1} \frac{x^2-3x-4}{x+1}$
3. $\lim_{x\to 5} \sqrt{5x}-12$
4. $\lim_{x\to 0} \frac{x^3+3x^2-x}{5x}$
5. $\lim_{x\to 1} \frac{3x|x-4|}{x+1}$
6. $\lim_{x\to 2} \frac{x^2+5x-14}{x-2}$
7. $\lim_{x\to 1} \frac{x^2-8x+7}{x-1}$
8. $\lim_{x\to 0} \frac{5x-1}{2x^2+3}$
9. $\lim_{x\to 1} 4x^2-2x+5$
10. $\lim_{x\to 0} \frac{x^2+5x}{x}$
11. $\lim_{x\to -3} \frac{x^2-9}{x+3}$
12. $\lim_{x\to 0} \frac{5x+1}{x}$
13. $\lim_{x\to 1} \frac{5x+1}{x}$
14. $\lim_{x\to 5} \frac{x^2-25}{x^3-125}$
15. $\lim_{x\to -1} \frac{x-2}{x+1}$

# 14.5 Rationalization to Find Limits

Here you will evaluate limits analytically using rationalization.

Some limits cannot be evaluated directly by substitution and no factors immediately cancel. In these situations there is another algebraic technique to try called rationalization. With rationalization, you make the numerator and the denominator of an expression rational by using the properties of conjugate pairs.

How do you evaluate the following limit using rationalization?

$$\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$$

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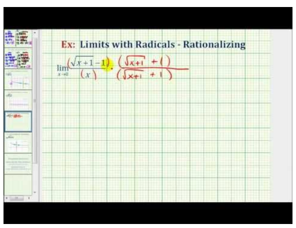


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http://youtu.be/ouWAhqeAaik James Sousa: Find a Limit Requiring Rationalizing

### Guidance

The properties of conjugates are used in a variety of places in PreCalculus.

Conjugates can be used to simplify expressions with a radical in the denominator:

$$\frac{5}{1+\sqrt{3}} = \frac{5}{\left(1+\sqrt{3}\right)} \cdot \frac{\left(1-\sqrt{3}\right)}{\left(1-\sqrt{3}\right)} = \frac{5-5\sqrt{3}}{1-3} = \frac{5-5\sqrt{3}}{-2}$$

Conjugates can be used to simplify complex numbers with $i$ in the denominator:

$$\frac{4}{2+3i} = \frac{4}{(2+3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{8-12i}{4+9} = \frac{8-12i}{13}$$

Here, they can be used to transform an expression in a limit problem that does not immediately factor to one that does immediately factor.

$$\lim_{x \to 16} \frac{\left(\sqrt{x}-4\right)}{(x-16)} \cdot \frac{\left(\sqrt{x}+4\right)}{\left(\sqrt{x}+4\right)} = \lim_{x \to 16} \frac{(x-16)}{(x-16)\left(\sqrt{x}+4\right)}$$

Now you can cancel the common factors in the numerator and denominator and use substitution to finish evaluating the limit.

The rationalizing technique works because when you algebraically manipulate the expression in the limit to an equivalent expression, the resulting limit will be the same. Sometimes you must do a variety of different algebraic manipulations in order avoid a zero in the denominator when using the substitution method.

**Example A**

Evaluate the following limit: $\lim_{x \to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}}$.

**Solution:**

$$\begin{aligned}\lim_{x\to 3}\frac{(x-3)(x+3)}{\left(\sqrt{x}-\sqrt{3}\right)}\cdot\frac{\left(\sqrt{x}+\sqrt{3}\right)}{\left(\sqrt{x}+\sqrt{3}\right)} &= \lim_{x\to 3}\frac{(x-3)(x+3)\left(\sqrt{x}+\sqrt{3}\right)}{(x-3)}\\ &= \lim_{x\to 3}(x+3)\left(\sqrt{x}+\sqrt{3}\right)\\ &= 6\cdot 2\sqrt{3}\\ &= 12\sqrt{3}\end{aligned}$$

**Example B**

Evaluate the following limit: $\lim_{x\to 25}\frac{x-25}{\sqrt{x}-5}$.

**Solution:**

$$\begin{aligned}\lim_{x\to 25}\frac{x-25}{\sqrt{x}-5} &= \lim_{x\to 25}\frac{(x-25)}{\left(\sqrt{x}-5\right)}\cdot\frac{\left(\sqrt{x}+5\right)}{\left(\sqrt{x}+5\right)}\\ &= \lim_{x\to 25}\frac{(x-25)\left(\sqrt{x}+5\right)}{(x-25)}\\ &= \lim_{x\to 25}\left(\sqrt{x}+5\right)\\ &= \sqrt{25}+5\\ &= 10\end{aligned}$$

**Example C**

Evaluate the following limit: $\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}$.

**Solution:**

$$\begin{aligned}\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7} &= \lim_{x\to 7}\frac{\left(\sqrt{x+2}-3\right)}{(x-7)}\cdot\frac{\left(\sqrt{x+2}+3\right)}{\left(\sqrt{x+2}+3\right)}\\ &= \lim_{x\to 7}\frac{(x+2-9)}{(x-7)\cdot\left(\sqrt{x+2}+3\right)}\\ &= \lim_{x\to 7}\frac{(x-7)}{(x-7)\cdot\left(\sqrt{x+2}+3\right)}\\ &= \lim_{x\to 7}\frac{1}{\left(\sqrt{x+2}+3\right)}\\ &= \frac{1}{\sqrt{7+2}+3}\\ &= \frac{1}{6}\end{aligned}$$

**Concept Problem Revisited**

In order to evaluate the limit of the following rational expression, you need to multiply by a clever form of 1 so that when you substitute there is no longer a zero factor in the denominator.

$$\begin{aligned}
\lim_{x\to 16}\frac{\sqrt{x}-4}{x-16} &= \lim_{x\to 16}\frac{(\sqrt{x}-4)}{(x-16)}\cdot\frac{(\sqrt{x}+4)}{(\sqrt{x}+4)}\\
&= \lim_{x\to 16}\frac{(x-16)}{(x-16)(\sqrt{x}+4)}\\
&= \lim_{x\to 16}(\sqrt{x}+4)\\
&= 4+4\\
&= 8
\end{aligned}$$

### Vocabulary

***Rationalization*** generally means to multiply a rational function by a clever form of one in order to eliminate radical symbols or imaginary numbers in the denominator. ***Rationalization*** is also a technique used to evaluate limits in order to avoid having a zero in the denominator when you substitute.

### Guided Practice

1. Evaluate the following limit: $\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x}$.

2. Evaluate the following limit: $\lim_{x\to -3}\frac{\sqrt{x^2-5}-2}{x+3}$.

3. Evaluate the following limit: $\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right)$.

**Answers:**

1.

$$\begin{aligned}
\lim_{x\to 0}\frac{(2+x)^{-1}-2^{-1}}{x} &= \lim_{x\to 0}\frac{\frac{1}{x+2}-\frac{1}{2}}{x}\cdot\frac{(x+2)\cdot 2}{(x+2)\cdot 2}\\
&= \lim_{x\to 0}\frac{2-(x+2)}{2x(x+2)}\\
&= \lim_{x\to 0}\frac{-x}{2x(x+2)}\\
&= \lim_{x\to 0}\frac{-1}{2(x+2)}\\
&= -\frac{1}{2(0+2)}\\
&= -\frac{1}{4}
\end{aligned}$$

2.

$$\begin{aligned}
\lim_{x\to -3}\frac{\sqrt{x^2-5}-2}{x+3} &= \lim_{x\to -3}\frac{\left(\sqrt{x^2-5}-2\right)}{(x+3)}\cdot\frac{\left(\sqrt{x^2-5}+2\right)}{\left(\sqrt{x^2-5}+2\right)}\\
&= \lim_{x\to -3}\frac{(x^2-5-4)}{(x+3)}\\
&= \lim_{x\to -3}\frac{(x^2-9)}{(x+3)}\\
&= \lim_{x\to -3}\frac{(x-3)(x+3)}{(x+3)}\\
&= \lim_{x\to -3}(x-3)\\
&= -3-3\\
&= -6
\end{aligned}$$

3.

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right) &= \lim_{x\to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{\sqrt{9-x}}{x\sqrt{9-x}}\right)\\
&= \lim_{x\to 0}\left(\frac{3-\sqrt{9-x}}{x\sqrt{9-x}}\right)\\
&= \lim_{x\to 0}\left(\frac{\left(3-\sqrt{9-x}\right)}{x\sqrt{9-x}}\cdot\frac{\left(3+\sqrt{9-x}\right)}{\left(3+\sqrt{9-x}\right)}\right)\\
&= \lim_{x\to 0}\left(\frac{9-(9-x)}{x\sqrt{9-x}}\right)\\
&= \lim_{x\to 0}\frac{x}{x\sqrt{9-x}}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{9-x}}\\
&= \frac{1}{\sqrt{9-0}}\\
&= \frac{1}{3}
\end{aligned}$$

## Practice

Evaluate the following limits:

1. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$

2. $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$

3. $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1}$

4. $\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$

5. $\lim_{x\to4}\frac{\sqrt{3x+4}-x}{4-x}$

6. $\lim_{x\to0}\frac{2-\sqrt{x+4}}{x}$

7. $\lim_{x\to0}\frac{\sqrt{x+7}-\sqrt{7}}{x}$

8. $\lim_{x\to16}\frac{16-x}{4-\sqrt{x}}$

9. $\lim_{x\to0}\frac{x^2}{\sqrt{x^2+12}-\sqrt{12}}$

10. $\lim_{x\to2}\frac{\sqrt{2x+5}-\sqrt{x+7}}{x-2}$

11. $\lim_{x\to1}\frac{1-\sqrt{x}}{1-x}$

12. $\lim_{x\to\frac{1}{9}}\frac{9x-1}{3\sqrt{x}-1}$

13. $\lim_{x\to4}\frac{4x^2-64}{2\sqrt{x}-4}$

14. $\lim_{x\to9}\frac{9x^2-90x+81}{9-3\sqrt{x}}$

15. When given a limit to evaluate, how do you know when to use the rationalization technique? What will the function look like?

# 14.6 One Sided Limits and Continuity

Here you will determine one sided limits graphically, numerically and algebraically and use the concept of a one sided limit to define continuity.

A one sided limit is exactly what you might expect; the limit of a function as it approaches a specific $x$ value from either the right side or the left side. One sided limits help to deal with the issue of a jump discontinuity and the two sides not matching.

Is the following piecewise function continuous?

$$f(x) = \begin{cases} -x-2 & x<1 \\ -3 & x=1 \\ x^2-4 & 1<x \end{cases}$$

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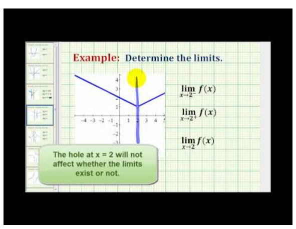


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http://www.youtube.com/watch?v=3iZUK15aPE0 James Sousa: Determining Limits and One-Sided Limits Graphically

### Guidance

A one sided limit can be evaluated either from the left or from the right. Since left and right are not absolute directions, a more precise way of thinking about direction is "from the negative side" or "from the positive side". The notation for these one sided limits is:

$\lim_{x \to a^-} f(x), \lim_{x \to a^+} f(x)$

The negative in the superscript of $a$ is not an exponent. Instead it indicates **from the negative side**. Likewise the positive superscript is not an exponent, it just means **from the positive side**. When evaluating one sided limits, it does not matter what the function is doing at the actual point or what the function is doing on the other side of the number. Your job is to determine what the height of the function should be using only evidence on one side.

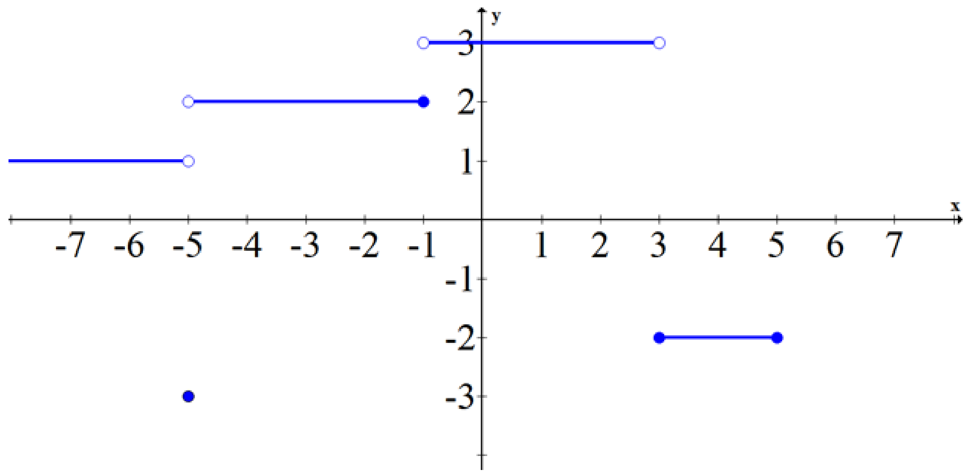
You have defined continuity in the past as the ability to draw a function completely without lifting your pencil off of the paper. You can now define a more rigorous definition of continuity.

If both of the one sided limits equal the value of the function at a given point, then the function is continuous at that point. In other words, a function is continuous at $a$ if:

$\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

**Example A**

What are the one sided limits at -5, -1, 3 and 5?




**Solution:**

$\lim_{x \to -5^-} f(x) = 1$

$\lim_{x \to -5^+} f(x) = 2$

$\lim_{x \to -1^-} f(x) = 2$

$\lim_{x \to -1^+} f(x) = 3$

$\lim_{x \to 3^-} f(x) = 3$

$\lim_{x \to 3^+} f(x) = -2$

$\lim_{x \to 5^-} f(x) = -2$

$\lim_{x \to -5^+} f(x) = DNE$

**Example B**

Evaluate the one sided limit at 4 from the negative direction numerically.

$f(x) = \frac{x^2-7x+12}{x-4}$

**Solution:** When creating the table, only use values that are smaller than 4.

**TABLE 14.11:**

| | | | |
|---|---|---|---|
| $x$ | 3.9 | 3.99 | 3.999 |
| $f(x)$ | 0.9 | 0.99 | 0.999 |

$\lim_{x \to 4^-} \left( \frac{x^2-7x+12}{x-4} \right) = 1$

**Example C**

Evaluate the following limits.

a. $\lim_{x \to 3^-} (4x-3)$

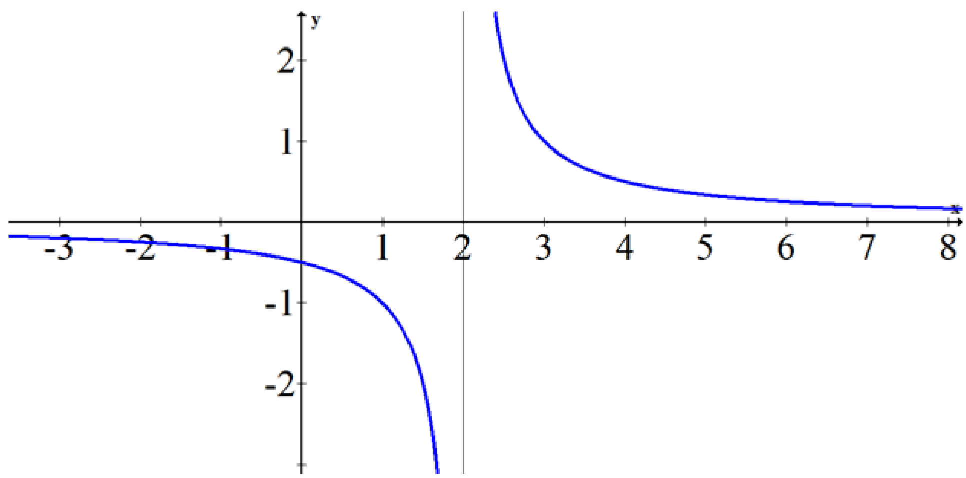
b. $\lim_{x \to 2^+} \left(\frac{1}{x-2}\right)$
c. $\lim_{x \to 1^+} \left(\frac{x^2+2x-3}{x-1}\right)$

**Solution:** Most of the time one sided limits are the same as the corresponding two sided limit. The exceptions are when there are jump discontinuities, which normally only happen with piecewise functions, and infinite discontinuities, which normally only happen with rational functions.

a. $\lim_{x \to 3^-} (4x-3) = 4 \cdot 3 - 3 = 12 - 3 = 9$

b. $\lim_{x \to 2^+} \left(\frac{1}{x-2}\right) = DNE$ or $\infty$

The reason why $\infty$ is preferable in this case is because the two sides of the limit disagree. One side goes to negative infinity and the other side goes to positive infinity (see the graph below). If you just indicate DNE then you are losing some perfectly good information about the nature of the function.




c. $\lim_{x \to 1^+} \left(\frac{x^2+2x-3}{x-1}\right) = \lim_{x \to 1^+} \left(\frac{(x-1)(x+3)}{(x-1)}\right) = \lim_{x \to 1^+} (x+3) = 1+3 = 4$

**Concept Problem Revisited**

In order to confirm or deny that the function is continuous, graphical tools are not accurate enough. Sometimes jump discontinuities can be off by such a small amount that the pixels on the display of your calculator will not display a difference. Your calculator will certainly not display removable discontinuities.

$$f(x) = \begin{cases} -x-2 & x < 1 \\ -3 & x = 1 \\ x^2-4 & 1 < x \end{cases}$$

You should note that on the graph, everything to the left of 1 is continuous because it is just a line. Next you should note that everything to the right of 1 is also continuous for the same reason. The only point to check is at $x = 1$. To check continuity, explicitly use the definition and evaluate all three parts to see if they are equal.

- $\lim_{x \to a^-} f(x) = -1-2 = -3$
- $f(1) = -3$
- $\lim_{x \to 1^+} f(x) = 1^2 - 4 = -3$

Therefore, $\lim_{x \to 1^-} f(x) = f(1) = \lim_{x \to 1^+} f(x)$ and the function is continuous at $x = 1$ and everywhere else.

### Vocabulary

A ***one sided limit*** is a limit of a function when the evidence from only the positive or only the negative side is used to evaluate the limit.

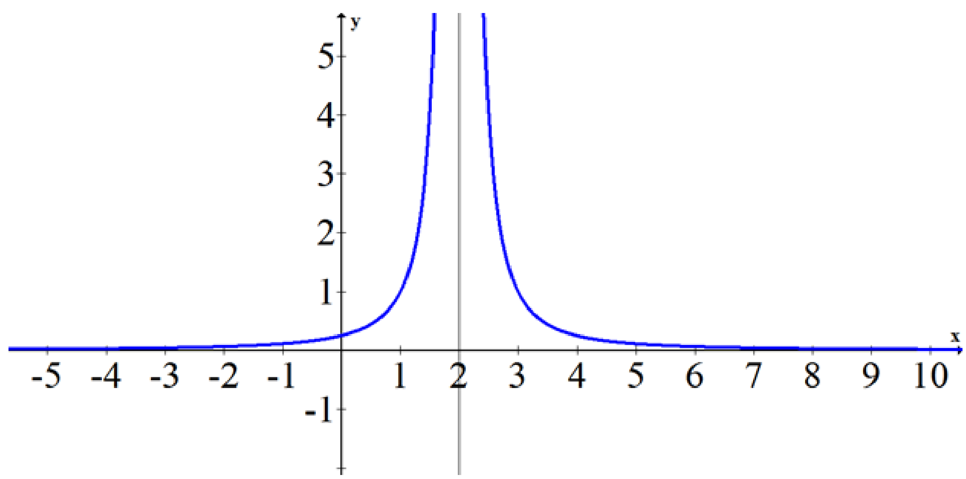
***Continuity for a point*** exists when the left and right sided limits match the function evaluated at that point. For an entire function to be continuous, the function must be continuous at every single point in an unbroken domain.

### Guided Practice

1. Megan argues that according to the definition of continuity, the following function is continuous. She says

- $\lim_{x \to 2^-} f(x) = \infty$
- $\lim_{x \to 2^+} f(x) = \infty$
- $f(2) = \infty$

Thus since $\lim_{x \to 2^-} f(x) = f(2) = \lim_{x \to 2^+} f(x)$, it meets the definition of continuous. How could you use the graph below to clarify Megan's reasoning?




2. Evaluate the following limits.

a. $\lim_{x \to 1^-} (2x - 1)$
b. $\lim_{x \to -3^+} \left(\frac{2}{x+2}\right)$
c. $\lim_{x \to 2^+} \left(\frac{x^3-8}{x-2}\right)$

3. Is the following function continuous?

$$f(x) = \begin{cases} x^2 - 1 & x < -1 \\ 3 & x = -1 \\ -x+3 & -1 < x \end{cases}$$

**Answers:**

1. Megan is being extremely liberal with the idea of "$=\infty$" because what she really means for the two limits is "DNE". For the function evaluated at 2 the correct response is "undefined". Two things that do not exist cannot be equal to one another.

2.

a. $\lim_{x\to 1^-}(2x-1)=2\cdot 1-1=2-1=1$
b. $\lim_{x\to -3^+}\left(\frac{2}{x+2}\right)=\frac{2}{-3+2}=\frac{2}{-1}=-2$
c. $\lim_{x\to 2^+}\left(\frac{x^3-8}{x-2}\right)=\lim_{x\to 2^+}\left(\frac{(x-2)(x^2+2x+4)}{(x-2)}\right)=\lim_{x\to 2^+}(x^2+2x+4)=2^2+2\cdot 2+4=12$

3. Use the definition of continuity.

- $\lim_{x\to 1^-} f(x)=(-1)^2-1=1-1=0$
- $f(-1)=3$
- $\lim_{x\to -1^+} f(x)=-1+3=2$

$\lim_{x\to a^-} f(x)\neq f(a)\neq \lim_{x\to a^+} f(x)$ so this function is discontinuous at $x=-1$. It is continuous everywhere else.

## Practice

Evaluate the following limits.

1. $\lim_{x\to 6^-}(3x^2-4)$
2. $\lim_{x\to 0^-}\frac{3x-1}{x}$
3. $\lim_{x\to 0^+}\frac{3x-1}{x}$
4. $\lim_{x\to 0^+}\frac{x}{|x|}$
5. $\lim_{x\to 0^-}\frac{x}{|x|}$
6. $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{1+\sqrt{x}-1}}$

Consider

$$f(x)=\begin{cases}2x^2-1 & x<1\\ 1 & x=1\\ -x+2 & 1<x\end{cases}$$

7. What is $\lim_{x\to 1^-} f(x)$?

8. What is $\lim_{x\to 1^+} f(x)$?

9. Is $f(x)$ continuous at $x=1$?

Consider

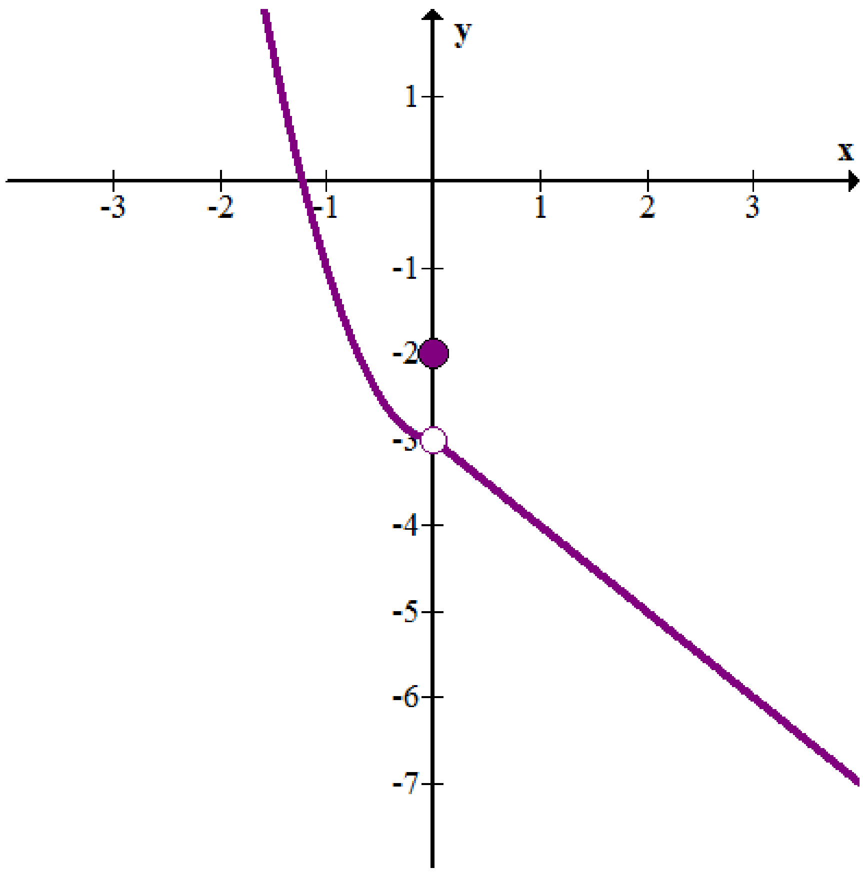
$$g(x)=\begin{cases}4x^2+2x-1 & x<-2\\ 8 & x=-2\\ -3x+5 & -2<x\end{cases}$$

10. What is $\lim_{x \to -2^-} g(x)$?

11. What is $\lim_{x \to -2^+} g(x)$?

12. Is $g(x)$ continuous at $x = -2$?

Consider $h(x)$ shown in the graph below.




13. What is $\lim_{x \to 0^-} h(x)$?

14. What is $\lim_{x \to 0^+} h(x)$?

15. Is $h(x)$ continuous at $x = 0$?

# 14.7 Intermediate and Extreme Value Theorems

Here you will use continuity to explore the intermediate and extreme value theorems.

While the idea of continuity may seem somewhat basic, when a function is continuous over a closed interval like $x \in [1,4]$, you can actually draw some major conclusions. The conclusions may be obvious when you understand the statements and look at a graph, but they are powerful nonetheless.

What can you conclude using the Intermediate Value Theorem and the Extreme Value Theorem about a function that is continuous over the closed interval $x \in [1,4]$?

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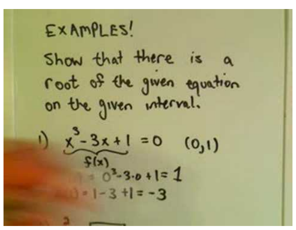


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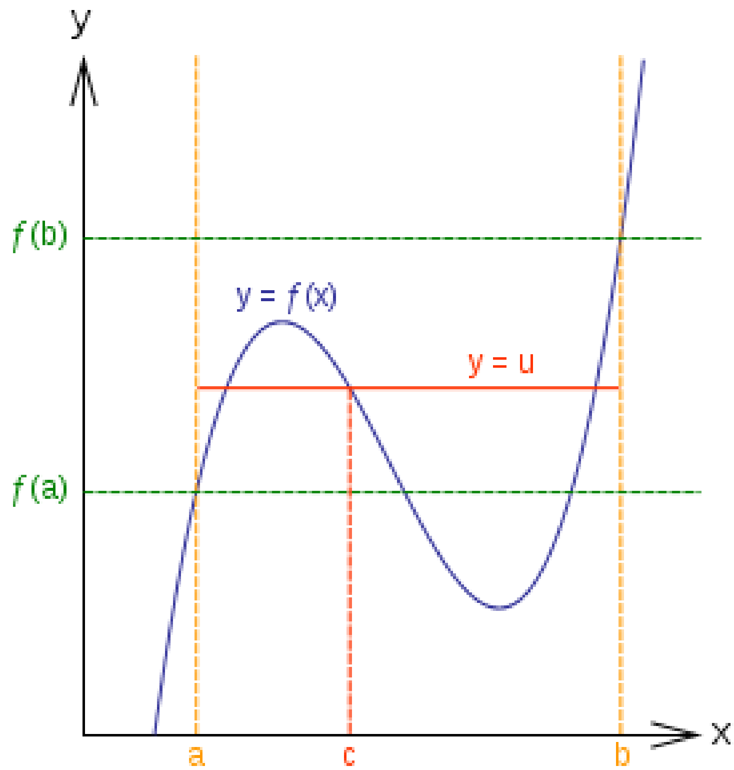
http://www.youtube.com/watch?v=6AFT1wnId9U Intermediate Value Theorem

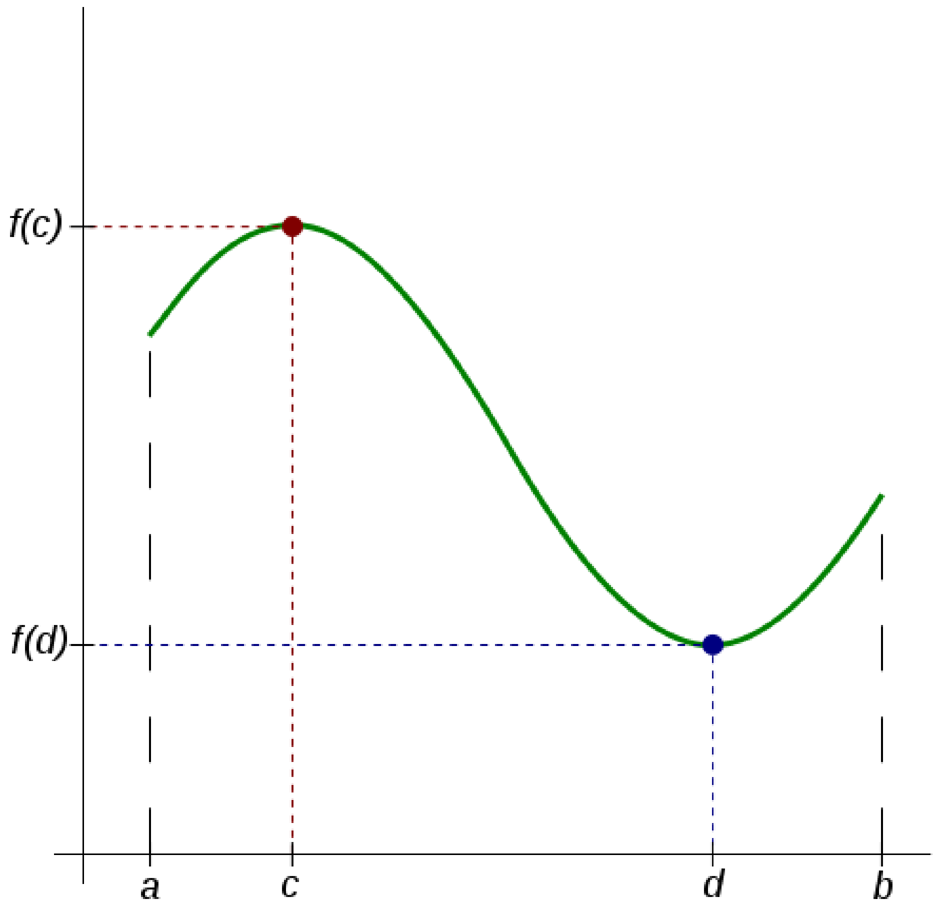
### Guidance

The **Intermediate Value Theorem** states that if a function is continuous on a closed interval and $u$ is a value between $f(a)$ and $f(b)$ then there exists a $c \in [a,b]$ such that $f(c) = u$.




Simply stated, if a function is continuous between a low point and a high point, then it must be valued at each intermediate height in between the low and high points.

The Extreme Value Theorem states that in every interval $[a,b]$ where a function is continuous there is at least one maximum and one minimum. In other words, it must have at least two extreme values.

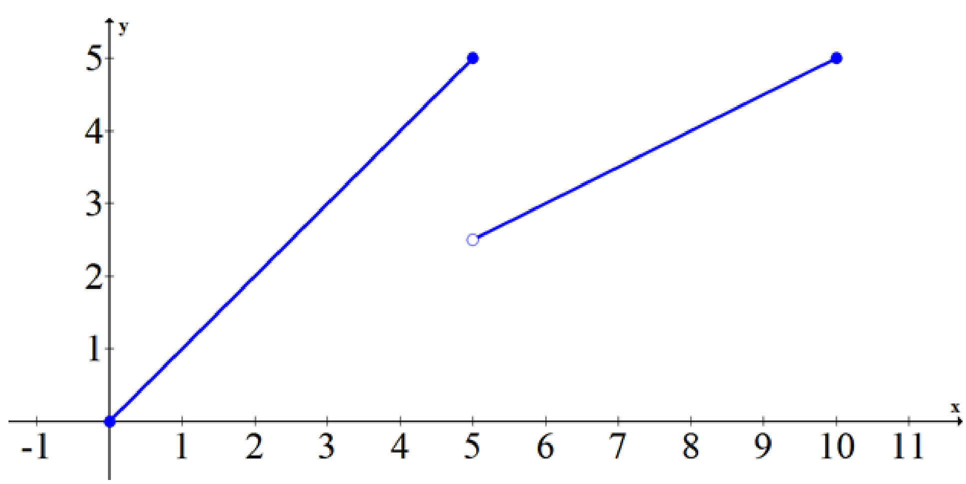



**Example A**

Show that the converse of the Intermediate Value Theorem is false.

**Solution:** The converse of the Intermediate Value Theorem is: *If there exists a value* $c \in [a,b]$ such that $f(c) = u$ for every $u$ between $f(a)$ and $f(b)$ then the function is continuous.

In order to show the statement is false, all you need is one counterexample where every intermediate value is hit and the function is discontinuous.



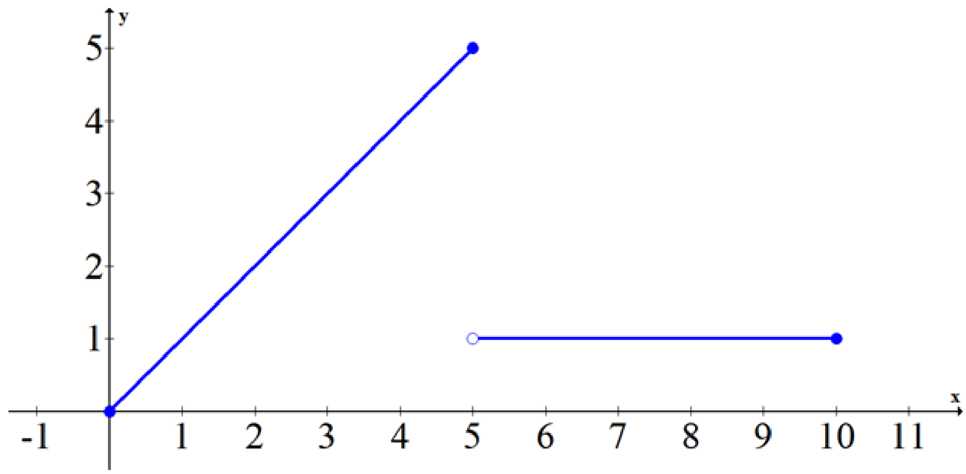

This function is discontinuous on the interval $[0,10]$ but every intermediate value between the first height at $(0,0)$ and the height of the last point $(10,5)$ is hit.

**Example B**

Show that the converse of the Extreme Value Theorem is false.

**Solution:** The converse of the Extreme Value Theorem is*:* If there is at least one maximum and one minimum in the closed interval $[a,b]$ *then the function is continuous on* $[a,b]$.

In order to show the statement is false, all you need is one counterexample. The goal is to find a function on a closed interval $[a,b]$ that has at least one maximum and one minimum and is also discontinuous.




On the interval $[0,10]$, the function attains a maximum at $(5,5)$ and a minimum at $(0,0)$ but is still discontinuous.

**Example C**

Use the Intermediate Value Theorem to show that the function $f(x) = (x+1)^3 - 4$ has a zero on the interval $[0,3]$.

**Solution:** First note that the function is a cubic and is therefore continuous everywhere.

- $f(0) = (0+1)^3 - 4 = 1^3 - 4 = -3$
- $f(3) = (3+1)^3 - 4 = 4^3 - 4 = 60$

By the Intermediate Value Theorem, there must exist a $c \in [0,3]$ such that $f(c) = 0$ since 0 is between -3 and 60.

**Concept Problem Revisited**

If a function is continuous on the interval $x \in [1,4]$, then you can conclude by the Intermediate Value Theorem that there exists a $c \in [1,4]$ such that $f(c) = u$ for every $u$ between $f(1)$ and $f(4)$. You can also conclude that on this interval the function has both a maximum and a minimum value.

### Vocabulary

The ***converse*** of an if then statement is a new statement with the hypothesis of the original statement switched with the conclusion of the original statement. In other words, the converse is when the if part of the statement and the then part of the statement are swapped. In general, the converse of a statement is not true.

A ***counterexample*** to an if then statement is when the hypothesis (the if part of the sentence) is true, but the conclusion (the then part of the statement) is not true.

### Guided Practice

1. Use the Intermediate Value Theorem to show that the following equation has at least one real solution.

$x^8 = 2^x$

2. Show that there is at least one solution to the following equation.

$\sin x = x + 2$

3. When are you not allowed to use the Intermediate Value Theorem?

**Answers:**

1. First rewrite the equation: $x^8 - 2^x = 0$

Then describe it as a continuous function: $f(x) = x^8 - 2^x$

This function is continuous because it is the difference of two continuous functions.

- $f(0) = 0^8 - 2^0 = 0 - 1 = -1$
- $f(2) = 2^8 - 2^2 = 256 - 4 = 252$

By the Intermediate Value Theorem, there must exist a $c$ such that $f(c) = 0$ because $-1 < 0 < 252$. The number $c$ is one solution to the initial equation.

2. Write the equation as a continuous function: $f(x) = \sin x - x - 2$

The function is continuous because it is the sum and difference of continuous functions.

- $f(0) = \sin 0 - 0 - 2 = -2$
- $f(-\pi) = \sin(-\pi) + \pi - 2 = 0 + \pi - 2 > 0$

By the Intermediate Value Theorem, there must exist a $c$ such that $f(c) = 0$ because $-2 < 0 < \pi - 2$. The number $c$ is one solution to the initial equation.

3. The Intermediate Value Theorem should not be applied when the function is not continuous over the interval.

### Practice

Use the Intermediate Value Theorem to show that each equation has at least one real solution.

1. $\cos x = -x$

2. $\ln(x) = e^{-x} + 1$

3. $2x^3 - 5x^2 = 10x - 5$

4. $x^3 + 1 = x$

5. $x^2 = \cos x$

6. $x^5 = 2x^3 + 2$

7. $3x^2 + 4x - 11 = 0$

8. $5x^4 = 6x^2 + 1$

9. $7x^3 - 18x^2 - 4x + 1 = 0$

10. Show that $f(x) = \frac{2x-3}{2x-5}$ has a real root on the interval $[1, 2]$.

11. Show that $f(x) = \frac{3x+1}{2x+4}$ has a real root on the interval $[-1, 0]$.

12. True or false: A function has a maximum and a minimum in the closed interval $[a, b]$; therefore, the function is continuous.

13. True or false: A function is continuous over the interval $[a, b]$; therefore, the function has a maximum and a minimum in the closed interval.

14. True or false: If a function is continuous over the interval $[a,b]$, then it is possible for the function to have more than one relative maximum in the interval $[a,b]$.

15. What do the Intermediate Value and Extreme Value Theorems have to do with continuity?

# 14.8 Instantaneous Rate of Change

Here you will learn about instantaneous rate of change and the concept of a derivative.

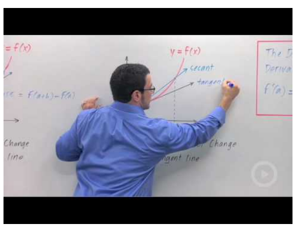
When you first learned about slope you learned the mnemonic device "rise over run" to help you remember that to calculate the slope between two points you use the following formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In Calculus, you learn that for curved functions, it makes more sense to discuss the slope at one precise point rather than between two points. The slope at one point is called the slope of the tangent line and the slope between two separate points is called a secant line.

Consider a car driving down the highway and think about its speed. You are probably thinking about speed in terms of going a given distance in a given amount of time. The units could be miles per hour or feet per second, but the units always have time in the denominator. What happens when you consider the instantaneous speed of the car at one instant of time? Wouldn't the denominator be zero?

### Watch This



MEDIA

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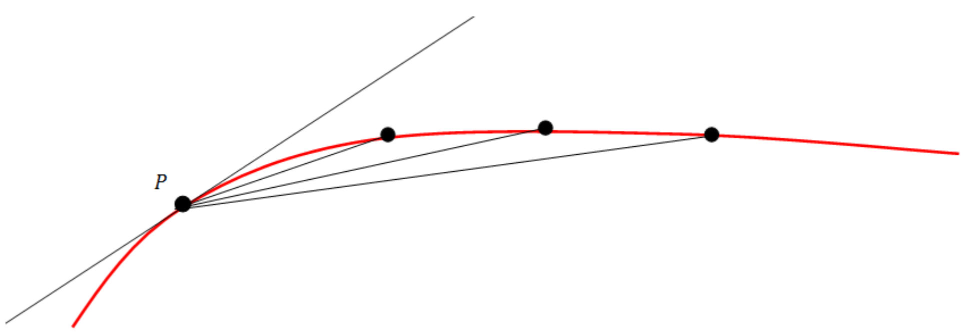
http://www.youtube.com/watch?v=7CvLzpzGhJI Brightstorm: Definition of a Derivative

### Guidance

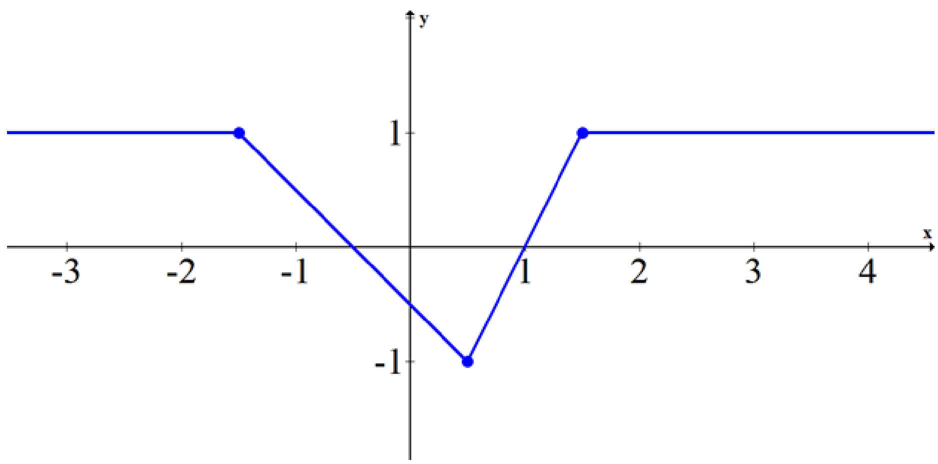
The slope at a point $P$ (also called the slope of the tangent line) can be approximated by the slope of secant lines as the "run" of each secant line approaches zero.



Because you are interested in the slope as the "run" approaches zero, this is a limit question. One of the main reasons that you study limits in calculus is so that you can determine the slope of a curve at a point (the slope of a tangent line).

**Example A**

Estimate the slope of the following function at -3, -2, -1, 0, 1, 2, 3. Organize the slopes in a table.




**Solution:** By mentally drawing a tangent line at the following $x$ values you can estimate the following slopes.

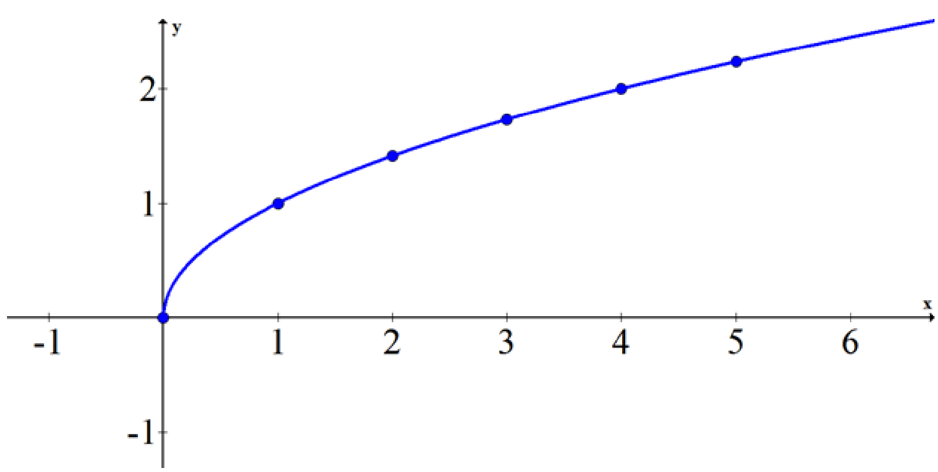
TABLE 14.12:

| $x$ | *slope* |
|---|---|
| -3 | 0 |
| -2 | 0 |
| -1 | -1 |
| 0 | -1 |
| 1 | 2 |
| 2 | 0 |
| 3 | 0 |

If you graph these points you will produce a graph of what's known as the derivative of the original function.

**Example B**

Estimate the slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ by calculating 4 successively close secant lines.



**Solution:** Calculate the slope between $(1,1)$ and 4 other points on the curve:

- The slope of the line between $\left(5,\ \sqrt{5}\right)$ and $(1,1)$ is: $m_1 = \frac{\sqrt{5}-1}{5-1} \approx 0.309$
- The slope of the line between $(4,\ 2)$ and $(1,1)$ is: $m_2 = \frac{2-1}{4-1} \approx 0.333$
- The slope of the line between $\left(3,\ \sqrt{3}\right)$ and $(1,1)$ is: $m_3 = \frac{\sqrt{3}-1}{3-1} \approx 0.366$
- The slope of the line between $\left(2,\ \sqrt{2}\right)$ and $(1,1)$ is: $m_4 = \frac{\sqrt{2}-1}{2-1} \approx 0.414$

If you had to guess what the slope was at the point $(1,1)$ what would you guess the slope to be?

**Example C**

Evaluate the following limit and explain its connection with Example B.

$\lim_{x \to 1} \left(\frac{\sqrt{x}-1}{x-1}\right)$

**Solution:** Notice that the pattern in the previous problem is leading up to $\frac{\sqrt{1}-1}{1-1}$ . Unfortunately, this cannot be computed directly because there is a zero in the denominator. Luckily, you know how to evaluate using limits.

$$\begin{aligned} m &= \lim_{x \to 1} \left( \frac{(\sqrt{x}-1)}{(x-1)} \cdot \frac{(\sqrt{x}+1)}{(\sqrt{x}+1)} \right) \\ &= \lim_{x \to 1} \left( \frac{(x-1)}{(x-1)(\sqrt{x}+1)} \right) \\ &= \lim_{x \to 1} \left( \frac{1}{(\sqrt{x}+1)} \right) \\ &= \frac{1}{\sqrt{1}+1} \\ &= \frac{1}{2} \\ &= 0.5 \end{aligned}$$

The slope of the function $f(x) = \sqrt{x}$ at the point $(1,1)$ is exactly $m = \frac{1}{2}$.

**Concept Problem Revisited**

If you write the ratio of distance to time and use limit notation to allow time to go to zero you do seem to get a zero in the denominator.

$\lim_{time \to 0} \left(\frac{distance}{time}\right)$

The great thing about limits is that you have learned techniques for finding a limit even when the denominator goes to zero. Instantaneous speed for a car essentially means the number that the speedometer reads at that precise moment in time. You are no longer restricted to finding slope from two separate points.

### Vocabulary

A ***tangent line*** to a function at a given point is the straight line that just touches the curve at that point. The slope of the tangent line is the same as the slope of the function at that point.

A ***secant line*** is a line that passes through two distinct points on a function.

A ***derivative*** is a function of the slopes of the original function.

### Guided Practice

1. Sketch a complete cycle of a sine graph. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

2. Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

3. Approximate the slope of $y = x^3$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

**Answers:**

1.

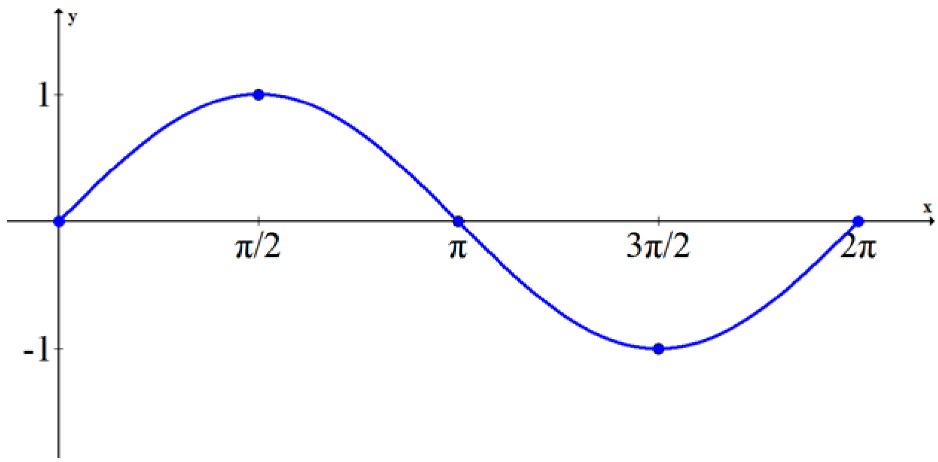
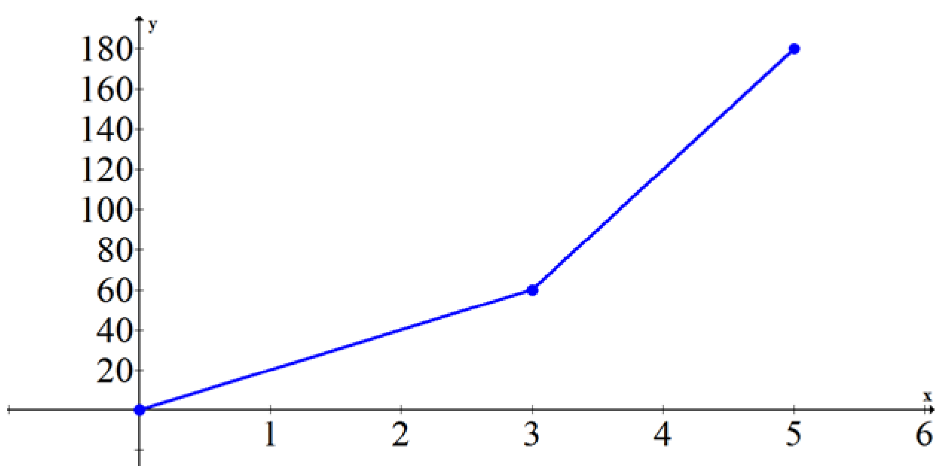



TABLE 14.13:

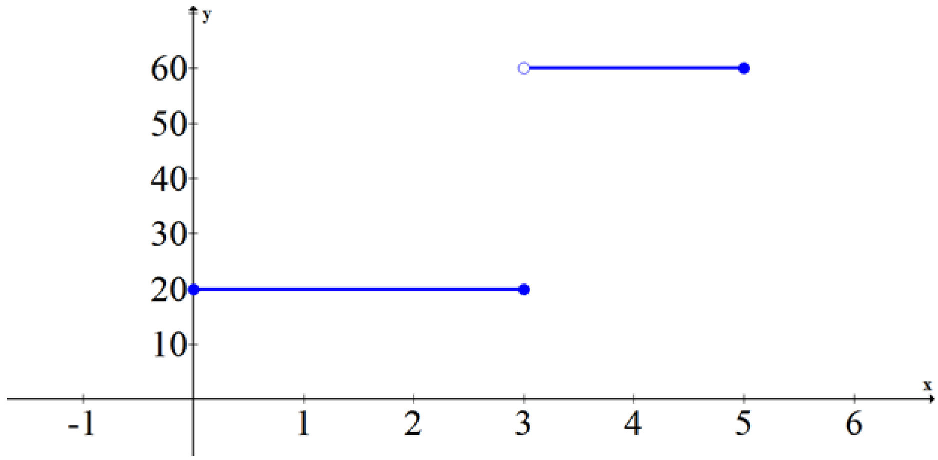
| $x$ | ***Slope*** |
|---|---|
| 0 | 1 |
| $\frac{\pi}{2}$ | 0 |
| $\pi$ | -1 |
| $\frac{3\pi}{2}$ | 0 |
| $2\pi$ | 1 |

You should notice that these are the exact values of cosine evaluated at those points.

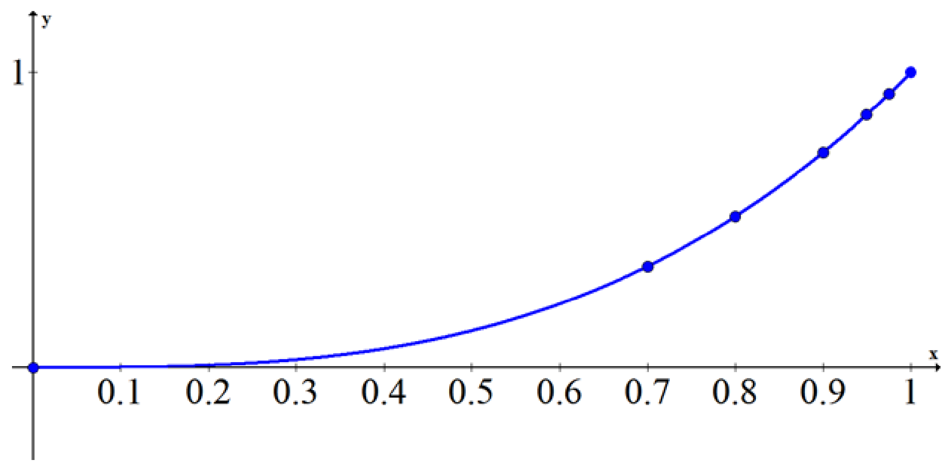
2. Distance vs. Time:



Rate vs. Time: (this is the graph of the derivative of the original function shown above)




3.




- The slope of the line between $(0.7, 0.7^3)$ and $(1, 1)$ is: $m_1 = \frac{0.7^3-1}{0.7-1} \approx 2.19$
- The slope of the line between $(0.8, 0.8^3)$ and $(1, 1)$ is: $m_2 = \frac{0.8^3-1}{0.8-1} \approx 2.44$
- The slope of the line between $(0.9, 0.9^3)$ and $(1, 1)$ is: $m_3 = \frac{0.9^3-1}{0.9-1} \approx 2.71$
- The slope of the line between $(0.95, 0.95^3)$ and $(1, 1)$ is: $m_1 = \frac{0.95^3-1}{0.95-1} \approx 2.8525$
- The slope of the line between $(0.975, 0.975^3)$ and $(1, 1)$ is: $m_1 = \frac{0.975^3-1}{0.975-1} \approx 2.925625$

The slope at $(1, 1)$ will be slightly greater than the estimates because of the way the slope curves. The slope at $(1, 1)$ appears to be about 3.

### Practice

1. Approximate the slope of $y = x^2$ at $(1, 1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

2. Evaluate the following limit and explain how it confirms your answer to #1.

$$\lim_{x \to 1} \left( \frac{x^2-1}{x-1} \right)$$

3. Approximate the slope of $y = 3x^2 + 1$ at $(1,4)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

4. Evaluate the following limit and explain how it confirms your answer to #3.

$\lim_{x \to 1} \left( \frac{3x^2+1-4}{x-1} \right)$

5. Approximate the slope of $y = x^3 - 2$ at $(1,-1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

6. Evaluate the following limit and explain how it confirms your answer to #5.

$\lim_{x \to 1} \left( \frac{x^3-2-(-1)}{x-1} \right)$

7. Approximate the slope of $y = 2x^3 - 1$ at $(1,1)$ by using secant lines from the left. Will the actual slope be greater or less than the estimates?

8. What limit could you evaluate to confirm your answer to #7?

9. Sketch a complete cycle of a cosine graph. Estimate the slopes at $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

10. How do the slopes found in the previous question relate to the sine function? What function do you think is the derivative of the cosine function?

11. Sketch the line $y = 2x + 1$. What is the slope at each point on this line? What is the derivative of this function?

12. Logan travels by bike at 30 mph for 2 hours. Then she gets in a car and drives 65 mph for 3 hours. Sketch both the distance vs. time graph and the rate vs. time graph.

13. Explain what a tangent line is and how it relates to derivatives.

14. Why is finding the slope of a tangent line for a point on a function the same as the instantaneous rate of change at that point?

15. What do limits have to do with finding the slopes of tangent lines?

# 14.9 Area Under a Curve

Here you will estimate the area under a curve and interpret its meaning in context.

Calculating the area under a straight line can be done with geometry. Calculating the area under a curved line requires calculus. Often the area under a curve can be interpreted as the accumulated amount of whatever the function is modeling. Suppose a car's speed in meters per second can be modeled by a quadratic for the first 8 seconds of acceleration:

$s(t) = t^2$

How far has the car traveled in 8 seconds?

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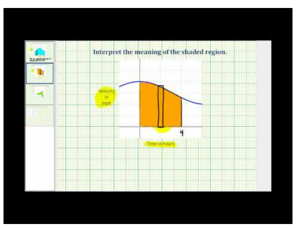


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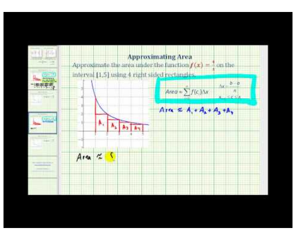
http://www.youtube.com/watch?v=Z_OHgubPJKA James Sousa: Interpret the Meaning of Area Under a Function



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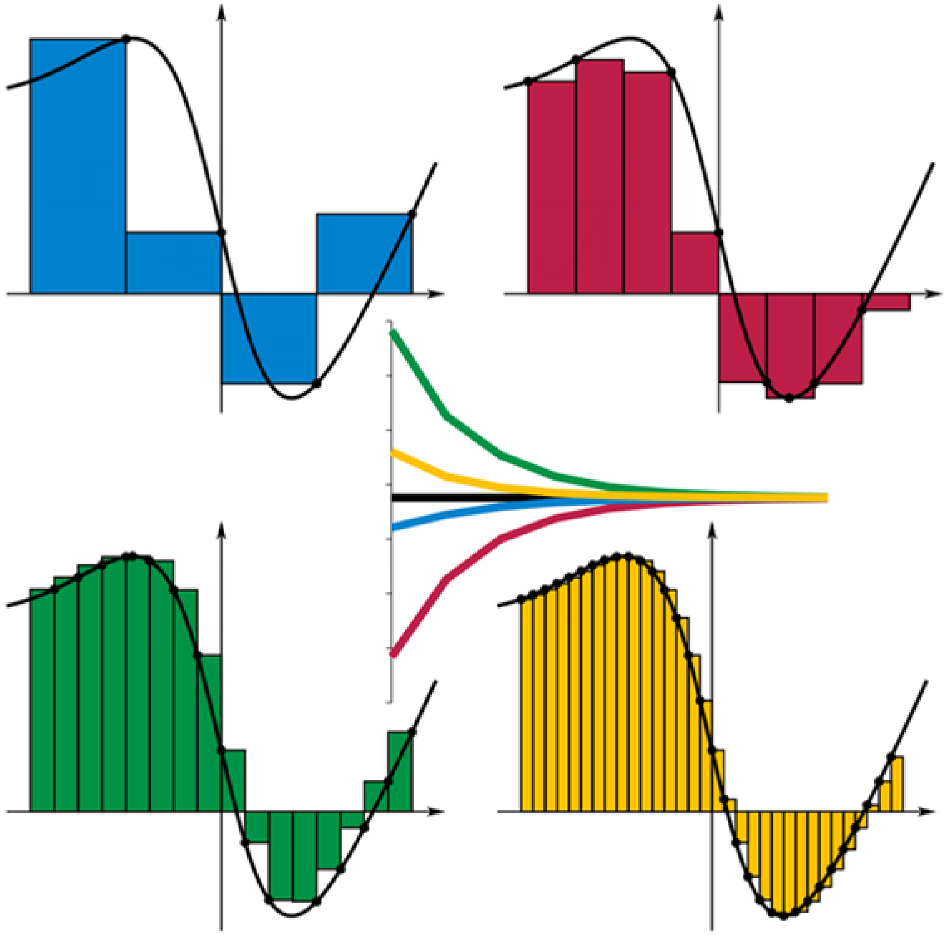
http://www.youtube.com/watch?v=BvwyTXeuLS0 James Sousa: Approximate the Area Under a Curve with 4 Right Sided Rectangles

### Guidance

The area under a curve can be approximated with rectangles equally spaced under a curve as shown below. For consistency, you can choose whether the boxes should hit the curve on the left hand corner, the right hand corner, the maximum value, or the minimum value. The more boxes you use the narrower the boxes will be and thus, the more accurate your approximation of the area will be.



The blue approximation uses right handed boxes. The red approximation assigns the height of the box to be the minimum value of the function in each subinterval. The green approximation assigns the height of the box to be the maximum value of the function in each subinterval. The yellow approximation uses left handed boxes. Rectangles above the $x$-axis will have positive area and rectangles below the $x$-axis will have negative area in this context.

All four of these area approximations get better as the number of boxes increase. In fact, the limit of each approximation as the number of boxes increases to infinity is the precise area under the curve.

This is where the calculus idea of an integral comes in. An integral is the limit of a sum as the number of summands increases to infinity.

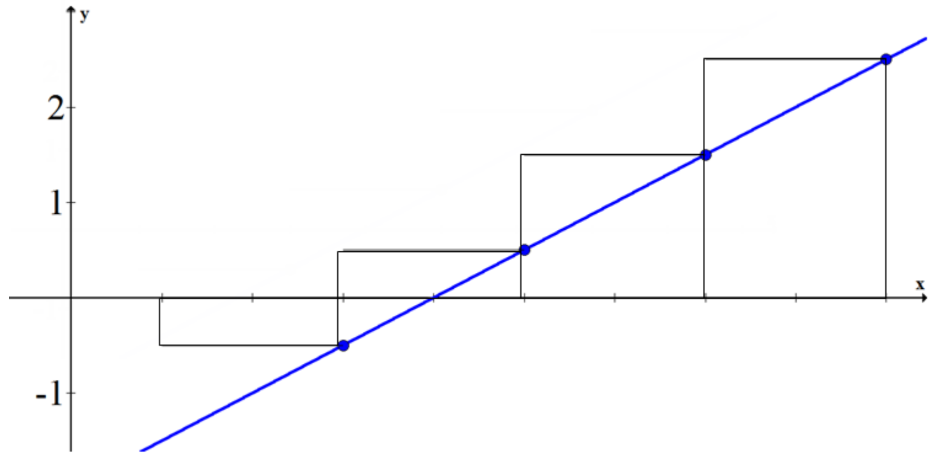
$\int f(x) = \lim_{n \to \infty} \sum_{i=1}^{n} (Area\ of\ box\ i)$

The symbol on the left is the calculus symbol of an integral. Using boxes to estimate the area under a curve is called a Riemann Sum.

**Example A**

Use four right handed boxes to approximate the area between 1 and 9 of the function $f(x) = \frac{1}{2}x - 2$.

**Solution:**




The area of the first box is 2 times the height of the function evaluated at 3:

$2 \cdot \left(\frac{1}{2} \cdot 3 - 2\right) = 3 - 4 = -1$

Because this box is under the $x$-axis, its area is negative.

The area for each of the rest of the boxes is 2 times the height of the function evaluated at 5, 7 and 9.

$$2 \cdot \left(\frac{1}{2} \cdot 5 - 2\right) = 5 - 4 = 1$$

$2 \cdot \left(\frac{1}{2} \cdot 7 - 2\right) = 7 - 4 = 3$

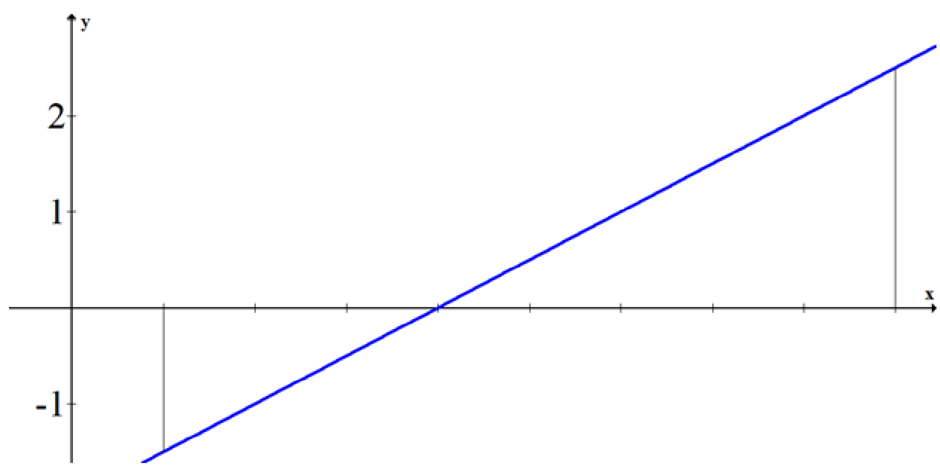
$2 \cdot \left(\frac{1}{2} \cdot 9 - 2\right) = 9 - 4 = 5$

The approximate sum of the total area under the curve is: $-1 + 1 + 3 + 5 = 8$ square units.

**Example B**

Evaluate the exact area under the curve in Example A using the area formula for a triangle.

**Solution:** Remember that the area below the $x$ axis is negative while the area above the $x$ axis is positive.




Negative Area: $\frac{1}{2} \cdot 3 \cdot 1.5 = \frac{9}{4}$

Positive Area: $\frac{1}{2} \cdot 5 \cdot 2.5 = \frac{25}{4}$

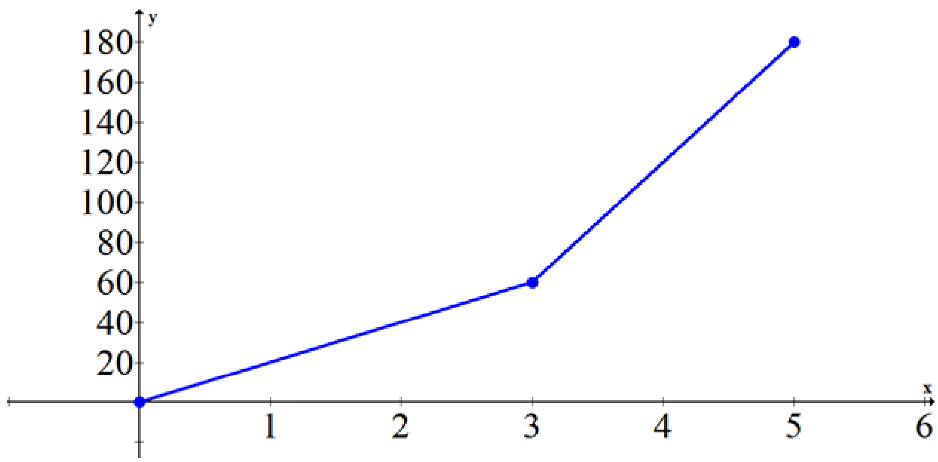
Area under the curve between 1 and 8: $\frac{25}{4} - \frac{9}{4} = \frac{16}{4} = 4$

It appears that approximations that are 2 units wide produce an area with significant error.

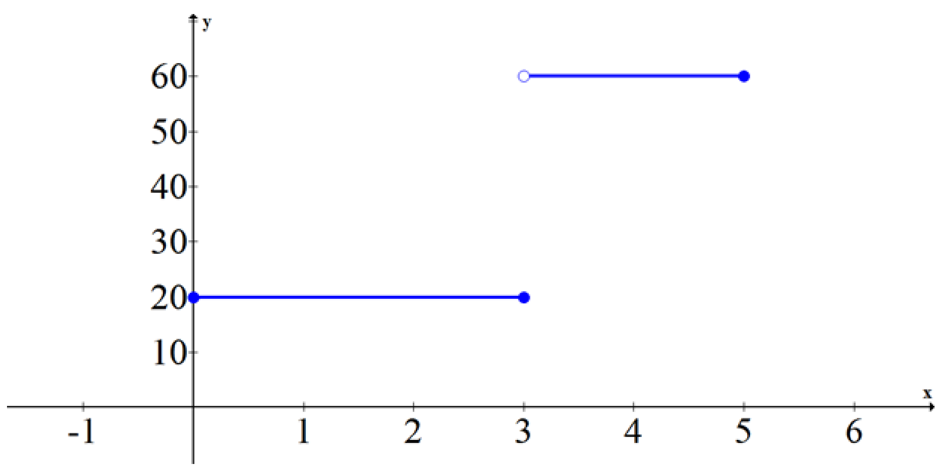
**Example C**

Logan travels by bike at 20 mph for 3 hours. Then she gets in a car and drives 60 mph for 2 hours. Sketch both the distance vs. time graph and the rate vs. time graph. Use an area under the curve argument to connect the two graphs.

**Solution:** Distance vs. Time:




Rate vs. Time:



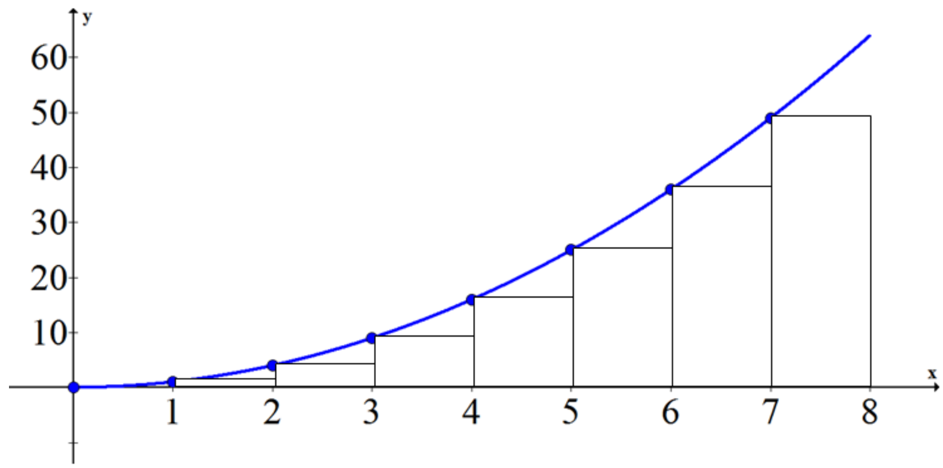

The slope of the first graph is 20 from 0 to 3 and then 60 from 3 to 5. The second graph is a graph of the slopes from the first graph. If you calculate the area of the second graph at the key points 0, 1, 2, 3, 4 and 5 you will see that they align perfectly with the points on the first graph.

**TABLE 14.14:**

| $x$ | ***Area under curve from 0 to*** $x$ |
|---|---|
| 0 | 0 |
| 1 | 20 |
| 2 | 40 |
| 3 | 60 |
| 4 | 120 |
| 5 | 180 |

**Concept Problem Revisited**

You can use the area under the curve to find the total distance traveled in the first 8 seconds. Since the quadratic is a curve you must choose the number of subintervals you want to use and whether you want right or left handed boxes for estimating. Suppose you choose 8 left handed boxes of width one.




### TABLE 14.15:

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| *Area of box to the right* | $1 \cdot 0$ | $1 \cdot 1$ | $1 \cdot 4$ | $1 \cdot 9$ | $1 \cdot 16$ | $1 \cdot 25$ | $1 \cdot 36$ | $1 \cdot 49$ |

The approximate sum is $1+4+9+16+25+36+49 = 140$. This means that the car traveled approximately 140 meters in the first 8 seconds.

## Vocabulary

***Subintervals*** are created when an interval is broken into smaller equally sized intervals.

An ***integral*** is the limit of a sum as the number of summands increases to infinity.

Using boxes to estimate the area under a curve is called a ***Riemann Sum***.

A ***summand*** is one of many pieces being summed together.

## Guided Practice

1. Approximate the area under the curve using eight subintervals and right endpoints.

$f(x) = 3x^2 - 1, \ -1 \le x \le 7$

2. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = \frac{4}{x} + 3, \ 2 \le x \le 6$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = x^x, \ 1 \le x \le 3$

**Answers:**

1. While a graph is helpful to visualize the problem and drawing each box can help give meaning to each summand, it is not always necessary. Since there are going to be 8 subintervals over the total interval of $-1 \le x \le 7$, each interval is going to have a width of 1. The height of each interval is going to be at the right hand endpoints of each subinterval (0, 1, 2, 3, 4, 5, 6, 7).

$$\sum height \cdot width = \sum_{i=0}^{7} (3i^2 - 1) \cdot 1 = 412$$

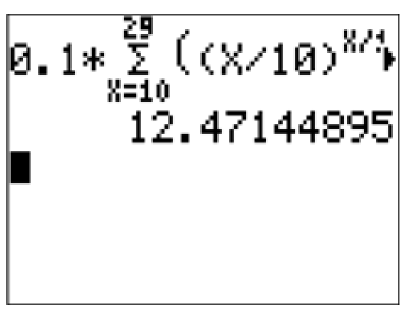
2. Each interval will be only $\frac{1}{2}$ wide which means that the left endpoints have $x$ values of: 2, 2.5, 3, 3.5, 4, 4.5, 5, 5.5. Since the index of summation notation does not work with decimals, double each of these to get a good counting sequence: 4, 5, 6, 7, 8, 9, 10, 11 and remember to halve them in the argument of the summation.

$$x = \frac{i}{2}$$
$$\sum height \cdot width = \sum_{i=4}^{11} \left( \frac{4}{\left(\frac{i}{2}\right)} \right) \cdot \frac{1}{2} \approx 4.7462$$

3. When the number of subintervals gets large and the subintervals get extremely narrow it will be impossible to draw an accurate picture. This is why using summation notation and thinking through what the indices and the argument will be is incredibly important. With 20 subintervals between $[1,3]$, each interval will be 0.1 wide. Left endpoints means that the first box has a height of $f(1)$ and the second box has a height of $f(1.1)$.

$$\begin{aligned}
\sum height \cdot width &= f(1) \cdot 0.1 + f(1.1) \cdot 0.1 + f(1.2) \cdot 0.1 + \cdots + f(2.9) \cdot 0.1 \\
&= 0.1(f(1) + f(1.1) + \cdots f(2.9)) \\
&= 0.1 \cdot \sum_{i=10}^{29} f\left(\frac{i}{10}\right) \\
&= 0.1 \cdot \sum_{i=10}^{29} \left(\frac{i}{10}\right)^{\left(\frac{i}{10}\right)} \\
&\approx 12.47144
\end{aligned}$$

Your calculator can compute summations when you go under the math menu.




## Practice

1. Approximate the area under the curve using eight subintervals and right endpoints.

$f(x) = x^2 - x + 1,\ 0 \leq x \leq 8$

2. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = x^2 - 2x + 1,\ -4 \leq x \leq 4$

3. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = \sqrt{x+3},\ 0 \leq x \leq 4$

4. Approximate the area under the curve using 100 subintervals and left endpoints. Compare to your answer from #3.

$f(x) = \sqrt{x+3},\ 0 \leq x \leq 4$

5. Approximate the area under the curve using eight subintervals and left endpoints.

$f(x) = \cos(x),\ 0 \leq x \leq 4$

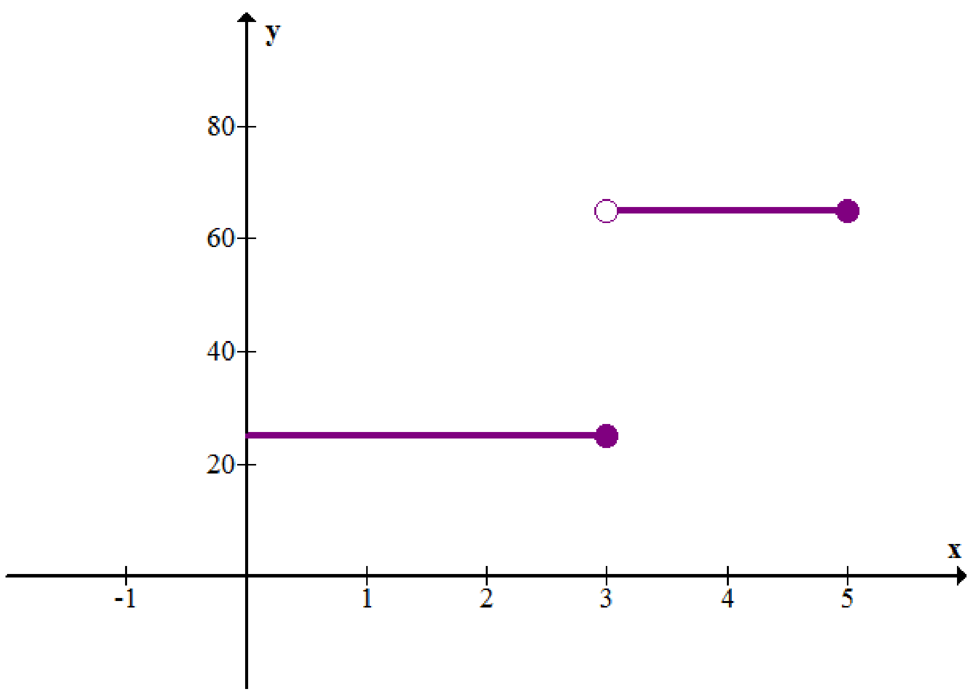
6. Approximate the area under the curve using twenty subintervals and left endpoints.

$f(x) = \cos(x),\ 0 \leq x \leq 4$

7. Approximate the area under the curve using 100 subintervals and left endpoints.

$f(x) = \cos(x),\ 0 \leq x \leq 4$

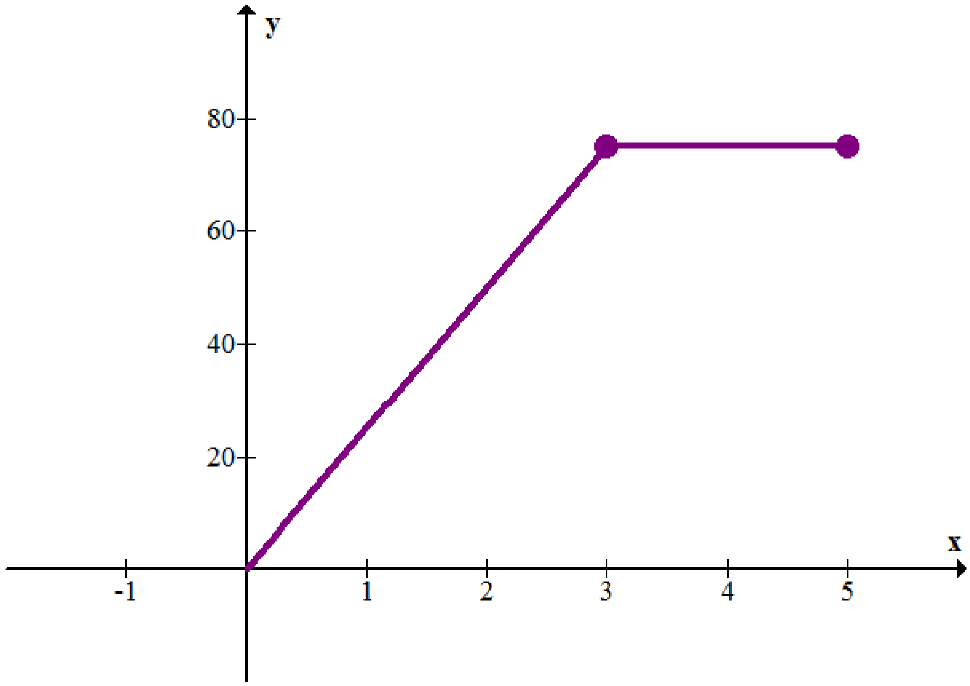
The following graph shows the rate (in miles per hour) vs. time (in hours) for a car.




8. Describe what is happening with the car.

9. How far did the car travel in 5 hours?

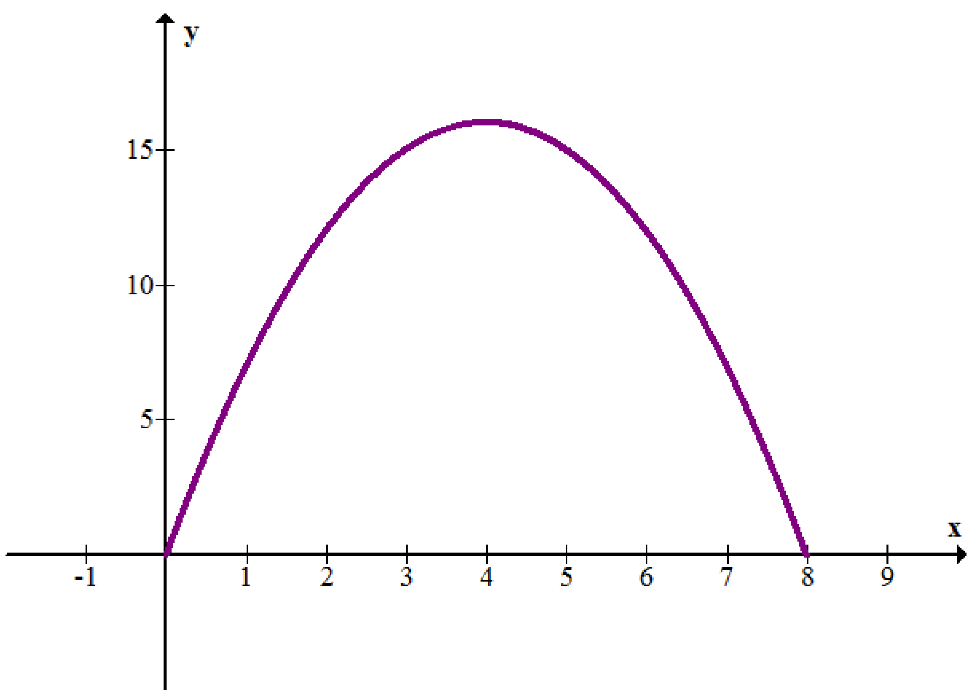
The following graph shows the rate (in feet per second) vs. time (in seconds) for a car.




10. Describe what is happening with the car. In particular, what is happening in the first 3 seconds?

11. How far did the car travel in 5 seconds?

The following graph shows the function $f(x) = -(x-4)^2 + 16$, which represents the rate (in feet per second) vs. time (in seconds) for a runner.




12. Describe what is happening with the runner. In particular, what happens after 4 seconds?

13. Use rectangles to approximate the total distance (in feet) that the runner traveled in the 8 seconds. Try to get as good an approximation as possible.

14. Explain how an integral is like the opposite of a derivative.

15. How do integrals relate to sums?

You learned that limits enable you to work infinitely close to a point without caring what happens at the point itself. Using this subtle approach, you began to answer two of the biggest questions in Calculus. 1) What is the slope of the tangent line at a point on a curve? 2) What is the area under a curve?

# 14.10 References


1. CK-12 Foundation. . CCSA
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# CHAPTER 15 Concepts of Statistics

## Chapter Outline



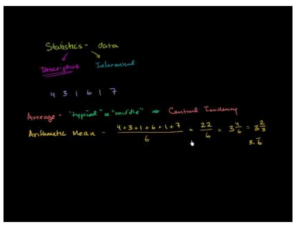
Statistics is hugely important for understanding, describing and predicting the world around you. Descriptive statistics is using summaries to present information that you have found to a reader. Summaries can be graphs or small groups of numbers that are easier to understand than long lists of numbers. Inferential statistics is using data to make predictions. Both inferential statistics and descriptive statistics help you understand the world around you and communicate it effectively.

# 15.1 Mean, Median and Mode

Here you will calculate three measures of the center of univariate data and decide which measure is best based on context.

The three measures of central tendency are mean, median, and mode. When would it make sense to use one of these measures and not the others?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=h8EYEJ32oQ8 Khan Academy: Statistics Intro: Mean, Median, and Mode

### Guidance

With **descriptive statistics**, your goal is to describe the data that you find in a sample or is given in a problem. Because it would not make sense to present your findings as long lists of numbers, you summarize important aspects of the data. One important aspect of the data is the **measure of central tendency**, which is a measure of the "middle" value of a set of data. There are three ways to measure central tendency:

1. Use the **mean**, which is the arithmetic average of the data.
2. Use the **median**, which is the number exactly in the middle of the data. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the arithmetic average of the two most central numbers.
3. Use the **mode**, which is the most often occurring number in the data. If there are two or more numbers that occur equally frequently, then the data is said to be bimodal or multimodal.

Calculating the mean, median and mode is straightforward. What is challenging is determining when to use each measure and knowing how to interpret the data using the relationships between the three measures.

**Example A**

Five people were called on a phone survey to respond to some political opinion questions. Two people were from the zip code 94061, one person was from the zip code 94305 and two people were from 94062.

Which measure of central tendency makes the most sense to use if you want to state where the average person was from?

**Solution:** None of the measures of central tendency make sense to apply to this situation. Zip codes are categorical data rather than quantitative data even though they happen to be numbers. Other examples of categorical data are hair color or gender. You could argue that mode is applicable in a broad sense, but in general remember that mean, median, and mode can only be applied to quantitative data.

**Example B**

Compute the mean, median and mode for the following numbers.

3, 5, 1, 6, 8, 4, 5, 2, 7, 8, 4, 2, 1, 3, 4, 6, 7, 9, 4, 3, 2

**Solution:**

**Mean:** The sum of all these numbers is 94 and there are 21 numbers total so the mean is $\frac{94}{21} \approx 4.4762$.

**Median:** When you order the numbers from least to greatest you get:

1, 1, 2, 2, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9

The $11^{th}$ number has ten numbers to the right and ten numbers to the left so it is the median. The median is the number 4.

**Mode:** the most frequently occurring number is the number 4.

*Note: it is common practice to round to 4 decimals in AP Statistics.*

**Example C**

You write a computer code to produce a random number between 0 and 10 with equal probability. Unfortunately, you suspect your code doesn't work perfectly because in your first few attempts at running the code, it produces the following numbers:

1, 9, 1, 1, 9, 2, 9, 1, 9, 9, 9, 2, 2

How would you argue using mean, median, or mode that this code is probably not producing a random number between 0 and 10 with equal probability?

**Solution:** This question is very similar to questions you will see when you study statistical inference.

First you would note that the mean of the data is 4.9231. If the data was truly random then the mean would probably be right around the number 5 which it is. This is not strong evidence to suggest that the random number generating code is broken.

Next you would note that the median of the data is 2. This should make you suspect that something is wrong. You would expect that the median is of random numbers between 0 and 10 to be somewhere around 5.

Lastly, you would note that the mode of the data is 9. By itself this is not strong data to suggest anything. Every sample will have to have at least one mode. What should make you suspicious, however, is the fact that only two other numbers were produced and were almost as frequent as the number 9. You would expect a greater variety of numbers to be produced.

**Concept Problem Revisited**

In order to decide which measure of central tendency to use, it is a good idea to calculate and interpret all three of the numbers.

For example, if someone asked you how many people can sit in the typical car, it would make more sense to use mode than to use mean. With mode, you could find out that a five person car is the most frequent car driven and determine that the answer to the question is 5. If you calculate the mean for the number of seats in all cars, you will end up with a decimal like 5.4, which makes less sense in this context.

On the other hand, if you were finding the central heights of NBA players, using mean might make a lot more sense than mode.

## Vocabulary

The ***mean*** is the arithmetic average of the data.

The ***median*** is the number in the middle of a data set. When the data has an odd number of counts, the median is the middle number after the data has been ordered. When the data has an even number of counts, the median is the average of the two most central numbers.

The ***mode*** is the most often occurring number in the data. If there are two or more numbers which occur equally frequently, then the data is said to be ***bimodal*** or ***multimodal***.

With ***descriptive statistics***, your goal is to describe the data that you find in a sample or is given in a problem.

With ***inference statistics***, your goal is use the data in a sample to draw conclusions about a larger population.

### Guided Practice

1. Ross is with his friends and they want to play basketball. They decide to choose teams based on the number of cousins everyone has. One team will be the team with fewer cousins and the other team will be the team with more cousins. Should they use the mean, median or mode to compute the cutoff number that will separate the two teams?

2. Compute the mean, median, and mode for the following numbers.

1, 4, 5, 7, 6, 8, 0, 3, 2, 2, 3, 4, 6, 5, 7, 8, 9, 0, 6, 5, 3, 1, 2, 4, 5, 6, 7, 8, 8, 8, 4, 3, 2

3. The cost of fresh blueberries at different times of the year are:

$2.50, $2.99, $3.20, $3.99, $4.99

If you bought blueberries regularly what would you typically pay?

**Answers:**

1. Ross and his friends should use the median number of cousins as the cutoff number because this will allow each team to have the same number of players. If there are an odd number of people playing, then the extra person will just join either team or switch in later.

2. The mean is 4.6061. The median is 5. The mode is 8.

3. The word "typically" is used instead of average to allow you to make your own choice as to whether mean, median, or mode would make the most sense. In this case, mean does make the most sense. The average cost is $3.53.

### Practice

You surveyed the students in your English class to find out how many siblings each student had. Here are your results:

0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 10, 12

1. Find the mean, median, and mode of this data.

2. Why does it make sense that the mean number of siblings is greater than the median number of siblings?

3. Which measure of central tendency do you think is best for describing the typical number of siblings?

4. So far in math you have taken 10 quizzes this semester. The mean of the scores is 88.5. What is the sum of the scores?

5. Find $x$ if 5, 9, 11, 12, 13, 14, 16, and $x$ have a mean of 12.

6. Meera drove an average of 22 miles a day last week. How many miles did she drive last week?

7. Find $x$ if 2, 6, 9, 8, 4, 5, 8, 1, 4, and $x$ have a median of 5.

Calculate the mean, median, and mode for each set of numbers:

8. 11, 15, 19, 12, 21, 34, 15, 28, 24, 15, 27, 19, 20, 13, 15

9. 3, 5, 7, 5, 5, 17, 8, 9, 11, 5, 3, 7

10. -3, 0, 5, 8, 12, 4, 2, 1, 6

Calculate the mean and median for each set of numbers:

11. 12, 88, 89, 90

12. 16, 17, 19, 20, 20, 98

13. For which of the previous two questions was the median **less than** the mean? What in the set of numbers caused this?

14. For which of the previous two questions was the median **greater than** the mean? What in the set of numbers caused this?

15. In each of the sets of numbers for problems 11 and 12, there is one number that could be considered an **outlier**. Which numbers do you think are the outliers and why? What would happen to the mean and median if you removed the outliers?

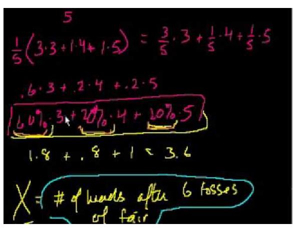
# 15.2 Expected Value and Payoffs

Here you will apply what you know about mean and averages to weighted averages and expected value.

When playing a game of chance there are three basic elements. There is the cost to play the game (usually), the probability of winning the game, and the amount you receive if you win. If games of chance with these three elements are played repeatedly, you can use probability and averages to calculate how much you can expect to win or lose in the long run.

Consider a dice game that pays you triple your bet if you roll a six and double your bet if you roll a five. If you roll anything else you lose your bet. What is your expected return on a one dollar wager?

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=j__Kredt7vY Khan Academy: Expected Value

### Guidance

A weighted average is like a regular average except the data is often given to you in summary form.

**Data in Raw Form:**

1, 3, 5, 3, 2, 1, 2, 5, 6, 4, 5, 2, 6, 1, 4, 3, 6, 1, 2, 4, 6, 1, 3, 1, 3, 5, 6

**Data in Summary Form:**

TABLE 15.1:

| *Number* | *Occurrence Count* |
|---|---|
| 1 | 6 |
| 2 | 4 |
| 3 | 5 |
| 4 | 3 |
| 5 | 4 |
| 6 | 5 |
| Total Occurrences: | 27 |

Notice that the summary data indicates, for example, how many times a 1 was rolled (6 times). To calculate the total number of occurrences of data:

- In raw form: count how many data points you have

- In summary form: find the sum the occurrence column

To calculate the average:

- In raw form: find the sum of the data points and divide by the total number of occurrences.
- In summary form: find the sum of the data points by finding the sum of the product of each number and its occurrence:

$$1 \cdot 6 + 2 \cdot 4 + 3 \cdot 5 + 4 \cdot 3 + 5 \cdot 4 + 6 \cdot 5 = 91$$

Then, divide that sum by the total number of occurrences. *In a sense, you are assigning a weight to each of the six numbers based on their frequency in your 27 trials.*

The same logic of finding the average of data given in summary form applies when doing theoretical expected value for a game or a weighted average. Consider a game of chance with 4 prizes ($1, $2, $3, and $4) where each outcome has a specific probability of happening, shown in the table below:

### TABLE 15.2:

| ***Number*** | ***Probability*** |
|---|---|
| $1 | 50% |
| $2 | 20% |
| $3 | 20% |
| $4 | 10% |

***Note that the probabilities must add up to 100%!*** In order to calculate the expected value of this game, weight the outcomes by their assigned probabilities.

$\$1 \cdot 0.50 + \$2 \cdot 0.20 + \$3 \cdot 0.30 + \$4 \cdot 0.10 = \$2.20$

This means that if you were to play this game many times, your average amount of winnings should be $2.20. Note that there will be no game that you actually get $2.20, because that was none of the options. Expected value is a measure of what you should expect to get per game in the long run.

The **payoff** of a game is the expected value of the game minus the cost. If you expect to win about $2.20 on average if you play a game repeatedly and it costs only $2 to play, then the expected payoff is $0.20 per game.

In general, to find the **expected value** for a game or other scenario, find the sum of all possible outcomes, each multiplied by the probability of its occurrence.

**Example A**

A teacher has five categories of grades that each make up a specific percentage of the final grade. Calculate Owen's grade.

### TABLE 15.3:

| **Category** | **Weight** | **Owen's grade** |
|---|---|---|
| Quizzes and Tests | 30% | 78% |
| Homework | 25% | 100% |
| Final | 20% | 74% |
| Projects | 20% | 90% |
| Participation | 5% | 100% |

**Solution:** Using the concept of weighted average, weight each of Owen's grades by the weight of the category.

$0.78 \cdot 0.3 + 1 \cdot 0.25 + 0.74 \cdot 0.20 + 0.90 \cdot 0.20 + 1 \cdot 0.05 = 0.862$

Owen gets an 86.2%.

**Example B**

Courtney plays a game where she flips a coin. If the coin comes up heads she wins $2. If the coin comes up tails she loses $3. What is Courtney's expected payoff each game?

**Solution:** The probability of getting heads is 50% and the probability of getting tails is 50%. Using the concept of weighted averages, you should weight winning 2 dollars and losing 3 dollars by 50% each. In this case there is no initial cost to the game.

$2 \cdot 0.50 - 3 \cdot 0.50 = -0.50$

This means that while sometimes she might win and sometimes she might lose, on average she is expected to lose about 50 cents per game.

**Example C**

Paul is deciding whether or not to pay the parking meter when he is going to the movies. He knows that a parking ticket costs $30 and he estimates that there is a 40% chance that the traffic police spot his car and write him a ticket. If he chooses to pay the meter it will cost 4 dollars and he will have a 0% chance of getting a ticket.

Is it cheaper to pay the meter or risk the fine?

**Solution:** Since there are two possible scenarios, calculate the expected cost in each case.

$Paying\ the\ meter : \$4 \cdot 100\% = \$4$

$Risking\ the\ fine : \$0 \cdot 60\% + \$30 \cdot 40\% = \$12$

Risking the fine has an expected cost three times that of paying the meter.

**Concept Problem Revisited**

In a game that pays you triple your bet if you roll a six and double your bet if you roll a five, the expected return on a one dollar wager is:

$\$0 \cdot \frac{2}{3} + \$2 \cdot \frac{1}{6} + \$3 \cdot \frac{1}{6} = \frac{5}{6}$

If you spend $1 to play the game and you play the game multiple times, you can expect a return of $\frac{5}{6}$ of one dollar or about 83 cents on average.

## Vocabulary

A ***weighted average*** is an average that multiplies each component by a factor representing its frequency or probability.

The ***expected value*** is the return or cost you can expect on average, given many trials.

The ***payoff*** of a game is the expected value of the game minus the cost.

## Guided Practice

1. What is the payoff of a slot machine that costs a dollar to play and pays out $5 with probability 4%, $10 with probability of 2%, and $30 with probability 0.5%?

2. What is the expected value of an experiment with the following outcomes and corresponding probabilities?

### TABLE 15.4:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Outcome | 31 | 35 | 37 | 39 | 43 | 47 | 49 |
| Probability | 0.1 | 0.1 | 0.1 | 0.2 | 0.2 | 0.2 | 0.1 |

3. Every day you record about how long it takes to get to school.

### TABLE 15.5:

| Time | Number of Days |
|---|---|
| 5-7 minutes | 1 |
| 7-8 minutes | 4 |
| 8-9 minutes | 7 |
| 9-10 minutes | 9 |
| 10-12 minutes | 2 |

How long does it take you to get to school on average?

**Answers:**

1. $0 \cdot 0.935 + 5 \cdot 0.04 + 10 \cdot 0.02 + 30 \cdot 0.005 - 1 = -0.45$. You will lose \$0.45 on average if you play the slot machine many times.

2. $31 \cdot 0.1 + 35 \cdot 0.1 + 37 \cdot 0.1 + 39 \cdot 0.2 + 43 \cdot 0.2 + 47 \cdot 0.2 + 49 \cdot 0.1 = 41$

3. To answer this question, you could find the weighted average of the expected value. If you choose expected value, consider that the situation gives you frequency rather than probability. You can calculate the probability of each of the categories by dividing each frequency by the total number of days (23). Since the time occurs in intervals, it is reasonable to use the average time in each interval as representative of the category when calculating the expected value.

### TABLE 15.6:

| Time | Number of Days | Probability |
|---|---|---|
| 5-7 minutes | 1 | $\frac{1}{23}$ |
| 7-8 minutes | 4 | $\frac{4}{23}$ |
| 8-9 minutes | 7 | $\frac{7}{23}$ |
| 9-10 minutes | 9 | $\frac{9}{23}$ |
| 10-12 minutes | 2 | $\frac{2}{23}$ |

$6 \cdot \frac{1}{23} + 7.5 \cdot \frac{4}{23} + 8.5 \cdot \frac{7}{23} + 9.5 \cdot \frac{9}{23} + 11 \cdot \frac{2}{23} = \frac{203}{23} \approx 8.8$

On average, it takes you 8-9 minutes to get to school.

## Practice

1. Explain how to calculate expected value.

2. True or false: If the expected value of a game is \$0.50, then you can expect to win \$0.50 each time you play.

3. True or false: The greater the number of games played, the closer the average winnings will be to the theoretical expected value.

4. A player rolls a standard pair of dice. If the sum of the numbers is a 6, the player wins \$6. If the sum of the numbers is anything else, the player has to pay \$1. What is the expected value for this game?

5. What is the payoff of a slot machine that costs 25 cents to play and pays out \$1 with probability 10%, \$50 with probability of 1%, and \$100 with probability 0.01%?

6. A slot machine pays out \$1 with probability 5%, \$100 with probability of 0.5%, and \$1000 with probability 0.01%? If the casino wants to guarantee that they won't lose money on this machine, how much should they charge people to play?

7. What is the expected value of an experiment with the following outcomes and corresponding probabilities?

## TABLE 15.7:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Outcome | 12 | 14 | 18 | 20 | 21 | 22 | 23 |
| Probability | 0.05 | 0.1 | 0.6 | 0.1 | 0.1 | 0.03 | 0.02 |

Calculate the final grades for each of the students given the information in the table.

## TABLE 15.8:

| **Category** | **Weight** | **Sarah** | **Jason** | **Kimy** | **Maria** | **Kayla** |
|---|---|---|---|---|---|---|
| Quizzes and Tests | 30% | 74% | 85% | 90% | 80% | 75% |
| Homework | 25% | 95% | 40% | 100% | 90% | 95% |
| Final | 20% | 68% | 80% | 85% | 70% | 50% |
| Projects | 20% | 85% | 70% | 95% | 75% | 85% |
| Participation | 5% | 95% | 100% | 100% | 80% | 60% |

8. What is Sarah's final grade?

9. What is Jason's final grade?

10. What is Kimy's final grade?

11. What is Maria's final grade?

12. What is Kayla's final grade?

13. Look back at the grades and final grades for the five students. Do the grades seem fair to you given how each student performed in each of the areas? Do you think the category weights should be changed?

14. You are in charge of a booth for a game at the fair. In the game, players pick a card at random from the deck. If the card is a J, Q, or K, the player wins \$5. What is the minimum amount you should charge in order to feel confident you will make a profit by the end of the fair?

15. Make up your own game that has at least 2 possible outcomes with an expected payoff of \$0.50.

16. Explain why it makes sense for a casino to consider the concept of expected value when designing their games.

# 15.3 Five Number Summary

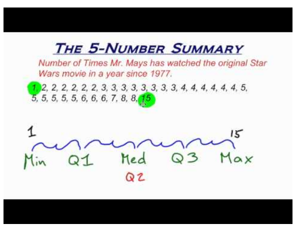
Here you will calculate quartiles and produce five number summaries for data sets.

When given a long list of numbers, it is useful to summarize the data. One way to summarize the data is to give the lowest number, the highest number and the middle number. In addition to these three numbers it is also useful to give the median of the lower half of the data and the median of the upper half of the data. These five numbers give a very concise summary of the data.

What is the five number summary of the following data?

0, 0, 1, 2, 63, 61, 27, 13

### Watch This




MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=XDS5TgZ4CJA The 5 Number Summary

### Guidance

Suppose you have ordered data with $m$ observations. The rank of each observation is shown by its index.

$y_1 \leq y_2 \leq y_3 \leq \cdots \leq y_m$

In data sets that are large enough, you can divide the numbers into four parts called quartiles. The quartiles of interest are the first quartile, $Q1$, the second quartile, $Q2$, and the third quartile $Q3$. The second quartile, $Q2$, is defined to be the median of the data. The first quartile, $Q1$, is defined to be the median of the lower half of the data. The third quartile, $Q3$, is similarly defined to be the median of the upper half of the data.

These three numbers in addition to the minimum and maximum values are the five number summary. Note that there are variations of the five number summary that you can study in a statistics course.

**Example A**

Compute the five number summary for the following data.

2, 7, 17, 19, 25, 26, 26, 32

**Solution:** There are 8 observations total.

- Lowest value (minimum) : 2
- $Q1 : \frac{7+17}{2} = 12$ *(Note that this is the median of the first half of the data - 2, 7, 17, 19)*
- $Q2 : \frac{19+25}{2} = 22$ *(Note that this is the median of the full set of data)*
- $Q3 : 26$ *(Note that this is the median of the second half of the data - 25, 26, 26, 32)*
- Upper value (maximum) : 32

**Example B**

Compute the five number summary for the following data:

4, 8, 11, 11, 12, 14, 16, 20, 21, 25

**Solution:** There are 10 observations total.

- Lowest value (minimum) : 4
- $Q1 : 11$ *(Note that this is the median of the first half of the data - 4, 8, 11, 11, 12)*
- $Q2 : \frac{12+14}{2} = 13$ *(Note that this is the median of the full set of data)*
- $Q3 : 20$ *(Note that this is the median of the second half of the data - 14, 16, 20, 21, 25)*
- Upper value (maximum) : 25

**Example C**

Compute the five number summary for the following data:

3, 7, 10, 14, 19, 19, 23, 27, 29

**Solution:** There are 9 observations total. To calculate $Q1$ and $Q3$, you should include the median in both the lower half and upper half calculations.

- Lowest value (minimum) : 3
- $Q1 : 10$ *(this is the median of 3, 7, 10, 14, 19)*
- $Q2 : 19$
- $Q3 : 23$ *(this is the median of 19, 19, 23, 27, 29)*
- Upper value (maximum) : 29

**Concept Problem Revisited**

To compute the five number summary, it helps to order the data.

0, 0, 1, 2, 13, 27, 61, 63

- Since there are 8 observations, the median is the average of the $4^{th}$ and $5^{th}$ observations: $\frac{2+13}{2} = 7.5$
- The lowest observation is 0.
- The highest observation is 63.
- The middle of the lower half is $\frac{0+1}{2} = 0.5$
- The middle of the upper half is $\frac{27+61}{2} = 44$

The five number summary is 0, 0.5, 7.5, 44, 63

## Vocabulary

The ***rank*** of an observation is the number of observations that are less than or equal to the value of that observation.

Data is divided into four parts by the **first quartile** $(Q1)$**, second quartile** $(Q2)$ and **third quartile** $(Q3)$**.** The **second quartile** is also known as the median.

## Guided Practice

1. Create a set of data that meets the following five number summary:

{2, 5, 9, 18, 20}

2. Compute the five number summary for the following data:

1, 1, 1, 2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 6, 6, 7, 8, 9, 10, 15

3. Compute the five number summary for the following data:

1, 4, 96, 356, 2557, 9881, 14420, 20100

**Answers:**

1. Suppose there are 8 data points. The lowest point must be 2 and the highest point must be 20. The middle two points must average to be 9 so they could be 8 and 10. The second and third points must average to be 5 so they could be 4 and 6. The sixth and seventh points need to average to be 18 so they could be 18 and 18. Here is one possible answer:

2, 4, 6, 8, 10, 18, 18, 20

2. There are 20 observations.

- Lower : 1
- $Q1 : \frac{2+3}{2} = 2.5$
- $Q2 : \frac{4+5}{2} = 4.5$
- $Q3 : \frac{6+7}{2} = 6.5$
- Upper : 15

3. There are 8 observations.

- Lower : 1
- $Q1 : \frac{4+96}{2} = 50$
- $Q2 : \frac{356+2557}{2} = 1456.5$
- $Q3 : \frac{9881+14420}{2} = 12150.5$
- Upper: 20100

## Practice

Compute the five number summary for each of the following sets of data:

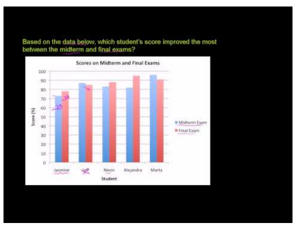
1. 0.16, 0.08, 0.27, 0.20, 0.22, 0.32, 0.25, 0.18, 0.28, 0.27
2. 77, 79, 80, 86, 87, 87, 94, 99
3. 79, 53, 82, 91, 87, 98, 80, 93
4. 91, 85, 76, 86, 96, 51, 68, 92, 85, 72, 66, 88, 93, 82, 84
5. 335, 233, 185, 392, 235, 518, 281, 208, 318
6. 38, 33, 41, 37, 54, 39, 38, 71, 49, 48, 42, 38
7. 3, 7, 8, 5, 12, 14, 21, 13, 18
8. 6, 22, 11, 25, 16, 26, 28, 37, 37, 38, 33, 40, 34, 39, 23, 11, 48, 49, 8, 26, 18, 17, 27, 14
9. 9, 10, 12, 13, 10, 14, 8, 10, 12, 6, 8, 11, 12, 12, 9, 11, 10, 15, 10, 8, 8, 12, 10, 14, 10, 9, 7, 5, 11, 15, 8, 9, 17, 12, 12, 13, 7, 14, 6, 17, 11, 15, 10, 13, 9, 7, 12, 13, 10, 12
10. 49, 57, 53, 54, 49, 67, 51, 57, 56, 59, 57, 50, 49, 52, 53, 50, 58
11. 18, 20, 24, 21, 5, 23, 19, 22
12. 900, 840, 880, 880, 800, 860, 720, 720, 620, 860, 970, 950, 890, 810, 810, 820, 800, 770, 850, 740, 900, 1070, 930, 850, 950, 980, 980, 880, 960, 940, 960, 940, 880, 800, 850, 880, 760, 740, 750, 760, 890, 840, 780, 810, 760, 810, 790, 810, 820, 850
13. 13, 15, 19, 14, 26, 17, 12, 42, 18
14. 25, 33, 55, 32, 17, 19, 15, 18, 21
15. 149, 123, 126, 122, 129, 120

# 15.4 Graphic Displays of Data

Here you will display data using bar charts, histograms, pie charts and boxplots.

Two common types of graphic displays are bar charts and histograms. Both bar charts and histograms use vertical or horizontal bars to represent the number of data points in each category or interval. The main difference graphically is that in a bar chart there are spaces between the bars and in a histogram there are not spaces between the bars. Why does this subtle difference exist and what does it imply about graphic displays in general?

### Watch This



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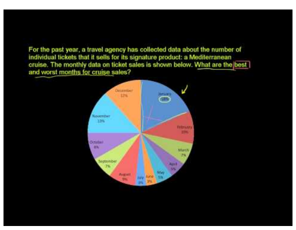
http://www.youtube.com/watch?v=kiQ6MUQZHSs Khan Academy: Reading Bar Graphs



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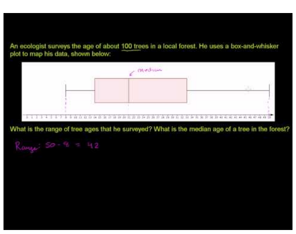
http://www.youtube.com/watch?v=4JqH55rLGKY Khan Academy: Reading Pie Graphs



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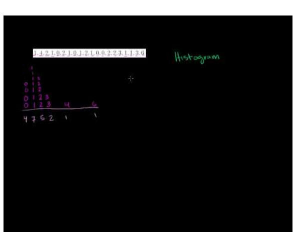
http://www.youtube.com/watch?v=b2C9I8HuCe4 Khan Academy: Reading Box-and-Whisker Plots



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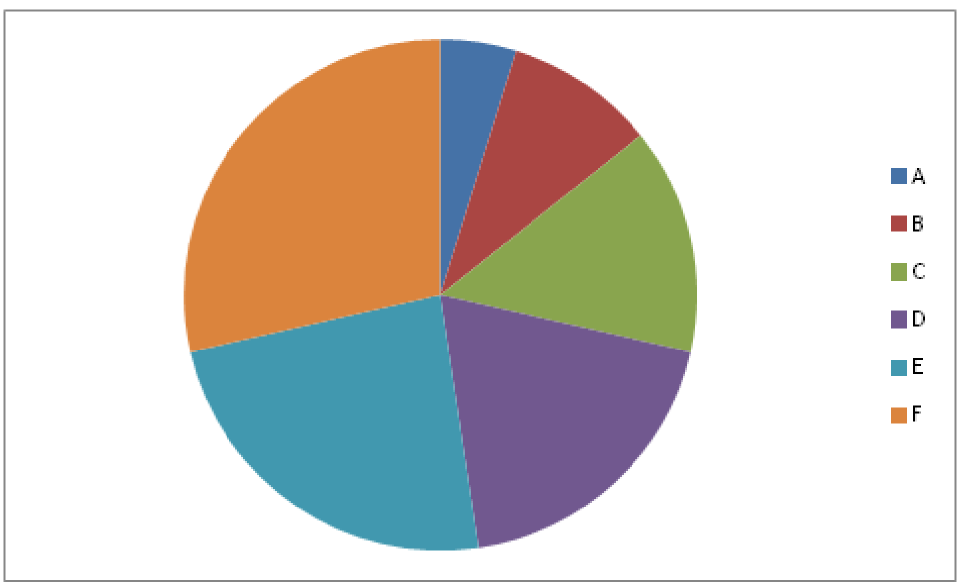
http://www.youtube.com/watch?v=4eLJGG2Ad30 Khan Academy: Histogram

### Guidance

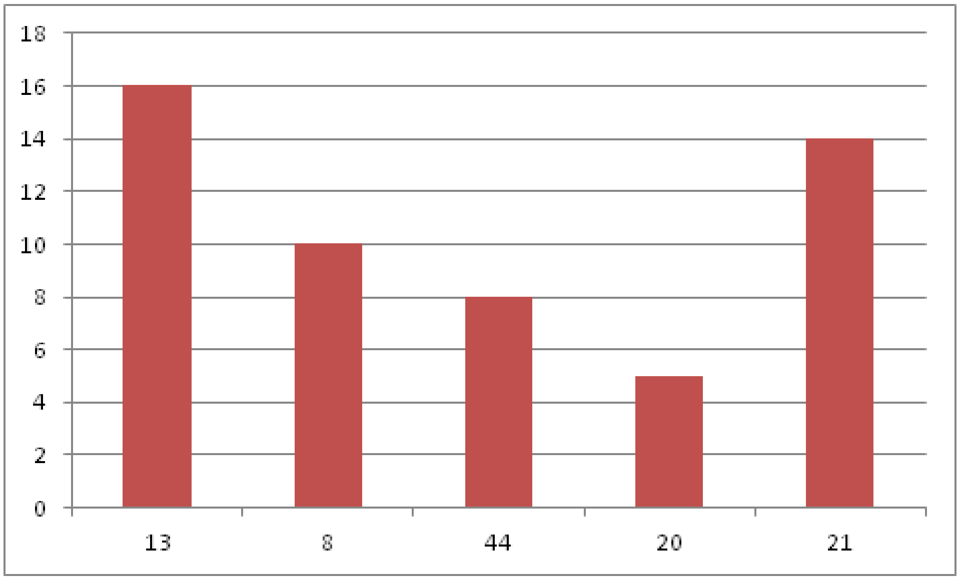
It is often easier for people to interpret relative sizes of data when that data is displayed graphically. There are a few common ways of displaying data graphically that you should be familiar with.

1) A **pie chart** shows the relative proportions of data in different categories. Pie charts are excellent ways of displaying categorical data with easily separable groups. The following pie chart shows six categories labeled $A-F$. The size of each pie slice is determined by the central angle. Since there are 360° in a circle, the size of the central angle $\theta_A$ of category $A$ can be found by:

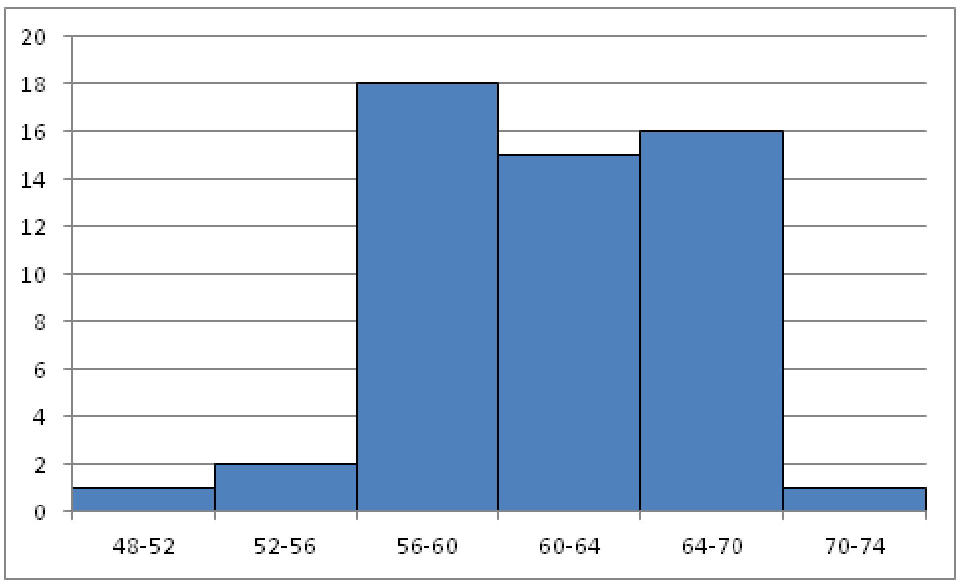
$$\frac{\theta_A}{360} = \frac{\text{data points in category } A}{\text{Total number of data points}}$$



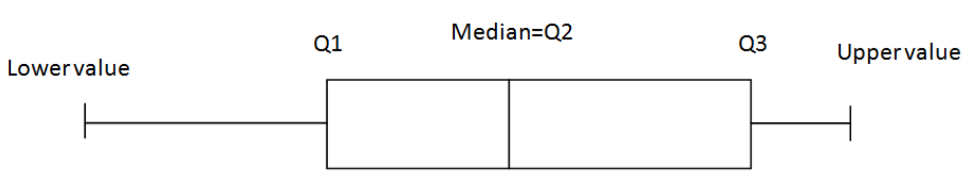

2) A **bar chart** displays frequencies of categories of data. The bar chart below has 5 categories, and shows the TV channel preferences for 53 adults. The horizontal axis could have also been labeled *News, Sports, Local News, Comedy, Action Movies.* The reason why the bars are separated by spaces is to emphasize the fact that they are categories and not continuous numbers. For example, just because you split your time between channel 8 and channel 44 does not mean on average you watch channel 26. Categories can be numbers so you need to be very careful.




3) A **histogram** displays frequencies of quantitative data that has been sorted into intervals. The following is a histogram that shows the heights of a class of 53 students. Notice the largest category is 56-60 inches with 18 people.




4) A **boxplot** (also known as a **box and whiskers plot**) is another way to display quantitative data. It displays the five 5 number summary (minimum, $Q1$, median, $Q3$, maximum). The box can either be vertically or horizontally displayed depending on the labeling of the axis. The box does not need to be perfectly symmetrical because it represents data that might not be perfectly symmetrical.

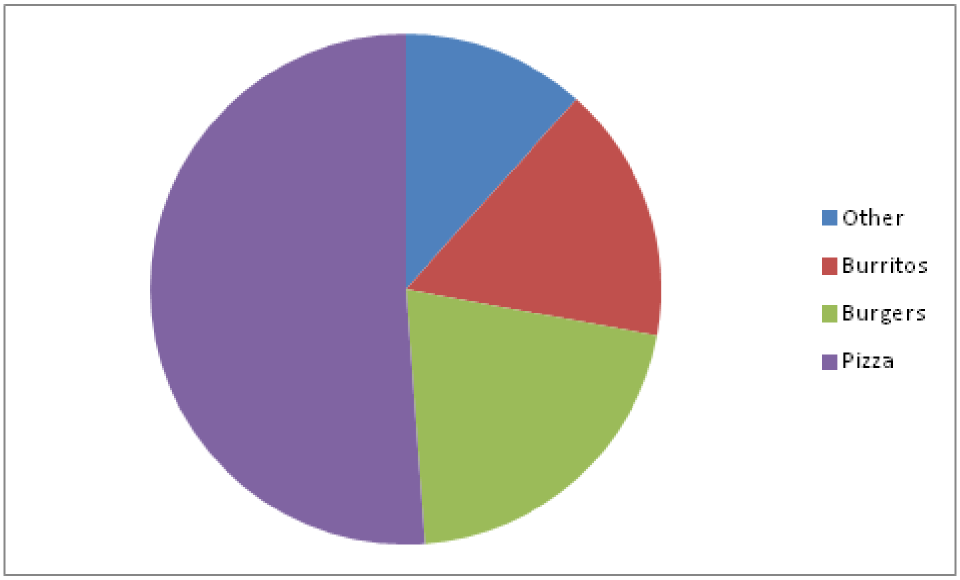



#### Example A

Create a pie chart to represent the preferences of 43 hungry students.

- Other – 5
- Burritos – 7
- Burgers – 9
- Pizza – 22

**Solution:**

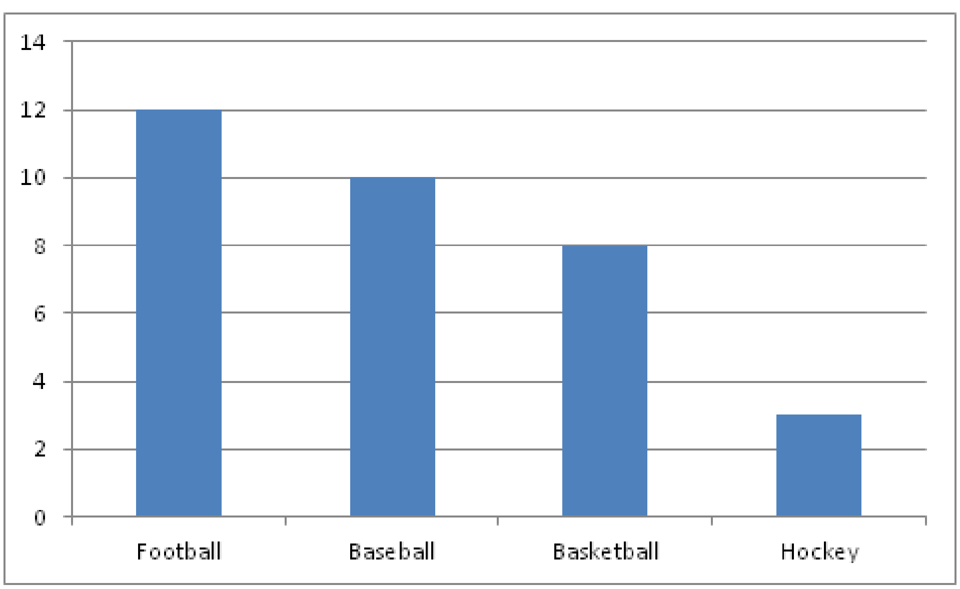



### Example B

Create a bar chart representing the preference for sports of a group of 23 people.

- Football – 12
- Baseball – 10
- Basketball – 8
- Hockey – 3

**Solution:**

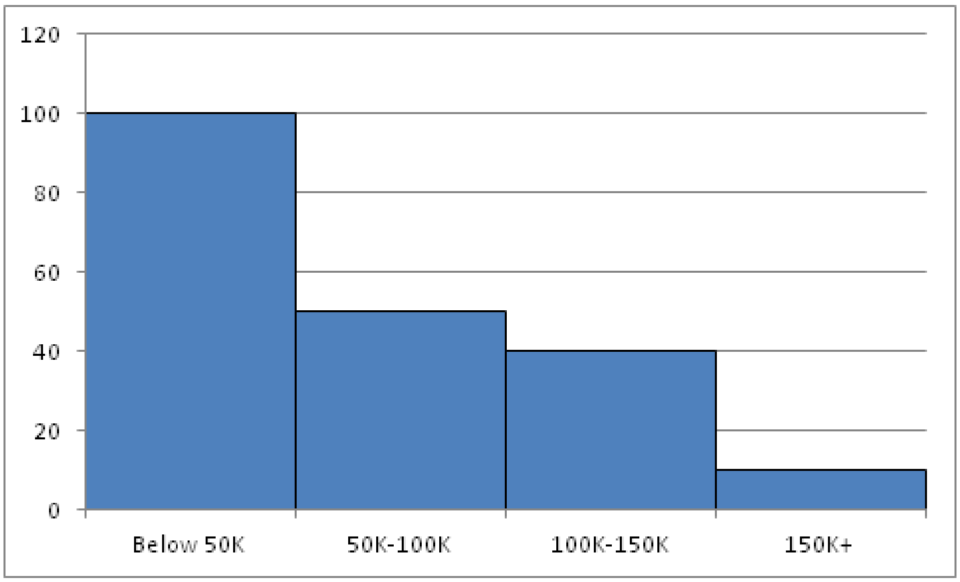


**Example C**

Create a histogram for the income distribution of 200 million people.

- Below $50,000 is 100 million people
- Between $50,000 and $100,000 is 50 million people
- Between $100,000 and $150,000 is 40 million people
- Above $150,000 is 10 million people

**Solution:**




**Concept Problem Revisited**

The reason for the space in bar charts but no space in histograms is bar charts graph categorical variables while histograms graph quantitative variables. It would be extremely improper to forget the space with bar charts because you would run the risk of implying a spectrum from one side of the chart to the other. Note that in the bar chart where TV stations where shown, the station numbers were not listed horizontally in order by size. This was to emphasize the fact that the stations were categories.

## Vocabulary

A ***categorical variable*** is a variable that can take on one of a limited number of values. Examples of categorical variables are tv stations, the state someone lives in, and eye color.

A ***quantitative variable*** is a variable that takes on numerical values that represent a measureable quantity. Examples of quantitative variables are the height of students or the population of a city.

A ***bar chart*** is a graphic display of categorical variables that uses bar to represent the frequency of the count in each category.

A ***histogram*** is a graphic display of quantitative variables that uses bars to represent the frequency of the count of the data in each interval.

A ***pie chart*** is a graphic display of categorical data where the relative size of each pie slice corresponds to the frequency of each category.

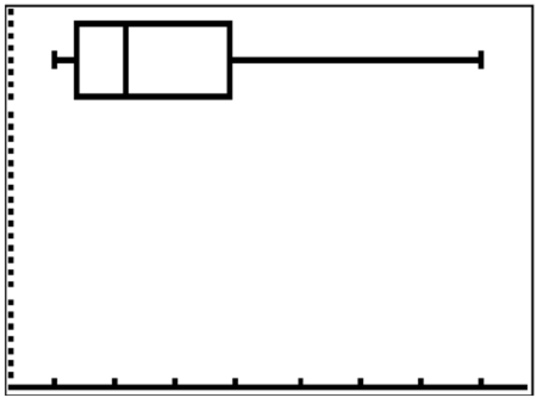
A ***boxplot*** is a graphic display of quantitative data that demonstrates the five number summary.

### Guided Practice

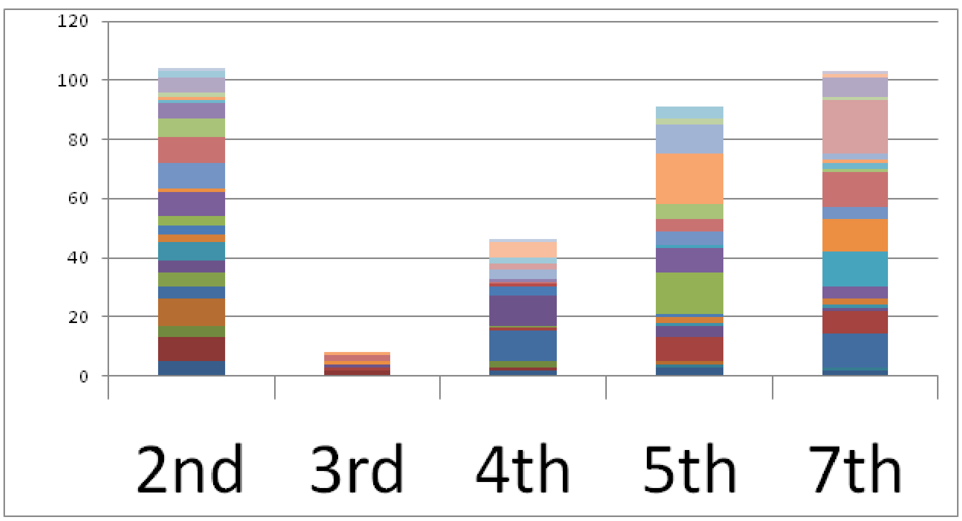
1. Create a boxplot of the following numbers in your calculator.

8.5, 10.9, 9.1, 7.5, 7.2, 6, 2.3, 5.5

2. Identify the interesting characteristics of the following boxplot.

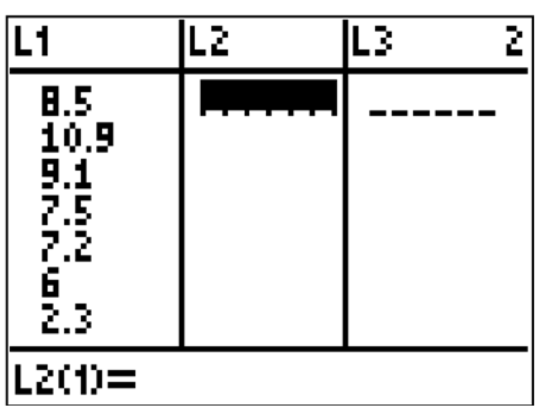


3. Interpret the following bar chart that represents the number of tardy students in 5 class periods over the course of a year.

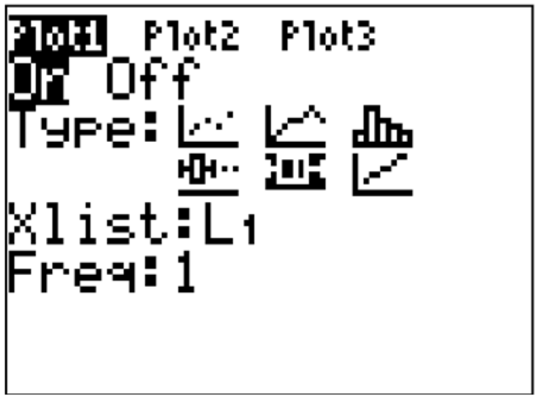



**Answers:**

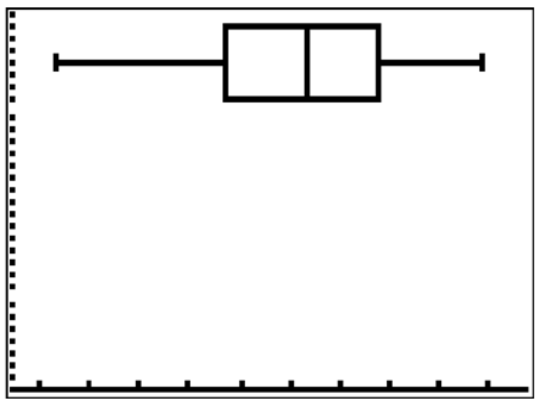
1. Enter the data into $L_1$ by going into the Stat menu.



Then turn the statplot on and choose boxplot.




Use Zoomstat to automatically center the window on the boxplot.



2. The lower bound, $Q1$ and $Q2$ all seem to be relatively close together. $Q3$ seems to be stretched a little to the right and the upper bound is significantly stretched to the right.

3. The bar chart has 5 categories representing each of the five periods. Within each category there are bands of different colors. Each band represents the number of times an individual student was tardy. For periods 5 and periods 7 there seem to be fewer students who were tardy more often. In period 2 there seems to be more students tardy a handful of times each.

### Practice

1. What types of graphs show categorical data?
2. What types of graphs show quantitative data?

A math class of 30 students had the following grades:

**TABLE 15.9:**

| Grade | Number of Students with Grade |
|---|---|
| A | 10 |
| B | 10 |
| C | 5 |
| D | 3 |
| F | 2 |

3. Create a bar chart for this data.

4. Create a pie chart for this data.

5. Which graph do you think makes a better visual representation of the data?

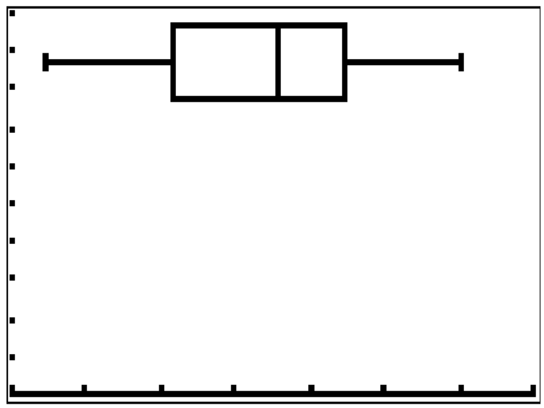
A set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98

6. Create a histogram for this data. Use your best judgment to decide what the intervals should be.

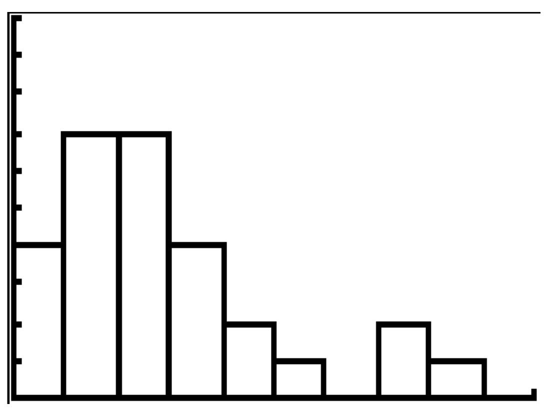
7. Find the five number summary for this data.

8. Use the five number summary to create a boxplot for this data.

9. Describe the data shown in the boxplot below.



10. Describe the data shown in the histogram below.



A math class of 30 students has the following eye colors:

**TABLE 15.10:**

| Grade | Number of Students with Grade |
|---|---|
| Brown | 20 |
| Blue | 5 |
| Green | 3 |
| Other | 2 |

11. Create a bar chart for this data.

12. Create a pie chart for this data.

13. Which graph do you think makes a better visual representation of the data?

14. Suppose you have data that shows the breakdown of registered republicans by state. What types of graphs could you use to display this data?

15. From which types of graphs could you obtain information about the spread of the data? Note that spread is a measure of how spread out all of the data is.

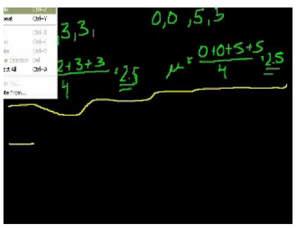
# 15.5 Variance

Here you will calculate population variance, sample variance and standard deviation from univariate data.

Two groups of students that each have an average test score of 75 might have a score distribution that looks remarkably different. One class might be made up entirely of grades between 72 and 78 while the other class may have half the group around 50, with the other half getting near 100. Variance is a way of measuring the variation in a set of data. What is the mean and variance for the following sample test scores taken from a larger student population?

75, 73, 78, 90, 60, 51, 87, 79, 80, 77

### Watch This



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http://www.youtube.com/watch?v=6JFzI1DDyyk Khan Academy: Statistics: Variance of a Population

### Guidance

The thought process of a person trying to describe the spread of some data for the first time must have been something like this.

*Well, the average is 75. What if I try to just add up how different each number is from 75?*

As the person calculates the numbers, they realize pretty quickly that this sum will be zero, essentially by definition. This is because the numbers that occur below 75 precisely cancel out with the numbers above 75.

*Since I cannot add the differences directly, why don't I just sum the absolute value of the differences?*

This is a legitimate method for describing the spread of data. It is called absolute deviation and is simply the sum of the absolute values of each of the differences.

*If I take the average absolute difference, I will be able to judge on average how far away each data point is from the mean. A larger difference means more spread out.*

If you take the average of the absolute deviation, you get the mean absolute deviation. The mean absolute variation is a legitimate, but limited, way of describing the spread of data. Eventually, a person trying to describe the spread of data for the first time might consider a method called population variance.

*What if instead of using absolute value to solve the issue, I square each difference and then add them together? Of course I'd have to divide by the number of data points to get the average difference squared*

This method turns out to be extraordinarily powerful in statistics. One downside is that most of the time you cannot get data from the entire population, you usually only get it from a sample. Over time people realized that samples were typically less variable than their populations and dividing by the number of data points was consistently underestimating the true variance of the population. In other words, if $n$ is the size of the sample then multiplying the sum of the square differences by $\frac{1}{n}$ makes the variance too small. Research and theory progressed until it was realized that multiplying the sum of the square differences by $\frac{1}{n-1}$ made the fraction slightly larger and properly estimated the variance of the population. Thus, there are two ways to calculate variance, one for populations and one for samples.

*Hey wait, by squaring the differences, doesn't that mean that the units are squared? What if I want to describe the spread in the regular units? Should I just take the square root of the variance?*

This is why the Greek letter lowercase sigma, $\sigma$, is used for standard deviation of a population (which is the square root of the variance) and $\sigma^2$ is the symbol for variance of a population. The letters $s$, $s^2$ are used for sample standard deviation and sample variance. The Greek letter mu, $\mu$, is the symbol used for mean of a population, while $\bar{x}$ is the symbol used for mean of a sample.

**Mean and variance for the population:** $x_1, x_2, x_3, \ldots, x_n$

$$\mu = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$$
$$\sigma^2 = \frac{1}{n} \cdot \sum_{i=1}^{n} (\mu - x_i)^2$$

**Mean and variance for a sample from a population:** $x_1, x_2, x_3, \ldots, x_m$

$$\bar{x} = \frac{1}{m} \cdot \sum_{i=1}^{m} x_i$$
$$s^2 = \frac{1}{m-1} \cdot \sum_{i=1}^{m} (\bar{x} - x_i)^2$$

Remember that variance is a measure of the spread of data. The bigger the variance, the more spread out the data points.

**Example A**

Calculate the variance and mean for rolling a fair six sided die.

**Solution:** $\mu = \frac{1}{6}(1+2+3+4+5+6) = \frac{1}{6} \cdot 21 = 3.5$

Since the population for a six sided die is entirely known, you would use the population variance.

$$\begin{aligned}\sigma^2 &= \frac{1}{6}\left[(3.5-1)^2 + (3.5-2)^2 + (3.5-3)^2 + (3.5-4)^2 + (3.5-5)^2 + (3.5-6)^2\right] \\ &= \frac{1}{6}[6.25 + 2.25 + 0.25 + 0.25 + 2.25 + 6.25] \\ &\approx 2.9167\end{aligned}$$

**Example B**

Calculate the mean and variance of the following data sample of lap times.

59.8, 57.1, 58.2, 58.6, 57.8, 57.9, 58.0, 57.3

**Solution:** $\bar{x} = \frac{1}{8}(59.8+57.1+58.2+58.6+57.8+57.9+58.0+57.3) = 58.0875$

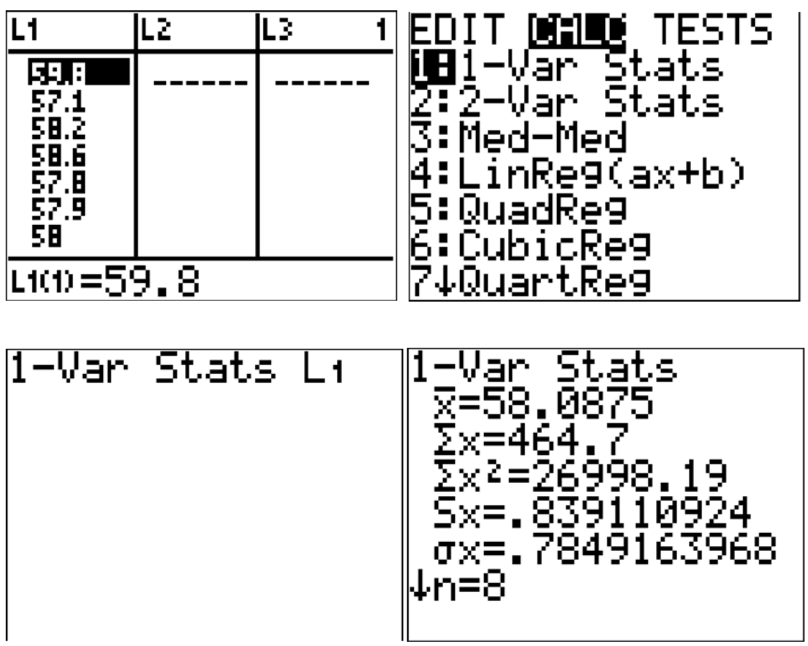
This is a *sample*, so you should use the sample variance formula.

$$\begin{aligned} s^2 &= \frac{1}{8-1} \cdot \left[(\mu-59.8)^2+(\mu-57.1)^2+(\mu-58.2)^2+(\mu-58.6)^2+(\mu-57.8)^2 \right. \\ &\quad \left. +(\mu-57.9)^2+(\mu-58.0)^2+(\mu-57.3)^2\right] \\ &= \frac{1}{7}\left[(-1.7125)^2+0.9875^2+(-0.1125)^2+(-0.5125)^2+0.2875^2+0.1875^2+0.0875^2+0.7875^2\right] \\ &\approx \frac{1}{7}[2.9327+0.9751+0.0126+0.2626+0.0826+0.0351+0.0076+0.6201] \\ &\approx \frac{1}{7}[4.9288] \\ &\approx 0.7041 \end{aligned}$$

**Example C**

Use a calculator to calculate the variance from Example B.

**Solution:** To calculate variance on your calculator, enter the data in a list, choose 1-Var Stats and run the 1-Var Stats on the list you entered the data.




The two outputs that are important for you to interpret are:

$Sx = 0.839110924$

$\sigma x = 0.7848163968$

Since the calculator does not know whether the data is a population or a sample, it produces both. Since this problem is about a sample, the number of interest is $Sx$. This number does not match the variance from Example B because it is the sample standard deviation which means it is the square root of the sample variance. The calculator produces standard deviation. You need to square that number to produce the appropriate variance.

$0.8391^2 \approx 0.7041$

**Concept Problem Revisited**

The mean of the test scores is 75. The variance is calculated by taking the difference of each number from the mean, squaring and summing these differences.

$0^2+2^2+3^2+15^2+15^2+24^2+12^2+4^2+5^2+2^2=1228$

Since this data is a sample, you divide the sum by one fewer than the number of terms.

$\frac{1228}{10-1}\approx 136.4444$

If you knew the variances for two samples, each from a different class, you could quickly determine which class had test scores that were more spread out.

## Vocabulary

***Variance*** is a measure of how spread out the data is.

The square root of the variance is the ***standard deviation***.

Both the variance and the standard deviation can be calculated from a ***sample*** or from the whole ***population***. The formulas are slightly different in each case so it is important to know whether your data is just a sample or is from the whole population.

The ***absolute deviation*** is the sum total of how different each number is from the mean.

The ***mean absolute deviation*** is an alternate measure of how spread out the data is. While this method might seem more intuitive, in statistics it has been found to be too limited and is not commonly used.

## Guided Practice

1. Calculate the standard deviation for the following 6 numbers by hand. Assume the numbers are a population.

2, 4, 6, 8, 12, 19

2. Use a spreadsheet to organize your calculations for computing the variance of the following numbers. Assume these numbers are a true population.

14, 15, 7, 15, 2, 0, 6, 5, 12, 3

**Answers:**

1.

$$\begin{aligned}
\mu &= \frac{1}{6}(2+4+6+8+12+17)=8\\
\sigma^2 &= \frac{1}{6}((8-2)^2+(8-4)^2+(8-6)^2+0+(8-12)^2+(8-17)^2)\\
&= \frac{1}{6}(6^2+4^2+2^2+4^2+9^2)\\
&= \frac{1}{6}(36+16+4+16+81)\\
&= \frac{1}{6}(153)\\
&= 25.5\\
\sigma &\approx 5.0498
\end{aligned}$$

2. After entering the data in a column, you can use the power of the embedded programming of the spreadsheet to make a second column of just the average.

- The average command is: "= average(A2:A11)"

You can subtract one cell from another cell to find the difference. You can then square the difference to find the difference squared. You can then sum these values using the sum command.

- The sum command is: "= sum(D2:D11)"

Finally, just divide the sum by the number of observations (which is 10) to get the variance.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | Data | Average | Difference | Difference squared | | Sum of difference squared | | |
| 2 | 14 | 7.9 | -6.1 | 37.21 | | 288.9 | | |
| 3 | 15 | 7.9 | -7.1 | 50.41 | | | | |
| 4 | 7 | 7.9 | 0.9 | 0.81 | | | | |
| 5 | 15 | 7.9 | -7.1 | 50.41 | | Variance | | |
| 6 | 2 | 7.9 | 5.9 | 34.81 | | 28.89 | | |
| 7 | 0 | 7.9 | 7.9 | 62.41 | | | | |
| 8 | 6 | 7.9 | 1.9 | 3.61 | | | | |
| 9 | 5 | 7.9 | 2.9 | 8.41 | | | | |
| 10 | 12 | 7.9 | -4.1 | 16.81 | | | | |
| 11 | 3 | 7.9 | 4.9 | 24.01 | | | | |

### Practice

1. What are the similarities and differences between standard deviation and variance?

2. Data Set A has a mean of 30 and a standard deviation of 10. Data Set B also has a mean of 30, but a standard deviation of 2. What does this mean about Data Set A compared to Data Set B?

Calculate the variance of each set of data by hand.

3. Sample: 1, 4, 7, 10, 3, 6, 12, 5, 8, 16, 21, 3, 1, 5

4. Population: 23, 27, 19, 24, 20, 22, 31, 30, 28

5. Sample: 64, 62, 60, 58, 54, 60, 61, 63, 47, 100, 29, 59

Calculate the variance of each set of data using your calculator. Compare your answers to your answers to 3-5.

6. Sample: 1, 4, 7, 10, 3, 6, 12, 5, 8, 16, 21, 3, 1, 5

7. Population: 23, 27, 19, 24, 20, 22, 31, 30, 28

8. Sample: 64, 62, 60, 58, 54, 60, 61, 63, 47, 100, 29, 59

9. If $\sigma^2 = 16$, what is the population standard deviation?

10. Which data set has the largest standard deviation?

a. 10 10 10 10 10
b. 0 0 10 10 10

c. 0 9 10 11 20
d. 20 20 20 20 20

11. What will a large variance look like on a histogram? What will a small variance look like on a histogram?

12. You find some data organized in a bar graph. Could you calculate the variance of this data? Explain.

13. A sample set of 20 exam scores is 67, 94, 88, 76, 85, 93, 55, 87, 80, 81, 80, 61, 90, 84, 75, 93, 75, 68, 100, 98. Calculate the mean, variance, and standard deviation for this data.

14. All of Mike's bowling scores are: 1, 1, 2, 10, 12, 1, 9, 6, 7, 8, 4, 3, 4, 1, 4, 1, 6, 7, 11, 5. Calculate the mean, variance, and standard deviation for this data.

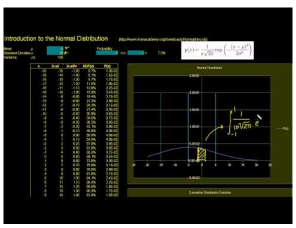
15. Why can't you always calculate the population variance and standard deviation? Why do you sometimes have to calculate the sample variance and standard deviation?

# 15.6 The Normal Curve

Here you will define the standard normal distribution and learn how standard deviation and the area under the curve are connected.

When students ask their teachers to curve exams, what they often mean is they want everyone to simply get a higher grade. Curving a grade can also mean fitting to a bell curve where lots of people get Cs, some people get Ds and Bs and very few people get As and Fs. Even though this second interpretation is not what most students mean, the normal curve is one of the most widely used and applied probability distributions. What are other examples that follow a normal distribution?

### Watch This

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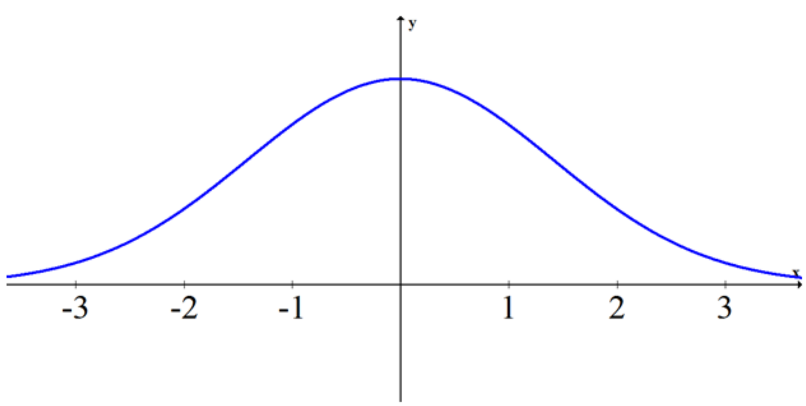
Click image to the left for more content.

http://www.youtube.com/watch?v=hgtMWR3TFnY Khan Academy: Introduction to the Normal Distribution

### Guidance

The **Standard Normal Distribution** is graphed from the following function and is represented by the Greek letter phi, $\phi$.

$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$




This distribution represents a population with a mean of 0 and a standard deviation of 1. The numbers along the $x$-axis represent standard deviations. For data that is normally distributed, the **empirical rule** states that:

- Approximately 68% of the data will be within 1 standard deviation of the mean.

- Approximately 95% of the data will be within 2 standard deviations of the mean.
- Approximately 99.7% of the data will be within 3 standard deviations of the mean.

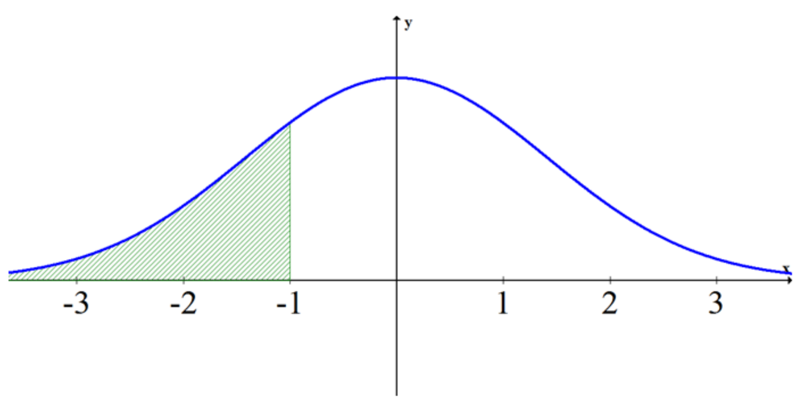
Some other important points about the normal distribution:

- The total area between the normal curve and the $x$ axis is 1 and this area represents all possible probabilities.
- If data is distributed normally, you can use the normal distribution to determine the percentage of the data between any two values by calculating the area under the curve between those two values. *When you take calculus, you will learn how to calculate this area analytically, but for now you can use the normalcdf function on your calculator.*
- Many histograms approximate a normal curve, but a true normal curve is infinitely smooth.

**Example A**

The amount of rain each year in Connecticut follows a normal distribution. What is the probability of getting one standard deviation below the normal amount of rain?

**Solution:** You are looking for the area of the shaded portion of the normal distribution shown below. By the empirical rule, you know that approximately 34% of the data is in between -1 and 0. Also, 50% of the data is above 0. Therefore, approximately 84% of the data is unshaded. Therefore, $100\% - 84\% = 16\%$ of the data is shaded. The approximate probability is 16%.

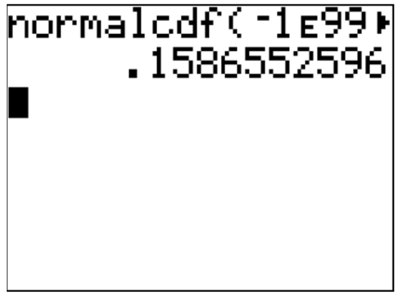



To get the exact probability, use the normal cdf function on your calculator to calculate the exact area under the curve. Go to [DISTR] (which is $[2^{nd}]$ [VARS]) and choose normalcdf. This is the *normal cumulative distribution function* and calculates the area under the curve between two $x$-values. The syntax (how you will type it in) for normal cdf is:

normalcdf(lower, upper, mean, standard deviation)

The lower bound for this shaded region is technically $-\infty$, but the TI-84 cannot handle that so use -1E99. -1E99 is $-1 \times 10^{99}$, an extremely small number, and will give identical results that are correct to many decimal places. The upper bound is -1. For a standard normal distribution with a mean of zero and a standard deviation of 1 you don't need to type anything else in, but since you will be working with normal distributions with means and standard deviations that are different, it will make sense to get used to using the whole syntax.

normalcdf(-1E99, -1, 0, 1)

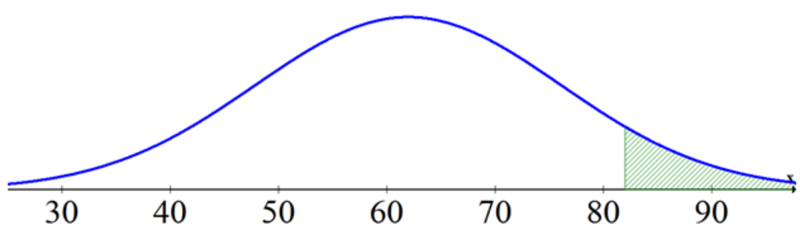



The exact answer is closer to 15.87%.

**Example B**

On your first college exam, you score an 82. After the exam the professor tells the class that the mean was a 62 and the standard deviation was 10. What percentage of the class did better than you?

**Solution:** An 82 is 20 away from the mean so is 2 standard deviations from the mean. Therefore, this question is asking for the percentage of students that are above +2 standard deviations above the mean.




In future statistics courses you will learn how to create the equation for this distribution and then transform it to standard normal. For now, you can use the fact that your score was exactly 2 standard deviations above the mean. Or, you can calculate the probability using the actual numbers.

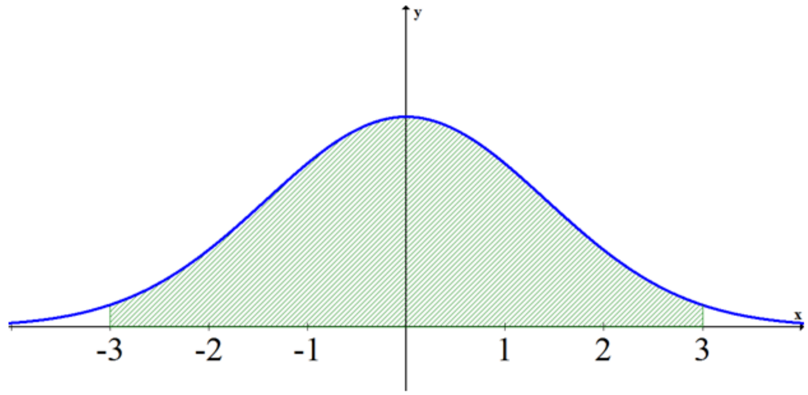
- normalcdf(2, 1E99, 0, 1) = 0.022750 or 2.750%
- normalcdf(82, 1E99, 62, 10) = 0.022750 or 2.750%

2.75% of the class did better than you on the exam. Even though you seemed to score a B-, the professor would probably note that you were near the top of the class and adjust grades accordingly.

**Example C**

The quality control technician of a widget making factory observes that widgets that are three standard deviations too large or three standard deviations too small from the precise widget size are unusable. What is the probability of producing a usable widget?

**Solution:** This question is essentially asking for the area between -3 standard deviations and positive 3 standard deviations. The empirical rule says this should be 99.7%. Use the normalcdf function to find the exact value.




normalcdf(-3, 3, 0, 1) = 0.997300 or 99.73%

The quality control technician would decide if this is a high enough success rate for producing a usable widget.

**Concept Problem Revisited**

Height, weight and other measures of people, animals or plants are normally distributed.

### Vocabulary

A ***standard normal distribution*** is a normal distribution with mean of 0 and a standard deviation of 1.

The ***empirical rule*** states that for data that is normally distributed, approximately 68% of the data will fall within one standard deviation of the mean, approximately 95% of the data will fall within two standard deviations of the mean, and approximately 99.7% of the data will fall within three standard deviations of the mean. It is a good way to quickly approximate probabilities.

***Normalcdf*** is the normal cumulative distribution function and calculates the area between any two values for data that is normally distributed as long as you know the mean and standard deviation for the data. Your calculator has this function built in, and it produces an exact answer as opposed to the empirical rule.

### Guided Practice

1. What is the probability that a person in Texas is exactly 6 feet tall?

2. Two percent of high school football players are invited to play at a competitive college level. How many standard deviations above the average player would someone need to be to have this opportunity?

3. On average, a pumpkin at your local farm weighs 10 pounds with a standard deviation of 6 pounds. You go and find a pumpkin weighing 26 pounds. Of all the pumpkins at the farm, what percent weigh less than this enormous pumpkin?

**Answers:**

1. Since height is a continuous variable, meaning any number within a reasonable domain interval is possible, the probability of choosing any single number is zero. Many people may be close to 6 feet tall, but in reality they are 5.9 or 6.0001 feet tall. There must be someone in Texas who is the closest to being exactly 6 feet tall, but even that person when measured accurately enough will still be slightly off from 6 feet. This is why instead of calculating the probability for a single outcome, you calculate the probability between a certain interval, like between 5.9 feet and 6.1 feet. For continuous variables, the probability of any specific outcome, like 6 feet, will always be 0.

2. This situation is the inverse of the previous questions. Instead of being given the standard deviation and asked to find the probability, you are given the probability and asked to find the standard deviation.

There is a second programmed feature in the distribution menu that performs this calculation. You are looking for how many standard deviations above the mean will include 98% of the data.

invNorm(0.98) = 2.0537

A person would have to be greater than about 2 standard deviations above the mean to be in the top 2 percent.

3. Normalcdf(-1E99, 26, 10, 6) = 0.9961 or 99.61%

The vast majority of the pumpkins weigh less than the 26 pound pumpkin you found.

### Practice

Consider the standard normal distribution for the following questions.

1. What is the mean?

2. What is the standard deviation?

3. What is the percentage of the data below 1?

4. What is the percentage of the data below -1?

5. What is the percentage of the data above 2?

6. What is the percentage of the data between -2 and 2?

7. What is the percentage of the data between -0.5 and 1.7?

8. What is the probability of a value of 2?

Assume that the mean weight of 1 year old girls in the USA is normally distributed, with a mean of about 9.5 kilograms and a standard deviation of approximately 1.1 kilograms.

9. What percent of 1 year old girls weigh between 8 and 12 kilograms?

10. What percent of girls weigh above 12 kilograms?

11. Girls in the bottom 5% by weight need their weight monitored every 2 months. How many standard deviations below the mean would a girl need to be to have their weight monitored?

Suppose that adult women's heights are normally distributed with a mean of 65 inches and a standard deviation of 2 inches.

12. What percent of adult women have heights between 60 inches and 65 inches?

13. Use the empirical rule to describe the range of heights for women within one standard deviation of the mean.

14. What is the probability that a randomly selected adult woman is more than 64 inches tall?

15. What percent of adult women are either less than 60 inches or greater than 72 inches tall?

# 15.7 Linear Correlation

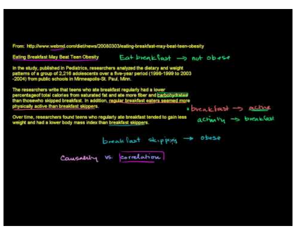
Here you will begin to work with bivariate data as you learn about linear correlation, correlation coefficients, and regression.

Statistics is largely concerned with how a change in one variable relates to changes in a second variable. Bivariate data is two lists of data that are paired up. Is there any relationship between the following data? If there is, does it mean that doctors cause cancer?

TABLE 15.11:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Number of Doctors | 27 | 30 | 36 | 60 | 81 | 90 | 156 | 221 | 347 |
| Cancer Rate | 0.02 | 0.07 | 0.16 | 0.20 | 0.43 | 0.87 | 1.21 | 2.80 | 3.91 |

### Watch This



MEDIA

Click image to the left for more content.

http://www.youtube.com/watch?v=ROpbdO-gRUo Khan Academy: Correlation vs. Causality

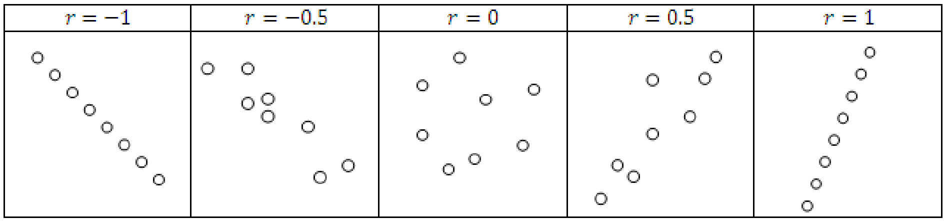
### Guidance

A **scatterplot** creates an $(x,y)$ point from each data pair. When making a scatterplot, you can try to assign the independent variable to $x$ and the dependent variable to $y$; however, it will often not be obvious which variable is the dependent variable, so you will just have to pick one.

Once you plot the data and zoom appropriately you will see the points scattered about. Sometimes there will be a clear linear relationship and sometimes it will appear random. The **correlation coefficient**, $r$, is a number that quantifies two aspects of the relationship between the data:

- The correlation coefficient is either negative, zero or positive. This tells you whether the data is negatively correlated, uncorrelated or positively correlated.
- The correlation coefficient is a number between $-1 \leq r \leq 1$ indicating the strength of correlation. If $r = 1$ or $r = -1$ then the data is perfectly linear. Note that a perfectly linear relationship includes lines with slopes other than 1.

Consider the examples below to see what different correlation coefficients will look like in data:




In PreCalculus you will not learn how to calculate the correlation coefficient (you will if you take future statistics courses!). For now, the calculator will calculate it for you and your job will be to interpret the result. See Example C.

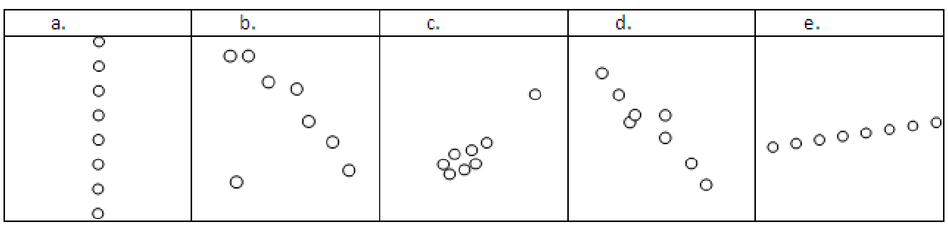
If the data is sufficiently linear, then your calculator can perform a regression to produce the equation of a line that attempts to model the trend of the data. The regression line may actually pass through all, some or none of the data points. This regression line is represented in statistics by:

$\hat{y} = a + bx$

The symbol $\hat{y}$ is pronounced "$y$-hat" and is the predicted $y$ value based on a given $x$ value. Occasionally, you may also calculate the predicted $x$ value given a $y$ value, however this is less mathematically sound. Also notice that the linear regression model is simply a rearrangement of the standard equation of a line, $y = mx + b$.

**Example A**

Estimate the correlation coefficient for the following scatterplots.

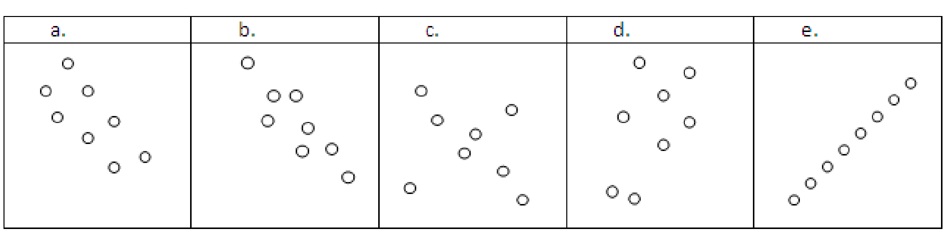



**Solution:**

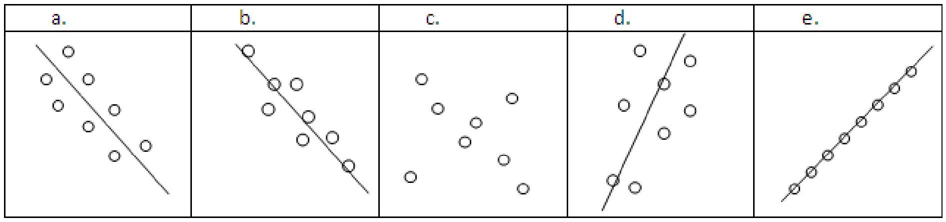
a. $r \approx 0$. Because the height $(y)$ does not seem to be dependent on the $x$, the data is uncorrelated. Another way to see this is that the slope appears to be undefined.
b. $r \approx -0.7$. If the solo point in the bottom left is an outlier, you could choose to not include it in the data. Then, the $r$ value would be closer to -1.
c. $r \approx +0.8$. The clump of data seems to be slightly positive correlated and the single point in the upper left has a strong effect indicating positive slope.
d. $r \approx -0.8$. The data seems to be fairly strongly negatively correlated.
e. $r \approx 1$. The data seems to be perfectly linearly correlated.

**Example B**

Estimate the regression line through the following scatterplots.



**Solution:** Visualize and sketch the "line of best fit" for each set of points.




Note that in part a, the regression line does not touch any point. Instead, it captures the general trend of the data. In part c, the correlation is not high enough in any direction to produce a regression line. The calculator may give a regression line for scatterplots that look like part c, but you need to be very skeptical that there is actually a relationship between the two variables.

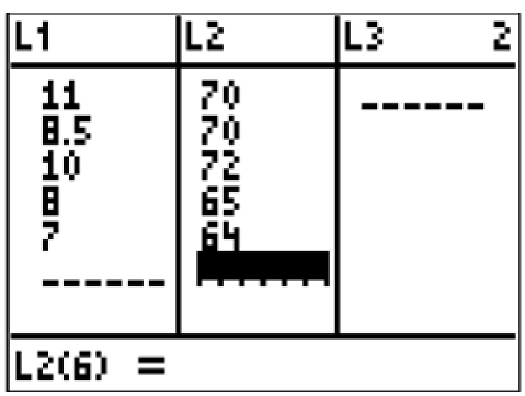
**Example C**

Use your calculator to perform a linear regression on the following data. Then, predict the height of someone who has shoe size 9.

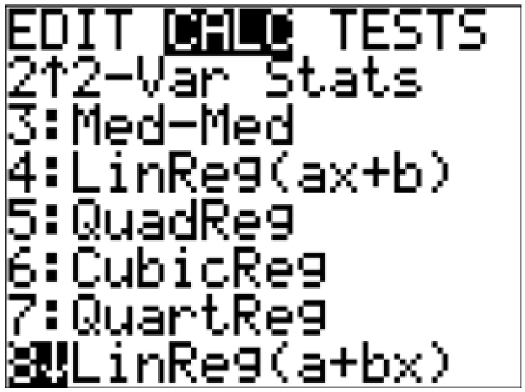
**TABLE 15.12:**

| Shoe Size | Height (in) |
|---|---|
| 11 | 70 |
| 8.5 | 70 |
| 10 | 72 |
| 8 | 65 |
| 7 | 64 |

**Solution:** First enter the data.



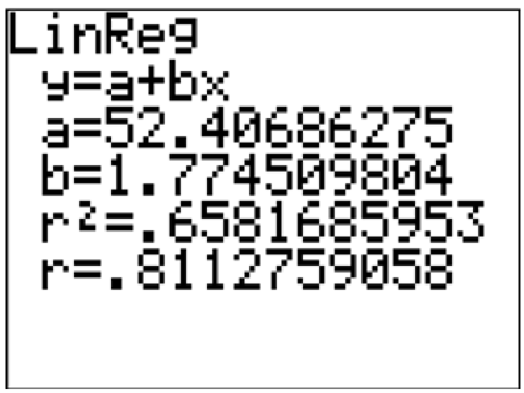

Next perform the regression. Notice that the calculator can perform linear regression in two ways that are essentially the same. To keep consistent with $\hat{y} = a + bx$, use linear regression. This is option 8 in the [STATS], [CALC] menu.




Now you need to tell the calculator to perform the regression on the two lists you want and where to copy the equation. The syntax is:

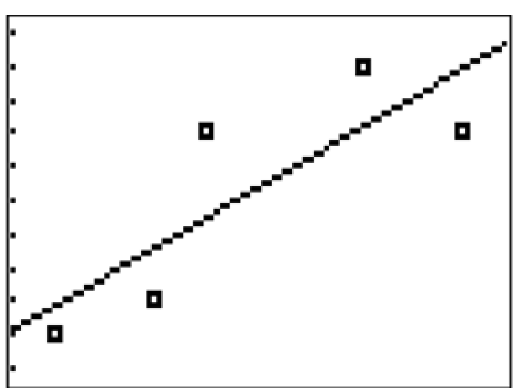
- LinReg$(a+bx)L_1,L_2,Y_1$

*Note: to find Y1, go to – [VARS], [Y-VARS], [FUNCTION], [Y1].*




Notice that the $r$ value is about 0.8. This indicates that there is a fairly strong positive correlation between shoe size and height. *If you calculator does not display the r and $r^2$ lines then you need to go into the catalog and run the program "DiagnosticOn". This will enable the display of the correlation coefficient.*

You can then graph the scatterplot and the regression line:



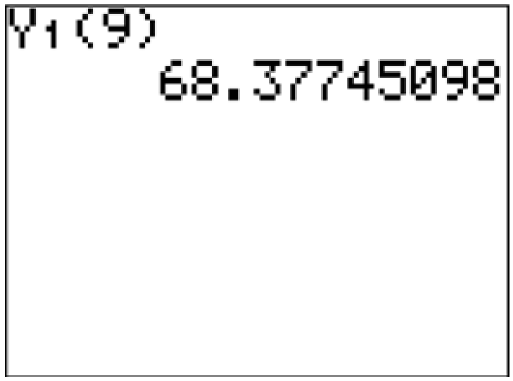
The regression equation is:

$\hat{y} = 52.4069 + 1.7745x$

Where $x$ represents shoe size and $\hat{y}$ represents predicted height. The predicted height for someone with size 9 shoe is 68.3774:

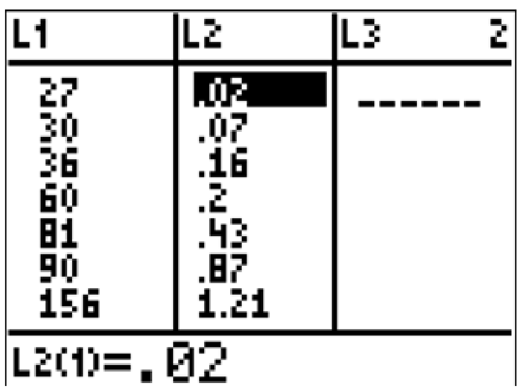
$\hat{y} = 52.4069 + 1.7745 \cdot 9 = 68.3774$

An easy way to use the power of the calculator is to use function notation from the home screen:

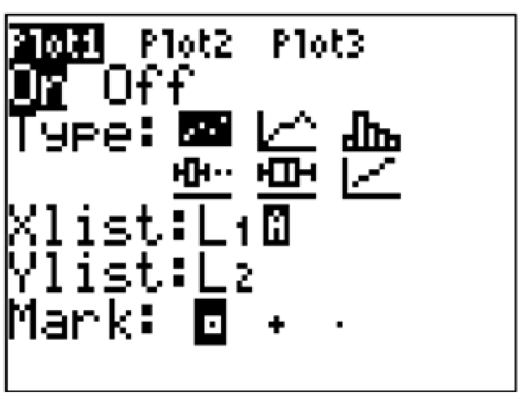



**Concept Problem Revisited**

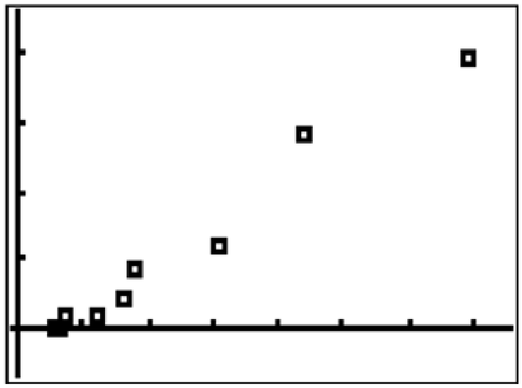
Enter the data onto lists in your calculator:




Turn the [STAT PLOT] on that compares the two lists of data:




You should note that the data is extremely linear with a positive correlation coefficient:



A naïve conclusion would be to say that doctors cause cancer. One of the most misunderstood concepts in statistics is that correlation does not imply causation. Just because there is a correlation between the number of doctors and the cancer rate doesn't mean that the number of doctors *causes* the cancer. There are dozens of reasons why more doctors might correlate with higher cancer rates. In general, remember that correlation is not the same as causation. Be careful before making any conclusions about change in one variable *causing* change in another variable.

### Vocabulary

A ***scatterplot*** creates an $(x, y)$ point from each data pair.

***Bivariate data*** is two sets of data that are paired.

The ***correlation coefficient***, $r$, is a number in the interval [-1, 1]. It indicates the strength of the correlation between two variables.

### Guided Practice

1. The data below represents the average number of working words in an elementary student's vocabulary as it relates to their shoe size. Perform a linear regression that models the data.

**TABLE 15.13:**

| Shoe Size | 1.0 | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 | 4.5 |
|---|---|---|---|---|---|---|---|---|
| Vocabulary | 1135 | 1983 | 2501 | 4113 | 5431 | 7891 | 9320 | 11041 |

2. Use the equation from Guided Practice 1 to predict the vocabulary for someone who has a 1.0 shoe size. Does this prediction seem reasonable given the data? Why or why not?

3. Shaquille O'Neal has size 23 shoes. What, if anything can you infer about his vocabulary? Does a larger shoe size cause a larger vocabulary?

**Answers:**

1. Let $x$ represent shoe size and $y$ represent vocabulary.

$\hat{y} = -2660.4167 + 2940.9333x$

$r = 0.9865$

The correlation coefficient is very close to positive one. This is a strong indication that the data can be modeled by a linear relationship.

2. $\hat{y} = -2660.4167 + 2940.9333 \cdot 1$

$\hat{y} = 280.4167$

This number seems remarkably low considering the data. This point is very close to the $x$ intercept, which can be found using algebra:

$0 = -2660.4167 + 2940.9333x$

$0.9046 = x$

The interpretation of the point (0.9046, 0) from the model is that when a person has a shoe size of just under 1.0, then their predicted vocabulary is zero. Shoe sizes below 0.9046 will have a negative vocabulary. Is this reasonable? It certainly does not make sense that someone could have a negative number of words in their vocabulary. Newborn babies are born without knowing any words and this number stays flat at 0 for some length of time. Therefore, this model is not accurate for very low shoe sizes.

3. Shaquille's shoe size is significantly beyond the scope of the data that the model is based on. The data relates to elementary school students and a size 23 shoe is beyond the relevant domain. This means it wouldn't make sense to use this model to predict Shaquille's shoe size. Shoe size does not cause vocabulary, but the two variables are strongly correlated because over time both tend to grow.

### Practice

For each correlation coefficient, describe what it means for data to have that correlation coefficient and sketch a scatterplot with that correlation coefficient.

1. $r = 1$

2. $r = -0.5$

3. $r = -1$

4. $r = 0$

5. $r = 0.8$

The data below shows the SAT math score and GPA for 7 different students.

**TABLE 15.14:**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| SAT math score | 595 | 520 | 715 | 405 | 680 | 490 | 565 |
| GPA | 3.4 | 3.2 | 3.9 | 2.3 | 3.9 | 2.5 | 3.5 |

6. Use your calculator to perform a linear regression that models the data. What is the regression equation? What is the correlation coefficient?

7. Use the equation from #6 to predict the GPA for a student with an SAT score of 500. Does this prediction seem reasonable given the data? Why or why not?

8. What is the relevant domain of this data?

9. Does a high SAT math score cause a high GPA?

The data below shows scores from two different quizzes for 10 different students.

**TABLE 15.15:**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Quiz 1 Score | 15 | 12 | 10 | 14 | 10 | 8 | 6 | 15 | 16 | 13 |
| Quiz 2 Score | 20 | 15 | 12 | 18 | 10 | 13 | 12 | 10 | 18 | 15 |

TABLE 15.15: (continued)

10. Use your calculator to perform a linear regression that models the data. What is the regression equation? What is the correlation coefficient?

11. Use the equation from #10 to predict the Quiz 2 score for a student with a Quiz 1 score of 19. Does this prediction seem reasonable given the data? Why or why not?

12. What conclusions can you make about this data?

13. Explain in your own words the difference between causation and correlation.

14. Explain in your own words what the correlation coefficient measures.

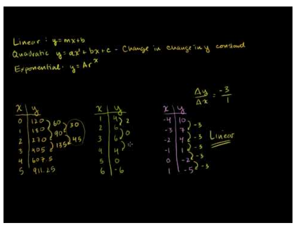
15. Explain why a larger sample size will cause a more accurate correlation coefficient.

# 15.8 Modeling with Regression

Here you will use regression on a variety of different types of data to make reasonable predictions.

Linear correlation is the simplest type of relationship between two variables. Your calculator has the power to use a variety of different function families to find other relationships and create many different types of models. How do you choose which function family is best for a given situation?

### Watch This



MEDIA
Click image to the left for more content.

http://www.youtube.com/watch?v=CxEFOozrMSE Khan Academy: Linear, Quadratic, and Exponential Models

### Guidance

Once you understand how to do linear regression with your calculator, you already know the technical mechanics to perform other regressions in the [STAT] [CALC] menu. The most common regressions correspond to the function families.

- QuadReg - Quadratic function family
- CubicReg - Cubic function family
- QuarticReg - Quartic function family or $4^{th}$ degree polynomial
- LnReg - Natural Log function family
- ExpReg - Exponential function family
- PwrReg - Power function family
- Logistic - Logistic function family
- SinReg - Sinusoidal function family.

When you perform these types of regressions, it will be incredibly important for you to interpret and explain parts of the graph. Here are some points to keep in mind:

1. The $y$-intercept may have a particular meaning that may or may not be reasonable.
2. When you use your model to make predictions it is important for you to remember the relevant domain of your model. If your data is about elementary school students then it might extend to middle and high school students, but it might not.
3. The calculator may produce a correlation coefficient for each of these non-linear regressions, but you should be very careful. Technically, the correlation coefficient is only supposed to be calculated with linear regression, so the calculator is doing some fancy linearization to produce it. You can learn more about this process in future statistics courses.

In general, at this point you should use your best judgment when choosing a function family to model a given set of data and deciding how good a fit the model is based on context.

**Example A**

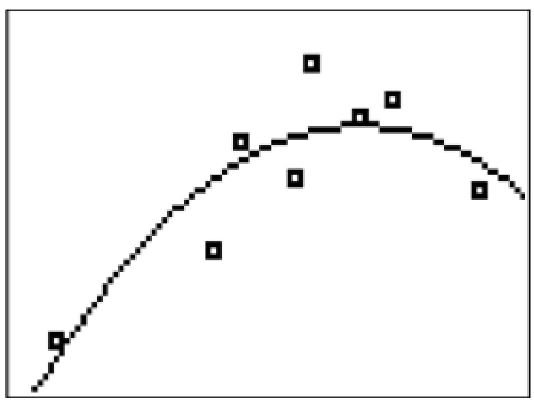
Given the following data about SAT scores and number of hours slept the night before, use an appropriate function family to produce a reasonable model. Defend your choice of function families.

**TABLE 15.16:**

| # Hours Slept | SAT Score |
|---|---|
| 8.5 | 1840 |
| 10.9 | 1510 |
| 9.1 | 1900 |
| 7.5 | 2070 |
| 7.2 | 1550 |
| 6.0 | 1720 |
| 2.3 | 840 |
| 5.5 | 1230 |

**Solution:** Let $x$ be the number of hours slept and $y$ be the SAT score.

After plotting the points, you should choose a function family to use as a model. In this case, it would be appropriate to try a quadratic relationship.



A quadratic model makes sense because there seems to be a peak in the model and in the data around 8.5 hours of sleep. It makes sense that someone who does not get enough sleep will do worse and someone who gets too much sleep might also do worse.

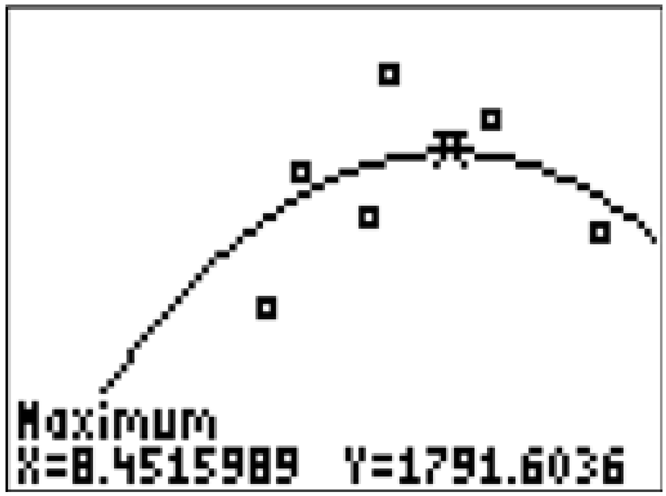
**Example B**

Using the model from Example A, answer the following questions.

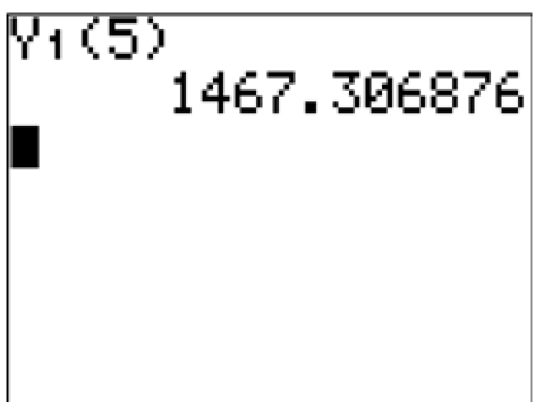
a. What is the perfect amount of sleep to get before the SATs?
b. Calculate the score you are predicted to get if you get 5 hours of sleep.
c. What is the relevant domain of the model?
d. The average SAT score is about 1500. According to the model, what amount of sleep predicts this score? Does this number represent the average number of hours that people sleep before the SATs?
e. Compare the actual and predicted score for someone with 6 hours of sleep.

**Solution:**

a. Use the calculator to find the maximum of the parabola. The $x$ coordinate represents the "perfect" amount of sleep.

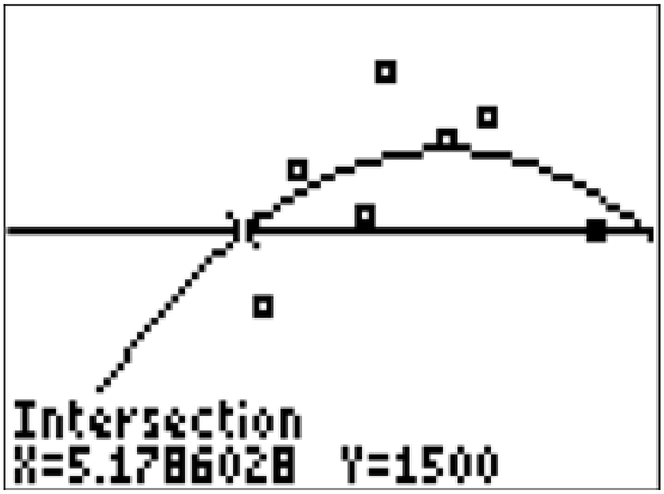



b. You can substitute $x = 5$ into the equation, or you can let the function you created and stored in $y_1$ simply act on the 5.




c. The relevant domain is between about 2 hours and 10 hours of sleep. Beyond those numbers of sleep, the model will probably not make a whole lot of sense. How could someone get negative hours of sleep?

d. You can substitute $\hat{y} = 1500$ into the equation and solve for $x$ using the quadratic formula, or you can graph the line $y = 1500$ and use the calculator to produce the two intersecting points.



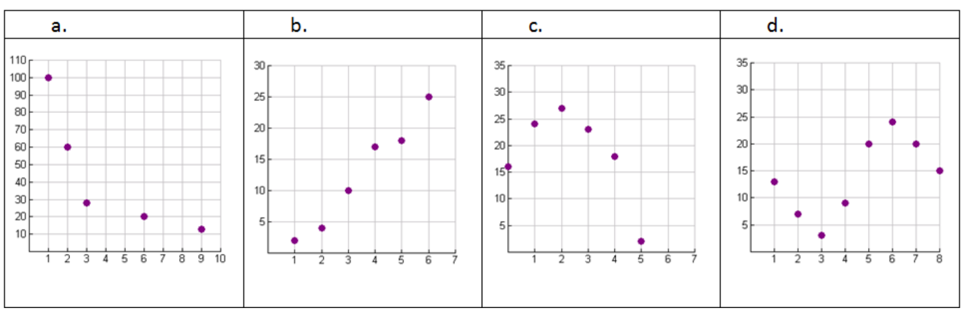

5.1786 hours and 11.7246 hours are the number hours of sleep that predict a score of 1500.

When using the model in this direction, the results do not make as much sense and you need to be extremely careful about what you say.

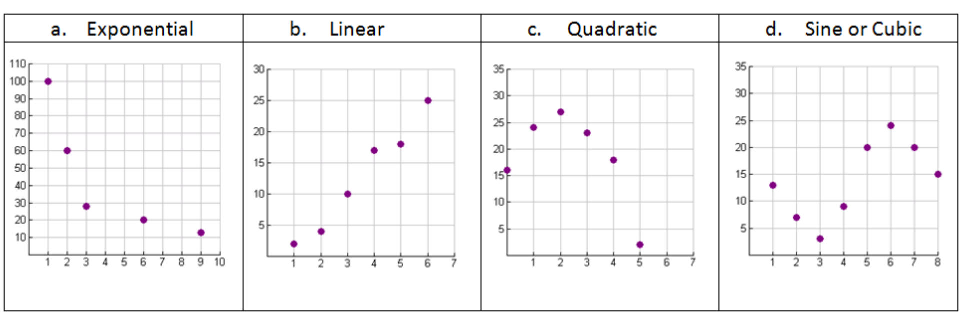
e. The actual score for someone who got 6 hours of sleep can be found in the original data to be 1720. The model predicts 1627.9970. The difference between the model and what actually happened is $1720 - 1627.9970 = 92.0030$

**Example C**

Use your knowledge of function families to predict the best model for each of the following scatterplots.

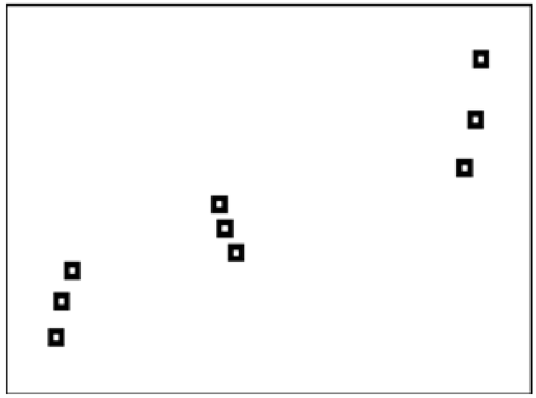


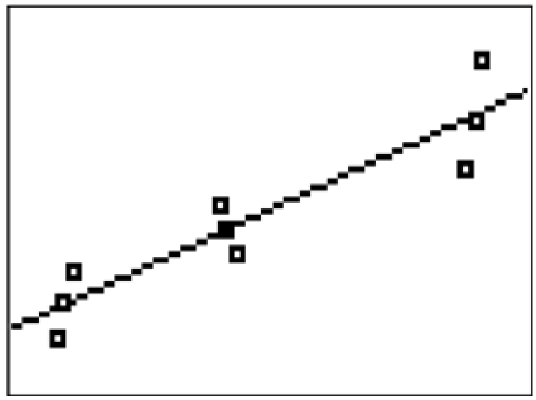

**Solution:**

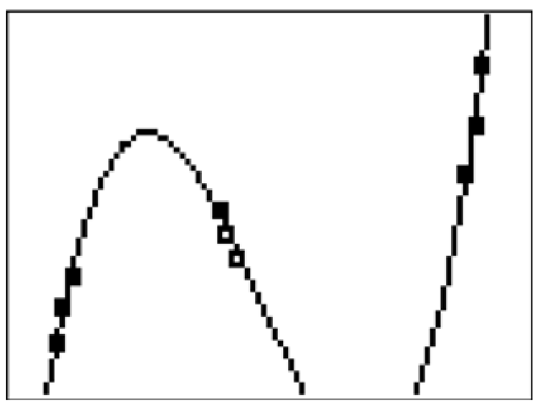



**Concept Problem Revisited**

For some data sets it is possible to use a polynomial or other complicated shape to exactly intersect every data point. The downside is that the model will miss the overall relationship. Consider the following data and modeling a linear relationship or cubic relationship.







The linear relationship describes the upward positive relationship in the data very well, but some points are slightly off of the line. The cubic relationship is much more accurate at the specific data points; however, there are features of the cubic relationship that differ significantly from reality when interpreted in context. In order to choose the best regression model you need to use context clues and the reasonableness of the various features of the model that fit each situation.

### Vocabulary

The ***residual*** is the difference between the actual height and the predicted height using the model.

### Guided Practice

1. The following data represents the height of an elephant over time. Determine the best regression function to use and determine its equation.

### TABLE 15.17:

| Age(Years) | Height (ft) |
|---|---|
| 0 | 2 |
| 2 | 2.8 |
| 4 | 4 |
| 8 | 7.5 |
| 12 | 10 |
| 16 | 10.4 |
| 20 | 10.45 |

2. The following data represents the speed that Ben can kick a soccer ball at different ages. Determine the best regression function to use and determine its equation.

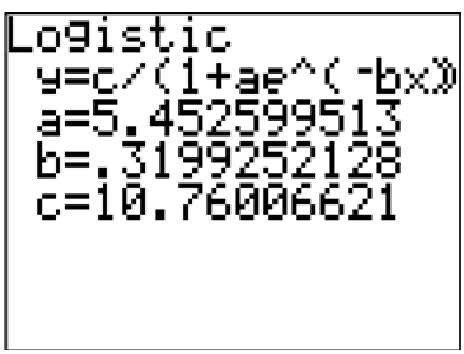
### TABLE 15.18:

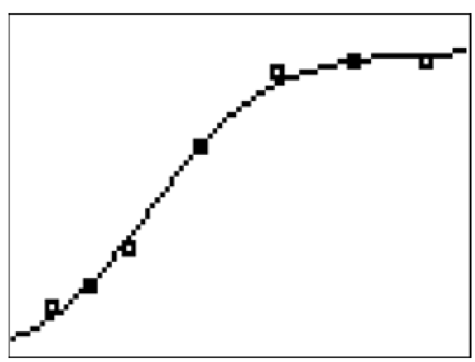
| Age(Years) | Speed (mph) |
|---|---|
| 4 | 15 |
| 10 | 32 |
| 20 | 65 |
| 30 | 70 |
| 50 | 45 |
| 60 | 35 |

3. What are two weaknesses and two strengths of the model used to predict Ben's kicking speed from Guided Practice 2?

**Answers:**

1. Logistic is the best function family because it levels off over time indicating that the elephant ceases to grow once it matures.



$\hat{y} = \frac{10.7601}{1+5.4526 \cdot e^{-0.3199x}}$

2. The best regression to use is a quadratic relationship because when Ben is little he cannot kick the ball very fast and when he is old he also cannot kick the ball very fast. Ben can kick the ball the fastest when he is an adult between the ages of 20 and 40.

$\hat{y} = -0.05981x^2 + 4.0679x + 0.6191$

3. One strength is that a quadratic model correctly describes the peak of kicking speed occurring in the middle of Ben's life. A linear regression might forecast Ben's kicking speed increasing forever and a logistic regression might forecast Ben's kicking speed always staying fast despite his old age. A second strength of the model could be the

$y$-intercept of 0.6191. Even though this number is not really in the relevant domain, it implies that as a newborn baby Ben could kick the ball very slowly which is arguably true.

One weakness of the model is that it predicts that Ben will kick the ball at 0 miles per hour at age 68.1660. This implies that Ben will not be able to kick the ball at all which isn't necessarily true.

A second weakness of the model is that it predicts negative speed at either age extreme which doesn't make sense. A better model would be flat at 0 when Ben is born and also at the end of Ben's life when he is no longer able to kick the ball.

## Practice

The table below shows the average height of an American female by age.

### TABLE 15.19:

| Age (Years) | Height (inches) |
|---|---|
| 2 | 34 |
| 8 | 50 |
| 11 | 57 |
| 15 | 63 |
| 23 | 64 |
| 35 | 64 |

1. Determine two different equations that model the height over time using two different function families.

2. Which function is a better fit for this data? Why?

3. Use both equations to predict the $y$-intercepts. What does the $y$-intercept represent in each case? Are your predictions reasonable for this part of the graph?

4. Use your "better fit" equation to predict the height of a 70 year-old woman. Is your prediction reasonable for this part of the graph? Why or why not? What do you really need your model to do for the domain [16,100]?

Alice is in Wonderland and drinks a potion that approximately halves her height for each sip she takes, as shown in the table below.

### TABLE 15.20:

| # of sips | Height (inches) |
|---|---|
| 0 | 60 |
| 1 | 29 |
| 2 | 16 |
| 3 | 8 |
| 4 | 4.1 |

5. Do an exponential regression to determine an appropriate model. What is the equation?

6. Explain why exponential regression is a good choice in this case.

7. How many sips did she take if she is 2 inches tall?

8. How tall will she be if she has 6 sips?

A rumor is spreading around your 400 person school. The following table shows the number of people who know the rumor each day.

### TABLE 15.21:

| Day | # of people who know the rumor |
|---|---|
| 1 | 2 |
| 2 | 8 |
| 3 | 29 |
| 5 | 161 |
| 6 | 372 |
| 7 | 378 |
| 8 | 391 |

9. Use logistic regression to determine an equation that models the number of people who know the rumor over time.

10. Why is the logistic model appropriate in this case?

11. Use your regression equation to predict the time when only one person knew the rumor. Does this make sense?

The data table below represents how the tide changes the depth of the ocean water at a beach. At a certain place in the water, a scientist measures the depth of the water for ten consecutive hours.

### TABLE 15.22:

| Hours | Depth of Ocean Water (ft) |
|---|---|
| 0 | 9 |
| 1 | 11.2 |
| 2 | 12.4 |
| 3 | 12.9 |
| 4 | 12.5 |
| 5 | 11 |
| 6 | 8.9 |
| 7 | 7 |
| 8 | 5.5 |
| 9 | 4.9 |
| 10 | 5.4 |

12. Choose the function family that is the best model for this situation and determine the regression equation.

13. Use your regression equation to predict the depth of the water at 10 hours. What is the difference between the actual depth from the data and the predicted depth from your equation (residual)?

14. Do a cubic regression on the calculator. What is the cubic regression equation? Is this a better or worse model than the model you originally chose?

15. Why might statisticians do modeling with regression for their data?

You started by working with univariate data and learned how to display it graphically and summarize it numerically. You learned how to calculate mean, median, mode and variance, and when to use each. You also explored bivariate data and used the regression capabilities of your calculator to create mathematical models for real world phenomenon.

# 15.9 References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# CHAPTER 16 Logic and Set Theory

## Chapter Outline



Mathematics involves intricate notation, numbers, graphs and theorems, but at its core is logical reasoning. Most explanations, proofs, solutions and arguments involve step by step conclusions that follow from previous statements. Inherent to this structure is reasoning about when and how the truth of a conclusion follows from previous statements. Most students take this reasoning for granted, but to make mathematics truly objective it makes sense to study the nature of logic so that you can be sure that your reasoning truly is airtight. Here you will explore the basics of logic and set theory.

# 16.1 And and Or Statements

Here you will use the words "and" and "or" as logical connectives in complex statements. You will also use set theory and logical notation to diagram logical statements.

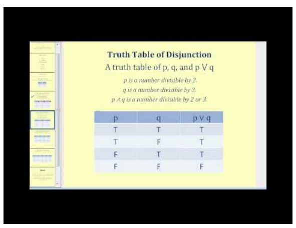
The words "and" and "or" are common in everyday language. In mathematics, there are some subtle differences that you need to watch out for, especially considering the word "or".

*It will rain or it will snow.*

When is this statement true and when is it false?

## Watch This

Watch the portion of this video focusing on truth tables of conjunction and disjunction:




MEDIA
Click image to the left for more content.

http://www.youtube.com/watch?v=_5Z_0824RHw James Sousa: Truth Tables for Compound Statements

## Guidance

An **atomic statement** is a **declarative statement** without logical connectives that has a truth value. Here are two declarative statements:

- $P = It\ is\ snowing.$
- $Q = I\ am\ cold.$

The **truth value** of a statement is whether the statement is true or false. As a mathematician, your job is to determine when a logical statement is true and when it is false. If you don't have enough information to determine whether the original statements are true or false, you can build a truth table to organize all the possible cases.

Consider the atomic statement $P$ joined with the atomic statement $Q$. The following sentence can be written using the symbol "$\vee$" for the logical connective "or".

$$It\ is\ snowing\ or\ I\ am\ cold.$$
$$P \vee Q$$

This statement is a little strange because it seems to imply that it is always the case that one or both of those atomic statements is happening. Your common sense may dictate that this statement isn't true because of course there are times when it is sunny and you are warm. It's important to remember that not all statements are true! Your job is to determine what has to be true for the above statement to be true. To organize your work, you should construct a truth table. A truth table considers all possible combinations of the original declarative statements being true or false, and then uses logic to deduce the truth value of the compound statement in each case.

**Here is the truth table for OR:**

## TABLE 16.1:

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

Notice that there are four possible truth combinations of $P$ and $Q$ (both true, first true/second false, first false/second true, both false). Only one of these combinations yields a false statement for $P \vee Q$. What this means is that the statement " *It is snowing or I am cold*" is only false if "it is snowing" is false and "I am cold" is false. Note that if "it is snowing" is true and "I am cold" is also true, then "It is snowing or I am cold" is true. In mathematics, the word "or" does not mean exactly one or the other. It means "one or the other or both".

Next consider the truth table for the following statement that uses the connective "and". The following sentence can be written using the symbol "$\wedge$" for the logical connective "and".

$$It\ is\ snowing\ and\ I\ am\ cold.$$
$$P \wedge Q$$

**Here is the truth table for AND:**

## TABLE 16.2:

| $P$ | $Q$ | $P \wedge Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

Notice that a compound statement using "and" is true only if each atomic statement is individually true.

**Example A**

Identify the atomic statements in the following compound sentence. Then, use logical connectives to rewrite the sentence with symbols.

*I am tired and hungry and I want a burger or a nap.*

**Solution:** The proper way to interpret this sentence is to identify the "or" as relating to just the burger and the nap.

- $P = I\ am\ tired.$
- $Q = I\ am\ hungry.$
- $R = I\ want\ a\ burger.$
- $S = I\ want\ a\ nap.$

The sentence could be rewritten with symbols as: $(P \wedge Q) \wedge (R \vee S)$

**Example B**

You play a game with a friend where you try to guess the number that your friend is thinking by asking yes or no questions. You ask your friend the following question.

*Is the number evenly divisible by 3 or 5?*

a. What should your friend say if their number is 8?
b. What should your friend say if their number is 9?
c. What should your friend say if their number is 10?
d. What should your friend say if their number is 15?

**Solution:**

a. If the number is 8 then your friend should say no, because 8 is not divisible by 3 or 5.
b. If the number is 9 your friend should say yes, because 9 is divisible by 3.
c. If the number is 10 your friend should say yes, because 10 is divisible by 5.
d. If the number is 15 then your friend should say yes, because 15 is divisible by 3 and 5. *Remember that the word "or" does not mean exclusively one or the other.*

**Example C**

Identify the atomic statements in the following compound sentence. Then, use logical connectives to rewrite the sentence with symbols.

*For lunch you had a ham and cheese sandwich and an apple or an orange.*

**Solution:** Not all sentences will be easy to break down into atomic statements. In this case, the ham and cheese sandwich is inseparable even though it contains the word "and". You have to use your prior knowledge to know that "ham and cheese sandwich" is a type of sandwich.

- $A = You\ had\ a\ ham\ and\ cheese\ sandwich\ for\ lunch.$
- $B = You\ had\ an\ apple\ for\ lunch.$
- $C = You\ had\ an\ orange\ for\ lunch.$

The sentence could be rewritten with symbols as: $A \wedge (B \vee C)$.

Note that each statement $A, B, C$ contains the words "you had" and "for lunch" and is a complete sentence.

**Concept Problem Revisited**

In English, most people use the word "or" to mean exclusive "or". If you were told *"you can have a brownie or a cookie for dessert"*, you would assume you had to choose just one and couldn't have both the brownie and the cookie. In mathematics, the word "or" means "one or the other or both". Therefore in logic, "or" includes the case when both atomic parts of the state are true.

$P = It\ will\ rain.$

$Q = It\ will\ snow.$

$P \vee Q$

### TABLE 16.3:

| $P$ | $Q$ | $P \vee Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

The statement is only false when both parts of the statement are false. In other words, the statement is only false if "it will rain" is false and "it will snow" is also false. When one or both parts of an "or" statement are true then the whole statement is true.

### Vocabulary

An ***atomic statement*** is a declarative statement without logical connectives that has a truth value.

The ***truth value*** of a statement is whether the statement is true or false.

An ***"or"*** statement combines two logical statements and is only false when both statements are false. The symbol for "or" is "$\vee$". "Or" statements are also called ***disjunctions.***

An ***"and"*** statement combines two logical statements and is only true when both statements are true. The symbol for "and" is "$\wedge$". "And" statements are also called ***conjunctions.***

### Guided Practice

1. Identify the atomic statements in the following sentence.

*You run a marathon, build a house and become a doctor or you consume too much TV and junk food.*

2. Diagram the sentence from Guided Practice #1 using the logical connectives "$\vee$" for "or" and "$\wedge$" for "and".

3. Use a truth table to identify all cases when the statement in Guided Practice #1 is true or false.

**Answers:**

1. $M = You\ run\ a\ marathon.$ $B = You\ build\ a\ house.$ $D = You\ become\ a\ doctor.$ $W = You\ watch\ too\ much\ TV.$ $J = You\ eat\ too\ much\ junk\ food.$

2. The hardest part in diagramming the logical connectives is often determining which parts of the sentence should be grouped together. In this case there is a clear separation between the three positive outcomes and with the two negative outcomes:

$(M \wedge B \wedge D) \vee (W \wedge J)$

3. Truth tables of complex sentences can be overwhelming, especially since 5 atomic statements means that there should be $2^5$ rows in the truth table to account for all of the T/F combinations. To save time and space you can note that the statement $M \wedge B \wedge D$ is only true when $M, B$, and $D$ are all true and $W \wedge J$ is only true when both $W$ and $J$ are true. This means that you now only need 4 rows in the truth table.

#### TABLE 16.4:

| $M \wedge B \wedge D$ | $W \wedge J$ | $(M \wedge B \wedge D) \vee (W \wedge J)$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ |

The statement is true if:

1. $M, B$, and $D$ are all true.
2. $W$ and $J$ are both true.
3. $M, B, D, W$, and $J$ are all true.

The statement is false if:

1. Not all of $M, B$, and $D$ are true and not both of $W$ and $J$ are true.

### Practice

I go to school and do my work or stay home and play games.

1. Identify the atomic statements in the above compound sentence.

2. Use logical connectives to rewrite the sentence with symbols.

I have macaroni and cheese or steak and green beans or potatoes.

3. Identify the atomic statements in the above compound sentence.

4. Use logical connectives to rewrite the sentence with symbols.

I wear flip flops and either shorts and a t-shirt or a dress.

5. Identify the atomic statements in the above compound sentence.

6. Use logical connectives to rewrite the sentence with symbols.

It is dark outside and I light a candle.

7. Identify the atomic statements in the above compound sentence.

8. Use logical connectives to rewrite the sentence with symbols.

We will go to the beach and have a picnic or go to the movies and eat popcorn.

9. Identify the atomic statements in the above compound sentence.

10. Use logical connectives to rewrite the sentence with symbols.

Make a truth table for each of the following statements.

11. $(P \wedge Q) \vee R$

12. $P \wedge (Q \vee R)$

13. $(P \vee Q) \vee R$

14. $P \vee (Q \vee R)$

15. How does the placement of parentheses affect the truth values of compound statements?

# 16.2 If-Then Statements

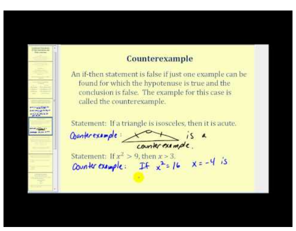
Here you will learn to identify different parts of if-then statements and determine when an if-then statement is true or false. You will also use traditional set theory graphical representations.

If-then statements are very common outside formal mathematics. In many cases, your familiarity with these types of statements will help you to interpret their meaning and truth value. In other cases, the truth value of some statements may be unclear without a formal understanding of logic. Consider the following two statements:

- If it rains today then you will stay at home and read a book.
- You stayed at home and read a book.

Did it or did it not rain?

### Watch This




MEDIA

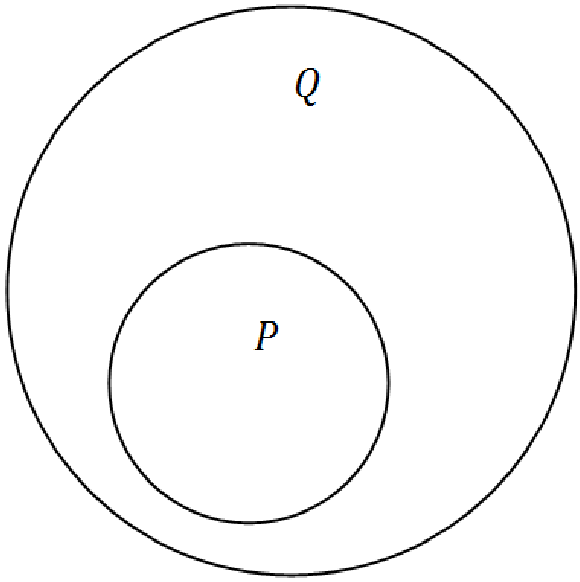
Click image to the left for more content.

http://www.youtube.com/watch?v=oEr27P1bX9o James Sousa: If-Then Statements and Converses

### Guidance

A mathematical set is just a group of things. The group can include anything: letters, numbers, objects or monkeys. Set theory focuses on the relationships between sets as they overlap or are completely within each other. For the purposes of if-then statements, set theory provides a perfect framework in which to reason.

Consider set $P$ and set $Q$ that are just collections of things represented by circles. If something is not in the set, then it is not in the circle.




In this case set $P$ is a subset of set $Q$ since it is entirely included within set $Q$. Mathematically we write the statement "$P$ is a subset of $Q$" as:

$P \subseteq Q$

This can be translated to an if-then statement, and simplified using symbols:

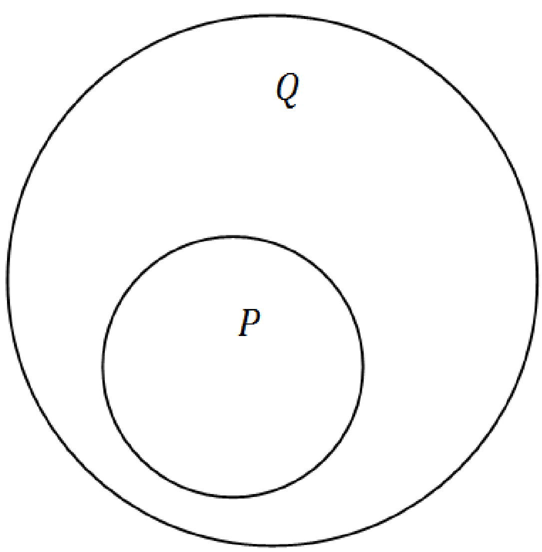
*If it is an element in P, then it is an element in Q.*

*If P, then Q.*

$P \to Q$

If-then statements are examples of **conditional statements**. Sometimes conditional statements are written without an "if" or a "then", but can be rewritten. The "if" part of the statement (represented by $P$ above) is called the **hypothesis, antecedent** or **protasis**. The "then" part of the statement (represented by $Q$ above) is called the **conclusion, consequent** or **apodosis**.

In order to precisely define the truth value of a conditional statement, we need to consider the four different combinations of the truth value for $P$ and $Q$ in relation to the diagram




1. *If P is true, then Q is true.* This statement is true because if an object is inside circle $P$, then it is definitely

inside circle $Q$.
2. *If P is true, then Q is false.* This statement is false because there is no possible way an object could be inside circle $P$ and yet outside circle $Q$.
3. *If P is false, then Q is true.* This statement is considered true because if an object is outside circle $P$ then it may or may not be in circle $Q$. **There is no contradiction**.
4. *If P is false, then Q is false.* This statement is also considered true because if an object is outside circle $P$, then it can be outside circle $Q$. Like the previous statement, **there is no contradiction**.

The truth values of the four combinations can be summarized in a truth table. Recall that truth tables are extremely useful for summarizing complicated logical sentences and identifying whether the statements are true or false.

### TABLE 16.5:

| $P$ | $Q$ | $P \to Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

Note that a conditional statement is only false when **the hypothesis is true** and **the conclusion is false**. Also note that **any conditional statement with a false hypothesis is trivially true**. The following statement is trivially true because the hypothesis is false.

*If pigs can fly then butterflies eat elephants.*

The truth of this statement confuses many people the first time they look at it. One way to frame it in your mind is to realize that a statement is false only when it results in a logical contradiction. In a world where pigs could fly perhaps butterflies could eat elephants, who knows? It would be ridiculous for a person to argue that in the hypothetical world where pigs could fly that there is no way that butterflies could eat elephants.

**Example A**

Rewrite the following conditional statements in if-then form.

a. If you go to the show, you will be amazed.
b. Unless you buy firewood you will be cold.
c. Come here and you will get a present.
d. Kicking a soccer ball makes it bounce.
e. Give me your lunch money or I'll put you in a locker.
f. Anyone who wears orange likes Halloween.
g. Without my sunglasses on I can't drive.
h. Buy this product and you'll be beautiful and popular.

**Solution:** Even though these statements have words like "and", "or" and "not" they are still just conditional statements. In each case, consider which action or event leads to another action or event.

a. If you go to the show, then you will be amazed.
b. If you do not buy firewood, then you will be cold.
c. If you come here, then you will get a present.
d. If you kick a soccer ball, then it will bounce.
e. If you don't give me your lunch money, then I'll put you in a locker.
f. If a person wears orange, then that person likes Halloween.
g. If I do not wear my sunglasses, then I can't drive.
h. If you buy this product, then you will be beautiful and popular.

**Example B**

Evaluate the truth value of the following conditional statement using a truth table.

*If when I go somewhere I always run, then when I run I always go somewhere.*

**Solution:** This statement actually has two layers of conditional statements because both the hypothesis and the conclusion are conditional statements themselves.

- Let $R$ be the statement "I run".
- Let $S$ be the statement "I go somewhere".

The original sentence can be rewritten in symbols as: $(S \to R) \to (R \to S)$. To make the truth table for the sentence, start with all possible truth combinations of $S$ and $R$ (both true, $S$ true/$R$ false, $S$ false/$R$ true, both false). Then, find the truth values for each piece working up to the full sentence.

**TABLE 16.6:**

| $S$ | $R$ | $S \to R$ | $R \to S$ | $(S \to R) \to (R \to S)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

The statement is always true unless you run but do not go anywhere.

**Example C**

One day a student went to her logic professor and said, "I don't believe that when the hypothesis of a conditional statement is true, then the whole statement is trivially true".

The professor replied, "Alright then. Give me any false hypothesis you can think of and I will mathematically prove that you are a goat."

The student thought about it and said, "Okay, prove the statement: 'If $1 = 2$ then I am a goat'."

How would you prove this ridiculous (but true!) statement?

**Solution:** If $1 = 2$, then any group of 2 things must also be a group of just 1 thing. Therefore in the two element set containing the student and the goat, there must be only one element. Thus, the goat and the student must be the same thing. $\therefore$

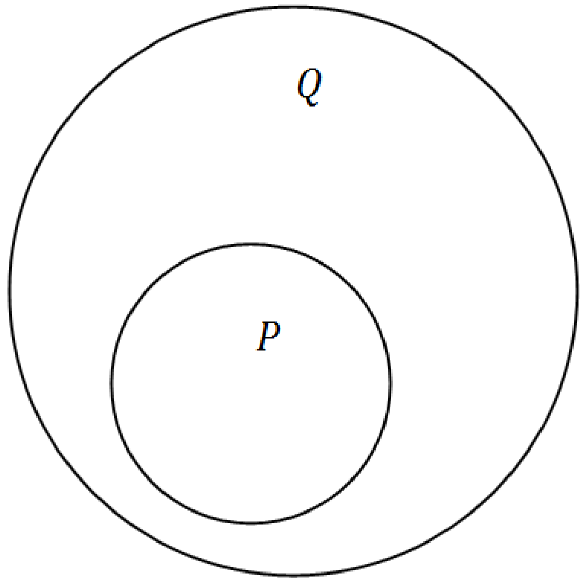
**Concept Problem Revisited**

Use a diagram to represent the conditional statement.

If it rains today then you will stay at home and read a book.

- $P =$ *it rains today*
- $Q =$ *you will stay at home and read a book.*

You stayed at home and read a book. This implies that $Q$ is true. According to the diagram, if an object is inside $Q$ it may or may not be inside $P$. Thus, you can conclude nothing about the rain. Many people will want to incorrectly conclude that it must have rained, but conditional statements only flow in one direction.




### Vocabulary

A ***set*** is a collection of numbers, letters or anything.

A ***subset*** is a collection of objects within a larger set.

***Set theory*** studies the relationships of sets and subsets.

A ***conditional statement*** is a statement that can be written as an ***if-then statement***.

The "if" part of an if-then statement is called the ***hypothesis, antecedent*** or ***protasis***.

The "then" part of an if-then statement is called the ***conclusion, consequent*** or ***apodosis***.

### Guided Practice

1. Consider the four possible scenarios of the following statement and explain why there is only one way the statement is false.

*If Brian is promised cookies then Brian will eat cookies soon.*

2. If all of the following statements are true, what can you conclude?

- *All babies are cute.*
- *Laura likes cute people.*
- *Laura is a baby.*

3. There are 53 people in the marching band and 49 people on the swim team. If 84 people belong to either or both teams, how many people are on both teams?

**Answers:**

1. Scenario A – both hypothesis and conclusion are true. Brian is promised cookies and then later eats cookies. Life is as it should be so the statement is true.

Scenario $B$ – hypothesis is true but the conclusion is false. Brian is promised cookies, but he does not end up eating cookies. Life is tough for Brian. The statement is false in this case.

Scenario $C$ – hypothesis is false but the conclusion is true. Brian is never promised cookies, but he's lucky and gets to eat some anyways. The statement is true.

Scenario $D$ – hypothesis is false and the conclusion is false. Brian is never promised cookies and never gets any, so the promise isn't broken. The statement is true.

2. First translate each of the statements into conditional statements (even if they sound awkward!). This is helpful for determining an if-then chain of events.

- $A$: *If a person is a baby then the person is cute*.
- $B$: *If a person is cute then Laura likes that person*.
- $C$: *If a person is named Laura then that person is a baby*.

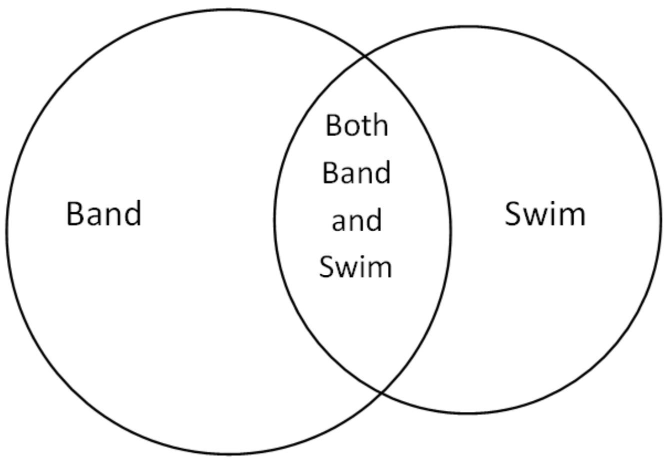
You should notice the circular structure of these three statements.

$A \to B, B \to C, C \to A$

$A \to B \to C \to A$

While many conclusions could be made, one conclusion about Laura is that she likes herself.

3. While this problem is not specific to if-then statements, it can be solved using a set theory representation and if-then logic.




Students on both teams would be double counted if you simply added up the number of students in band and the number of students on the swim team.

$53 + 49 = 102$

Since there are only 84 people total, then $102 - 84 = 18$ students must have been counted twice. Therefore, there are 18 students on both squads.

### Practice

1. What are the three names for the "if" part of an if-then statement?

2. What are the three names for the "then" part of an if-then statement?

Rewrite each of the following statements in if-then form.

3. If you like Pepsi, you will like Coke.

4. Do your homework and you will get candy.

5. Anyone who goes to the mall likes to shop.

6. Unless you cook dinner, you will be hungry.

7. Join this program and you will lose weight.

8. Shoveling snow makes your back sore.

9. Without knowing how to drive, you will not get your license.

10. Be nice to your sister or you will be punished.

11. When is the following statement false? *If you are a kid, then you like pizza*.

12. If all of the following statements are true, what can you conclude?

- *All kids like pizza.*
- *Sam is 8 years old.*
- *8 year olds are kids.*

13. There are 15 people in the math club and 27 people in the debate club. If 33 people belong to either or both clubs, how many people are just in the debate club?

14. There are 32 people on the bus in the morning and 40 people on the bus in the afternoon. If 20 people only ride the bus in the afternoon, how many people only ride the bus in the morning?

15. There are 24 people in your math class and 27 people in your English class. If 7 people are in your English class but not your math class, how many people are in your math class but not your English class?

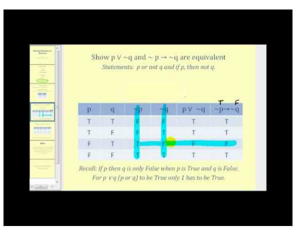
# 16.3 Negative Statements

Here you will negate statements and use graphical representations and set theory to explore the implications of negative statements. You will also learn about De Morgan's Law.

In everyday speech, negative statements are often ambiguous or unclear. Mathematically, you need a precise way to negate statements so that you can accurately determine whether statements are true or false. When is the negation of the following sentence true?

*If I am not cold then it is not snowing.*

### Watch This



MEDIA

Click image to the left for more content.

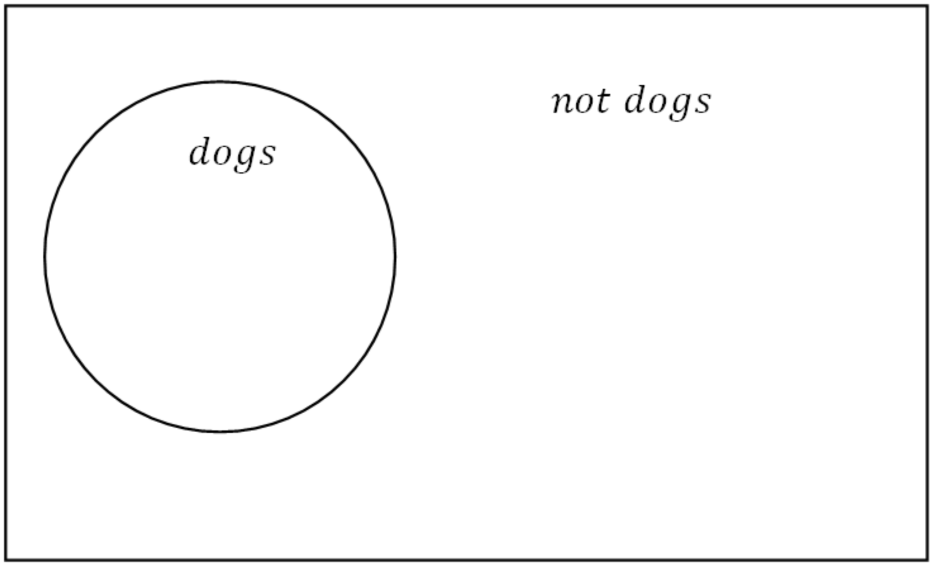
http://www.youtube.com/watch?v=hfz1gAoNd1w James Sousa: Showing Statements are Equivalent

### Guidance

While in everyday language the opposite of "dog" might be "cat", in mathematics the opposite of "dog" is "not a dog". Using the word "not" is the basic way to negate an atomic sentence. An atomic sentence is a logical statement without logical connectives that has a truth value.

- Original sentence $(D)$: *That thing is a dog*.
- Negation of sentence $(\sim D)$: *That thing is not a dog*.

The box below represents the universe of all things. This universe can be separated into things that are dogs and things that are not dogs.




To negate complex statements that involve logical connectives like or, and, or if-then, you should start by constructing a truth table and noting that negation completely switches the truth value.

The negation of a conditional statement is only true when the original if-then statement is false.

### TABLE 16.7:

| $P$ | $Q$ | $P \rightarrow Q$ | $\sim (P \rightarrow Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |

The negation of a conjunction is only false when the original two statements are both true.

### TABLE 16.8:

| $P$ | $Q$ | $P \wedge Q$ | $\sim (P \wedge Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ |

The negation of a disjunction is only true when both of the original statements are false.

### TABLE 16.9:

| $P$ | $Q$ | $P \vee Q$ | $\sim (P \vee Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $F$ | $T$ |

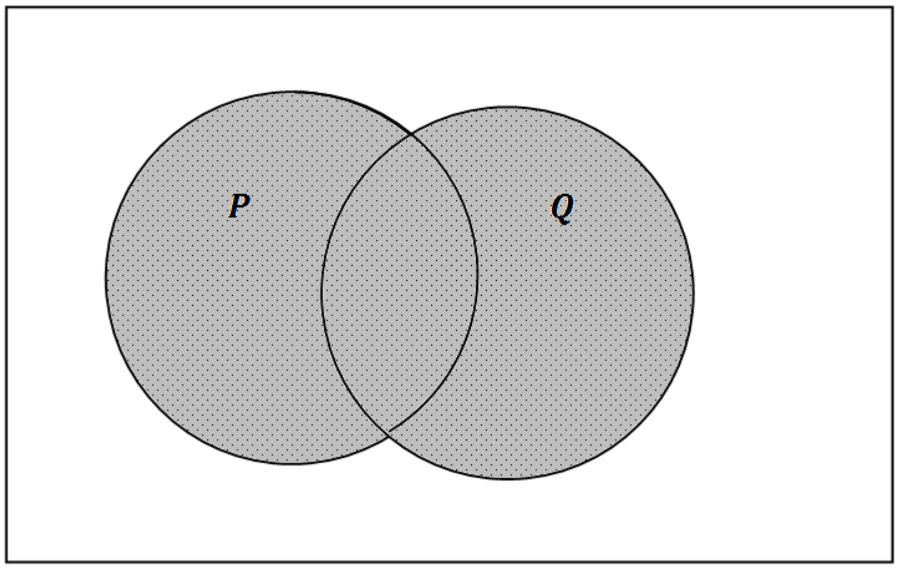
As mathematical sentences become more complex with additional connectives, truth tables and set theory circles are good ways to interpret when the statements are true and when the statements are false.

**Example A**

Use set theory circles to interpret the negation of a disjunction and explain how the negation of a disjunction can be written in a different way.

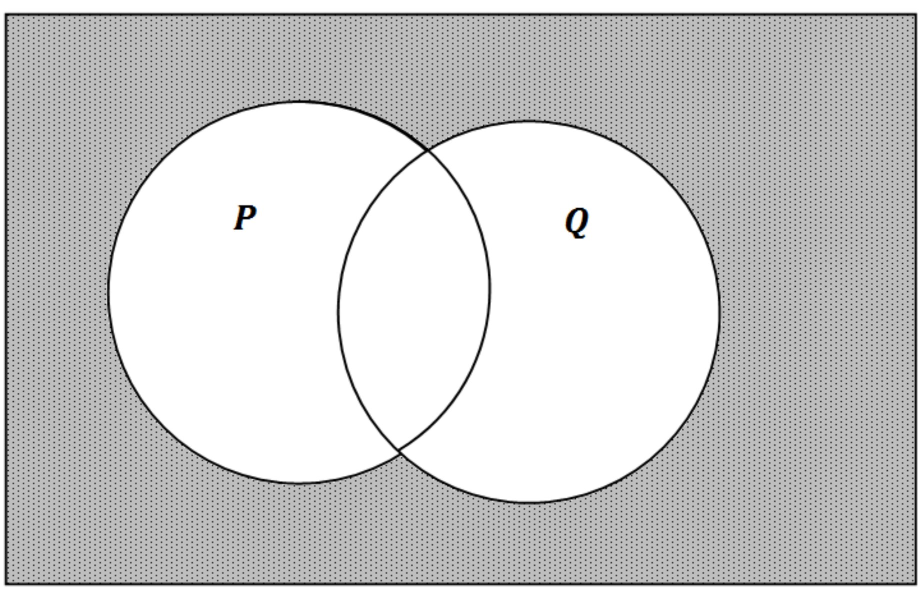
**Solution:** The shaded portion in the box represents the area that is within $P$ or within $Q$. Recall that in mathematical logic, this is written as $P \vee Q$. In set theory this area is represented similarly as $P \cup Q$ where the symbol $\cup$ stands for union. While the notation is slightly different, the reasoning about the relationships and implications is identical.

$$P \vee Q$$




When you negate the statement, you completely switch what is shaded.

$$\sim(P \vee Q)$$




A different way to think about this shaded region is that it is the region that is **not in** $P$ **and also not in** $Q$. This means that $\sim (P \vee Q)$ is equivalent to $\sim P \wedge \sim Q$.

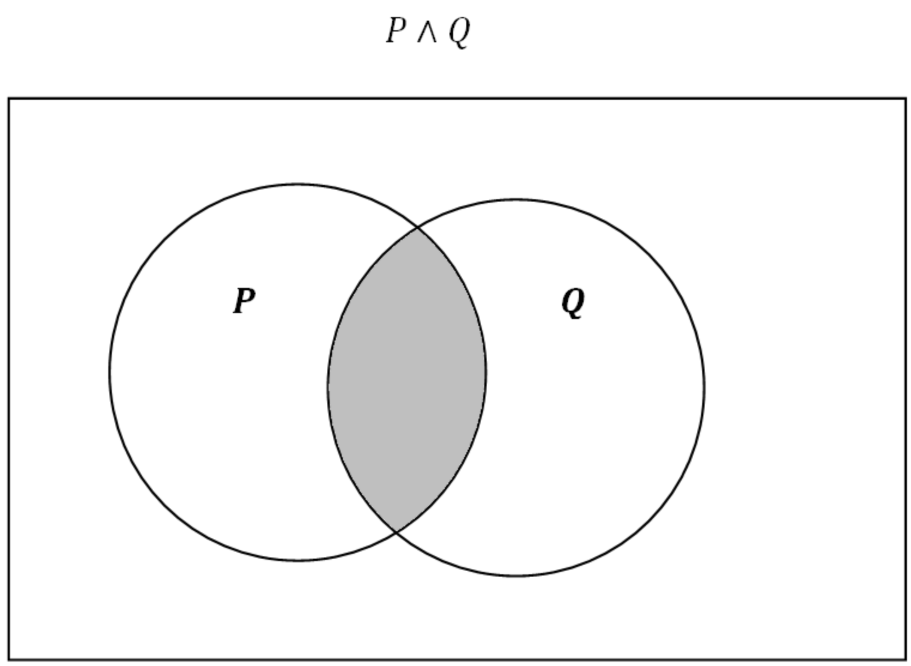
$\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

The symbol "$\Leftrightarrow$" works in mathematical logic and set theory the same way "=" works in arithmetic and algebra. In this case, the negative appears to distribute throughout the or statement by negating each statement individually and changes the or statement to an and statement. This is called **De Morgan's Law**.

**Example B**

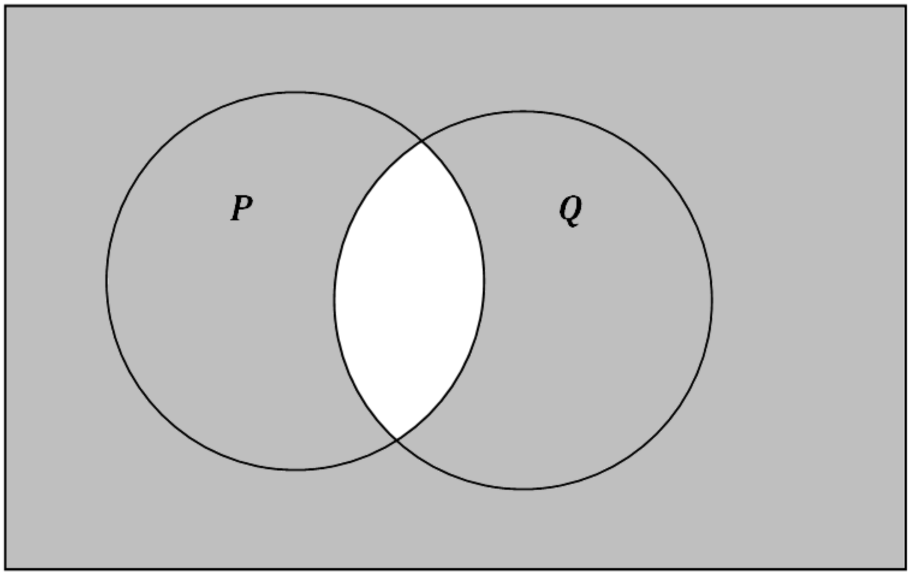
Use set theory circles to interpret the negation of a conjunction and explain how the negation of a conjunction can be written in a different way.

**Solution:** The shaded portion in the box represents the area that is within $P$ **and** within $Q$. In mathematical logic this is written as $P \wedge Q$. In set theory this area is represented similarly as $P \cap Q$ where the symbol $\cap$ stands for intersection. As before, the notation between mathematical logic and set theory is slightly different. However, the reasoning about the relationships and the logical implications are identical.




When you negate the statement, you completely switch what is shaded.

$$\sim(P \wedge Q)$$




A different way to think about this shaded region is to consider that it represents everything that isn't in $P$ or isn't in $Q$.

$\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$

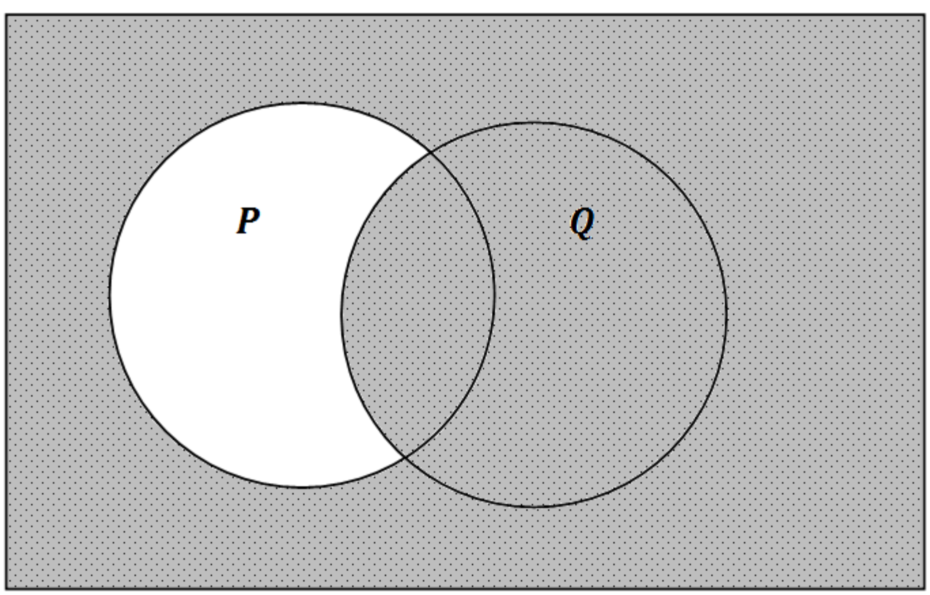
This is a second representation of De Morgan's Law.

**Example C**

Use set theory circles to interpret the negation of a conditional statement and explain how the negation of a conditional can be written in a different way.

**Solution:** The shaded portion in the box represents the area that makes the following statement true.

$$P \rightarrow Q$$



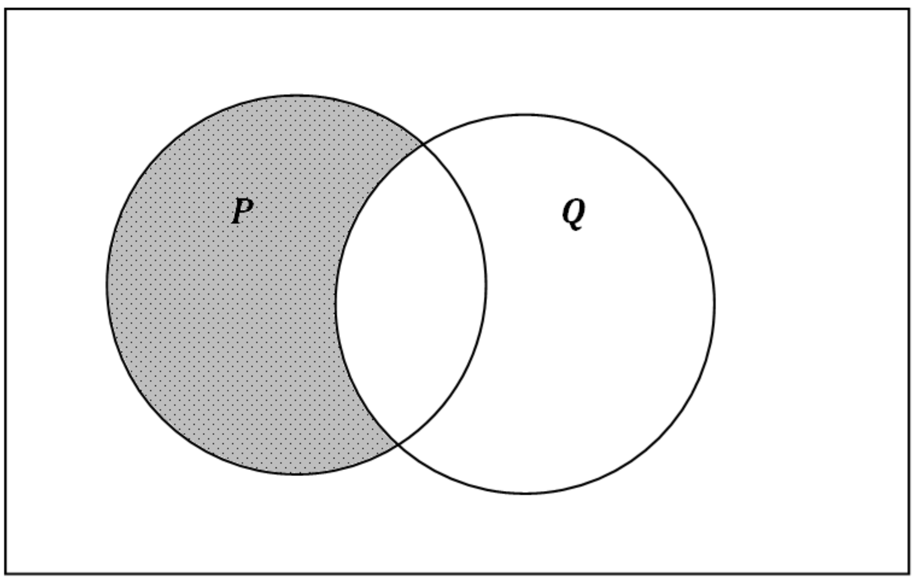
Notice that there are four different regions in the Venn Diagram which correspond to the four different rows of the conditional truth table. The only space that does not make the statement true is when $P$ is true (inside circle $P$) and $Q$ is false (outside circle $Q$).

### TABLE 16.10:

| $P$ | $Q$ | $P \to Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

To negate this statement, you switch the values in the truth table and switch the shaded region in the Venn Diagram.

$$\sim(P \to Q)$$




A different way to think about this shaded region is to see it as the space that is in $P$ but not in $Q$.

$\sim (P \to Q) \Leftrightarrow P \wedge \sim Q$

**Concept Problem Revisited**

The statement already has several negative parts, so it is incorrect to simply switch one or both of the negations.

*If I am not cold, then it is not snowing.*

- $P = I\ am\ cold.$
- $Q = It\ is\ snowing.$

### TABLE 16.11:

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $\sim P \to \sim Q$ | $\sim (\sim P \to \sim Q)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ |

Start by building up to the **original** statement. Then, add a column that completely negates the original statement. Notice that there is only one row where the final negated statement is true. That is when $P$ is false and $Q$ is true. Therefore, the negation of the original sentence is true when "I am not cold" and "it is snowing".

### Vocabulary

***De Morgan's Law*** transforms a conjunction into a disjunction using negation.

A ***tautology*** is a logical statement that is always true. A tautology is a type of basic theorem.

### Guided Practice

1. Show that the following statements are equivalent in a truth table. The symbol $\equiv$ means equivalent. To be equivalent in this case means to be true at the same time and false at the same time.

$P \wedge Q \equiv \sim (\sim P \vee \sim Q)$

2. Write a sentence in two different ways illustrating the mathematical statement in Guided Practice #1.

3. Demonstrate the following tautology in a truth table.

$B \rightarrow (A \vee \sim A)$

**Answers:**

1. While a truth table is not a proof, it can help you recognize when two statements have the same truth values.

**TABLE 16.12:**

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $\sim P \vee \sim Q$ | $\sim (\sim P \vee \sim Q)$ | $P \wedge Q$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $F$ | $F$ |

Notice that the final two columns are identical.

2. $P \wedge Q$: I like movies and I like TV. $\sim (\sim P \vee \sim Q)$: It is not the case that either I don't like movies or I don't like TV.

3. A tautology is a logical statement that is always true.

**TABLE 16.13:**

| $A$ | $\sim A$ | $B$ | $A \vee \sim A$ | $B \rightarrow (A \vee \sim A)$ |
|---|---|---|---|---|
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ |

### Practice

I'm either going to go skiing or snowboarding next weekend.

1. Identify the atomic statements in the above sentence and use logical connectives to rewrite the sentence with

symbols.

2. Write the negation of the sentence with symbols and write the negation of the sentence in words in a natural way.

Mike and John both ate lunch with me.

3. Identify the atomic statements in the above sentence and use logical connectives to rewrite the sentence with symbols.

4. Write the negation of the sentence with symbols and write the negation of the sentence in words in a natural way.

Neither my brother nor my sister wants to play with me.

5. Identify the atomic statements in the above sentence and use logical connectives to rewrite the sentence with symbols.

6. Write the negation of the sentence with symbols and write the negation of the sentence in words in a natural way.

Write negations for the following statements.

7. All dogs go to heaven.

8. My teacher is seldom wrong.

9. Everyone likes pizza.

Make truth tables for each of the following.

10. $(P \wedge Q) \vee \sim Q$

11. $P \wedge (Q \vee \sim Q)$

12. $(P \vee Q) \vee \sim R$

13. $(\sim P \wedge \sim Q) \vee \sim R$

14. What is the simplest statement that is equivalent to #11: $P \wedge (Q \vee \sim Q)$?

15. Use De Morgan's Law to find a statement equivalent to the following statement: $\sim (Q \vee \sim Q)$

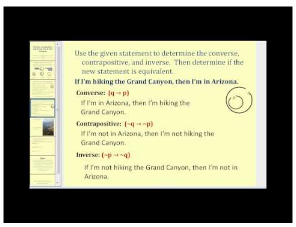
16. Use De Morgan's Law to find a statement equivalent to the following statement: $\sim (P \vee Q) \vee \sim R$

# 16.4 Inverse, Converse, and Contrapositive

Here you will examine the implications of the inverse, converse and contrapositive statements as they relate to a conditional statement.

The typical conditional statement is$P \to Q$. The inverse, converse and contrapositive are each related to the conditional statement but are not always identical. Which type of statement is always equivalent to the original?

### Watch This



MEDIA
Click image to the left for more content.

http://www.youtube.com/watch?v=IHd8jiUF3Lk James Sousa: The Converse, Contrapositive, and Inverse of an If-Then Statement

### Guidance

Consider a conditional statement $P \to Q$ where:

- $P = It\ is\ raining.$
- $Q = The\ driveway\ is\ wet.$

**Original Conditional:** $P \to Q =$ *If it is raining, then the driveway is wet.*

TABLE 16.14:

| $P$ | $Q$ | $P \to Q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

The inverse of a conditional negates both the hypothesis and the conclusion.

**Inverse:** $\sim P \to \sim Q =$ *If it is not raining, then the driveway is not wet.*

TABLE 16.15:

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $\sim P \to \sim Q$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |

TABLE 16.15: (continued)

| | | | | |
|---|---|---|---|---|
| $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

Notice that the truth values of the inverse are not identical to the truth values of the original conditional. The absence of rain does not guarantee a dry driveway. Some children could have had a water balloon party in the summer. ***Just because a statement is true does not mean that its inverse will be true!***

The converse of a conditional statement switches the order of the hypothesis and the conclusion.

**Converse:** $Q \rightarrow P =$ *If the driveway is wet, then it is raining.*

TABLE 16.16:

| $P$ | $Q$ | $Q \rightarrow P$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $T$ |

Notice that the truth values of the converse are also not identical to the truth values of the original conditional. If children playing with water balloons made the driveway wet then it isn't necessarily raining. ***Just because a statement is true does not mean that its converse will be true!***

The contrapositive of a conditional statement switches the hypothesis with the conclusion and negates both parts.

**Contrapositive:** $\sim Q \rightarrow \sim P =$ *If the driveway is not wet, then it is not raining.*

TABLE 16.17:

| $P$ | $Q$ | $\sim P$ | $\sim Q$ | $\sim Q \rightarrow \sim P$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

The contrapositive of a conditional statement is functionally equivalent to the original conditional. This is because you can logically conclude that a dry driveway means no rain. ***This means that if a statement is a true then its contrapositive will also be true.***

**Example A**

Assume the statement "everyone with blonde hair is smart" is true. Use the contrapositive to write another statement that is related and also true.

**Solution:** The statement "everyone with blonde hair is smart" can be rewritten as "if a person has blonde hair then the person is smart". The contrapositive is "if a person is not smart, then the person does not have blonde hair". This statement must be true if the original statement is true.

**Example B**

Write the inverse, converse and contrapositive of the following conditional statement.

*If you buy our product, then you are attractive.*

**Solution:** Note that advertisers regularly imply certain results about their products that may or may not be true. If you listen carefully you will notice that ironclad conditional statements are always avoided so they are not technically false advertising. At the same time, advertisers prey on the fact that many people mistakenly believe that the inverse and converse are equivalent to the original conditional.

- **Inverse:** *If you do not buy our product, then you are not attractive.*
- **Converse:** *If you are attractive, then you will buy our product.*
- **Contrapositive:** *If you are not attractive, then you will not buy our product.*

**Example C**

Assume each of the following is true. Do you end up doing your homework?

- *If it is raining then you will be tired.*
- *If you are tired you will nap.*
- *If you do not do your homework then you will take a nap.*
- *If you nap or it rains then you will not do your homework.*
- *If you do not do your homework then it will rain.*
- *Either it will rain, you will take a nap or you will not be tired.*

**Solution:** Start by identifying the individual statements for each conditional.

- $R = It\ is\ raining.$
- $T = You\ will\ be\ tired.$
- $H = You\ will\ do\ your\ homework.$
- $N = You\ will\ take\ a\ nap.$

Now, rewrite each sentence using mathematical symbols.

- Statement 1: $R \rightarrow T$
- Statement 2: $T \rightarrow N$
- Statement 3: $\sim H \rightarrow N$
- Statement 4: $(N \vee R) \rightarrow \sim H$
- Statement 5: $\sim H \rightarrow R$
- Statement 6: $R \vee N \vee T$

Next, create a train of conditional statements until you reach either "$H$" or "$\sim H$". From statement 6 you know that either $R, N$, or $T$ must be true. Consider three cases, one for $R$ being true, one for $N$ being true, and one for $T$ being true.

- Assume $R : R \rightarrow N \vee R \rightarrow \sim H$
- Assume $N : N \rightarrow N \vee R \rightarrow \sim H$
- Assume $T : T \rightarrow N \rightarrow N \vee R \rightarrow \sim H$

In all cases you do not end up doing your homework.

**Concept Problem Revisited**

The only transformation of a conditional statement that is equivalent to the original statement is the contrapositive. Being comfortable with the contrapositive is absolutely essential for logical reasoning about puzzles and riddles.

## Vocabulary

A ***premise*** is a starting statement that you use to make logical conclusions.

The ***inverse*** of a conditional statement negates both the hypothesis and the conclusion.

The ***converse*** of a conditional statement switches the order of the hypothesis and the conclusion.

The ***contrapositive*** of a conditional statement switches the hypothesis with the conclusion and negates both parts.

### Guided Practice

1. State the inverse, converse, and contrapositive of the following statement.

*If a relation passes the vertical line test, then it is a function.*

2. You and two logical friends stand in a line so that you cannot see anyone, the friend behind you can see you and the friend in the back can see both people. The three of you are shown three black hats and two white hats. The hats are mixed up and placed on you and your friends' heads in such a way that nobody knows or can see their own hat. The three of you are told when you know the color of your hat to shout out. Nobody says anything for a long time, but eventually you figure out what color hat you must have even though you cannot see it.

What color hat do you have and how do you know?

3. In a land where some people always lie and some people always tell the truth, you are approached by 2 people who both call the other a liar. You want to know if you are on the right path, so if you can only ask one of them one question what should you ask?

**Answers:**

1. **Inverse:** *If a relation does not pass the vertical line test, then the relation is not a function.*

**Converse:** *If a relation is a function, then the relation passes the vertical line test.*

**Contrapositive:** *If a relation is not a function, then the relation does not pass the vertical line test.*

2. The way to solve this puzzle is to consider what each person sees starting with the person in the back and then consider what would make them sure and what would make them unsure.

The friend in the back does not shout out. This means that he does not see two white hats because otherwise he would know that he has a black hat.

Fact #1: You and the friend behind you have either BB, BW or WB.

The friend in the middle does not shout out. This person also has figured out Fact #1. This means that if they see a white hat they know they must have a black hat. Since they don't shout out, they must not see a white hat

Fact #2: You cannot have a white hat.

**Conclusion:** Your hat is black because that is the only scenario where nobody else is sure about their own hat color.

In this question, both Fact #1 and Fact #2 require contrapositive thinking.

3. This type of question is extremely common in riddles. The trick is to use the fact that a double negative is a positive and a double positive is also a positive. Anything else is negative.

**You should ask person A if person B would tell you that you are on the right path.**

There will be 4 scenarios that you can organize in a table. Each scenario needs to be carefully thought through. For example, the upper right hand response is yes because if you are on the wrong path the person telling the truth would tell you that no, you are not on the right path. The liar would respond oppositely telling you yes.

**TABLE 16.18:**

| | **Right path** | **Wrong Path** |
|---|---|---|
| A is liar | No | Yes |
| A tells truth | No | Yes |

You can be sure you are on the right path if the response is no and you can be sure you are on the wrong path if the response is yes.

### Practice

Assume each statement is true. Use the contrapositive to write another statement that is related and also true.

1. All unripe fruit is bad.

2. All bears like honey.

3. No desserts are healthy.

4. Music by Taylor Swift is good.

5. Everyone who is overweight is unhealthy.

Write the inverse, converse, and contrapositive for each of the following statements.

6. Puppies like to play.

7. If I don't like something, then I won't buy it.

8. Everyone at the party is popular.

9.You like music if you go to a concert.

10. All apples have cores.

I. My pants are the only things I have that are made of jean material.
II. All the clothes you gave me are the right size.
III. None of my pants are the right size.

11. Write each of the above statements and its contrapositive symbolically.

12. Determine the final conclusion about "the clothes from you" by stringing the statements/contrapositives together.

I. Nobody who is experienced is unsuccessful.
II. Mike is always confused.
III. No successful person is always confused.

13. Write each of the above statements and its contrapositive symbolically.

14. Determine the final conclusion about "Mike" by stringing the statements/contrapositives together.

I. All the plates that got shipped are cracked.
II. None of the plates from your mother got shipped.
III. Plates that didn't get shipped should not go in the trash.

15. Write each of the above statements and its contrapositive symbolically.

16. Can you determine whether or not the "plates from your mother" should go in the trash?

You learned that declarative statements can be joined with other declarative statements using the words not, and, or, if and then. You saw that the relationship between all but the simplest statements requires investigation in a precise way. You formalized the definitions of the words "or" and "not". You practiced logical reasoning through puzzles, proofs and riddles. The focus was on metamathematical logic, the how and the why of thinking mathematically.

# 16.5 References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