CK-12 Algebra I - Honors

Concept Collection

CK-12 Algebra I Concepts - Honors

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Kaitlyn Spong

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CHAPTER 1

The Real Number System

Chapter Outline

Introduction

Here you'll review how to perform the basic operations of addition, subtraction, multiplication and division with real numbers. You will simplify expressions using the order of operations. You will also learn how to express a fraction as a decimal and a decimal as a fraction. Finally, you will represent real numbers on a number line.

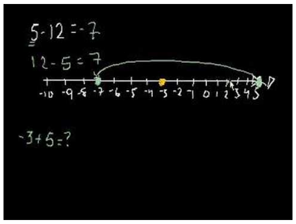
1.1 Addition of Integers

Here you'll learn to add integers using different representations including a number line. These methods will lead to the formation of two rules for adding integers.

On Monday, Marty borrows \$50.00 from his father. On Thursday, he gives his father \$28.00. Can you write an addition statement to describe Marty's financial transactions?

Watch This

Khan Academy Adding/Subtracting Negative Numbers



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Guidance

When adding integers, you need to make sure you follow two rules:

1. Integers with unlike signs must be subtracted. The answer will have the same sign as that of the higher number.
2. Integers with the same sign must be added. The answer will have the same sign as that of the numbers being added.

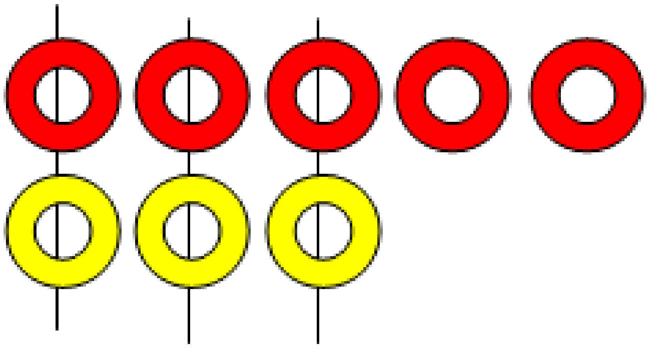
In order to understand why these rules work, you can represent the addition of integers with manipulatives such as color counters or algebra tiles. A number line can also be used to show the addition of integers. The following examples show how to use these manipulatives to understand the rules for adding integers.

Example A

$5+(-3)=?$

Solution: This problem can be represented with color counters. In this case, the red counters represent positive numbers and the yellow ones represent the negative numbers.

One positive counter and one negative counter equals zero because $1+(-1)=0$. Draw a line through the counters that equal zero.

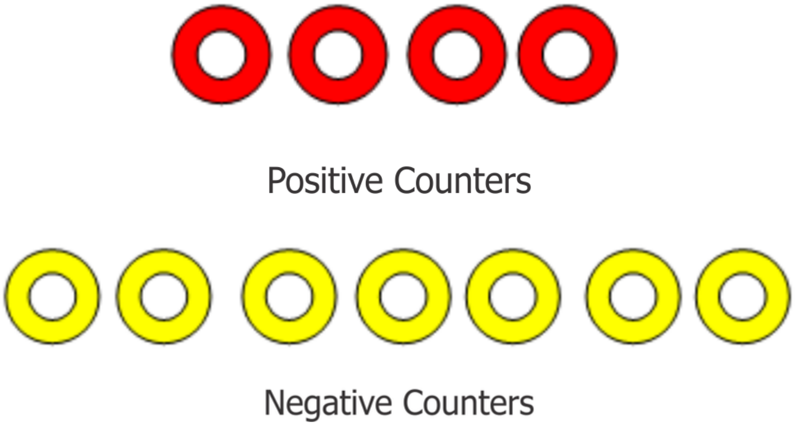


The remaining counters represent the answer. Therefore, $5+(-3)=2$. The answer is the difference between 5 and 3. The answer takes on the sign of the larger number. In this case, the five has a positive value and it is greater than 3.

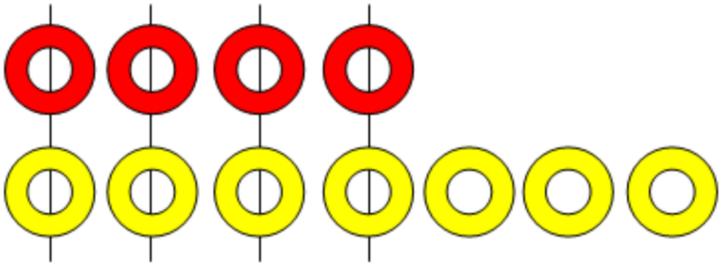
Example B

$4+(-7)=?$

Solution:



Draw a line through the counters that equal zero.

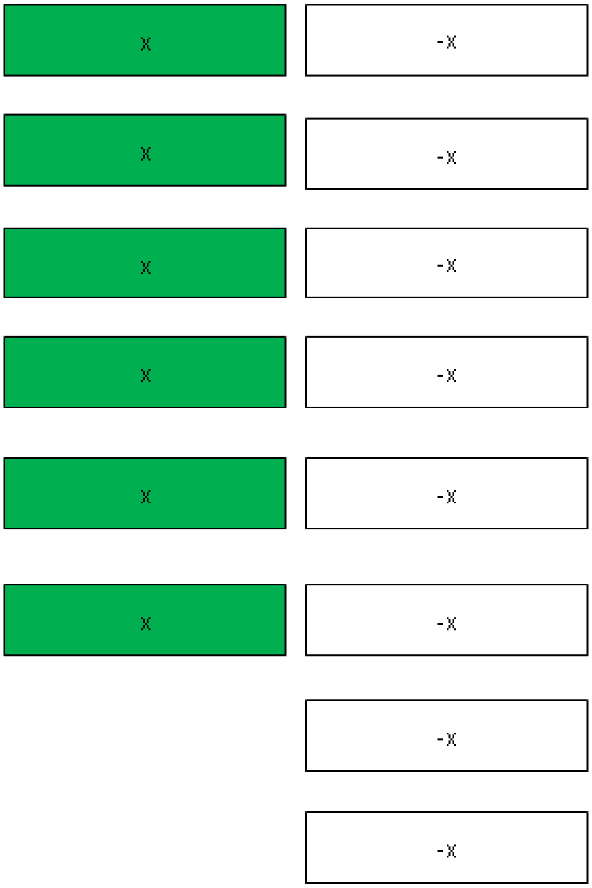


The remaining counters represent the answer. Therefore, $4+(-7)=-3$. The answer is the difference between 7 and 4. The answer takes on the sign of the larger number, which is 7 in this case.

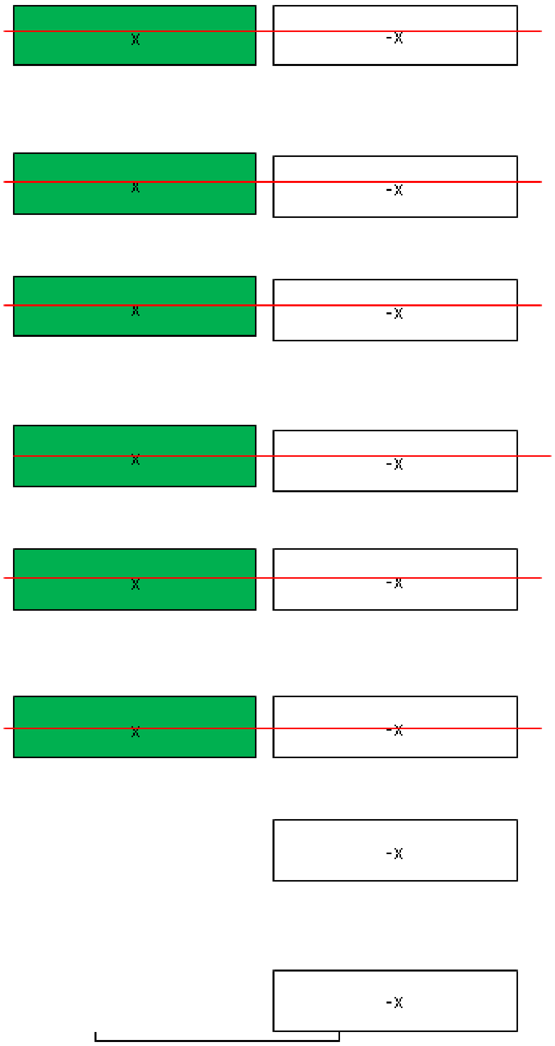
Example C

$6x + (-8x) = ?$

Solution: This same method can be extended to adding variables. Algebra tiles can be used to represent positive and negative values.



The green algebra tiles represent positive x and the white tiles represent negative x. There are 6 positive x *tiles* and 8 negative x *tiles*.

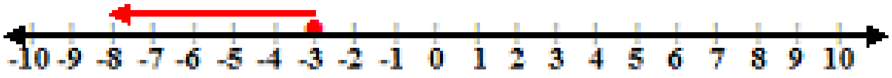


The remaining algebra tiles represent the answer. There are two negative x tiles remaining. Therefore, $(6x)+(-8x)=-2x$. The answer is the difference between $8x$ and $6x$. The answer takes on the sign of the larger coefficient, which in this case is 8.

Example D

$(-3)+(-5)=?$

Solution: You can solve this problem with a number line. Indicate the starting point of -3 by using a dot. From this point, add a -5 by moving five places to the left. You will stop at -8.



The point where you stopped is the answer to the problem. Therefore, $(-3)+(-5)=-8$

Concept Problem Revisited

On Monday, Marty borrows \$50.00 from his father. On Thursday, he gives his father \$28.00.

Marty borrowed \$50.00 which he must repay to his father. Therefore Marty has $-\$50.00$.

He returns \$28.00 to his father. Now Marty has $-\$50.00+(\$28.00)=-\$22.00$. He still owes his father \$22.00.

Vocabulary

Integer

All natural numbers, their opposites, and zero are ***integers***. A number in the list ..., -3, -2, -1, 0, 1, 2, 3...

Irrational Numbers

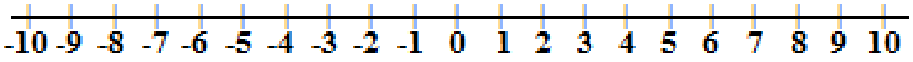
The ***irrational numbers*** are those that cannot be expressed as the ratio of two numbers.

Natural Numbers

The ***natural numbers*** are the counting numbers and consist of all positive, whole numbers. The ***natural numbers*** are the numbers in the list 1, 2, 3... and are often referred to as positive integers.

Number Line

A ***number line*** is a line that matches a set of points and a set of numbers one to one.



Rational Numbers

The ***rational numbers*** are numbers that can be written as the ratio of two numbers $\frac{a}{b}$ with $b \neq 0$.

Real Numbers

The rational numbers and the irrational numbers make up the ***real numbers***.

Guided Practice

1. $(-7)+(+5)=?$

2. $8+(-2)=?$

3. Determine the answer to $(-6)+(-3)=?$ and $(2)+(-5)=?$ by using the rules for adding integers.

Answers:

1. $(-7)+(+5)=5-7=-2$

2. $8+(-2)=8-2=6$

3. $(-6)+(-3)=-9.$

$(2)+(-5)=2-5=-3.$

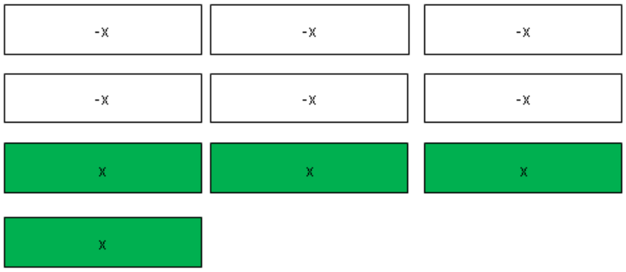
Practice

Complete the following addition problems using any method.

1. $(-7)+(-2)$
2. $(6)+(-8)$
3. $(5)+(4)$
4. $(-7)+(9)$
5. $(-1)+(5)$
6. $(8)+(-12)$
7. $(-2)+(-5)$
8. $(3)+(4)$
9. $(-6)+(10)$
10. $(-1)+(-7)$
11. $(-13)+(9)$
12. $(-3)+(-8)+(12)$
13. $(14)+(-6)+(5)$
14. $(15)+(-8)+(-9)$
15. $(7)+(6)+(-9)+(-8)$

For each of the following models, write an addition problem and answer the problem.

16.



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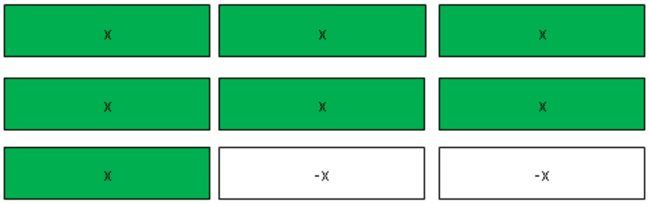

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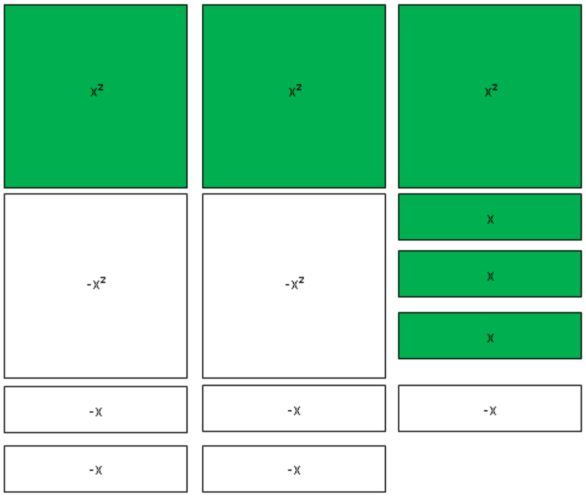
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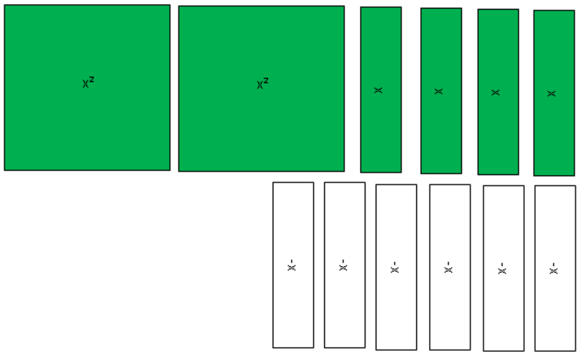
19.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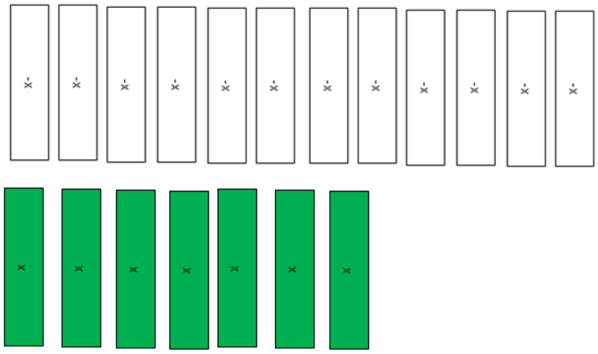
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1.2 Addition of Fractions

Here you will review how to add fractions using different representations.

Lily and Howard ordered a pizza that was cut into 8 slices. Lily ate 3 slices and Howard ate 4 slices. What fraction of the pizza did each person eat? What fraction of the pizza did they eat all together?

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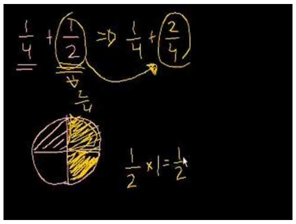
Khan Academy Adding and Subtracting Fractions



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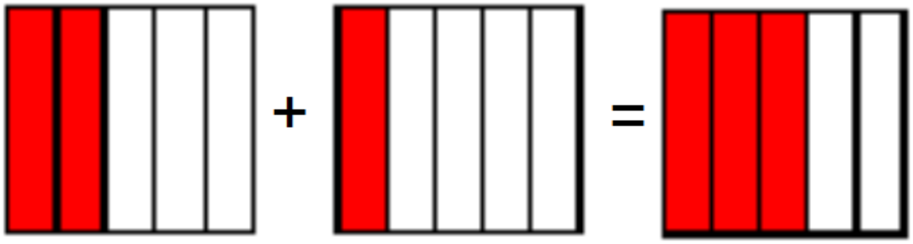
Guidance

$\frac{2}{5}+\frac{1}{5}=?$

The problem above can be represented using fraction strips.



$$\boxed{\frac{2}{5}+\frac{1}{5}=\frac{2+1}{5}=\frac{3}{5}}$$

To add fractions, the fractions must have the same bottom numbers (denominators). In this case, both fractions have a denominator of 5. The answer is the result of adding the top numbers (numerators). The numbers in the numerator are 1 and 2. The sum of 1 and 2 is 3. This sum is written in the numerator over the denominator of 5. Therefore $\frac{2}{5}+\frac{1}{5}=\frac{3}{5}$.

A number line can also be used to show the addition of fractions, as you will explore in Example C.

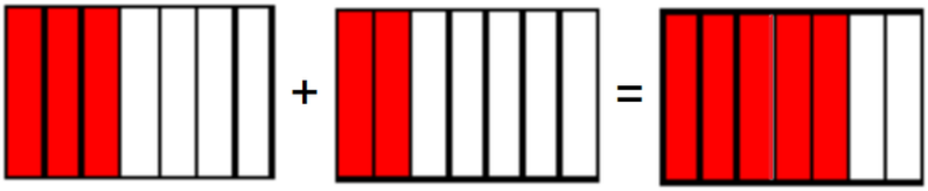
The sum of two fractions will sometimes result in an answer that is an improper fraction. An improper fraction is a fraction that has a larger numerator than denominator. This answer can be written as a mixed number. A mixed number is a number made up of a whole number and a fraction.

In order to add fractions that have different denominators, the fractions must be expressed as equivalent fractions with a least common denominator (LCD). The sum of the numerators can be written over the common denominator.

Example A

$\frac{3}{7}+\frac{2}{7}=?$

Solution:

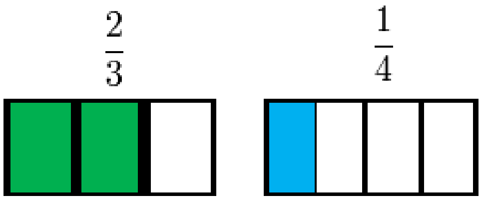


$$\boxed{\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}=\frac{5}{7}}$$

Example B

Louise is taking inventory of the amount of water in the water coolers located in the school. She estimates that one cooler is $\frac{2}{3}$ full and the other is $\frac{1}{4}$ full. What single fraction could Louise use to represent the amount of water of the two coolers together?

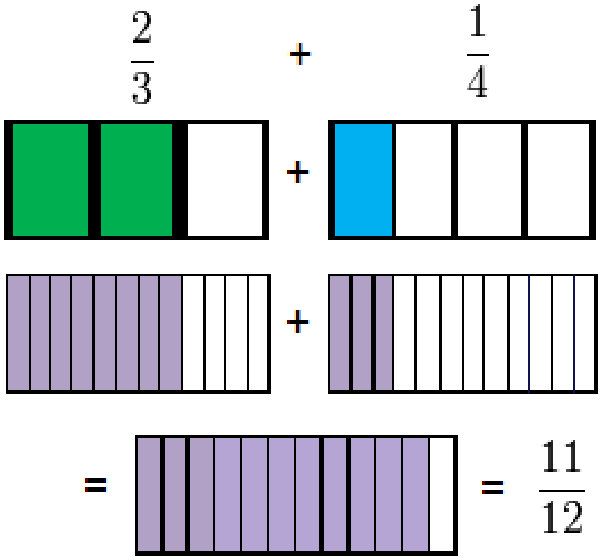
Solution: Use fraction strips to represent each fraction.



$\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions. $\frac{2}{3}\left(\frac{4}{4}\right)=\frac{8}{12}$.

$\frac{1}{4}$ and $\frac{3}{12}$ are equivalent fractions. $\frac{1}{4}\left(\frac{3}{3}\right)=\frac{3}{12}$.

The two green pieces will be replaced with eight purple pieces and the one blue piece will be replaced with three purple pieces.



The amount of water in the two coolers can be represented by the single fraction $\frac{11}{12}$.

$$\frac{2}{3}+\frac{1}{4}=\frac{8}{12}+\frac{3}{12}=\frac{11}{12}$$

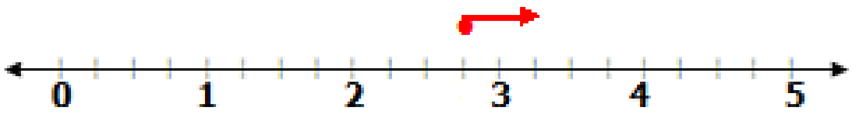
The denominator of 12 is the LCD (least common denominator) of $\frac{2}{3}$ and $\frac{1}{4}$ because it is the LCM (least common multiple) of the numbers 3 and 4.

Example C

What is $2\frac{3}{4}+\frac{1}{2}$?

Solution: The number line is labeled in intervals of 4 which indicates that each interval represents $\frac{1}{4}$. From zero, move to the number 2 plus 3 more intervals to the right. Mark the location. This represents $2\frac{3}{4}$.

From here, move to the right $\frac{1}{2}$ or $\frac{1}{2}$ of 4, which is 2 intervals.



The sum of $2\frac{3}{4}$ and $\frac{1}{2}$ is $3\frac{1}{4}$.

Concept Problem Revisited

Lily ate $\frac{3}{8}$ of the pizza because she ate 3 out of the 8 slices. Howard ate $\frac{4}{8}$ (or $\frac{1}{2}$) of the pizza. Together they ate 7 slices which is $\frac{7}{8}$ of the pizza.

Vocabulary

Denominator

The ***denominator*** of a fraction is the number on the bottom that indicates the total number of equal parts in the whole or the group. $\frac{5}{8}$ has ***denominator*** 8.

Fraction

A ***fraction*** is any rational number that is not an integer.

Improper Fraction

An ***improper fraction*** is a fraction in which the numerator is larger than the denominator.
$\frac{8}{3}$ is an ***improper fraction.***

LCD

The ***least common denominator*** is the lowest common multiple of the denominators of two or more fractions. The ***least common denominator*** of $\frac{3}{4}$ and $\frac{1}{5}$ is 20.

LCM

The ***least common multiple*** is the lowest common multiple that two or more numbers share. The ***least common multiple*** of 6 and 4 is 12.

Mixed Number

A ***mixed number*** is a number made up of a whole number and a fraction such as $4\frac{3}{5}$.

Numerator

The ***numerator*** of a fraction is the number on top that is the number of equal parts being considered in the whole or the group. $\frac{5}{8}$ has ***numerator*** 5.

Guided Practice

1. $\frac{1}{2}+\frac{1}{6}=?$
2. $\frac{1}{6}+\frac{3}{4}=?$
3. $\frac{2}{5}+\frac{2}{3}=?$

Answers:

1.

$$\frac{1}{2}+\frac{1}{6}=\frac{3}{6}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

2.

$$\frac{1}{6}+\frac{3}{4}=\frac{2}{12}+\frac{9}{12}=\frac{11}{12}$$

3.

$$\frac{2}{5}+\frac{2}{3}=\frac{6}{15}+\frac{10}{15}=\frac{16}{15}=1\frac{1}{15}$$

$\frac{16}{15}$ is an improper fraction. An improper fraction is one with a larger numerator than denominator. $\frac{15}{15}=1$ plus there is $\frac{1}{15}$ left over. This can be written as a whole number and a fraction $1\frac{1}{15}$. This representation is called a mixed number.

Practice

Complete the following addition problems using any method.

1. $\frac{1}{4}+\frac{5}{8}$
2. $\frac{2}{5}+\frac{1}{3}$
3. $\frac{2}{9}+\frac{2}{3}$
4. $\frac{3}{7}+\frac{2}{3}$
5. $\frac{7}{10}+\frac{1}{5}$
6. $\frac{2}{3}+\frac{1}{2}$
7. $\frac{2}{5}+\frac{3}{10}$
8. $\frac{5}{9}+\frac{2}{3}$
9. $\frac{3}{8}+\frac{3}{4}$
10. $\frac{3}{5}+\frac{3}{10}$
11. $\frac{7}{11}+\frac{1}{2}$
12. $\frac{7}{8}+\frac{5}{12}$
13. $\frac{3}{4}+\frac{5}{6}$
14. $\frac{5}{6}+\frac{2}{5}$
15. $\frac{4}{5}+\frac{3}{4}$

For each of the following questions, write an addition statement and find the result. Express all answers as either proper fraction or mixed numbers.

16. Karen used $\frac{5}{8}$ *cups* of flour to make cookies. Jenny used $\frac{15}{16}$ *cups* of flour to make a cake. How much flour did they use altogether?
17. Lauren used $\frac{3}{4}$ *cup* of milk, $1\frac{1}{3}$ *cups* of flour and $\frac{3}{8}$ *cup* of oil to make pancakes. How many cups of ingredients did she use in total?
18. Write two fractions with different denominators whose sum is $\frac{5}{6}$.
19. Allan's cat ate $2\frac{2}{3}$ *cans* of food in one week and $3\frac{1}{4}$ *cans* the next week. How many cans of food did the cat eat in two weeks?
20. Amanda and Justin each solved the same problem.

Amanda's Solution:

$$\frac{1}{6}+\frac{3}{4}$$
$$\frac{2}{12}+\frac{9}{12}$$
$$=\frac{11}{24}$$

Justin's Solution:

$$\frac{1}{6}+\frac{3}{4}$$
$$\frac{2}{12}+\frac{9}{12}$$
$$=\frac{11}{12}$$

Who is correct? What would you tell the person who has the wrong answer?

1.3 Addition of Decimals

Here you review how to add decimals.

Stephen went shopping to buy some new school supplies. He bought a backpack that cost $28.67 and a scientific calculator for $34.88. How much money did Stephen spend altogether?

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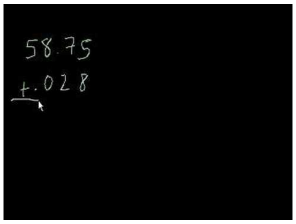
Khan Academy Adding Decimals



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Guidance

When you add decimals you are adding whole numbers and like fraction parts. To make this process simpler, you should add decimal numbers using the vertical alignment method. The decimal points must be kept directly under each other and the digits in the same place value must be kept in line with each other. If the decimal numbers are signed numbers, the rules for adding integers are applied to the problem.

Example A

$87.296 + 48.6 = ?$

Solution: Begin by writing the question using the vertical alignment method.

$$\begin{array}{r} 87.296 \\ \underline{+48.6} \end{array}$$

Don't forget that the decimal points must be kept directly under one another. To ensure that the digits are aligned correctly, add zeros to 48.6.

$$\begin{array}{r} 87.296 \\ \underline{+48.600} \end{array}$$

Add the numbers.

$$\begin{array}{r} 87.296 \\ \underline{+48.6\textcolor{blue}{00}} \\ 135.896 \end{array}$$

Example B

$(97.38)+(-45.17)$

Solution: The first step is to write the problem using the vertical alignment method. The two decimal numbers that are being added have opposite signs. Apply the same rule that you used when adding integers that had opposite signs – subtract the numbers and use the sign of the larger number in the answer.

$$\begin{array}{r} 97.38 \\ \underline{-45.17} \\ 52.21 \end{array}$$

The larger number is 97.38 and it has a positive sign. This means that the sign of the answer will also be a positive value.

Example C

$(-168.8)+(-217.4536)$

Solution: The first step is to write the problem using the vertical alignment method.

$$\begin{array}{r} -168.8 \\ \underline{+-217.4536} \end{array}$$

To ensure that the digits are aligned correctly, add zeros to 168.8.

$$\begin{array}{r} -168.8\textcolor{blue}{000} \\ \underline{+-217.4536} \end{array}$$

Add the numbers.

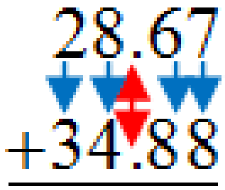
$$\begin{array}{r} -168.8\textcolor{blue}{000} \\ \underline{+-217.4536} \\ -386.2536 \end{array}$$

The numbers being added each had negative signs. This means that the sign of the answer is also a negative value.

Concept Problem Revisited

Stephen went shopping to buy some new school supplies. He bought a backpack that cost \$28.67 and a scientific calculator for \$34.88.

Stephen bought two items. To determine the total amount of money he spent, add the prices of the items.



The numbers and the decimal points have been correctly aligned. Now add the numbers.

$$\begin{array}{r} ^{1}2\,^{1}8.\,^{1}6\,7 \\ +34.88 \\ \hline 63.55 \end{array}$$

Stephen spent \$63.55 altogether.

Vocabulary

Decimal Point

A ***decimal point*** is the place marker in a decimal number that separates the whole number and the fraction part. The decimal number 326.45 has the decimal point between the six and the four.

Guided Practice

1. $45.36+15+137.692+32.8=?$

2. $(53.69)+(-33.7)+(6.298)=?$

3. $14.68+39.217=?$

Answers:

1.You can add all three numbers at once.

$$\begin{array}{r} {\scriptstyle 21\ \ 1\ \ \ \ } \\ 45.360 \\ 15.000 \\ 137.692 \\ +32.800 \\ \hline 230.852 \end{array}$$

2. Add the two positive numbers together:

$$\begin{array}{r} 53.690 \\ +\ \ 6.298 \\ \hline 59.988 \end{array}$$

Subtract the negative number from the result:

$$\begin{array}{r} 59.988 \\ -33.700 \\ \hline 26.288 \end{array}$$

3. $14.68+39.217=53.897$

Practice

Add the following numbers.

1. $14.36+9.42$
2. $52.72+27.163$
3. $0.26+4.5+1.137$
4. $37.231+14.567$
5. $78.32+6.2+19.46$
6. $65.23+12.75$
7. $148.067+53.78+6.9$
8. $56.75+14.9294+17.854$
9. $18+26.87+65.358$
10. $23.067+268.93+9.4$
11. $(-24.69)+(-39.87)$
12. $(76.35)+(-36.68)$
13. $(-12.5)+(47.97)+(-21.653)$
14. $(62.462)+(254.69)+(-427.9)$
15. $(-37.76)+(-45.8)+(53.92)$

Determine the answer to the following problems.

16. When the owners of the Finest Fixer Co. completed a small construction job, they found that the following expenses had been incurred: labour, \$975.75; gravel, \$88.79; sand, \$43.51; cement, \$284.96; and bricks \$2214.85. What bill should they give the customer if they want to make a profit of \$225 for the job?

17. A tile setter purchases the following supplies for the day:
    - One bag of thin-set mortar - @$5.67 per bag
    - 44 sq ft of tile - @$107.80 for 44 sq ft of tile
    - One gallon of grout - @$17.97 per gallon
    - One container of grout sealer - @$32.77 per container
    - 3 containers of grout and tile cleaner - @$5.99 per container
    - 4 scrub pads - @$2.78 each
    - One trowel - @ $3.95 each
    - 2 packages of tile spacers - @$2.27 each
    - One grout bag - @$2.79 each
    - One grout float - @$10.45 each

    What is the cost of these items before tax is added?
18. The four employees of the Broken Body Shop earned the following amounts last week: $815.86, $789.21, $804.18 and $888.35. What is the average weekly pay for the employees?
19. Jennifer bought the following school supplies:
    - 1000 sheets of paper - @$14.67
    - 36 pencils - @ $6.55
    - 1 binder - @$18.48
    - 1 backpack - @ $22.74
    - 1 lunch bag - @ $4.64

    How much did Jennifer spend on these supplies before taxes?
20. A local seamstress needs to purchase fabric to sew curtains for the local theatre. She needs 123.75 yd. of black cotton for a backdrop, 217.4 yd. of white linen for stage curtains, 75 yd. for accessory curtains and 98.5 yd. for costumes. How many yards of fabric must be purchased to fill this order?

1.4 Subtraction of Integers

Here you will review how to subtract integers using different representations.

Molly goes shopping with \$20. She buys a new notebook for \$4 and a soda for \$2. How much money does she have left?

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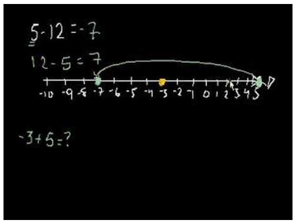
Khan Academy Adding/Subtracting Negative Numbers



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Guidance

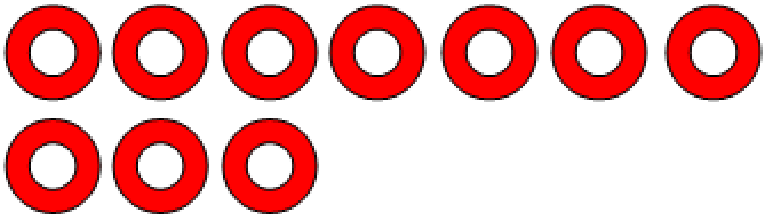
To subtract one signed number from another, change the problem from a subtraction problem to an addition problem and change the sign of the number that was originally being subtracted. In other words, to subtract signed numbers simply add the opposite. Then, follow the rules for adding signed numbers.

The subtraction of integers can be represented with manipulatives such as color counters and algebra tiles. A number line can also be used to show the subtraction of integers.

Example A

$7-(-3)=?$

Solution: This is the same as $7+(+3)=?$. The problem can be represented with color counters. In this case, the red counters represent positive numbers.



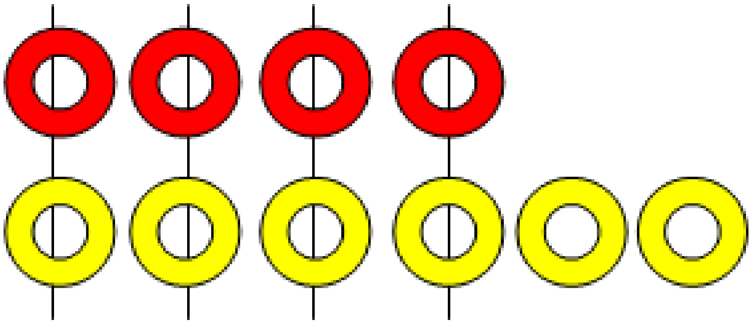
The answer is the sum of 7 and 3. $7+(+3)=10$

Example B

$4-(+6)=$

Solution: Change the problem to an addition problem and change the sign of the original number that was being subtracted.

$4-(+6)=4+(-6)=?$

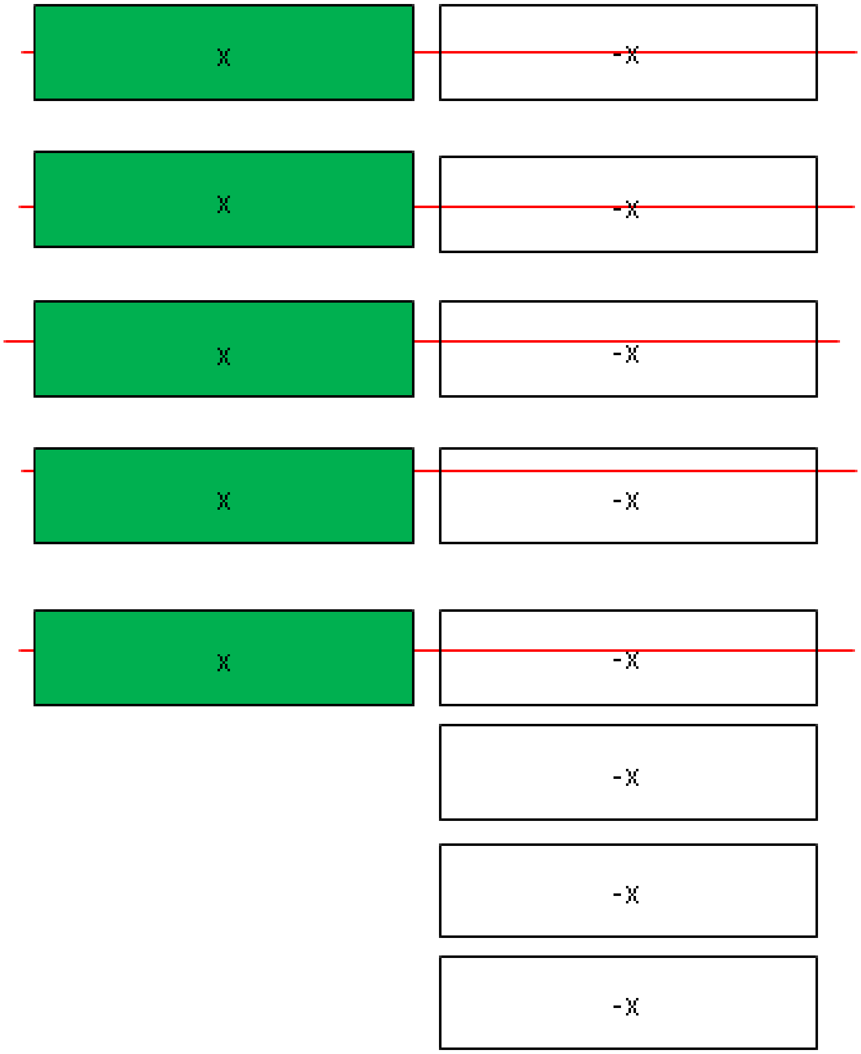


The remaining counters represent the answer. Therefore, $4-(+6)=-2$. The answer is the difference between 6 and 4 and takes the sign of the larger number.

Example C

$5x-(+8x)=?$

Solution: You can rewrite the problem: $5x-(+8x)=5x+(-8x)=?$



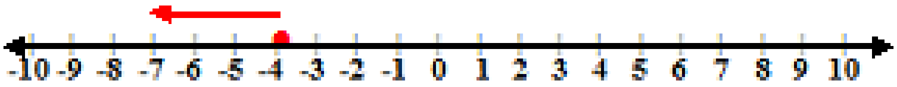
The remaining algebra tiles represent the answer. There are three negative x tiles remaining. Therefore, $(6x)-(+8x)=-3x$.

Example D

$(-4)-(+3)=?$

Solution: This is the same as $(-4)+(-3)=?$. The solution to this problem can be determined by using the number line.

Indicate the starting point of -4 by using a dot. From this point, add a -3 by moving three places to the left. You will stop at -7.



The point where you stopped is the answer to the problem. Therefore, $(-4)-(+3)=-7$

Concept Problem Revisited

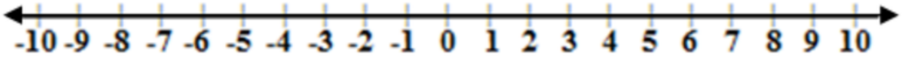
Molly goes shopping with \$20. She buys a new notebook for \$4 and a soda for \$2. You can figure out how much money she has left by subtracting.

$\$20 - \$4 - \$2 = \$14$

Vocabulary

Integer
All natural numbers, their opposites, and zero are ***integers***. A number in the list ..., -3, -2, -1, 0, 1, 2, 3...

Number Line
A ***number line*** is a line that matches a set of points and a set of numbers one to one.



Guided Practice

1. $(-2) - (-6) = ?$

2. $7 - (+5) = ?$

3. $(-8) - (-5) = ?$

4. $(-4) - (+9) = ?$

Answers:

1. $(-2) - (-6) = -2 + 6 = 6 - 2 = 4.$

2. $7 - (+5) = 7 - 5 = 2.$

3. $(-8) - (-5) = -8 + 5 = 5 - 8 = -3$

4. $(-4) - (+9) = -4 - 9 = -13$

Practice

Subtract.

1. $(-9) - (-2)$
2. $(5) - (+8)$
3. $(5) - (-4)$
4. $(-7) - (-9)$
5. $(6) - (+5)$
6. $(8) - (+4)$
7. $(-2) - (-7)$
8. $(3) - (+5)$
9. $(-6) - (-10)$
10. $(-4) - (-7)$
11. $(-13) - (-19)$

12. $(-6)-(+8)-(-12)$
13. $(14)-(+8)-(-6)$
14. $(18)-(+8)-(+3)$
15. $(10)-(-6)-(+4)-(+2)$

For each of the following models, write a subtraction problem and answer the problem.

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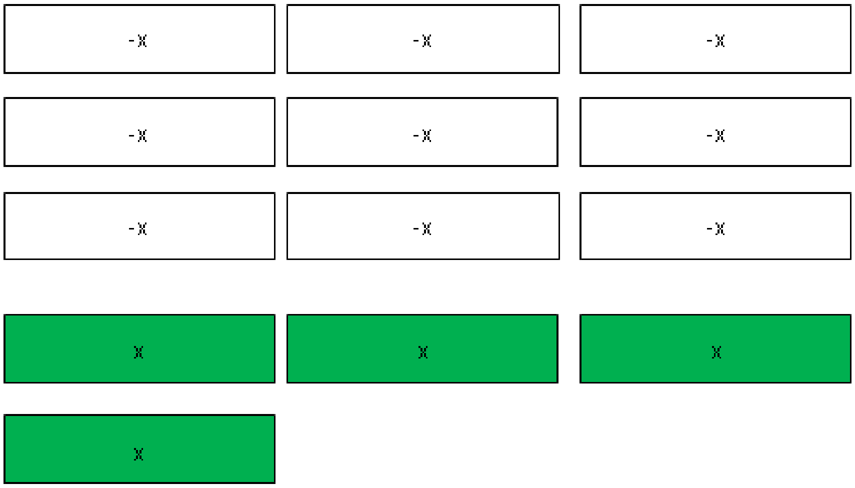

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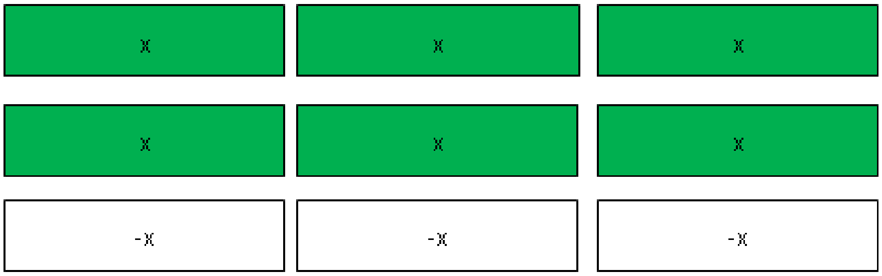

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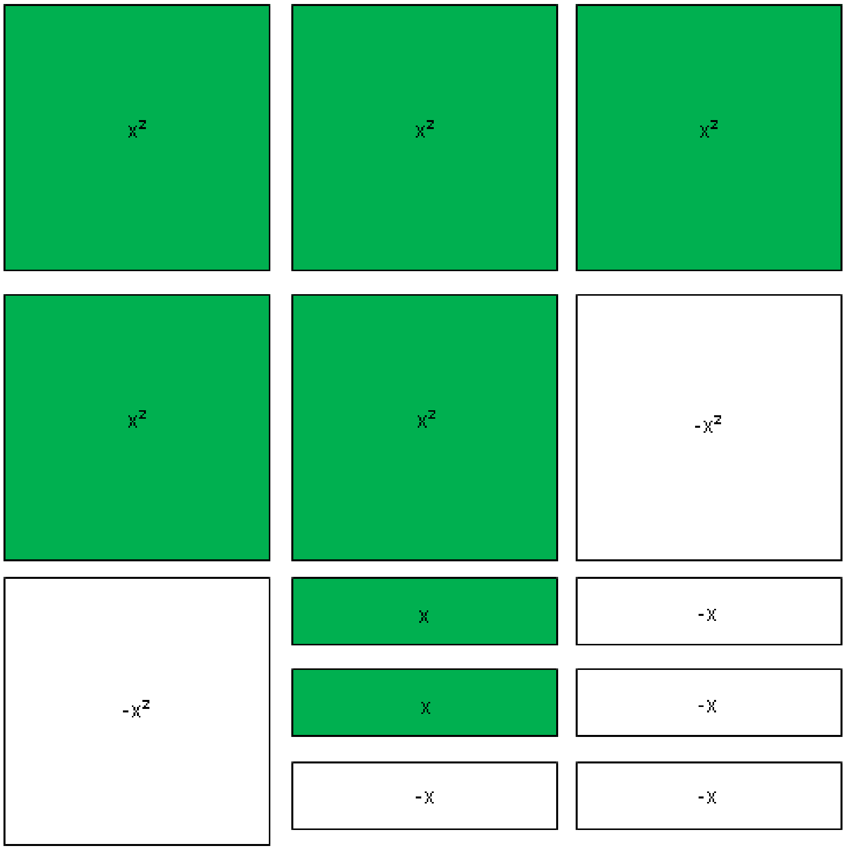

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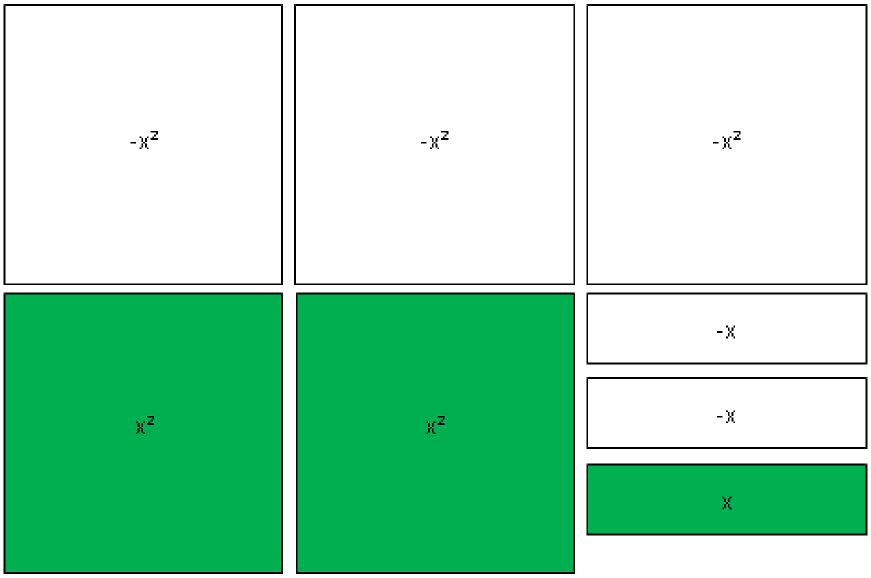

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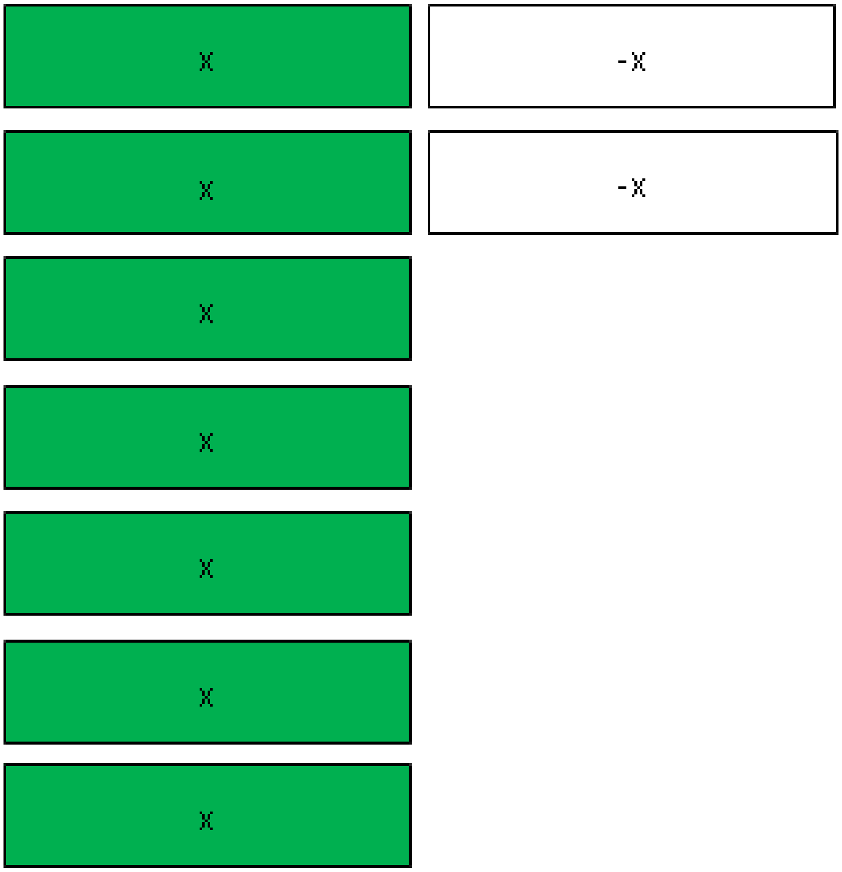
x
-x
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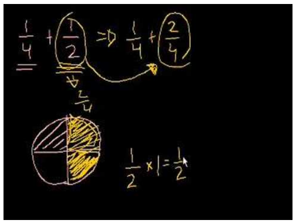
1.5 Subtraction of Fractions

In this concept you will review the rules for subtracting fractions.

Julian and Suz ordered a pizza that was cut into 10 slices. Suz ate 3 slices and Julian ate 4 slices. What fraction of the pizza did each person eat? What fraction of the pizza is left?

Watch This

Khan Academy Adding and Subtracting Fractions



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Guidance

$\frac{5}{7}-\frac{2}{7}=?$

The problem above can be represented with fraction strips:

$$\boxed{\frac{5}{7}-\frac{2}{7}=\frac{5-2}{7}=\frac{3}{7}}$$

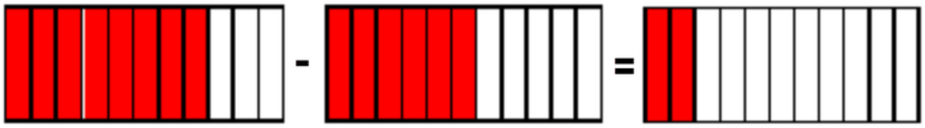
To subtract fractions, the fractions must have the same bottom numbers (denominators). In this case, both fractions have a denominator of 7. The answer is the result of subtracting the top numbers (numerators).

In order to subtract fractions that have different denominators, the fractions must be expressed as equivalent fractions with a least common denominator (LCD). The difference of the numerators can be written over the common denominator.

Example A

$\frac{8}{11}-\frac{6}{11}=?$

Solution:

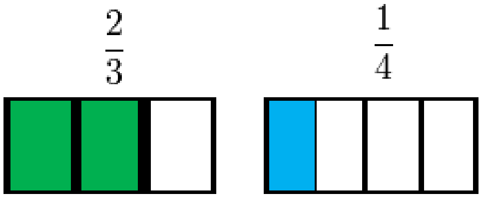


$$\boxed{\frac{8}{11} - \frac{6}{11} = \frac{8-6}{11} = \frac{2}{11}}$$

Example B

Bessie is measuring the amount of soda in the two coolers in the cafeteria. She estimates that the first cooler is $\frac{2}{3}$ full and the second cooler is $\frac{1}{4}$ full. What single fraction could Bessie use to represent how much more soda is in the first cooler than in the second cooler?

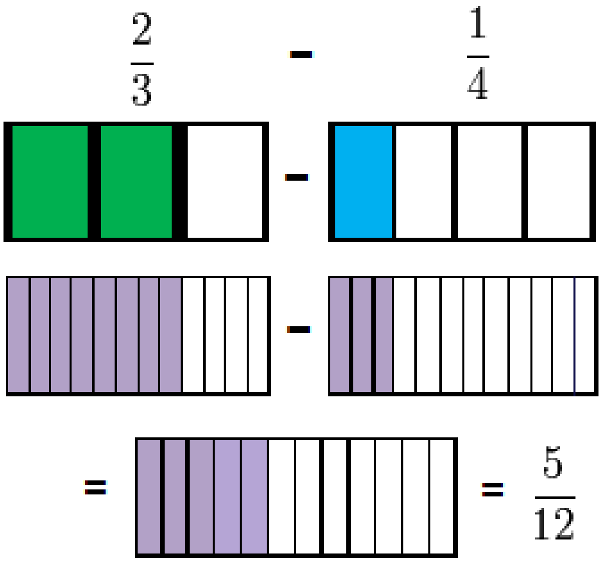
Solution: Use fraction strips to represent each fraction.



$\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions. $\frac{2}{3}\left(\frac{4}{4}\right) = \frac{8}{12}$.

$\frac{1}{4}$ and $\frac{3}{12}$ are equivalent fractions. $\frac{1}{4}\left(\frac{3}{3}\right) = \frac{3}{12}$.

The two green pieces will be replaced with eight purple pieces and the one blue piece will be replaced with three purple pieces.



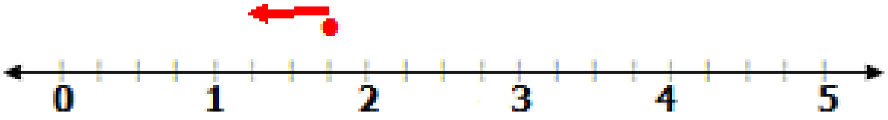
The denominator of 12 is the LCD (least common denominator) of $\frac{2}{3}$ and $\frac{1}{4}$ because it is the LCM (least common multiple) of the denominators 3 and 4.

Therefore, there is $\frac{5}{12}$ more soda in the first cooler than in the second.

Example C

$1\frac{3}{4} - \frac{1}{2}$

Solution: The number line is labeled in intervals of 4. This indicates that each interval represents $\frac{1}{4}$. From zero, move to the number 1 plus 3 more intervals to the right. Mark the location. This represents $1\frac{3}{4}$. From there, move to the left $\frac{1}{2}$ or $\frac{1}{2}$ of 4, which is 2 intervals. An equivalent fraction for $\frac{1}{2}$ is $\frac{2}{4}$.



The difference of $1\frac{3}{4}$ and $\frac{1}{2}$ is $1\frac{1}{4}$.

Concept Problem Revisited

Julian and Suz ordered a pizza that was cut into 10 slices. Suz ate 3 slices and Julian ate 4 slices. What fraction of the pizza did each person eat? What fraction of the pizza is left?

Suz ate $\frac{3}{10}$ of the pizza because she ate 3 out of the 10 slices. Julian ate $\frac{4}{10}$ of the pizza. Together they ate $\frac{7}{10}$ of the pizza. $\frac{10}{10} - \frac{7}{10} = \frac{3}{10}$. Therefore, $\frac{3}{10}$ of the pizza is left.

Vocabulary

Denominator
The ***denominator*** of a fraction is the number on the bottom that indicates the total number of equal parts in the whole or the group. $\frac{5}{8}$ has ***denominator*** 8.

Fraction
A ***fraction*** is any rational number that is not an integer.

LCD
The ***least common denominator*** is the lowest common multiple of the denominators of two or more fractions. The ***least common denominator*** of $\frac{3}{4}$ and $\frac{1}{5}$ is 20.

LCM
The ***least common multiple*** is the lowest common multiple that two or more numbers share. The ***least common multiple*** of 6 and 5 is 30.

Numerator
The ***numerator*** of a fraction is the number on top that is the number of equal parts being considered in the whole or the group. $\frac{5}{8}$ has **’numerator** 5.

Guided Practice

1. $\frac{7}{10} - \frac{2}{5} = ?$
2. $\frac{7}{8} - \frac{1}{2}$.
3. $\frac{5}{8} - \frac{1}{3} = ?$
4. $\frac{4}{5} - \frac{1}{4} = ?$

Answers:

1.

$$\frac{7}{10} - \frac{4}{10} = \frac{7-4}{10} = \frac{3}{10}$$

2. $\frac{7}{8} - \frac{4}{8} = \frac{7-4}{8} = \frac{3}{8}$.
3. $\frac{5}{8} - \frac{1}{3} = \frac{7}{24}$
4. $\frac{4}{5} - \frac{1}{4} = \frac{11}{20}$

Practice

Complete the following subtraction problems using any method.

1. $\frac{3}{4} - \frac{5}{8}$
2. $\frac{4}{5} - \frac{2}{3}$
3. $\frac{5}{9} - \frac{2}{3}$
4. $\frac{6}{7} - \frac{2}{3}$
5. $\frac{7}{10} - \frac{1}{5}$
6. $\frac{2}{3} - \frac{1}{2}$
7. $\frac{3}{5} - \frac{3}{10}$
8. $\frac{7}{9} - \frac{1}{3}$
9. $\frac{5}{8} - \frac{1}{4}$
10. $\frac{2}{5} - \frac{2}{10}$
11. $\frac{7}{11} - \frac{1}{2}$
12. $\frac{5}{8} - \frac{5}{12}$
13. $\frac{5}{6} - \frac{3}{4}$
14. $\frac{5}{6} - \frac{2}{5}$
15. $\frac{4}{5} - \frac{3}{4}$

For each of the following questions, write a subtraction statement and find the result.

16. Sally used $\frac{2}{3}$ *cups* of flour to make cookies. Terri used $\frac{1}{2}$ *cups* of flour to make a cake. Who used more flour? How much more flour did she use?
17. Lauren used $\frac{3}{4}$ *cup* of milk, $1\frac{1}{3}$ *cups* of flour and $\frac{3}{8}$ *cup* of oil to make pancakes. Alyssa used $\frac{3}{8}$ *cup* of milk, $2\frac{1}{4}$ *cups* of flour and $\frac{1}{3}$ *cup* of melted butter to make waffles. Who used more cups of ingredients? How many more cups of ingredients did she use?
18. Write two fractions with different denominators whose difference is $\frac{3}{8}$.
19. Jake's dog ate $12\frac{2}{3}$ *cans* of food in one week and $9\frac{1}{4}$ *cans* the next week. How many more cans of dog food did Jake's dog eat in week one?
20. Sierra and Clark each solved the same problem.

Sierra's Solution

$$\begin{gathered}\frac{3}{4}-\frac{1}{6}\\ \frac{9}{12}-\frac{2}{12}\\ =\frac{7}{12}\end{gathered}$$

Clark's Solution

$$\begin{gathered}\frac{3}{4}-\frac{1}{6}\\ \frac{9}{12}-\frac{2}{12}\\ =\frac{7}{0}\end{gathered}$$

Who is correct? What would you tell the person who has the wrong answer?

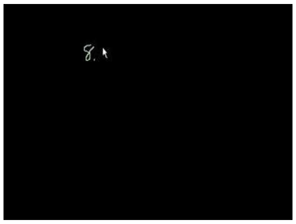
1.6 Subtraction of Decimals

Here you will review how to subtract decimals.

Jeremy and his family are driving to visit his grandparents. On the first day they drove 234.8 miles and on the second day they drove 251.6 miles. How many more miles did they drive on the second day?

Watch This

Khan Academy Subtracting Decimals



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Guidance

To subtract decimals, first write the decimals using the vertical alignment method. The decimal points must be kept directly under each other and the digits in the same place value must be kept in line with each other. If the decimal numbers are signed numbers, the rules for adding integers are applied to the problem. The number of greater magnitude should be placed above the number of smaller magnitude. Magnitude is simply the size of the number without respect to its sign. The number -42.8 has a magnitude of 42.8.

Example A

Subtract: 57.62 – 6.18

Solution: Subtracting decimals is similar to subtracting whole numbers. Line up the decimal points so that you can subtract corresponding place value digits (e.g. tenths from tenths, hundredths from hundredths, and so on). As with whole numbers, start from the right and work toward the left remembering to borrow when it is necessary.

$$\begin{array}{r} 57.\overset{5}{\cancel{6}}\;{}^{1}2 \\ -6.1\;\;8 \\ \hline 51.4\;\;4 \end{array}$$

Example B

(98.04) – (32.801)

Solution: Begin by writing the question using the vertical alignment method. To ensure that the digits are aligned correctly, add zero to 98.04.

$$\begin{array}{r} 98.040 \\ \underline{-32.801} \end{array}$$

Subtract the numbers.

$$\begin{array}{r} 9\overset{7}{\cancel{8}}.{}^{1}0\overset{3}{\cancel{4}}\,{}^{1}0 \\ -32.\ 80\ \ 1 \\ \hline 65.\ 23\ \ 9 \end{array}$$

Example C

$(67.65)-(-25.43)$

Solution: The first step is to write the problem as an addition problem and to change the sign of the original number being subtracted. In other words, add the opposite.

$(67.65)+(+25.43)$

Now, write and solve the problem using the vertical alignment method.

$$\begin{array}{r} 67.65 \\ \underline{+25.43} \\ +93.08 \end{array}$$

Example D

$(137.4)-(+259.687)$

Solution: The first step is to write the problem as an addition problem and to change the sign of the original number being subtracted. In other words, add the opposite.

$(137.4)+(-259.687)$

Now write the problem using the vertical alignment method. Remember to put 259.687 above 137.4 because 259.687 is the number of greater magnitude. The two numbers that are being added have opposite signs. Apply the same rule that you used when adding integers that had opposite signs – subtract the numbers and use the sign of the larger number in the answer.

$$\begin{array}{l} -259.687 \\ \underline{+137.4} \end{array}$$

To ensure that the digits are aligned correctly, add zeros to 137.4.

$$\begin{array}{r} -259.687 \\ \underline{+137.400} \end{array}$$

Subtract the numbers.

$$\begin{array}{r} -259.687 \\ \underline{+137.400} \\ -122.287 \end{array}$$

The numbers being added have opposite signs. This means that the sign of the answer will be the same sign as that of the number of greater magnitude. In this problem the answer has a negative sign.

Concept Problem Revisited

Jeremy and his family are driving to visit his grandparents. On the first day they drove 234.8 miles and on the second day they drove 251.6 miles.

The decimal number 251.6 is of greater magnitude than 234.8. The numbers must be vertically aligned with the larger one above the smaller one. Now the numbers can be subtracted.

$$\begin{array}{r} 2\overset{4}{\cancel{5}}\,\overset{10}{\cancel{1}}.\overset{1}{6} \\ \underline{-23\;\;4.8} \\ 1\;\;6.8 \end{array}$$

They drove 16.8 miles more on the second day.

Vocabulary

Decimal Point

A ***decimal point*** is the place marker in a number that separates the whole number and the fraction part. The number 326.45 has the decimal point between the six and the four.

Magnitude

A ***magnitude*** is the size of a number without respect to its sign. The number -35.6 has a ***magnitude*** of 35.6.

Guided Practice

1. Subtract these decimal numbers: $(243.67)-(196.3579)$

2. $(32.47)-(-28.8)-(19.645)$

3. Josie has \$59.27 in her bank account. She went to the grocery store and wrote a check for \$62.18 to pay for the groceries. Describe Josie's balance in her bank account now.

Answers:

1. $(243.67)-(196.3579)=47.3121$

2. $(32.47)-(-28.8)-(19.645)=41.625$

Write the question as an addition problem and change the sign of the original number being subtracted.

$(32.47) + (+28.8) + (-19.645)$

Follow the rules for adding integers.

3. \$59.27-\$62.18=\$-2.91. The account will have a negative value. This means that her account is overdrawn.

Practice

Subtract the following numbers:

1. $42.37 - 15.32$
2. $37.891 - 7.2827$
3. $579.237 - 45.68$
4. $4.2935 - 0.327$
5. $16.074 - 7.58$
6. $(-17.39) - (-49.68)$
7. $(92.75) + (-106.682)$
8. $(-72.5) - (-77.57) - (31.724)$
9. $(-82.456) - (279.83) + (-567.3)$
10. $(-57.76) - (-85.9) - (33.84)$

Determine the answer to the following problems.

11. The diameter of No. 12 bare copper wire is 0.08081 in., and the diameter of No. 15 bare copper wire is 0.05707 in. How much larger is the diameter of the No.12 wire compared to the diameter of the No. 15 wire?
12. The resistance of an armature while it is cold is 0.208 ohm. After running for several minutes, the resistance increases to 1.340 ohms. Find the increase in resistance of the armature.
13. The highest temperature recorded in Canada this year was $114.8°F$. The lowest temperature of $-62.9°F$ was recorded in February this year. Find the difference between the highest and lowest temperatures recorded in Canada this year.
14. The temperature in Alaska was recorded as $-78.64°F$ in January of 2010 and as $-59.8°F$ on the same date in 2011. What is the difference between the two recorded temperatures?
15. Laurie has a balance of -\$32.16 in her bank account. Write a problem that could represent this balance.

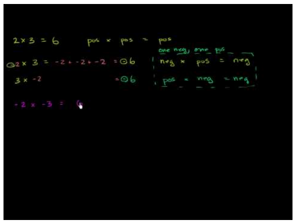
1.7 Multiplication of Real Numbers

Here you will learn to multiply integers, fractions and decimals.

Jacob received tips of $4.00 each from three of his paper route customers. How much did he receive in total?

Watch This

Khan Academy Multiplying Positive and Negative Numbers



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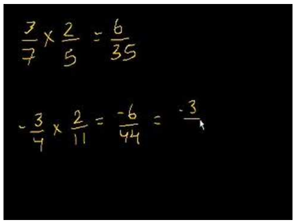
Khan Academy Multiplying Fractions



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Khan Academy Multiplication 8: Multiplying Decimals

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Guidance

Multiplication of two integers with the same signs produces a positive result and multiplication of two integers with unlike signs results in a negative answer.

These rules can be applied to the multiplication of all real numbers. To multiply fractions, you multiply the numerators and then you multiply the denominators. The product of the numerators over the product of the denominators is the answer to the problem. Sometimes the answer can be expressed as an equivalent fraction.

The rules for multiplying integers also apply to multiplying decimals. The sum of the number of digits after the decimal points determines the placement of the decimal point in the answer.

Example A

Sam spent $2.00 for a bottle of chocolate milk at the school cafeteria every school day. At the end of the week, how does this affect his net worth?

Solution: The result of $(+5) \times (-2)$ is –10. The product of a positive integer and a negative integer is always negative.

Example B

What is $(-2) \times (-3)$?

Solution: The result of $(-2) \times (-3)$ is +6. The product of two negative integers is always positive.

Example C

i) $\left(\frac{2}{3}\right) \times \left(\frac{5}{7}\right)$

ii) $\left(\frac{7}{8}\right) \times \left(3\frac{3}{4}\right)$

iii) $\left(5\frac{3}{4}\right) \times \left(2\frac{3}{5}\right)$

Solution: Remember, there are three simple steps to follow to multiply fractions:

1. Multiply the numerators of the fractions
2. Multiply the denominators of the fractions.
3. Simplify the fraction if necessary.

i) $\left(\frac{2}{3}\right) \times \left(\frac{5}{7}\right)$

$$\begin{aligned} &= \frac{2 \times 5}{3 \times 7} \\ &= \frac{10}{21} \end{aligned}$$

ii) $\left(\frac{7}{8}\right) \times \left(3\frac{3}{4}\right)$ Express the mixed number as an improper fraction.

$$\begin{aligned} &= \left(\frac{7}{8}\right) \times \left(\frac{15}{4}\right) \rightarrow \frac{(4 \times 3) + 3}{4} \\ &= \frac{7 \times 15}{8 \times 4} \\ &= \frac{105}{32} = 3\frac{9}{32} \end{aligned}$$

iii) $\left(5\frac{3}{4}\right) \times \left(2\frac{3}{5}\right)$ Express the mixed numbers as improper fractions.

$$\begin{aligned} &= \left(\frac{23}{4}\right) \times \left(\frac{13}{5}\right) \rightarrow \frac{(4 \times 5) + 3}{4} \text{ and } \frac{(5 \times 2) + 3}{5} \\ &= \frac{23 \times 13}{4 \times 5} \\ &= \frac{299}{20} = 14\frac{19}{20} \end{aligned}$$

Example D

$(14.65)\times(2.7)$

Solution: Multiply the numbers as you would whole numbers. To place the decimal point in the answer, count the number of digits after the decimal points in the problem. There are two digits after the decimal point in 14.65 and one digit after the decimal point in 2.7. This is a total of three digits after the decimal points. From the right of the answer, count three places to the left and insert the decimal point.

$$\begin{array}{r} 14.65 \\ \times\ 2.7 \\ \hline 10255 \\ +29300 \\ \hline 39.\underset{\longleftarrow}{555} \end{array}$$

Concept Problem Revisited

Jacob received tips of \$4.00 each from three of his paper route customers. How much did he receive in total?

The result of $(+3)\times(+4)$ is +12. The product of two positive integers is always positive.

Vocabulary

Denominator

The ***denominator*** of a fraction is the number on the bottom that indicates the total number of equal parts in the whole or the group. $\frac{5}{8}$ has ***denominator*** 8.

Fraction

A ***fraction*** is any rational number that is not an integer.

Improper Fraction

An ***improper fraction*** is a fraction in which the numerator is larger than the denominator. $\frac{8}{3}$ is an ***improper fraction***.

Integer

All natural numbers, their opposites, and zero are ***integers***. A number in the list ..., –3, –2, –1, 0, 1, 2, 3...

Mixed Number

A ***mixed number*** is a number made up of a whole number and a fraction such as $4\frac{3}{5}$.

Numerator

The ***numerator*** of a fraction is the number on top that is the number of equal parts being considered in the whole or the group. $\frac{5}{8}$ has ***numerator*** 5.

Guided Practice

Multiply the following fractions:

1. $\left(\frac{5}{9}\right) \times \left(\frac{-4}{7}\right)$
2. $\left(3\frac{2}{3}\right) \times \left(4\frac{1}{5}\right)$
3. Determine the answer to $(-135.697) \times (-34.32)$

Answers:

1. Multiply the numerators. Multiply the denominators. Simplify the fraction.

$$\left(\frac{5}{9}\right) \times \left(\frac{-4}{7}\right) = \frac{5 \times (-4)}{9 \times 7} = -\frac{20}{63}$$

The answer can be written as $\frac{-20}{63}$ or $-\frac{20}{63}$.

2. Write the two mixed numbers as improper fractions. Multiply the denominator and the whole number. Add the numerator to this product. Write the answer over the denominator. Follow the steps for multiplying fractions. Simplify the fraction if necessary.

$$\begin{gathered}\left(3\frac{2}{3}\right) \times \left(4\frac{1}{5}\right) \\ \left(\frac{11}{3}\right) \times \left(\frac{21}{5}\right) \\ \left(\frac{11}{3}\right) \times \left(\frac{21}{5}\right) = \frac{231}{15} = 15\frac{2}{5}\end{gathered}$$

3. Multiply the numbers as you would whole numbers. Remember the rule for multiplying integers. When you multiply two integers that have the same sign, the product will always be positive.

$$\begin{array}{r} -135.697 \\ \times\ -34.32 \\ \hline 271394 \\ 4070910 \\ 54278800 \\ 407091000 \\ \hline 4657.\underleftarrow{12}104 \end{array}$$

There are three digits after the decimal point in 135.697 and two digits after the decimal point in 34.32. Beginning at the right of the product, count five places to the left and insert the decimal point.

Practice

Multiply.

1. $(-7) \times (-2)$
2. $(+3) \times (+4)$
3. $(-5) \times (+3)$
4. $(+2) \times (-4)$

5. $(+4) \times (-1)$

Match each given phrase with the correct multiplication statement. Then, determine each product.

6. take away six groups of 3 balls
7. net worth after losing seven $5 bills
8. take away nine sets of 8 forks
9. take away four sets of four plates
10. receive eight groups of 4 glasses
11. buy seven sets of 12 placemats
a) $(+8) \times (+4)$
b) $(+7) \times (-5)$
c) $(-4) \times (+4)$
d) $(-9) \times (+8)$
e) $(+7) \times (+12)$
f) $(-6) \times (+3)$

Use the rules that you have learned for multiplying real numbers to answer the following problems.

12. $(-13) \times (-9)$
13. $(-3.68) \times (82.4)$
14. $\left(\frac{4}{9}\right) \times \left(\frac{5}{7}\right)$
15. $\left(7\frac{2}{3}\right) \times \left(6\frac{1}{2}\right)$
16. $(15.734) \times (-8.1)$

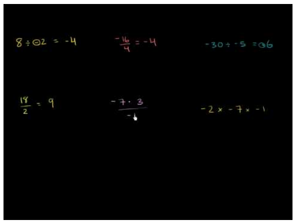
1.8 Division of Real Numbers

Here you will review how to divide real numbers including fractions and decimals.

The meteorologist on the local radio station just announced that a cold front caused the temperature to drop $12^\circ C$ in just four hours. What was the mean temperature change per hour over these four hours?

Watch This

Khan Academy Dividing Positive and Negative Numbers



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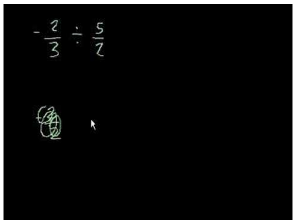
Khan Academy Dividing Fractions



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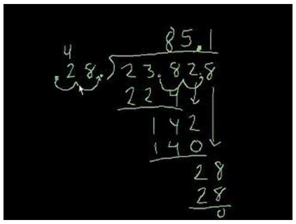
Khan Academy Dividing Decimals



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Guidance

There are two rules for dividing real numbers:

1. When you divide two integers that have the same signs, the answer is always positive.
2. When you divide two integers that have opposite signs, the answer is always negative.

Dividing fractions is like multiplying fractions with one additional step. To divide fractions, multiply the first fraction by the reciprocal of the second fraction. For example, $\frac{3}{5} \div \frac{2}{9} = \frac{3}{5} \times \frac{9}{2}$.

To divide decimals, use the following steps:

1. Write the divisor and the dividend in standard long-division form.
2. Move the decimal point of the divisor to the right so that the divisor is a whole number.
3. Move the decimal point of the dividend to the right the same number of places that you moved the decimal point of the divisor. If necessary, add zeros in the dividend.
4. Place the decimal point in the quotient directly above the new decimal point in the dividend.
5. The decimal points can now be ignored. Divide the numbers the same as you would divide whole numbers.

Example A

Miguel was doing a science project on weather and he reported a total temperature change of $-15^\circ F$ and a mean hourly change of $-3^\circ C$. How many hourly temperature changes did Miguel record?

Solution: The result of $(-15) \div (-3)$ is 5.

Example B

i) $\left(\frac{6}{11}\right) \div \left(\frac{5}{7}\right)$

ii) $\left(4\frac{1}{3}\right) \div \left(2\frac{5}{7}\right)$

Solution:

i)

$$\begin{aligned}&\left(\frac{6}{11}\right) \div \left(\frac{5}{7}\right)\\&\frac{6}{11} \times \frac{7}{5}\\&\frac{6 \times 7}{11 \times 5}\\&= \frac{42}{55}\end{aligned}$$

ii)

$$\begin{aligned}&\left(4\frac{1}{3}\right) \div \left(2\frac{5}{7}\right) \text{ Write the mixed numbers as improper fractions.}\\&\left(\frac{13}{3}\right) \div \left(\frac{19}{7}\right) \text{ Multiply by the reciprocal of } \frac{19}{7}.\\&\frac{13}{3} \times \frac{7}{19}\\&= \frac{91}{57} = 1\frac{34}{57} \text{ Simplify the fraction.}\end{aligned}$$

Example C

i) $(0.68) \div (1.7)$

ii) $0.365 \div -18.25$

Solution:

i) $(0.68) \div (1.7)$

$$\begin{array}{r} 0.4 \\ 1\underset{\rightarrow}{.7}\overline{)0\underset{\rightarrow}{.6}\,8} \\ \underline{-6\,8} \\ 0 \end{array}$$

The decimal point of the divisor was moved one place to the right. The decimal point of the dividend was moved one place to the right. The decimal point was placed in the quotient directly above the new decimal point of the dividend.

ii) $0.365 \div -18.25$

You have learned that when you divide a positive number by a negative number, the answer will always be negative.

$$\begin{array}{r} -.02 \\ -18\underset{\longrightarrow}{.25}\overline{)0\underset{\longrightarrow}{.36}\,50} \\ \underline{-36\;\;50} \\ 0 \end{array}$$

The decimal point of the divisor was moved two places to the right. The decimal point of the dividend was moved two place to the right. The decimal point was placed in the quotient directly above the new decimal point of the dividend.

Concept Problem Revisited

The meteorologist on the local radio station just announced that a cold front caused the temperature to drop $12^\circ C$ in just four hours.

The mean temperature change per hour is the result of $(-12) \div (+4)$, which is –3.

Vocabulary

Dividend

In a division problem, the ***dividend*** is the number that is being divided. The ***dividend*** is written under the division sign. In $4\overline{)38}$, 38 is the ***dividend.***

Divisor

In a division problem, the ***divisor*** is the number that is being divided into the dividend. The ***divisor*** is written in front of the division sign. In $4\overline{)38}$, 4 is the ***divisor.***

Mixed Number

A mixed number is a number made up of a whole number and a fraction such as $4\frac{3}{5}$.

Reciprocal

The ***reciprocal*** of a number is the inverse of that number. If $\frac{a}{b}$ is a nonzero number, then $\frac{b}{a}$ is its ***reciprocal.*** The product of a number and its ***reciprocal*** is one.

Quotient

The ***quotient*** is the answer of a division problem.

Guided Practice

1. $\left(\frac{7}{10}\right) \div \left(\frac{5}{6}\right) =?$
2. $\left(6\frac{2}{5}\right) \div \left(1\frac{2}{3}\right) =?$
3. How many pieces of plywood 0.375 in. thick are in a stack of 30 in. high?

Answers:

1. $\left(\frac{7}{10}\right) \div \left(\frac{5}{6}\right)$

$= \frac{7}{10} \times \frac{6}{5}$

$= \frac{7\times6}{10\times5}$

$= \frac{42}{50} = \frac{21}{25}$

2. $\left(6\frac{2}{5}\right) \div \left(1\frac{2}{3}\right)$

$= \left(\frac{32}{5}\right) \div \left(\frac{5}{3}\right)$

$= \left(\frac{32}{5}\right) \times \left(\frac{3}{5}\right)$

$= \frac{32\times3}{5\times5}$

$$\begin{aligned} &= \frac{96}{25} \\ &= 3\frac{21}{25} \end{aligned}$$

3. To determine the number of pieces of plywood in the stack, divide the thickness of one piece into the height of the pile.

$$\begin{array}{r} 80 \\ 0.\underrightarrow{375})\overline{30.\underrightarrow{000}} \\ \underline{-3000} \\ 0 \\ \underline{-\quad 0} \\ 0 \end{array}$$

There are 80 pieces of plywood in the pile.

Practice

Find each quotient or product.

1. $(+14) \div (+2)$
2. $(-14) \div (+2)$
3. $(-9) \div (-3)$
4. $(+16) \div (+4)$
5. $(+25) \div (-5)$
6. $(-9) \times (7)$
7. $(-8) \times (-8)$
8. $(+4) \times (-7)$
9. $(-10) \times (-3)$
10. $(+5) \times (+2)$
11. $\left(\frac{5}{16}\right) \div \left(\frac{3}{7}\right)$
12. $(-8.8) \div (-3.2)$
13. $(7.23) \div (0.6)$
14. $\left(2\frac{3}{4}\right) \div \left(1\frac{1}{8}\right)$
15. $(-30.24) \div (-0.42)$

For each of the following questions, write a division statement and find the result.

16. A truck is delivering fruit baskets to the local food banks for the patrons. Each fruit basket weighs 3.68 lb. How many baskets are in a load weighing 5888 lb?
17. A wedding invitation must be printed on card stock measuring $4\frac{1}{4}\ in.$ wide. If the area of the invitation is $23\frac{3}{8}\ in^2$, what is its length? (Hint: The area of a rectangle is found by multiplying the length times the width.)
18. A seamstress needs to divide $32\frac{5}{8}\ ft.$ of piping into 3 equal pieces. Calculate the length of each piece.
19. The floor area of a room on a house plan measures 3.5 in. by 4.625 in. If the house plan is drawn to the scale 0.25 in. represents 1 ft, what is the actual size of the room?
20. How many hair bows of $3\frac{1}{2}\ in.$ can be cut from $24\frac{3}{4}\ in.$ of ribbon?

1.9 Properties of Real Number Addition

Here you will learn the properties of addition that apply to real numbers: the commutative property, the closure property, the associative property, the identity property and the inverse property.

On the first day of school, you are all dressed in your new clothes. When you got dressed, you put one sock on your left foot and one sock on your right foot. Would it have made a difference if you had put one sock on your right foot first and then one sock on your left foot?

Guidance

There are five properties of addition that are important for you to know.

Commutative Property

In algebra, the operation of addition is commutative. The order in which you add two real numbers does not change the result, as shown below:

$$\begin{aligned} &(+7)+(+20)=? && (+20)+(+7)=? \\ &(+7)+(+20)=+27 && (+20)+(+7)=+27 \end{aligned}$$

The order in which you added the numbers did not affect the answer. This is called the commutative property of addition. In general, the commutative property of addition states that the order in which two numbers are added does not affect the sum. If a and b are real numbers, then $a+b=b+a$.

Closure Property

The sum of any two real numbers will result in a real number. This is known as the closure property of addition. The result will always be a real number. In general, the closure property states that the sum of any two real numbers is a unique real number. If a, b and c are real numbers, then $a+b=c$.

Associative Property

The order in which three or more real numbers are grouped for addition will not affect the sum. This is known as the associative property of addition. The result will always be the same real number. In general, the associative property states that the order in which the numbers are grouped for addition does not change the sum. If a, b and c are real numbers, then $(a+b)+c=a+(b+c)$.

Additive Identity

If zero is added to any real number the answer is always the real number. Zero is known as the additive identity or the identity element of addition. The sum of a number and zero is the number. This is called the identity property of addition. If a is a real number, then $a+0=a$.

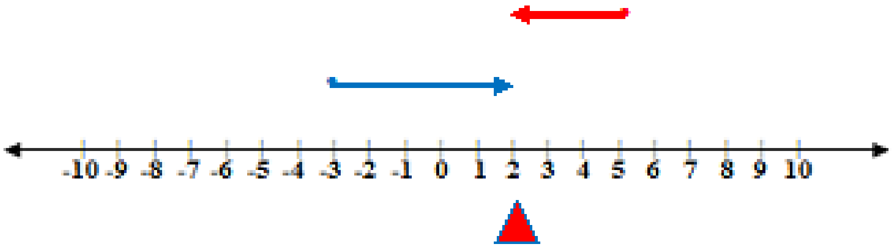
Additive Inverse

The sum of any real number and its additive inverse is zero. This is called the inverse property of addition. If a is a real number, then $a+(-a)=0$.

Example A

Use a number line to show that $(5)+(-3)=(-3)+(5)$.

Solution: On a number line, you add a positive number by moving to the right on the number line and you add a negative number by moving to the left on the number line.



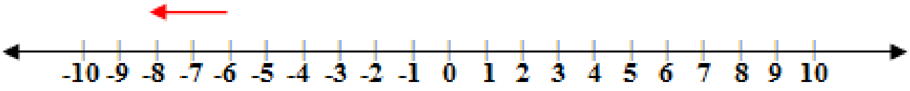
$(5)+(-3)=+2$ *The red dot is placed at +5. Then the (–3) is added by moving three places to the left. The result is +2.*

$(-3)+(5)=+2$ *The blue dot is placed at –3. Then the (+5) is added by moving five places to the right. The result is +2.*

Example B

Does $(-6)+(-2)=$ a real number?

Solution:

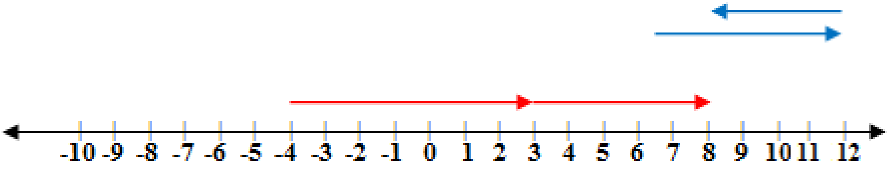


The result is –8. This is an integer. An integer is a real number. This is an example of the closure property.

Example C

Does $(-4+7)+5=-4+(7+5)$?

Solution:



$(-4+7)+5 =$ *The red dot is placed at –4. Then the (+7) is added by moving seven places to the left. Then (+5) is added by moving five places to the right. The result is +8.*

$-4+(7+5) =$ *The blue dot is placed at +7. Then the (+5) is added by moving five places to the left. Then (–4) is added by moving four places to the left. The result is +8.*

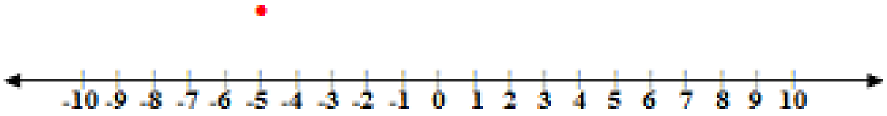
$(-4+7)+5 = -4+(7+5)$

The numbers in the problem were the same, but were grouped differently. The answer was the same in both cases. This is an example of the associative property of addition.

Example D

Does $(-5)+0 = -5$?

Solution:

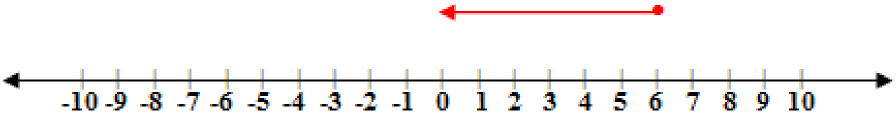


$(-5)+0 = -5$ *The red dot is placed at –5. If zero is being added to the number, there is no movement to the right and no movement to the left. Therefore the result is –5.* This is an example of using the additive identity.

Example E

Does $(+6)+(-6) = 0$?

Solution:



$(+6)+(-6) = 0$ *The red dot is placed at +6. Then the (–6) is added by moving six places to the left. The result is 0.*

When any real number is added to its opposite, the result is always zero. If a is any real number, its opposite is $-a$. The opposite, $-a$, is also known as the additive inverse of a.

Concept Problem Revisited

Think back to the question about putting on socks. The order in which you put on the socks does not affect the outcome – you have one sock on each foot.

This is like the commutative property in algebra. The order in which you add two real numbers does not change the result.

Vocabulary

Additive Identity

The ***additive identity*** for addition of real numbers is zero.

Additive Inverse

The ***additive inverse*** of addition is the opposite of the real number and the sum of the real number and its additive inverse is zero. If a is any real number, its additive inverse is $-a$.

Associative Property

The ***associative property of addition*** states the order in which three or more real numbers are grouped for addition will not affect the sum. If a, b and c are real numbers, then $(a+b)+c = a+(b+c)$.

Closure Property

The ***closure property of addition*** states that the sum of any two real numbers is a unique real number. If a, b and c are real numbers, then $a+b=c$.

Commutative Property

The ***commutative property*** of addition states that the order in which two numbers are added does not affect the sum. If a and b are real numbers, then $a+b=b+a$.

Identity Element of Addition

The ***identity element of addition*** is another term for the additive identity of addition. Therefore, the identity element of addition is zero.

Identity Property

The ***identity property of addition*** states that the sum of a number and zero is the number. If a is a real number, then $a+0=a$.

Inverse Property

The ***inverse property of addition*** states that the sum of any real number and its additive inverse is zero. If a is a real number, then $a+(-a)=0$.

Guided Practice

1. Add using the properties of addition: $-1.6+4.2+1.6$
2. What property justifies the statement? $(-21+6)+8 = -21+(6+8)$
3. Apply the commutative property of addition to the following problem. $17x-15y$

Answers:

1. $-1.6+4.2+1.6 = -1.6+1.6+4.2 = 0+4.2 = 4.2$
2. associative property
3. This is the same as $-15y+17x$.

Practice

Match the following addition statements with the correct property of addition.

1. $(-5)+5=0$

2. $(-16+4)+5=-16+(4+5)$

3. $-9+(-7)=-16$

4. $45+0=45$

5. $9+(-6)=(-6)+9$

a) Commutative Property

b) Closure Property

c) Inverse Property

d) Identity Property

e) Associative Property

Add the following using the properties of addition:

6. $24+(-18)+12$
7. $-21+34+21$
8. $5+\left(-\frac{2}{5}\right)+\left(-\frac{3}{5}\right)$
9. $19+(-7)+9$
10. $8+\frac{3}{7}+\left(-\frac{3}{7}\right)$

Name the property of addition that is being shown in each of the following addition statements:

11. $(-12+7)+10=-12+(7+10)$
12. $-18+0=-18$
13. $16.5+18.4=18.4+16.5$
14. $52+(-75)=-23$
15. $(-26)+(26)=0$

1.10 Properties of Real Number Multiplication

Here you will learn the properties of multiplication that apply to real numbers: the commutative property, the closure property, the associative property, the identity property and the inverse property.

Does $(-2) \times (-3)$ give the same result as $(-3) \times (-2)$?

Guidance

There are five properties of multiplication that are important for you to know.

Commutative Property

The commutative property of multiplication states that the order in which two numbers are multiplied does not affect the sum. If a and b are real numbers, then $a \times b = b \times a$.

Closure Property

The product of any two real numbers will result in a real number. This is known as the closure property of multiplication. In general, the closure property states that the product of any two real numbers is a unique real number. If a, b and c are real numbers, then $a \times b = c$.

Associative Property

The order in which three or more real numbers are grouped for multiplication will not affect the product. This is known as the associative property of multiplication. The result will always be the same real number. In general, the associative property states that the order in which the numbers are grouped for multiplication does not change the product. If a, b and c are real numbers, then $(a \times b) \times c = a \times (b \times c)$.

Multiplicative Identity

When any real number is multiplied by the number one, the real number does not change. This is true whether the real number is positive or negative. The number 1 is called the multiplicative identity or the identity element of multiplication. The product of a number and one is the number. This is called the identity property of multiplication. If a is a real number, then $a \times 1 = a$.

Multiplicative Inverse

If a is a nonzero real number, then the reciprocal or multiplicative inverse of a is $\frac{1}{a}$. The product of any nonzero real number and its reciprocal is always one. The number 1 is called the multiplicative identity or the identity element of multiplication. Therefore, the product of a and its reciprocal is the identity element of multiplication (one). This is known as the inverse property of multiplication. If a is a nonzero real number, then $a \times \frac{1}{a} = 1$.

Example A

Does $(-3)\times(+2)=(+2)\times(-3)$?

Solution: $(-3)\times(+2)=(+2)\times(-3)=-6$.

This is an example of the commutative property of multiplication.

Example B

Does $(-6)\times(+3)=$ a real number?

Solution: $(-6)\times(+3)=-18$, a real number. This is an example of the closure property of multiplication.

Example C

Does $(-3\times 2)\times 2=-3\times(2\times 2)$?

Solution: $(-3\times 2)\times 2=-3\times(2\times 2)=-12$. Even though the numbers are grouped differently, the result is the same. This is an example of the associative property of multiplication.

Example D

Does $8\times 1=8$?

Solution: Yes. This is an example of the identity property of multiplication.

Example E

Does $7\times\frac{1}{7}=1$?

Solution: Yes. This is an example of the inverse property of multiplication.

Concept Problem Revisited

$(-2)\times(-3)=(-3)\times(-2)=6$.

The order in which you multiplied the numbers did not affect the answer. This is an example of the commutative property of multiplication.

Vocabulary

Multiplicative Identity

The ***multiplicative identity*** for multiplication of real numbers is one.

Multiplicative Inverse

The ***multiplicative inverse*** of multiplication is the reciprocal of the nonzero real number and the product of the real number and its multiplicative inverse is one. If a is any nonzero real number, its multiplicative inverse is $\frac{1}{a}$.

Associative Property

The ***associative property of multiplication*** states the order in which three or more real numbers are grouped for multiplication will not affect the product. If a, b and c are real numbers, then $(a\times b)\times c=a\times(b\times c)$.

Closure Property

The ***closure property of multiplication*** states that the product of any two real numbers is a unique real number. If a, b and c are real numbers, then $a \times b = c$.

Commutative Property

The ***commutative property of multiplication*** states that the order in which two numbers are multiplied does not affect the product. If a and b are real numbers, then $a \times b = b \times a$.

Identity Element of Multiplication

The ***identity element of multiplication*** is another term for the multiplicative identity of multiplication. Therefore, the identity element of multiplication is one.

Identity Property

The ***identity property of multiplication*** states that the product of a number and one is the number. If a is a real number, then $a \times 1 = a$.

Inverse Property

The ***inverse property of multiplication*** states that the product of any real number and its multiplicative inverse is one. If a is a nonzero real number, then $a \times \left(\frac{1}{a}\right) = 1$.

Guided Practice

1. Multiply using the properties of multiplication: $\left(6 \times \frac{1}{6}\right) \times (3 \times -1)$

2. What property of multiplication justifies the statement $(-9 \times 5) \times 2 = -9 \times (5 \times 2)$?

3. What property of multiplication justifies the statement $-176 \times 1 = -176$?

Answers:

1. $\left(6 \times \frac{1}{6}\right) \times (3 \times -1) = \frac{6}{6} \times -3 = 1 \times -3 = -3$

2. associative property of multiplication

3. identity property of multiplication

Practice

Match the following multiplication statements with the correct property of multiplication.

1. $9 \times \frac{1}{9} = 1$

2. $(-7 \times 4) \times 2 = -7 \times (4 \times 2)$

3. $-8 \times (4) = -32$

4. $6 \times (-3) = (-3) \times 6$

5. $-7 \times 1 = -7$

a) Commutative Property

b) Closure Property

c) Inverse Property

d) Identity Property

e) Associative Property

In each of the following, circle the correct answer.

6. What does $-5(4)\left(-\frac{1}{5}\right)$ equal?
   a. –20
   b. –4
   c. +20
   d. +4

7. What is another name for the reciprocal of any real number?
   a. the additive identity
   b. the multiplicative identity
   c. the multiplicative inverse
   d. the additive inverse

8. What is the multiplicative identity?
   a. –1
   b. 1
   c. 0
   d. $\frac{1}{2}$

9. What is the product of a nonzero real number and its multiplicative inverse?
   a. 1
   b. –1
   c. 0
   d. there is no product

10. Which of the following statements is NOT true?
    a. The product of any real number and negative one is the opposite of the real number.
    b. The product of any real number and zero is always zero.
    c. The order in which two real numbers are multiplied does not affect the product.
    d. The product of any real number and negative one is always a negative number.

Name the property of multiplication that is being shown in each of the following multiplication statements:

11. $(-6\times 7)\times 2=-6\times(7\times 2)$
12. $-12\times 1=-12$
13. $25\times 3=3\times 25$
14. $10\times\frac{1}{10}=1$
15. $-12\times 3=-36$

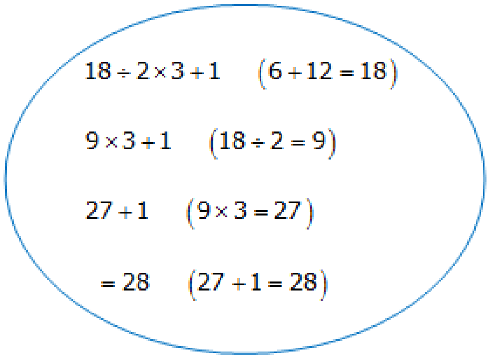
1.11 Order of Operations with Positive Real Numbers

Here you will learn the standard order of operations for arithmetic calculations.

Rosa walked into Math class and saw the following question on the board.

$$\boxed{6+12\div 2\times 3+1}$$

Her teacher, Ms. Black, directed the class to evaluate the mathematical expression. When the students had completed the task, Ms. Black then asked several students to put their work on the board. Here are the results:

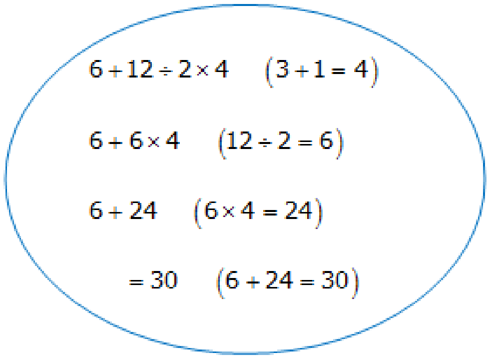


$18\div 2\times 3+1 \quad (6+12=18)$

$9\times 3+1 \quad (18\div 2=9)$

$27+1 \quad (9\times 3=27)$

$=28 \quad (27+1=28)$



$6+12\div 2\times 4 \quad (3+1=4)$

$6+6\times 4 \quad (12\div 2=6)$

$6+24 \quad (6\times 4=24)$

$=30 \quad (6+24=30)$

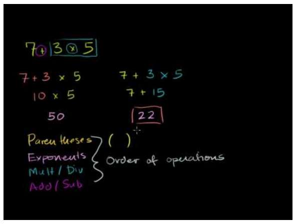
$$\begin{array}{ll} 6+6\times 3+1 & (12\div 2=6) \\ 12\times 3+1 & (6+6=12) \\ 36+1 & (12\times 3=36) \\ =37 & (36+1=37) \end{array}$$

$$\begin{array}{ll} 6+6\times 3+1 & (12\div 2=6) \\ 6+18+1 & (6\times 3=18) \\ 24+1 & (6+18=24) \\ =25 & (24+1=25) \end{array}$$

Which answer is correct?

Watch This

Khan Academy Introduction to Order of Operations



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Guidance

$$\boxed{6+12\div 2\times 3+1}$$

To avoid confusion in evaluating mathematical expressions like the one shown above, mathematicians have adopted a standard order of operations for arithmetic calculations. This adopted standard of operations consists of the following rules:

1. Perform any calculations shown inside **p**arentheses first.
2. Perform any calculations with terms that have **e**xponents.
3. Perform all **m**ultiplication and **d**ivision, in the order they occur, working from left to right.
4. Perform all **a**ddition and **s**ubtraction, in the order they occur, working from left to right.

If you look at the letters above that have been highlighted, you will see that they form the word **PEMDAS** – parentheses, exponents, multiplication, division, addition, subtraction. The word PEMDAS serves as a method for you to remember the order in which to perform the arithmetic calculations.

Example A

Perform the following calculations, using PEMDAS.

$360 \div (18 + 6 \times 2) - 2$

Solution: When performing the calculations in parentheses, follow the rules for order of operations.

$360 \div (18 + 12) - 2$

When the calculations in parentheses have been completed, the parentheses are no longer necessary.

$360 \div 30 - 2$

There are no exponents in this problem. The next step is to perform the division.

$12 - 2$

The final step is to subtract 2 from 12. The final answer is 10.

$= 10$

Example B

$\left(\frac{3}{4} + \frac{1}{6}\right) \times (5 \times 3^2 - 5)$

Solution: There are two sets of parentheses. Work from left to right in the first set of parentheses.

$\frac{11}{12} \times (5 \times 3^2 - 5)$

$\frac{11}{12} \times (5 \times 9 - 5)$

$\frac{11}{12} \times (45 - 5)$

$\frac{11}{12} \times 40$

$= 36\frac{2}{3}$

Example C

$(1 + 6)^2 - \frac{2+4\times12}{18-4\times2} + (72 \div 8)$

Solution: To start, add the numbers in the first parentheses.

$(7)^2 - \frac{2+4\times12}{18-4\times2} + (72 \div 8)$

$49 - \frac{2+4\times12}{18-4\times2} + (72 \div 8)$

$49 - \frac{2+4\times12}{18-4\times2} + 9$

Remember that the line of a fraction means divide. Before the division can be completed, you must obtain an answer for the calculations in the numerator and in the denominator. PEMDAS must be applied when doing the calculations.

$49 - \frac{2+48}{18-8} + 9$

$49 - \frac{50}{10} + 9$

$49 - 5 + 9$

$= 53$

Example D

$6.12+8.6\times 0.9-(10.26\div 3.8)$

Solution:

$$\begin{aligned}
&= 6.12+8.6\times 0.9-2.7\\
&= 6.12+7.74-2.7\\
&= 13.86-2.7\\
&= = 11.16
\end{aligned}$$

Example E

If $m=2$ and $n=3$, evaluate m^2+3n-7.

Solution: The first step is to substitute the values into the given statement.

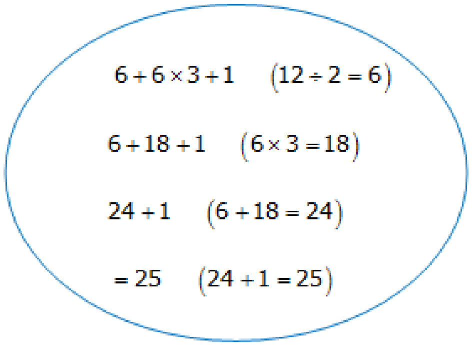
m^2+3n-7

$=(2)^2+3\times 3-7$

$$\begin{aligned}
&= 4+3\times 3-7\\
&= 4+9-7\\
&= 13-7\\
&= 6
\end{aligned}$$

Concept Problem Revisited

The final solution is correct.



$6+6\times 3+1 \quad (12\div 2=6)$

$6+18+1 \quad (6\times 3=18)$

$24+1 \quad (6+18=24)$

$=25 \quad (24+1=25)$

Ms. Black could have minimized the confusion by writing the statement with parentheses.

$$\boxed{6+(12\div 2\times 3)+1}$$

Vocabulary

Parentheses

Parentheses, ( ), are symbols that are used to group numbers in mathematics.

PEMDAS

The letters ***PEMDAS*** represent the standard order of operations for calculating mathematical statements.

P - Parentheses E - Exponents M - Multiplication D - Division A - Addition S - Subtraction

Guided Practice

1. Perform the following operations using PEMDAS: $8 \times 9 + 19 \div (30 - 11) - 6$

2. A remodeling job requires 132 square feet of countertops. Two options are being considered. The more expensive option is to use all Corian at \$66 per sq ft. The less expensive option is to use 78 sq ft of granite at \$56 per sq ft and 54 sq ft of laminate at \$23 per sq ft. Write a mathematical statement to calculate the difference in cost between the more expensive option and the less expensive option. What is the cost difference?

3. Determine the answer to $\frac{12+6}{6+3} + \frac{36}{4} - (12 \div 12)$ by using the rules for the standard order of operations.

Answers:

1.

$$\begin{aligned}
&8 \times 9 + 19 \div (30 - 11) - 6\\
&8 \times 9 + 19 \div 19 - 6\\
&72 + 19 \div 19 - 6\\
&72 + 1 - 6\\
&73 - 6\\
&= 67
\end{aligned}$$

2. The first option is \$3102 more than the second option.

$$\begin{aligned}
&(132 \times \$66) - (78 \times \$56 + 54 \times \$23)\\
&\$8712 - (78 \times \$56 + 54 \times \$23)\\
&\$8712 - (\$4368 + \$1242)\\
&\$8712 - \$5610\\
&= \$3102
\end{aligned}$$

3.

$$\begin{aligned}
&\frac{12+6}{6+3}+\frac{36}{4}-(12\div 12)\\
&\frac{12+6}{6+3}+\frac{36}{4}-1\\
&\frac{12+6}{6+3}+9-1\\
&\frac{18}{9}+9-1\\
&2+9-1\\
&11-1\\
&=10
\end{aligned}$$

Problem Set

Perform the indicated calculations, using PEMDAS to determine the answer.

1. $\frac{4^2(8+7)}{6}$
2. $\frac{2\times 6}{4}(5-2)$
3. $\frac{15\times 3}{5}+4(7\times 1)-2\times 3$
4. $4+27\div 3\times 2-6$
5. $7^2-3\times 2^3-5$

For each of the following problems write a single mathematical statement to represent the problem. Then use the statement to determine the answer.

6. At the beginning of the day on Monday, the cafeteria has 520 tortilla wraps. The supervisor estimates that she will need 68 wraps each day. A new shipment of 300 wraps will arrive on Thursday. Calculate the number of wraps she will have at the end of the day on Friday.
7. The students enrolled in the masonry course are estimating the cost for building a stone wall and gate. They estimate that the job will require 40 hours to complete. They will need the services of two laborers and they will be paid \$12 per hour. They will also need three masons who will be paid \$16 per hour. The cost of the materials is \$2140. What is the estimated cost of the job?
8. Mrs. Forsythe purchased 15 scientific calculators at \$19 each and received \$8 credit for each of the seven regular calculators that she returned. How much money did she spend to buy the scientific calculators?
9. A landscaper charged a customer \$472 for labor and \$85 each for eight flats of Bedding plants. What was the total cost of the job?
10. A painter had a one hundred dollar bill when he went to the hardware store to purchase supplies for a job. He bought 2 quarts of white latex paint for \$8 a quart and 4 gallons of white enamel paint for \$19 a gallon. How much change did he receive?

If $a=2, b=3$ and $c=5$, evaluate, using PEMDAS to determine the answer.

11. $6a-3b+4c$
12. $2a^2-3a+b^2$
13. $3ac-2ab+bc$
14. $a^2+b^2+c^2$
15. $3a^2(4c-3b)$

1.12 Order of Operations with Negative Real Numbers

Here you will practice the standard order of operations for arithmetic calculations involving negative real numbers.

Ginny walked into Math class and saw the following question on the board.

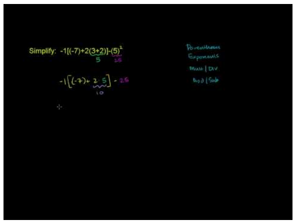
Find the value of the following expression when $a = -2, b = 3, c = -1, d = 1$

$$\boxed{(4a+c) \div b - (bd) \div (ac)}$$

Can you answer this question?

Watch This

Khan Academy Order of Operations1



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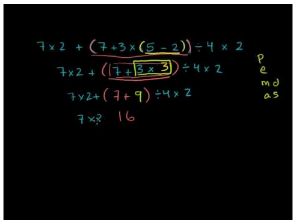
Khan Academy More Complicated Order of Operations Example



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Guidance

The standard order of operations involves specific steps for performing the mathematical calculations presented in a mathematical statement. These steps are represented by the letters PEMDAS.

P – Parentheses – Do all the calculations within parentheses.

E – Exponents – Do all calculations that involve exponents.

M/D – Multiplication/Division – Do all multiplication and division, in the order it occurs, working from left to right.

A/S – Addition/Subtraction – Do all addition and subtraction, in the order it occurs, working from left to right.

These steps do not change whether they are being applied to positive real numbers or to negative real numbers. The rules for adding, subtracting, multiplying and dividing real negative numbers must be applied when evaluating expressions that require PEMDAS to be used.

Example A

Perform the following calculations: $32 \div 4^2 \times 2 - 21$

Solution: There are no calculations inside parentheses. The first step is to evaluate the number with the exponent.

$= 32 \div 16 \times 2 - 21$

The next step is to perform any division or multiplication, in the order they occur, working from left to right.

$= 2 \times 2 - 21$

$= 4 - 21$

The final step is to rewrite the expression as an addition problem and to change the sign of the original number being subtracted.

$= 4 + -21$

When adding two numbers with unlike signs, subtract the numbers and the sign of the number with the greater magnitude will be the sign of the answer.

$= -17$

Example B

Find the value of the following expression when $a = -2, b = 3, c = -1, d = 1$.

$(4a^2c^2) - (3ac^3)$

Solution: Begin by substituting the variables with the given values. Use brackets to group the operations with parentheses.

$= [4(-2)^2(-1)^2] - [3(-2)(-1)^3]$

In the first set of brackets, do the calculations with exponents.

$= [4(4)(1)] - [3(-2)(-1)^3]$

In the second set of brackets, do the calculations with exponents.

$= [4(4)(1)] - [3(-2)(-1)]$

In the first set of brackets, do the multiplication. The brackets can now be removed.

$= 16 - [3(-2)(-1)]$

In the second set of brackets, do the multiplication. The brackets can now be removed.

$= 16 - 6$

Subtract the numbers.

$= 10$

Example C

What is the value of $3 - 2\left[\frac{8(-1)-7}{-3(2)-4}\right]$?

Solution: $=3 - 2\left[\frac{-8-7}{-3(2)-4}\right]$

$= 3 - 2\left[\frac{-8-7}{-6-4}\right]$

$= 3 - 2\left[\frac{-8+-7}{-6+-4}\right]$

$= 3 - 2\left[\frac{-15}{-10}\right]$

$= 3 - \frac{-30}{-10}$

$= 3 - 3$

$= 0$

Concept Problem Revisited

Find the value of the following expression when $a = -2, b = 3, c = -1, d = 1$

$$\boxed{(4a + c) \div b - (bd) \div (ac)}$$

Ginny felt good about answering the problem because she remembered the steps involved in the standard order of operations. When she noticed that the values for two of the variables were negative numbers, she realized that she would have to be careful doing the calculations because she would also have to apply the rules that she had learned for adding, subtracting, multiplying and dividing negative real numbers.

$(4a + c) \div b - (bd) \div (ac)$

Ginny began by substituting the variables with the given values.

$(4(-2) + (-1)) \div 3 - ((3)(1)) \div ((-2)(-1))$

To reduce errors in her calculations, Ginny wrote all of the values in parentheses. The statement now has parentheses within parentheses. This may seem confusing and the order in which to perform the operations may become skewed. Ginny asked her teacher about writing the expression another way. Her teacher advised her to replace the outer parentheses with brackets [ ]. Brackets are another type of grouping symbol. When evaluating an expression that has grouping symbols (parentheses) within grouping symbols (brackets), perform the operations within the innermost set of symbols first. This is not necessary, but it is a good rule to follow.

Ginny rewrote the expression using both brackets and parentheses.

$[4(-2) + (-1)] \div 3 - [(3)(1)] \div [(-2)(-1)]$

$= [-8 + (-1)] \div 3 - [(3)(1)] \div [(-2)(-1)]$

$= [-8 + (-1)] \div 3 - [3] \div [2]$

$= [-9] \div 3 - [3] \div [2]$

$= -9 \div 3 - 3 \div 2$

$= -3 - 3 \div 2$

$= -3 - 1.5$

$= -3 - 1.5 \rightarrow -3 + -1.5$

$= -3 + -1.5$

$= -4.5$

Vocabulary

Brackets

Brackets, [ ], are symbols that are used to group numbers in mathematics.

Parentheses

Parentheses, ( ), are symbols that are used to group numbers in mathematics.

PEMDAS

The letters ***PEMDAS*** represent the standard order of operations for calculating mathematical statements.

P - Parentheses E - Exponents M - Multiplication D - Division A - Addition S - Subtraction

Guided Practice

1. Perform the following operations: $8 \times -9 + 19 \div (-30 + 11) - 14 \times (-1)^2$

2. Determine the answer to: $\left(\frac{-12-6}{6+3}\right) + \left(\frac{-36}{-4}\right) + (-8 \times 2)$

3. A formula from geometry is $V = \frac{h}{6}(B + 4M + b)$. Find V when $h = -15, B = 12, M = 8, b = 4$.

Answers:

1.

$$\begin{aligned}
&8 \times -9 + 19 \div (-30 + 11) - 14 \times (-1)^2\\
&= 8 \times 9 + 19 \div -19 - 14 \times (-1)^2\\
&= 8 \times 9 + 19 \div -19 - 14 \times (-1)^2\\
&= 72 + 19 \div -19 - 14 \times 1\\
&= 72 + -1 - 14 \times 1\\
&= 72 + -1 - 14\\
&= 71 - 14\\
&= 57
\end{aligned}$$

2.

$$\begin{aligned}
&\left(\frac{-12-6}{6+3}\right) + \left(\frac{-36}{-4}\right) + (-8 \times 2)\\
&= \left(\frac{-18}{9}\right) + \left(\frac{-36}{-4}\right) + (-8 \times 2)\\
&= -2 + \left(\frac{-36}{-4}\right) + (-8 \times 2)\\
&= -2 + 9 + (-8 \times 2)\\
&= -2 + 9 + -16\\
&= -9
\end{aligned}$$

3.

$$
\begin{aligned}
V &= \frac{h}{6}(B+4M+b). \\
V &= \frac{-15}{6}(12+4(8)+4) \\
V &= \frac{-15}{6}(12+32+4) \\
V &= \frac{-15}{6}(48) \\
V &= -2.5(48) \\
V &= -120
\end{aligned}
$$

Practice

1. If $a=-3, b=1$ and $c=2$, what is the value of $2a^3-3b+2c^2$? _______
   - a. 221
   - b. 59
   - c. –49
   - d. 43

   (b) Evaluate $a-bc$ when $a=\frac{1}{2}, b=\frac{1}{3}$ and $c=\frac{5}{4}$: _______
   - a. 1
   - b. $\frac{1}{12}$
   - c. $\frac{2}{5}$
   - d. $\frac{5}{2}$

   iii. Evaluate: $a(-b^2-a^2)$ when $a=-3$ and $b=4$ _______
   - . 75
   - . 2
   - . 3
   - . 4

   D. Simplify the following: $[3-[5-(6-8)]+4]-2$ _______
   E. –6
   F. –2
   G. 2
   H. –19
   I. Perform the following operations and evaluate: $(5\times3-7)^2\div4+9$ _______
   J. 109
   K. 5
   L. 11
   M. 25
   N. Which expression has the greatest value if $a=-2$ and $b=3$?
   O. $3[a^2+b^2-2ab-2(a^2-b^2)]$
   P. $3(a-b)-2(b-a)+3(a-2b)$
   Q. $b(a-b)-a(b-a)-ab$
   Perform the indicated calculations.
   R. $\frac{6-(24-14)}{-10-[2-(-4)^2]}$
   S. $\frac{[12-(-15+6)]\times4-16}{-4}$
   T. $3\left(-2\times\frac{20}{-4\times-1}-5-7\right)$
   U. $-3\times5-[3(3+9)-9\times-3]$

V. $4^2-(4+8-2-4-2\times7)$
W. $2\times-18+9\div-3-5\times-2$
X. $-4\times(-12-8)\div-2$
Y. $-15-6\div3$
Z. $\frac{-2-(15-5)}{-4\times-2-12+-6\div3}$

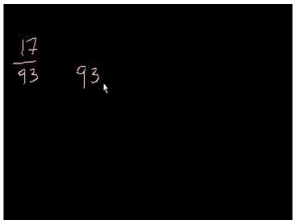
1.13 Decimal Notation

Here you will learn the difference between a terminating decimal number and a periodic decimal number. You will also learn how to express a given fraction in decimal form and how to express a number in decimal form as a fraction.

Which is greater, $\frac{18}{99}$ or $\frac{15}{80}$? How can you use decimals to help you with this problem?

Watch This

Khan Academy Converting Fractions to Decimals



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Guidance

To change a fraction to a number in decimal form, the numerator must be divided by the denominator. You can use long division or a calculator for this calculation. If, after the division, the numbers after the decimal point end, it is a terminating decimal. If, after the division, the numbers after the decimal point repeat in a pattern forever, it is a periodic decimal (also known as a repeating decimal).

- $\frac{3}{4} = 3 \div 4 = 0.75$. This is a terminating decimal.
- $\frac{3}{13} = 3 \div 13 = 0.230769230769....$ This is a periodic or repeating decimal. The period is 230769 because this is the set of numbers that repeats.

Keep in mind that a rational number is any number that can be written in the form $\frac{a}{b}$ where $b \neq 0$. Therefore, periodic decimals and terminating decimals are both rational numbers.

Example A

What fraction is equal to 0.45454545... ?

Solution: This is a periodic or repeating decimal. The period has a length of two because the pattern that is repeating consists of 2 digits. To express the number as a fraction, follow these steps:

Step 1: Let $x = 0.45454545$

Step 2: The repeating digit is 45. Place the repeating digit to the left of the decimal point by moving the decimal point 2 places to the right.

45.454545

To move the decimal point two places to the right, the decimal number was multiplied by 100. If you go back to step 1, you will see the equation $x = 0.45454545$. If you multiply one side of the equation by 100, you must multiply the other side by 100. You now have $100x = 45.454545$

Step 3: Subtract the two equations and solve for x.

$$\begin{aligned} 100x &= 45.454545 \\ \underline{-x} &\underline{= 0.45454545} \\ \frac{99x}{99} &= \frac{45}{99} \\ x &= \frac{45}{99} = \frac{5}{11} \end{aligned}$$

Example B

What fraction is equal to 0.125?

Solution: This number appears to be a terminating decimal number. The steps to follow to express 0.125 as a fraction are:

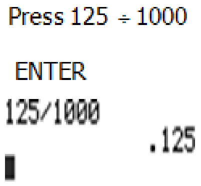
Step 1: Express the number as a whole number by moving the decimal point to the right. In this case, the decimal must be moved three places to the right.

Step 2: $0.125 = 125$

Express 125 as a fraction with a denominator of 1 and three zeros. The three zeros represent the number of places that the decimal point was moved.

$\frac{125}{1000}$

Step 3: If possible, simplify the fraction. If you are not sure of the simplified form, your graphing calculator can do the calculations.



Press 125 ÷ 1000

ENTER

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125/1000
                .125
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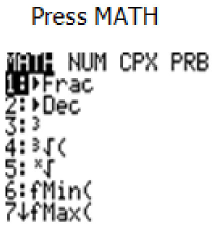

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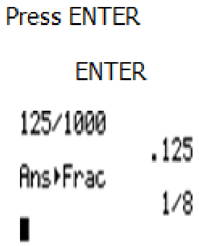


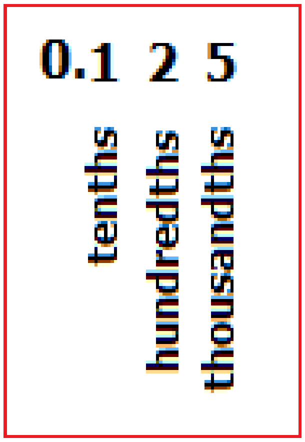
Press ENTER

ENTER

```
125/1000
              .125
Ans▸Frac
               1/8
▮
```

Therefore, the decimal number of 0.125 is equivalent to the fraction $\frac{1}{8}$.

The method shown above is one that can be used if you can't remember the place value associated with the decimal numbers. If you remember the place values, you can simply write the decimal as a fraction and simplify that fraction.



$$\frac{125}{1000}=\frac{1}{8}$$

Example C

Are the following decimal numbers terminating or periodic? If they are periodic, what is the period and what is its length?

i) 0.318181818...

ii) 0.375

iii) 0.3125

iv) 0.1211221112...

Solution:

i) A periodic decimal with a period of 18. The length of the period is 2.

ii) 0.375 A terminating decimal.

iii) 0.3125 A terminating decimal.

iv) 0.1211221112 This decimal is not a terminating decimal nor is it a periodic decimal. Therefore, the decimal is not a rational number. Decimals that are non-periodic belong to the set of irrational numbers.

Concept Problem Revisited

You can convert both fractions to decimals in order to figure out which is greater.

$\frac{18}{99} = .1818...$

$\frac{15}{80} = .1875$

You can see that $\frac{15}{80}$ is greater.

Vocabulary

Irrational Numbers

An ***irrational number*** is the set of non-periodic decimal numbers. Some examples of ***irrational numbers*** are $\sqrt{3}$, $\sqrt{2}$ and π.

Periodic Decimal

A ***periodic decimal*** is a decimal number that has a pattern of digits that repeat. The decimal number 0.1465325325..., is a ***periodic decimal.***

Rational Numbers

A rational number is any number that be written in the form $\frac{a}{b}$ where $b \neq 0$. Therefore, periodic decimal numbers and terminating decimal numbers are ***rational numbers.***

Terminating Decimal

A ***terminating decimal*** is a decimal number that ends. The process of dividing the fraction ends when the remainder is zero. The decimal number 0.25 is a ***terminating decimal.***

Guided Practice

1. Express 2.018181818 in the form $\frac{a}{b}$.

2. Express $\frac{15}{11}$ in decimal form.

3. If one tablet of micro K contains 0.5 grams of potassium, how much is contained in $2\frac{3}{4}$ tablets?

Answers:

1. Let $x = 2.018181818$ The period is 18.

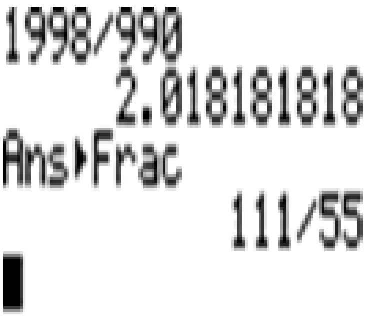
$1000x = 2018.181818$

$10x = 20.18181818$

$$\begin{aligned} 1000x &= 2018.181818 && \text{These are the two equations that must be subtracted.} \\ \underline{-10x} &\underline{= 20.18181818} && \\ \frac{990x}{990} &= \frac{1998}{990} && \text{Solve for } x. \\ x &= \frac{1998}{990} && \end{aligned}$$

Use your calculator to simplify the fraction.

$x = \frac{1998}{990}$



```
1998/990
        2.018181818
Ans▸Frac
             111/55
```

$x = \frac{111}{55}$

2. $\frac{15}{11} = 1.3636...$

3. The number of tablets is given as a mixed number. $2\frac{3}{4} = 2.75$. $2.75 \times 0.5 = 1.375$ *grams*.

Practice

Express the following fractions in decimal form.

1. $\frac{1}{12}$
2. $\frac{6}{11}$
3. $\frac{3}{20}$
4. $\frac{1}{13}$
5. $\frac{3}{8}$

Express the following numbers in the form $\frac{a}{b}$.

6. 0.325
7. 3.72727272...
8. 0.245454545...
9. 0.618
10. 0.36363636...

Complete the following table.

TABLE 1.1:

	Fraction	**Decimal**
11.	$\frac{5}{64}$	
12.	$\frac{11}{32}$	
13.	$\frac{1}{20}$	
14.		0.0703125
15.		0.1875

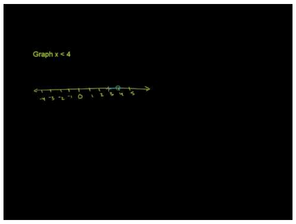
1.14 Real Number Line Graphs

Here you will review the number sets that make up the real number system. In addition, you will graph inequalities on a real number line.

Can you describe the number 13? Can you say what number sets the number 13 belongs to?

Watch This

Khan Academy Inequalities on a Number Line



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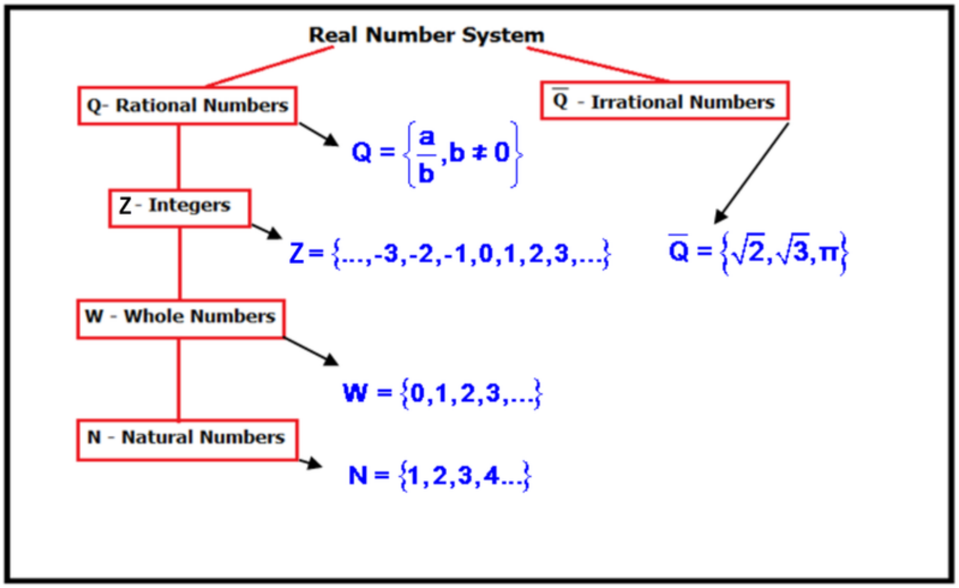
Guidance

All of the numbers you have learned about so far in math belong to the real number system. Positives, negatives, fractions, and decimals are all part of the real number system. The diagram below shows how all of the numbers in the real number system are grouped.



Any number in the real number system can be plotted on a real number line. You can also graph inequalities on a real number line. In order to graph inequalities, make sure you know the following symbols:

- The symbol >means "is greater than."

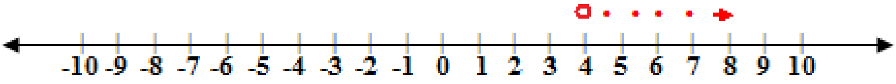
- The symbol <means "is less than."
- The symbol $\geq$ means "is greater than or equal to."
- The symbol $\leq$ means "is less than or equal to."

The inequality symbol indicates the type of dot that is placed on the beginning point and the number set indicates whether an arrow is drawn on the number line or if points are used.

Example A

Represent $x > 4$ where x is an integer, on a number line.

Solution:



The open dot on the four means that 4 is not included in the graph of all integers greater than 4. The closed dots on 5, 6, 7, 8 means that these numbers are included in the set of integers greater than 4. The arrow pointing to the right means that all integers to the right of 8 are also included in the graph of all integers greater than 4.

Example B

Represent this inequality statement on a number line $\{x \geq -2 | x \in R\}$.

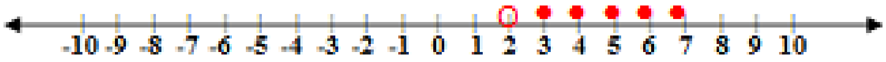
Solution: The statement can be read as "x is greater than or equal to –2, such that x belongs to or is a member of the real numbers." In other words, represent all real numbers greater than or equal to –2.

The inequality symbol says that x is greater than or equal to –2. This means that –2 is included in the graph. A solid dot is placed on –2 and on all numbers to the right of –2. The line is on the number line to indicate that all real numbers greater than –2 are also included in the graph.

Example C

Represent this inequality statement, also known as set notation, on a number line $\{x | 2 < x \leq 7, x \in N\}$.

Solution: This inequality statement can be read as x such that x is greater than 2 and less than or equal to 7 and x belongs to the natural numbers. In other words, all natural numbers greater than 2 and less than or equal to 7.



The inequality statement that was to be represented on the number line had to include the natural numbers greater than 2 and less than or equal to 7. These are the only numbers to be graphed. There is no arrow on the number line.

Concept Problem Revisited

To what number set(s) does the number 13 belong?

The number 13 is a natural number because it is in the set $N = \{1,2,3,4\ldots\}$.

The number 13 is a whole number because it is in the set $W = \{0,1,2,3\ldots\}$.

The number 13 is an integer because it is in the set $Z = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$.

The number 13 is a rational number because it is in the set $Q = \{\frac{a}{b}, b \neq 0\}$.

The number 13 belongs to the real number system.

Vocabulary

Inequality
An ***inequality*** is a mathematical statement relating expressions by using one or more inequality symbols. The inequality symbols are $>,<,\geq,\leq$.

Integer
All natural numbers, their opposites, and zero are ***integers***. A number in the list $\ldots,-3,-2,-1,0,1,2,3\ldots$

Irrational Numbers
The ***irrational numbers*** are those that cannot be expressed as the ratio of two numbers. The ***irrational numbers*** include decimal numbers that are both non-terminating decimals as well as non-periodic decimal numbers.

Natural Numbers
The ***natural numbers*** are the counting numbers and consist of all positive, whole numbers. The ***natural numbers*** are numbers in the list $1,2,3\ldots$ and are often referred to as positive integers.

Number Line
A ***number line*** is a line that matches a set of points and a set of numbers one to one. It is often used in mathematics to show mathematical computations.

Rational Numbers
The ***rational numbers*** are numbers that can be written as the ratio of two numbers $\frac{a}{b}$ and $b \neq 0$. The ***rational numbers*** include all terminating decimals as well as periodic decimal numbers.

Real Numbers
The rational numbers and the irrational numbers make up the ***real numbers.***

Set Notation
Set notation is a mathematical statement that shows an inequality and the set of numbers to which the variable belongs. $\{x|x \geq -3, x \in Z\}$ is an example of ***set notation.***

Guided Practice

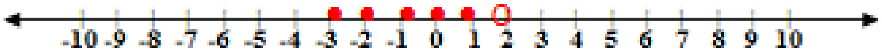
1. Check the set(s) to which each number belongs. The number may belong to more than one set.

TABLE 1.2:

Number	N	W	Z	Q	$\overline{Q}$	R
5						
$-\frac{47}{3}$						
1.48						
$\sqrt{7}$						
0						
π						

2. Graph $\{x|-3 \le x \le 8, x \in R\}$ on a number line.

3. Use set notation to describe the set shown on the number line.



Answers:

1. Review the definitions for each set of numbers.

TABLE 1.3:

Number	N	W	Z	Q	$\overline{Q}$
5	X	X	X	X	
$-\frac{47}{3}$	X	X	X	X	
1.48	X	X	X	X	
$\sqrt{7}$					X
0		X	X	X	
π					X

2. $\{x|-3 \le x \le 8, x \in R\}$

The set notation means to graph all real numbers between –3 and +8. The line joining the solid dots represents the fact that the set belongs to the real number system.

3. The closed dot means that –3 is included in the answer. The remaining dots are to the right of –3. The open dot means that 2 is not included in the answer. This means that the numbers are all less than 2. Graphing on a number line is done from smallest to greatest or from left to right. There is no line joining the dots so the variable does not belong to the set of real numbers. However, negative whole numbers, zero and positive whole numbers make up the integers. The set notation that is represented on the number line is $\{x|-3 \le x < 2, x \in Z\}$.

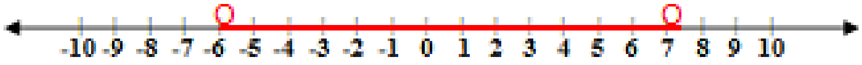
Practice

Describe each set notation in words.

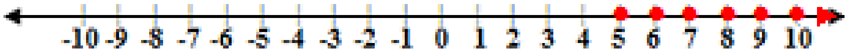
1. $\{x|x > 8, x \in R\}$
2. $\{x|x \leq -3, x \in Z\}$
3. $\{x|-4 \leq x \leq 6, x \in R\}$
4. $\{x|5 \leq x \leq 11, x \in W\}$
5. $\{x|x \geq 6, x \in N\}$

Represent each graph using set notation

6\.

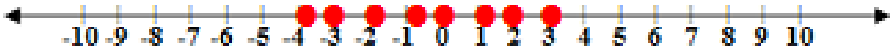


7\.

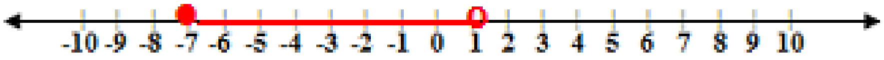


8\.

9\.



10\.



For each of the following situations, use set notations to represent the limits.

11. To ride the new tilt-a whirl at the fairgrounds, a child can be no taller than 4.5 feet.
12. A dance is being held at the community hall to raise money for breast cancer. The dance is only for those people 19 years of age or older.
13. A sled driver in the Alaska Speed Quest must start the race with no less than 10 dogs and no more than 16 dogs.

14. The residents of a small community are planning a skating party at the local lake. In order for the event to take place, the outdoor temperature needs to be above $-6^\circ C$ and not above $-1^\circ C$.
15. Juanita and Hans are planning their wedding supper at a local venue. To book the facility, they must guarantee that at least 100 people will have supper but no more than 225 people will eat.

Represent the following set notations on a number line.

16. $\{x|x > 6, x \in N\}$
17. $\{x|x \leq 8, x \in R\}$
18. $\{x|-3 \leq x < 6, x \in Z\}$

1.15 Real Number Comparisons

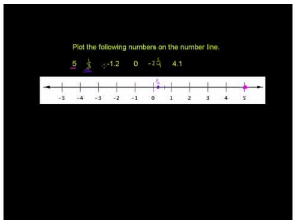
Here you will revisit the number sets that make up the real number system. You will also apply the skills you have learned for changing fractions to decimals. In addition, you will learn to order real numbers from least to greatest and to place these numbers on a number line.

Can you order the following real numbers from least to greatest?

$\frac{22}{7}, 1.234234\ldots, -\sqrt{7}, -5, -\frac{21}{4}, 7, -1.666, 0, 8.32, \frac{\pi}{2}, -\pi, -5.38$

Watch This

Khan Academy Points on a Number Line



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Guidance

The simplest way to order numbers is to express them all in the same form – all fractions or all decimals. With a calculator, it is easy to express every number in its decimal form. Watch your signs – don't drop any of the negative signs.

When plotting numbers on a number line, keep in mind that it is impossible to place decimals in the exact location on the number line. Place them as close as you can to the appropriate spot on the line.

Example A

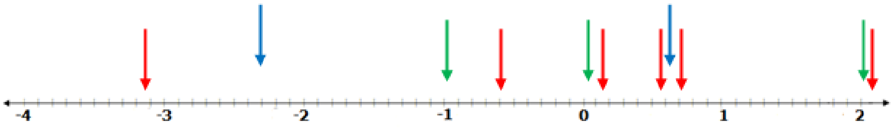
Draw a number line and place these numbers on the line.

$\sqrt{\frac{2}{5}}, 0.6467, -\frac{3}{5}, \frac{1}{8}, 0, \sqrt{0.5}, -2.34, \pi, \frac{2\pi}{3}, -1, 2$

Solution: $\sqrt{\frac{2}{5}} = 0.6324\ldots \quad -\frac{3}{5} = -0.6 \quad \frac{1}{8} = 0.125 \quad \sqrt{0.5} = 0.7071\ldots \quad -\pi = -3.1416\ldots \quad \frac{2\pi}{3} = 2.0944\ldots$

Start by placing the **integers** on the line first. Next place the **decimal numbers** on the line.

Use your calculator to convert **the remaining numbers** to decimal numbers. Place these on the line last.



Example B

For each given pair of real numbers, find another real number that is between the pair of numbers.

i) $-2, 1$

ii) $3.5, 3.6$

iii) $\frac{1}{2}, \frac{1}{3}$

iv) $-\frac{1}{3}, -\frac{1}{4}$

Solution: There are multiple possible answers. Here are possible answers.

i) The number must be greater than –2 and less than 1. $-2, 0, 1$

ii) The number must be greater than 3.5 and less than 3.6. $3.5, 3.54, 3.6$

iii) The number must be greater than $\frac{1}{3}$ and less than $\frac{1}{2}$. Write each fraction with a common denominator. $\frac{1}{2} = \frac{3}{6}, \frac{1}{3} = \frac{2}{6}$. If you look at $\frac{2}{6}$ and $\frac{3}{6}$, there is no fraction, with a denominator of 6, between these values. Write the fractions with a larger common denominator. $\frac{1}{2} = \frac{6}{12}, \frac{1}{3} = \frac{4}{12}$. If you look at $\frac{4}{12}$ and $\frac{6}{12}$, the fraction $\frac{5}{12}$ is between them. $\frac{1}{3}, \frac{5}{12}, \frac{1}{2}$

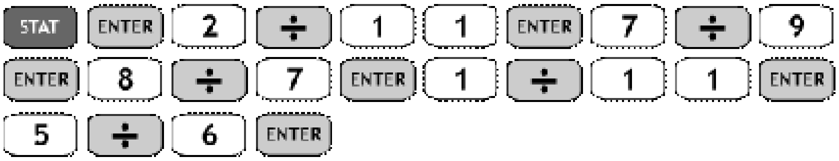
iv) The number must be greater than $-\frac{1}{3}$ and less than $-\frac{1}{4}$. Write each fraction with a common denominator. $-\frac{1}{3} = -\frac{4}{12}, -\frac{1}{4} = -\frac{3}{12}$. If you look at $-\frac{3}{12}$ and $-\frac{4}{12}$, there is no fraction, with a denominator of 12, between these values. Write the fractions with a larger common denominator. $-\frac{1}{3} = -\frac{8}{24}, -\frac{1}{4} = -\frac{6}{24}$. If you look at $-\frac{6}{24}$ and $-\frac{8}{24}$, the fraction $-\frac{7}{24}$ is between them. $-\frac{8}{24}, -\frac{7}{24}, -\frac{6}{24}$

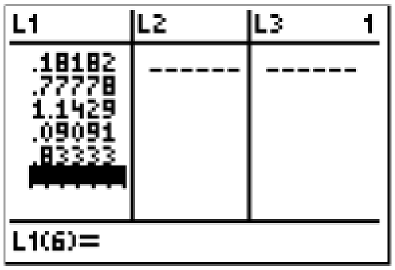
Example C

Order the following fractions from least to greatest.

$\frac{2}{11}, \frac{7}{9}, \frac{8}{7}, \frac{1}{11}, \frac{5}{6}$

Solution: The fractions do not have a common denominator. Let's use the TI-83 to order these fractions.

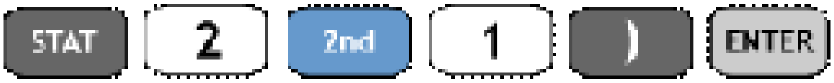


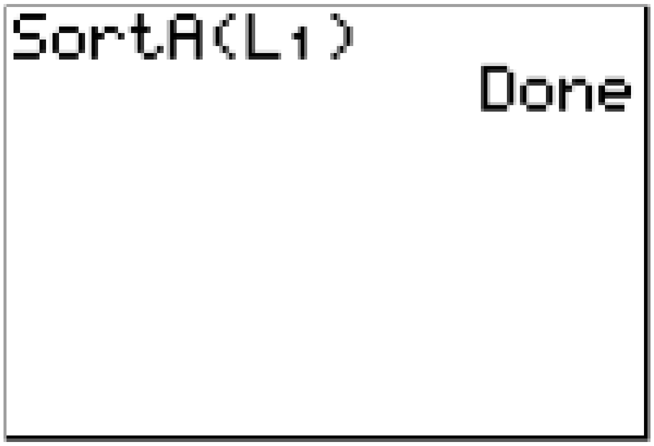


```
L1      L2      L3     1
.18182  ------  ------
.77778
1.1429
.09091
.83333

L1(6)=
```

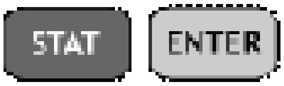
The fractions were entered into the calculator as division problems. The decimal forms of the numbers were entered into List 1.

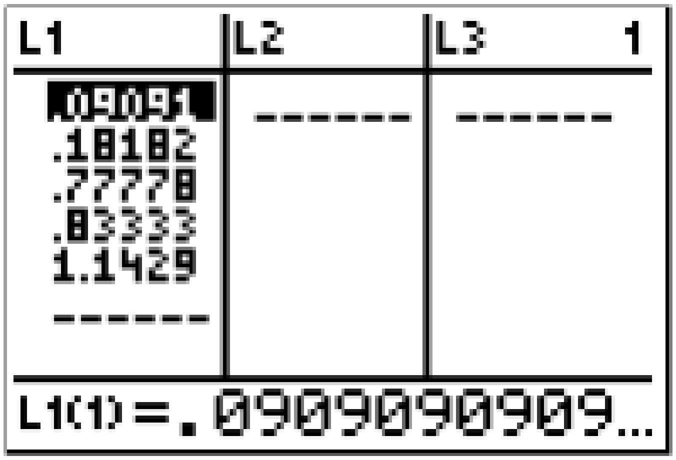




```
SortA(L₁)
                Done
```

The calculator has sorted the data from least to greatest.





```
L1      L2      L3      1
.09091  ------  ------
.18182
.77778
.83333
1.1429
------
L1(1)=.0909090909...
```

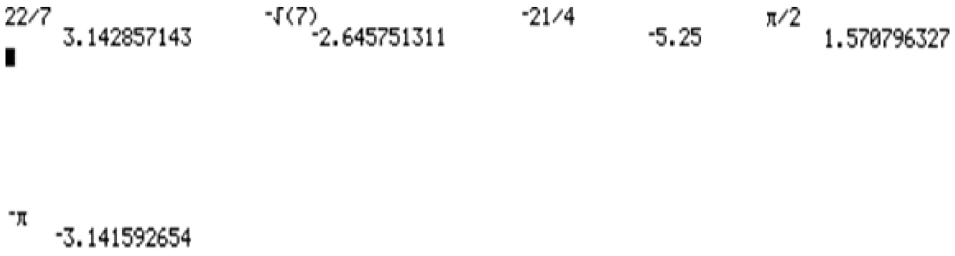
The data is sorted. The decimal numbers and the corresponding fractions can now be matched from the screen where they were first entered.

$\frac{1}{11}, \frac{2}{11}, \frac{7}{9}, \frac{5}{6}, \frac{8}{7}$

Concept Problem Revisited

$\frac{22}{7}, 1.234234\ldots, -\sqrt{7}, -5, -\frac{21}{4}, 7, -1.666, 0, 8.32, \frac{\pi}{2}, -\pi, -5.38$

As you examine the above numbers, you can see that there are natural numbers, whole numbers, integers, rational numbers and irrational numbers. These numbers, as they are presented here, would be very difficult to order from least to greatest.



```
22/7                -√(7)               -21/4               π/2
     3.142857143         -2.645751311             -5.25          1.570796327
▮

-π
    -3.141592654
```

Now that all the numbers are in decimal form, order them from least to greatest.

$-5.38, -\frac{21}{4}, -5, -\pi, -\sqrt{7}, -1.666, 0, 1.234234, \frac{\pi}{2}, \frac{22}{7}, 7, 8.32$

Vocabulary

Inequality

An ***inequality*** is a mathematical statement relating expressions by using one or more inequality symbols. The inequality symbols are $>, <, \geq, \leq$.

Integer

All natural numbers, their opposites, and zero are ***integers***. A number in the list $\ldots, -3, -2, -1, 0, 1, 2, 3\ldots$.

Irrational Numbers

The ***irrational numbers*** are those that cannot be expressed as the ratio of two numbers. The ***irrational numbers*** include decimal numbers that are both non-terminating decimals as well as non-periodic decimal numbers.

Natural Numbers

The ***natural numbers*** are the counting numbers and consist of all positive, whole numbers. The ***natural numbers*** are numbers in the list $1, 2, 3\ldots$ and are often referred to as positive integers.

Number Line

A ***number line*** is a line that matches a set of points and a set of numbers one to one. It is often used in mathematics to show mathematical computations.

Rational Numbers

The ***rational numbers*** are numbers that can be written as the ratio of two numbers $\frac{a}{b}$ and $b \neq 0$. The ***rational numbers*** include all terminating decimals as well as periodic decimal numbers.

Real Numbers

The rational numbers and the irrational numbers make up the ***real numbers.***

Guided Practice

1. Arrange the following numbers in order from least to greatest and place them on a number line.

$-3.78, -\frac{11}{4}, -4, \frac{\pi}{2}, -\sqrt{6}, -1.888, 0, 0.2424, \pi, \frac{21}{15}, 2, 1.75$

2. For each given pair of real numbers, find another real number that is between the pair of numbers.

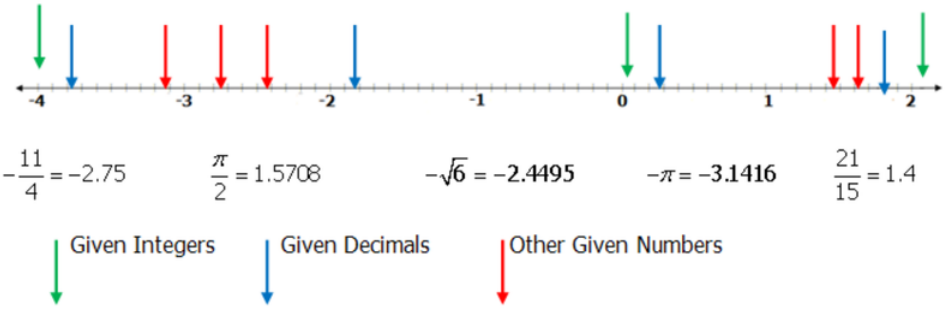
i) $-3, -5$
ii) $-3.4, -3.5$
iii) $\frac{1}{5}, \frac{3}{10}$
iv) $-\frac{3}{4}, -\frac{11}{6}$

3. Use technology to sort the following numbers:

$\sqrt{\frac{3}{5}}, \frac{15}{38}, -\frac{7}{12}, \frac{1}{4}, 0, \sqrt{8}, -\frac{13}{21}, -\pi, \frac{3\pi}{5}, -6, 3$

Answers:

1. $-3.78, -\frac{11}{4}, -4, \frac{\pi}{2}, -\sqrt{6}, -1.888, 0, 0.2424, \pi, \frac{21}{15}, 2, 1.75$



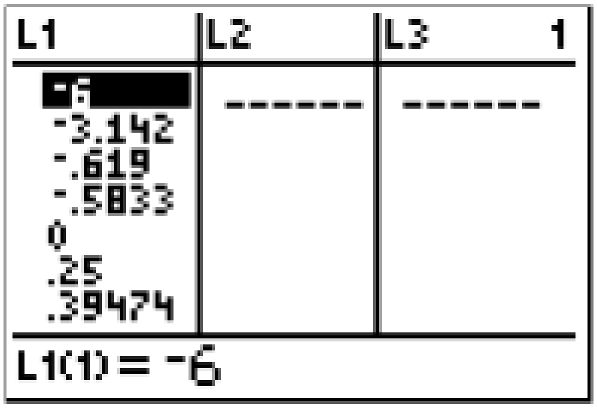
2. i) The number must be greater than –5 and less than –3. $-5, -4, -3$

ii) The number must be greater than –3.5 and less than –3.4. $-3.5, -3.45, -3.4$

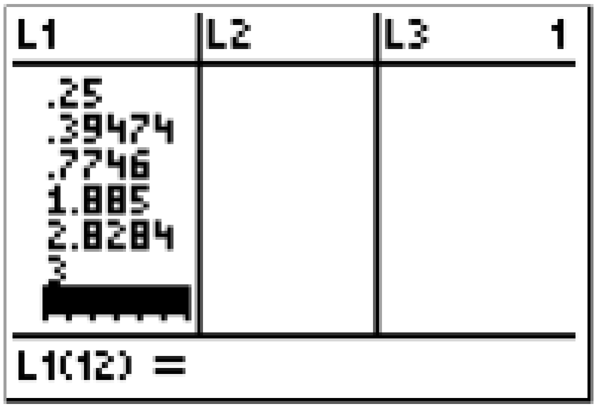
iii) The number must be greater than $\frac{1}{5}$ and less than $\frac{3}{10}$. Write each fraction with a common denominator. $\frac{1}{5} = \frac{2}{10}$. If you look at $\frac{2}{10}$ and $\frac{3}{10}$, there is no fraction, with a denominator of 10, between these values. Write the fractions with a larger common denominator. $\frac{1}{5} = \frac{4}{20}, \frac{3}{10} = \frac{6}{20}$. If you look at $\frac{4}{20}$ and $\frac{6}{20}$, the fraction $\frac{5}{20} = \frac{1}{4}$ is between them. $\frac{1}{5}, \frac{1}{4}, \frac{3}{10}$

iv) The number must be greater than $-\frac{3}{4}$and less than $-\frac{11}{16}$. Write each fraction with a common denominator. $-\frac{3}{4} = -\frac{12}{16}$. If you look at $-\frac{12}{16}$ and $-\frac{11}{16}$, there is no fraction, with a denominator of 16 between these values. Write the fractions with a larger common denominator. $-\frac{3}{4} = -\frac{24}{32}, -\frac{11}{16} = -\frac{22}{32}$. If you look at $-\frac{24}{32}$ and $-\frac{22}{32}$, the fraction $-\frac{23}{32}$ is between them. $-\frac{3}{4}, -\frac{23}{32}, -\frac{11}{16}$

3. $\sqrt{\frac{3}{5}}, \frac{15}{38}, -\frac{7}{12}, \frac{1}{4}, 0, \sqrt{8}, -\frac{13}{21}, -\pi, \frac{3\pi}{5}, -6, 3$



```
L1        L2        L3       1
⁻6        ------    ------
⁻3.142
⁻.619
⁻.5833
0
.25
.39474
L1(1)= ⁻6
```



```
L1        L2        L3       1
.25
.39474
.7746
1.885
2.8284
3
------
L1(12) =
```

The numbers have been sorted. The numbers from least to greatest are:

$-6, -\pi, -\frac{13}{21}, -\frac{7}{12}, 0, \frac{1}{4}, \frac{15}{38}, \sqrt{\frac{3}{5}}, \frac{3\pi}{5}, \sqrt{8}, 3$

Practice

Arrange the following numbers in order from least to greatest and place them on a number line.

1. $\{0.5, 0.45, 0.65, 0.33, 0, 2, 0.75, 0.28\}$
2. $\{0.3, 0.32, 0.21, 0.4, 0.3, 0, 0.31\}$
3. $\{-0.3, -0.32, -0.21, -0.4, -0.3, 0, -0.31\}$
4. $\{\frac{1}{2}, -2, 0, -\frac{1}{3}, 3, \frac{2}{3}, -\frac{1}{2}\}$
5. $\{0.3, -\sqrt{2}, 1, -0.25, 0, 1.8, -\frac{\pi}{3}\}$

For each given pair of real numbers, find another real number that is between the pair of numbers.

6. $8, 10$
7. $-12, -13$
8. $-12.01, -12.02$
9. $-7.6, -7.5$
10. $\frac{1}{7}, \frac{4}{21}$
11. $\frac{2}{5}, \frac{7}{9}$

12. $-\frac{2}{9}, -\frac{3}{18}$
13. $-\frac{3}{5}, -\frac{1}{2}$

Use technology to sort the following numbers:

14. $\{-2, \frac{2}{3}, 0, \frac{3}{8}, -\frac{7}{5}, \frac{1}{2}, 4, -3.6\}$
15. $\{\sqrt{10}, -1, \frac{7}{12}, 3, -\frac{5}{4}, -\sqrt{7}, 0, -\frac{2\pi}{3}, -\frac{3}{5}\}$

Summary

You reviewed how to add real numbers using two rules:

- Real numbers with unlike signs must be subtracted. The answer will have the same sign as that of the higher number.
- Real numbers with the same sign must be added. The answer will have the same sign as that of the numbers being added.

Then you reviewed how to subtract real numbers by adding the opposite. You learned that a number has both direction and magnitude. This direction is determined by the number's location with respect to zero on the number line. The magnitude of a number is simply its size with no regard to its sign.

Next you reviewed multiplication and division of real numbers using the following rules:

- The product/quotient of two positive numbers is always positive.
- The product/quotient of two negative numbers is always positive.
- The product/quotient of a positive number and a negative number is always negative.

You reviewed the standard order of operations represented by the letters PEMDAS.

- P – Parentheses – Do all the calculations within parentheses.
- E – Exponents – Do all calculations that involve exponents.
- M/D – Multiplication/Division – Do all multiplication/division, in the order it occurs, working from left to right.
- A/S – Addition/Subtraction – Do all addition/subtraction, in the order it occurs, working from left to right.
- S – Subtraction – Do all subtraction, in the order it occurs, working from left to right.

Next you learned how to distinguish between a terminating decimal and a periodic decimal. You were also shown that a fraction can be changed to a decimal by long division.

Last you revisited the real number system (R). The real number system is made up of rational and irrational numbers. The rational numbers include the natural numbers, whole numbers, integers and rational numbers. The natural numbers are the counting numbers and consist of all positive, whole numbers.

CHAPTER 2 Linear Equations and Inequalities

Chapter Outline

Introduction

Here you'll learn how to solve equations in one variable as well as inequalities in one variable. You will learn how to translate words into mathematical symbols in order to solve word problems using equations. You will also learn about absolute value and how to solve equations and inequalities that contain absolute value symbols. For inequalities, you will learn how to represent your solutions graphically on a number line.

2.1 Equations with Variables on One Side

Here you will begin your study of mathematical equations by learning how to solve equations with a variable on only one side.

Erin, Jillian, Stephanie and Jacob went to the movies. The total bill for the tickets and snacks came to $72.00. What is an equation that represents this situation? How much should each teen pay to split the bill evenly?

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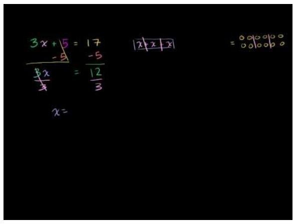
Khan Academy Slightly More Complicated Equations



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Guidance

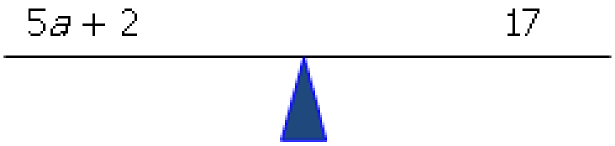
When solving any equation, your job is to find the value for the letter that makes the equation true. Solving equations with variables on one side can be done with the help of models such as a balance or algebra tiles.

When solving equations with variables on one side of the equation there is one main rule to follow: whatever you do to one side of the equals sign you must do the same to the other side of the equals sign. For example, if you add a number to the left side of an equals sign, you must add the same number to the right side of the equals sign.

Example A

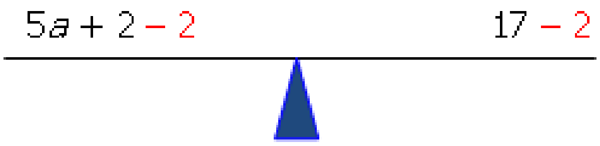
$5a+2=17$

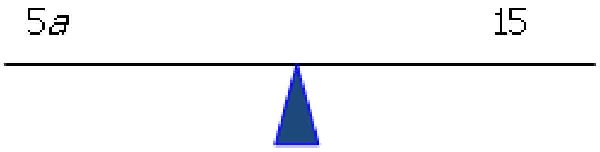
Solution: The problem can be solved if you think about the problem in terms of a balance. You know that the two sides are equal so the balance has to stay horizontal. You can place each side of the equation on each side of the balance.



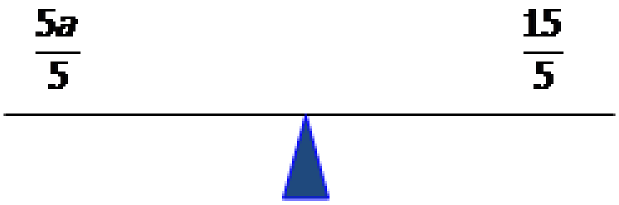
In order to solve the equation, you have to get the variable a all by itself. Always remember that you need to keep the balance horizontal. This means that whatever you do to one side of the equation, you ***have*** to do to the other side.

First subtract 2 from both sides to get rid of the 2 on the left.

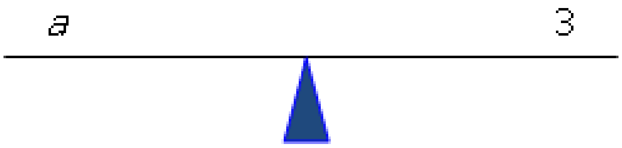




Since 5 is multiplied by a, you can get a by itself (or isolate it) by dividing by 5. Remember that whatever you do to one side, you have to do to the other.



If you simplify this expression, you get:



Therefore $a = 3$.

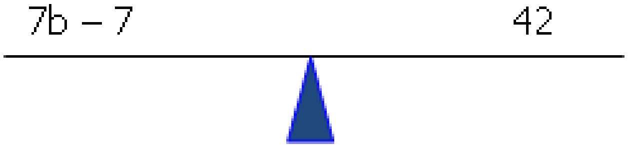
You can check your answer to see if you are correct by substituting your answer back into the original equation.

$$\begin{aligned} 5a+2 &= 17 \\ 5(3)+2 &= 17 \\ 15+2 &= 17 \\ 17 &= 17 \end{aligned}$$

Example B

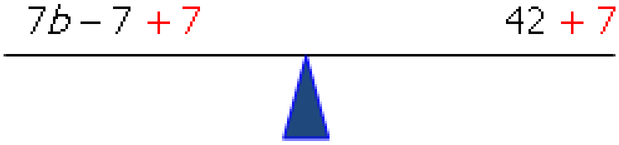
$7b-7 = 42$

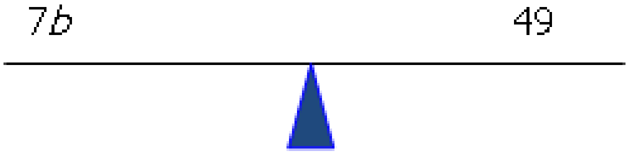
Solution: Again, you can solve the problem if you think about the problem in terms of a balance (or a seesaw). You know that the two sides are equal so the balance has to stay horizontal. You can place each side of the equation on each side of the balance.



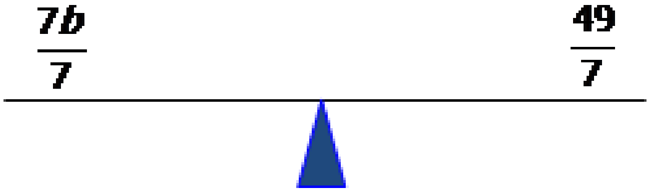
In order to solve the equation, you have to get the variable b all by itself. Always remember that you need to keep the balance horizontal. This means that whatever you do to one side of the equation, you ***have*** to do to the other side.

First add 7 from both sides to get rid of the 7 on the left.

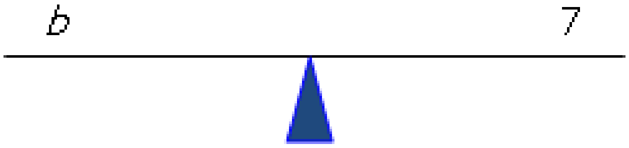




Since 7 is multiplied by b, you can get b by itself (or isolate it) by dividing by 7. Remember that whatever you do to one side, you have to do to the other.



If you simplify this expression, you get:



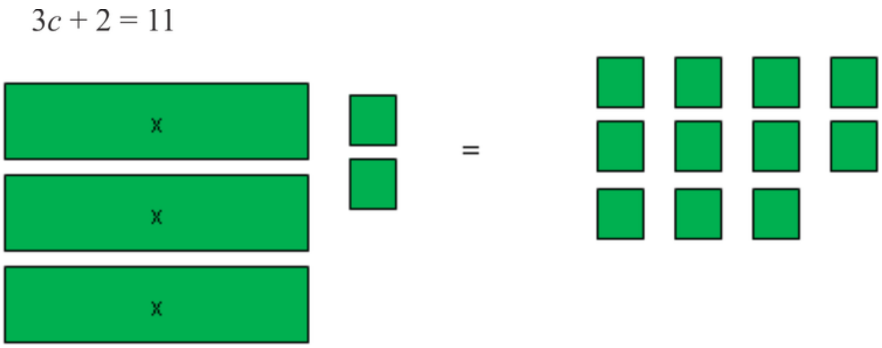
Therefore $b = 7$.

You can check your answer to see if you are correct.

$$
\begin{aligned}
7b-7&=42\\
7(7)-7&=42\\
49-7&=42\\
42&=42
\end{aligned}
$$

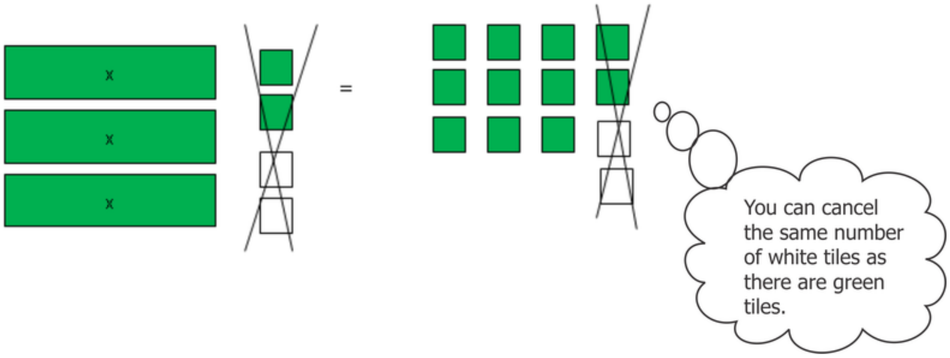
Example C

This same method can be extended by using algebra tiles. If you let rectangular tiles represent the variable, square tiles represent one unit, green tiles represent positives numbers, and white tiles represent the negative numbers, you can solve the equations using an alternate method.

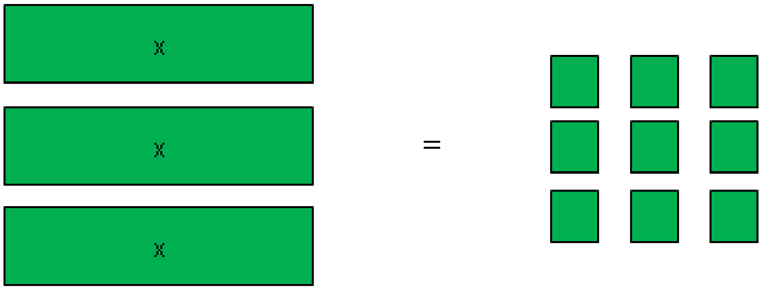


The green algebra $x-$tiles represent variables; therefore, there are 3 c blocks for the equation. The other green blocks represent the numbers or constants. There is a 2 on the left side of the equation so there are 2 square green blocks. There is an 11 on the right side of the equation so there are 11 square green blocks on the right side of the equation.

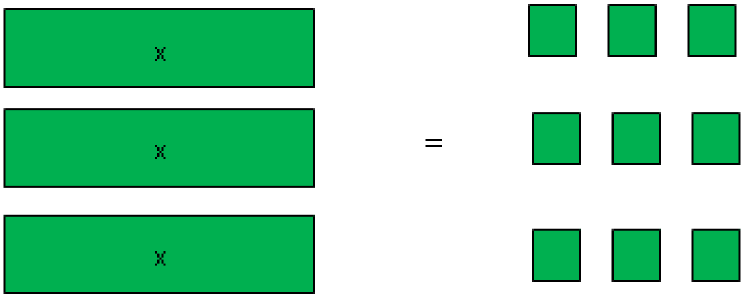
Solution: To solve, add two negative tiles to the right and left hand sides. The same rule applies to this problem as to all of the previous problems. Whatever you do to one side you have to do to the other.



This leaves us with the following:



You can reorganize these to look like the following:



Organizing the remaining algebra tiles allows us to realize the answer to be $x = 3$ or for your example $c = 3$.

Let's do your check as with the previous two problems.

$$\begin{aligned}3c+2&=11\\3(3)+2&=11\\9+2&=11\\11&=11\end{aligned}$$

Concept Problem Revisited

There are four teens going to the movies (Erin, Jillian, Stephanie, and Jacob). The total bill was \$72.00. Therefore your equation is $4x = 72$. you divide by 4 to find your answer.

$$\begin{aligned}x&=\frac{72}{4}\\x&=18\end{aligned}$$

Therefore each teen will have to pay \$18.00 for their movie ticket and snack.

Vocabulary

Constant

A ***constant*** is a numerical coefficient. For example in the equation $4x+72 = 0$, the 72 is a constant.

Equation

An ***equation*** is a mathematical statement with expressions separated by an equals sign.

Numerical Coefficient

In mathematical equations, the ***numerical coefficients*** are the numbers associated with the variables. For example, with the expression $4x$, 4 is the numerical coefficient and x is the variable.

Variable

A ***variable*** is an unknown quantity in a mathematical expression. It is represented by a letter. It is sometimes referred to as the literal coefficient.

Guided Practice

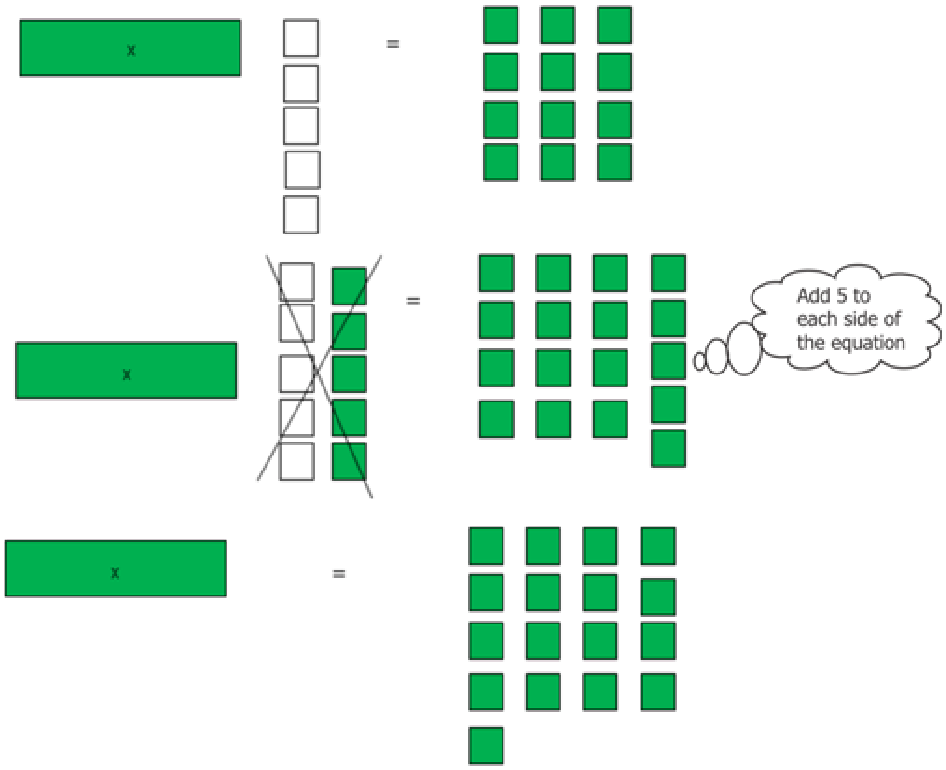
1. Use a model to solve for the variable in the equation $x-5=12$.

2. Use a different model to solve for the variable in the equation $3y+9=12$.

3. Solve for x in the equation $3x-2x+16=-3$.

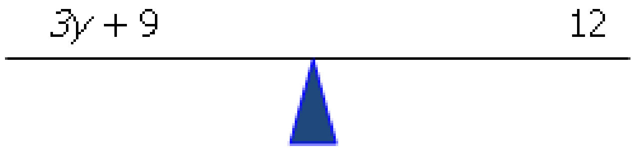
Answers:

1. $x-5=12$

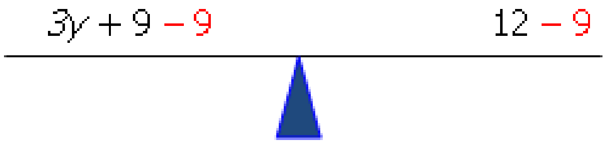


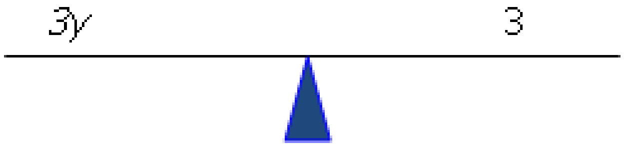
Therefore, $x=17$.

2. $3y+9=12$

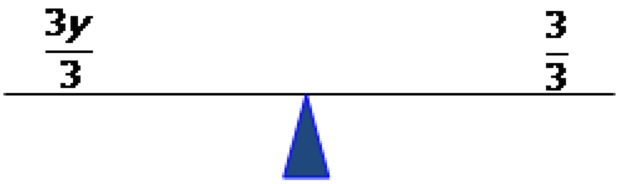


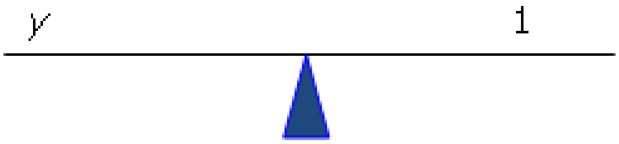
First you have to subtract 9 from both sides of the equation in order to start to isolate the variable.





Now, in order to get y all by itself, you have to divide both sides by 3. This will isolate the variable y.





Therefore, $y = 1$.

3. $3x - 2x + 16 = -3$

You can use any method to solve this equation. Remember to isolate the x variable. You will notice here that there are two x values on the left. First let's combine these terms.

$$\begin{aligned} 3x - 2x + 16 &= -3 \\ x + 16 &= -3 \end{aligned}$$

Now you can use any method to solve the equation. You now should just have to subtract 16 from both sides to isolate the x variable.

$$\begin{aligned} x+16-16 &= -3-16 \\ x &= -19 \end{aligned}$$

Practice

Use the models that you have learned to solve for the variables in the following problems.

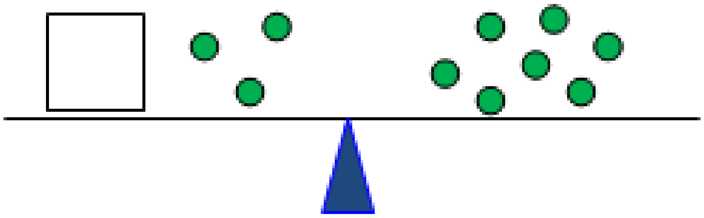
1. $a+3=-5$
2. $2b-1=5$
3. $4c-3=9$
4. $2-d=3$
5. $4-3e=-2$

6. $x+3=14$
7. $2y-7=5$
8. $3z+6=9$
9. $5+3x=-3$
10. $2x+2=-4$

11. $-4x+13=5$
12. $3x-5=22$
13. $11-2x=5$
14. $2x-4=4$
15. $5x+3=28$

For each of the following models, write a problem involving an equation with a variable on one side of the equation expressed by the model and then solve for the variable.

16.



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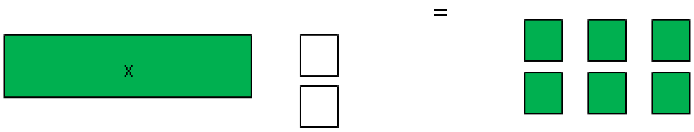
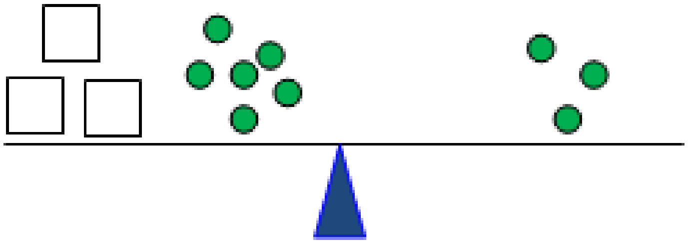
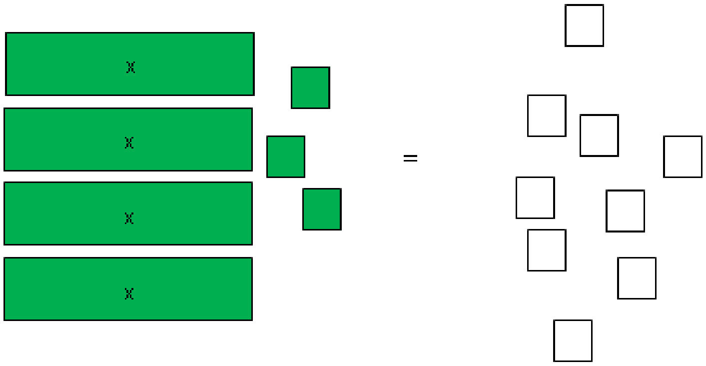


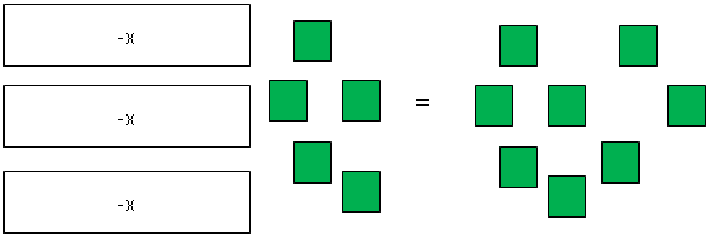
18.



19.



20.



2.2 Equations with Variables on Both Sides

Here you will learn to solve equations where there are variables on both sides of the equals sign.

Thomas has \$50 and Jack has \$100. Thomas is saving \$10 per week for his new bike. Jack is saving \$5 a week for his new bike. Can you represent this situation with an equation? How long will it be before the two boys have the same amount of money?

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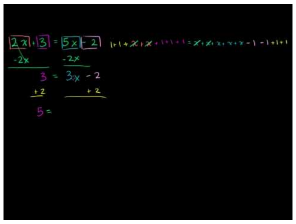
Khan Academy Equations with Variables on Both Sides



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Guidance

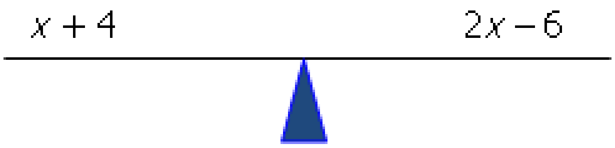
The methods used for solving equations with variables on both sides of the equation are the same as the methods used to solve equations with variables on one side of the equation. What differs is that first you must add or subtract a term from both sides in order to have the variable on only one side of the equals sign.

Remember that your goal for solving any equation is to get the variables on one side and the constants on the other side. You do this by adding and subtracting terms from both sides of the equals sign. Then you isolate the variables by multiplying or dividing. You must remember in these problems, as with any equation, whatever operation (addition, subtraction, multiplication, or division) you do to one side of the equals sign, you must do to the other side. This is a big rule to remember in order for equations to remain equal or to remain in balance.

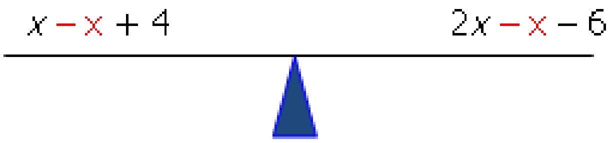
Example A

$x+4=2x-6$

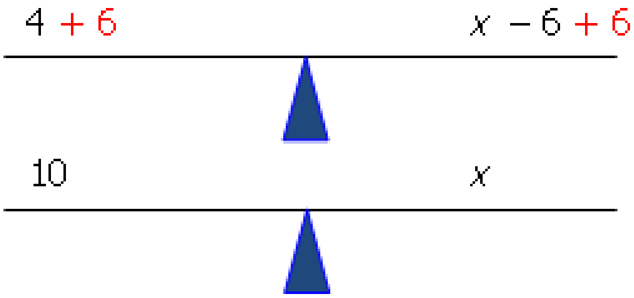
Solution: You can solve this problem using the balance method.



You could first try to get the variables all on one side of the equation. You do this by subtracting x from both sides of the equation.



Next, isolate the x variable by adding 6 to both sides.



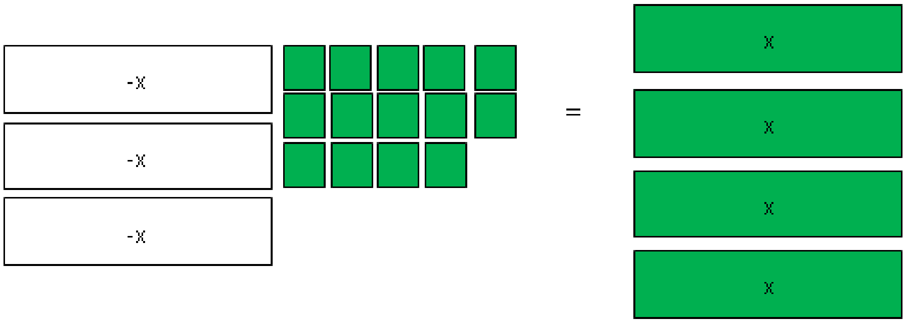
Therefore $x = 10$.

$$\begin{aligned}
&\text{Check:}\\
x + 4 &= 2x - 6\\
(\textcolor{red}{10}) + 4 &= 2(\textcolor{red}{10}) - 6\\
14 &= 20 - 6\\
14 &= 14
\end{aligned}$$

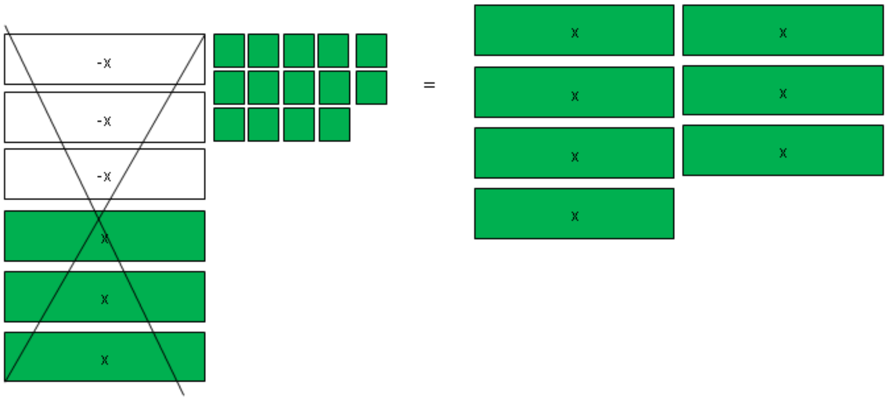
Example B

$14 - 3y = 4y$

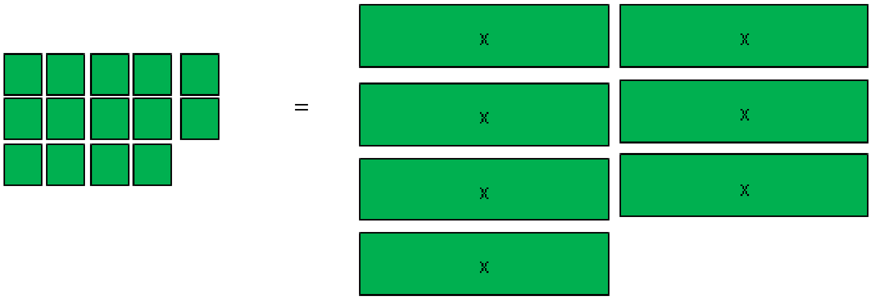
Solution: You can solve this equation using algebra tiles.



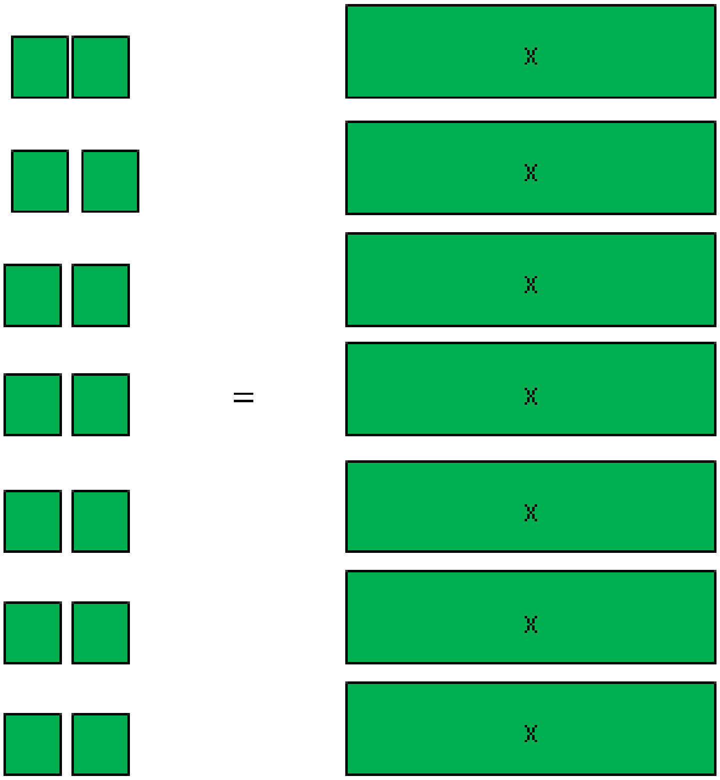
You first have to combine our variables (x) tiles onto the same side of the equation. You do this by adding $3x$ tiles to both sides of the equals sign. In this way the $-3y$ will be eliminated from the left hand side of the equation.



By isolating the variable (y) you are left with these algebra tiles.



Rearranging you will get the following.



Check:

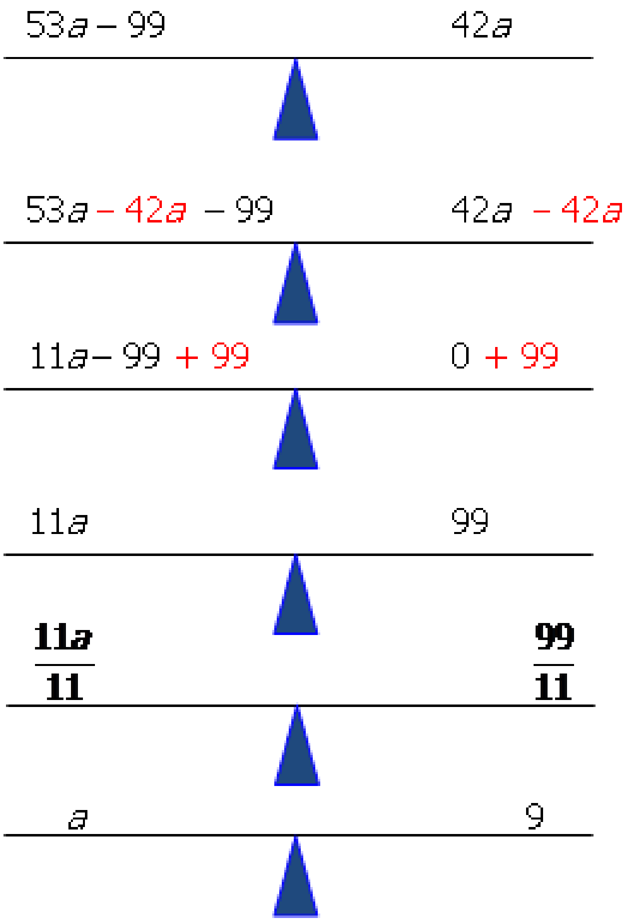
$$\begin{aligned}14-3y&=4y\\14-3(2)&=4(2)\\14-6&=8\\8&=8\end{aligned}$$

Therefore $y=2$.

Example C

$53a-99=42a$

Solution: To solve this problem, you would need to have a large number of algebra tiles! It might be more efficient to use the balance method to solve this problem.



$$\begin{aligned}
&\text{Check:}\\
53a-99&=42a\\
53({\color{red}9})-99&=42({\color{red}9})\\
477-99&=378\\
378&=378
\end{aligned}$$

Therefore, $a=9$.

Concept Problem Revisited

Thomas has \$50 and Jack has \$100. Thomas is saving \$10 per week for his new bike. Jack is saving \$5 a week for his new bike.

If you let x be the number of weeks, you can write the following equation.

$$\underbrace{10x+50}_{\text{Thomas's money: \$10 per week+\$50}} = \underbrace{5x+100}_{\text{Jack's money: \$5 per week+\$100}}$$

You can solve the equation now by first combining like terms.

$$\begin{aligned}
10x+50&=5x+100\\
10x{\color{red}-5x}+50&=5x{\color{red}-5x}+100 && \text{-moving the } x \text{ variables to left side of the equation}\\
5x+50{\color{red}-50}&=100{\color{red}-50} && \text{-moving the constants to right side of the equation}\\
5x&=50
\end{aligned}$$

You can now solve for x to find the number of weeks until the boys have the same amount of money.

$$\begin{aligned}
5x&=50\\
\frac{5x}{5}&=\frac{50}{5}\\
x&=10
\end{aligned}$$

Therefore, in 10 weeks Jack and Thomas will each have the same amount of money.

Vocabulary

Variable

A ***variable*** is an unknown quantity in a mathematical expression. It is represented by a letter. It is sometimes referred to as the literal coefficient.

Guided Practice

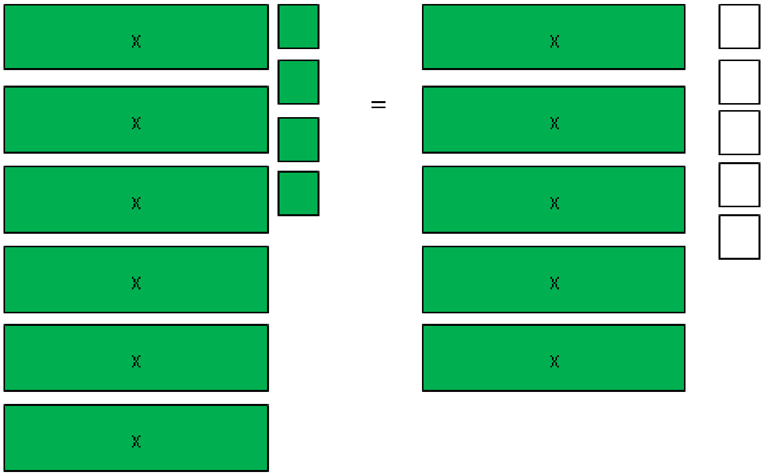
1. Solve for the variable in the equation $6x+4=5x-5$.

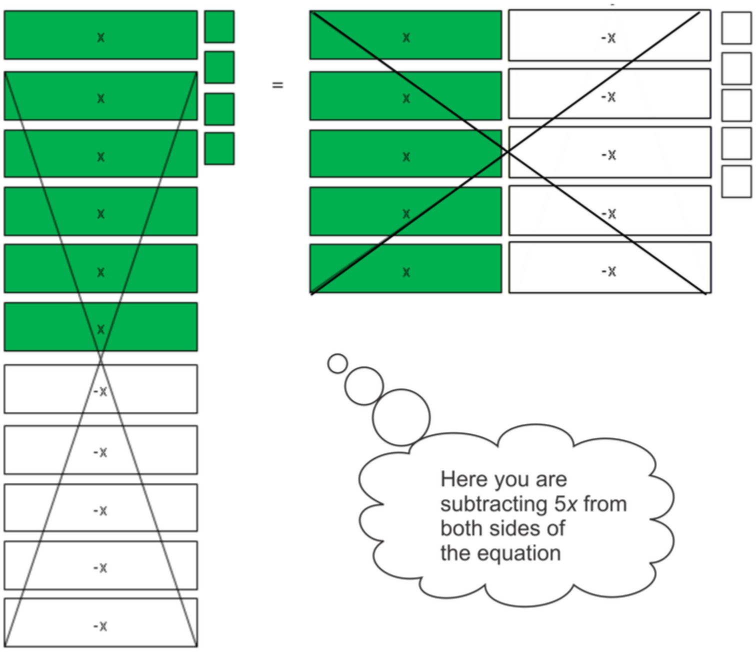
2. Solve for the variable in the equation $7r-4=3+8r$.

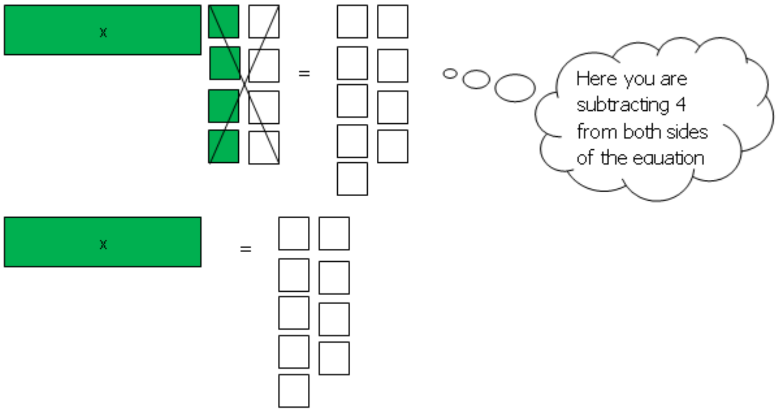
3. Determine the most efficient method to solve for the variable in the problem $10b-22=29-7b$. Explain your choice of method for solving this problem.

Answers:

1. $6x+4=5x-5$

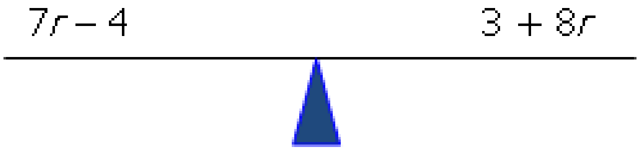




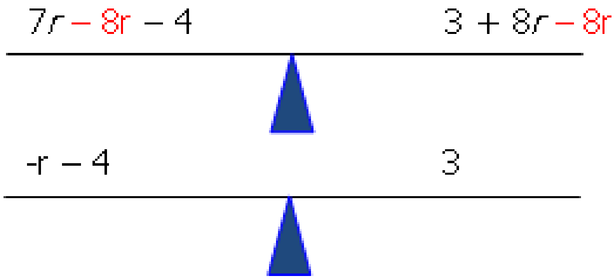


Therefore $x = -9$.

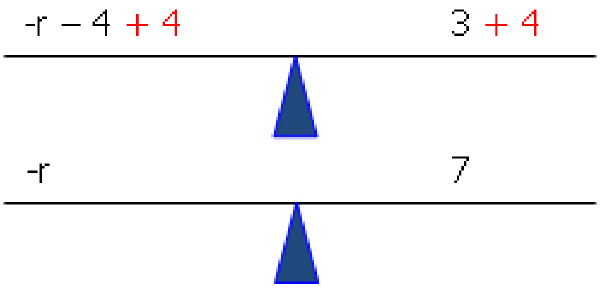
2. $7r - 4 = 3 + 8r$



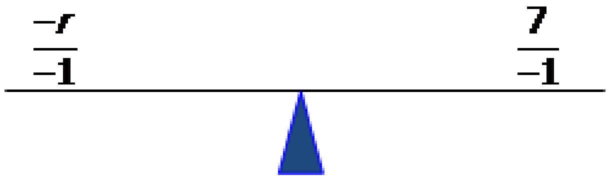
You can begin by combining the r terms. Subtract $8r$ from both sides of the equation.

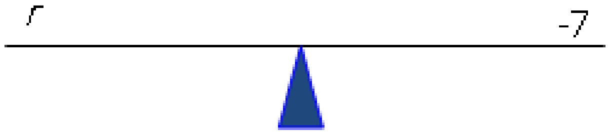


You next have to isolate the variable. To do this, add 4 to both sides of the equation.



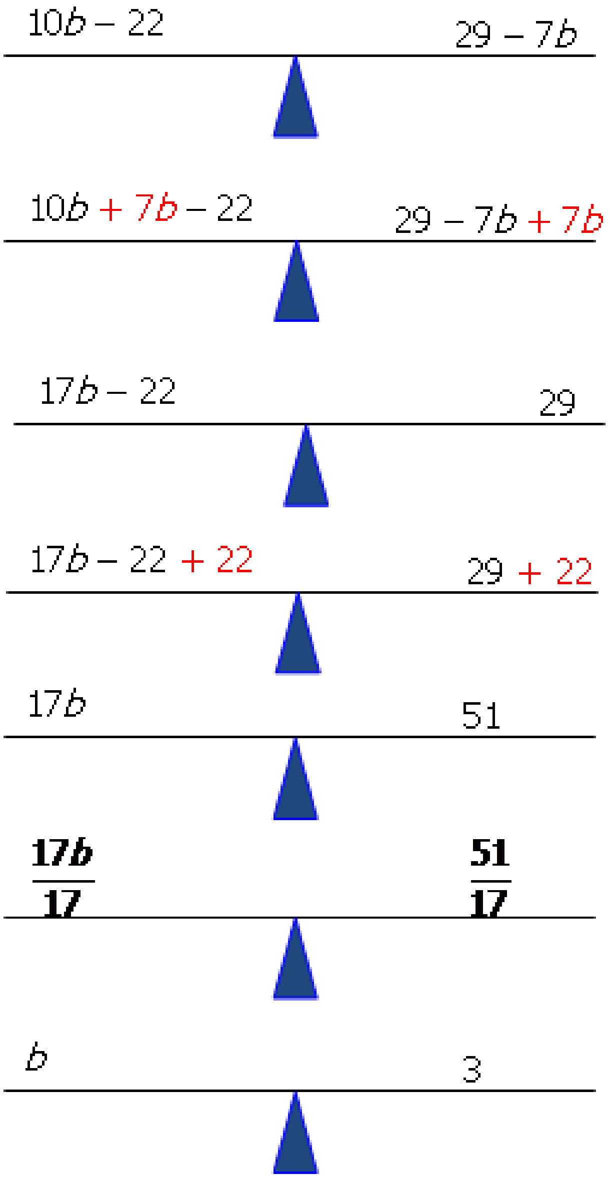
But there is still a negative sign with the *r* term. You now have to divide both sides by –1 to finally isolate the variable.





Therefore $r = -7$.

3. You could choose either method but there are larger numbers in this equation. With larger numbers, the use of algebra tiles is not an efficient manipulative. You should solve the problem using the balance method. Work through the steps to see if you can follow them.



Therefore $b = 3$.

Practice

Use the balance method to find the solution for the variable in each of the following problems.

1. $5p+3=-3p-5$
2. $6b-13=2b+3$
3. $2x-5=x+6$
4. $3x-2x=-4x+4$
5. $4t-5t+9=5t-9$

Use algebra tiles to find the solution for the variable in each of the following problems.

6. $6-2d=15-d$

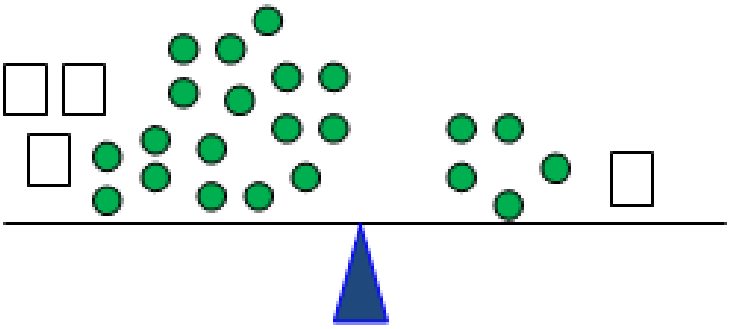
7. $8-s=s-6$
8. $5x+5=2x-7$
9. $3x-2x=-4x+4$
10. $8+t=2t+2$

Use the methods that you have learned for solving equations with variables on both sides to solve for the variables in each of the following problems. Remember to choose an efficient method to solve for the variable.

11. $4p-7=21-3p$
12. $75-6x=4x-15$
13. $3t+7=15-t$
14. $5+h=11-2h$
15. $9-2e=3-e$

For each of the following models, write a problem to represent the model and solve for the variable for the problem.

16.



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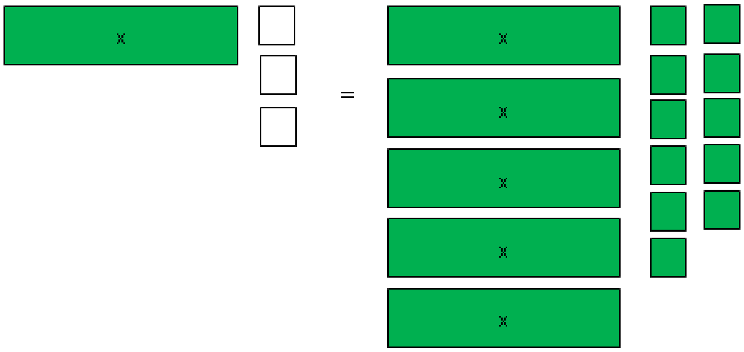

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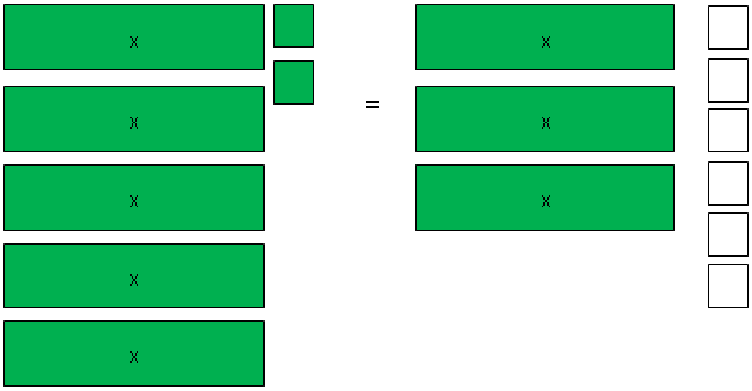

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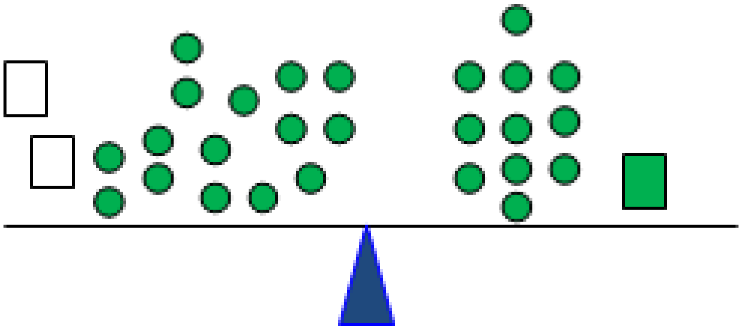

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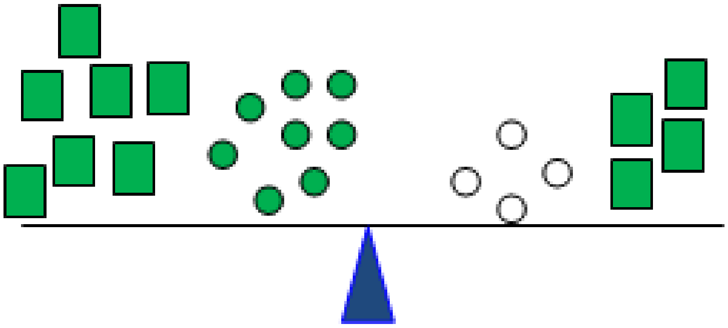


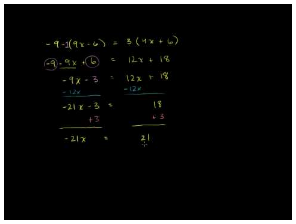
2.3 Equations with the Distributive Property

Here you will learn to solve equations that require the distributive property.

Morgan, Connor, and Jake are going on a class field trip with 12 of their class mates. Each student is required to have a survival kit with a flashlight, a first aid kit, and enough food rations for the trip. A flashlight costs \$10. A first aid kit costs \$9. Each day's food ration costs \$7. If the class has only \$1000 for the survival kits, how many days can they go on their trip?

Watch This

Khan Academy Solving Equations with the Distributive Property



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Guidance

The distributive property is a mathematical way of grouping terms. The **distributive property** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends.

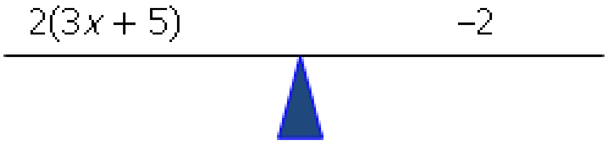
Consider $3(x+5)$. The distributive property states that the product of a number (3) and a sum $(x+5)$ is equal to the sum of the individual products of the number (3) and the addends (x and 5). So, $3(x+5) = 3x+15$.

When solving equations that require you to use the distributive property, you simply have one more step to follow. You still have the same goal of trying to get the variables on one side and the constants on the other side. When there are parentheses in the equation, your first step in solving the equation will be to use the distributive property to remove them.

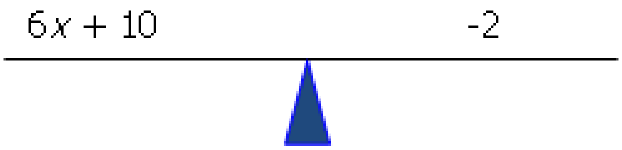
Example A

$2(3x+5) = -2$

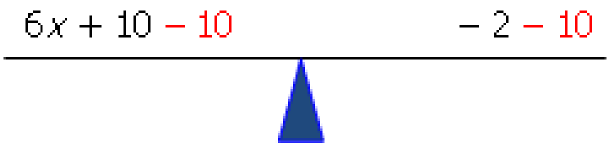
Solution: You can solve this problem using the balance method.



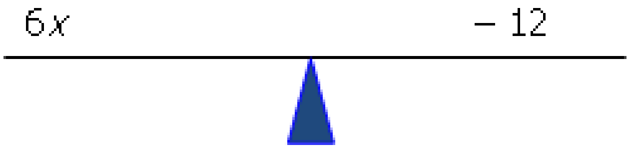
Your first step is to remove the parentheses. To do this multiply the 2 by the numbers inside the parentheses. Therefore, multiply 2 by $3x$ and 2 by 5.



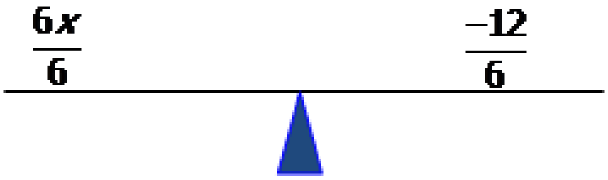
Next, isolate the *x* variable by subtracting 10 from both sides.



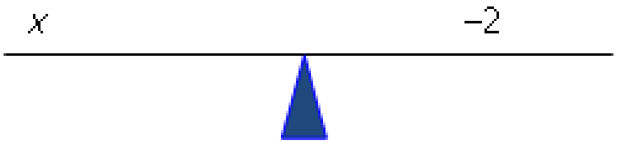
Simplifying you get:



Now divide by 6 to solve for the *x* variable.



To finally solve for the variable, simplify each side.



Therefore $x = -2$.

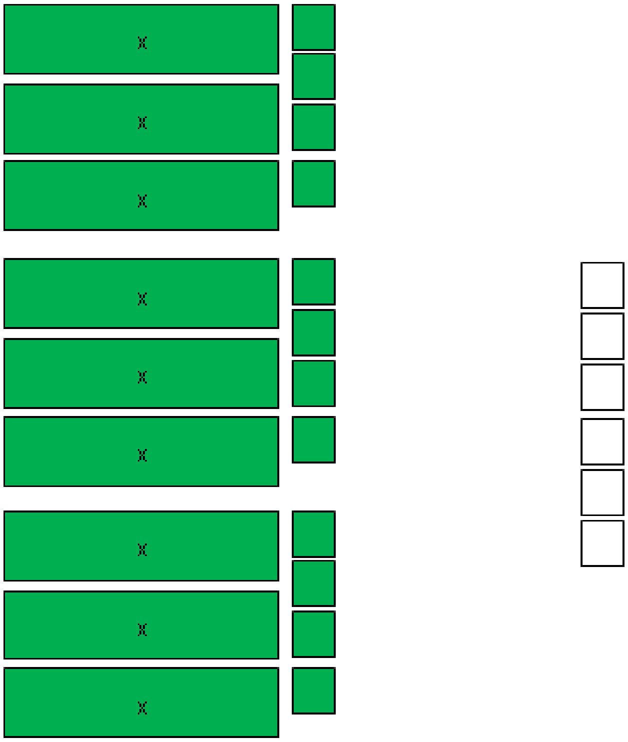
Check:

$$\begin{aligned}2(3x+5) &= -2\\ 2(3(-2)+5) &= -2\\ 2(-6+5) &= -2\\ 2(-1) &= -2\\ -2 &= -2\end{aligned}$$

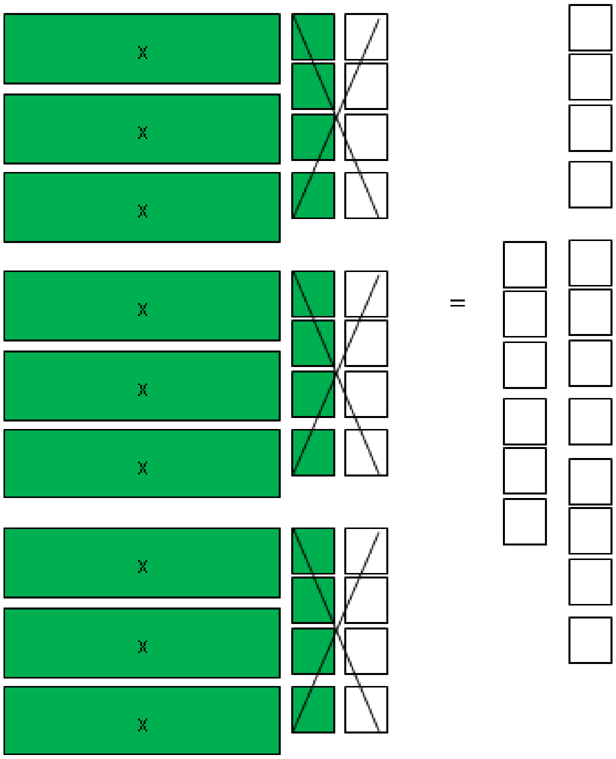
Example B

$3(4+3y) = -6$

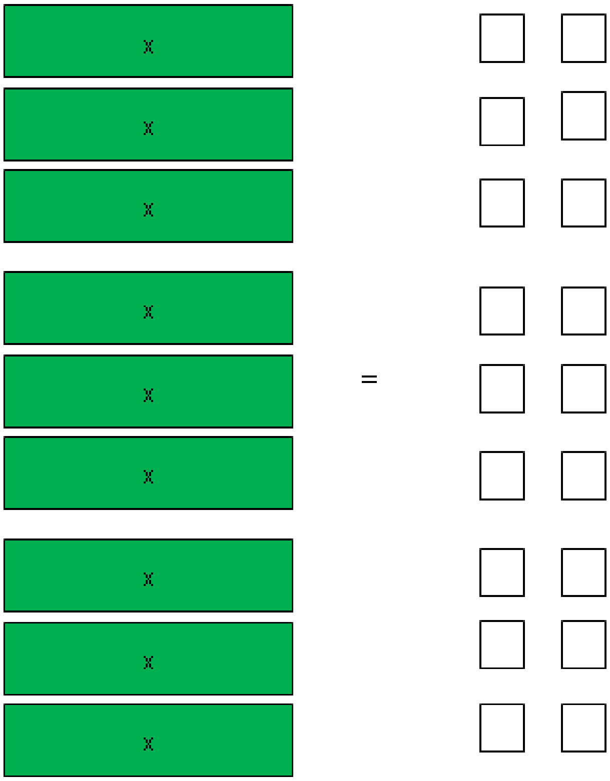
Solution: You can solve this problem using algebra tiles.



Since all of our variables (x tiles) are on the same side, you only need to subtract the 3 groups of 4 from both sides of the equal sign to isolate the variable.



Simplifying the right side of the equation and rearranging the tiles leave the following:



Therefore $y = -2$.

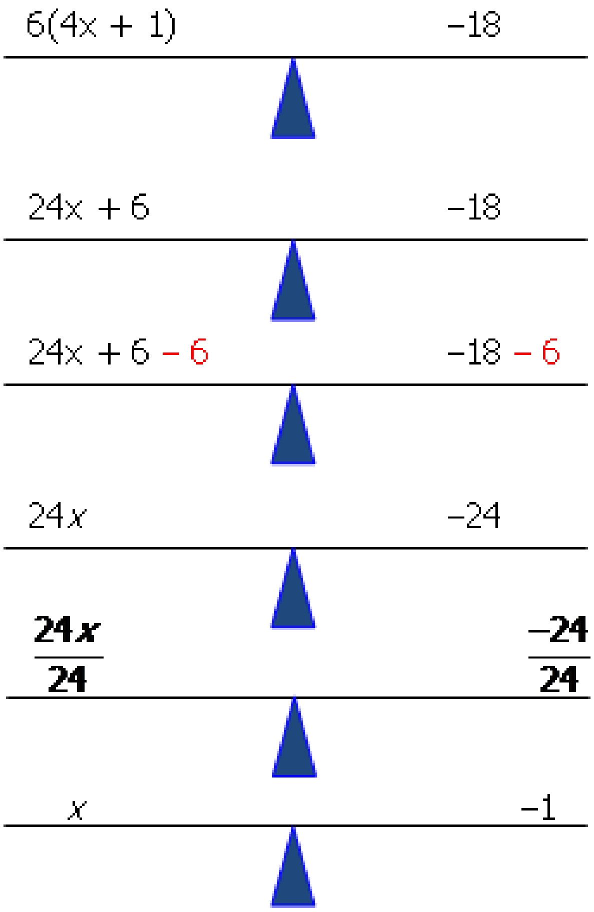
Check:

$$\begin{aligned}3(4+3y)&=-6\\3(4+3(-2))&=-6\\3(4-6)&=-6\\3(-2)&=-6\\-6&=-6\end{aligned}$$

Example C

$6(4x+1)=-18$

Solution:



Therefore $x=-1$.

$$\begin{aligned}
&\text{Check:}\\
6(4x+1) &= -18\\
6(4(-1)+1) &= -18\\
6(-4+1) &= -18\\
6(-3) &= -18\\
-18 &= -18
\end{aligned}$$

Concept Problem Revisited

Morgan, Connor, and Jake are going on a class field trip with 12 of their class mates. Each student is required to have a survival kit with a flashlight, a first aid kit, and enough food rations for the trip. A flashlight costs \$10. A first aid kit costs \$9. Each day's food ration costs \$7. If the class has only \$1000 for the survival kits, how many days can they go on their trip?

Write down what you know:

- Number of students going on the trip = 15 (Morgan, Connor, Jake + 12 others)
- Total money available = \$500
- Each flashlight costs \$10
- Each first aid kit costs \$9
- Each day's food ration costs \$7

One survival kit contains 1 flashlight + 1 first aid kit + x day's rations. Therefore, the cost of each survival kit is: $\$10+\$9+\$7x = \$19+\$7x$.

For the 15 students, the total cost of the survival kits would be:

$$15(\$19+\$7x) = \$285+\$105x$$

Since the class has \$1000 to spend, you can calculate how many days they can go on their trip.

$$\begin{aligned}
\$1000 &= \$285+\$105x\\
\$1000-\$285 &= \$285-\$285+\$105x\\
\$715 &= \$105x\\
\frac{\$715}{\$105} &= \frac{\$105x}{\$105}\\
6.81 &= x
\end{aligned}$$

Since the class does not have enough money to buy 7 days of food rations for each student, they will buy six days of food rations and the class will go on a class trip for six days.

Vocabulary

Distributive Property

The ***distributive property*** is a mathematical way of grouping terms. It states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $3(x+5)$, the distributive property states that the product of a number (3) and a sum $(x+5)$ is equal to the sum of the individual products of the number (3) and the addends (x and 5). $3(x+5) = 3x+15$.

Variable

A ***variable*** is an unknown quantity in a mathematical expression. It is represented by a letter. It is sometimes referred to as the literal coefficient.

Guided Practice

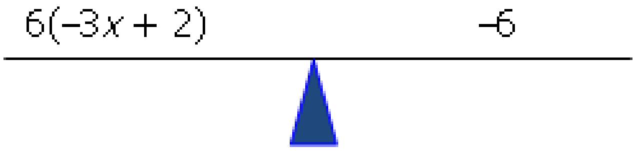
1. Use the balance method to solve for the variable in the equation $6(-3x+2) = -6$.

2. Use algebra tiles to solve for the variable in the equation $2(r-2) = -6$.

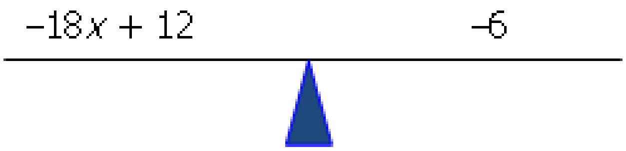
3. Determine the most efficient method to solve for the variable in the equation $2(j+1) = 3(j-1)$. Explain your choice of method for solving this problem.

Answers:

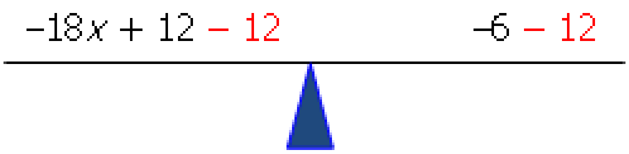
1. $6(-3x+2) = -6$.



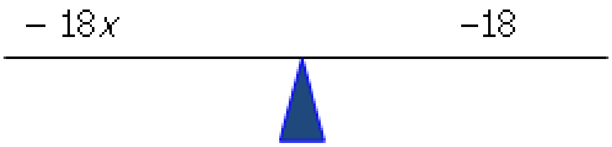
You can begin by using the distributive property on the left side of the equation.



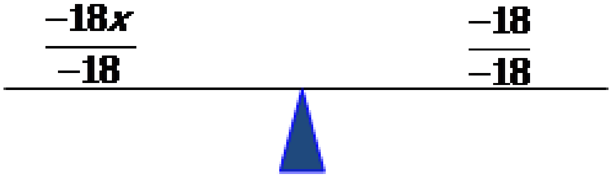
Next subtract 12 from both sides of the equation to get the variable term alone.



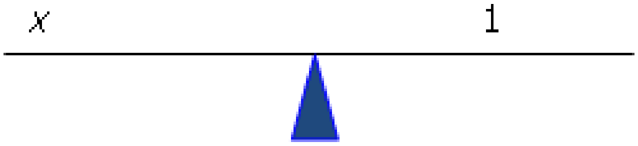
Simplifying, you get:



You next have to isolate the variable. To do this, divide both sides of the equation by –18.

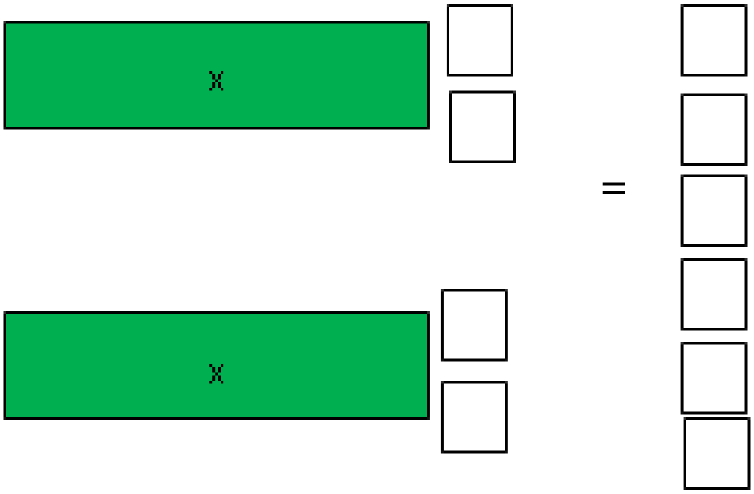


Finally, simplifying leaves you with:

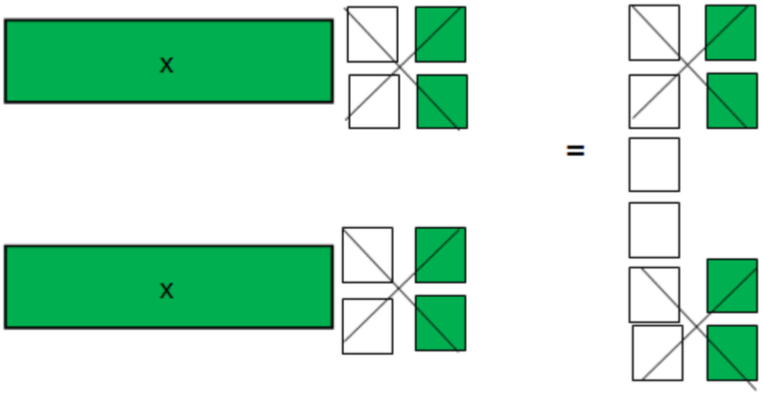


Therefore $x = 1$.

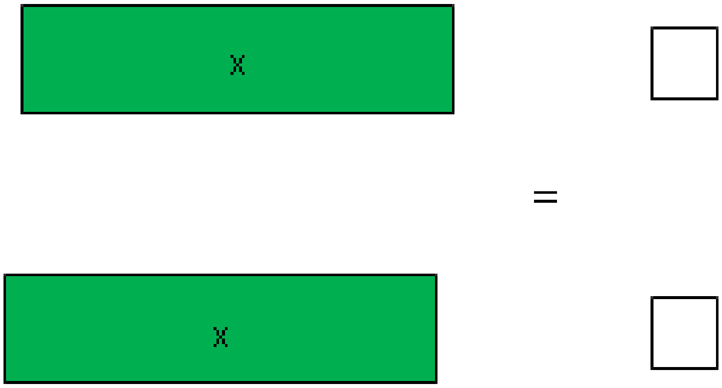
2. $2(r-2) = -6$



First, you need to add 4 to both sides to get the variables alone on the one side of the equal sign.



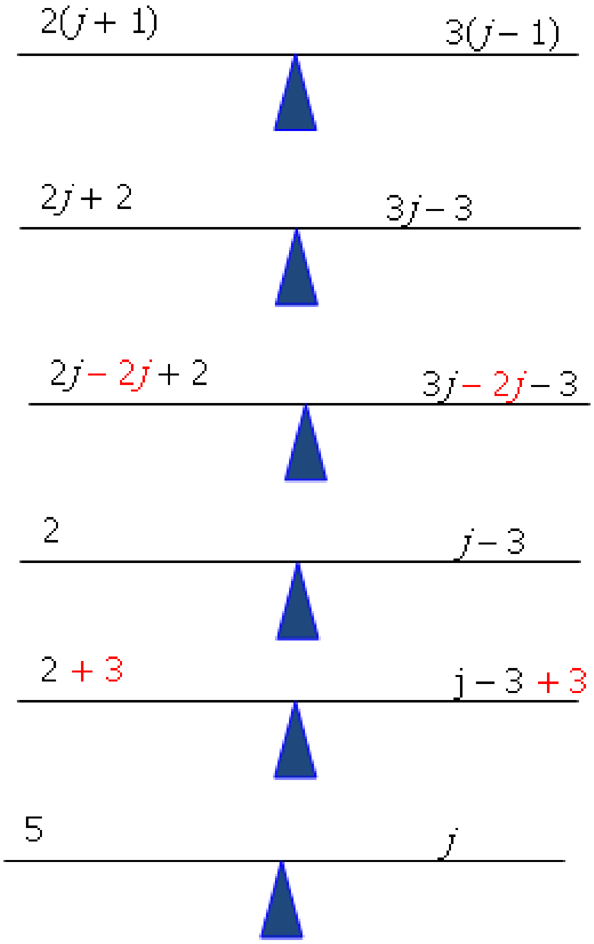
Simplifying leaves you with:



Therefore $r = -1$.

3. $2(j+1) = 3(j-1)$.

You could choose either method for this problem. Below you will see the balance method used to solve the problem. Work through the steps to see if you can follow them. Remember since there are parentheses, you must start with the distributive property and remove the parentheses.



Therefore $j = 5$.

Practice

Use the balance method to find the solution for the variable in each of the following equations.

1. $5(4x+3) = 75$
2. $3(s-4) = 15$
3. $5(k-4) = 10$
4. $43 = 4(t+6) - 1$
5. $6(x+4) = 3(5x+2)$

Use algebra tiles to find the solution for the variable in each of the following equations.

6. $2(d-3) = 4$
7. $5+2(x+7) = 20$
8. $2(3x-4) = 22$
9. $2(3x+2) - x = -6$
10. $2(x+4) - x = 9$

Use the methods that you have learned for solving equations with variables to solve for the variables in each of the following equations. Remember to choose an efficient method to solve for the variable.

11. $-6 = -6(3x - 8)$
12. $-2(x - 2) = 11$
13. $2 + 3(-2x + 1) = 20$
14. $3(x + 2) - x = 12$
15. $5(2 - 3x) = -8 - 6x$

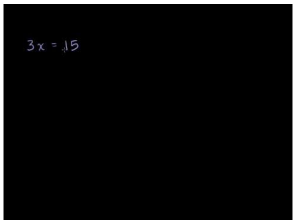
2.4 Equations with Decimals

Here you will solve equations that have decimals.

Karen wants to design a garden for her back yard. She knows she only has space for a rectangular garden of a perimeter equal to 46.5 feet. She needs to know the dimensions. The width will be half the length. What will be the dimensions of the garden?

Watch This

Khan Academy Simple Equations



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Guidance

You can solve equations that contain decimals in the same way that you solve any equation. You can also multiply both sides of the equation by a multiple of 10 in order to get rid of the decimals if you prefer. Consider the equation $0.1x+0.4=0.5$. At first this looks difficult because of the decimals. But multiply all of the numbers by 10 and see what happens:

$$\begin{aligned}(10)0.1x+(10)0.4&=(10)0.5\\1x+4&=5\\&or\\x+4&=5\end{aligned}$$

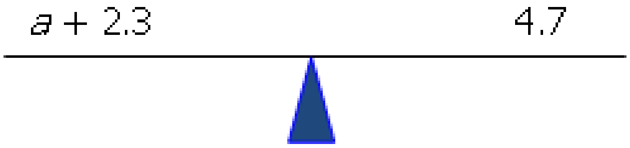
Now you can see that the answer is $x=1$.

When you have decimals in an equation, you can get rid of them by multiplying by 10 (if one decimal place), 100 (if two decimal places), or 1000 (if three decimal places). Then, solve the equation as usual. Remember that if you feel confident with adding, subtracting, multiplying, and dividing decimals, you can also solve equations with decimals by isolating the variable and solving without first removing the decimals.

Example A

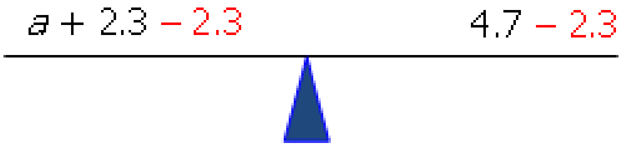
$a+2.3=4.7$

Solution: You can think about this problem with the balance method. You know that the two sides are equal so the balance has to stay horizontal. You can place each side of the equation on each side of the balance.

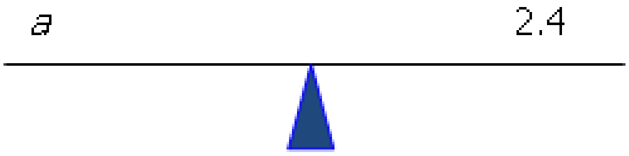


Like always, in order to solve the equation, you have to get the a all by itself. Always remember that you need to keep the balance horizontal. This means that whatever you do to one side of the equation, you ***have*** to do to the other side.

Subtract 2.3 from both sides to get rid of the 2.3 on the left and isolate the variable.



If you simplify this expression, you get:



Therefore $a = 2.4$.

You can, as always, check your answer to see if you are correct.

$$\begin{aligned} a + 2.3 &= 4.7 \\ ({\color{red}2.4}) + 2.3 &= 4.7 \\ 4.7 &= 4.7 \end{aligned}$$

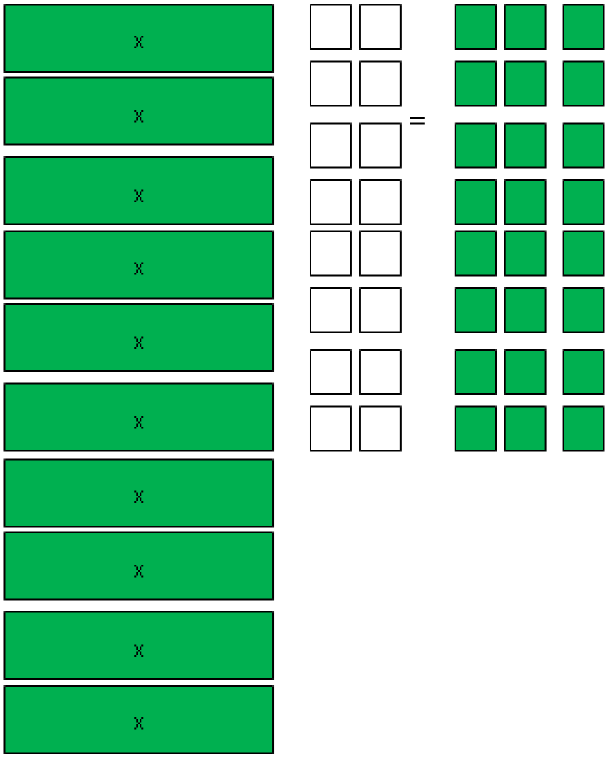
Example B

$b - 1.6 = 2.4$

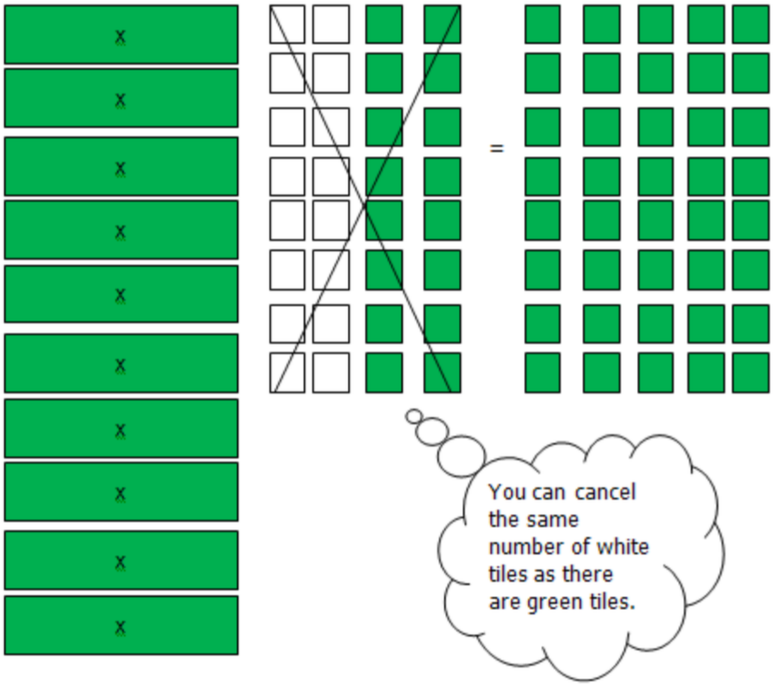
Solution: This time try solving the problem using algebra tiles. Multiply this problem by 10 to get rid of the decimals.

$$\begin{aligned} {\color{red}(10)}b - {\color{red}(10)}1.6 &= {\color{red}(10)}2.4 \\ 10b - 16 &= 24 \end{aligned}$$

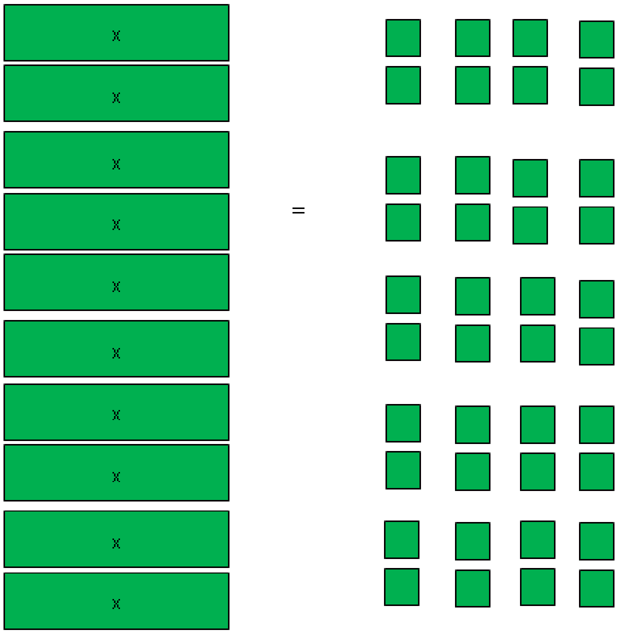
Now you can use algebra tiles to solve for the variable.



Your first step is to add 16 to both sides of the equal sign.



Simplify and rearrange and you get:



Therefore $b = 4$.

You should always check your answer:

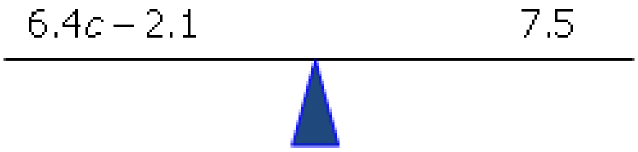
$$\begin{aligned} b - 1.6 &= 2.4 \\ (4) - 1.6 &= 2.4 \\ 2.4 &= 2.4 \end{aligned}$$

Note: since the numbers were so large with this problem, the balance method is more efficient. This is true with most problems involving decimals. In fact, algebra tiles are rarely more efficient when solving problems with variables involving decimals.

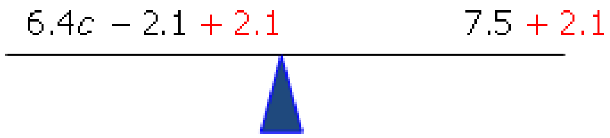
Example C

$6.4c - 2.1 = 7.5$

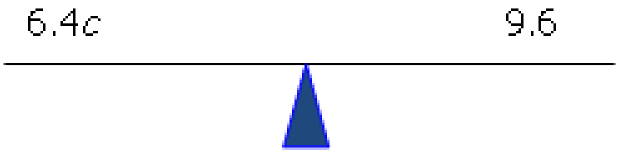
Solution: You can again use the balance method to solve this problem.



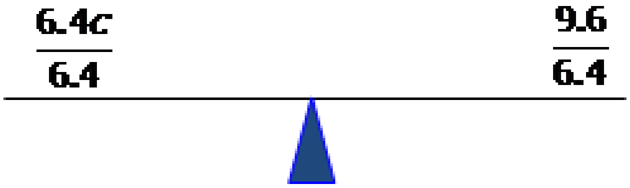
Let's first add 2.1 to both sides to get rid of the 2.1 on the left.



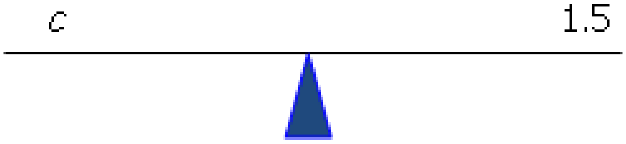
Simplifying you get:



Since 6.4 is multiplied by c, you can get a by itself (or isolate it) by dividing by 6.4. Remember that whatever you do to one side, you have to do to the other.



If you simplify this expression, you get:



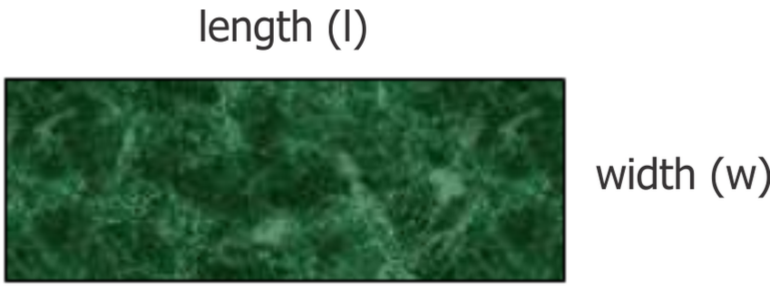
Therefore $c = 1.5$.

You can check your answer to see if you are correct.

$$\begin{aligned}6.4c - 2.1 &= 7.5\\ 6.4(1.5) - 2.1 &= 7.5\\ 9.6 - 2.1 &= 7.5\\ 7.5 &= 7.5\end{aligned}$$

Concept Problem Revisited

Karen wants to design a garden for her back yard. She knows she only has space for a rectangular garden of a perimeter equal to 46.5 feet. She needs to know the dimensions. The width will be half the length. What will be the dimensions of the garden?



$$\begin{aligned} Perimeter\ (P) &= 46.5\ feet \\ P &= 2l + 2w \\ w &= \frac{1}{2}l \end{aligned}$$

Therefore: $P = 2l + 2\left(\frac{1}{2}l\right)$

or

$$\begin{aligned} &P = 2l + \cancel{2}\left(\frac{1}{\cancel{2}}l\right) \\ &P = 2l + l \\ &P = 3l \end{aligned}$$

Since you know that Karen has 60 feet for her perimeter, you can substitute 46.5 in for P.

$46.5 = 3l$

Now you can solve for l (the length).

$$\begin{aligned} \frac{46.5}{3} &= \frac{3l}{3} \\ l &= 15.5\ feet \end{aligned}$$

You now know that the length is 15.5 feet. You also know that the width is $\frac{1}{2}$ the length. So you can solve for the width.

$$\begin{aligned} &w = \frac{1}{2}l \\ &w = \frac{1}{2}(15.5) \\ &w = 7.75\ feet \end{aligned}$$

Therefore the dimensions of Karen's garden are $15.5\ feet \times 7.75\ feet$.

Vocabulary

Equation

An ***equation*** is a mathematical statement with expressions separated by an equals sign.

Variable

A ***variable*** is an unknown quantity in a mathematical expression. It is represented by a letter. It is sometimes referred to as the literal coefficient.

Guided Practice

1. Use a model to solve for the variable in the equation $t - 1.4 = 0.6$.

2. Use a model to solve for the variable in the equation $1.2s + 3.9 = 4.1$.

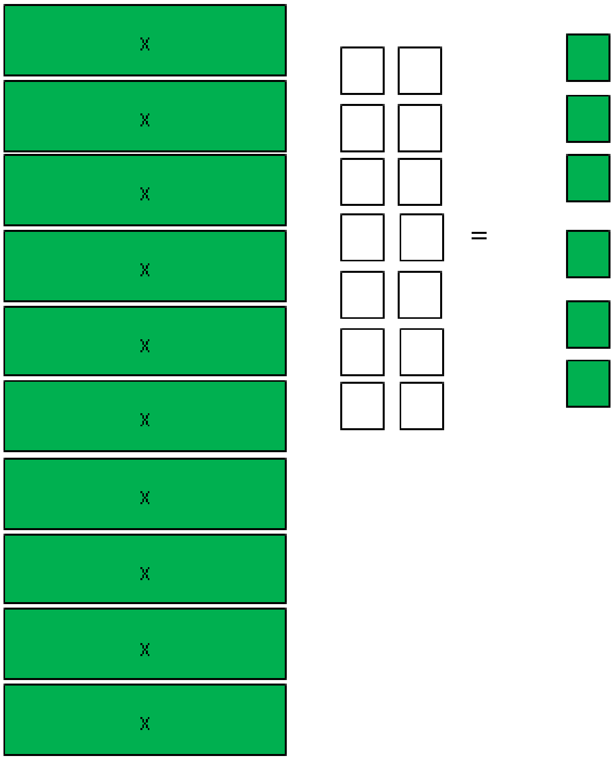
3. Solve for x in the equation $7 - 0.03x = 1.72x + 1.75$.

Answers:

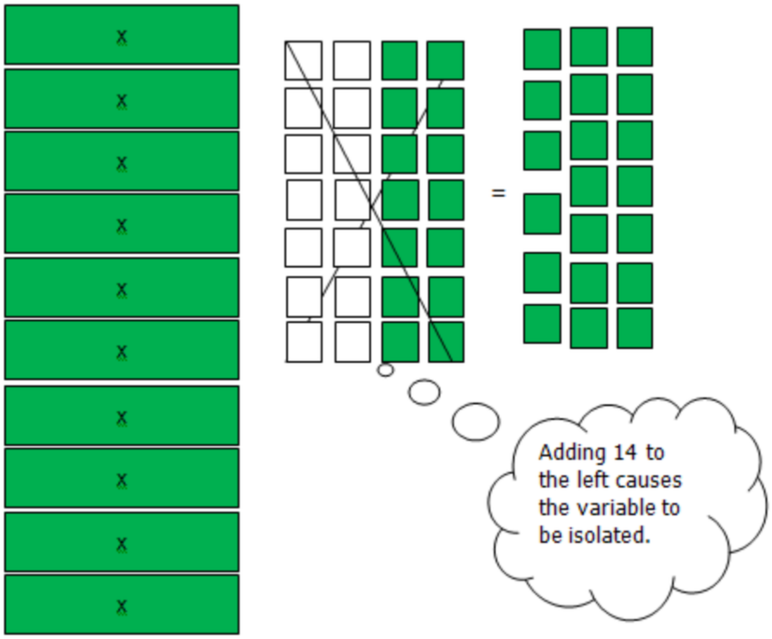
1. $t - 1.4 = 0.6$

You can use either method to solve this problem but let's use algebra tiles for this one. Remember since there are decimals (to one decimal place to be exact), you first must multiply the equation by 10 to get rid of the decimals.

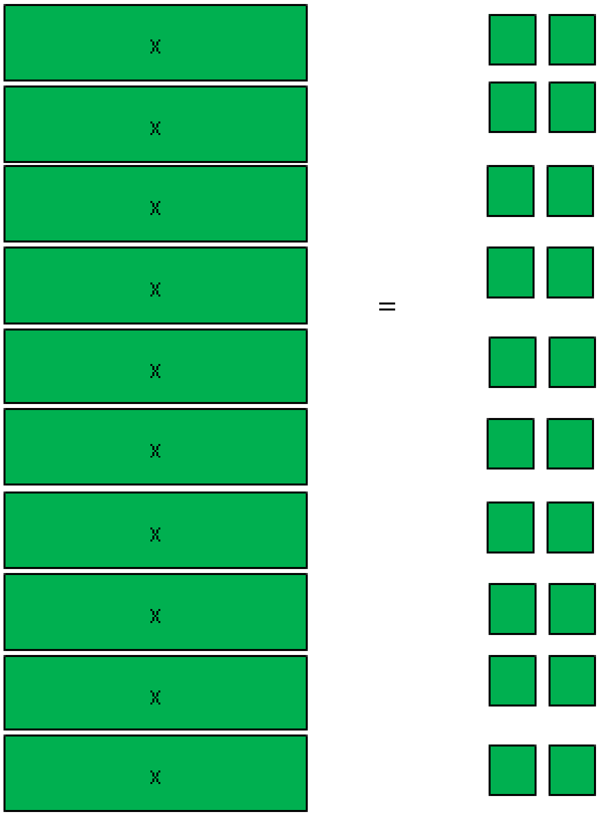
$$\begin{gathered}(10)t - (10)1.4 = (10)0.6 \\ 10t - 14 = 6\end{gathered}$$



You now can add 14 to each side to isolate the variable.

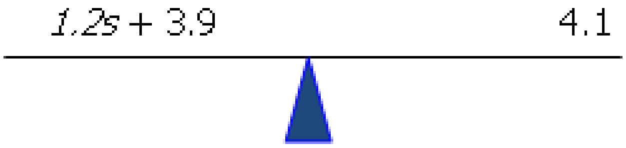


Simplifying and rearranging leaves you with:

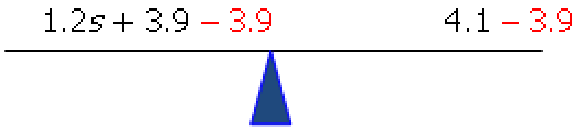


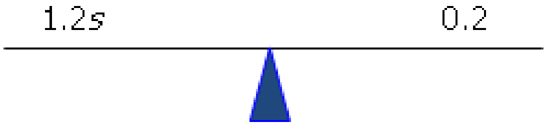
Therefore, $t = 2$.

2. $1.2s + 3.9 = 4.1$

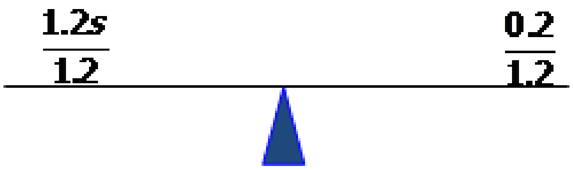


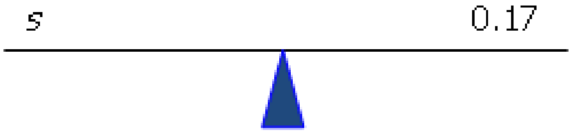
First you have to subtract 3.9 from both sides of the equation in order to start to isolate the variable.





Now, in order to get s all by itself, you have to divide both sides by 1.2. This will isolate the variable s





Therefore, rounded to the nearest hundredth, $s = 0.17$.

3. $7 - 0.03x = 1.72x + 1.75$

You can use any method to solve this equation. Remember to isolate the x variable. You will notice here that there are two x values, one on each side of the equation. First combine these terms by adding $0.03x$ to both sides of the equation.

$7 - 0.03x + 0.03x = 1.72x + 0.03x + 1.75$

Simplifying you get:

$7 = 1.75x + 1.75$

Now you can use any method to solve the equation. You now should just have to subtract 1.75 from both sides to isolate the x variable.

$7-1.75=1.75x+1.75-1.75$

Simplifying you get:

$5.25=1.75x$

Now to solve for the variable, you need to divide both sides by 1.75.

$\frac{5.25}{1.75}=\frac{1.75x}{1.75}$

You can now solve for x.

$x=3$

Practice

Use the balance model to solve for each of the following variables.

1. $a+0.3=-0.5$
2. $2b-1.5=6.3$
3. $1.4c-2.8=4.9$
4. $2.1-1.5d=3.2$
5. $4.4-3.3e=-2.2$

Use algebra tiles to solve for each of the following variables.

6. $0.5x+0.2=0.7$
7. $0.2y-0.9=0.5$
8. $0.2z-0.3=0.7$
9. $0.07+0.05x=-0.03$
10. $0.05x+0.16=-0.04$

Use the rules that you have learned to solve for the variables in the following problems.

11. $0.87+0.15x=-0.03$
12. $0.52x+0.12=-0.4$
13. $0.25z-3.3=0.7$
14. $0.6x-1.25=0.4x+0.35$
15. $x-0.35-0.05x=2-1.4x$

2.5 Equations with Fractions

Here you will learn to solve equations that contain fractions.

In this year's student election for president, there were two candidates. The winner received $\frac{1}{3}$ more votes than the loser. If there were 584 votes cast for president, how many votes did each of the two candidates receive?

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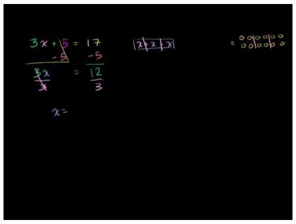
Khan Academy Slightly More Complicated Equations



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Guidance

When introducing fractions into an equation, the same rules for solving any equation apply. You need to keep the equations in balance by adding, subtracting, multiplying, or dividing on both sides of the equals sign in order to isolate the variable. The goal still remains to get your variable alone on one side of the equals sign with your constant terms on the other in order to solve for this variable.

With fractions, there is sometimes an added step of multiplying and dividing the equation by the numerator and denominator in order to solve for the variable. Or, if there are multiple fractions that do not have the same denominator, you must first find the least common denominator (LCD) before combining like terms.

Example A

Solve: $\frac{1}{3}t+5=-1$.

Solution:

$$\begin{aligned}
\frac{1}{3}t+5&=-1 \\
\frac{1}{3}t+5-5&=-1-5 && \text{(Subtract 5 from both sides to isolate the variable)} \\
\frac{1}{3}t&=-6 && \text{(Simplify)} \\
(\cancel{3})\frac{1}{\cancel{3}}t&=-6(3) && \text{(Multiply both sides by the denominator (3) in the fraction)} \\
t&=-18 && \text{(Simplify)}
\end{aligned}$$

Therefore $t=-18$.

$$\begin{aligned} &\text{Check:}\\ \frac{1}{3}t+5&=-1\\ \frac{1}{3}(-18)+5&=-1\\ -6+5&=-1\\ -1&=-1 \end{aligned}$$

Example B

Solve: $\frac{3}{4}x-3=2$.

Solution:

$$\begin{aligned} \frac{3}{4}x-3&=2\\ \frac{3}{4}x-3+3&=2+3 && \text{(Add 3 to both sides to isolate the variable)}\\ \frac{3}{4}x&=5 && \text{(Simplify)}\\ (\cancel{4})\frac{3}{\cancel{4}}x&=5(4) && \text{(Multiply both sides by the denominator (4) in the fraction)}\\ 3x&=20 && \text{(Simplify)}\\ \frac{\cancel{3}x}{\cancel{3}}&=\frac{20}{3}\\ x&=\frac{20}{3} \end{aligned}$$

Therefore $x=\frac{20}{3}$.

$$\begin{aligned} &\text{Check:}\\ \frac{3}{4}x-3&=2\\ \frac{3}{4}\left(\frac{20}{3}\right)-3&=2\\ \frac{20}{4}-3&=2\\ 5-3&=2\\ 2&=2 \end{aligned}$$

Example C

Solve: $\frac{2}{5}x-4=-\frac{1}{5}x+8$.

Solution:

$$\begin{aligned}
\frac{2}{5}x-4&=-\frac{1}{5}x+8 && \\
\frac{2}{5}x+\frac{1}{5}x-4&=-\frac{1}{5}x+\frac{1}{5}x+8 && \left(\text{Add } \frac{1}{5}x \text{ to both sides of the equal sign to combine variables}\right)\\
\frac{3}{5}x-4&=8 && (\text{Simplify})\\
\frac{3}{5}x-4+4&=8+4 && (\text{Add 4 to both sides of the equation to isolate the variable})\\
\frac{3}{5}x&=12 && (\text{Simplify})\\
(\cancel{5})\frac{3}{\cancel{5}}x&=12(5) && (\text{Multiply both sides by the denominator } (5) \text{ in the fraction})\\
3x&=60 && (\text{Simplify})\\
\frac{\cancel{3}x}{\cancel{3}}&=\frac{60}{3} && (\text{Divide both sides by the numerator } (3) \text{ in the fraction})\\
x&=20 && (\text{Simplify})
\end{aligned}$$

Therefore $x=20$.

Check:

$$\begin{aligned}
\frac{2}{5}x-4&=-\frac{1}{5}x+8\\
\frac{2}{5}(20)-4&=-\frac{1}{5}(20)+8\\
\frac{40}{5}-4&=-\frac{20}{5}+8\\
8-4&=-4+8\\
4&=4
\end{aligned}$$

Concept Problem Revisited

In this year's student election for president, there were two candidates. The winner received $\frac{1}{3}$ more votes. If there were 584 votes cast for president, how many votes did each of the two candidates receive?

Let $x=$ votes for candidate 1 (the winner)

Let $y=$ votes for candidate 2

$x+y=584$

You must have only one variable in the equation in order to solve it. Let's look at another relationship from the problem.

$$\begin{aligned}
x&=y+\frac{1}{3}y && \left(\text{Candidate 1 received } \frac{1}{3} \text{ more votes than candidate 2}\right)\\
x&=\frac{3}{3}y+\frac{1}{3}y && (\text{Make denominator common for both } y \text{ variables})\\
x&=\frac{4}{3}y && (\text{Simplify})
\end{aligned}$$

Now substitute into the original problem.

$$\begin{aligned}
\frac{4}{3}y+y&=584 && \text{(Substitute for } x \text{ into the equation)}\\
\frac{4}{3}y+\frac{3}{3}y&=584 && \text{(Make denominator common for both } y \text{ variables)}\\
\frac{7}{3}y&=584 && \text{(Combine like terms)}\\
(\cancel{3})\frac{7}{\cancel{3}}y&=584(3) && \text{(Multiply both sides by the denominator in the fraction)}\\
7y&=1752 && \text{(Simplify)}\\
\frac{\cancel{7}y}{\cancel{7}}&=\frac{1752}{7} && \text{(Divide both sides by the numerator in the fraction)}\\
y&=250.28 && \text{(Simplify)}
\end{aligned}$$

So candidate 2 received 250 votes. Candidate 1 would then receive $584-250=334$ votes.

Vocabulary

Fraction

A ***fraction*** is a part of a whole consisting of a numerator divided by a denominator. For example, if a pizza is cut into eight slices and you ate 3 slices, you would have eaten $\frac{3}{8}$ of the pizza. $\frac{3}{8}$ is a fraction with 3 being the numerator and 8 being the denominator.

Least Common Denominator

The ***least common denominator*** or lowest common denominator is the smallest number that all of the denominators (or the bottom numbers) can be divided into evenly. For example with the fractions $\frac{1}{2}$ and $\frac{1}{3}$, the smallest number that both 2 and 3 will divide into evenly is 6. Therefore the least common denominator is 6.

Guided Practice

1. Solve for x: $\frac{2}{3}x=12$.
2. Solve for x: $\frac{3}{4}x-5=19$.
3. Solve for x: $\frac{1}{4}w-3=\frac{2}{3}w$.

Answers:

1.

$$\begin{aligned}
\frac{2}{3}x&=12 && \\
(\cancel{3})\frac{2}{3}x&=12(3) && \text{(Multiply both sides by the denominator (3) in the fraction)}\\
2x&=36 && \text{(Simplify)}\\
\frac{\cancel{2}x}{\cancel{2}}&=\frac{36}{2} && \text{(Divide both sides by the numerator (2) in the fraction)}\\
x&=18 && \text{(Simplify)}
\end{aligned}$$

Therefore $x = 18$.

Check:

$$\begin{aligned}\frac{2}{3}x &= 12\\ \frac{2}{3}(18) &= 12\\ \frac{36}{3} &= 12\\ 12 &= 12\end{aligned}$$

2.

$$\begin{aligned}\frac{3}{4}x-5 &= 19 \\ \frac{3}{4}x-5+5 &= 19+5 && \text{(Add 5 to both sides of the equal sign to isolate the variable)}\\ \frac{3}{4}x &= 24 && \text{(Simplify)}\\ (\cancel{4})\frac{3}{\cancel{4}}x &= 24(4) && \text{(Multiply both sides by the denominator(4) in the fraction)}\\ 3x &= 96 && \text{(Simplify)}\\ \frac{\cancel{3}x}{\cancel{3}} &= \frac{96}{3} && \text{(Divide both sides by numerator (3) in the fraction)}\\ x &= 32 && \text{(Simplify)}\end{aligned}$$

Therefore $x = 32$.

Check:

$$\begin{aligned}\frac{3}{4}x-5 &= 19\\ \frac{3}{4}(32)-5 &= 19\\ \frac{96}{4}-5 &= 19\\ 24-5 &= 19\\ 19 &= 19\end{aligned}$$

3.

$$\begin{aligned}
\frac{1}{4}w-3&=\frac{2}{3}w\\
\frac{1}{4}w-3+3&=\frac{2}{3}w+3 && \text{(Add 3 to both sides of the equal sign to start)}\\
\frac{1}{4}w&=\frac{2}{3}w+3 && \text{(Simplify)}\\
\frac{1}{4}w-\frac{2}{3}w&=\frac{2}{3}w-\frac{2}{3}w+3 && \text{(Subtract }\frac{2}{3}w\text{ from both sides of the equal sign to get variables on same side)}\\
\frac{1}{4}w-\frac{2}{3}w&=3 && \text{(Simplify)}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{3}{3}\right)\frac{1}{4}w-\left(\frac{4}{4}\right)\frac{2}{3}w&=\left(\frac{12}{12}\right)3 && \text{(Multiply by the LCD)}\\
\frac{3}{12}w-\frac{8}{12}w&=\frac{36}{12} && \text{(Simplify)}
\end{aligned}$$

Since all the denominators are the same (12), we can simplify further:

$$\begin{aligned}
3w-8w&=36 && \text{(Combine like terms)}\\
-5w&=36 && \text{(Simplify)}\\
\frac{-5w}{-5}&=\frac{36}{-5} && \text{(Divide by }-5\text{ to solve for the variable)}\\
w&=-\frac{36}{5} && \text{(Simplify)}
\end{aligned}$$

Therefore $w=-\frac{36}{5}$.

$$\begin{aligned}
&\text{Check:}\\
\frac{1}{4}w-3&=\frac{2}{3}w\\
\frac{1}{4}\left(\frac{-36}{5}\right)-3&=\frac{2}{3}\left(\frac{-36}{5}\right)\\
\frac{-36}{20}-3&=\frac{-72}{15}\\
\frac{-108}{60}-\frac{180}{60}&=\frac{-288}{60}\\
\frac{-288}{60}&=\frac{-288}{60}
\end{aligned}$$

Practice

Solve for the variable in each of the following equations.

1. $\frac{1}{3}p=5$
2. $\frac{3}{7}j=8$
3. $\frac{2}{5}b+4=6$

4. $\frac{2}{3}x-2=1$
5. $\frac{1}{3}x+3=-3$

6. $\frac{1}{8}k+\frac{2}{3}=5$
7. $\frac{1}{6}c+\frac{1}{3}=-2$
8. $\frac{4}{5}x+3=\frac{2}{3}$
9. $\frac{3}{4}x-\frac{2}{5}=\frac{1}{2}$
10. $\frac{1}{4}t+\frac{2}{3}=\frac{1}{2}$

11. $\frac{1}{3}x+\frac{1}{4}x=1$
12. $\frac{1}{5}d+\frac{2}{3}d=\frac{5}{3}$
13. $\frac{1}{2}x-1=\frac{1}{3}x$
14. $\frac{1}{3}x-\frac{1}{2}=\frac{3}{4}x$
15. $\frac{2}{3}j-\frac{1}{2}=\frac{3}{4}j+\frac{1}{3}$

2.6 Equations with Decimals, Fractions and Parentheses

Here you will learn to solve equations that contain fractions and/or decimals and require to use the distributive property to get rid of parentheses.

Pens are \$9 per dozen and pencils are \$6 per dozen. Janet needs to buy a half dozen of each for school. How much is the total cost of her purchase?

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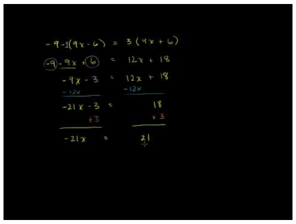
Khan Academy Solving Equations with the Distributive Property



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Guidance

Recall that the distributive property is a mathematical way of grouping terms. It states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. Here, you will use the distributive property with equations that contain decimals and/or fractions. The same rules apply. If the equation has parentheses, your first step is to multiply what is on the outside of the parentheses by what is on the inside of the parentheses. After you remove parentheses, you then solve the equation by combining like terms, moving constants to one side of the equals sign and variables to the other side of the equals sign, and finally isolating the variable to find the solution.

Example A

Solve: $\frac{2}{5}(d+4)=6$.

Solution:

$$\begin{aligned}\frac{2}{5}(d+4)&=6\\ \frac{2}{5}d+\frac{8}{5}&=6 && \text{(Apply the distributive property to remove the parentheses)}\end{aligned}$$

Find the LCD for 5, 5, and 1. Since it is 5, multiply the last number by $\frac{5}{5}$, to get the same denominator.

$$\begin{aligned}\frac{2}{5}d+\frac{8}{5}&=\left(\frac{5}{5}\right)6\\ \frac{2}{5}d+\frac{8}{5}&=\frac{30}{5} && \text{(Simplify)}\end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned}
2d+8&=30 \\
2d+8-8&=30-8 && \text{(Subtract 8 from both sides of the equals sign to isolate the variable)} \\
2d&=22 && \text{(Simplify)} \\
\frac{2d}{2}&=\frac{22}{2} && \text{(Divide by 2 to solve for the variable)} \\
d&=11
\end{aligned}$$

Therefore $d = 11$.

$$\begin{aligned}
&\text{Check:} \\
\frac{2}{5}(d+4)&=6 \\
\frac{2}{5}(11+4)&=6 \\
\frac{2}{5}(15)&=6 \\
\frac{30}{5}&=6 \\
6&=6
\end{aligned}$$

Example B

Solve: $\frac{1}{4}(3x+7)=2$.

Solution:

$$\begin{aligned}
\frac{1}{4}(3x+7)&=-2 \\
\frac{3}{4}x+\frac{7}{4}&=-2 && \text{(Apply the distributive property to remove the parentheses)}
\end{aligned}$$

Find the LCD for 4, 4, and 1. Since it is 4, multiply the last number by $\frac{4}{4}$, to get the same denominator.

$$\begin{aligned}
\frac{3}{4}x+\frac{7}{4}&=\left(\frac{4}{4}\right)-2 \\
\frac{3}{4}x+\frac{7}{4}&=\frac{-8}{4} && \text{(Simplify)}
\end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned} 3x+7&=-8 \\ 3x+7-7&=-8-7 && \text{(Subtract 7 from both sides of the equals sign to isolate the variable)} \\ 3x&=-15 && \text{(Simplify)} \\ \frac{3x}{3}&=\frac{-15}{3} && \text{(Divide by 3 to solve for the variable)} \\ x&=-5 \end{aligned}$$

Therefore $x=-5$.

$$\begin{aligned} &\text{Check:} \\ \frac{1}{4}(3x+7)&=-2 \\ \frac{1}{4}(3(-5)+7)&=-2 \\ \frac{1}{4}(-15+7)&=-2 \\ \frac{1}{4}(-8)&=-2 \\ \frac{-8}{4}&=-2 \\ -2&=-2 \end{aligned}$$

Example C

Solve: $\frac{1}{3}(x-2)=-\frac{2}{3}(2x+4)$.

Solution:

$$\begin{aligned} \frac{1}{3}(x-2)&=-\frac{2}{3}(2x+4) \\ \frac{1}{3}x-\frac{2}{3}&=-\frac{4}{3}x-\frac{8}{3} && \text{(Apply the distributive property to remove the parentheses)} \end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned} x-2&=-4x-8 \\ x+4x-2&=-4x+4x-8 && \text{(Add $4x$ to both sides of the equals sign to combine variables)} \\ 5x-2&=-8 && \text{(Simplify)} \\ 5x-2+2&=-8+2 && \text{(Add 2 to both sides of the equation to isolate the variable)} \\ 5x&=-6 && \text{(Simplify)} \\ \frac{5x}{5}&=\frac{-6}{5} && \text{(Divide both sides by 5 to solve for the variable)} \\ x&=\frac{-6}{5} && \text{(Simplify)} \end{aligned}$$

Therefore $x = \frac{-6}{5}$.

$$\begin{aligned}
&\text{Check:}\\
\frac{1}{3}(x-2) &= -\frac{2}{3}(2x+4)\\
\frac{1}{3}\left(\left(\frac{-6}{5}\right)-2\right) &= -\frac{2}{3}\left(2\left(\frac{-6}{5}\right)+4\right)\\
0.33(-1.2-2) &= -0.67(2(-1.2)+4)\\
0.33(-3.2) &= -0.67(1.6)\\
-1.1 &= -1.1
\end{aligned}$$

Concept Problem Revisited

Pens are \$9 per dozen and pencils are \$6 per dozen. Janet needs to buy a half dozen of each for school. How much is the total cost of her purchase?

First you should write down what you know:

Let $x =$ total cost

Cost of pens: \$9/dozen

Cost of pencils: \$6/dozen

Janet needs one half dozen of each.

The total cost would therefore be:

$$\begin{aligned}
\frac{1}{2}(\$9+\$6) &= x\\
\frac{\$9}{2}+\frac{\$6}{2} &= x\\
\$4.50+\$3.00 &= x\\
\$7.50 &= x
\end{aligned}$$

Therefore Janet would need \$7.50 to buy these supplies.

Vocabulary

Distributive Property

The ***distributive property*** is a mathematical way of grouping terms. It states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $\frac{2}{3}(x+5)$, the distributive property states that the product of a number $(\frac{2}{3})$ and a sum $(x+5)$ is equal to the sum of the individual products of the number $(\frac{2}{3})$ and the addends (x and 5).

Guided Practice

1. Solve for x: $\frac{1}{2}(5x+3)=1$.

2. Solve for x: $\frac{2}{3}(9x-6)=2$.

3. Solve for x: $\frac{2}{3}(3x+9) = \frac{1}{4}(2x+5)$.

Answers:

1.

$$\begin{aligned}\frac{1}{2}(5x+3) &= 1\\ \frac{5}{2}x+\frac{3}{2} &= 1 && \text{(Apply the distributive property to remove the parentheses)}\end{aligned}$$

Find the LCD for 2, 2, and 1. Since it is 2, multiply the last number by $\frac{2}{2}$, to get the same denominator.

$$\begin{aligned}\frac{5}{2}x+\frac{3}{2} &= 1\left(\frac{2}{2}\right)\\ \frac{5}{2}x+\frac{3}{2} &= \frac{2}{2} && \text{(Simplify)}\end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned}5x+3 &= 2\\ 5x+3-3 &= 2-3 && \text{(Subtract 3 from both sides of the equals sign to isolate the variable)}\\ 5x &= -1 && \text{(Simplify)}\\ \frac{5x}{5} &= \frac{-1}{5} && \text{(Divide both sides by the 5 to solve for the variable)}\\ x &= \frac{-1}{5} && \text{(Simplify)}\end{aligned}$$

Therefore $x = \frac{-1}{5}$.

2.

$$\begin{aligned}\frac{2}{3}(9x-6) &= 2\\ \frac{18}{3}x-\frac{12}{3} &= 2 && \text{(Apply the distributive property to remove the parentheses)}\end{aligned}$$

Find the LCD for 3, 3, and 1. Since it is 3, multiply the last number by $\frac{3}{3}$, to get the same denominator.

$$\begin{aligned}\frac{18}{3}x-\frac{12}{3} &= 2\left(\frac{3}{3}\right)\\ \frac{18}{3}x-\frac{12}{3} &= \frac{6}{3} && \text{(Simplify)}\end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned}
18x-12&=6 \\
18x-12+12&=6+12 && \text{(Add 12 to both sides of the equals sign to isolate the variable)}\\
18x&=18 && \text{(Simplify)}\\
\frac{18x}{18}&=\frac{18}{18} && \text{(Divide both sides by the 18 to solve for the variable)}\\
x&=1 && \text{(Simplify)}
\end{aligned}$$

Therefore $x = 1$.

3.

$$\begin{aligned}
\frac{2}{3}(3x+9)&=\frac{1}{4}(2x+5) \\
\frac{6}{3}x+\frac{18}{3}&=\frac{2}{4}x+\frac{5}{4} && \text{(Apply the distributive property to remove the parentheses)}
\end{aligned}$$

Find the LCD for 3, 3, and 4, 4. Since it is 12, multiply the first two fractions by $\frac{4}{4}$ and the second two fractions by $\frac{3}{3}$, to get the same denominator.

$$\begin{aligned}
\left(\frac{4}{4}\right)\frac{6}{3}x+\left(\frac{4}{4}\right)\frac{18}{3}&=\left(\frac{3}{3}\right)\frac{2}{4}x+\left(\frac{3}{3}\right)\frac{5}{4} \\
\frac{24}{12}x+\frac{72}{12}&=\frac{6}{12}x+\frac{15}{12} && \text{(Simplify)}
\end{aligned}$$

Since all of the denominators are the same, the equation becomes:

$$\begin{aligned}
24x+72&=6x+15 \\
24x+72-72&=6x+15-72 && \text{(Subtract 72 from both sides of the equals sign to isolate the variable)}\\
24x&=6x-57 && \text{(Simplify)}\\
24x-6x&=6x-6x-57 && \text{(Subtract } 6x \text{ from both sides of the equals sign to get variables on same side)}\\
18x&=-57 && \text{(Simplify)}\\
\frac{18x}{18}&=\frac{-57}{18} && \text{(Divide both sides by 18 to solve for the variable)}\\
x&=\frac{-57}{18} && \text{(Simplify)}
\end{aligned}$$

Practice

Solve for the variable in each of the following equations.

1. $\frac{1}{2}(x+5)=6$
2. $\frac{1}{4}(g+2)=8$
3. $0.4(b+2)=2$
4. $0.5(r-12)=4$
5. $\frac{1}{4}(x-16)=7$

6. $26.5-k=0.5(50-k)$
7. $2(1.5c+4)=-1$
8. $-\frac{1}{2}(3x-5)=7$
9. $0.35+0.10(m-1)=5.45$
10. $\frac{1}{4}+\frac{2}{3}(t+1)=\frac{1}{2}$

11. $\frac{1}{2}x-3(x+4)=\frac{2}{3}$
12. $-\frac{5}{8}x+x=\frac{1}{8}$
13. $0.4(12-d)=18$
14. $0.25(x+3)=0.4(x-5)$
15. $\frac{2}{3}(t-2)=\frac{3}{4}(t+2)$

2.7 Mathematical Symbols to Represent Words

Here you are going to explore mathematical symbols and their word translations.

Rob is describing his weight training to his friend James. He said that when he started training he weighed 185 pounds. He gained 8 pounds in the first month of training. How much did he weigh at the end of the first month of training?

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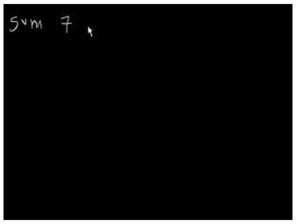
Khan Academy Problem Solving Word Problems 2



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Guidance

Knowing how to translate key words from English into mathematical symbols is important in problem solving. The first step in any problem solving situation in mathematics is always to read the problem. Translating the words into mathematical symbols is next.

Words such as gain, more, sum, total, increase, plus all mean to add. Words such as difference between, minus, decrease, less, fewer, and loss all mean to subtract. Words such as the product of, double $(2x)$, twice $(2x)$, triple $(3x)$, a fraction of, a percent of, or times all mean to multiply. And finally, words such as the quotient of, divided equally, and per mean to divide.

Experience and practice with problem solving will help better acquaint you with the key words that translate into these operations.

Example A

What is the sum of five and seventeen?

Solution: Break apart the sentence. It is often helpful to underline the words before and after the word AND. Also, it is helpful to circle the mathematical symbol.

$$\begin{array}{cccccc} \text{What is the} & \boxed{\text{sum}} & \text{of} & \underline{\text{five}} & \text{and} & \underline{\text{seventeen}}? \\ & \uparrow & & \uparrow & & \uparrow \\ & + & & 5 & & 17 \end{array}$$

Then translate the symbols together into a mathematical equation and solve it.

$5+17=22$

Example B

Thomas had twenty-four dollars and after shopping his money decreased by four dollars.

Solution: Break apart the sentence. Underline the numerical words and circle the mathematical symbol.

$$\text{Thomas had } \underset{\substack{\uparrow \\ 24}}{\underline{\text{twenty-four}}} \text{ dollars and after shopping his money } \underset{\substack{\uparrow \\ -}}{\boxed{\text{decreased}}} \text{ by } \underset{\substack{\uparrow \\ 4}}{\underline{\text{four}}} \text{ dollars.}$$

Then translate the symbols together into a mathematical equation and solve it.

$24-4=20$

Therefore Thomas had \$20.00 left after shopping.

Example C

Nick, Chris, and Jack are sharing a bag of jelly beans. There are 30 jelly beans for the three boys to share equally. How many would each get?

Solution: Again, break apart the sentence. Underline the numerical words and circle the mathematical symbol.

$$\text{There are } \underset{\substack{\uparrow \\ 30}}{\underline{\text{30}}} \text{ jelly beans for the } \underset{\substack{\uparrow \\ 3}}{\underline{\text{three}}} \text{ boys to } \underset{\substack{\uparrow \\ \div}}{\boxed{\text{share equally.}}} \text{ How many would each get?}$$

$30 \div 3 = 10$

Therefore each boy would get 10 jelly beans.

Concept Problem Revisited

Rob is describing his weight training to his friend James. He said that when he started training he weighed 185 pounds. He gained 8 pounds in the first month of training. How much did he weigh at the end of the first month of training?

The word gain is the same as saying add.

Therefore Rob weighs $185+8=193$ *pounds*.

Vocabulary

Addition

Words such as gain, more, sum, total, increase, plus all mean to use ***addition*** or to add.

Subtraction

Words such as the difference between, minus, decrease, less, fewer, loss all mean to use ***subtraction*** or to subtract.

Multiplication

Words such as the product of, double $(2x)$, twice $(2x)$, triple $(3x)$, fraction of, percent of, times all mean to use ***multiplication*** or to multiply.

Division

Words such as the quotient of, divided equally, and per all mean to use ***division*** or to divide.

Guided Practice

1. What is twelve increased by eighteen?

2. Joanne and Jillian were each going to share their babysitting money for the week. They made \$45.00 in total. How much does each girl receive?

3. The number five is increased by seven. Three-fourths of this number is then decreased from twenty. What is the result?

Answers:

1. Break apart the sentence. Underline the numerical words and circle the mathematical symbol.

$$\begin{array}{cccc} \text{What is} & \underline{\text{twelve}} & \boxed{\text{increased}} & \text{by } \underline{\text{eighteen}}? \\ & \uparrow & \uparrow & \uparrow \\ & 12 & + & 18 \end{array}$$

$12 + 18 = 30$

2. Break apart the sentence. Underline the numerical words and circle the mathematical symbol.

$$\begin{array}{cccc} \underline{\text{Joanne and Jillian}} & \text{were each going to } \boxed{\text{share}} & \text{their babysitting money for the week. They made} & \underline{\$45.00} \text{ in total?} \\ \uparrow & \uparrow & & \uparrow \\ 2 & \div & & 45 \end{array}$$

$45 \div 2 = 22.5$

Therefore each girl would get \$22.50.

3. Break apart the sentence. Underline the numerical words and circle the mathematical symbol.

$$\begin{array}{cccccccc} \text{The number} & \underline{\text{five}} \text{ is} & \boxed{\text{increased}} & \text{by } \underline{\text{seven}}. & \boxed{\text{Three-fourths}} & \text{of this } \underline{\text{number}} \text{ is then} & \boxed{\text{decreased}} & \text{from } \underline{\text{twenty}}. \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & 5 & + & 7 & \times & 5+7 & - & 20 \end{array}$$

Step 1: $5 + 7 = 12$

Step 2: $20 - \frac{3}{4}(12) = 11$

Practice

1. Six less than fifty-three is what number?
2. Twice the sum of eight and nine is what number?
3. Twenty-five is diminished by four times five. What is the result?
4. The product of five times four plus seven is what number?
5. The sum of forty-four and fifty-two then divided by twelve results in what number?
6. Four less than twice 15 is what number?
7. The sum of 12 and the product of 2 and 3 is what number?
8. The number 12 is increased by 4. Three-fourths of this number is then decreased from 20. What is the result?
9. What is 17 decreased by the product of 2 and 4?
10. Mike had $100. His money increased by $25 after his tutoring job. How much money does he have now?
11. Kathryn had $20 saved and doubled her money after working on Saturday. How much money does she have now?
12. Jen and Olivia together sold 300 boxes of cookies. Each box of cookies cost $4. If the money is divided equally, how much money did each girl make?
13. The difference between 212 and the product of 15 and 18 is what number?
14. Lindsey got 5 donations on Saturday. Over the course of the next week she got 12 more donations. How many donations did she get total?
15. The quotient of 12 and the product of 2 and 3 is increased by 15. What is the result?

2.8 Algebraic Equations to Represent Words

Here you will learn to translate words into algebraic equations.

The sum of two consecutive even integers is 34. What are the integers?

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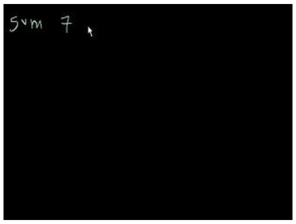
Khan Academy Problem Solving Word Problems 2



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Guidance

To translate a problem from words into an equation, look for key words to indicate the operation used in the problem.

Once the equation is known, to solve the problem you use the same rules as when solving equations with one variable. Isolate the variable and then solve for it making sure that whatever you do to one side of the equals sign you do to the other side. Drawing a diagram is also helpful in solving some word problems.

Example A

Two consecutive integers have a sum of 173. What are those numbers?

Solution: Let $x =$ integer 1

Then $x+1 =$ integer 2 (Because they are consecutive, they must be separated by only one number. For example: 1, 2, 3, 4,... all are consecutive.)

Translate the sentence into an equation and solve:

$$\begin{aligned}
x+(x+1) &= 173 \\
x+x+1 &= 173 && (\text{Remove the parentheses}) \\
2x+1 &= 173 && (\text{Combine like terms}) \\
2x+1-1 &= 173-1 && (\text{Subtract 1 from both sides to isolate the variable}) \\
2x &= 172 && (\text{Simplify}) \\
\frac{2x}{2} &= \frac{172}{2} && (\text{Divide both sides by 2 to solve for the variable}) \\
x &= 86 && (\text{Simplify})
\end{aligned}$$

Therefore the first integer is 86 and the second integer is $(86+1) = 87$. Check: $86+87 = 173$.

Example B

When a number is subtracted from 35, the result is 11. What is the number?

Solution: Let $x =$ the number

Translate the sentence into an equation and solve:

$$\begin{aligned} 35 - x &= 11 \\ 35 {\color{red}-35} - x &= 11 {\color{red}-35} && \text{(Subtract 35 from both sides to isolate the variable)} \\ -x &= -24 && \text{(Simplify)} \\ \frac{-x}{\color{red}-1} &= \frac{-24}{\color{red}-1} && \text{(Divide both sides by } -1 \text{ to solve for the variable)} \\ x &= 24 && \text{(Simplify)} \end{aligned}$$

Therefore the number is 24.

Example C

When one third of a number is subtracted from one half of a number, the result is 14. What is the number?

Solution: Let $x =$ number

Translate the sentence into an equation and solve:

$\frac{1}{2}x - \frac{1}{3}x = 14$

You need to get a common denominator in this problem in order to solve it. For this problem, the denominators are 2, 3, and 1. The LCD is 6. Therefore multiply the first fraction by $\frac{3}{3}$, the second fraction by $\frac{2}{2}$, and the third number by $\frac{6}{6}$.

$$\begin{aligned} \left(\frac{\color{red}3}{\color{red}3}\right)\frac{1}{2}x - \left(\frac{\color{red}2}{\color{red}2}\right)\frac{1}{3}x &= \left(\frac{\color{red}6}{\color{red}6}\right)14 \\ \frac{3}{6}x - \frac{2}{6}x &= \frac{84}{6} && \text{(Simplify)} \end{aligned}$$

Now that the denominator is the same, the equation can be simplified to be:

$$\begin{aligned} 3x - 2x &= 84 \\ x &= 84 && \text{(Combine like terms)} \end{aligned}$$

Therefore the number is 84.

Concept Problem Revisited

The sum of two consecutive even integers is 34. What are the integers?

Let $x =$ integer 1

Then $x + 2 =$ integer 2 (Because they are even, they must be 2 numbers apart. For example: 2, 4, 6, 8,... are all consecutive even numbers.)

Translate the sentence into an equation and solve:

$$\begin{aligned}
x+(x+2) &= 34 && \\
x+x+2 &= 34 && \text{(Remove the parentheses)}\\
2x+2 &= 34 && \text{(Combine like terms)}\\
2x+2-2 &= 34-2 && \text{(Subtract 2 from both sides to isolate the variable)}\\
2x &= 32 && \text{(Simplify)}\\
\frac{2x}{2} &= \frac{32}{2} && \text{(Divide both sides by 2 to solve for the variable)}\\
x &= 16 && \text{(Simplify)}
\end{aligned}$$

Therefore the first integer is 16 and the second integer is $(16+2)=18$. Note that $16+18$ is indeed 34.

Vocabulary

Algebraic Equation

An algebraic equation contains numbers, variables, operations, and an equals sign.

Consecutive

The term consecutive means in a row. Therefore an example of consecutive numbers is 1, 2, and 3. An example of consecutive even numbers would be 2, 4, and 6. An example of consecutive odd numbers would be 1, 3, and 5.

Guided Practice

1. What is a number that when doubled would equal sixty?

2. The sum of two consecutive odd numbers is 176. What are these numbers?

3. The perimeter of a square frame is 48 in. What are the lengths of each side?

Answers:

1. The number is 30.

$$\begin{aligned}
2x &= 60 && \\
\frac{2x}{2} &= \frac{60}{2} && \text{(Divide by 2 to solve for the variable)}\\
x &= 30 && \text{(Simplify)}
\end{aligned}$$

2. The first number is 87 and the second number is $(87+2)=89$.

$$
\begin{aligned}
x+(x+2) &= 176 \\
x+x+2 &= 176 && \text{(Remove parentheses)} \\
2x+2 &= 176 && \text{(Combine like terms)} \\
2x+2-2 &= 176-2 && \text{(Subtract 2 from both sides of the equals sign to isolate the variable)} \\
2x &= 174 && \text{(Simplify)} \\
\frac{2x}{2} &= \frac{174}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &= 87
\end{aligned}
$$

3. The side length is 12 inches.

$$
\begin{aligned}
s+s+s+s &= 48 && \text{(Write initial equation with four sides adding to the perimeter)} \\
4s &= 48 && \text{(Simplify)} \\
\frac{4s}{4} &= \frac{48}{4} && \text{(Divide by 4 to solve for the variable)} \\
s &= 12
\end{aligned}
$$

Practice

1. The sum of two consecutive numbers is 159. What are these numbers?
2. The sum of three consecutive numbers is 33. What are these numbers?
3. A new computer is on sale for 30% off. If the sale price is $500, what was the original price?
4. Jack and his three friends are sharing an apartment for the next year while at university (8 months). The rent costs $1200 per month. How much does Jack have to pay if they split the cost evenly?
5. You are designing a triangular garden with an area of 168 square feet and a base length of 16 feet. What would be the height of the triangular garden shape?
6. If four times a number is added to six, the result is 50. What is that number?
7. This week, Emma earned ten more than half the number of dollars she earned last week babysitting. If this week, she earned 100 dollars, how much did she earn last week?
8. Three is twenty-one divided by the sum of a number plus five. What is the number?
9. Five less than three times a number is forty-six. What is the number?
10. Hannah had $237 in her bank account at the start of the summer. She worked for four weeks and now she has $1685 in the bank. How much did Hannah make each week in her summer job?
11. The formula to estimate the length of the Earth's day in the future is found to be twenty–four hours added to the number of million years divided by two hundred and fifty. In five hundred million years, how long will the Earth's day be?
12. Three times a number less six is one hundred twenty-six. What is the number?
13. Sixty dollars was two-thirds the total money spent by Jack and Thomas at the store. How much did they spend total?
14. Ethan mowed lawns for five weekends over the summer. He worked ten hours each weekend and each lawn takes an average of two and one-half hours. How many lawns did Ethan mow?
15. The area of a rectangular pool is found to be 280 square feet. If the base length of the pool is 20 feet, what is the width of the pool?
16. A cell phone company charges a base rate of $10 per month plus 5¢ per minute for any long distance calls. Sandra gets her cell phone bill for $21.20. How many long distance minutes did she use?

2.9 One Variable Inequalities

Here you are going to learn about one variable linear inequalities.

Janet holds up a card that reads $2x+6=16$. Donna holds up a card that reads $2x+6>16$. Andrew says they are not the same but Donna argues with him. Show, using an example, that Andrew is correct.

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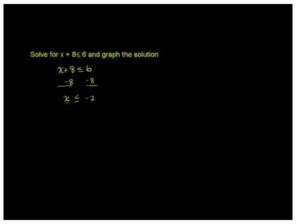
Khan Academy One Step Inequalities



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Guidance

One variable linear inequalities have a different form than one variable linear equations. Linear equations have the general form of $ax+b=c$, where $a\neq 0$. Linear inequalities can have one of four forms: $ax+b>c, ax+b<c, ax+b\geq c$, or $ax+b\leq c$. You should notice the difference is that instead of an equals sign, there is an inequality symbol.

When you solve for a linear inequality, you follow the same rules as you would for a linear equation; however, you must remember one big rule: ***If you divide or multiply by a negative number while solving, you must reverse the sign of the inequality.***

Example A

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 8 in for p?

TABLE 2.1:

Equation	Inequality	Is the inequality still true?
$2p+4=20$	$2p+4<20$	?
$2p+4-4=20-4$		
$2p=16$		
$\frac{2p}{2}=\frac{16}{2}$		
$p=8$		

Solution:

TABLE 2.2:

Equation	**Inequality**	**Is the inequality still true?**
$2p+4=20$	$2p+4<20$	no
$2p+4-4=20-4$	$2p+4-4<20-4$	
$2p=16$	$2p<16$	
$\frac{2p}{2}=\frac{16}{2}$	$\frac{2p}{2}<\frac{16}{2}$	
$p=8$	$p<8$	

No, there is no difference in the steps used to find the two solutions.

Example B

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 6 in for x?

TABLE 2.3:

Equation	**Inequality**	**Is the inequality still true?**
$3x+5=23$	$3x+5\geq 23$	?
$3x+5-5=23-5$		
$3x=18$		
$\frac{3x}{3}=\frac{18}{3}$		
$x=6$		

Solution:

TABLE 2.4:

Equation	**Inequality**	**Is the inequality still true?**
$3x+5=23$	$3x+5\geq 23$	yes
$3x+5-5=23-5$	$3x+5-5\geq 23-5$	
$3x=18$	$3x\geq 18$	
$\frac{3x}{3}=\frac{18}{3}$	$\frac{3x}{3}\geq\frac{18}{3}$	
$x=6$	$x\geq 6$	

No, there is no difference in the steps used to find the two solutions.

Example C

In the following table, a linear equation has been solved. Solve for the inequality using the similar steps. Are the steps the same? Is the inequality still true if you substitute 3 in for c?

TABLE 2.5:

Equation	**Inequality**	**Is the inequality still true?**
$5-3c=-4$	$5-3c\leq -4$	?
$5-5-3c=-4-5$		
$-3c=-9$		
$\frac{-3c}{-3}=\frac{-9}{-3}$		
$c=3$		

Solution:

TABLE 2.6:

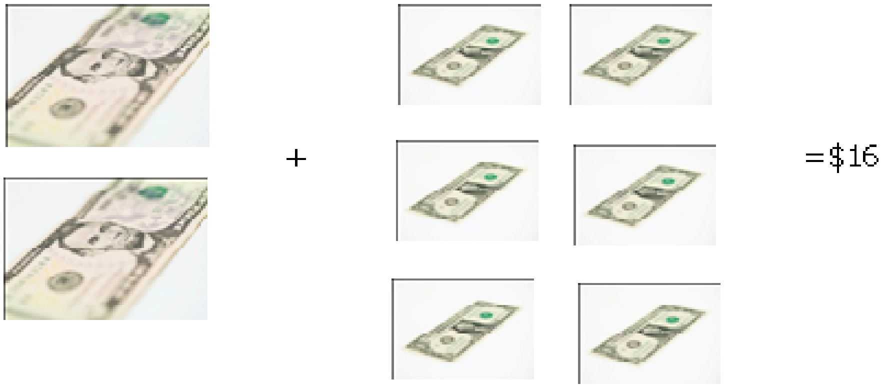
Equation	Inequality	Is the inequality still true?
$5-3c=-4$	$5-3c \leq -4$	yes
$5-5-3c=-4-5$	$5-5-3c \leq -4-5$	
$-3c=-9$	$-3c \leq -9$	
$\frac{-3c}{-3}=\frac{-9}{-3}$	$\frac{-3c}{-3} \geq \frac{-9}{-3}$	
$c=3$	$c \geq 3$	

Yes, there was a difference in the steps used for the two solutions. When dividing by –3, the sign of the inequality was reversed.

Concept Problem Revisited

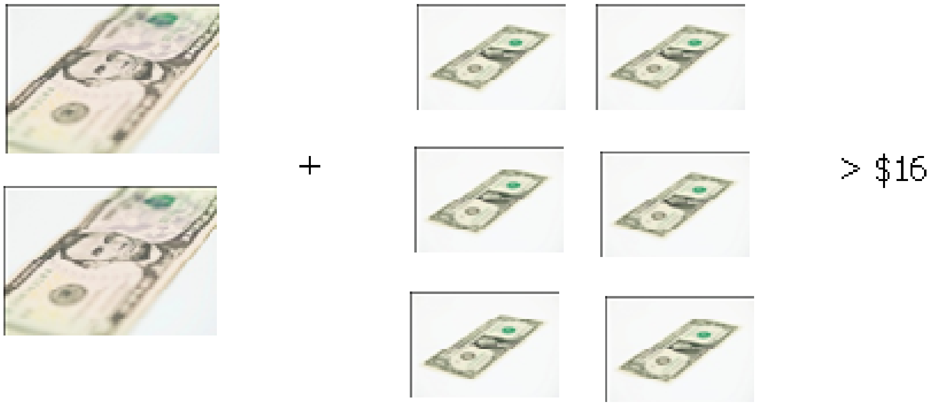
Janet holds up a card that reads $2x+6=16$. Donna holds up a card that reads $2x+6>16$. Andrew says they are not the same but Donna argues with him. Show, using an example, that Andrew is correct.

Andrew could use a real world example. For example, say Andrew held out two \$5 bills and six \$1 bills. Andrew holds Janet's card and says, "Is this true?"



The answer would be yes.

Now let's try it with Donna's inequality.



This amount of money is not greater than $16; it is just equal to $16. The two mathematical statements are not the same.

Vocabulary

Linear Inequality

Linear inequalities can have one of four forms: $ax+b > c, ax+b < c, ax+b \geq c$, or $ax+b \leq c$.

Guided Practice

1. In the following table, a linear equation has been solved. Solve for the inequality using similar steps, but remember if you multiply or divide by a negative number you should reverse the inequality sign. Is the inequality still true if you substitute –10 in for a?

Table 2.7:

Equation	Inequality	Is the inequality still true?
$4.6a+8.2 = 2.4a-13.8$	$4.6a+8.2 > 2.4a-13.8$	?
$4.6a+8.2+13.8 = 2.4a-13.8+13.8$		
$4.6a+22 = 2.4a$		
$4.6a-4.6a+22 = 2.4a-4.6a$		
$22 = -2.2a$		
$\frac{22}{-2.2} = \frac{-2.2a}{-2.2}$		
$a = -10$		

2. In the following table, a linear equation has been solved. Solve for the inequality using similar steps, but remember if you multiply or divide by a negative number you should reverse the inequality sign. Is the inequality still true if you substitute 6 in for w?

Table 2.8:

Equation	Inequality	Is the inequality still true?
$3(w+4) = 2(3+2w)$	$3(w+4) < 2(3+2w)$	?
$3w+12 = 6+4w$		
$3w+12-12 = 6-12+4w$		

TABLE 2.8: (continued)

Equation	Inequality	Is the inequality still true?
$3w = -6 + 4w$		
$3w - 4w = -6 + 4w - 4w$		
$-w = -6$		
$\frac{-w}{-1} = \frac{-6}{-1}$		
$w = 6$		

3. In the following table, a linear equation has been solved. Solve for the inequality using similar steps, but remember if you multiply or divide by a negative number you should reverse the inequality sign. Is the inequality still true if you substitute –10 in for h?

TABLE 2.9:

Equation	Inequality	Is the inequality still true?
$\frac{1}{3}(2-h) = 4$	$\frac{1}{3}(2-h) \geq 4$	?
$\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$		
$\frac{1}{3}(2-h) = \frac{12}{3}$		
$2 - h = 12$		
$2 - 2 - h = 12 - 2$		
$-h = 10$		
$\frac{-h}{-1} = \frac{10}{-1}$		
$h = -10$		

Answers:

1.

TABLE 2.10:

Equation	Inequality	Is the inequality still true?
$4.6a + 8.2 = 2.4a - 13.8$	$4.6a + 8.2 > 2.4a - 13.8$	no
$4.6a + 8.2 + 13.8 = 2.4a - 13.8 + 13.8$	$4.6a + 8.2 + 13.8 > 2.4a - 13.8 + 13.8$	
$4.6a + 22 = 2.4a$	$4.6a + 22 > 2.4a$	
$4.6a - 4.6a + 22 = 2.4a - 4.6a$	$4.6a - 4.6a + 22 > 2.4a - 4.6a$	
$22 = -2.2a$	$22 > -2.2a$	
$\frac{22}{-2.2} = \frac{-2.2a}{-2.2}$	$\frac{22}{-2.2} < \frac{-2.2a}{-2.2}$	
$a = -10$	$a > -10$	

2.

TABLE 2.11:

Equation	Inequality	Is the inequality still true?
$3(w+4) = 2(3+2w)$	$3(w+4) < 2(3+2w)$	no
$3w + 12 = 6 + 4w$	$3w + 12 = 6 < 4w$	
$3w + 12 - 12 = 6 - 12 + 4w$	$3w + 12 - 12 < 6 - 12 + 4w$	
$3w = -6 + 4w$	$3w < -6 + 4w$	
$3w - 4w = -6 + 4w - 4w$	$3w - 4w < -6 + 4w - 4w$	
$-w = -6$	$-w < -6$	
$\frac{-w}{-1} = \frac{-6}{-1}$	$\frac{-w}{-1} > \frac{-6}{-1}$	

TABLE 2.11: (continued)

Equation	Inequality	Is the inequality still true?
$w = 6$	$w > 6$	

3.

TABLE 2.12:

Equation	Inequality	Is the inequality still true?
$\frac{1}{3}(2-h) = 4$	$\frac{1}{3}(2-h) \geq 4$	yes
$\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$	$\frac{1}{3}(2-h) = 4\left(\frac{3}{3}\right)$	
$\frac{1}{3}(2-h) = \frac{12}{3}$	$\frac{1}{3}(2-h) \geq \frac{12}{3}$	
$2-h = 12$	$2-h \geq 12$	
$2-2-h = 12-2$	$2-2-h \geq 12-2$	
$-h = 10$	$-h \geq 10$	
$\frac{-h}{-1} = \frac{10}{-1}$	$\frac{-h}{-1} \leq \frac{10}{-1}$	
$h = -10$	$h \leq -10$	

Practice

In the following table, a linear equation has been solved.

TABLE 2.13:

Equation	Inequality	Is the inequality still true?
$5.2+x+3.6 = 4.3$	$5.2+x+3.6 \geq 4.3$	?
$8.8+x = 4.3$		
$8.8-8.8+x = 4.3-8.8$		
$x = -4.5$		

1. Solve for the inequality using similar steps.
2. Were all of the steps the same? Why or why not?
3. Is the inequality still true if you substitute –4.5 in for x?

In the following table, a linear equation has been solved.

TABLE 2.14:

Equation	Inequality	Is the inequality still true?
$\frac{n}{4} - 5 = -3$	$\frac{n}{4} - 5 < -3$	?
$\frac{n}{4} - 5\left(\frac{4}{4}\right) = -3\left(\frac{4}{4}\right)$		
$\frac{n}{4} - \frac{20}{4} = \frac{-12}{4}$		
$n - 20 = -12$		
$n-20+20 = -12+20$		
$n = 8$		

4. Solve for the inequality using similar steps.
5. Were all of the steps the same? Why or why not?
6. Is the inequality still true if you substitute 8 in for n?

In the following table, a linear equation has been solved.

TABLE 2.15:

Equation	Inequality	Is the inequality still true?
$1-z=5(3+2z)+8$	$1-z<5(3+2z)+8$	?
$1-z=15+10z+8$		
$1-z=23+10z$		
$1-z+z=23+10z+z$		
$1=23+11z$		
$1-23=23-23+11z$		
$-22=11z$		
$\frac{-22}{11}=\frac{11z}{11}$		
$z=-2$		

7. Solve for the inequality using similar steps.
8. Were all of the steps the same? Why or why not?
9. Is the inequality still true if you substitute –2 in for z?

10. The sum of two numbers is greater than 764. If one of the numbers is 416, what could the other number be?
11. 205 less a number is greater than or equal to 112. What could that number be?
12. Five more than twice a number is less than 20. If the number is a whole number, what could the number be?
13. The product of 7 and a number is greater than 42. If the number is a whole number less than 10, what could the number be?
14. Three less than 5 times a number is less than or equal to 12. If the number is a whole number, what could the number be?
15. Double a number and add 12 and the result will be greater than 20. The number is less than 6. What is the number?

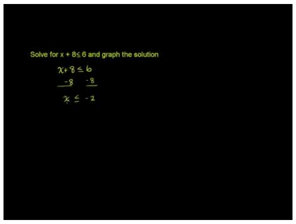
2.10 Algebraic Solutions to One Variable Inequalities

Here you'll learn how to solve one variable inequalities.

The Morgan Silver Dollar is a very valuable American dollar minted between 1978 and 1921. When placed on a scale with twenty 1-gram masses, the scale tips toward the Morgan Dollar. Draw a picture to represent this scenario and then write the inequality.

Watch This

Khan Academy One Step Inequalities



Solve for x + 8≤ 6 and graph the solution

$x+8 \leq 6$

$-8 \quad -8$

$x \leq -2$

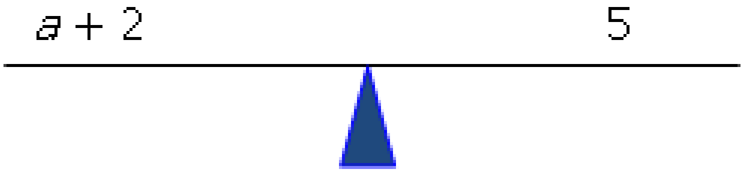
MEDIA

Click image to the left for more content.

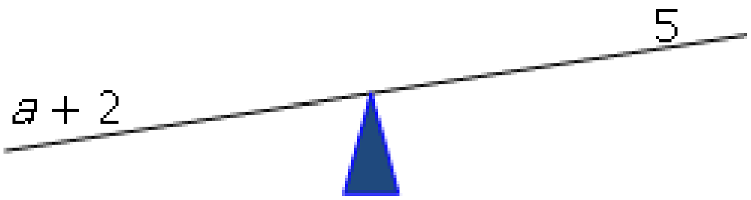
Guidance

Linear equations are of the form $ax+b=0$, where $a \neq 0$. With linear equations there is always an equals sign. Linear inequalities are mathematical statements relating expressions by using one or more inequality symbols $<, >, \leq$, or $\geq$. In other words, the left side no longer equals the right side, it is less than, greater than, less than or equal to, or greater than or equal to.

Recall that one method for solving an equation is to use the balance method. To solve for $a+2=5$, you would draw the following balance:



If you were to use the balance method to solve the linear inequality version, it would look more like this: $a+2>5$



Think about it this way. It is like having someone heavier on the $(a+2)$ side of the balance and someone light on the (5) side of the balance. The $(a+2)$ person has a weight greater than (>) the (5) person and therefore the balance moves down to the ground.

The rules for solving inequalities are basically the same as you used for solving linear equations. If you have parentheses, remove these by using the distributive property. Then you must isolate the variable by moving constants to one side and variables to the other side of the inequality sign. You also have to remember that whatever you do to one side of the inequality, you must do to the other. The same was true when you were working with linear equations. One additional rule is to reverse the sign of the inequality if you are multiplying or dividing both sides by a negative number.

It is important for you to remember what the symbols mean. Always remember that the mouth of the sign opens toward the larger number. So $8 > 5$, the mouth of the >sign opens toward the 8 so 8 is larger than 5. You know that's true. $6b-5 < 300$, the mouth opens toward the 300, so 300 is larger than $6b-5$.

Example A

Solve: $15 < 4+3x$

Solution: Remember that whatever you do to one side of the inequality sign, you do to the other.

$$\begin{aligned}
15-4 &< 4-4+3x && \text{Subtract 4 from both sides to isolate the variable}\\
11 &< 3x && \text{Simplify}\\
\frac{11}{3} &< \frac{3x}{3} && \text{Divide by 3 to solve for the variable}\\
x &> \frac{11}{3} && \text{Simplify}
\end{aligned}$$

Do a quick check to see if this is true. $\frac{11}{3}$ is approximately 3.67. Try substituting 0, 3, and 4 into the equation.

$15 < 4+3x$	$15 < 4+3x$	$15 < 4+3x$
$15 < 4+3(0)$	$15 < 4+3(3)$	$15 < 4+3(4)$
$15 < 4$ False	$15 < 13$ False	$15 < 16$ True

The only value out of 0, 3, and 4 where $x > \frac{11}{3}$ is 4. If you look at the statements above, it was the only inequality that gave a true statement.

Example B

Solve: $2y+3>7$

Solution: Use the same rules as if you were solving any algebraic expression. Remember that whatever you do to one side of the inequality sign, you do to the other.

$$\begin{aligned} 2y+3-3 &> 7-3 && \text{Subtract 3 from both sides to isolate the variable} \\ 2y &> 4 && \text{Simplify} \\ \frac{2y}{2} &> \frac{4}{2} && \text{Divide by 2 to solve for the variable} \\ y &> 2 && \text{Simplify} \end{aligned}$$

Do a quick check to see if this is true. Try substituting 0, 4, and 8 into the equation.

$$\begin{array}{ccc} 2y+3>7 & 2y+3>7 & 2y+3>7 \\ 2(0)+3>7 & 2(4)+3>7 & 2(8)+3>7 \\ 3>7 \text{ False} & 11>7 \text{ True} & 19>7 \text{ True} \end{array}$$

The values out of 0, 4, and 8 where $y>2$ are 4 and 8. If you look at the statements above, these values when substituted into the inequality gave true statements.

Example C

Solve: $-2c-5<8$

Solution: Again, to solve this inequality, use the same rules as if you were solving any algebraic expression. Remember that whatever you do to one side of the inequality sign, you do to the other.

$$\begin{aligned} -2c-5+5 &< 8+5 && \text{Add 5 to both sides to isolate the variable} \\ -2c &< 13 && \text{Simplify} \\ \frac{-2c}{-2} &< \frac{13}{-2} && \text{Divide by -2 to solve for the variable} \end{aligned}$$

Note: When you divide by a negative number, the inequality sign reverses.

$$c > \frac{-13}{2} \qquad \text{Simplify}$$

Do a quick check to see if this is true. $\frac{-13}{2}$ is equal to –6.5. Try substituting –8, 0, and 2 into the equation.

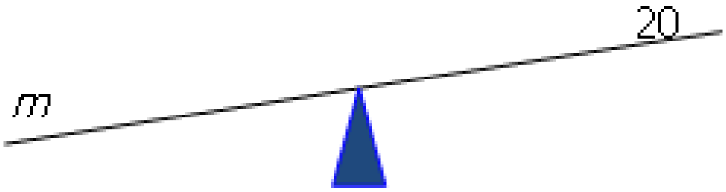
$$\begin{array}{ccc} -2c-5<8 & -2c-5<8 & -2c-5<8 \\ -2(-8)-5<8 & -2(0)-5<8 & -2(2)-5<8 \\ 11<8 \text{ False} & -5<8 \text{ True} & -9<8 \text{ True} \end{array}$$

The values out of –8, 0, and 2 where $c>\frac{-13}{2}$ are 0 and 2. If you look at the statements above, these values when substituted into the inequality gave true statements.

Concept Problem Revisited

The Morgan Silver Dollar is a very valuable American dollar minted between 1978 and 1921. When placed on a scale with twenty 1-gram masses, the scale tips toward the Morgan Dollar. Draw a picture to represent this scenario and then write the inequality.

Let $m =$ Morgan dollar



Since the weight of the Morgan dollar is greater than 20 g, the mouth of the inequality sign would open towards the variable, m. Therefore the inequality equation would be:

$m > 20$

Vocabulary

Linear Inequality

Linear inequalities are mathematical statements relating expressions by using one or more inequality symbols $<, >$ $, \leq$, or $\geq$.

Guided Practice

Solve each inequality.

1. $4t + 3 > 11$

2. $2z + 7 \leq 5z + 28$

3. $9(j - 2) \geq 6(j + 3) - 9$

Answers:

1. $t > 2$. Here are the steps:

$$\begin{aligned}
4t + 3 &> 11 \\
4t + 3 {\color{red}-3} &> 11 {\color{red}-3} && \text{Subtract 3 from both sides to isolate the variable} \\
4t &> 8 && \text{Simplify} \\
\frac{4t}{\color{red}4} &> \frac{8}{\color{red}4} && \text{Divide by 4 to solve for the variable} \\
t &> 2
\end{aligned}$$

2. $z \geq -7$. Here are the steps:

$$\begin{array}{rl}
2z+7 \leq 5z+28 & \\
2z-2z+7 \leq 5z-2z+28 & \text{Subtract } 2z \text{ from both sides to get variables on same side of the inequality sign.} \\
7 \leq 3z+28 & \text{Simplify} \\
7-28 \leq 3z+28-28 & \text{Subtract 28 from both sides to isolate the variable} \\
-21 \leq 3z & \text{Simplify} \\
\frac{3z}{3} \geq \frac{-21}{3} & \text{Divide by 3 to solve for the variable} \\
z \geq -7 &
\end{array}$$

3. $j \geq 9$. Here are the steps:

$$\begin{array}{rl}
9(j-2) \geq 6(j+3)-9 & \\
9j-18 \geq 6j+18-9 & \text{Remove parentheses} \\
9j-18 \geq 6j+9 & \text{Combine like terms on each side of inequality sign} \\
9j-6j-18 \geq 6j-6j+9 & \text{Subtract } 6j \text{ from both sides to get variables on same side of the inequality sign.} \\
3j-18 \geq 9 & \text{Simplify} \\
3j-18+18 \geq 9+18 & \text{Add 18 to both sides to isolate the variable} \\
3j \geq 27 & \text{Simplify} \\
\frac{3j}{3} \geq \frac{27}{3} & \text{Divide by 3 to solve for the variable} \\
j \geq 9 &
\end{array}$$

Practice

Solve for the variable in the following inequalities.

1. $a+8>4$
2. $4c-1>7$
3. $5-3k<6$
4. $3-4t \leq -11$
5. $6 \geq 11-2b$
6. $\frac{e}{5}-3>-1$
7. $\frac{1}{5}(r-3)<-1$
8. $\frac{1}{3}(f+2)<4$
9. $\frac{p+3}{4} \geq -2$
10. $\frac{1}{2}(5-w) \leq -3$
11. $3(2x-5)<2(x-1)+3$
12. $2(y+8)+5(y-1)>6$
13. $2(d-3)<-3(d+3)$
14. $3(g+3) \geq 2(g+1)-2$
15. $2(3s-4)+1 \leq 3(4s+1)$

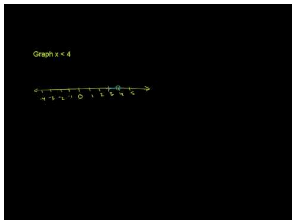
2.11 Graphical Solutions to One Variable Inequalities

Here you'll learn how to represent your solution to an inequality on a number line.

Jack is 3 years older than his brother. How old are they if the sum of their ages is greater than 17? Write an inequality and solve. Represent the solution set on a number line.

Watch This

Khan Academy Inequalities on a Number Line



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Guidance

When you solve a linear inequality you can represent your solution graphically with a number line. Your job is to show which of the numbers on the number line are solutions to the inequality.

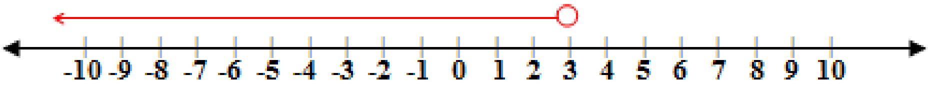
When graphing solutions to inequalities, use an open circle to show that the number is not included as part of the solution and a closed circle to show that the number is included as part of the solution. The line above the number line shows all of the numbers that are possible solutions to the inequality.

Example A

Represent the solution set to the following inequality on a number line: $x+2<5$.

Solution:

$$\begin{aligned} x+2&<5 \\ x+2-2&<5-2 && \text{(Subtract 2 from both sides to isolate the variable)} \\ x&<3 && \text{(Simplify)} \end{aligned}$$

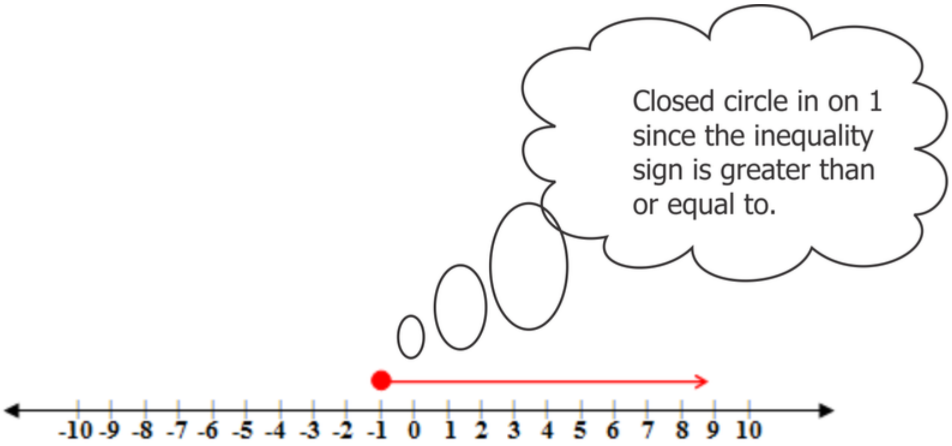


Example B

Represent the solution set to the following inequality on a number line: $2x+6\geq 4$.

Solution:

$$\begin{aligned}
2x+6 &\geq 4 \\
2x+6-6 &\geq 4-6 && \text{(Subtract 6 from both sides to isolate the variable)} \\
2x &\geq -2 && \text{(Simplify)} \\
\frac{2x}{2} &\geq \frac{-2}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &\geq -1
\end{aligned}$$

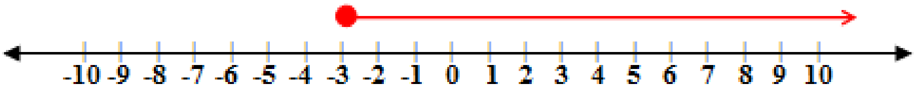


Example C

Represent the solution set to the following inequality on a number line: $-3x+8 \leq 17$.

Solution:

$$\begin{aligned}
-3x+8 &\leq 17 \\
-3x+8-8 &\leq 17-8 && \text{(Subtract 8 from both sides to isolate the variable)} \\
-3x &\leq 9 && \text{(Simplify)} \\
\frac{-3x}{-3} &\geq \frac{9}{-3} && \text{(Divide both sides by -3 to solve for the variable, reverse sign of inequality)} \\
x &\geq -3
\end{aligned}$$



Concept Problem Revisited

Jack is 3 years older than his brother. How old are they if the sum of their ages is greater than 17? Write an inequality and solve. Represent the solution set on a number line.

If first let's write down what you know:

Let $x =$ Jack's brother's age

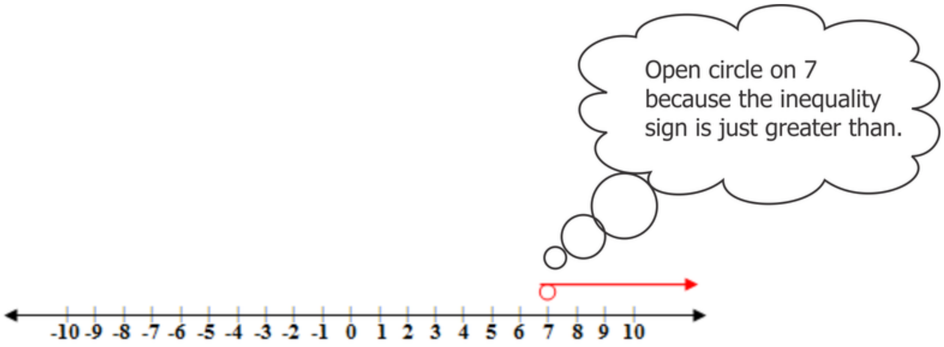
Let $x+3 =$ Jack's age

The equation would therefore be:

$$\begin{aligned}
x+x+3 &> 17 \\
2x+3 &> 17 && \text{(Combine like terms)} \\
2x+3-3 &> 17-3 && \text{(Subtract 3 from both sides to solve for the variable)} \\
2x &> 14 && \text{(Simplify)} \\
\frac{2x}{2} &> \frac{14}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &> 7
\end{aligned}$$

Therefore if Jack's brother is 8 (since $8 > 7$), Jack would be 11. If Jack's brother is 10, Jack would be 13.

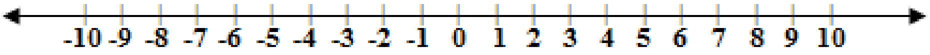
Representing Jack's brother's age on a number line:



Vocabulary

Number Line

A ***number line*** is a line that matches a set of points and a set of numbers one to one.



Guided Practice

1. Represent the solution set to the inequality $3(a-1) < 9$ on a number line.

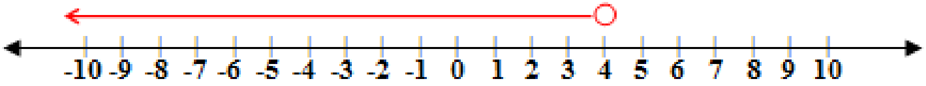
2. Represent the solution set to the inequality $2b+4 \geq 5b+19$ on a number line.

3. Represent the solution set to the inequality $0.6c+2 \geq 5.6$ on a number line.

Answers:

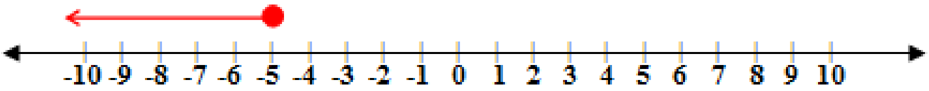
1.

$$\begin{aligned}
3(a-1) &< 9 \\
3a-3 &< 9 && \text{(Remove parentheses)} \\
3a-3+3 &< 9+3 && \text{(Add 3 to both sides to isolate the variable)} \\
3a &< 12 && \text{(Simplify)} \\
\frac{3a}{3} &< \frac{12}{3} && \text{(Divide both sides by 3 to solve for the variable)} \\
a &< 4
\end{aligned}$$



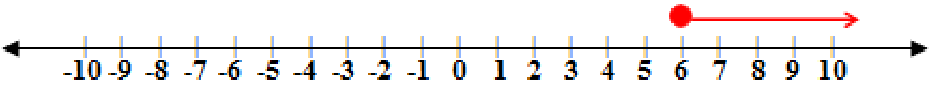
2.

$$\begin{aligned}
2b+4 &\geq 5b+19 \\
2b-5b+4 &\geq 5b-5b+19 && \text{(Subtract } 5b \text{ from both sides to get variables on same side)} \\
-3b+4 &\geq 19 && \text{(Simplify)} \\
-3b+4-4 &\geq 19-4 && \text{(Subtract 4 from both sides to isolate the variable)} \\
-3b &\geq 15 && \text{(Simplify)} \\
\frac{-3b}{-3} &\leq \frac{15}{-3} && \text{(Divide by -3 to solve for the variable, reverse sign of inequality)} \\
b &\leq -5
\end{aligned}$$



3.

$$\begin{aligned}
0.6c+2 &\geq 5.6 \\
0.6c+2-2 &\geq 5.6-2 && \text{(Subtract 2 from both sides to isolate the variable)} \\
0.6c &\geq 3.6 && \text{(Simplify)} \\
\frac{0.6c}{0.6} &\geq \frac{3.6}{0.6} && \text{(Divide both sides by 0.6 to solve for the variable)} \\
c &\geq 6
\end{aligned}$$



Practice

Solve each inequality and represent the solution set on a number line.

1. $-4v > 12$
2. $\frac{-2r}{3} > 4$
3. $4(t-2) \leq 24$
4. $\frac{1}{2}(x+5) > 6$
5. $\frac{1}{4}(g+2) \leq 2$
6. $0.4(b+2) \geq 2$
7. $0.5(r-1) < 4$
8. $\frac{1}{4}(x+16) > 2$
9. $2-k > 5(1-k)$
10. $2(1.5c+4) \leq -1$
11. $-\frac{1}{2}(3x-5) \geq 7$
12. $0.35+0.10(m-1) < 0.45$
13. $\frac{1}{4}+\frac{2}{3}(t+1) > \frac{1}{2}$
14. The prom committee is selling tickets for a fundraiser for the decorations. Each ticket costs \$3.50. What is the least number of tickets the committee needs to sell to make \$1000? Write an inequality and solve.
15. Brenda got 69%, 72%, 81%, and 88% on her last four major tests. How much does she need on her next test to have an average of at least 80%? Write an inequality and solve.

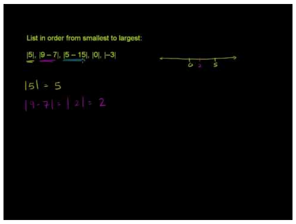
2.12 Absolute Value

Here you'll learn about absolute value.

What is the absolute value of the number –6?

Watch This

Khan Academy Absolute Value



List in order from smallest to largest:

|5|, |9 – 7|, |5 – 15|, |0|, |–3|

|5| = 5

|9 - 7| = |2| = 2

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Click image to the left for more content.

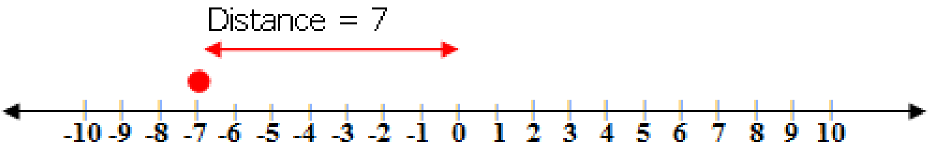
Guidance

Absolute value in the real number system is the distance from zero on the number line. It is always a positive number. Absolute value is always written as $|x|$. Using this notation can be translated as "the positive value of x".

Example A

$|-7| = ?$

Solution:



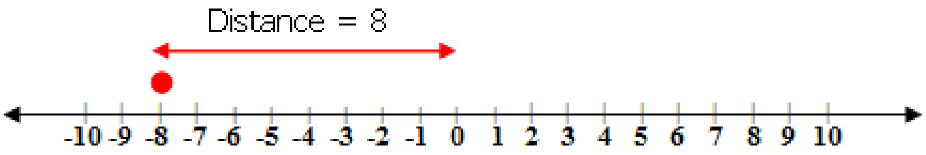
$|-7| = 7$

Example B

$|7-15| = ?$

Solution:

$$\begin{aligned}|7-15| &= ?\\ |-8| &= ?\end{aligned}$$



$$|-8| = 8$$

Example C

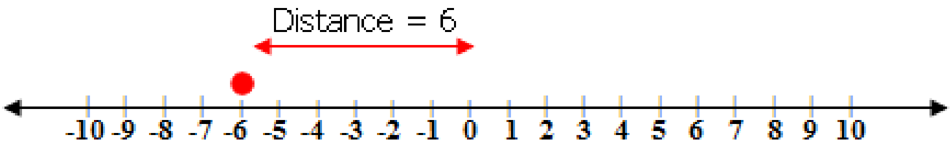
$|-9|-|-2|=?$

Solution:

$$\begin{aligned}|-9|-|-2| &= ?\\|-9|-|-2| &= 9-2\\|-9|-|-2| &= 7\end{aligned}$$

Concept Problem Revisited

Remember the definition of absolute value in the real number system is the distance from zero on the number line. Let's look at the distance from the –6 to zero.



Therefore $|-6| = 6$.

Vocabulary

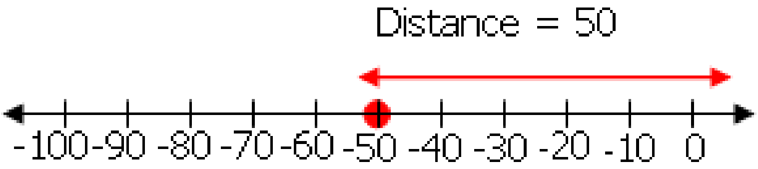
Absolute Value
Absolute value in the real number system is the distance from zero on the number line. It is always a positive number and is represented using the symbol $|x|$.

Guided Practice

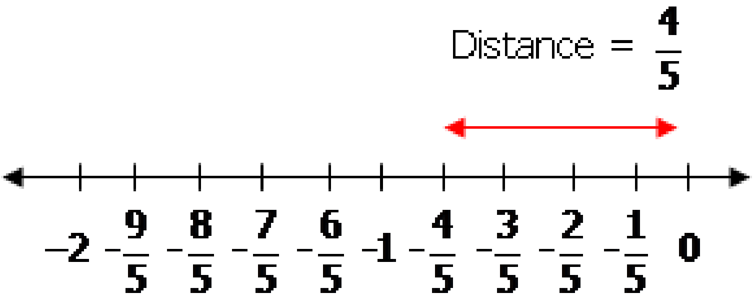
1. $|-50|=?$
2. $|-\frac{4}{5}|=?$
3. $|-3|-|-4|=?$

Answers:

1. $|-50| = 50$



2. $|-\frac{4}{5}| = \frac{4}{5}$



3. $|-3|-|-4| = 3-4 = -1$. Remember that just because there is an absolute value in the problem doesn't mean that the answer is automatically positive!

Practice

Evaluate each of the following:

1. $|-4.5| = ?$
2. $|-1| = ?$
3. $|-\frac{1}{3}| = ?$
4. $|4| = ?$
5. $|-2| = ?$
6. $|-1|+|3| = ?$
7. $|5|-|-2| = ?$
8. $|-\frac{1}{2}|+|-\frac{2}{3}| = ?$
9. $|-2.4|-|-1.6| = ?$
10. $|-3|-|-2.4| = ?$
11. $|2-4| = ?$
12. $|5-6| = ?$
13. $|-\frac{1}{2}-\frac{5}{6}| = ?$
14. $|2.3-3.7| = ?$
15. $|7.8-9.4| = ?$

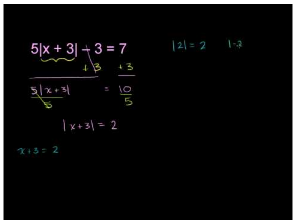
2.13 Solutions to Absolute Value Equations

Here you'll learn how to solve absolute value equations.

Solve for the variable in the equation: $|3x+1|=4$

Watch This

Khan Academy Absolute Value Equations



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Guidance

Recall that a linear equation relates mathematical expressions with an equals sign. To solve an absolute linear equation you have to remember the same rules that you have used to solve linear equations with one variable. The difference with absolute value equations is there will often be two solutions instead of just one solution. Consider the following equations:

1.

$$|x|=5$$

This means that x can be 5 or x can be –5. This is because $|5|=|-5|=5$.

2.

$$|x+1|=7$$

This means that $x+1$ can be 7 or $x+1$ can be –7. This is because $|7|=|-7|=7$.

3.

$$|x|=-1$$

The absolute value of x can never be equal to a negative number. Therefore if an absolute value equation is equal to a negative number, there is no solution.

Example A

$|d+3|=2.1$

Solution: Set up two equations to solve. You know that either $d+3=2.1$ OR $d+3=-2.1$. *The quantity inside the absolute value signs could be either the positive or negative of the value on the right side.*

$$\begin{aligned}
d+3&=2.1 \\
d+3-3&=2.1-3 && \text{Subtract 3 from both sides to isolate the variable} \\
d&=0.9 && \text{Simplify} \\
&OR \\
d+3&=-2.1 \\
d+3-3&=-2.1-3 && \text{Subtract 3 from both sides to isolate the variable} \\
d&=-5.1 && \text{Simplify}
\end{aligned}$$

Solutions $= 0.9, -5.1$

Example B

$|2(z+4)| = |5|$

Solution: First of all, you know that $|5| = 5$. Now, set up two equations to solve. You know that either $2(z+4) = 5$ OR $2(z+4) = -5$.

$$\begin{aligned}
2(z+4)&=5 \\
2z+8&=5 && \text{Remove parentheses} \\
2z+8-8&=5-8 && \text{Subtract 8 from both sides to isolate the variable} \\
2z&=-3 && \text{Simplify} \\
\frac{2z}{2}&=\frac{-3}{2} && \text{Divide by 2 to solve for the variable} \\
z&=\frac{-3}{2} && \text{Simplify} \\
&OR \\
2(z+4)&=-5 \\
2z+8&=-5 && \text{Remove parentheses} \\
2z+8-8&=-5-8 && \text{Subtract 8 from both sides to isolate the variable} \\
2z&=-13 && \text{Simplify} \\
\frac{2z}{2}&=\frac{-13}{2} && \text{Divide by 2 to solve for the variable} \\
z&=\frac{-13}{2} && \text{Simplify}
\end{aligned}$$

Solutions $= \frac{-3}{2}, \frac{-13}{2}$

Example C

$|\frac{1}{2}x+3| = |\frac{4}{5}|$

Solution: First of all, you know that $|\frac{4}{5}| = \frac{4}{5}$. Now, set up two equations to solve. You know that either $\frac{1}{2}x+3 = \frac{4}{5}$ OR $\frac{1}{2}x+3 = -\frac{4}{5}$.

$$\begin{aligned}
\frac{1}{2}x+3 &= \frac{4}{5} && \\
\left(\frac{5}{5}\right)\frac{1}{2}x+\left(\frac{10}{10}\right)3 &= \left(\frac{2}{2}\right)\frac{4}{5} && \text{Multiply to get common denominator (LCD} = 10) \\
\frac{5}{10}x+\frac{30}{10} &= \frac{8}{10} && \text{Simplify} \\
5x+30 &= 8 && \text{Simplify} \\
5x+30-30 &= 8-30 && \text{Subtract 30 from both sides to isolate the variable} \\
5x &= -22 && \text{Simplify} \\
\frac{5x}{5} &= \frac{-22}{5} && \text{Divide by 5 to solve for the variable} \\
x &= \frac{-22}{5} && \text{Simplify} \\
&\mathit{OR} && \\
\frac{1}{2}x+3 &= \frac{-4}{5} && \\
\left(\frac{5}{5}\right)\frac{1}{2}x+\left(\frac{10}{10}\right)3 &= \left(\frac{2}{2}\right)\frac{-4}{5} && \text{Multiply to get common denominator (LCD} = 10) \\
\frac{5}{10}x+\frac{30}{10} &= \frac{-8}{10} && \text{Simplify} \\
5x+30 &= -8 && \text{Simplify} \\
5x+30-30 &= -8-30 && \text{Subtract 30 from both sides to isolate the variable} \\
5x &= -38 && \text{Simplify} \\
\frac{5x}{5} &= \frac{-38}{5} && \text{Divide by 5 to solve for the variable} \\
x &= \frac{-38}{5} && \text{Simplify}
\end{aligned}$$

Solutions $= \frac{-22}{5}, \frac{-38}{5}$

Concept Problem Revisited

Solve for the variable in the expression: $|3x+1| = 4$

Because $|3x+1| = 4$, the expression $3x+1$ is equal to 4 ***or*** –4.

$$\begin{aligned}
3x+1 &= 4 && \\
3x+1-1 &= 4-1 && \text{Subtract 1 from both sides to isolate the variable}\\
3x &= 3 && \text{Simplify}\\
\frac{3x}{3} &= \frac{3}{3} && \text{Divide by 3 to solve for the variable}\\
x &= 1 && \text{Simplify}
\end{aligned}$$

OR

$$\begin{aligned}
3x+1 &= -4 && \\
3x+1-1 &= -4-1 && \text{Subtract 1 from both sides to isolate the variable}\\
3x &= -5 && \text{Simplify}\\
\frac{3x}{3} &= \frac{-5}{3} && \text{Divide by 3 to solve for the variable}\\
x &= \frac{-5}{3} && \text{Simplify}
\end{aligned}$$

Just like with regular linear equations, you can check both answers.

$$\begin{aligned}
&|3x+1|= 4 && |3x+1|= 4\\
&\left|3\left(\frac{-5}{3}\right)+1\right| = 4 && |3(1)+1|= 4\\
&|-5+1|= |-4|= 4 && |4|= 4
\end{aligned}$$

Vocabulary

Absolute Value

Absolute value in the real number system is the distance from zero on the number line. It is always a positive number and is represented using the symbol $|x|$.

Linear Equation

A ***linear equation*** relates mathematical expressions with the equals sign.

Guided Practice

Solve each equation.

1. $|4a-2|= 3$

2. $|2b-8|-3 = 4$

3. $|\frac{1}{2}c-5| = 3$

Answers:

1. The solutions are $\frac{5}{4}, \frac{-1}{4}$. Here are the steps:

$$\begin{aligned}
4a-2&=3 \\
4a-2{\color{red}+2}&=3{\color{red}+2} && \text{Add 2 to both sides to isolate the variable} \\
4a&=5 && \text{Simplify} \\
\frac{4a}{\color{red}4}&=\frac{5}{\color{red}4} && \text{Divide by 4 to solve for the variable} \\
a&=\frac{5}{4} \\
&OR \\
4a-2&=-3 \\
4a-2{\color{red}+2}&=-3{\color{red}+2} && \text{Add 2 to both sides to isolate the variable} \\
4a&=-1 && \text{Simplify} \\
\frac{4a}{\color{red}4}&=\frac{-1}{\color{red}4} && \text{Divide by 4 to solve for the variable} \\
a&=\frac{-1}{4}
\end{aligned}$$

2. The solutions are $\frac{15}{2},\frac{1}{2}$. First, isolate the part of the equation with the absolute value sign by adding 3 to both sides. The new equation is $|2b-8|=7$. Then, set up two equations and solve.

$$\begin{aligned}
2b-8&=7 \\
2b&=15 && \text{Add 8 to both sides and simplify} \\
\frac{2b}{\color{red}2}&=\frac{15}{\color{red}2} && \text{Divide by 2 to solve for the variable} \\
b&=\frac{15}{2} \\
&OR \\
2b-8&=-7 \\
2b-8{\color{red}+8}&=-7{\color{red}+8} && \text{Add 8 to both sides to isolate the variable} \\
2b&=1 && \text{Simplify} \\
\frac{2b}{\color{red}2}&=\frac{1}{\color{red}2} && \text{Divide by 2 to solve for the variable} \\
b&=\frac{1}{2}
\end{aligned}$$

3. The solutions are 16, 4. Here are the steps to solve:

$$\begin{aligned}
\frac{1}{2}c-5&=3 \\
\frac{1}{2}c-\left(\frac{2}{2}\right)5&=\left(\frac{2}{2}\right)3 && \text{Multiply to get common denominator. (LCD} = 2) \\
\frac{c}{2}-\frac{10}{2}&=\frac{6}{2} && \text{Simplify} \\
c-10&=6 && \text{Simplify} \\
c-10+10&=6+10 && \text{Add 10 to both sides to isolate the variable} \\
c&=16 \\
&OR \\
\frac{1}{2}c-5&=-3 \\
\frac{1}{2}c-\left(\frac{2}{2}\right)5&=\left(\frac{2}{2}\right)-3 && \text{Multiply to get common denominator. (LCD} = 2) \\
\frac{c}{2}-\frac{10}{2}&=\frac{-6}{2} && \text{Simplify} \\
c-10&=-6 && \text{Simplify} \\
c-10+10&=-6+10 && \text{Add 10 to both sides to isolate the variable} \\
c&=4
\end{aligned}$$

Practice

Solve each of the following absolute value linear equations.

1. $|t+2|=4$
2. $|r-2|=7$
3. $|5-k|=6$
4. $|6-y|=12$
5. $-6=|1-b|$
6. $|\frac{1}{5}x-3|=1$
7. $|\frac{1}{2}(r-3)|=2$
8. $|\frac{1}{3}(f+1)|=5$
9. $|3d-11|=-2$
10. $|5w+9|-6=68$
11. $|5(2t+5)+3(t-1)|=-3$
12. $|2.24x-24.63|=2.25$
13. $|6(5j-3)+2|=14$
14. $|7g-8(g+3)|=1$
15. $|e+4(e+3)|=17$

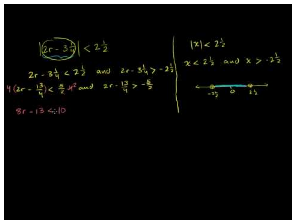
2.14 Algebraic Solutions to Absolute Value Inequalities

Here you'll learn how to solve an absolute value linear inequality.

A ball is fired from the cannon during the Independence Day celebrations. It is fired directly into the air with an initial velocity of 150 ft/sec. The speed of the cannon ball can be calculated using the formula $s = |-32t + 150|$, where s is the speed measure in ft/sec and t is the time in seconds. Calculate the times when the speed is less than 86 ft/sec.

Watch This

Khan Academy Absolute Value Inequalities



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Guidance

You have learned that a linear inequality is of the form $ax+b > c, ax+b < c, ax+b \geq c$, or $ax+b \leq c$. Linear inequalities, unlike linear equations, have more than one solution. They have a solution set. For example, if you look at the linear inequality $x+3 > 5$. You know that $2+3$ is equal to 5, therefore the solution set could be any number greater than 2.

Recall that when solving absolute value linear equations, you have to solve for the two related equations. Remember that for $|ax+b| = c$, you had to solve for $ax+b = c$ and $ax+b = -c$. The same is true for linear inequalities. If you have an absolute value linear inequality, you would need to solve for the two related linear inequalities.

The table below shows the four types of absolute value linear inequalities and the two related inequalities required to be solved for each one.

TABLE 2.16:

Absolute Value Inequality	$\|ax+b\| > c$	$\|ax+b\| < c$	$\|ax+b\| \geq c$	$\|ax+b\| \leq c$
Inequality 1	$ax+b > c$	$ax+b < c$	$ax+b \geq c$	$ax+b \leq c$
Inequality 2	$ax+b < -c$	$ax+b > -c$	$ax+b \leq -c$	$ax+b \geq -c$

Remember the rules to algebraically solve for the variable remain the same as you have used before.

Example A

Solve for the absolute value inequality $|g+5| < 3$.

Solution: Set up and solve two inequalities:

$$\begin{array}{rl}
g+5<3 & \\
g+5-5<3-5 & \text{Subtract 5 from both sides to isolate the variable} \\
g<-2 & \\
OR & \\
g+5>-3 & \\
g+5-5>-3-5 & \text{Subtract 5 from both sides to isolate the variable} \\
g>-8 &
\end{array}$$

Solution: $-8<g<-2$

Example B

Solve for the absolute value inequality $|j-\frac{1}{2}|>2$.

Solution: Set up and solve two inequalities:

$$\begin{array}{rl}
j-\frac{1}{2}>2 & \\
\left(\frac{2}{2}\right)j-\frac{1}{2}>\left(\frac{2}{2}\right)2 & \text{Multiply to get a common denominator (LCD} = 2) \\
\frac{2j}{2}-\frac{1}{2}>\frac{2}{2} & \text{Simplify} \\
2j-1>2 & \text{Simplify} \\
2j-1+1>2+1 & \text{Add 1 to both sides to isolate the variable} \\
2j>3 & \text{Simplify} \\
\frac{2j}{2}>\frac{3}{2} & \text{Divide by 2 to solve for the variable.} \\
j>\frac{3}{2} & \\
OR & \\
j-\frac{1}{2}<-2 & \\
\left(\frac{2}{2}\right)j-\frac{1}{2}<\left(\frac{2}{2}\right)(-2) & \text{Multiply to get a common denominator (LCD} = 2) \\
\frac{2j}{2}-\frac{1}{2}<\frac{-2}{2} & \text{Simplify} \\
2j-1<-2 & \text{Simplify} \\
2j-1+1<-2+1 & \text{Add 1 to both sides to isolate the variable} \\
2j<-1 & \text{Simplify} \\
\frac{2j}{2}<\frac{-1}{2} & \text{Divide by 2 to solve for the variable.} \\
j<\frac{-1}{2} &
\end{array}$$

Solution: $j>\frac{3}{2}$ or $j<\frac{-1}{2}$

Example C

Solve for the absolute value inequality $|t+1|-3 \geq 2$.

Solution: First, isolate the absolute value part of the inequality:

$$\begin{aligned}|t+1|-3 &\geq 2\\ |t+1|-3{\color{red}+3} &\geq 2{\color{red}+3}\\ |t+1| &\geq 5\end{aligned}$$

Now, set up and solve the two inequalities:

$$\begin{aligned}&t+1 \geq 5\\ &t+1{\color{red}-1} \geq 5{\color{red}-1}\\ &t \geq 4\\ &\qquad OR\\ &t+1 \leq -5\\ &t+1{\color{red}-1} \leq -5{\color{red}-1}\\ &t \leq -6\end{aligned}$$

Solution: $t \geq 4$ or $t \leq -6$.

Concept Problem Revisited

A ball is fired from the cannon during the Independence Day celebrations. It is fired directly into the air with an initial velocity of 150 ft/sec. The speed of the cannon ball can be calculated using the formula $s = |-32t+150|$, where s is the speed measure in ft/sec and t is the time in seconds. Calculate the times when the speed is less than 86 ft/sec.

$86 > |-32t+150|$

$$
\begin{aligned}
86 &> -32t+150 \\
86-150 &> -32t+150-150 && \text{Subtract 150 from both sides to isolate the variable} \\
-64 &> -32t && \text{Simplify} \\
\frac{-64}{-32} &< \frac{-32t}{-32} && \text{Divide by -32 to solve for the variable. Remember when} \\
& && \quad \text{dividing by a negative number to reverse the sign of the inequality.} \\
t &> 2 \\
& OR \\
-86 &< -32t+150 \\
-86-150 &< -32t+150-150 && \text{Subtract 150 from both sides to isolate the variable} \\
-236 &< -32t && \text{Simplify} \\
\frac{-236}{-32} &> \frac{-32t}{-32} && \text{Divide by -32 to solve for the variable. Remember when} \\
& && \quad \text{dividing by a negative number to reverse the sign of the inequality.} \\
t &< 7.375
\end{aligned}
$$

Therefore when $2 < t < 7.375$, the speed is greater than 86 ft/sec.

Vocabulary

Absolute Value Linear Inequality

Absolute Value Linear inequalities can have one of four forms: $|ax+b| > c, |ax+b| < c, |ax+b| \geq c$, or $|ax+b| \leq c$. Absolute value linear inequalities have two related inequalities. For example for $|ax+b| > c$, the two related inequalities are $ax+b > c$ and $ax+b < -c$.

Linear Inequality

Linear inequalities can have one of four forms: $ax+b > c, ax+b < c, ax+b \geq c$, or $ax+b \leq c$. In other words, the left side no longer equals the right side, it is less than, greater than, less than or equal to, or greater than or equal to.

Guided Practice

Solve each inequality:

1. $|x-1| \geq 9$

2. $|-2w+7| < 23$

3. $|-4+2b|+3 \leq 21$

Answers:

1. $|x-1| \geq 9$

$$\begin{aligned}
x-1 &\geq 9 \\
x-1+1 &\geq 9+1 && \text{(Add 1 to both sides to isolate and solve for the variable)} \\
x &\geq 10 \\
&OR \\
x-1 &\leq -9 \\
x-1+1 &\leq -9+1 && \text{(Add 1 to both sides to isolate and solve for the variable)} \\
x &\leq -8
\end{aligned}$$

Solution: $x \geq 10$ or $x \leq -8$.

2. $|-2w+7| < 23$

$$\begin{aligned}
-2w+7 &< 23 \\
-2w+7-7 &< 23-7 && \text{(Subtract 7 from both sides to get variables on same side)} \\
-2w &< 16 && \text{(Simplify)} \\
\frac{-2w}{-2} &> \frac{16}{-2} && \text{(Divide by -2 to solve for the variable, reverse sign of inequality)} \\
w &> -8 \\
&OR \\
-2w+7 &> -23 \\
-2w+7-7 &> -23-7 && \text{(Subtract 7 from both sides to get variables on same side)} \\
-2w &> -30 && \text{(Simplify)} \\
\frac{-2w}{-2} &< \frac{-30}{-2} && \text{(Divide by -2 to solve for the variable, reverse sign of inequality)} \\
w &< 15
\end{aligned}$$

Solution: $-8 < w < 15$

3. First, isolate the absolute value part of the inequality:

$$\begin{aligned}
|-4+2b|+3 &\leq 21 \\
|-4+2b|+3-3 &\leq 21-3 \\
|-4+2b| &\leq 18
\end{aligned}$$

Now, set up and solve the two inequalities:

$$\begin{aligned}
-4+2b &\leq 18\\
-4+2b+4 &\leq 18+4\\
2b &\leq 22\\
b &\leq 11\\
&OR\\
-4+2b &\geq -18\\
-4+2b+4 &\geq -18+4\\
2b &\geq -14\\
b &\geq -7
\end{aligned}$$

Solution: $-7 \leq b \leq 11$

Practice

Solve each of the following absolute value linear inequalities:

1. $|p-16| > 10$
2. $|r+2| < 5$
3. $|3-2k| \geq 1$
4. $|8-y| > 5$
5. $8 \geq |5d-2|$
6. $|s+2|-5 > 8$
7. $|10+8w|-2 < 16$
8. $|2q+1|-5 \leq 7$
9. $|\frac{1}{3}(g-2)| < 4$
10. $|-2(e+4)| > 17$
11. $|-5x-3(2x-1)| > 3$
12. $|2(a-1.2)| \geq 5.6$
13. $|-2(r+3.1)| \leq 1.4$
14. $|\frac{3}{4}(m-3)| \leq 8$
15. $|-2\left(e-\frac{3}{4}\right)| \geq 3$

2.15 Graphical Solutions to Absolute Value Inequalities

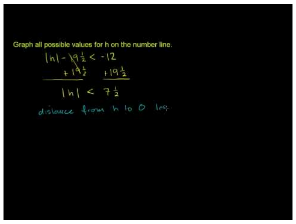
Here you'll learn how to represent the solutions of an absolute value inequality on a number line.

Solve the following inequality and graph the solution on a number line.

$|x+2| \leq 3$

Watch This

Khan Academy Absolute Value Inequalities on a Number Line



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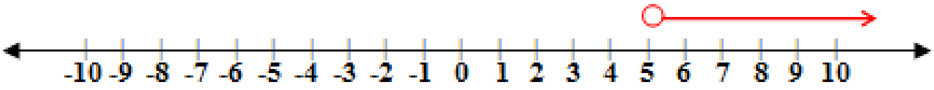
Guidance

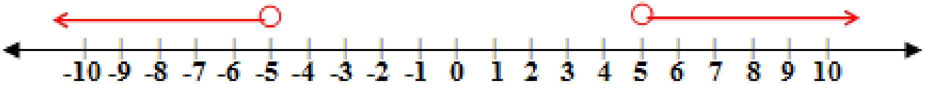
Recall that you can graph linear inequalities on number lines. For $x > 5$, the graph can be shown as:



Notice that there is only one solution set and therefore one section of the number line has the region shown in red.

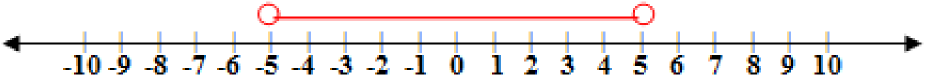
What do you think would happen with absolute value linear inequalities? With absolute value linear inequalities, there are two inequalities to solve. Therefore there can be two sections of the number line showing solutions.

For $|t| > 5$, you would actually solve for $t > 5$ and $t < -5$. If you were to graph this solution on a number line it would look like the following:



The solution is $t > 5$ OR $t < -5$.

For $|t| < 5$, you would actually solve for $t < 5$ and $t > -5$. If you were to graph this solution on a number line it would look like the following:



The solution is $-5 < t < 5$. This is the same as $t < 5$ AND $t > -5$.

Graphing the solution set to an absolute value linear inequality gives you the same visual representation as you had when graphing the solution set to linear inequalities. The same rules apply when graphing absolute values of linear inequalities on a real number line. Once the solution is found, the open circle is used for absolute value inequalities containing the symbols >and <. The closed circle is used for absolute value inequalities containing the symbols $\leq$ and $\geq$.

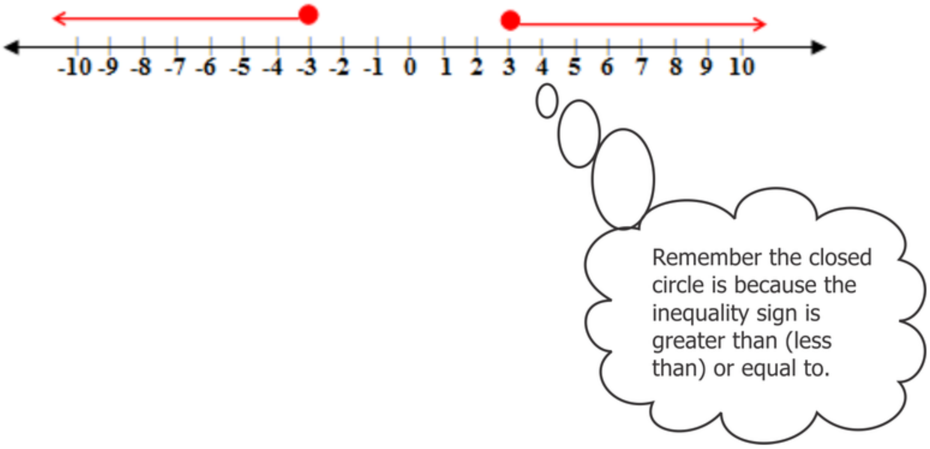
Example A

Represent the solution set to the following inequality on a number line: $|2x| \geq 6$.

Solution: First solve the inequality. Then, represent your solution on a number line.

$$\begin{aligned}
|2x| &\geq 6 \\
2x &\geq 6 \\
\frac{2x}{\color{red}2} &\geq \frac{6}{\color{red}2} && \text{(Divide by 2 to isolate and solve for the variable)} \\
x &\geq 3 && \text{(Simplify)} \\
&OR \\
2x &\leq -6 \\
\frac{2x}{\color{red}2} &\leq \frac{-6}{\color{red}2} && \text{(Divide by 2 to isolate and solve for the variable)} \\
x &\leq -3 && \text{(Simplify)}
\end{aligned}$$

The solution sets are $x \geq 3$ OR $x \leq -3$.



Remember the closed circle is because the inequality sign is greater than (less than) or equal to.

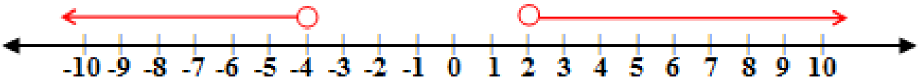
Example B

Solve the following inequality and graph the solution on a number line: $|x+1| > 3$

Solution: First solve the inequality. Then, represent your solution on a number line.

$$\begin{aligned}
|x+1| &> 3 && \text{(Divide both sides by 2 to solve for the variable)}\\
x+1 &> 3\\
x+1-1 &> 3-1 && \text{(Subtract 1 from both sides of the inequality sign)}\\
x &> 2\\
&OR\\
x+1 &< -3\\
x+1-1 &< -3-1 && \text{(Subtract 1 from both sides of the inequality sign)}\\
x &< -4
\end{aligned}$$

The solution sets are $x > 2$, OR $x < -4$.



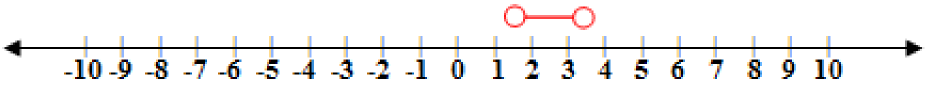
Example C

Solve the following inequality and graph the solution on a number line: $\left|x-\frac{5}{2}\right| < 1$

Solution: First solve the inequality. Then, represent your solution on a number line.

$$\begin{aligned}
\left|x-\frac{5}{2}\right| &< 1 \\
x-\frac{5}{2} &< 1 \\
\left(\frac{2}{2}\right)x-\frac{5}{2} &< \left(\frac{2}{2}\right)1 \\
\frac{2x}{2}-\frac{5}{2} &< \frac{2}{2} \\
2x-5 &< 2 && \text{(Simplify)} \\
2x-5+5 &< 2+5 && \text{(Add 5 to isolate the variable)} \\
2x &< 7 && \text{(Simplify)} \\
\frac{2x}{2} &< \frac{7}{2} \\
x &< \frac{7}{2} \\
& OR \\
x-\frac{5}{2} &> -1 \\
\left(\frac{2}{2}\right)x-\frac{5}{2} &> \left(\frac{2}{2}\right)(-1) && \text{(Multiply to get common denominator (LCD = 2))} \\
\frac{2x}{2}-\frac{5}{2} &< \frac{-2}{2} && \text{(Simplify)} \\
2x-5 &> -2 && \text{(Simplify)} \\
2x-5+5 &> -2+5 && \text{(Add 5 to isolate the variable)} \\
2x &> 3 && \text{(Simplify)} \\
\frac{2x}{2} &> \frac{3}{2} && \text{(Divide both sides by 2 to solve for the variable)} \\
x &> \frac{3}{2}
\end{aligned}$$

The solution is $\frac{3}{2} < x < \frac{7}{2}$.



Concept Problem Revisited

Solve the following inequality and graph the solution on a number line.

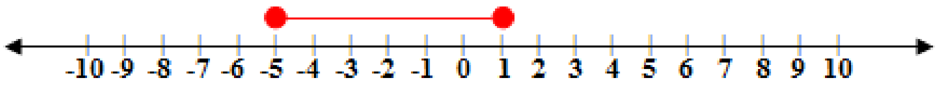
$|x+2| \leq 3$

First solve the inequality:

$$\begin{aligned} x+2 &\leq 3 && \\ x+2-2 &\leq 3-2 && \text{Subtract 2 from both sides to isolate the variable} \\ x &\leq 1 && \text{Simplify} \\ &OR && \\ x+2 &\geq -3 && \\ x+2-2 &\geq -3-2 && \text{Subtract 2 from both sides to isolate the variable} \\ x &\geq -5 && \text{Simplify} \end{aligned}$$

The solution is $-5 \leq x \leq 1$.

Representing on a number line:



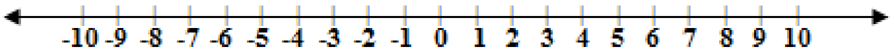
Vocabulary

Absolute Value Linear Inequality

Absolute Value Linear inequalities can have one of four forms: $|ax+b| > c, |ax+b| < c, |ax+b| \geq c$, or $|ax+b| \leq c$. Absolute value linear inequalities have two related inequalities. For example for $|ax+b| > c$, the two related inequalities are $ax+b > c$ and $ax+b < -c$.

Number Line

A ***number line*** is a line that matches a set of points and a set of numbers one to one.



Guided Practice

1. Represent the solution set to the inequality $|2x+3| > 5$ on a number line.

2. Represent the solution set to the inequality $|32x-16| \geq 32$ on a number line.

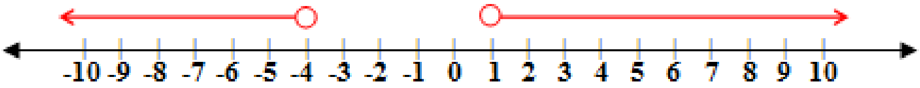
3. Represent the solution set to the inequality $|x-21.5| > 12.5$ on a number line.

Answers:

1. $|2x+3| > 5$

$$
\begin{aligned}
2x+3 &> 5 \\
2x+3-3 &> 5-3 && \text{(Subtract 3 from both sides of the inequality sign)} \\
2x &> 2 && \text{(Simplify)} \\
\frac{2x}{2} &> \frac{2}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &> 1 \\
&OR \\
2x+3 &< -5 \\
2x+3-3 &< -5-3 && \text{(Subtract 3 from both sides of the inequality sign)} \\
2x &< -8 && \text{(Simplify)} \\
\frac{2x}{2} &> \frac{-8}{2} && \text{(Divide by 2 to solve for the variable)} \\
x &< -4
\end{aligned}
$$

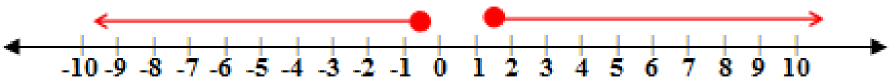
The solution sets are $x > 1$ or $x < -4$.



2. $|32x-16| \geq 32$

$$
\begin{aligned}
32x-16 &\geq 32 \\
32x-16+16 &\geq 32+16 && \text{(Add 16 to both sides of the inequality sign)} \\
32x &\geq 48 && \text{(Simplify)} \\
\frac{32x}{32} &\geq \frac{48}{32} && \text{(Divide by 32 to solve for the variable)} \\
x &\geq \frac{3}{2} \\
&OR \\
32x-16 &\leq -32 \\
32x-16+16 &\leq -32+16 && \text{(Add 16 to both sides of the inequality sign)} \\
32x &\leq -16 && \text{(Simplify)} \\
\frac{32x}{32} &\leq \frac{-16}{32} && \text{(Divide by 32 to solve for the variable)} \\
x &\leq -\frac{1}{2}
\end{aligned}
$$

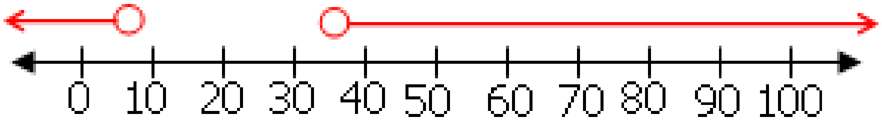
The solution sets are $x \geq \frac{3}{2}$ or $x \leq -\frac{1}{2}$.



3. $|x-21.5| > 12.5$

$$\begin{aligned} x-21.5 &> 12.5 && \\ x-21.5+21.5 &> 12.5+21.5 && \text{(Add 21.5 to both sides to isolate the variable)} \\ x &> 34 && \text{(Simplify)} \\ & OR && \\ x-21.5 &< -12.5 && \\ x-21.5+21.5 &< -12.5+21.5 && \text{(Add 21.5 to both sides to isolate the variable)} \\ x &< 9 && \text{(Simplify)} \end{aligned}$$

The solution sets are $x < 9$ or $x > 34$.



Practice

Represent the solution sets to each absolute value inequality on a number line.

1. $|3-2x| < 3$
2. $2|\frac{2x}{3}+1| \geq 4$
3. $|\frac{2g-9}{4}| < 1$
4. $|\frac{4}{3}x-5| \geq 7$
5. $|2x+5|+4 \geq 7$
6. $|p-16| > 10$
7. $|r+2| < 5$
8. $|3-2k| \geq 1$
9. $|8-y| > 5$
10. $8 \geq |5d-2|$
11. $|s+2|-5 > 8$
12. $|10+8w|-2 < 16$
13. $|2q+1|-5 \leq 7$
14. $|\frac{1}{3}(g-2)| < 4$
15. $|-2(e+4)| > 17$

Summary

You learned how to solve equations where variables were on one side of the equation, where variables were on both sides of the equation, and also when there were parentheses involved in the equations. The key to solving these equations was to make sure that whatever you did to one side of the equals sign, you did the other.

You learned how to translate from words into mathematical symbols in order to solve linear equations. Remember the key steps were to identify key words in the problem and from there write your linear equation. Sometimes it is helpful to circle mathematical operations and underline constants and variables to help you in the identification of key words.

Next you learned about one variable inequalities. Remember that inequalities do not have an equals sign but rather use the $>, <, \geq$, and $\leq$ signs to relate the two mathematical expressions. The rules for solving inequalities remained the same as for equations with one exception. The exception was if you have to multiply or divide by a negative number. If this happens, you have to reverse the sign of the inequality.

You learned to graph inequalities on a real number line. You used an open circle for >and <. You used a closed circle for $\leq$ and $\geq$.

Finally, you were introduced to absolute value. In solving with absolute value, for both equations and inequalities, the rules remain the same with one addition. Because you are dealing with the absolute value function, you have to solve for the two related expressions when solving an equation or inequality.

CHAPTER 3 Functions and Graphs

Chapter Outline

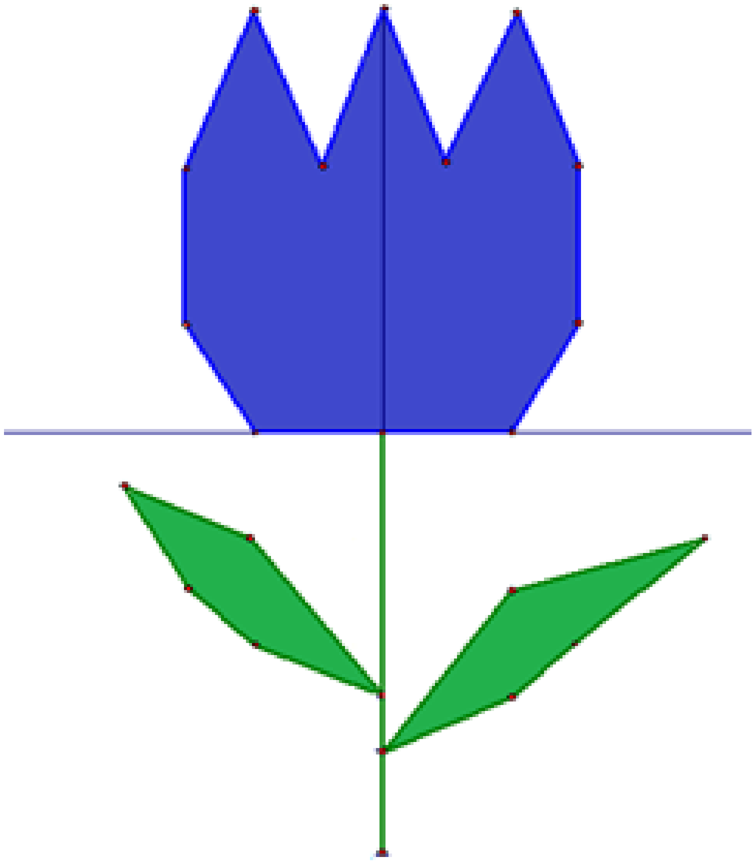
Introduction

Here you'll learn the difference between a function and a relation. You will explore both linear and quadratic functions.

3.1 The Cartesian Plane

Here you will learn about the Cartesian plane and review how to plot points on the Cartesian plane.

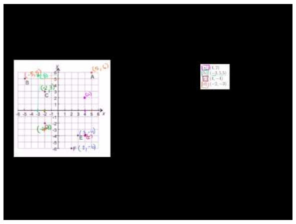
Kaitlyn walked into Math class and saw the following image displayed on the board. Her teacher asked everyone in the class to duplicate the picture on a blank sheet of paper.



When the teacher felt that the students had completed the drawing, she asked them to share their results with the class. Most of the students had difficulty reproducing the picture. Kaitlyn told the class that she could not make the picture the same size as the one shown. She also said that she had a problem locating the leaves in the same places on the stem. Her teacher said that she could offer a solution to these problems. What was the solution?

Watch This

Khan Academy The Coordinate Plane

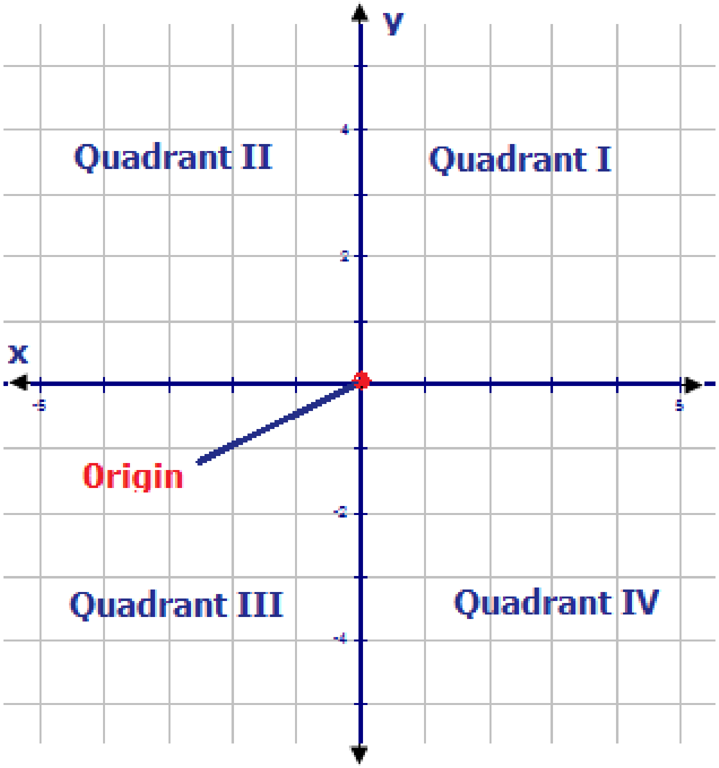


MEDIA

Click image to the left for more content.

Guidance

The ***Cartesian plane*** is a system of four areas or quadrants produced by the perpendicular intersection of two number lines. The two number lines intersect at right angles. The point of intersection is known as the ***origin***. One number line is a horizontal line and this is called the ***x-axis***. The other number line is a vertical line and it is called the ***y-axis***. The two number lines are referred to as the ***axes*** of the Cartesian plane. The Cartesian plane, also known as the ***coordinate plane***, has four quadrants that are labeled counterclockwise.



The value of the origin on the x-axis is zero. If you think of the x-axis as a number line, the numbers to the right of zero are positive values, and those to the left of zero are negative values. The same can be applied to the y-axis. The value of the origin on the y-axis is zero. The numbers above zero are positive values and those below zero are negative values.

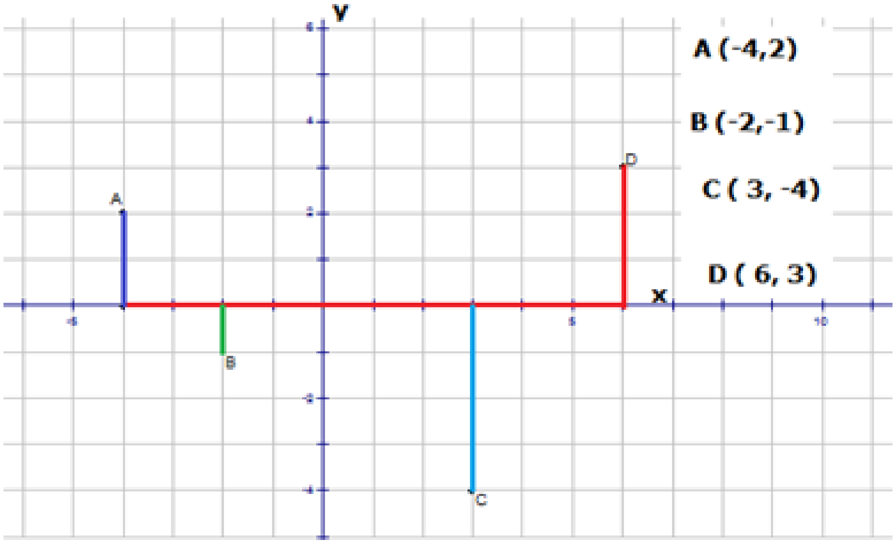
Every point that is plotted on a Cartesian plane has two values associated with it. The first value represents the x-value and the second value represents the y-value. These two values are called the ***coordinates*** of the point and are written as the ordered pair (x,y).

To plot a point on the Cartesian plane:

- Start at zero (the origin) and locate the $x-$coordinate on the x-axis.

- If the $x-$coordinate is positive, move to the right of the origin the number of units displayed by the number. If the $x-$coordinate is negative, move to the left of the origin the number of units displayed by the number.
- Once the $x-$coordinate (also called the ***abscissa***) has been located, move vertically the number of units displayed by the $y-$coordinate (also called the ***ordinate***). If the $y-$coordinate is positive, move vertically upward from the $x-$coordinate, the number of units displayed by the $y-$coordinate. If the $y-$coordinate is negative, move vertically downward from the $x-$coordinate, the number of units displayed by the $y-$coordinate.
- The point can now be plotted.

Examine the points A,B,C and D that have been plotted on the graph below.

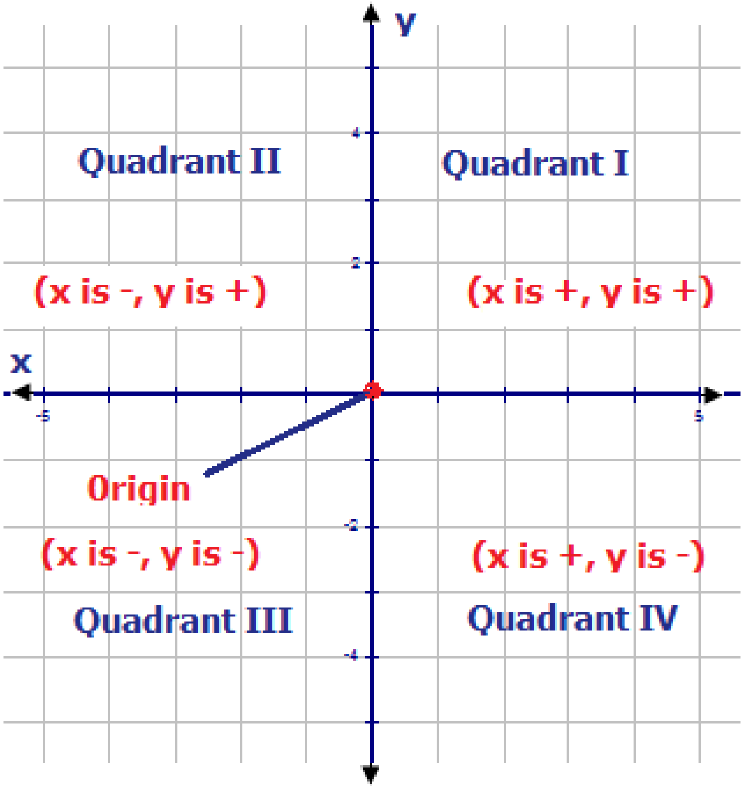


- $A(-4,2)$ – From the origin, move to the left four units (along the red line on the x-axis). Now, move vertically upward two units. Plot the point A.
- $B(-2,-1)$ – From the origin, move to the left two units (along the red line on the x-axis). Now, move vertically downward one unit. Plot the point B.
- $C(3,-4)$ – From the origin, move to the right three units (along the red line on the x-axis). Now, move vertically downward four units. Plot the point C.
- $D(6,3)$ – From the origin, move to the right six units (along the red line on the x-axis). Now, move vertically upward three units. Plot the point D.

Example A

For each quadrant, say whether the values of x and y are positive or negative.

Solution: The graph below shows where x and y values are positive and negative.

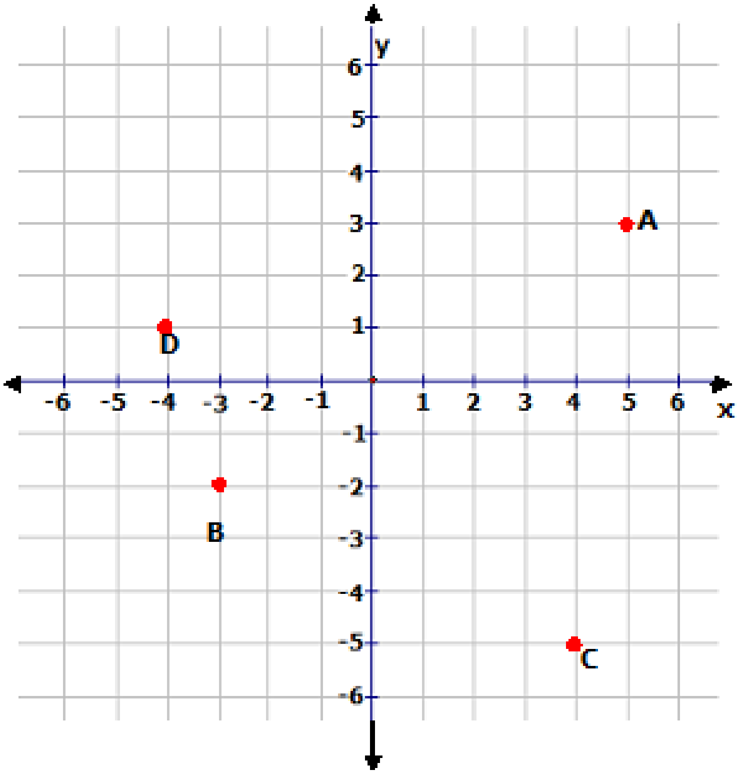


Example B

On a Cartesian plane, plot the following points:

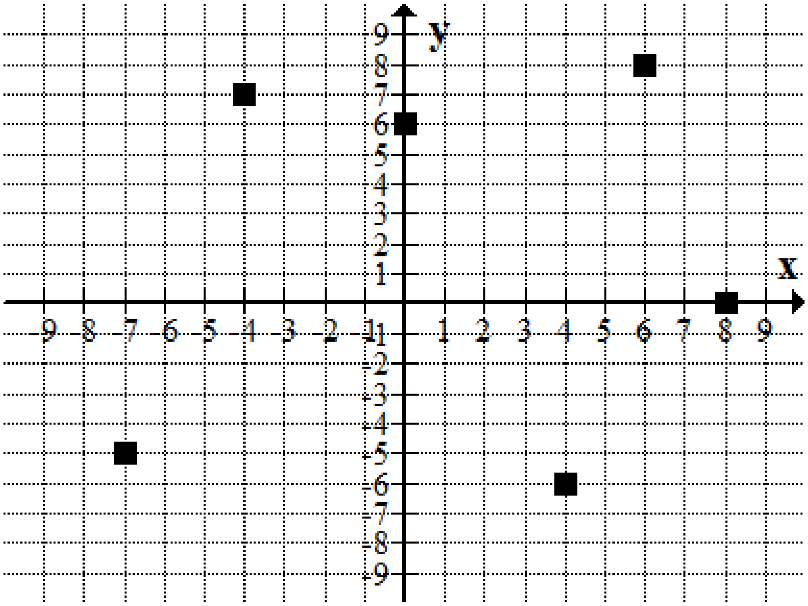
$A(5,3) \quad B(-3,-2) \quad C(4,-5) \quad D(-4,1)$

Solution:

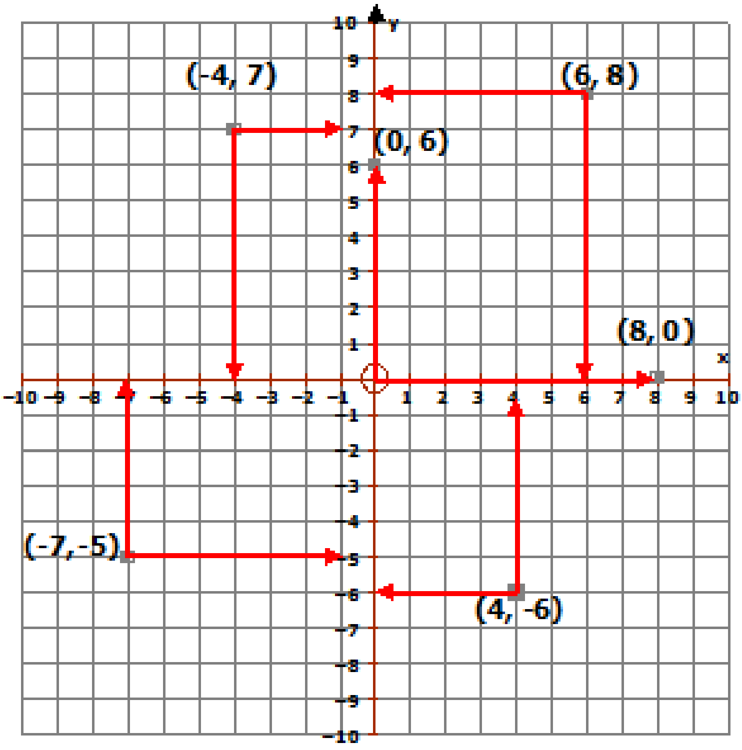


Example C

Determine the coordinates of each of the plotted points on the following graph.



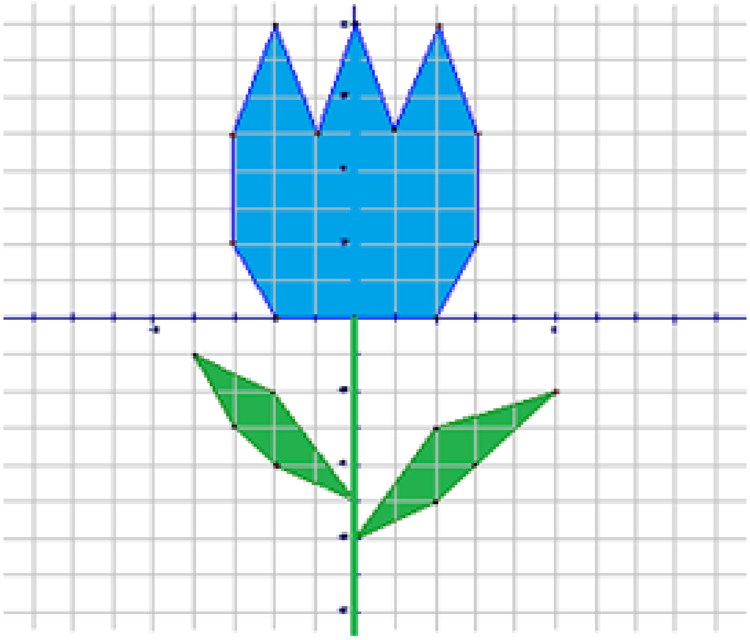
Solution:



Concept Problem Revisited

Now, let us return to the beginning of the lesson to find out the solution that the teacher had for the students.

If the teacher lets the students see the picture on a Cartesian plane, the reproduction process should be much easier.



Vocabulary

Abscissa

The ***abscissa*** is the $x-$coordinate of the ordered pair that represents a plotted point on a Cartesian plane. For the point (3, 7), **3** is the ***abscissa.***

Cartesian Plane

A ***Cartesian plane*** is a system of four areas or quadrants produced by the perpendicular intersection of two number lines. A ***Cartesian plane*** is the grid on which points are plotted.

Coordinates

The ***coordinates*** are the ordered pair (x, y) that represent a point on the Cartesian plane.

Coordinate Plane

The ***coordinate plane*** is another name for the Cartesian plane.

Ordinate

The ***ordinate*** is the y-coordinate of the ordered pair that represents a plotted point on a Cartesian plane. For the point (3, 7), **7** is the ***ordinate.***

Origin

The ***origin*** is the point of intersection of the x and y axes on the Cartesian plane. The coordinates of the origin are (0, 0).

x-axis

The ***x-axis*** is the horizontal number line of the Cartesian plane.

y-axis

The ***y-axis*** is the vertical number line of the Cartesian plane.

Guided Practice

1. Draw a Cartesian plane that displays only positive values. Number the x and y axes to twelve. Plot the following coordinates and connect them in order. Use a straight edge to connect the points. When the word "STOP" appears, begin the next line. Plot the points in the order they appear in each Line row.

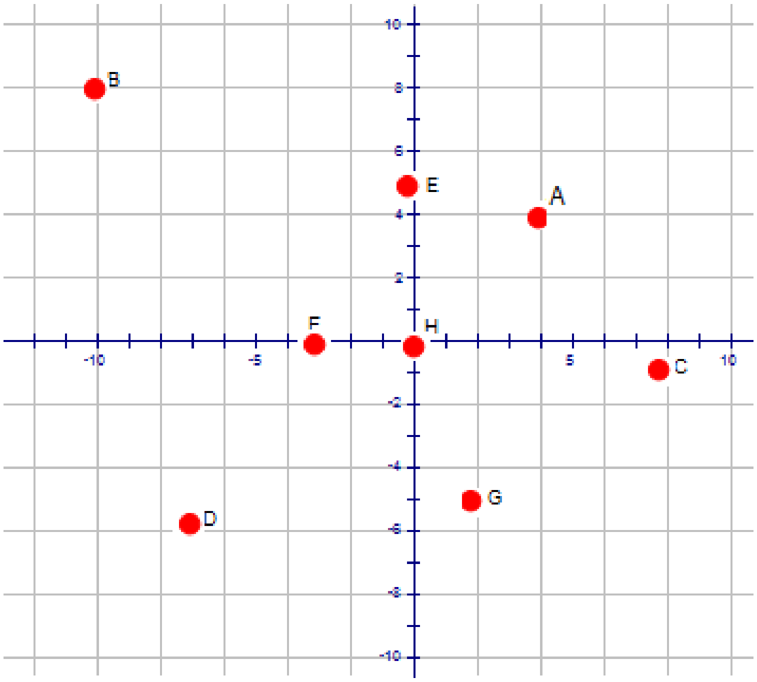
LINE 1 (6, 0) (8, 0) (9, 1) (10, 3) (10, 6) (9, 8) (7, 9) (5, 9) **STOP**
LINE 2 (6, 0) (4, 0) (3, 1) (2, 3) (2, 6) (3, 8) (5, 9) **STOP**
LINE 3 (7, 9) (6, 12) (4, 11) (5, 9) **STOP**
LINE 4 (4, 8) (3, 6) (5, 6) (4, 8) **STOP**
LINE 5 (8, 8) (7, 6) (9, 6) (8, 8) **STOP**
LINE 6 (5, 5) (7, 5) (6, 3) (5, 5) **STOP**
LINE 7 (3, 2) (4, 1) (5, 2) (6, 1) (7, 2) (8, 1) (9, 2) **STOP**
LINE 8 (4, 1) (6, 1) (8, 1) **STOP**

2. In which quadrant would the following points be located?

i) (3, –8)
ii) (–5, 4)

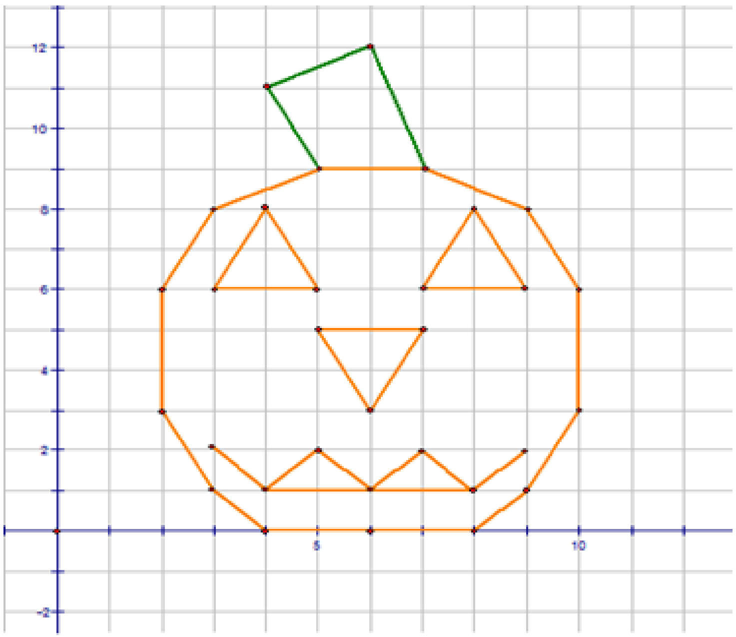
iii) (7, 2)
iv) (–6, –9)
v) (–3, 3)
vi) (9, –7)

3. State the coordinates of the points plotted on the following Cartesian plane.



Answers:

1. The following picture is the result of plotting the coordinates and joining them in the order in which they were plotted. Your pumpkin can be any color you like.



2. i) (3, –8) – the *x* coordinate is positive and the $y-$coordinate is negative. This point will be located in Quadrant IV.

ii) (–5, 4) – the *x* coordinate is negative and the $y-$coordinate is positive. This point will be located in Quadrant II.
iii) (7, 2) – the *x* coordinate is positive and the $y-$coordinate is positive. This point will be located in Quadrant I.
iv) (–6, –9) – the *x* coordinate is negative and the $y-$coordinate is negative. This point will be located in Quadrant III.
v) (–3, 3) – the *x* coordinate is negative and the $y-$coordinate is positive. This point will be located in Quadrant II.
vi) (9, –7) – the *x* coordinate is positive and the $y-$coordinate is negative. This point will be located in Quadrant IV.

3. $A(4,4)$ $B(-10,8)$ $C(8,-1)$ $D(-6,-6)$ $E(0,5)$ $F(-3,0)$ $G(2,-5)$ $H(0,0)$

Practice

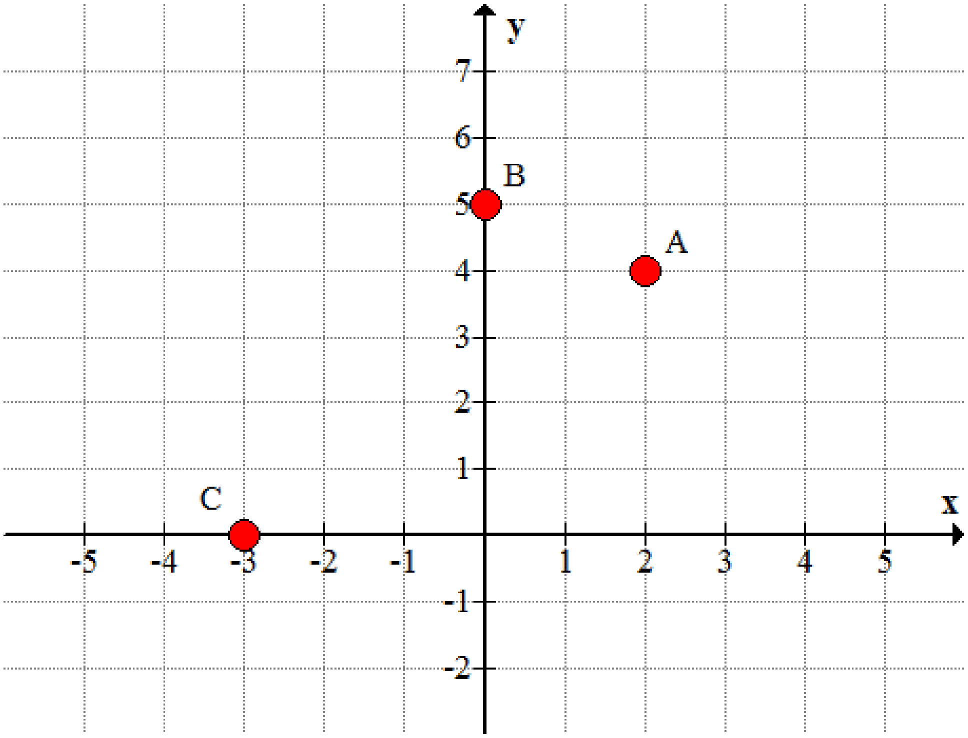
Answer the following questions with respect to the Cartesian plane:

1. What name is given to the horizontal number line on the Cartesian plane?
2. What name is given to the four areas of the Cartesian plane?
3. What are the coordinates of the origin?
4. What name is given to the vertical number line on the Cartesian plane?
5. What other name is often used to refer to the $x-$coordinate of a point on the Cartesian plane?

On a Cartesian plane, plot each of the following points. For each point, name the quadrant it is in or axis it is on.

6. $(2,0)$
7. $(-3,1)$
8. $(0,4)$
9. $(1,-2)$
10. $(5,5)$

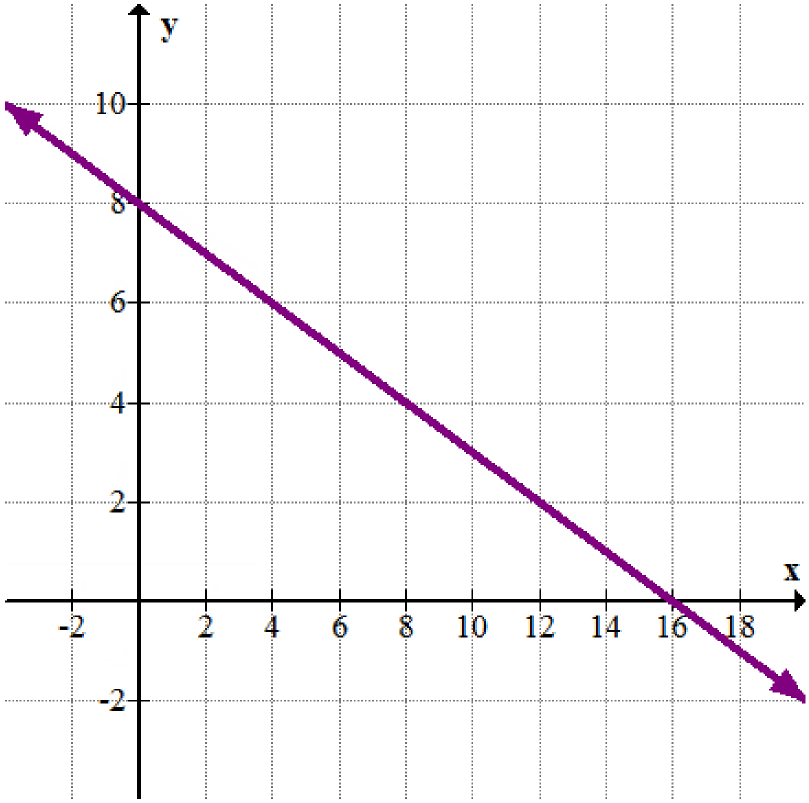
Use the graph below for #11-#13.



11. The coordinates of point A are ______.
12. The coordinates of point B are ______.
13. The coordinates of point C are ______.

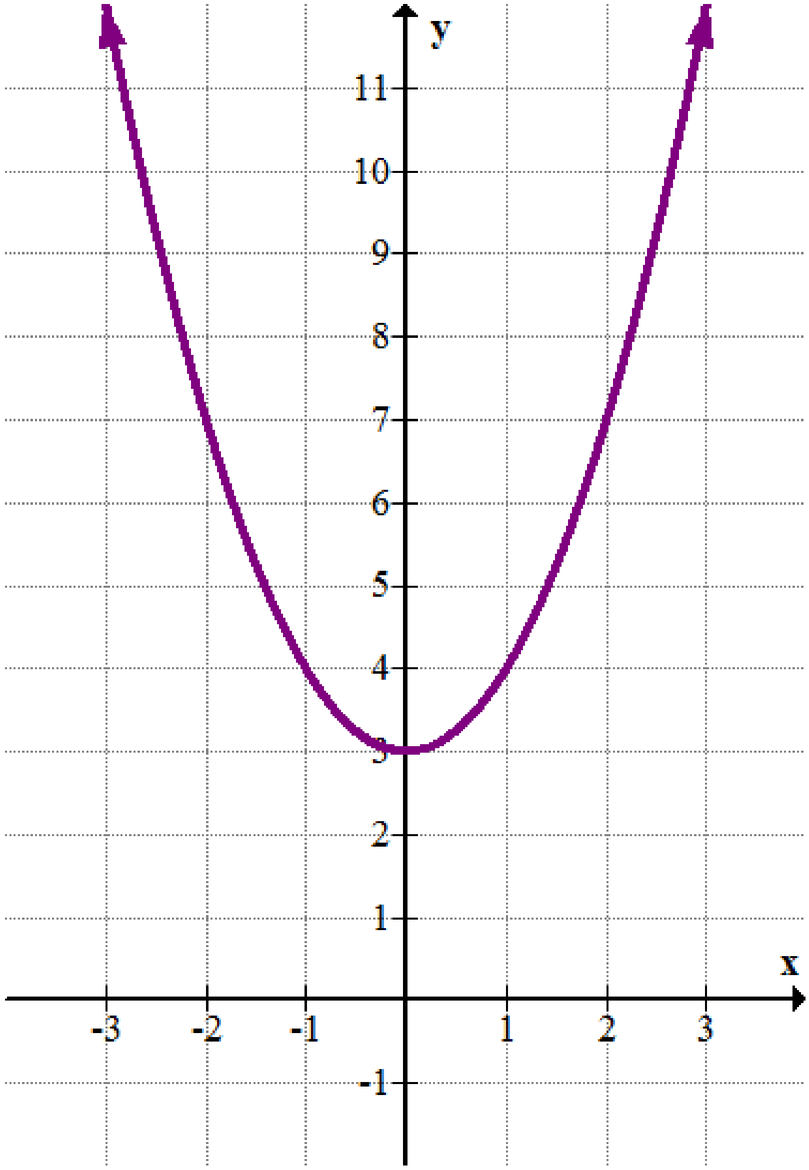
For each of the following graphs, select three points on the graph and state the coordinates of these points.

14.



}}

15.


y
11
10
9
8
7
6
5
4
3
2
1
-1
x
-3
-2
-1
1
2
3

}}

3.2 Relations and Functions

Here you will learn about relations, and what makes a relation a function.

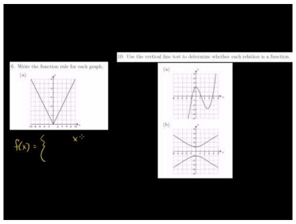
The following table of values represents data collected by a student in a math class.

x	5	10	15	10	5	0
y	12	25	37	55	72	0

Does this set of ordered pairs represent a function?

Watch This

Khan Academy Functions as Graphs



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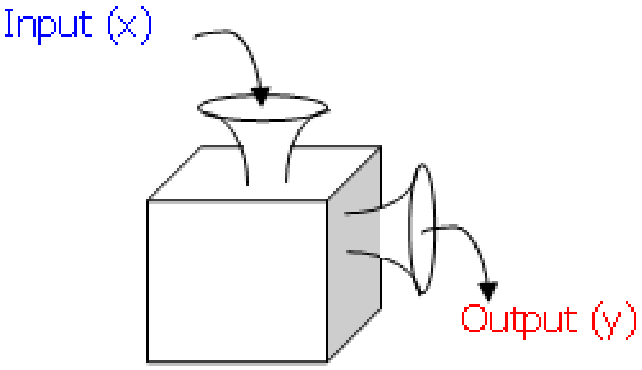
Guidance

Consider the relationship between two variables. You can think of this relationship in terms of an input/output machine.



If there is only one output for every input, you have a function. If not, you have a relation. Relations can have more than one output for every input. A relation is any set of ordered pairs. A function is a set of ordered pairs where there is only one value of y for every value of x.

Look at the two tables below. **Table A** shows a relation that is a function because every x value has only one y value. **Table B** shows a relation that is not a function because there are two different y values for the x value of 0.

TABLE 3.1: **Table A**

x	y
0	4
1	7
2	7
3	6

TABLE 3.2: **Table B**

x	y
0	4
0	2
2	6
2	7

When looking at the graph of a relation, you can determine whether or not it is a function using the vertical line test. If a vertical line can be drawn anywhere through the graph such that it intersects the graph more than once, the graph is not function.

Example A

Determine if the following relation is a function.

TABLE 3.3:

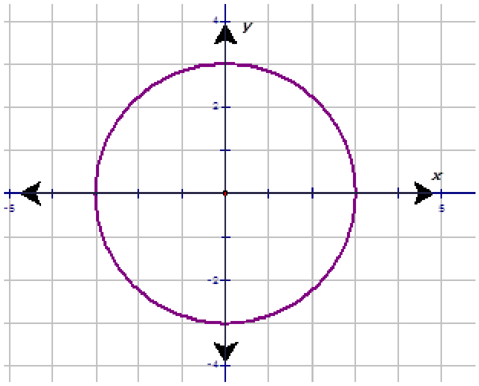
x	y
-3.5	-3.6
-1	-1
4	3.6
7.8	7.2

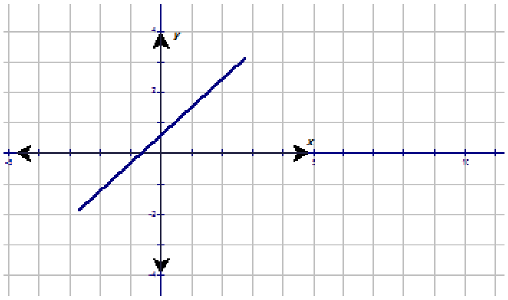
Solution:

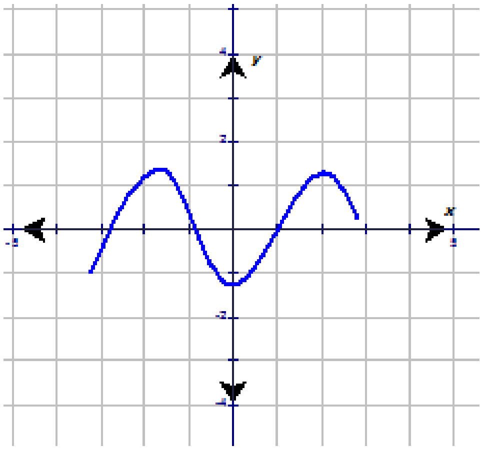
The relation is a function because there is only one value of y for every value of x.

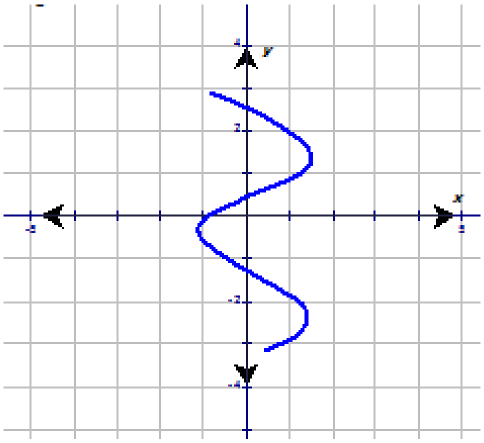
Example B

Which of the following graphs represent a function?


y
x
4
2
-2
-4
-5
5

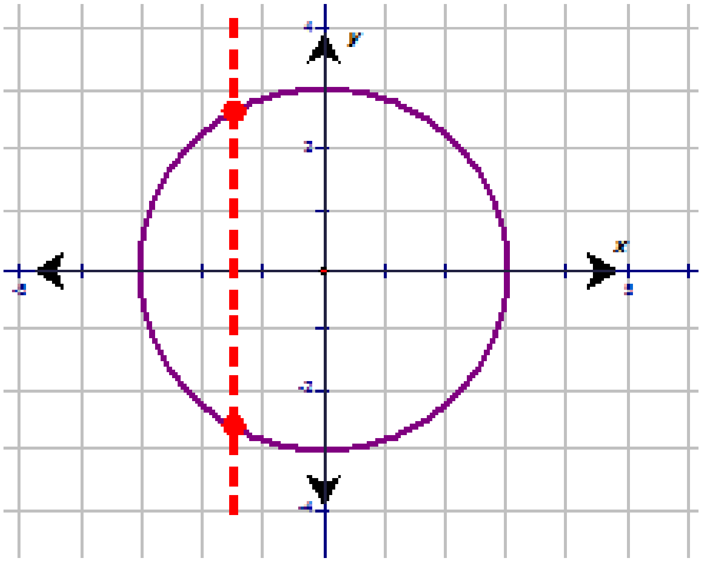

y
x


y
x
4
2
-2
-4


y
x
4
2
-2
-4

Solution:

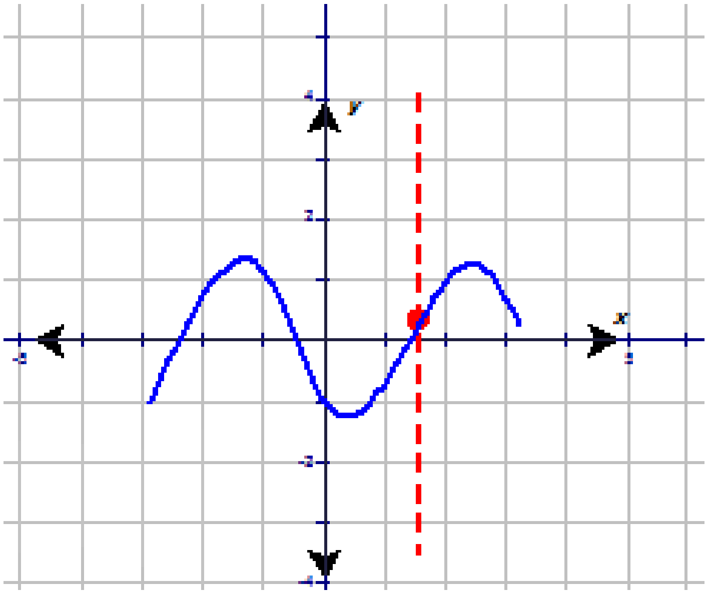
In order to answer this question, you need to use the vertical line test. A graph represents a function if no vertical line intersects the graph more than once. Let's look at the first graph. Draw a vertical line through the graph.



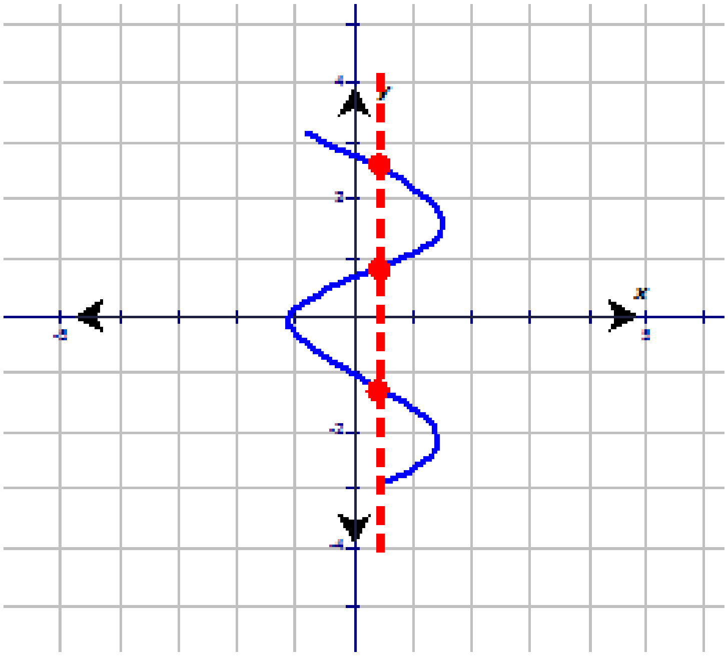
Since the vertical line hit the graph more than once (indicated by the two red dots), the graph ***does not*** represent a function.

IMAGE NOT AVAILABLE

Since the vertical line hit the graph only once (indicated by the one red dot), the graph ***does*** represent a function.



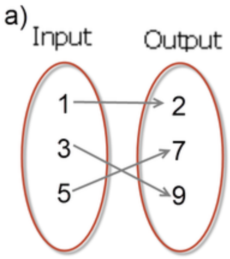
Since the vertical line hit the graph only once (indicated by the one red dot), the graph ***does*** represent a function.

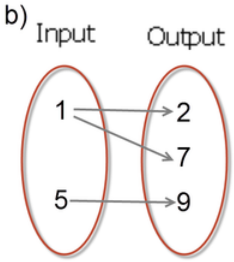


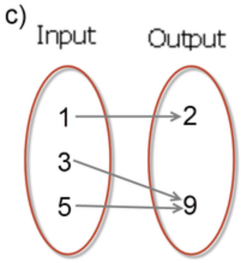
Since the vertical line hit the graph more than once (indicated by the three red dots), the graph ***does not*** represent a function.

Example C

Which of the following represent functions?







Solution:

a) This is a function because every input has only one output.

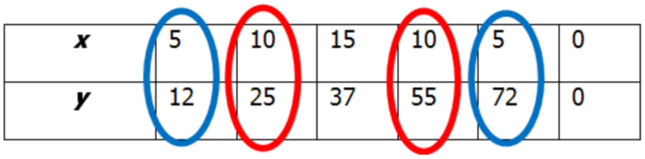
b) This is not a function because one input (1) has two outputs (2 and 7).

c) This is a function because every input has only one output.

Concept Problem Revisited

x	5	10	15	10	5	0
y	12	25	37	55	72	0

If you look at this table, there are two places where you see the more than one output for a single input.



x	5	10	15	10	5	0
y	12	25	37	55	72	0

You can conclude that this set of ordered pairs does not represent a function. It is just a relation.

Vocabulary

Function

A ***function*** is an example of a relation where there is only one output for every input. In other words, for every value of x, there is only one value for y.

Relation

A ***relation*** is any set of ordered pairs (x,y). A relation can have more than one output for an input.

Vertical Line Test

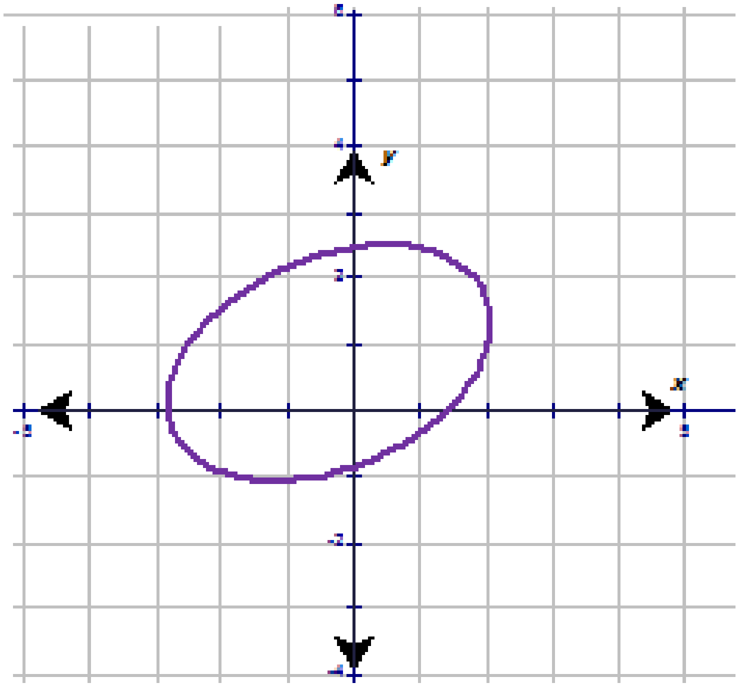
The ***Vertical Line Test*** is a test for functions. If you can take your pencil and draw a straight vertical line through any part of the graph, and the pencil hits the graph more than once, the graph is not a function.

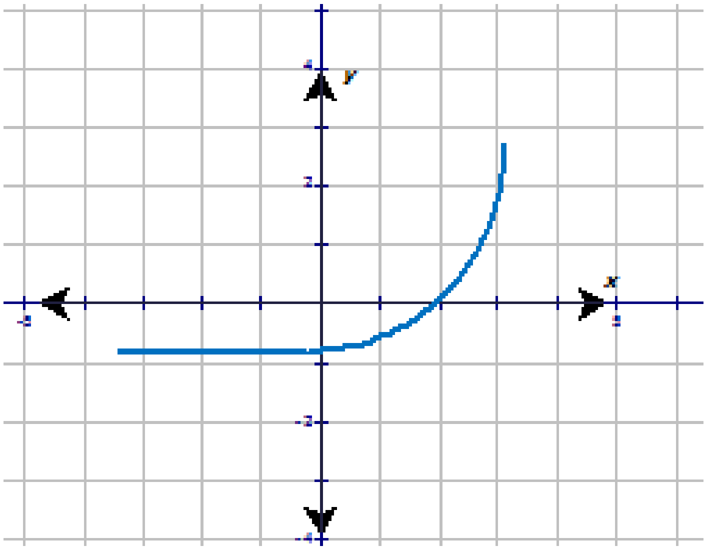
Guided Practice

1. Is the following a representation of a function? Explain.

$$s = \{(1,2),(2,2),(3,2),(4,2)\}$$

2. Which of the following relations represent a function? Explain.



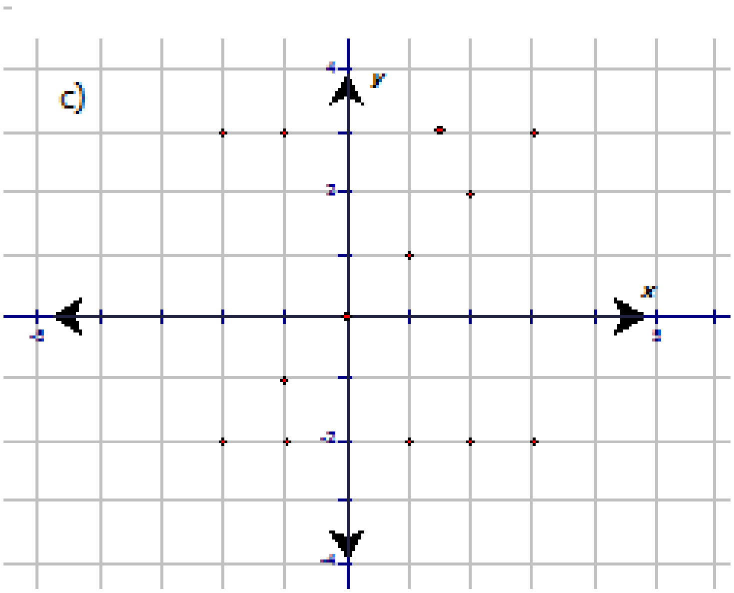


3. Which of the following relations represent a function? Explain.

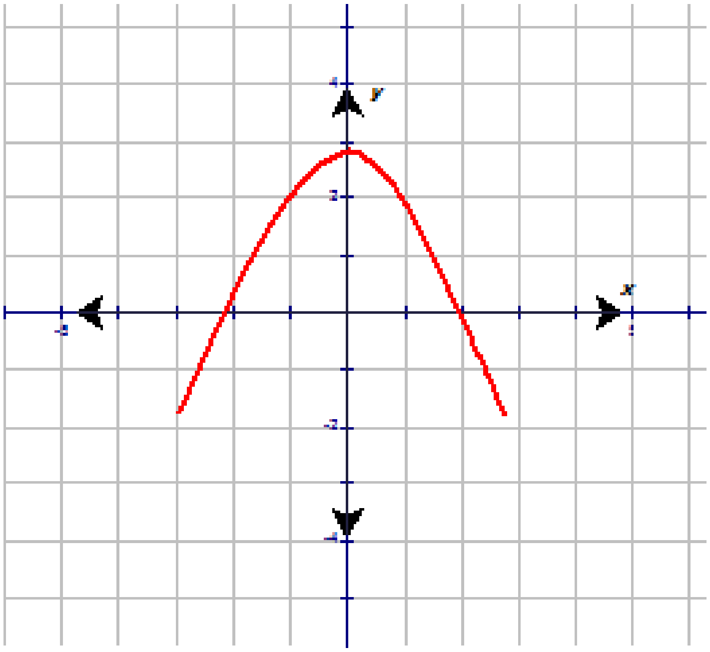
a)

x	2	4	6	8	10	12
y	3	7	11	15	19	23

b)



c)

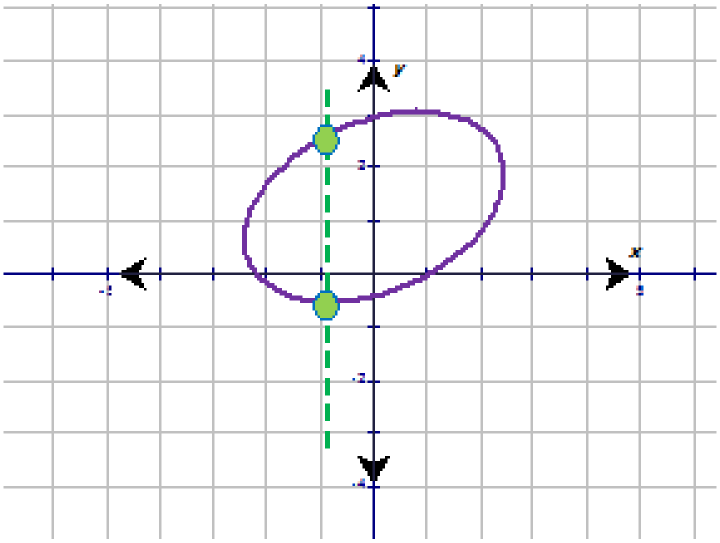


Answers:

1. $s = \{(1,2),(2,2),(3,2),(4,2)\}$

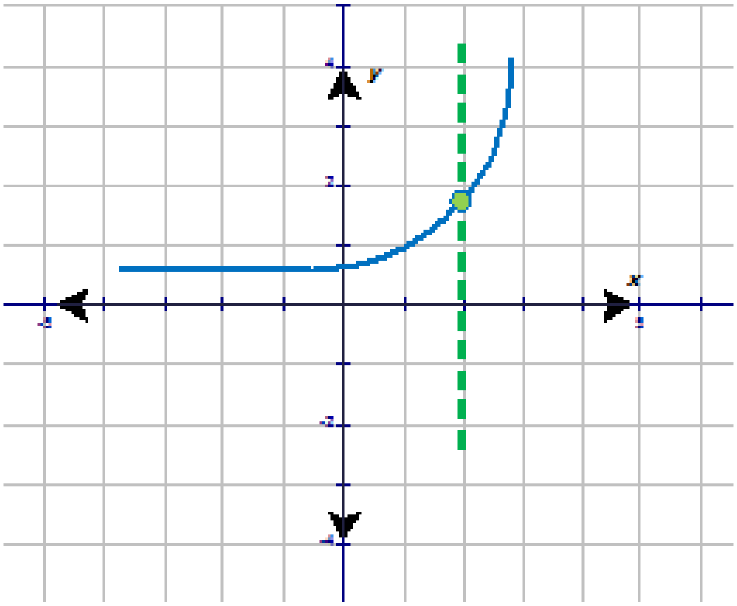
This is a function because there is one output for every input. In other words, if you think of these points as coordinate points (x, y), there is only one value for y given for every value of x.

2. a)



Since the vertical line hit the graph more than once (indicated by the two green circles), the graph ***does not*** represent a function.

b)



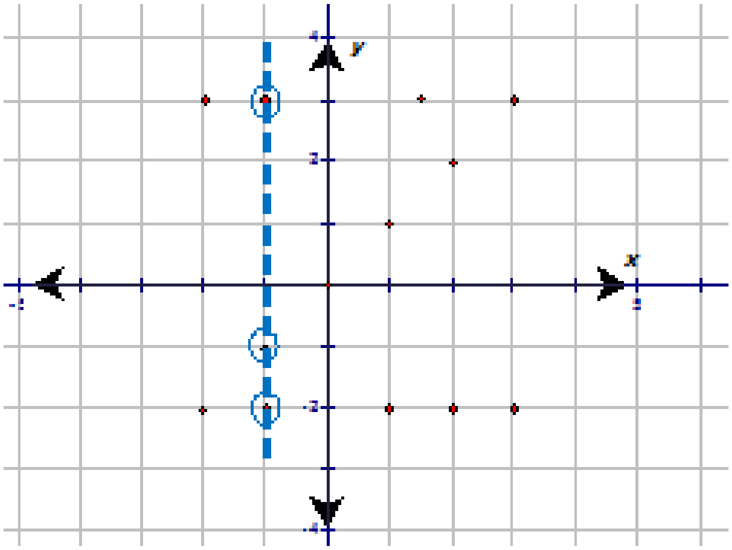
Since the vertical line hit the graph only once (indicated by the one green dot), the graph ***does*** represent a function.

3. a)

x	2	4	6	8	10	12
y	3	7	11	15	19	23

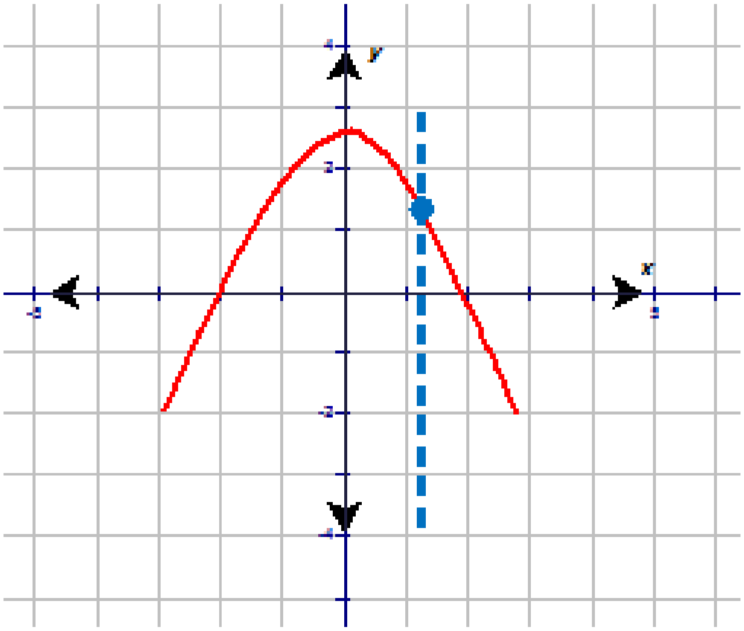
This ***is a function*** because there is only one output for a given input.

b)



Since the vertical line hit the graph more than once (indicated by the three blue circles), the graph ***does not*** represent a function.

c)

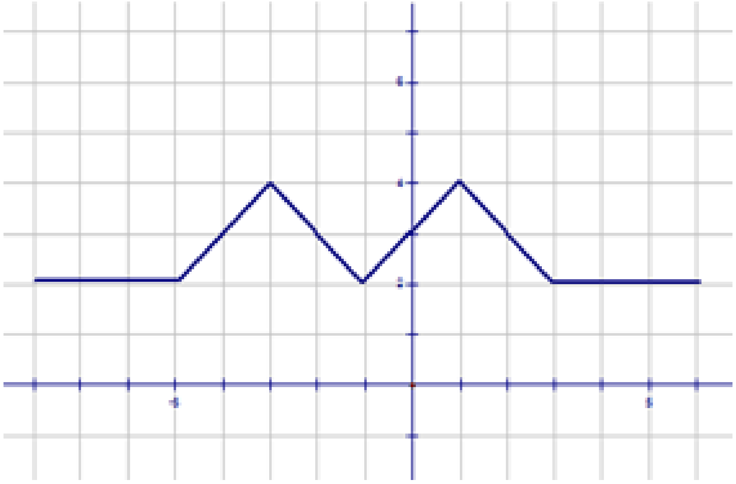


Since the vertical line hit the graph only once (indicated by the one blue dot), the graph ***does*** represent a function.

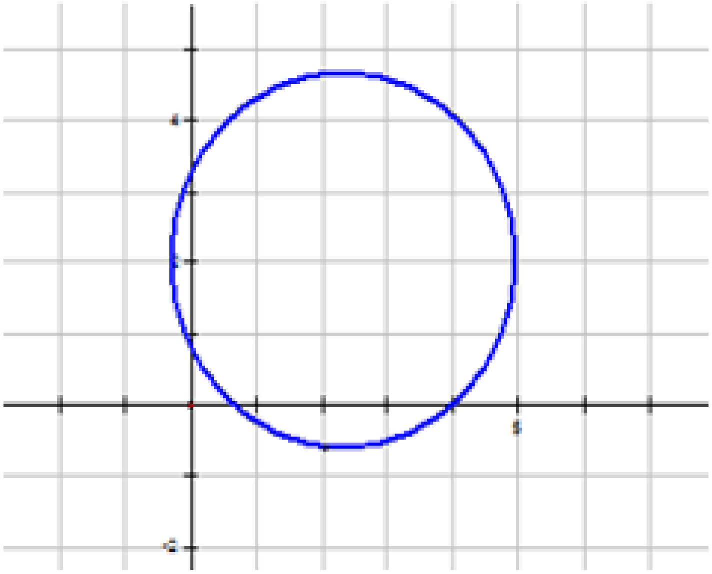
Practice

Determine whether or not each relation is a function. Explain your reasoning.

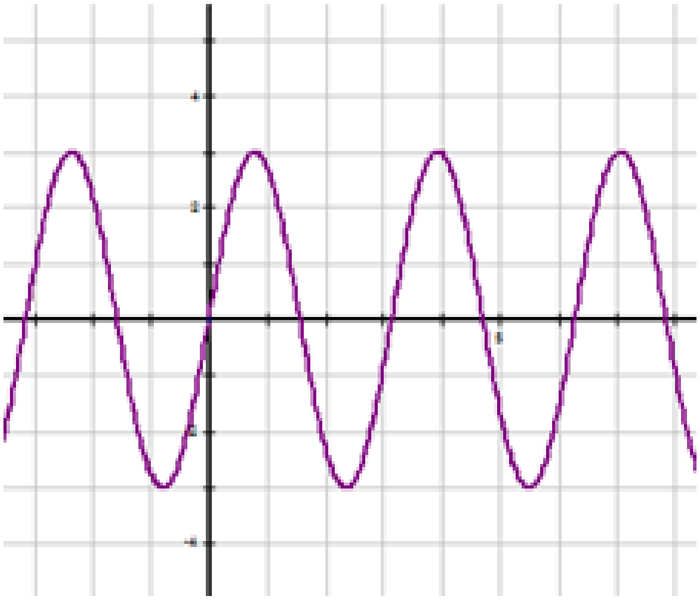
1.



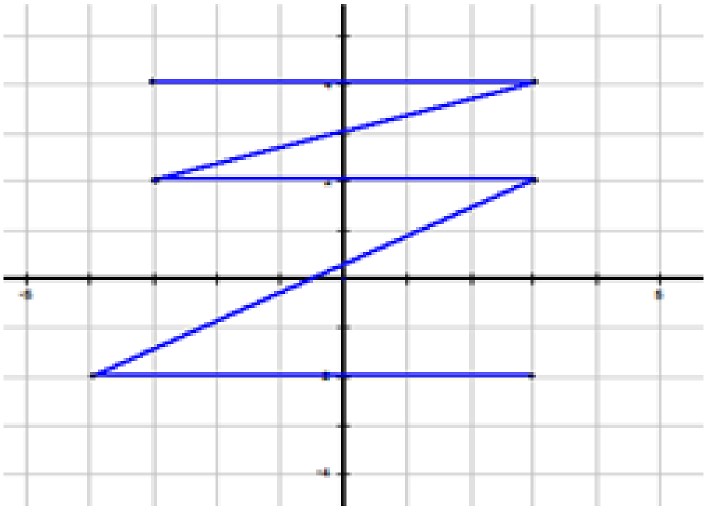
2.



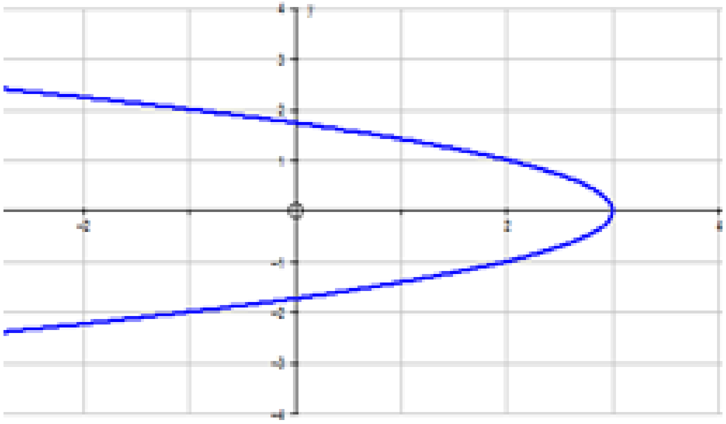
3.



4.



5.



Which of the following relations represent a function? Explain.

6.

X	2	3	2	5
Y	3	-1	5	-4

7.

X	4	2	6	-1
Y	2	4	-3	5

8.

X	1	2	3	4
Y	5	8	5	8

9.

X	-6	-5	-4	-3
Y	4	4	4	4

10.

X	-2	0	-2	4
Y	6	4	4	6

Which of the following relations represent a function? Explain.

11. $s = \{(-3,3),(-2,-2),(-1,-1),(0,0),(1,1),(2,2),(3,3)\}$
12. $s = \{(1,1),(1,2),(1,3),(1,4),(1,5)\}$
13. $s = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$
14. $s = \{(-3,9),(-2,4),(-1,1),(1,1),(2,4)\}$
15. $s = \{(3,-3),(2,-2),(1,-1),(0,0),(-1,1),(-2,2)\}$

3.3 Function Notation

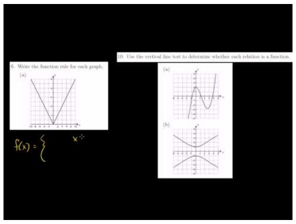
Here you'll learn how to use function notation when working with functions.

Suppose the value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$.

- Can you determine the value of $V(4)$ and explain what the solution means in the context of this problem?
- Can you determine the value of t when $V(t) = 525$ and explain what this situation represents?
- What was the original cost of the digital camera?

Watch This

Khan Academy Functions as Graphs



MEDIA

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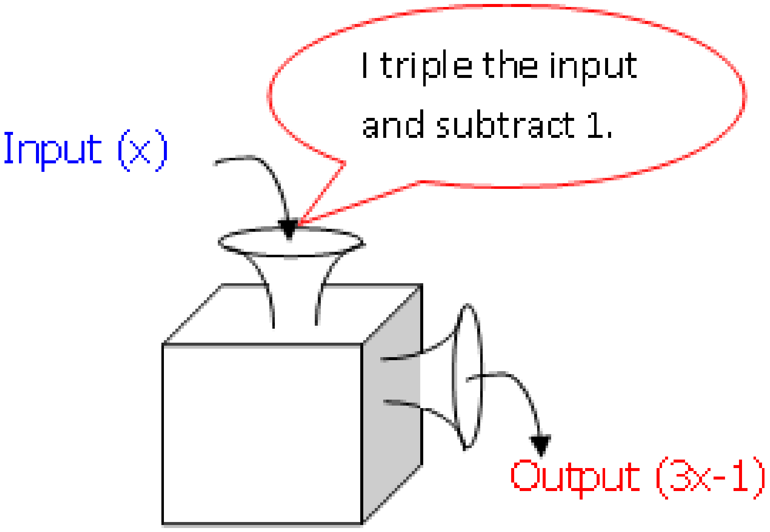
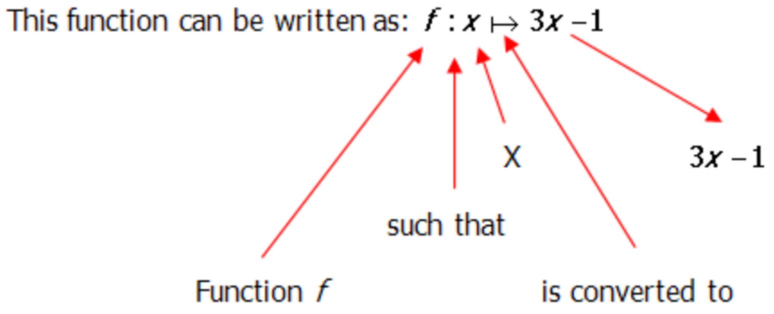
Guidance

A function machine shows how a function responds to an input. If I triple the input and subtract one, the machine will convert x into $3x - 1$. So, for example, if the function is named f, and 3 is fed into the machine, $3(3) - 1 = 8$ comes out.





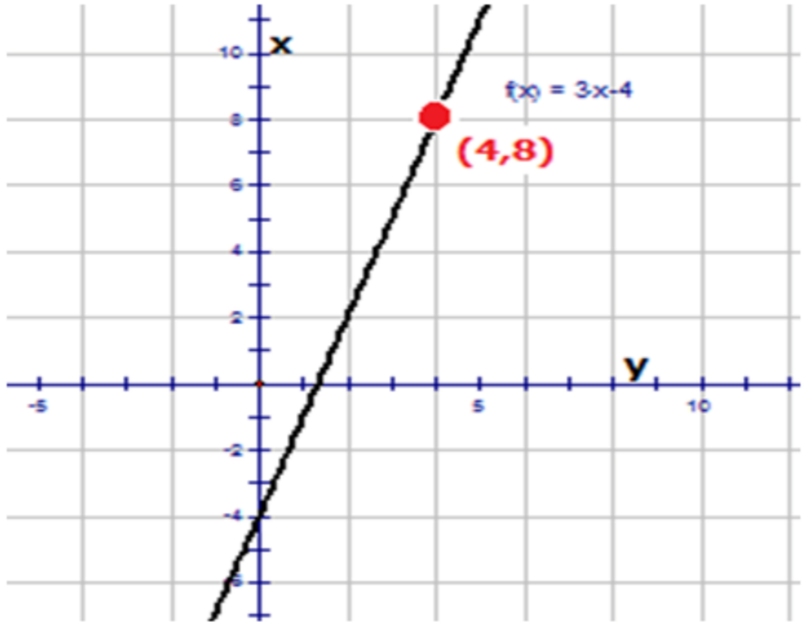
When naming a function the symbol $f(x)$ is often used. The symbol $f(x)$ is pronounced as "f of x." This means that the equation is a function that is written in terms of the variable x. An example of such a function is $f(x) = 3x + 4$. Functions can also be written using a letter other than f and a variable other than x. For example, $v(t) = 2t^2 - 5$ and $d(h) = 4h - 3$. In addition to representing a function as an equation, you can also represent a function:

- As a graph
- As ordered pairs
- As a table of values
- As an arrow or mapping diagram

When a function is represented as an equation, an ordered pair can be determined by evaluating various values of the assigned variable. Suppose $f(x) = 3x - 4$. To calculate $f(4)$, substitute:

$$\begin{aligned} f(4) &= 3(4) - 4 \\ f(4) &= 12 - 4 \\ f(4) &= 8 \end{aligned}$$

Graphically, if $f(4) = 8$, this means that the point (4, 8) is a point on the graph of the line.



Example A

If $f(x) = x^2 + 2x + 5$ find.

a) $f(2)$

b) $f(-7)$

c) $f(1.4)$

Solution:

To determine the value of the function for the assigned values of the variable, substitute the values into the function.

$$\begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \qquad \downarrow \qquad \searrow \\ f(2) = (2)^2 + 2(2) + 5 \\ f(2) = 4 + 4 + 5 \\ \boxed{f(2) = 13} \end{array}$$

$$\begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \qquad \downarrow \qquad \searrow \\ f(-7) = (-7)^2 + 2(-7) + 5 \\ f(-7) = 49 - 14 + 5 \\ \boxed{f(-7) = 40} \end{array}$$

$$\begin{array}{l} f(x) = x^2 + 2x + 5 \\ \downarrow \qquad \downarrow \qquad \searrow \\ f(1.4) = (1.4)^2 + 2(1.4) + 5 \\ f(1.4) = 1.96 + 2.8 + 5 \\ \boxed{f(1.4) = 9.76} \end{array}$$

Example B

Functions can also be represented as mapping rules. If $g : x \rightarrow 5 - 2x$ find the following in simplest form:

a) $g(y)$

b) $g(y-3)$

c) $g(2y)$

Solution:

a) $g(y) = 5 - 2y$

b) $g(y-3) = 5 - 2(y-3) = 5 - 2y + 6 = 11 - 2y$

c) $g(2y) = 5 - 2(2y) = 5 - 4y$

Example C

Let $P(a) = \frac{2a-3}{a+2}$.

a) Evaluate

i) $P(0)$
ii) $P(1)$
iii) $P\left(-\frac{1}{2}\right)$

b) Find a value of 'a' where $P(a)$ does not exist.

c) Find $P(a-2)$ in simplest form

d) Find 'a' if $P(a) = -5$

Solution:

a)

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
P(0) &= \frac{2(0)-3}{(0)+2} \\
\boxed{P(0) = \frac{-3}{2}}
\end{aligned}$$

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
P(1) &= \frac{2(1)-3}{(1)+2} \\
P(1) &= \frac{2-3}{1+2} \\
\boxed{P(1) = \frac{-1}{3}}
\end{aligned}$$

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
P\left(-\frac{1}{2}\right) &= \frac{2\left(-\frac{1}{2}\right)-3}{\left(-\frac{1}{2}\right)+2} \\
P\left(-\frac{1}{2}\right) &= \frac{{}^{1}\cancel{2}\left(-\frac{1}{\cancel{2}}\right)-3}{-\frac{1}{2}+\frac{4}{2}} \\
P\left(-\frac{1}{2}\right) &= \frac{-1-3}{\frac{3}{2}} \\
P\left(-\frac{1}{2}\right) &= -4 \div \frac{3}{2} \\
P\left(-\frac{1}{2}\right) &= -4\left(\frac{2}{3}\right) \\
\boxed{P\left(-\frac{1}{2}\right)} &= \frac{-8}{3}
\end{aligned}$$

b) The function will not exist if the denominator equals zero because division by zero is undefined.

$$\begin{aligned}
a+2 &= 0 \\
a+2-2 &= 0-2 \\
\boxed{a = -2}
\end{aligned}$$

Therefore, if $a = -2$, then $P(a) = \frac{2a-3}{a+2}$ does not exist.

c)

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} \\
P(a-2) &= \frac{2(a-2)-3}{(a-2)+2} && \text{Substitue } a-2 \text{ for } a \\
P(a-2) &= \frac{2a-4-3}{a-2+2} && \text{Remove parentheses} \\
P(a-2) &= \frac{2a-7}{a} && \text{Combine like terms} \\
P(a-2) &= \frac{2\cancel{a}}{\cancel{a}} - \frac{7}{a} && \text{Express the fraction as two separate fractions and reduce.} \\
\boxed{P(a-2) = 2 - \frac{7}{a}}
\end{aligned}$$

d)

$$\begin{aligned}
P(a) &= \frac{2a-3}{a+2} && \\
-5 &= \frac{2a-3}{a+2} && \text{Let } P(a) = -5 \\
-5(a+2) &= \left(\frac{2a-3}{a+2}\right)(a+2) && \text{Multiply both sides by } (a+2) \\
-5a-10 &= \left(\frac{2a-3}{\cancel{a+2}}\right)(\cancel{a+2}) && \text{Simplify} \\
-5a-10 &= 2a-3 && \text{Solve the linear equation} \\
-5a-10-2a &= 2a-2a-3 && \text{Move } 2a \text{ to the left by subtracting} \\
-7a-10 &= -3 && \text{Simplify} \\
-7a-10+10 &= -3+10 && \text{Move 10 to the right side by addition} \\
-7a &= 7 && \text{Simplify} \\
\frac{-7a}{-7} &= \frac{7}{-7} && \text{Divide both sides by -7 to solve for } a. \\
&\boxed{a = -1} &&
\end{aligned}$$

Concept Problem Revisited

The value V of a digital camera t years after it was bought is represented by the function $V(t) = 875 - 50t$

- Determine the value of $V(4)$ and explain what the solution means to this problem.
- Determine the value of t when $V(t) = 525$ and explain what this situation represents.
- What was the original cost of the digital camera?

Solution:

- The camera is valued at \$675, 4 years after it was purchased.

$$\begin{aligned}
V(t) &= 875 - 50t \\
V(4) &= 875 - 50(4) \\
V(4) &= 875 - 200 \\
&\boxed{V(4) = \$675}
\end{aligned}$$

- The digital camera has a value of \$525, 7 years after it was purchased.

$$\begin{aligned}
V(t) &= 875 - 50t && \text{Let } V(t) = 525 \\
525 &= 875 - 50t && \text{Solve the equation} \\
525 - 875 &= 875 - 875 - 50t && \\
-350 &= -50t && \\
\frac{-350}{-50} &= \frac{-50t}{-50} && \\
&\boxed{7 = t} &&
\end{aligned}$$

- The original cost of the camera was \$875.

$$\begin{aligned} V(t) &= 875-50t && \text{Let } t=0. \\ V(0) &= 875-50(0) \\ V(0) &= 875-0 \\ &\boxed{V(0)=\$875} \end{aligned}$$

Vocabulary

Function

A ***function*** is a set of ordered pairs (x,y) that shows a relationship where there is only one output for every input. In other words, for every value of x, there is only one value for y.

Guided Practice

1. If $f(x)=3x^2-4x+6$ find:

i) $f(-3)$
ii) $f(a-2)$

2. If $f(m)=\frac{m+3}{2m-5}$ find 'm' if $f(m)=\frac{12}{13}$

3. The emergency brake cable in a truck parked on a steep hill breaks and the truck rolls down the hill. The distance in feet, d, that the truck rolls is represented by the function $d=f(t)=0.5t^2$.

i) How far will the truck roll after 9 seconds?
ii) How long will it take the truck to hit a tree which is at the bottom of the hill 600 feet away? *Round your answer to the nearest second.*

Answers:

1. $f(x)=3x^2-4x+6$

i)

$$\begin{aligned} f(x) &= 3x^2-4x+6 && \text{Substitute } (-3) \text{ for } x \text{ in the function.} \\ f(-3) &= 3(-3)^2-4(-3)+6 && \text{Perform the indicated operations.} \\ f(-3) &= 3(9)+12+6 && \text{Simplify} \\ f(-3) &= 27+12+6 \\ f(-3) &= 45 \\ &\boxed{f(-3)=45} \end{aligned}$$

ii)

$$\begin{aligned}
f(x) &= 3x^2-4x+6 \\
f(a-2) &= 3(a-2)^2-4(a-2)+6 && \text{Write } (a-2)^2 \text{ in expanded form.} \\
f(a-2) &= 3(a-2)(a-2)-4(a-2)+6 && \text{Perform the indicated operations.} \\
f(a-2) &= (3a-6)(a-2)-4(a-2)+6 \\
f(a-2) &= 3a^2-6a-6a+12-4a+8+6 && \text{Simplify} \\
f(a-2) &= 3a^2-16a+26 \\
&\boxed{f(a-2) = 3a^2-16a+26}
\end{aligned}$$

2.

$$\begin{aligned}
f(m) &= \frac{m+3}{2m-5} \\
\frac{12}{13} &= \frac{m+3}{2m-5} && \text{Solve the equation for } m. \\
(13)(2m-5)\frac{12}{13} &= (13)(2m-5)\frac{m+3}{2m-5} \\
\cancel{(13)}(2m-5)\frac{12}{\cancel{13}} &= (13)\cancel{(2m-5)}\frac{m+3}{\cancel{2m-5}} \\
(2m-5)12 &= (13)m+3 \\
24m-60 &= 13m+39 \\
24m-60+60 &= 13m+39+60 \\
24m &= 13m+99 \\
24m-13m &= 13m-13m+99 \\
11m &= 99 \\
\frac{11m}{11} &= \frac{99}{11} \\
\frac{\cancel{11}m}{\cancel{11}} &= \frac{\overset{9}{\cancel{99}}}{\cancel{11}} \\
&\boxed{m=9}
\end{aligned}$$

3. $d = f(t) = 0.5^2$

i)

$$\begin{aligned}
d = f(t) &= 0.5^2 && \text{Substitute 9 for } t. \\
f(9) &= 0.5(9)^2 && \text{Perform the indicated operations.} \\
f(9) &= 0.5(81) \\
&\boxed{f(9) = 40.5\ feet}
\end{aligned}$$

After 9 seconds, the truck will roll 40.5 feet.

ii)

$$
\begin{aligned}
d = f(t) &= 0.5t^2 && \text{Substitute 600 for } d.\\
600 &= 0.5t^2 && \text{Solve for } t.\\
\frac{600}{0.5} &= \frac{0.5t^2}{0.5}\\
\frac{\overset{1200}{\cancel{600}}}{\cancel{0.5}} &= \frac{\cancel{0.5}t^2}{\cancel{0.5}}\\
1200 &= t^2\\
\sqrt{1200} &= \sqrt{t^2}\\
&\boxed{34.64 \ seconds \approx t}
\end{aligned}
$$

The truck will hit the tree in approximately 35 seconds.

Practice

If $g(x) = 4x^2 - 3x + 2$, find expressions for the following:

1. $g(a)$
2. $g(a-1)$
3. $g(a+2)$
4. $g(2a)$
5. $g(-a)$

If $f(y) = 5y - 3$, determine the value of 'y' when:

6. $f(y) = 7$
7. $f(y) = -1$
8. $f(y) = -3$
9. $f(y) = 6$
10. $f(y) = -8$

The value of a Bobby Orr rookie card n years after its purchase is $V(n) = 520 + 28n$.

11. Determine the value of $V(6)$ and explain what the solution means.
12. Determine the value of n when $V(n) = 744$ and explain what this situation represents.
13. Determine the original price of the card.

Let $f(x) = \frac{3x}{x+2}$.

14. When is $f(x)$ undefined?
15. For what value of x does $f(x) = 2.4$?

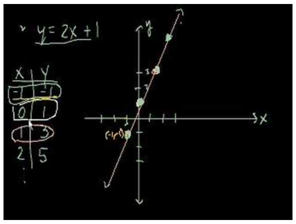
3.4 Graphs of Linear Functions from Tables

Here you will learn how to create a graph of a linear function by using a table.

Bonita will be celebrating her sixteenth birthday next month. Her parents would like to give her a surprise party at the local pool. To rent the pool for a private party costs \$100 plus \$55.00 for each hour the pool is rented. Write a linear function to represent the cost of the pool party and list five prices from which her parents can choose.

Watch This

Khan Academy Graphing Lines 1



MEDIA

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Guidance

One way to graph a linear function is to first create a table of points that work with the function and so must be on the graph of the function. A linear function will always result in a graph that is a straight line.

To create a table, substitute values for x into the function (you can choose values for x) and use the function to calculate the corresponding value for y. Each pair of values is one point on the graph. It is easier to create the table if you first solve the equation for y.

You can also use a graphing calculator to create a table of values and graph of the function. This will be explored in Example B.

Example A

Complete the table of values for the linear function $3x+2y=-6$.

Solution: Before completing the table of values, solve the given function in terms of 'y'. This step is not necessary, but it does simplify the calculations.

$$\begin{aligned}3x+2y&=-6\\3x-3x+2y&=-3x-6\\2y&=-3x-6\\\frac{2y}{2}&=\frac{-3x}{2}-\frac{6}{2}\\\boxed{y=\frac{-3x}{2}-3}\end{aligned}$$

Now pick a few values for x and substitute them into the equation to find the corresponding value for y. Here, pick $x = -4$, $x = 2$, $x = 0$, and $x = 6$.

$$y = \frac{-3x}{2} - 3 \qquad y = \frac{12}{2} - 3$$
$$y = \frac{-3(-4)}{2} - 3 \qquad y = 6 - 3$$
$$\boxed{y = 3}$$

$$y = \frac{-3x}{2} - 3 \qquad y = 0 - 3$$
$$y = \frac{-3(0)}{2} - 3 \qquad \boxed{y = -3}$$

$$y = \frac{-3x}{2} - 3 \qquad y = \frac{-6}{2} - 3$$
$$y = \frac{-3(2)}{2} - 3 \qquad y = -3 - 3$$
$$\boxed{y = -6}$$

$$y = \frac{-3x}{2} - 3 \qquad y = \frac{-18}{2} - 3$$
$$y = \frac{-3(6)}{2} - 3 \qquad y = -9 - 3$$
$$\boxed{y = -12}$$

Here is the table that shows the x and y pairs.

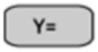
TABLE 3.4:

$y = -\frac{3}{2}x - 3$	
X	Y
-4	3
0	-3
2	-6
6	-12

Example B

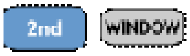
Use technology to create a table of values for the linear function $f(x) = -\frac{1}{2}x + 4$.

Solution:

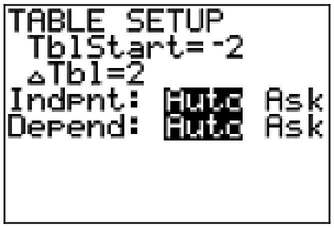


Type in the linear function

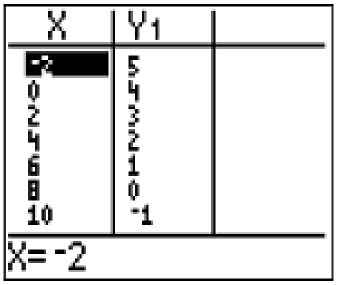
```
 Plot1  Plot2  Plot3
\Y1■-(1/2)X+4
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



This will bring you to Table Set.



```
TABLE SETUP
 TblStart=-2
 ΔTbl=2
Indpnt: Auto Ask
Depend: Auto Ask
```



X	Y1
-2	5
0	4
2	3
4	2
6	1
8	0
10	-1

X=-2

When the table is set up, you choose the beginning number as well as the pattern for the numbers in the table. In this table, the beginning value for 'x' was –2 and the difference between each number was +2.

Example C

Complete the table of values for $x-2y=4$, and use those values to graph the function.

Solution: First solve the equation for y.

$$\begin{aligned} x-2y&=4\\ -2y&=-x+4\\ \frac{-2y}{-2}&=\frac{-x}{-2}+\frac{4}{-2}\\ y&=\frac{1}{2}x-2 \end{aligned}$$

Next choose values for x to help make the table. Remember that you can choose any values for x.

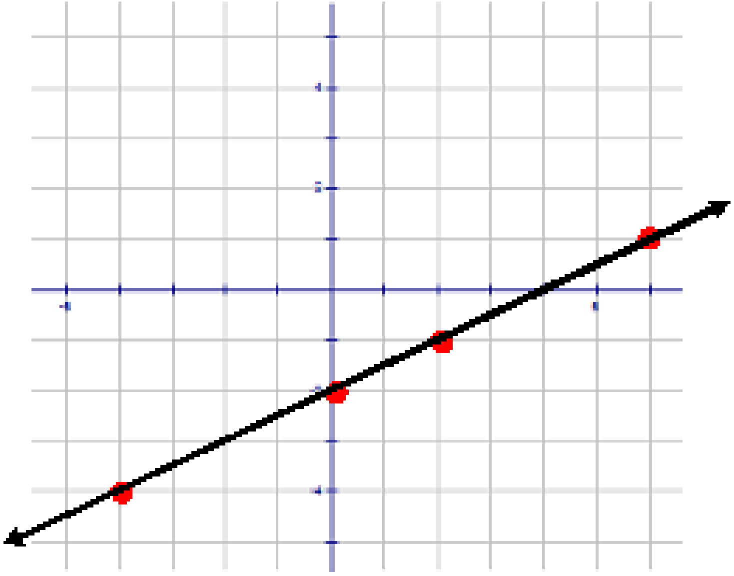
$$\begin{aligned} y&=\frac{1}{2}x-2\\ y&=\frac{1}{2}(-4)-2\\ y&=-2-2\\ &\boxed{y=-4} \end{aligned} \qquad \begin{aligned} y&=\frac{1}{2}x-2\\ y&=\frac{1}{2}(0)-2\\ y&=0-2\\ &\boxed{y=-2} \end{aligned} \qquad \begin{aligned} y&=\frac{1}{2}x-2\\ y&=\frac{1}{2}(2)-2\\ y&=1-2\\ &\boxed{y=-1} \end{aligned} \qquad \begin{aligned} y&=\frac{1}{2}x-2\\ y&=\frac{1}{2}(6)-2\\ y&=3-2\\ &\boxed{y=1} \end{aligned}$$

Next make the table.

TABLE 3.5:

$y=\frac{1}{2}x-2$	
X	Y
-4	-4
0	-2
2	-1
6	1

Last, plot the points from the table and connect to make the graph. You connect the points because there are more than just the four points in the table that work with the function and appear on the graph.



Concept Problem Revisited

Bonita will be celebrating her sixteenth birthday next month. Her parents would like to give her a surprise party at the local pool. To rent the pool for a private party costs \$100 plus \$55.00 for each hour the pool is rented. Write a linear function to represent the cost of the pool party and list five prices from which her parents can choose.

The cost of renting the pool is \$100. This amount is a fee that must be paid to rent the pool. In addition, Bonita's parents will also have to pay \$55.00 for each hour the pool is rented. Therefore, the linear function to represent this situation is $y = 55x + 100$ where 'y' represents the cost in dollars and 'x' represents the time, in hours, that the pool is rented.

$y = 55x + 100$ - To determine five options for her parents, replace 'x' with the values 1 to 5 and calculate the cost for each of these hours.

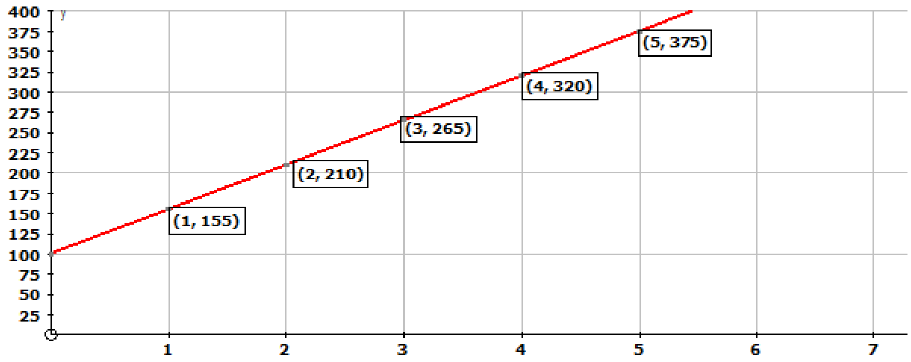
$$\begin{array}{lllll}
y = 55x + 100 & y = 55x + 100 & y = 55x + 100 & y = 55x + 100 & y = 55x + 100 \\
y = 55(1) + 100 & y = 55(2) + 100 & y = 55(3) + 100 & y = 55(4) + 100 & y = 55(5) + 100 \\
\boxed{y = \$155} & \boxed{y = \$210} & \boxed{y = \$265} & \boxed{y = \$320} & \boxed{y = \$375}
\end{array}$$

These results can now be represented in a table of values:

$X(hours)$	1	2	3	4	5
$Y(Cost)$	\$155	\$210	\$265	\$320	\$375

The values in the table represent the coordinates of points that are located on the graph of $y = 55x + 100$.

$$(1,155);(2,210);(3,265);(4,320);(5,375)$$



Bonita's parents can use the table of values and/or the graph to make their decision.

Vocabulary

Linear Function

The ***linear function*** is a relation between two variables, usually x and y, in which each value of the independent variable (x) is mapped to one and only one value of the dependent variable (y).

Guided Practice

1. Complete the following table of values for the linear function $3x-2y=-12$

TABLE 3.6:

$3x-2y=-12$	
X	Y
-6	
-4	
0	
6	

2. Use technology to complete a table of values for the linear function $2x-y=-8$. Use the coordinates to draw the graph.

3. A local telephone company charges a monthly fee of \$25.00 plus \$0.09 per minute for calls within the United States. If Sam talks for 200 minutes in one month, calculate the cost of his telephone bill.

Answers:

1. Solve the equation for 'y' to get $y=\frac{3}{2}x+6$.

Substitute the given values for 'x' into the function.

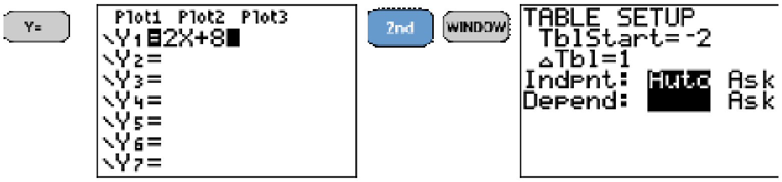
$$\begin{array}{llll} y=\dfrac{3}{2}x+6 & y=\dfrac{3}{2}x+6 & y=\dfrac{3}{2}x+6 & y=\dfrac{3}{2}x+6 \\ y=\dfrac{3}{2}(-6)+6 & y=\dfrac{3}{2}(-4)+6 & y=\dfrac{3}{2}(0)+6 & y=\dfrac{3}{2}(6)+6 \\ y=-9+6 & y=-6+6 & y=0+6 & y=9+6 \\ \boxed{y=-3} & \boxed{y=0} & \boxed{y=6} & \boxed{y=15} \end{array}$$

Complete the table.

TABLE 3.7:

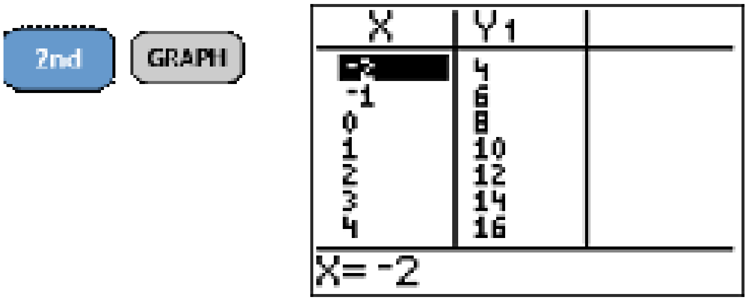
$3x-2y=-12$	
X	Y
-6	-3
-4	0
0	6
6	15

2. To enter the function into the calculator, it must be in the form $y =$ ________. Solve the function for 'y' to get $y = 2x+8$. Then, use your calculator.



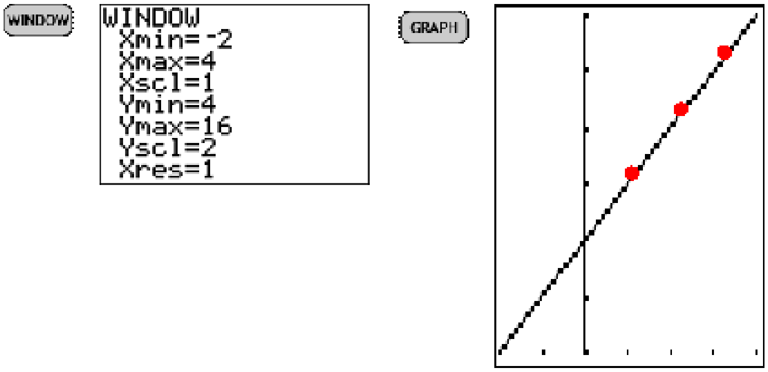
```
Plot1 Plot2 Plot3
\Y1=2X+8
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
TABLE SETUP
 TblStart=-2
 ΔTbl=1
Indpnt: Auto Ask
Depend: Auto Ask
```



```
 X    Y1
-2    4
-1    6
 0    8
 1    10
 2    12
 3    14
 4    16
X= -2
```

The graph can also be done using technology. The table can be used to set the window.



```
WINDOW
 Xmin=-2
 Xmax=4
 Xscl=1
 Ymin=4
 Ymax=16
 Yscl=2
 Xres=1
```

3. The function $y = 0.09x+25$ represents the word problem.

$y = 0.09(200)+25$ Substitute the time of 200 minutes for the variable x.

$y = \$43.00$

The cost of Sam's telephone bill is $43.00.

Practice

Solve each of the following linear functions for 'y'.

1. $2x-3y=18$
2. $4x-2y=10$
3. $3x-y=8$
4. $5x+3y=-12$
5. $3x-2y-2=0$

For each of the following linear functions, create a table of values that contains four coordinates.

6. $y=-4x+5$
7. $5x+3y=15$
8. $4x-3y=6$
9. $2x-2y+2=0$
10. $2x-3y=9$

For each of the linear functions, complete the table of values and use the values to draw the graph.

11. $y=-2x+1$

x	-3	0	1	5
y				

12. $x=2y-3$

x	-4	0	2	6
y				

13. $3x+2y=8$

x	-6	-2	0	4
y				

14. $4(y-1)=12x-7$

x	-2	0	3	7
y				

15. $\frac{1}{2}x+\frac{1}{3}y=6$

x	0	4	6	10
y				

Using technology, create a table of values for each of the following linear functions. Using technology, graph each of the linear functions.

16. $y = -2x + 3$
17. $y = -\frac{1}{2}x - 3$
18. $y = \frac{4}{3}x - 2$

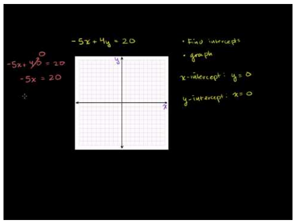
3.5 Graphs of Linear Functions from Intercepts

Here you will learn how to graph a linear function by first finding the x and y intercepts.

What are the intercepts of $4x+2y=8$? How could you use the intercepts to quickly graph the function?

Watch This

Khan Academy X and Y Intercepts



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Guidance

To graph a linear function, you need to plot only two points. These points can then be lined up with a straight edge and joined to graph the straight line. While any two points can be used to graph a linear function, two points in particular that can be used are the x-intercept and the y-intercept. Graphing a linear function by plotting the $x-$ and $y-$ intercepts is often referred to as the ***intercept method.***

The x-intercept is where the graph crosses the x-axis. Its coordinates are $(x,0)$. Because all x-intercepts have a y-coordinate equal to 0, you can find an x-intercept by substituting 0 for y in the equation and solving for x.

The y-intercept is where the graph crosses the y-axis. Its coordinates are $(0,y)$. Because all y-intercepts have a x-coordinate equal to 0, you can find an y-intercept by substituting 0 for x in the equation and solving for y.

Example A

Identify the $x-$ and y-intercepts for each line.

(a) $2x+y-6=0$

(b) $\frac{1}{2}x-4y=4$

Solution:

(a)

Let $y = 0$. Solve for 'x'.

$2x+y-6=0$

$2x+(0)-6=0$

$2x-6=0$

$2x-6+6=0+6$

$2x=6$

$\frac{2x}{2}=\frac{6}{2}$

$x=3$

The x-intercept is $(3,0)$

Let $x = 0$. Solve for 'y'.

$2x+y-6=0$

$2(0)+y-6=0$

$y-6=0$

$y-6+6=0+6$

$y=6$

The y-intercept is $(0,6)$

(b)

Let $y = 0$. Solve for 'x'.

$\frac{1}{2}x-4y=4$

$\frac{1}{2}x-4(0)=4$

$\frac{1}{2}x-0=4$

$\frac{1}{2}x=4$

$\overset{1}{\cancel{2}}\left(\frac{1}{\cancel{2}}\right)x=2(4)$

$x=8$

The x-intercept is $(8,0)$

Let $x = 0$. Solve for 'y'.

$\frac{1}{2}x-4y=4$

$\frac{1}{2}(0)-4y=4$

$0-4y=4$

$-4y=4$

$\frac{-4y}{-4}=\frac{4}{-4}$

$y=-1$

The y-intercept is $(0,-1)$

Example B

Use the intercept method to graph $2x-3y=-12$.

Solution:

Let $y = 0$. Solve for 'x'.

$2x-3y=-12$

$2x-3(0)=-12$

$2x-0=-12$

$2x=-12$

$\frac{2x}{2}=\frac{-12}{2}$

$x=-6$

The x-intercept is $(-6,0)$

Let $x = 0$. Solve for 'y'.

$2x-3y=-12$

$2(0)-3y=-12$

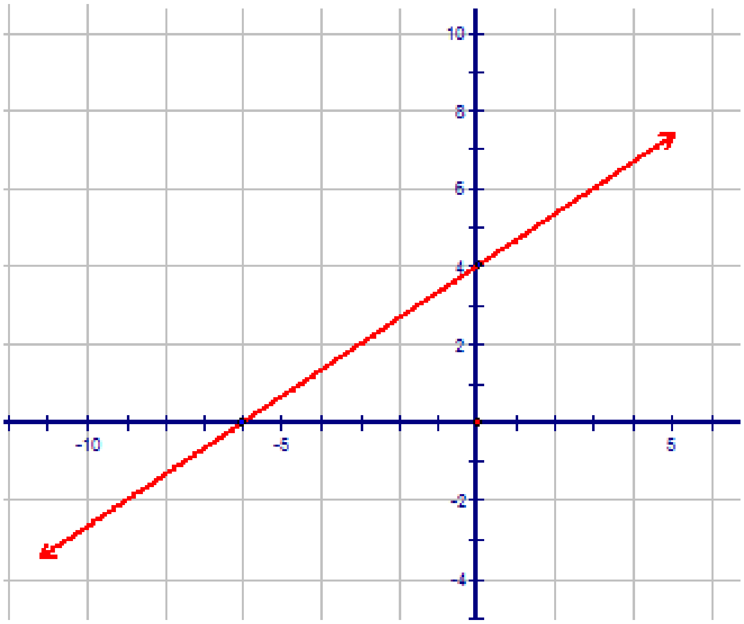
$0-3y=-12$

$-3y=-12$

$\frac{-3y}{-3}=\frac{-12}{-3}$

$y=4$

The y-intercept is $(0,4)$



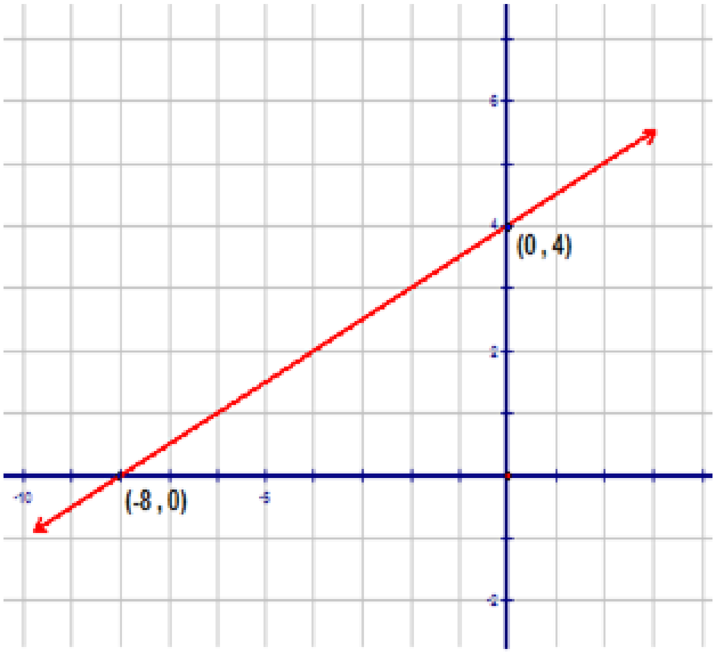
Example C

Use the $x-$ and y-intercepts of the graph to identify the linear function that matches the graph.

a) $y = 2x - 8$

b) $x - 2y + 8 = 0$

c) $2x + y - 8 = 0$



The x-intercept is (–8, 0) and the y-intercept is (0, 4).

Solution: Find the x and y intercepts for each equation and see which matches the graph.

a) x intercept: $0 = 2x - 8 \rightarrow x = 4$

y intercept: $y = 2(0) - 8 \rightarrow y = -8$

b) x intercept: $x - 2(0) + 8 = 0 \rightarrow x = -8$

y intercept: $0 - 2y + 8 = 0 \rightarrow y = 4$

c) x intercept: $2x + 0 - 8 = 0 \rightarrow x = 4$

y intercept: $2(0) + y - 8 = 0 \rightarrow y = 8$

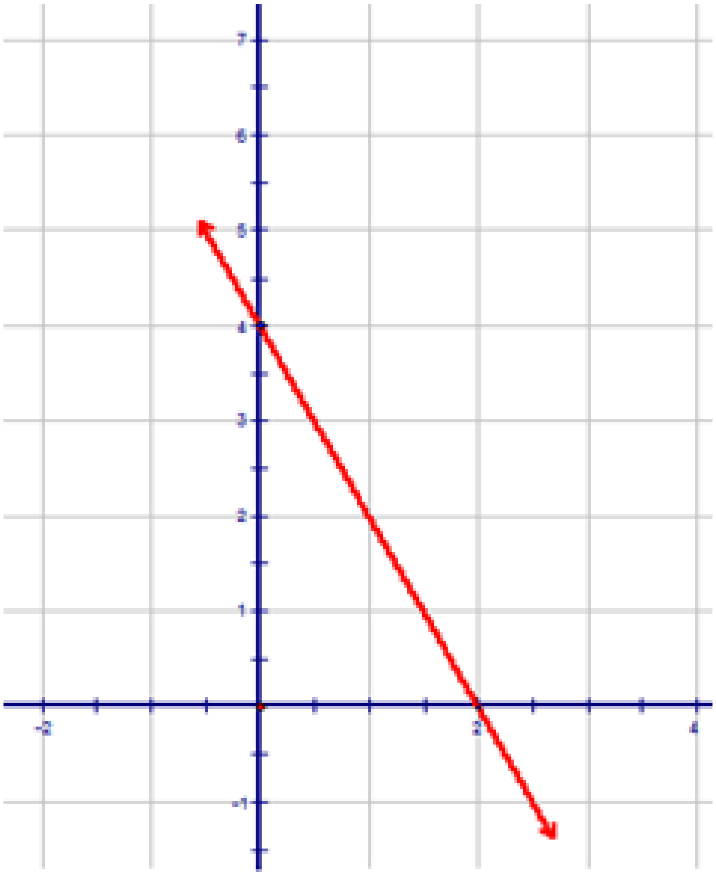
The x and y intercepts match for $x - 2y + 8 = 0$ so this is the equation of the line.

Concept Problem Revisited

The linear function $4x + 2y = 8$ can be graphed by using the intercept method.

To determine the x-intercept, let $y = 0$.	To determine the y-intercept, let $x = 0$.
Solve for 'x'.	Solve for 'y'.
$4x + 2y = 8$	$4x + 2y = 8$
$4x + 2(0) = 8$	$4(0) + 2y = 8$
$4x + 0 = 8$	$0 + 2y = 8$
$4x = 8$	$2y = 8$
$\frac{4x}{4} = \frac{8}{4}$	$\frac{2y}{2} = \frac{8}{2}$
$x = 2$	$y = 4$
The x-intercept is $(2, 0)$	The y-intercept is $(0, 4)$

Plot the x-intercept on the x-axis and the y-intercept on the y-axis. Join the two points with a straight line.



Vocabulary

Intercept Method

The **intercept method** is a way of graphing a linear function by using the coordinates of the $x-$ and y-intercepts. The graph is drawn by plotting these coordinates on the Cartesian plane and joining them with a straight line.

x-intercept

An x-intercept of a relation is the $x-$coordinate of the point where the relation intersects the x-axis.

y-intercept

A y-intercept of a relation is the $y-$coordinate of the point where the relation intersects the y-axis.

Guided Practice

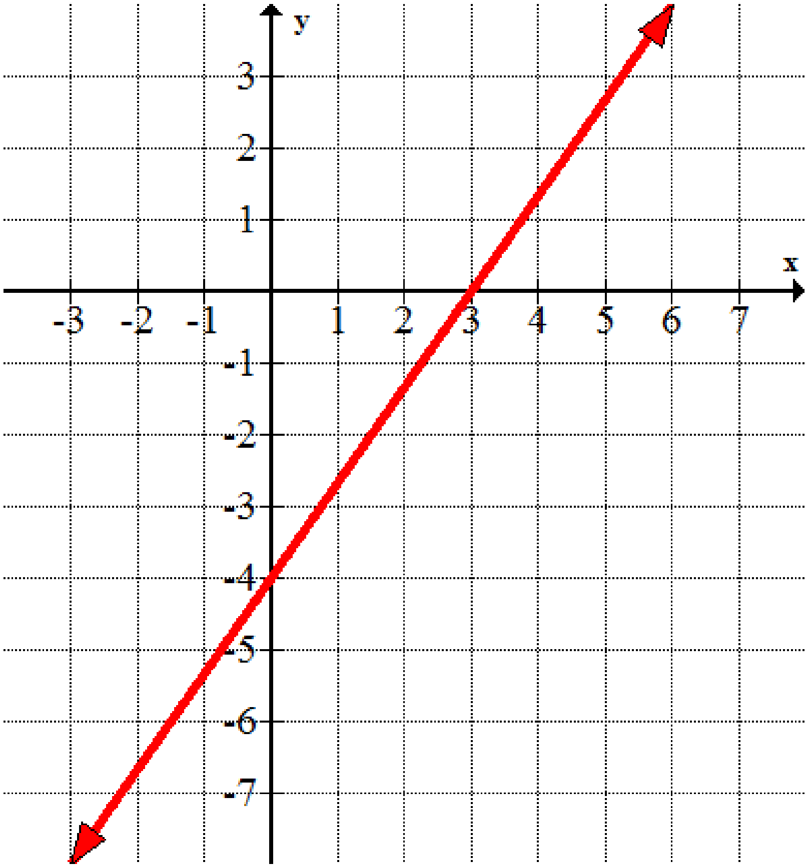
1. Identify the $x-$ and y-intercepts of the following linear functions:

(i) $2(x-3) = y+4$
(ii) $3x+\frac{2}{3}y-3=0$

2. Use the intercept method to graph the following relation:

(i) $5x+2y=-10$

3. Use the $x-$ and y-intercepts of the graph, to match the graph to its function.



(i) $2x+y=6$
(ii) $4x-3y-12=0$
(iii) $5x+3y=15$

Answers:

1. (i)

$$\begin{aligned} 2(x-3) &= y+4 && \text{Simplify the equation} \\ 2(x-3) &= y+4 \\ 2x-6 &= y+4 \\ 2x-6+6 &= y+4+6 \\ 2x &= y+10 && \text{You may leave the function in this form.} \\ 2x-y &= y-y+10 \\ 2x-y &= 10 \end{aligned}$$

If you prefer to have both variables on the same side of the equation, this form may also be used. The choice is your preference.

Let $y = 0$. Solve for x.

$2x - y = 10$

$2x - (0) = 10$

$2x = 10$

$\frac{2x}{2} = \frac{10}{2}$

$x = 5$

The x-intercept is $(5,0)$

Let $x = 0$. Solve for y.

$2x - y = 10$

$2(0) - y = 10$

$0 - y = 10$

$\frac{-y}{-1} = \frac{10}{-1}$

$y = -10$

The y-intercept is $(0,-10)$

(ii)

$$\begin{aligned}
3x + \frac{2}{3}y - 3 &= 0 && \text{Simplify the equation.}\\
3(3x) + 3\left(\frac{2}{3}\right)y - 3(3) &= 3(0) && \text{Multiply each term by 3.}\\
3(3x) + \cancel{3}\left(\frac{2}{\cancel{3}}\right)y - 3(3) &= 3(0)\\
9x + 2y - 9 &= 0\\
9x + 2y - 9 + 9 &= 0 + 9\\
9x + 2y &= 9
\end{aligned}$$

Let $y = 0$. Solve for x.

$9x + 2y = 9$

$9x + 2(0) = 9$

$9x + 0 = 9$

$\frac{9x}{9} = \frac{9}{9}$

$x = 1$

The x-intercept is $(1,0)$

Let $x = 0$. Solve for y.

$9x + 2y = 9$

$9(0) + 2y = 9$

$0 + 2y = 9$

$\frac{2y}{2} = \frac{9}{2}$

$y = 4.5$

The y-intercept is $(0,4.5)$

2.

Let $y = 0$. Solve for x.

$5x + 2y = -10$

$5x + 2(0) = -10$

$5x + 0 = -10$

$\frac{5x}{5} = \frac{-10}{5}$

$x = -2$

The x-intercept is $(-2,0)$

Let $x = 0$. Solve for y.

$5x + 2y = -10$

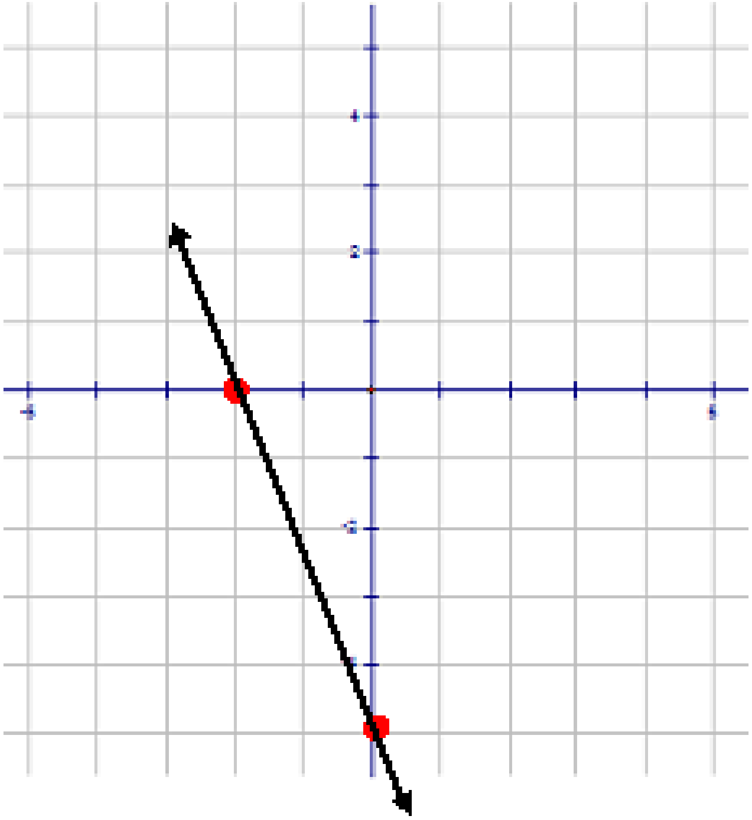
$5(0) + 2y = -10$

$0 + 2y = -10$

$\frac{2y}{2} = \frac{-10}{2}$

$y = -5$

The y-intercept is $(0,-5)$



3. Identify the $x-$ and y-intercepts from the graph.

The x-intercept is (3, 0)

The y-intercept is (0, -4)

Determine the $x-$ and y-intercept for each of the functions. If the intercepts match those of the graph, then the linear function will be the one that matches the graph.

(i)

Let $y = 0$. Solve for x.	Let $x = 0$. Solve for y.
$2x + y = 6$	$2x + y = 6$
$2x + (0) = 6$	$2(0) + y = 6$
$2x = 6$	$0 + y = 6$
$\frac{2x}{2} = \frac{6}{2}$	$y = 6$
$x = 3$	
The x-intercept is $(3, 0)$	The y-intercept is $(0, 6)$
This matches the graph.	This does not match the graph.

$2x + y = 6$ is not the linear function for the graph.

(ii)

Let $y = 0$. Solve for x.

$4x - 3y - 12 = 0$

$4x - 3y - 12 + 12 = 0 + 12$

$4x - 3y = 12$

$4x - 3(0) = 12$

$4x - 0 = 12$

$4x = 12$

$\frac{4x}{4} = \frac{12}{4}$

$x = 3$

The x-intercept is $(3, 0)$

This matches the graph.

Let $x = 0$. Solve for y.

$4x - 3y - 12 = 0$

$4x - 3y - 12 + 12 = 0 + 12$

$4x - 3y = 12$

$4(0) - 3y = 12$

$0 - 3y = 12$

$-3y = 12$

$\frac{-3y}{-3} = \frac{12}{-3}$

$y = -4$

The y-intercept is $(0, -4)$

This matches the graph.

$4x - 3y - 12 = 0$ is the linear function for the graph.

(iii)

Let $y = 0$. Solve for x.

$5x + 3y = 15$

$5x + 3(0) = 15$

$5x + 0 = 15$

$5x = 15$

$\frac{5x}{5} = \frac{15}{5}$

$x = 3$

The x-intercept is $(3, 0)$

This matches the graph.

Let $x = 0$. Solve for y.

$5x + 3y = 15$

$5(0) + 3y = 15$

$0 + 3y = 15$

$3y = 15$

$\frac{3y}{3} = \frac{15}{3}$

$y = 5$

The y-intercept is $(0, 5)$

This does not match the graph.

$5x + 3y = 15$ is not the linear function for the graph.

Practice

For 1-10, complete the following table:

TABLE 3.8:

Function	**x-intercept**	**y-intercept**
$7x - 3y = 21$	1.	2.
$8x - 3y + 24 = 0$	3.	4.
$\frac{x}{4} - \frac{y}{2} = 3$	5.	6.
$7x + 2y - 14 = 0$	7.	8.
$\frac{2}{3}x - \frac{1}{4}y = -2$	9.	10.

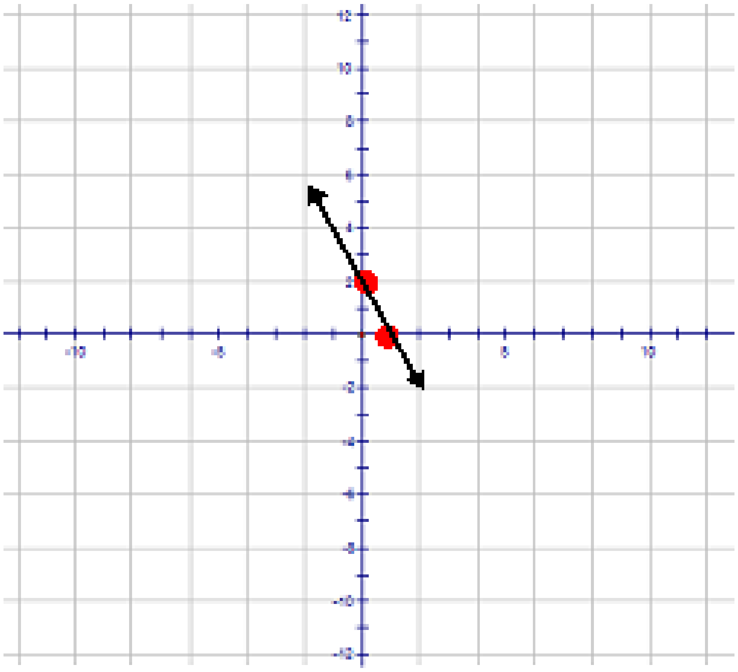
Use the intercept method to graph each of the linear functions in the above table.

11. $7x-3y=21$
12. $8x-3y+24=0$
13. $\frac{x}{4}-\frac{y}{2}=3$
14. $7x+2y-14=0$
15. $\frac{2}{3}x-\frac{1}{4}y=-2$

Use the $x-$ and y-intercepts to match each graph to its function.

a. $7x+5y-35=0$
b. $y=5x+10$
c. $2x+4y+8=0$
d. $2x+y=2$

16.



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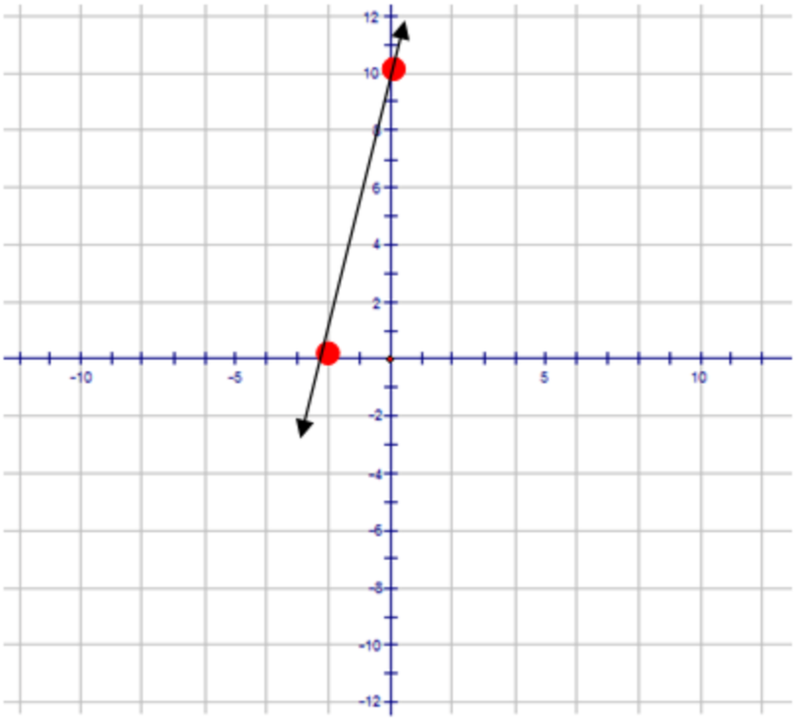

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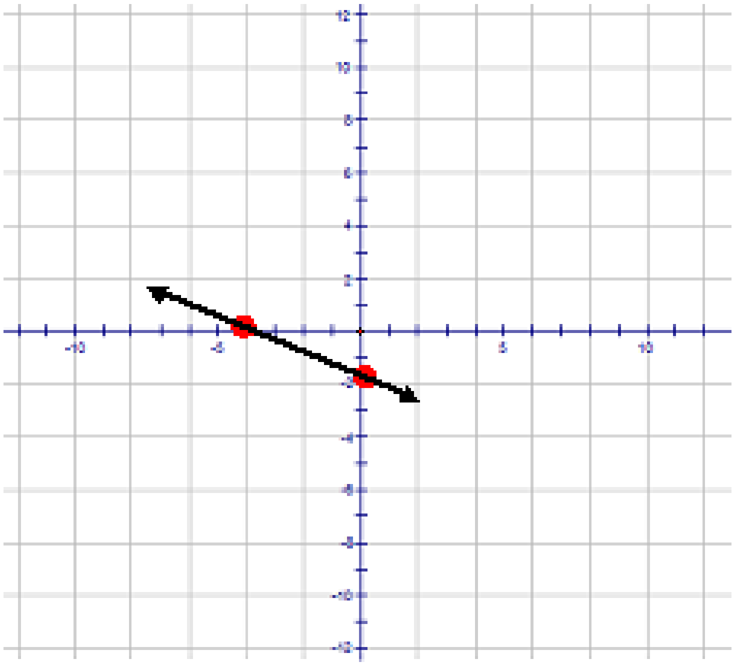

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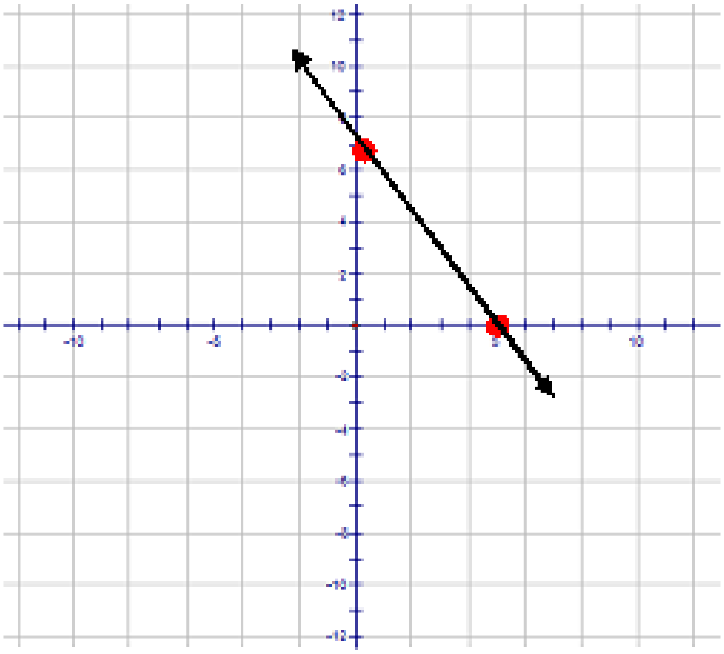
12
10
6
4
2
-2
-4
-6
-8
-10
-12
-10
-5
5
10

3.6 Domain and Range

Here you'll learn how to find the domain and range of a relation.

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.

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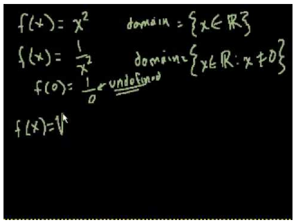
Khan Academy Domain and Range of a Function



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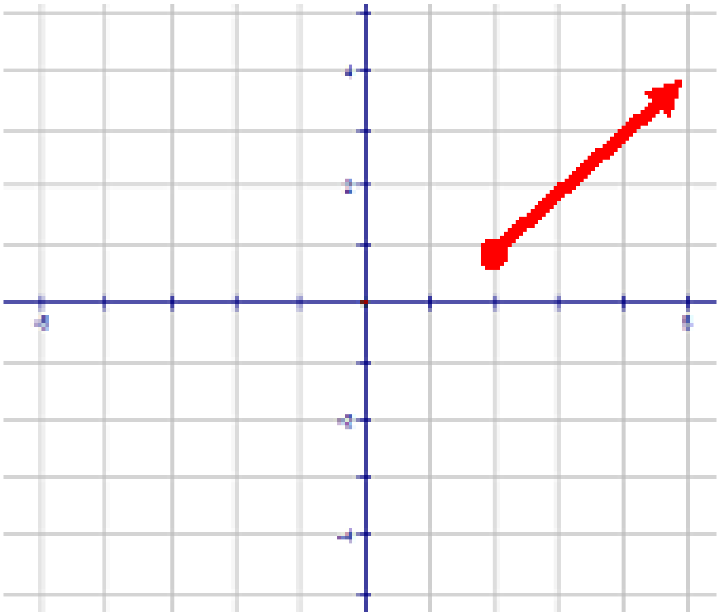
Guidance

The domain of a relation is the set of possible values that 'x' may have. The range of a relation is the set of possible values that 'y' may have. You can write the domain and range of a relation using interval notation and with respect to the number system to which it belongs. Remember:

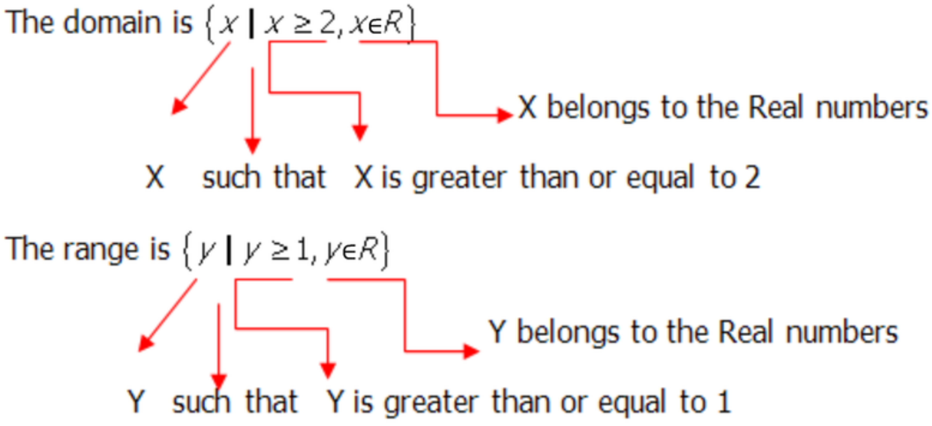
- $Z(\text{integers}) = \{-3,-2,-1,0,1,2,3,\ldots\}$
- $R(\text{real numbers}) = \{\text{all rational and irrational numbers}\}$.

These number systems are very important when the domain and range of a relation are described using interval notation.

A relation is said to be discrete if there are a finite number of data points on its graph. Graphs of discrete relations appear as dots. A relation is said to be continuous if its graph is an unbroken curve with no "holes" or "gaps." The graph of a continuous relation is represented by a line or a curve like the one below. Note that it is possible for a relation to be neither discrete nor continuous.



The relation is a straight line that that begins at the point (2, 1). The fact that the points on the line are connected indicates that the relation is continuous. The domain and the range can be written in interval notation, as shown below:



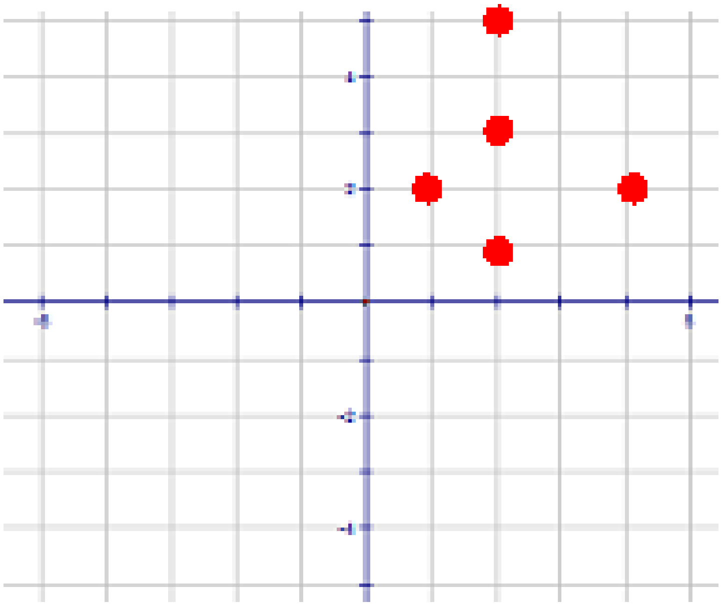
The domain is $\{x \mid x \geq 2, x \in R\}$

The range is $\{y \mid y \geq 1, y \in R\}$

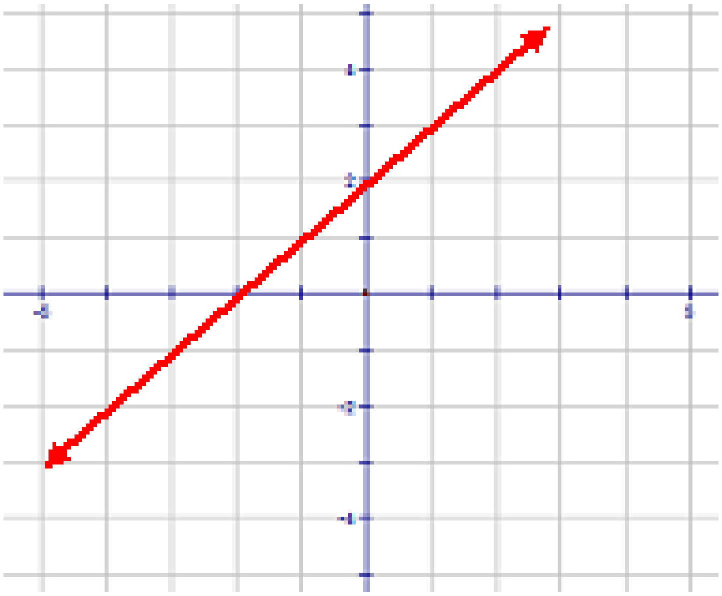
Example A

Which relations are discrete? Which relations are continuous? For each relation, find the domain and range.

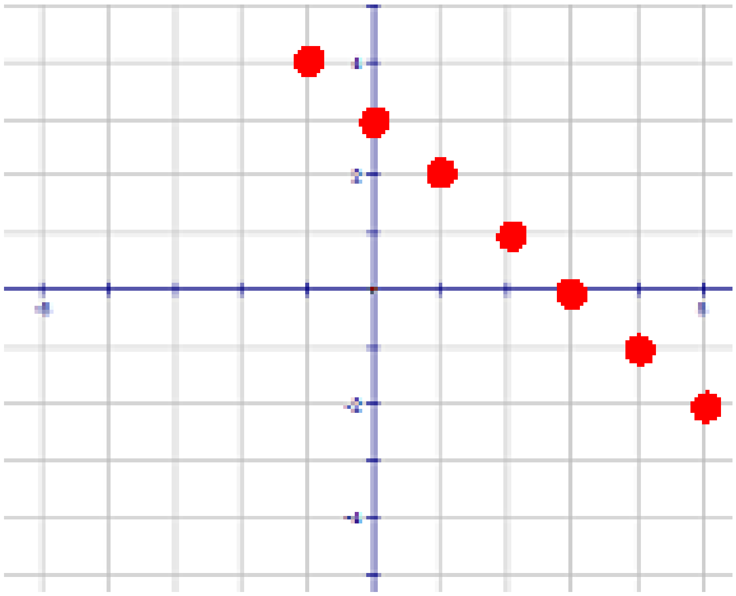
(i)



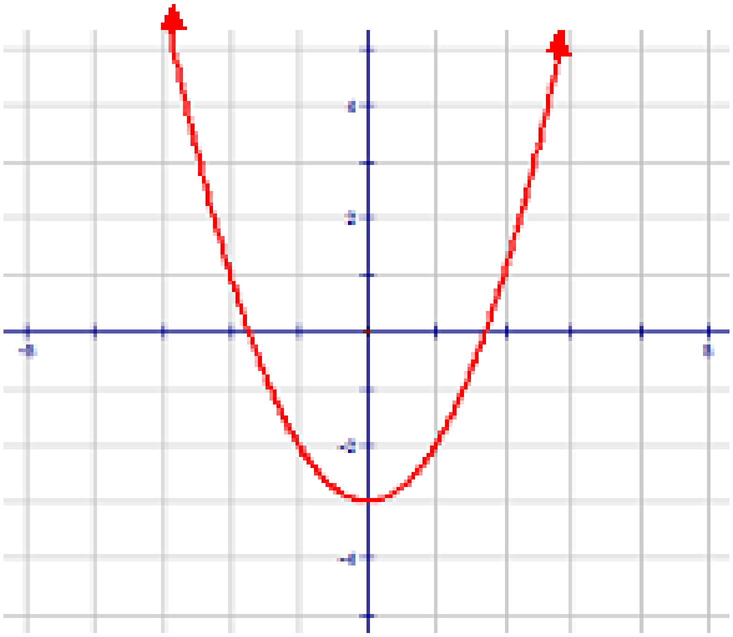
(ii)



(iii)



(iv)



Solution:

(i) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{1,2,4\}$. The range is $\{1,2,3,5\}$

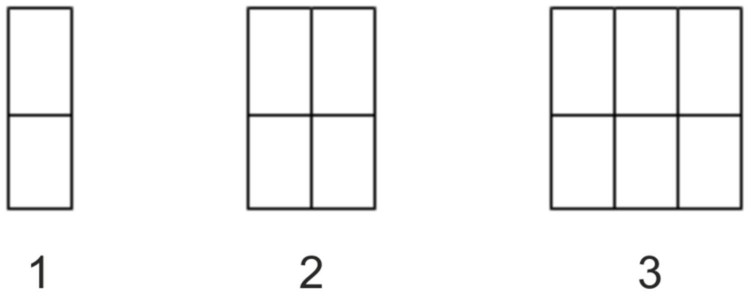
(ii) The graph appears as a straight line. Therefore, the relation is continuous. $D=\{x|x \in R\} \quad R=\{y|y \in R\}$

(iii) The graph appears as dots. Therefore, the relation is discrete. The domain is $\{-1,0,1,2,3,4,5\}$. The range is $\{-2,-1,0,1,2,3,4\}$

(iv) The graph appears as a curve. Therefore, the relation is continuous. $D=\{x|x \in R\} \quad R=\{y|y \geq -3, y \in R\}$

Example B

Whether a relation is discrete, continuous, or neither can often be determined without a graph. The domain and range can be determined without a graph as well. Examine the following toothpick pattern.



Complete the table below to determine the number of toothpicks needed for the pattern.

TABLE 3.9:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)									

Is the data continuous or discrete? Why?

What is the domain?

What is the range?

Solution:

TABLE 3.10:

Pattern number (n)	1	2	3	4	5	...	n	...	200
Number of toothpicks (t)	7	12	17	22	27		$5n+2$		1002

The number of toothpicks in any pattern number is the result of multiplying the pattern number by 5 and adding 2 to the product.

The number of toothpicks in pattern number 200 is:

$$\begin{aligned} t &= 5n+2 \\ t &= 5(200)+2 \\ t &= 1000+2 \\ t &= 1002 \end{aligned}$$

The data must be discrete. The graph would be dots representing the pattern number and the number of toothpicks. It is impossible to have a pattern number or a number of toothpicks that are not natural numbers. Therefore, the points would not be joined.

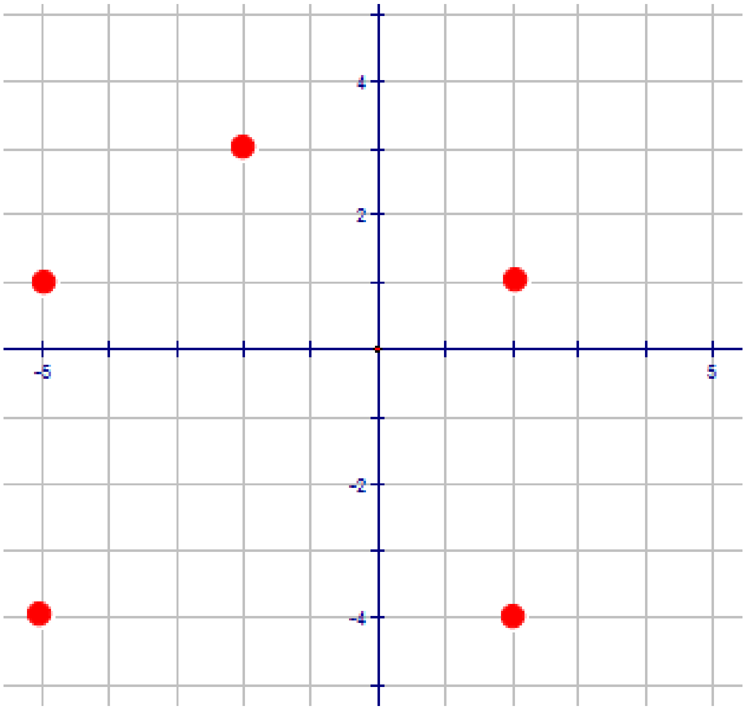
The domain and range are:

$$D = \{x|x \in N\} \quad R = \{y|y = 5x+2, x \in N\}$$

If the range is written in terms of a function, then the number system to which x belongs must be designated in the range.

Example C

Can you state the domain and the range of the following relation?



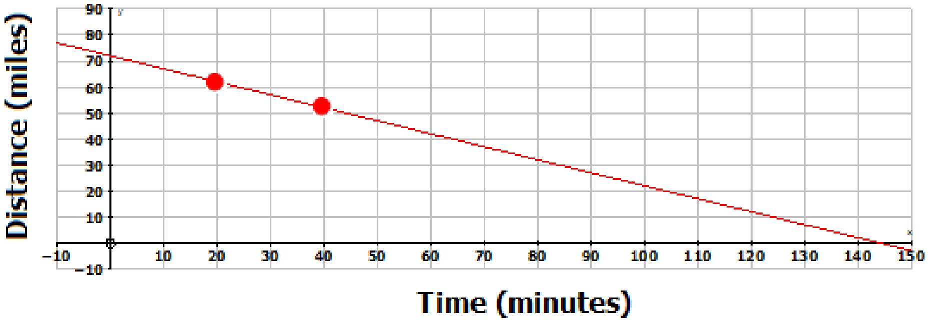
Solution:

The points indicated on the graph are $\{(-5,-4),(-5,1),(-2,3),(2,1),(2,-4)\}$

The domain is $\{-5,-2,2\}$ and the range is $\{-4,1,3\}$.

Concept Problem Revisited

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. Represent the problem on a graph and write a suitable domain and range for the situation.



To represent the problem on a graph, plot the points (20, 62) and (40, 52). The points can be joined with a straight line since the data is continuous. The distance traveled changes continuously as the time driving changes. The y-intercept represents the distance from Joseph's summer home to his place of work. This distance is approximately 72 miles. The x-intercept represents the time it took Joseph to drive from his summer home to work. This time is approximately 145 minutes.

Time cannot be a negative quantity. Therefore, the smallest value for the number of minutes would have to be zero. This represents the time Joseph began his trip. A suitable domain for this problem is $D = \{x|0 \le x \le 145, x \in R\}$

The distance from his summer home to work cannot be a negative quantity. This distance is represented on the y-axis as the y-intercept and is the distance before he begins to drive. A suitable range for the problem is $R = \{y|0 \le y \le 72, y \in R\}$

The domain and range often depend on the quantities presented in the problem. In the above problem, the quantities of time and distance could not be negative. As a result, the values of the domain and the range had to be positive.

Vocabulary

Continuous
A relation is said to be ***continuous*** if it is an unbroken curve with no "holes" or "gaps".

Discrete
A relation is said to be ***discrete*** if there are a finite number of data points on its graph. Graphs of ***discrete*** relations appear as dots.

Domain
The ***domain*** of a relation is the set of possible values that 'x' may have.

Range
The ***range*** of a relation is the set of possible values that 'y' may have.

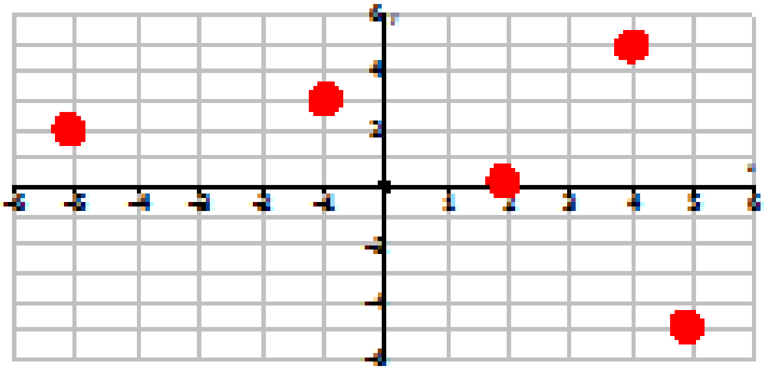
Coordinates
The ***coordinates*** are the ordered pair (x, y) that represents a point on the Cartesian plane.

Guided Practice

1. Which relation is discrete? Which relation is continuous?

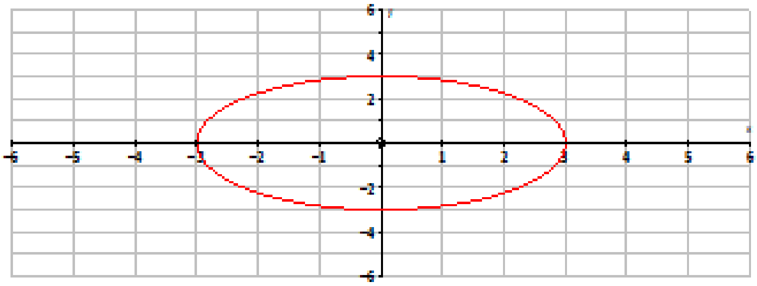
(i)

(ii)

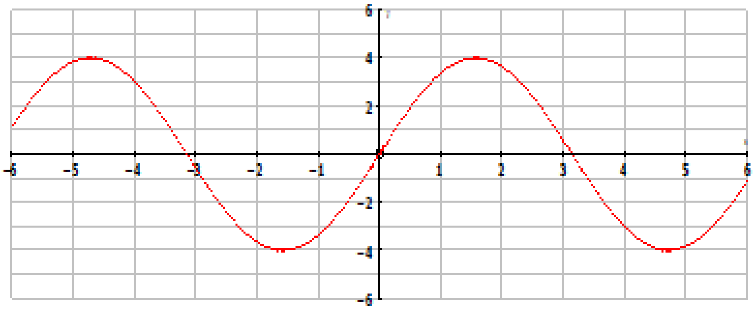


2. State the domain and the range for each of the following relations:

(i)



(ii)



3. A computer salesman's wage consists of a monthly salary of $200 plus a bonus of $100 for each computer sold.

(a) Complete the following table of values:

TABLE 3.11:

Number of computers sold	**0**	**2**	**5**	**10**	**18**
Wages in dollars for the month ($)					

(b) Sketch the graph to represent the monthly salary ($), against the number (N), of computers sold.
(c) Use the graph to write a suitable domain and range for the problem.

Answers:

1. (i) The graph clearly shows that the points are joined. Therefore the data is continuous.

(ii) The graph shows the plotted points as dots that are not joined. Therefore the data is discrete.

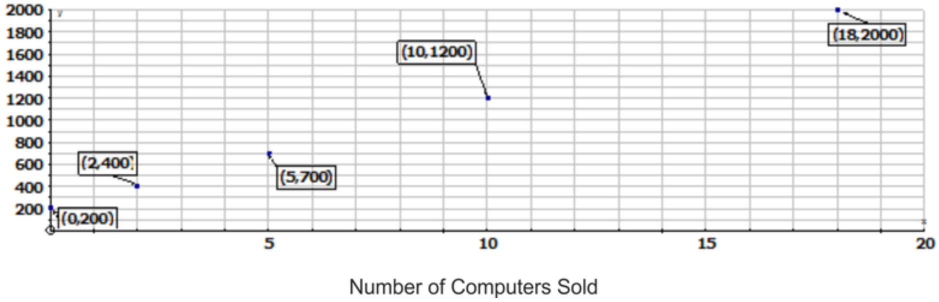
2. (i) The domain represents the values of 'x'. $D = \{x|-3 \leq x \leq 3, x \in R\}$

The range represents the values of 'y'. $R = \{y|-3 \leq y \leq 3, y \in R\}$
(ii) $D = \{x|x \in R\}$
$R = \{y|-4 \leq y \leq 4, y \in R\}$

3.

TABLE 3.12:

Number of computers sold	**0**	**2**	**5**	**10**	**18**
Wages in dollars for the month ($)	**$200**	**$400**	**$700**	**$1200**	**$2000**



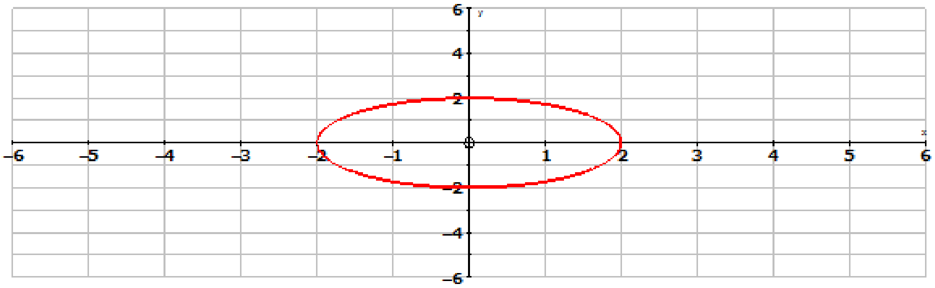
(c) The graph shows that the data is discrete. (The salesman can't sell a portion of a computer, so the data points can't be connected.) The number of computers sold and must be whole numbers. The wages must be natural numbers.

A suitable domain is $D = \{x|x \geq 0, x \ \in \ W\}$

A suitable domain is $R = \{y|y = 200 + 100x, x \ \in \ N\}$

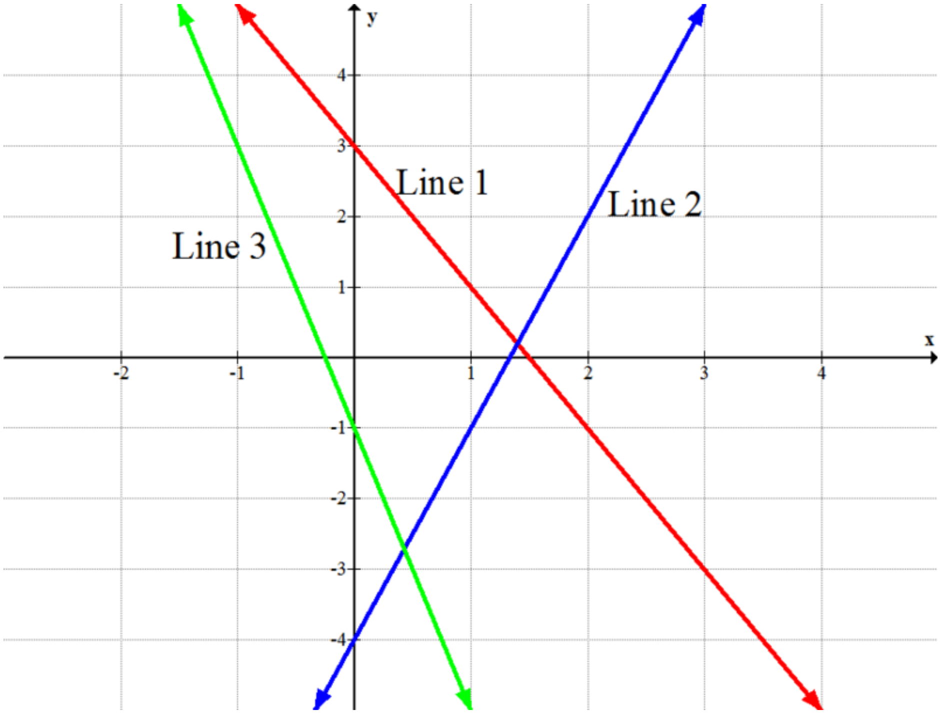
Practice

Use the graph below for #1 and #2.



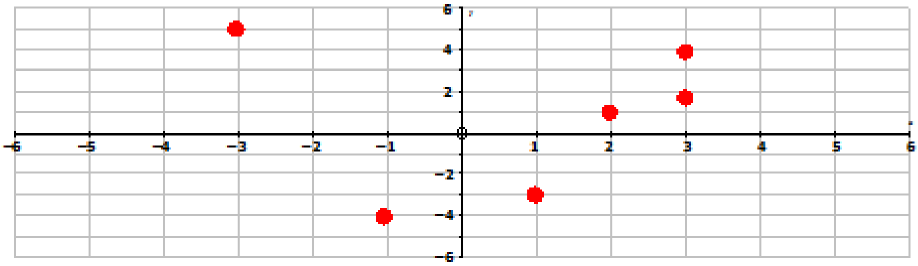
1. Is the relation discrete, continuous, or neither?
2. Find the domain and range for the relation.

Use the graph below for #3 and #4.



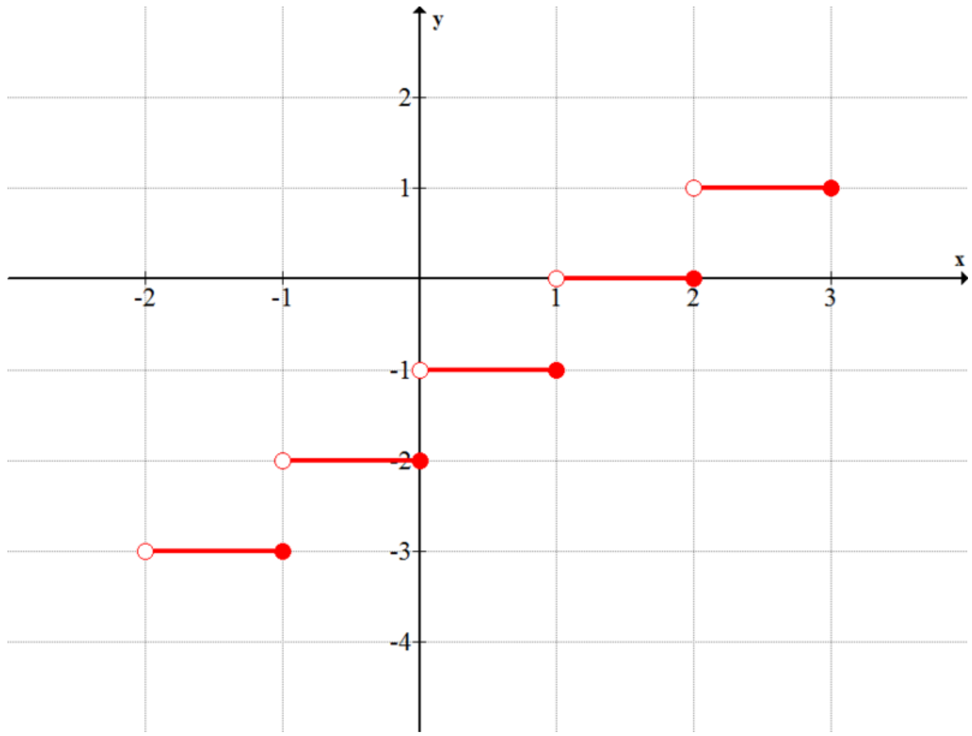
3. Is the relation discrete, continuous, or neither?
4. Find the domain and range for each of the three relations.

Use the graph below for #5 and #6.



5. Is the relation discrete, continuous, or neither?
6. Find the domain and range for the relation.

Use the graph below for #7 and #8.



7. Is the relation discrete, continuous, or neither?
8. Find the domain and range for the relation.

Examine the following pattern.

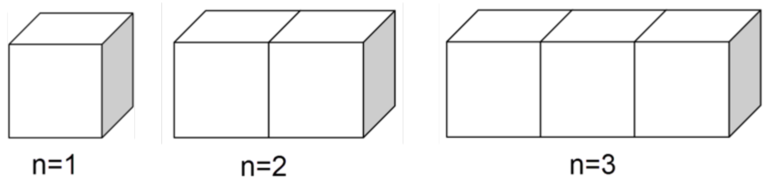


TABLE 3.13:

Number of Cubes (n)	**1**	**2**	**3**	**4**	**5**	**...**	n	**...**	**200**
Number of visible faces (f)	6	10	14						

9. Complete the table below the pattern.
10. Is the relation discrete, continuous, or neither?
11. Write a suitable domain and range for the pattern.

Examine the following pattern.

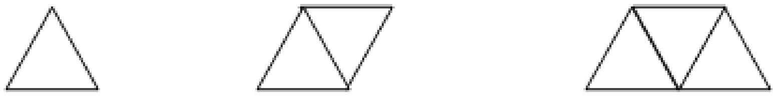


TABLE 3.14:

Number of triangles (n)	1	2	3	4	5	...	n	...	100
Number of toothpicks (t)									

12. Complete the table below the pattern.
13. Is the relation discrete, continuous, or neither?
14. Write a suitable domain and range for the pattern.

Examine the following pattern.

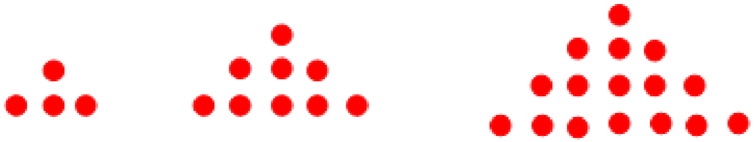


TABLE 3.15:

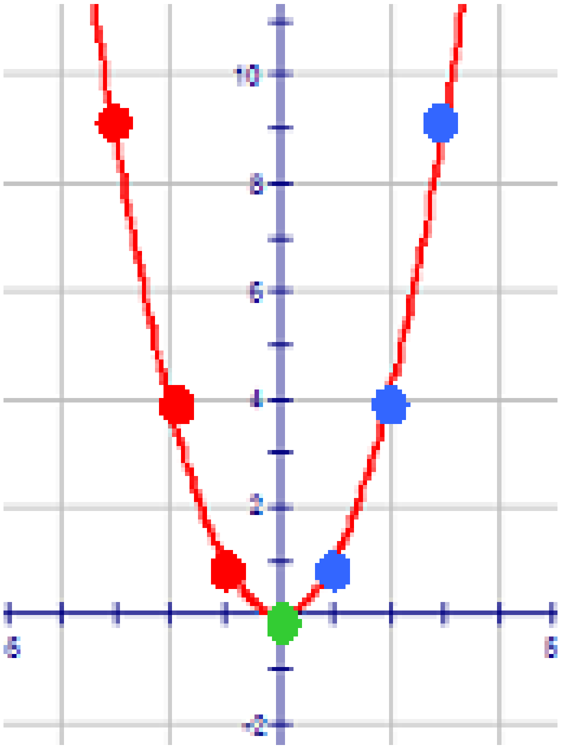
Pattern Number (n)	1	2	3	4	5	...	n	...	100
Number of dots (d)									

15. Complete the table below the pattern.
16. Is the relation discrete, continuous, or neither?
17. Write a suitable domain and range for the pattern.

3.7 Graphs of Basic Quadratic Functions

Here you'll learn how to graph and analyze the quadratic function, $y = x^2$.

Look at the graph below. Does the graph represent a function? Do you know the name of the graph? Do you know what makes the green point special? Do you notice any symmetry in the graph? Can you state the domain and range for the relation?



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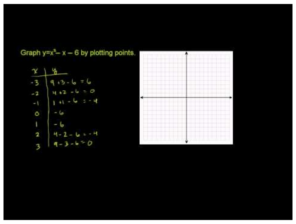
Khan Academy Quadratic Functions1



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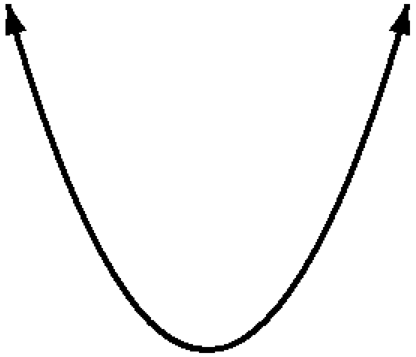
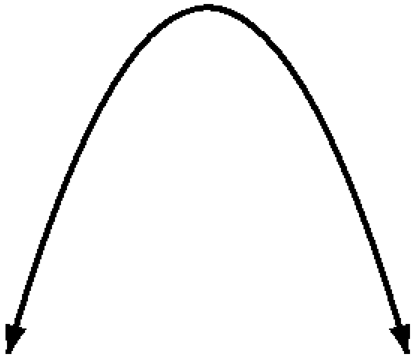
Click image to the left for more content.

Guidance

Until now you have been dealing with linear functions. The highest exponent of the independent variable (x) has been one and the graphs have been straight lines. Here you will be learning about quadratic functions. A quadratic function is one of the form $y = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$. The highest exponent of the independent variable is two. When graphed, a quadratic function creates a parabola that looks like this:



or like this:



You can create your own graph by plotting the points created from a table of values. The most basic quadratic function is $y = x^2$. The easiest way to make a table for this function is to use the domain $\{x|-3 \leq x \leq 3, x \in Z\}$ for the table.

A parabola has a turning point known as the vertex. The vertex is the minimum value of the parabola if it opens upward and the maximum value if the parabola opens downward. When the graph opens downward, the y-values in the base table change to negative values. The basic quadratic function that opens downward has the equation $y = -x^2$.

All parabolas have an axis of symmetry. The axis of symmetry is the vertical line that passes through the vertex of the parabola. The equation for the axis of symmetry is always $x = \alpha$, where α is the the $x-$coordinate of the vertex.

Example A

For the basic quadratic function $y = x^2$, complete a table such that $\{x|-3 \leq x \leq 3, x \in Z\}$.

Solution:

To complete the table of values, substitute the given x-values into the function $y = x^2$. If you are using a calculator, insert all numbers, especially negative numbers, inside parentheses before squaring them. The operation that needs to be done is $(-3)(-3)$ NOT $-(3)(3)$.

$$\begin{aligned} y &= x^2 \\ y &= (-3)^2 \\ y &= 9 \end{aligned} \qquad \begin{aligned} y &= x^2 \\ y &= (-2)^2 \\ y &= 4 \end{aligned} \qquad \begin{aligned} y &= x^2 \\ y &= (-1)^2 \\ y &= 1 \end{aligned} \qquad \begin{aligned} y &= x^2 \\ y &= (0)^2 \\ y &= 0 \end{aligned}$$

$$\begin{aligned} y &= x^2 \\ y &= (1)^2 \\ y &= 1 \end{aligned} \qquad \begin{aligned} y &= x^2 \\ y &= (2)^2 \\ y &= 4 \end{aligned} \qquad \begin{aligned} y &= x^2 \\ y &= (3)^2 \\ y &= 9 \end{aligned}$$

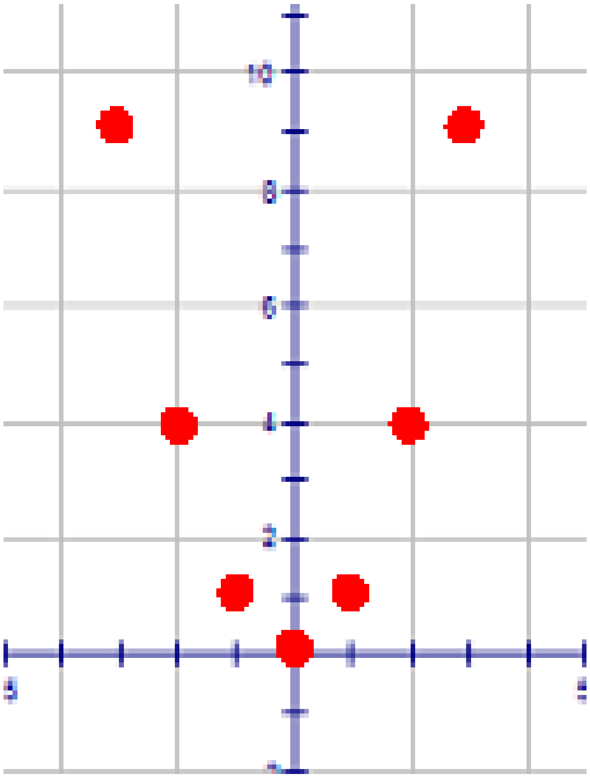
TABLE 3.16:

X	Y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

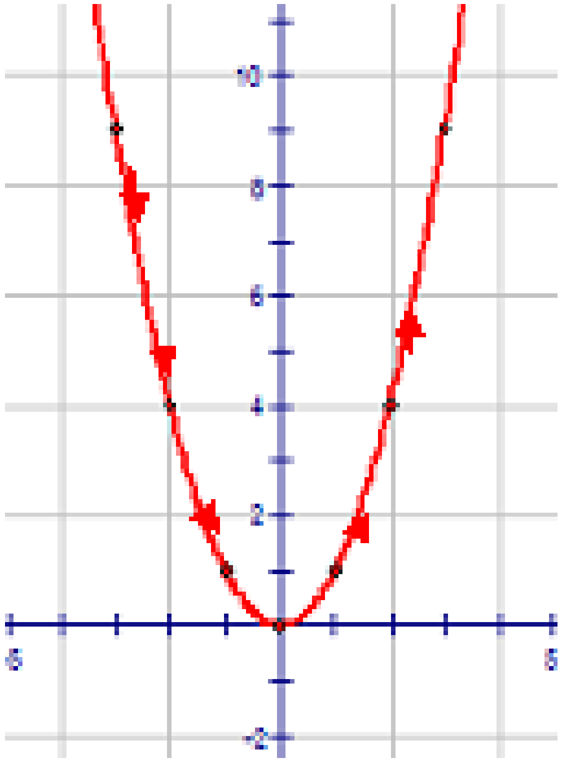
Example B

On a Cartesian plane, plot the points from the table for $y = x^2$.

Solution:



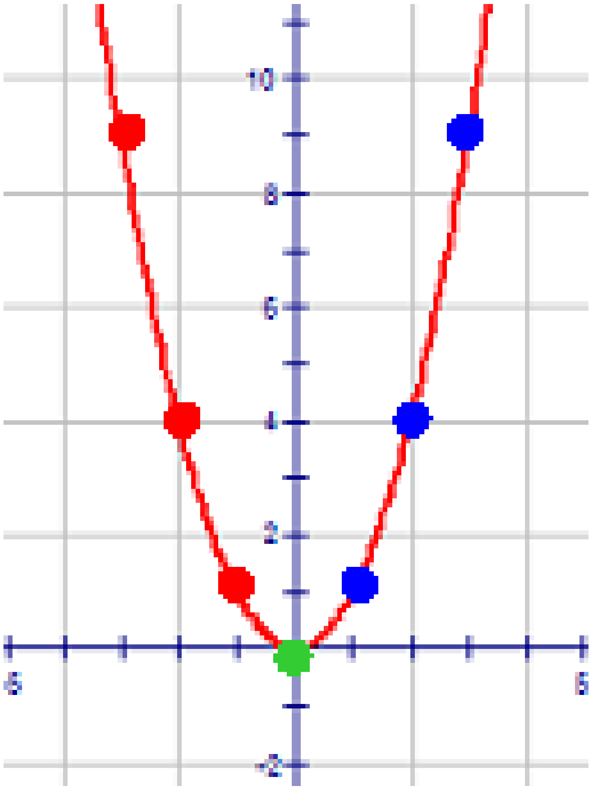
The plotted points cannot be joined to form a continuous curve. To join the points, begin with the point (–3, 9) or the point (3, 9) and without lifting your pencil, draw a smooth curve. The image should look like the following graph.



The arrows indicate the direction of the pencil as the points are joined. If the pencil is not moved off the paper, the temptation to join the points with a series of straight lines will be decreased. The points must be joined with a smooth curve that does not extend below the lowest point of the graph. In the above graph, the curve cannot go below the point (0, 0).

Example C

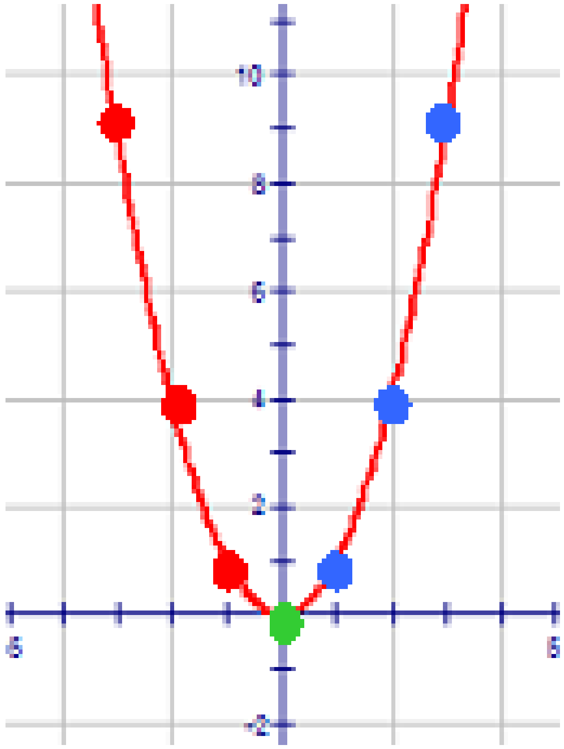
What are some unique characteristics of the graph of $y = x^2$?



Solution:

1. The green point is located at the lowest point on the image. The curve does not go below this point.
2. Every red point on the left side of the image has a corresponding blue point on the right side of the image.
3. If the image was folded left to right along the y-axis that passes through the green point, each red point would land on each corresponding blue point.
4. The sides of the image extend upward.
5. The red and the blue points are plotted to the right and to the left of the green point. The points are plotted left and right one and up one; left and right two and up four, left and right 3 and up nine.

Concept Problem Revisited



The green point is the lowest point on the curve. The smooth curve is called a *parabola* and it is the image produced when the basic quadratic function is plotted on a Cartesian grid. The green point is known as the *vertex* of the parabola. The *vertex* is the turning point of the graph.

For the graph of $y = x^2$, the vertex is (0, 0) and the parabola has a minimum value of zero which is indicated by the y-value of the vertex. The parabola opens upward since the y-values in the table of values are 0, 1, 4 and 9. The y-axis for this graph is actually the axis of symmetry. The axis of symmetry is the vertical line that passes through the vertex of the parabola. The parabola is symmetrical about this line. The equation for this axis of symmetry is $x = 0$. If the parabola were to open downward, the vertex would be the highest point of the graph. Therefore the image would have a maximum value of zero.

The domain for all parabolas is $D = \{x|x \in R\}$. The range for the above parabola is $R = \{y|y \geq 0, y \in R\}$.

Vocabulary

Axis of Symmetry

The ***axis of symmetry*** of a parabola is a vertical line that passes through the vertex of the parabola. The parabola is symmetrical about this line. The ***axis of symmetry*** has the equation $x = \alpha$ where α is the $x-$coordinate of the vertex.

Parabola

A ***parabola*** is the smooth curve that results from graphing a quadratic function of the form $y = ax^2 + bx + c$. The curve resembles a U-shape.

Quadratic Function

A ***quadratic function*** is a function of the form $y = ax^2 + bx + c$ where a, b and c are real numbers and $a \neq 0$.

Vertex

The ***vertex*** of a parabola is the point around which the parabola turns. The vertex is the maximum point of a parabola that opens downward and the minimum point of a parabola that opens upward.

Guided Practice

1. If the graph of $y = x^2$ opened downward, what changes would exist in the base table of values?

2. If the graph of $y = x^2$ opened downward, what changes would exist in the basic quadratic function?

3. Draw the image of the basic quadratic function that opens downward. State the domain and range for this parabola.

Answers:

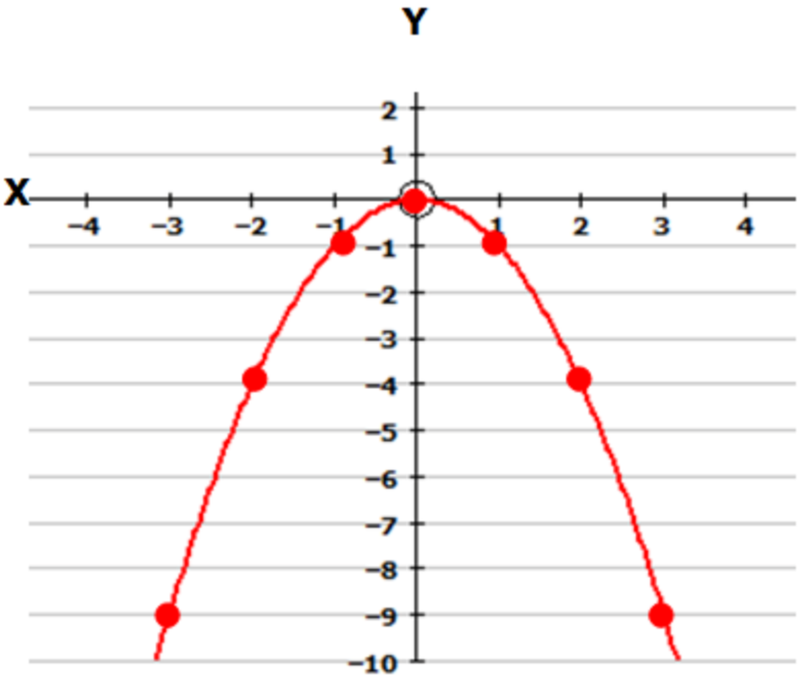
1. If the parabola were to open downward, the x-values would not change. The y-values would become negative values. The points would be plotted from the vertex as: right and left one and down one; right and left two and down four; right and left three and down nine. The table of values would be

TABLE 3.17:

X	Y
-3	-9
-2	-4
-1	-1
0	0
1	-1
2	-4
3	-9

2. To match the table of values, the basic quadratic function would have to be written as $y = -x^2$.

3.



The domain is $D = \{x|x \in R\}$. The range for this parabola is $R = \{y|y \leq 0, y \in R\}$.

Practice

Complete the following statements in the space provided.

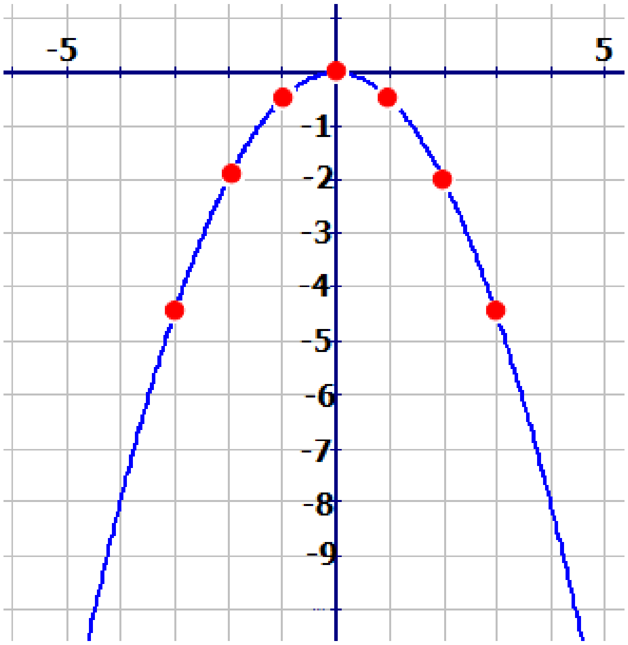
1. The name given to the graph of $y = x^2$ is ______________________.
2. The domain of the graph of $y = x^2$ is ______________________.
3. If the vertex of a parabola was (–3, 5), the equation of the axis of symmetry would be ______________________.
4. A parabola has a maximum value when it opens ______________________.
5. The point (–2, 4) on the graph of $y = x^2$ has a corresponding point at ______________________.
6. The range of the graph of $y = -x^2$ is ______________________.
7. If the table of values for the basic quadratic function included 4 and –4 as x-values, the y-value(s) would be ______________________.
8. The vertical line that passes through the vertex of a parabola is called ______________________.
9. A minimum value exists when a parabola opens ______________________.
10. The turning point of the graph of $y = x^2$ is called the ______________________.

11. Make a sketch of the function $y = x^2 - 2x + 1$ by first making a table and then plotting the points. What is the vertex of this parabola?
12. Make a sketch of the function $y = x^2 + 2x - 3$ by first making a table and then plotting the points. What is the vertex of this parabola?
13. Make a sketch of the function $y = x^2 - 4x + 5$ by first making a table and then plotting the points. What is the vertex of this parabola?
14. Make a sketch of the function $y = -x^2 + 2x + 1$ by first making a table and then plotting the points. What is the vertex of this parabola?
15. Make a sketch of the function $y = -x^2 - 4x$ by first making a table and then plotting the points. What is the vertex of this parabola?

3.8 Transformations of Quadratic Functions

Here you'll learn how to transform the basic quadratic functions ($y = x^2$ and $y = -x^2$) to make new quadratic functions.

Look at the parabola below. How is this parabola different from $y = -x^2$? What do you think the equation of this parabola is?



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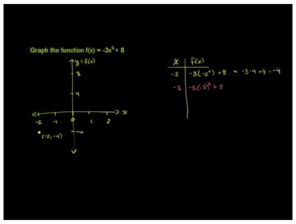
Khan Academy Graphing a Quadratic Function



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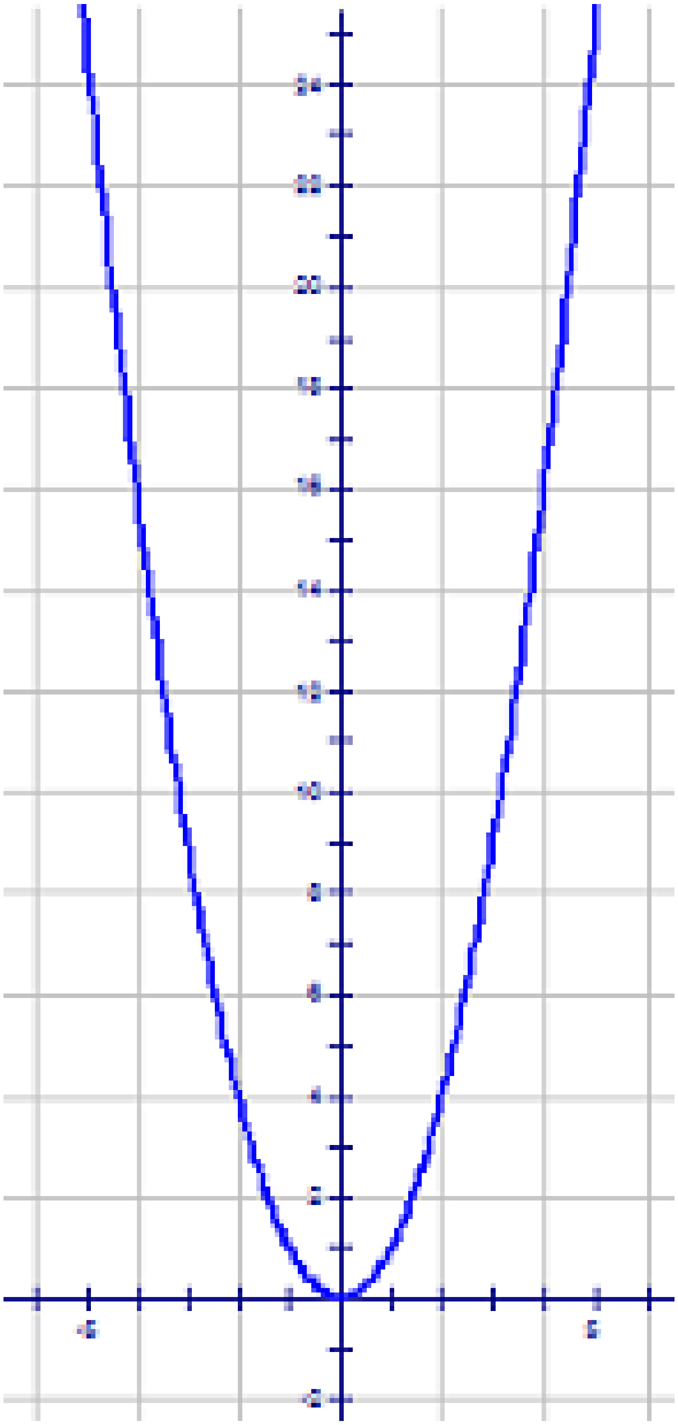
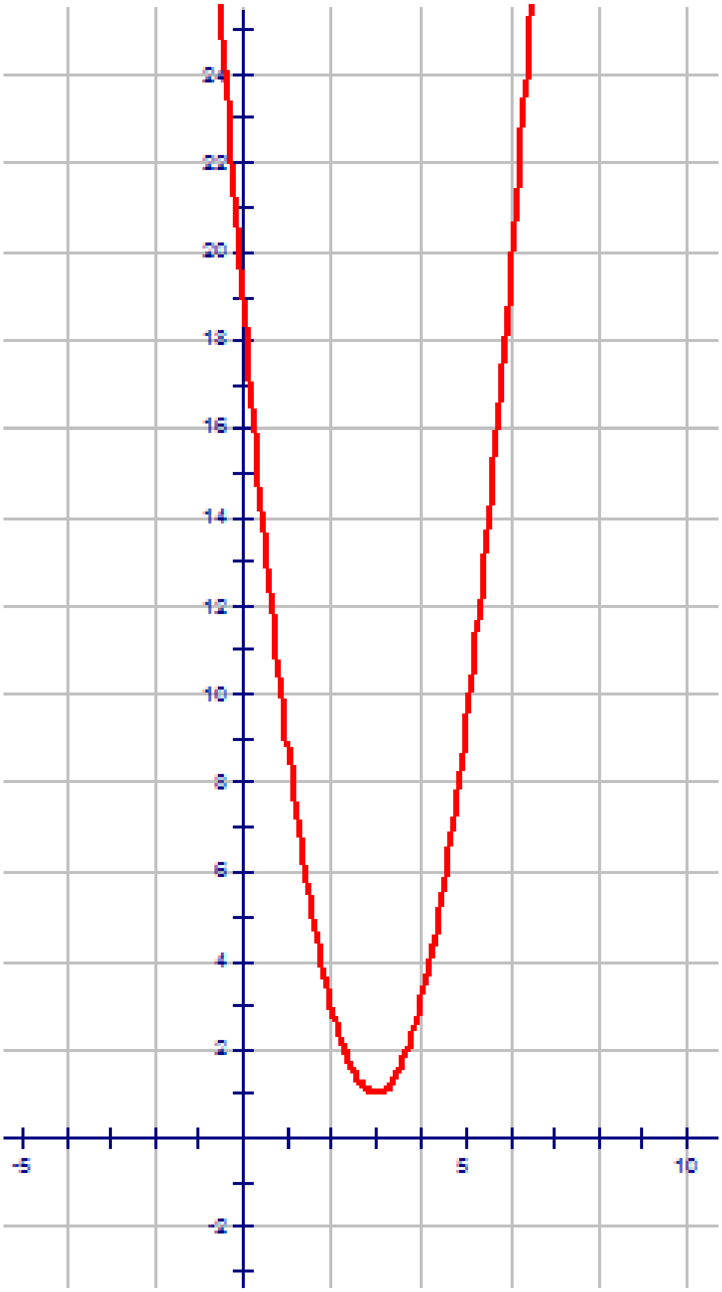
Guidance

This is the graph of $y = x^2$:



This is the graph of $y = x^2$ that has undergone transformations:



The vertex of the red parabola is (3, 1). The sides of the parabola open upward but they appear steeper and longer than those on the blue parabola.

As shown above, you can apply changes to the graph of $y = x^2$ to create a new parabola (still a 'U' shape) that no longer has its vertex at (0, 0) and no longer has y-values of 1, 4 and 9. These changes are known as transformations.

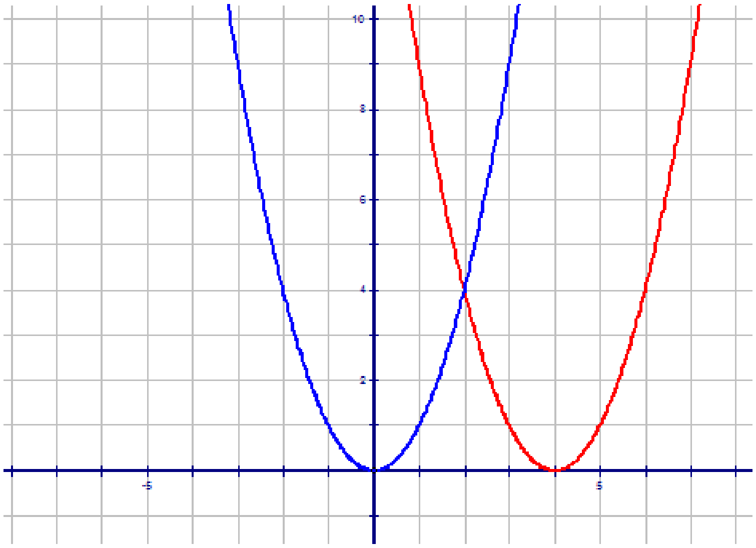
The vertex of (0, 0) will change if the parabola undergoes either a horizontal translation and/or a vertical translation. These transformations cause the parabola to slide left or right and up or down.

If the parabola undergoes a vertical stretch, the y-values of 1, 4 and 9 can increase if the stretch is a whole number. This will produce a parabola that will appear to be narrower than the original base graph. If the vertical stretch is a fraction less than 1, the values of 1, 4 and 9 will decrease. This will produce a parabola that will appear to be wider than the original base graph.

Finally, a parabola can undergo a vertical reflection that will cause it to open downwards as opposed to upwards. For example, $y = -x^2$ is a vertical reflection of $y = x^2$.

Example A

Look at the two parabolas below. Describe the transformation from the blue parabola to the red parabola. What is the coordinate of the vertex of the red parabola?

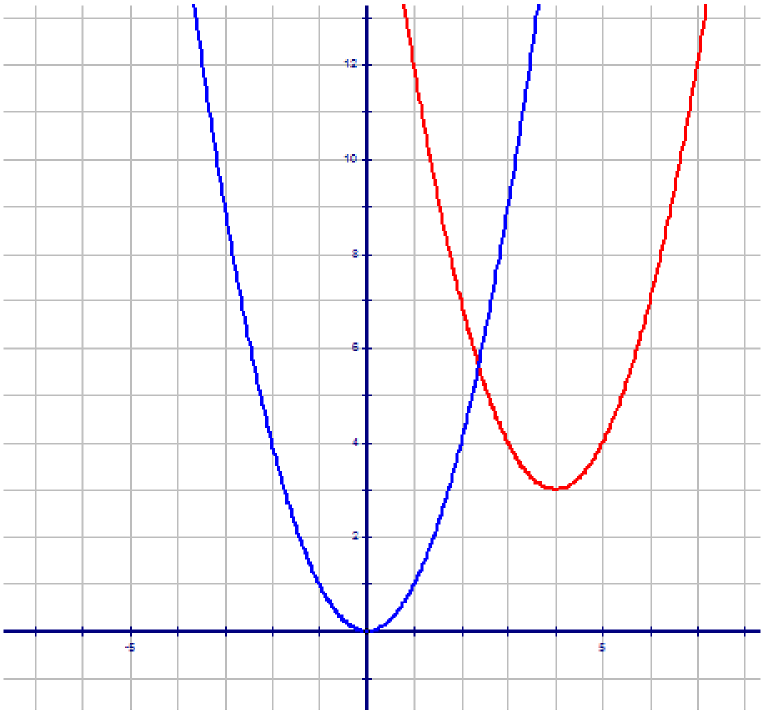


Solution:

The blue parabola is the graph of $y = x^2$. Its vertex is (0, 0). The red graph is the graph of $y = x^2$ that has been moved four units to the right. When the graph undergoes a slide of four units to the right, it has undergone a horizontal translation of +4. The vertex of the red graph is (4, 0). A horizontal translation changes the $x-$coordinate of the vertex of the graph of $y = x^2$.

Example B

Look at the two parabolas below. Describe the transformation from the blue parabola to the red parabola. What is the coordinate of the vertex of the red parabola?

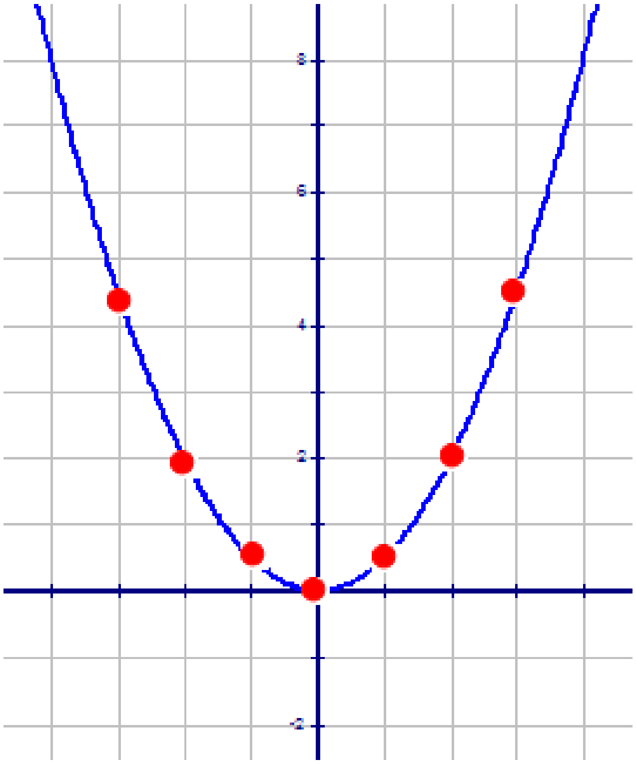


Solution:

The blue parabola is the graph of $y = x^2$. Its vertex is (0, 0). The red graph is the graph of $y = x^2$ that has been moved four units to the right and three units upward. When the graph undergoes a slide of four units to the right, it has undergone a horizontal translation of +4. When the graph undergoes a slide of three units upward, it has undergone a vertical translation of +3.The vertex of the red graph is (4, 3). A horizontal translation changes the $x-$coordinate of the vertex of the graph of $y = x^2$ while a vertical translation changes the $y-$coordinate of the vertex.

Example C

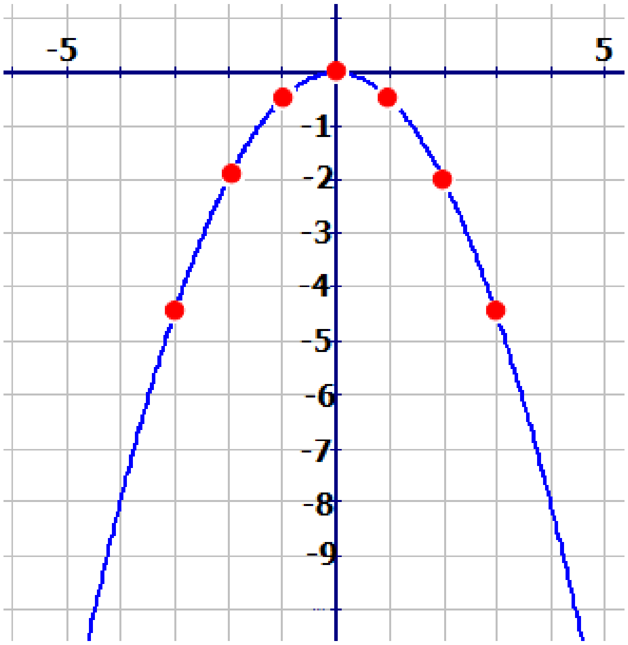
Look at the parabola below. How is this parabola different from $y = x^2$? What do you think the equation of this parabola is?



Solution:

This is the graph of $y = \frac{1}{2}x^2$. The points are plotted from the vertex as right and left one and up one-half, right and left 2 and up two, right and left three and up four and one-half. The original y-values of 1, 4 and 9 have been divided by two or multiplied by one-half. When the y-values are multiplied, the y-values either increase or decrease. This transformation is known as a vertical stretch.

Concept Problem Revisited



This is the graph of $y = -\frac{1}{2}x^2$. The points are plotted from the vertex as right and left one and down one-half, right and left 2 and down two, right and left three and down four and one-half. The original y-values of 1, 4 and 9 have been multiplied by one-half and then were made negative because the graph was opening downward. When the y-values become negative, the direction of the opening is changed from upward to downward. This transformation is known as a vertical reflection. The graph is reflected across the x-axis.

Vocabulary

Horizontal translation

The ***horizontal translation*** is the change in the base graph $y = x^2$ that shifts the graph right or left. It changes the $x-$coordinate of the vertex.

Transformation

A ***transformation*** is any change in the base graph $y = x^2$. The transformations that apply to the parabola are a horizontal translation, a vertical translation, a vertical stretch and a vertical reflection.

Vertical Reflection

The ***vertical reflection*** is the reflection of the image graph in the x-axis. The graph opens downward and the y-values are negative values.

Vertical Stretch

The ***vertical stretch*** is the change made to the base function $y = x^2$ by stretching (or compressing) the graph vertically. The vertical stretch will produce an image graph that appears narrower (or wider) then the original base graph of $y = x^2$.

Vertical Translation

The ***vertical translation*** is the change in the base graph $y = x^2$ that shifts the graph up or down. It changes the $y-$coordinate of the vertex.

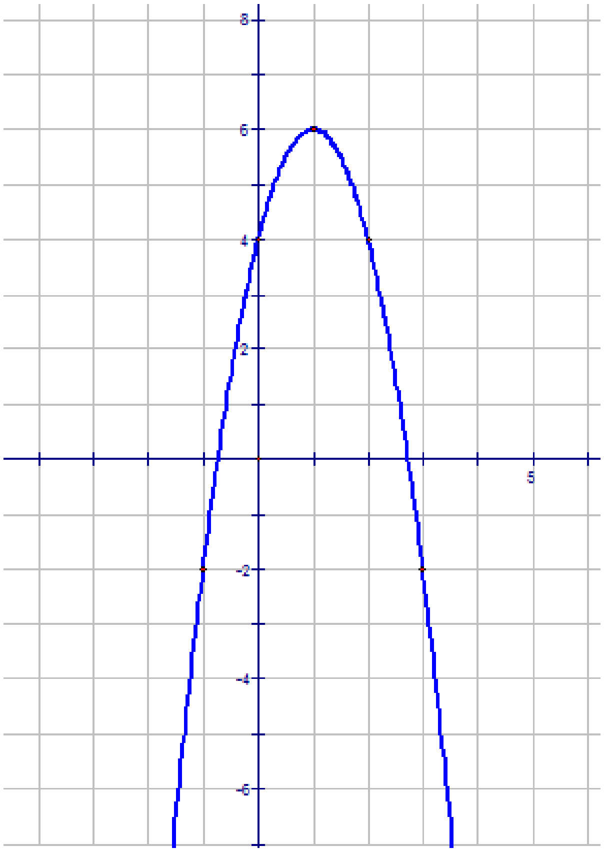
Guided Practice

1. Use the following tables of values and identify the transformations of the base graph $y = x^2$.

X	– 3	– 2	– 1	0	1	2	3
Y	9	4	1	0	1	4	9

X	– 4	– 3	– 2	– 1	0	1	2
Y	15	5	– 1	– 3	– 1	5	15

2. Identify the transformations of the base graph $y = x^2$.



3. Draw the image graph of $y = x^2$ that has undergone a vertical reflection, a vertical stretch by a factor of $\frac{1}{2}$, a vertical translation up 2 units, and a horizontal translation left 3 units.

Answers:

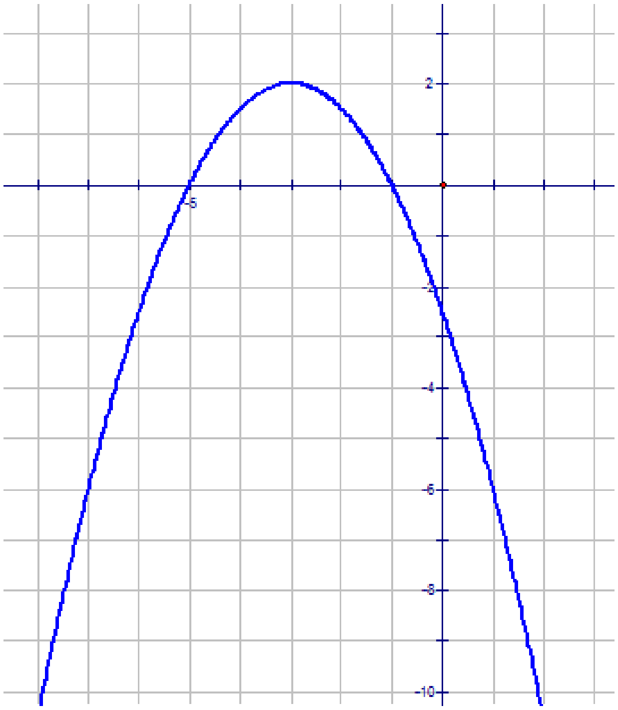
1. To identify the transformations from the tables of values, determine how the table of values for $y = x^2$ compare to the table of values for the new image graph.

- The x-values have moved one place to the left. This means that the graph has undergone a horizontal translation of –1.

- The $y-$coordinate of the vertex is –3. This means that the graph has undergone a vertical translation of –3. The vertex is easy to pick out from the tables since it is the point around which the corresponding points appear.
- The points from the vertex are plotted left and right one and up two, left and right two and up eight. This means that the base graph has undergone a vertical stretch of 2.
- The y-values move upward so the parabola will open upward. Therefore the image is not a vertical reflection.

2. The vertex is (1, 6). The base graph has undergone a horizontal translation of +1 and a vertical translation of +6. The parabola opens downward, so the graph is a vertical reflection. The points have been plotted such that the y-values of 1 and 4 are now 2 and 8. It is not unusual for a parabola to be plotted with five points rather than seven. The reason for this is the vertical stretch often multiplies the y-values such that they are difficult to graph on a Cartesian grid. If all the points are to be plotted, a different scale must be used for the y-axis.

3. The vertex given by the horizontal and vertical translations and is (–3, 2). The y-values of 1, 4 and 9 must be multiplied by $\frac{1}{2}$ to create values of $\frac{1}{2}$,2 and $4\frac{1}{2}$. The graph is a vertical reflection which means the graph opens downward and the y-values become negative.



Practice

The following table represents transformations to the base graph $y = x^2$. Draw an image graph for each set of transformations. VR = Vertical Reflection, VS = Vertical Stretch, VT = Vertical Translation, HT = Horizontal Translation.

TABLE 3.18:

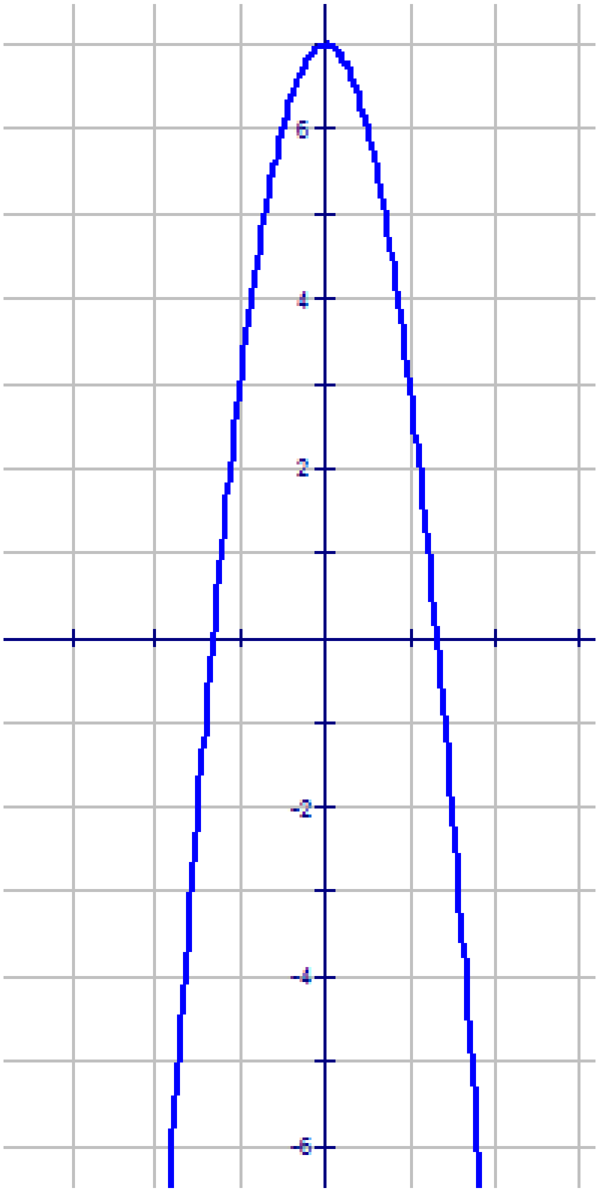
Number	VR	VS	VT	HT
1.	**NO**	3	-4	-8
2.	**YES**	2	5	6
3.	**YES**	$\frac{1}{2}$	3	-2
4.	**NO**	1	-2	4
5.	**NO**	$\frac{1}{4}$	1	-3

TABLE 3.18: (continued)

Number	VR	VS	VT	HT
6.	**YES**	1	-4	0
7.	**NO**	2	3	1
8.	**YES**	$\frac{1}{8}$	0	2

For each of the following graphs, list the transformations of $y = x^2$.

9.



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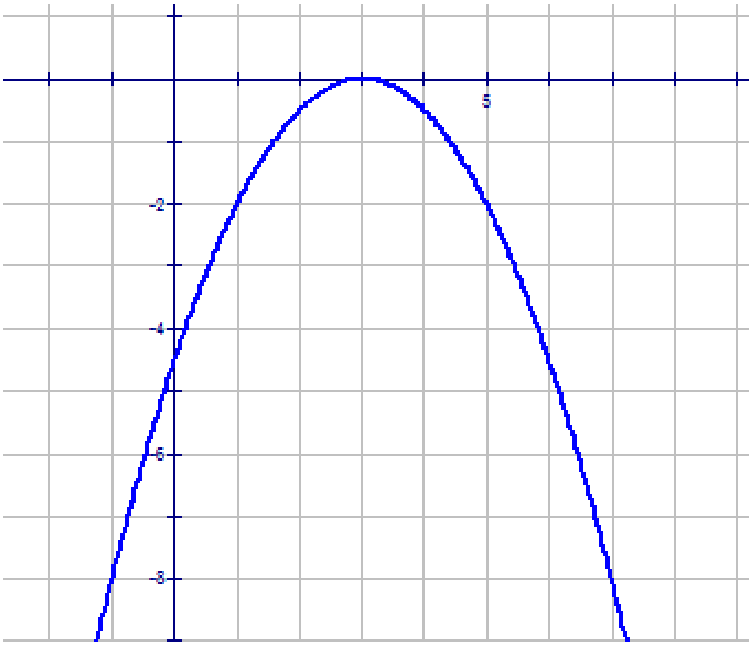

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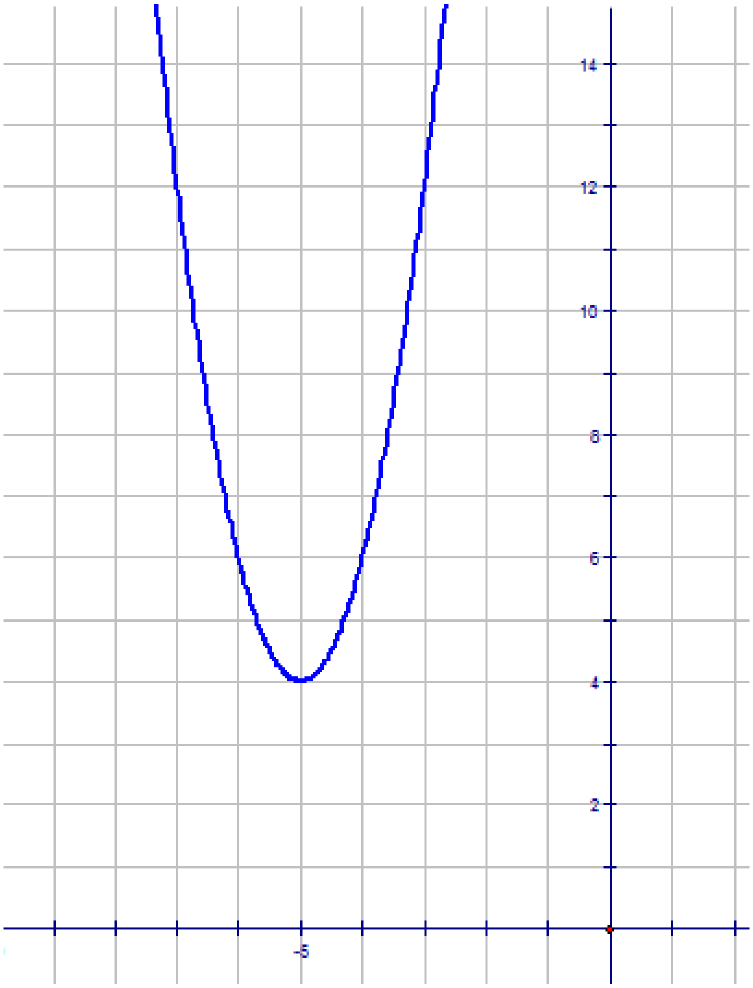

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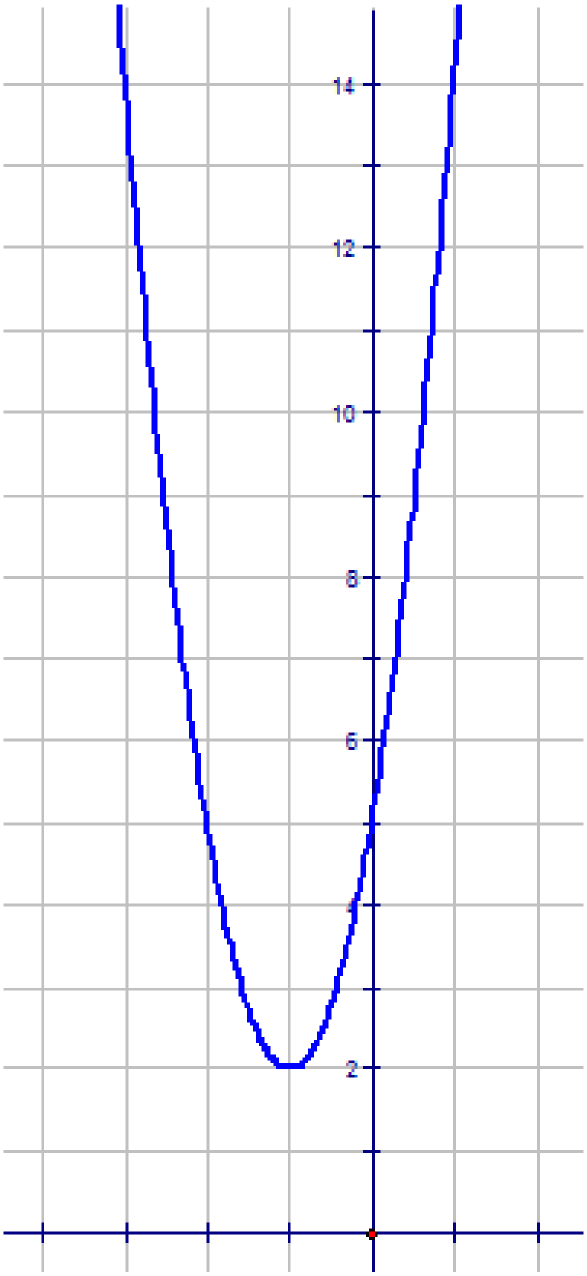
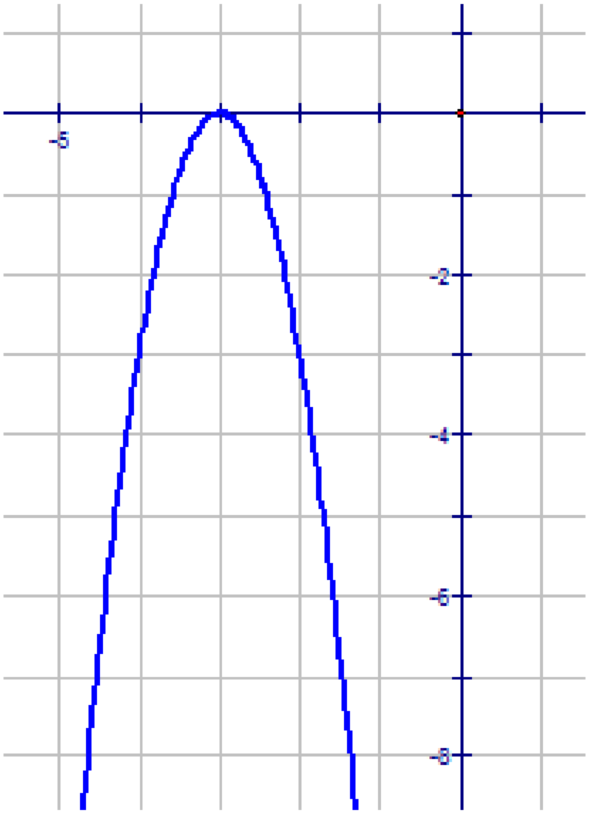


13.



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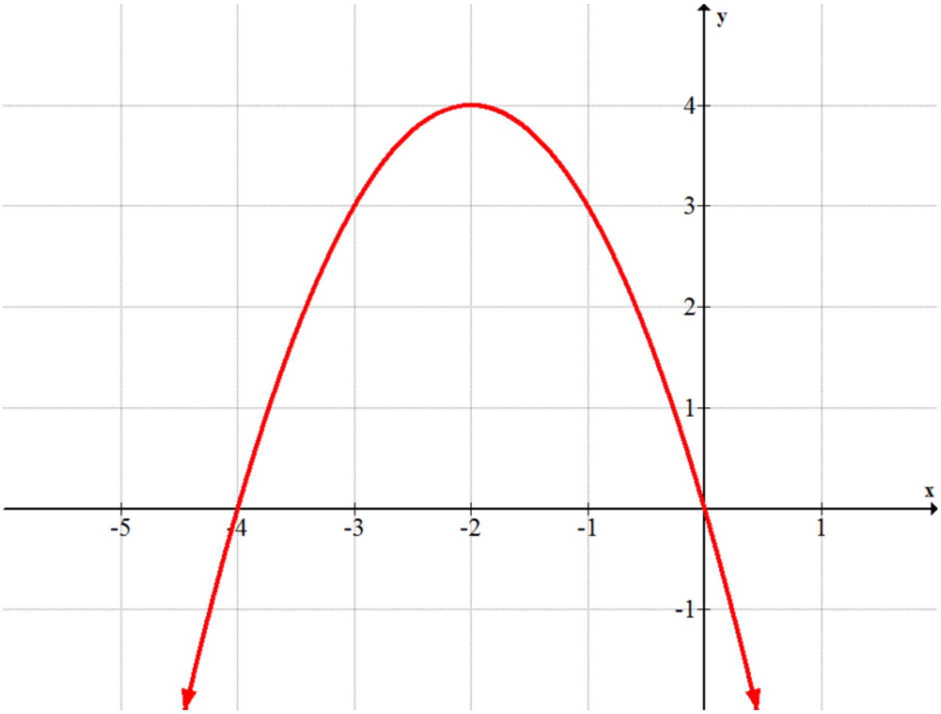
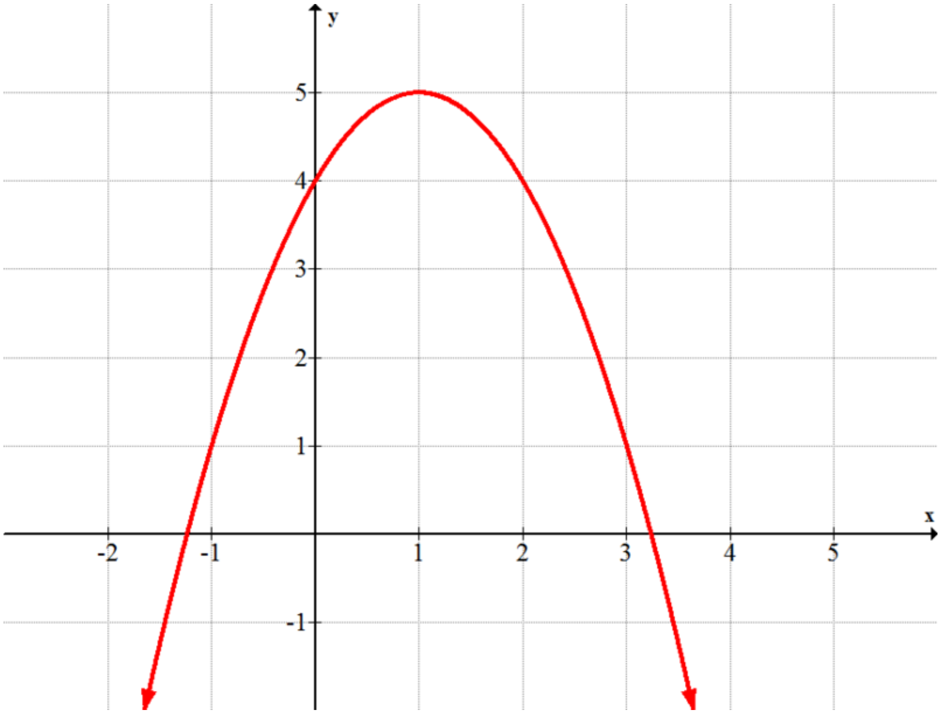


15.


y
x
5
4
3
2
1
-1
-2
-1
1
2
3
4
5

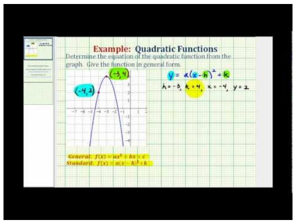
3.9 Vertex Form of a Quadratic Function

Here you will learn to write the equation for a parabola that has undergone transformations.

Given the equation $y = 3(x+4)^2 + 2$, list the transformations of $y = x^2$.

Watch This

James Sousa: Find the Equation of a Quadratic Function from a Graph



Example: Quadratic Functions

Determine the equation of the quadratic function from the graph. Give the function in general form.

$y = a(x-h)^2+k$

$h=-3,\ k=4,\ x=-4,\ y=2$

General: $f(x) = ax^2+bx+c$

Standard: $f(x) = a(x-h)^2+k$

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Guidance

The equation for a basic parabola with a vertex at $(0,0)$ is $y = x^2$. You can apply transformations to the graph of $y = x^2$ to create a new graph with a corresponding new equation. This new equation can be written in vertex form. The vertex form of a quadratic function is $y = a(x-h)^2 + k$ where:

- $|a|$ is the vertical stretch factor. If a is negative, there is a vertical reflection and the parabola will open downwards.
- k is the vertical translation.
- h is the horizontal translation.

Given the equation of a parabola in vertex form, you should be able to sketch its graph by performing transformations on the basic parabola. This process is shown in the examples.

Example A

Given the following function in vertex form, identify the transformations of $y = x^2$.

$y = -\frac{1}{2}(x-2)^2 - 1$

Solution:

- a – Is a negative? YES. The parabola will open downwards.

- a – Is there a number in front of the squared portion of the equation? YES. The vertical stretch factor is the absolute value of this number. Therefore, the vertical stretch of this function is $\frac{1}{2}$.

- k – Is there a number after the squared portion of the equation? YES. The value of this number is the vertical translation. The vertical translation is **–1**.

- *h* – Is there a number after the variable '*x*'? YES. The value of this number is the opposite of the sign that appears in the equation. The horizontal translation is **+2**.

Example B

Given the following transformations, determine the equation of the image of $y = x^2$ in vertex form.

- Vertical stretch by a factor of 3
- Vertical translation up 5 units
- Horizontal translation left 4 units

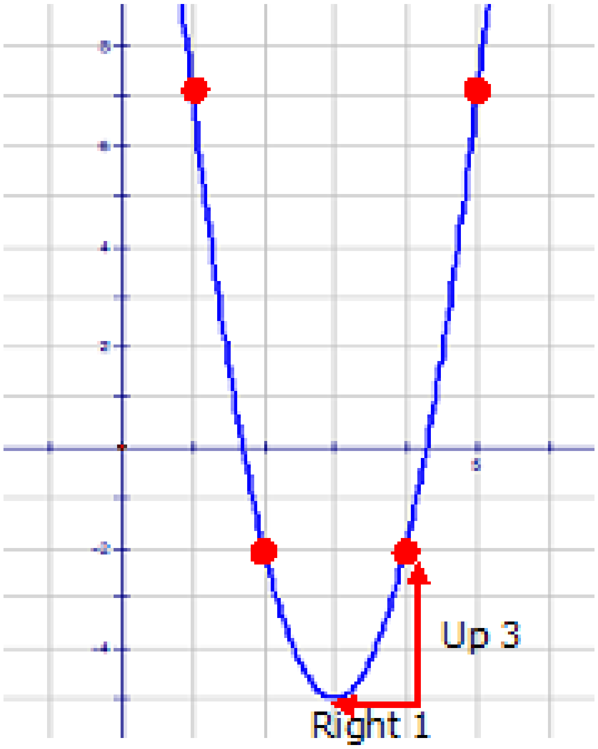
Solution:

- *a* – The image is not reflected in the *x*-axis. A negative sign is not required.
- *a* – The vertical stretch is 3, so $a = 3$.
- *k* – The vertical translation is 5 units up, so $k = 5$.
- *h* – The horizontal translation is 4 units left so $h = -4$.

The equation of the image of $y = x^2$ is $y = 3(x+4)^2 + 5$.

Example C

Using $y = x^2$ as the base function, identify the transformations that have occurred to produce the following image graph. Use these transformations to write the equation in vertex form.



Solution:

a – The parabola does not open downward so a will be positive.

a – The y-values of 1 and 4 are now up 3 and up 12. $a = 3$.

k – The $y-$coordinate of the vertex is –5 so $k = -5$.

h – The $x-$coordinate of the vertex is +3 so $h = 3$.

The equation is $y = 3(x-3)^2 - 5$.

Example D

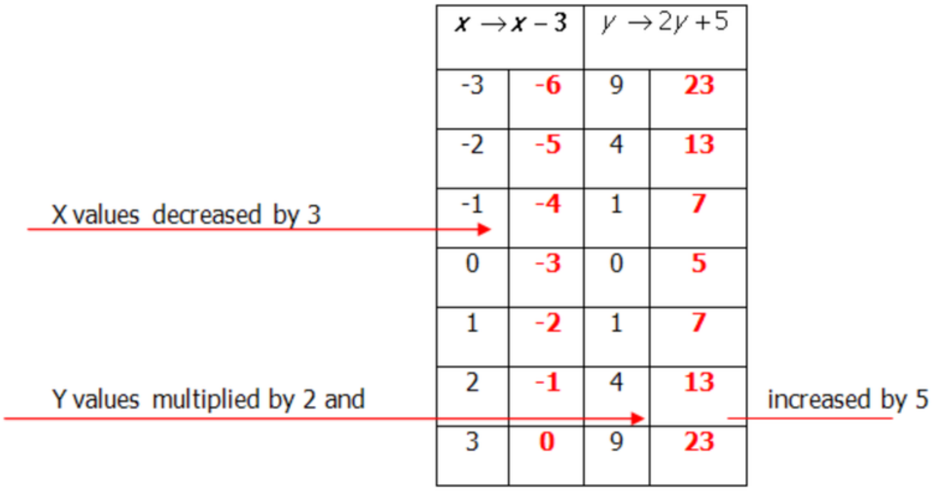
In general, the mapping rule used to generate the image of a function is $(x,y) \to (x',y')$ where (x',y') are the coordinates of the image graph. The resulting mapping rule from $y = x^2$ to the image $y = a(x-h)^2 + k$ is $(x,y) \to (x+h, ay+k)$. The mapping rule details the transformations that were applied to the coordinates of the base function $y = x^2$.

Given the following quadratic equation, $y = 2(x+3)^2 + 5$ write the mapping rule and create a table of values for the mapping rule.

Solution:

The mapping rule for this function will tell exactly what changes were applied to the coordinates of the base quadratic function.

$y = 2(x+3)^2 + 5 : \quad (x,y) \to (x-3, 2y+5)$



$x \to x - 3$		$y \to 2y + 5$	
-3	-6	9	23
-2	-5	4	13
-1	-4	1	7
0	-3	0	5
1	-2	1	7
2	-1	4	13
3	0	9	23

These new coordinates of the image graph can be plotted to generate the graph.

Concept Problem Revisited

Given the equation $y = 3(x+4)^2 + 2$, list the transformations of $y = x^2$.

$a = 3$ so the vertical stretch is **3**. $k = 2$ so the vertical translation is up **2**. $h = -4$ so the horizontal translation is left **4**.

Vocabulary

Horizontal translation

The ***horizontal translation*** is the change in the base graph $y = x^2$ that shifts the graph right or left. It changes the $x-$coordinate of the vertex.

Mapping Rule

The ***mapping rule*** defines the transformations that have occurred to a function. The mapping rule is $(x,y) \rightarrow (x',y')$ where (x',y') are the coordinates of the image graph.

Transformation

A ***transformation*** is any change in the base graph $y = x^2$. The transformations that apply to the parabola are a horizontal translation, a vertical translation, a vertical stretch and a vertical reflection.

Vertex form of $y = x^2$

The ***vertex form of*** $y = x^2$ is the form of the quadratic base function $y = x^2$ that shows the transformations of the image graph. The vertex form of the equation is $y = a(x-h)^2 + k$.

Vertical Reflection

The ***vertical reflection*** is the reflection of the image graph in the x-axis. The graph opens downward and the y-values are negative values.

Vertical Stretch

The ***vertical stretch*** is the change made to the base function $y = x^2$ by stretching (or compressing) the graph vertically. The vertical stretch will produce an image graph that appears narrower (or wider) then the original base graph of $y = x^2$.

Vertical Translation

The ***vertical translation*** is the change in the base graph $y = x^2$ that shifts the graph up or down. It changes the $y-$coordinate of the vertex.

Guided Practice

1. Identify the transformations of $y = x^2$ for the quadratic function $-2(y+3) = (x-4)^2$

2. List the transformations of $y = x^2$ and graph the function $= -(x+5)^2 + 4$

3. Graph the function $y = 2(x-2)^2 + 3$ using the mapping rule method.

Answers:

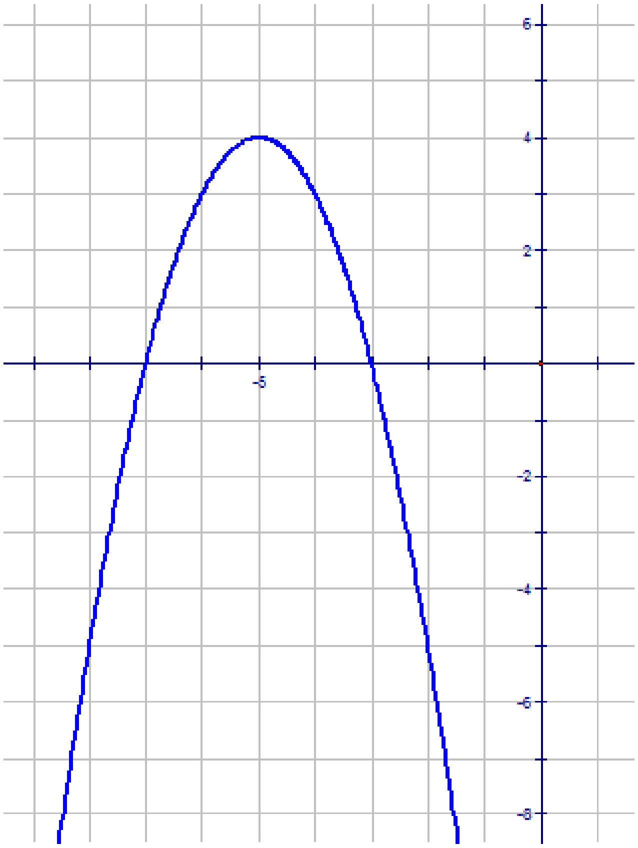
1. Rewrite the equation in vertex form. a – a is negative so the parabola opens downwards.

a – The vertical stretch of this function is $\frac{1}{2}$.

k – The vertical translation is **-3**.

h – The horizontal translation is **+4**.

2.



$$
\begin{aligned}
&a \rightarrow negative\\
&a \rightarrow 1\\
&k \rightarrow +4\\
&h \rightarrow -5
\end{aligned}
$$

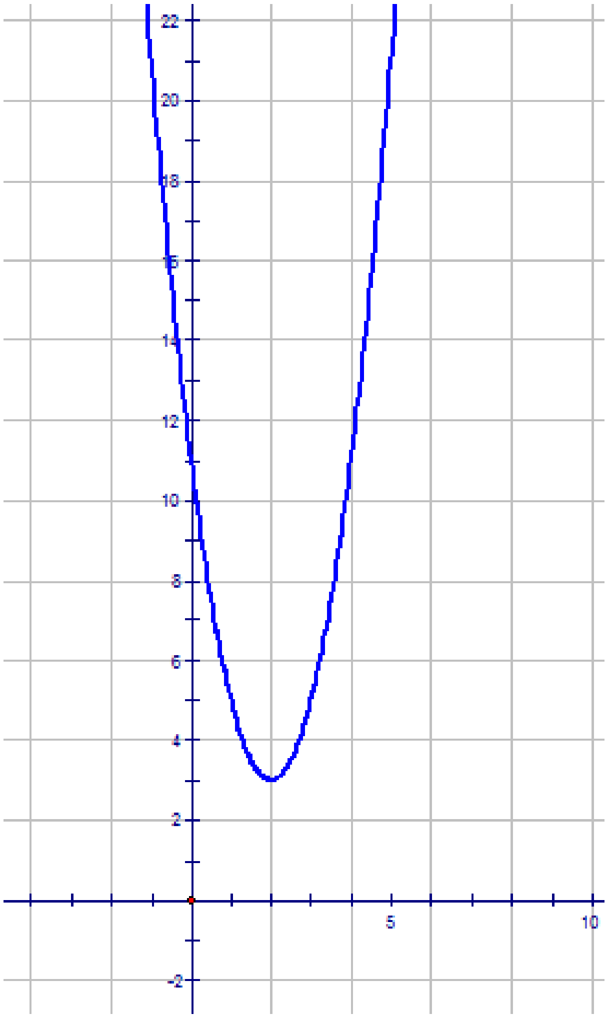
3. **Mapping Rule** $(x,y) \rightarrow (x+2, 2y+3)$

Make a table of values:

TABLE 3.19:

$x \rightarrow$	$x+2$	$y \rightarrow$	$2y+3$
–3	–1	9	21
–2	0	4	11
–1	1	1	5
0	2	0	3
1	3	1	5
2	4	4	11
3	5	9	21

Draw the graph:



Practice

Identify the transformations of $y = x^2$ in each of the given functions:

1. $y = 4(x-2)^2 - 9$
2. $y = -\frac{1}{6}x^2 + 7$
3. $y = -3(x-1)^2 - 6$
4. $y = \frac{1}{5}(x+4)^2 + 3$
5. $y = 5(x+2)^2$

Graph the following quadratic functions.

6. $y = 2(x-4)^2 - 5$
7. $y = -\frac{1}{3}(x-2)^2 + 6$
8. $y = -2(x+3)^2 + 7$
9. $y = -\frac{1}{2}(x+6)^2 + 9$
10. $y = \frac{1}{3}(x-4)^2$

Using the following mapping rules, write the equation, in vertex form, that represents the image of $y = x^2$.

11. $(x, y) \rightarrow \left(x+1, -\frac{1}{2}y\right)$

12. $(x,y) \rightarrow (x+6, 2y-3)$
13. $(x,y) \rightarrow \left(x-1, \frac{2}{3}y+2\right)$
14. $(x,y) \rightarrow (x+3, 3y+1)$
15. $(x,y) \rightarrow \left(x-5, -\frac{1}{3}y-7\right)$

Summary

You learned what makes a relation a function. You reviewed the Cartesian plane and how to plot points. You learned how to analyze a function's domain and range.

You explored linear functions and how to graph them using a table of values or intercepts. You also explored quadratic functions. You learned how to transform quadratic functions and write the equation of a parabola in vertex form.

CHAPTER 4

Linear Functions

Chapter Outline

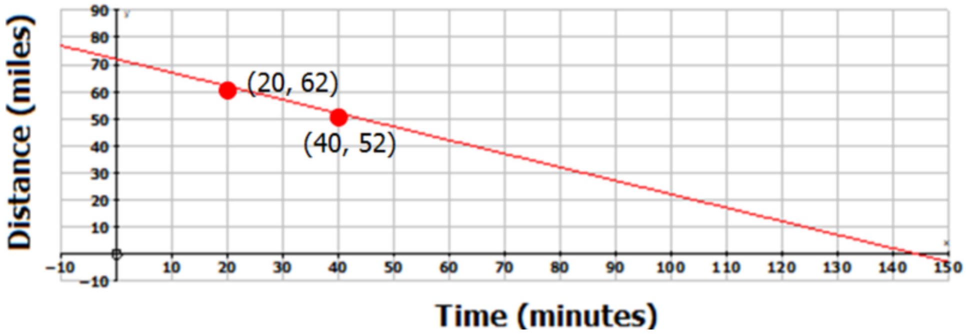
Introduction

Here you'll learn all about lines. You will learn how to find the slope of a line and the equation of a line. You will learn to graph a line directly from its equation without first having to make a table. You will learn about parallel and perpendicular lines and their relationship with slope. Finally, you will learn how to use your knowledge of lines to model and solve real-world problems.

4.1 Slopes of Lines from Graphs

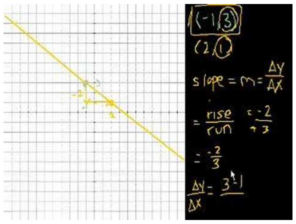
Here you'll learn what is meant by the "slope" of a line. You will learn how to find the slope of a line from its graph.

Joseph drove from his summer home to his place of work. To avoid the road construction, Joseph decided to travel the gravel road. After driving for 20 minutes he was 62 miles away from work and after driving for 40 minutes he was 52 miles away from work. This situation is shown in the graph below. Determine the slope of the line and tell what it means in this situation.



Watch This

Khan Academy Slope and Rate of Change



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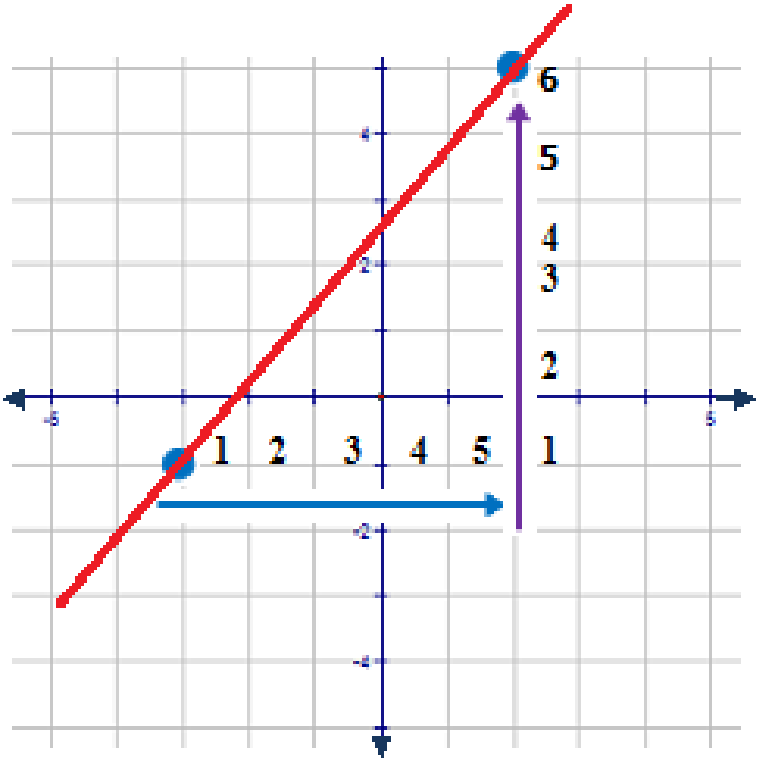
Guidance

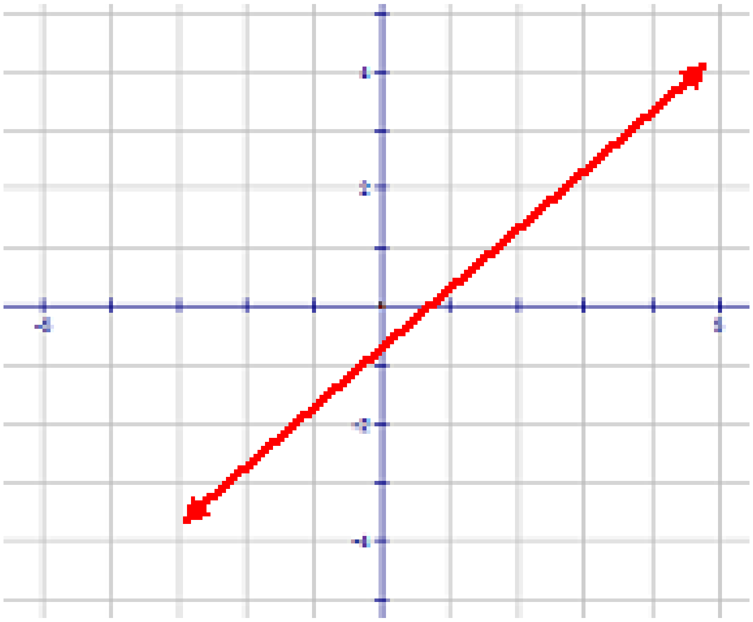
The **slope** of a line is the steepness, slant or gradient of that line. Slope is often defined as $\frac{\text{rise}}{\text{run}}$ (rise over run). The slope of a line is represented by the letter 'm' and its value is a real number.

You can determine the slope of a line from a graph by counting. Choose two points on the line that are exact points on the Cartesian grid. Exact points mean points that are located on the corner of a box or points that have coordinates that do not have to be estimated. On the graph below, two exact points are indicated by the blue dots.



Begin with the point that is farthest to the left and RUN to the right until you are directly below (in this case) the second indicated point. Count the number of spaces that you had to run to be below the second point and place this value in the run position in the denominator of the slope. Next count the number of spaces you have to move to reach the second point. In this case you have to rise upward which indicates a positive move. This value must be placed in the rise position in the numerator of the slope.

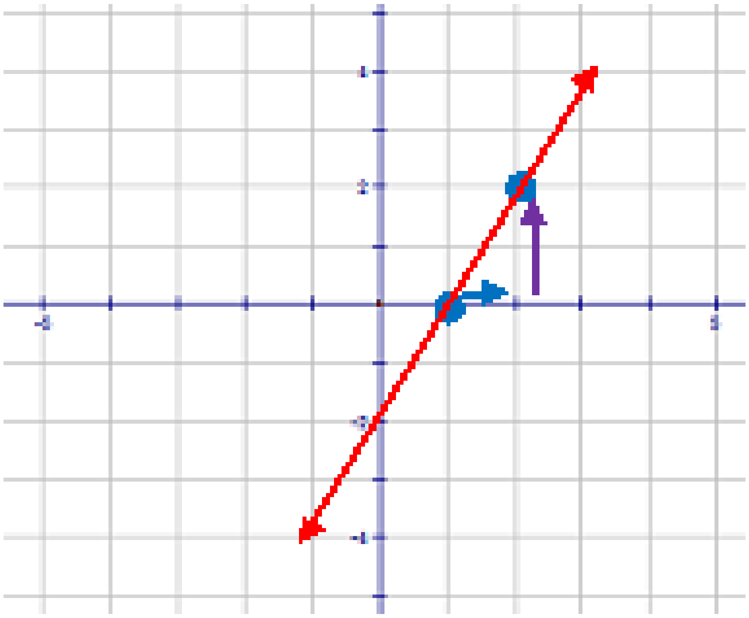
Here, you had to run 5 spaces to the right, which indicates moving 5 spaces in a positive direction. You now have $m = \frac{\text{rise}}{5}$. To reach the point directly above involved moving upward 6 spaces in a positive direction. You now have $m = \frac{6}{5}$. The slope of the above line is $\frac{6}{5}$.



In the above graph, there are not two points on the line that are exact points on the Cartesian grid. Therefore, the slope of the line cannot be determined by counting. The coordinates of points on this line would only be estimated values. When this occurs, the task of calculating the slope of the line must be presented in a different way. The slope would have to be determined from two points that are on the line and these points would have to be given.

Example A

What is the slope of the following line?

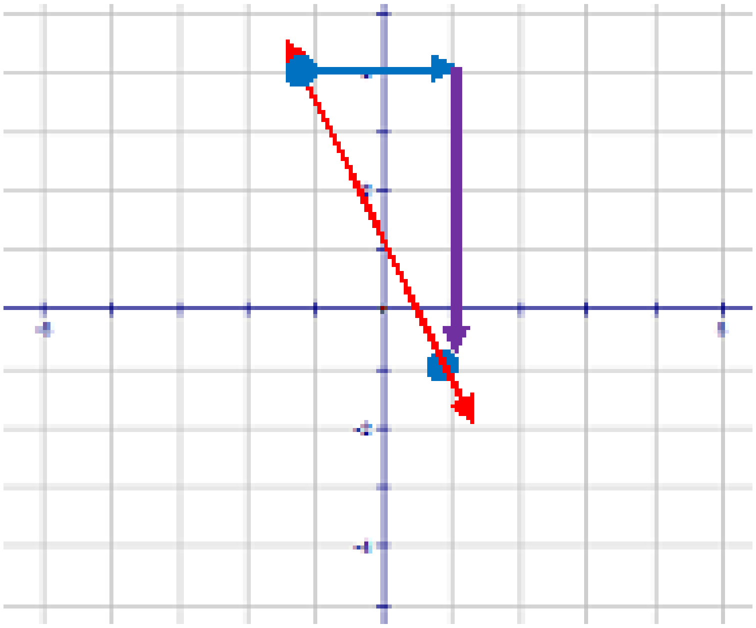


Solution: Two points have been indicated. These points are exact values on the graph. From the point to the left, run one space in a positive direction and rise upward 2 spaces in a positive direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{2}{1}$$

Example B

What is the slope of the following line?



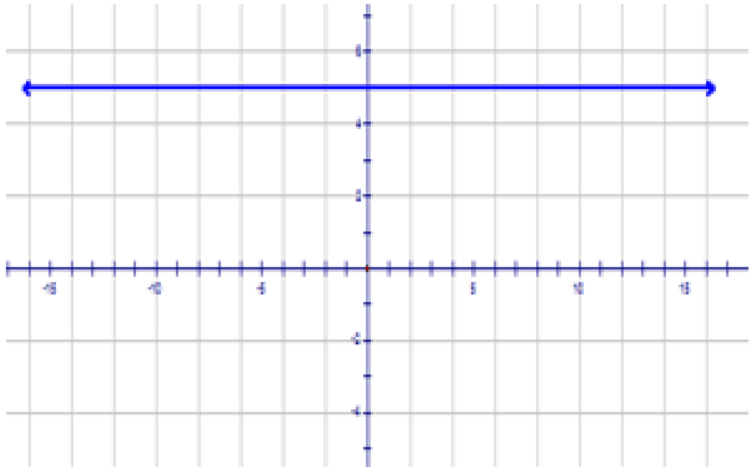
Solution: Two points have been indicated. These points are exact values on the graph. From the point to the left, run two spaces in a positive direction and move downward 5 spaces in a negative direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-5}{2}$$

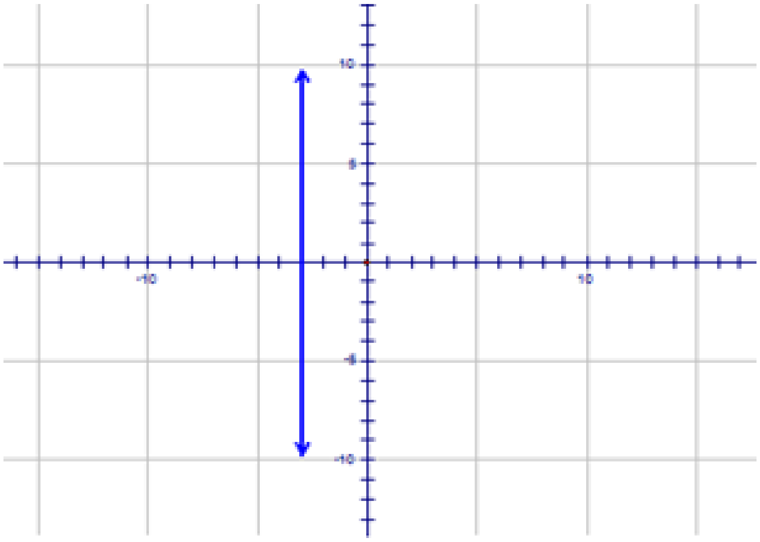
Example C

Find the slope of each of the following lines:

(a)

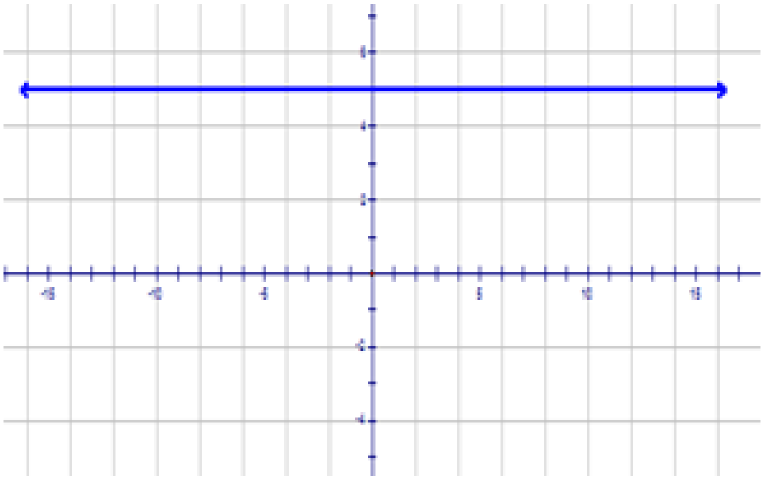


(b)



Solution:

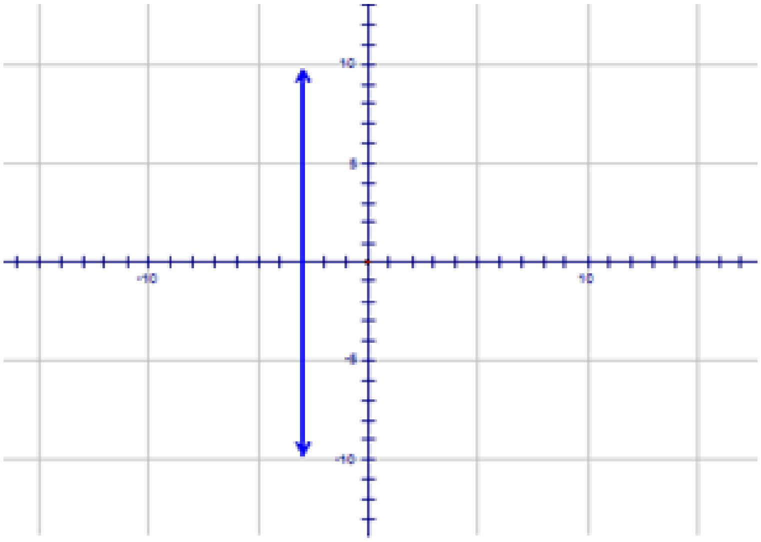
(a)



Two points on this line are (–5, 5) and (4, 5). The rise is 0 and the run is 9. The slope is $m = \frac{0}{9} = 0$.

All lines perpendicular to the y-axis (horizontal lines) will have a slope of 0.

(b)

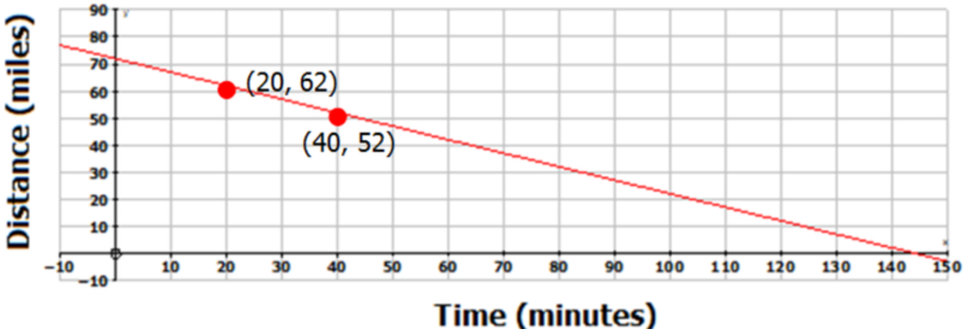


Two points on this line are (–3, 5) and (–3, –10). The rise is 15 and the run is 0. The slope is $m = \frac{15}{0} = undefined$.

All lines perpendicular to the x-axis (vertical lines) will have a slope that is undefined.

Note that having a slope of 0 is different from having a slope that is undefined.

Concept Problem Revisited



If the slope is calculated by counting, caution must be used to determine the correct values for both rise and run. The scale on both the x-axis and y-axis is increments of ten. Although these points are not exact values on the graph, knowing the coordinates makes counting an acceptable way to determine the slope of the line. The x-axis represents the time, in minutes, driving. The y-axis represents the distance, in miles, driving.

Two points have been indicated. These points are exact values on the graph. From the point to the left, run 20 spaces spaces in a positive direction and move downward 10 spaces.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-10}{20} = -\frac{1}{2}$$

The slope means that for every two minutes that Joseph is driving, he gets one mile closer to work.

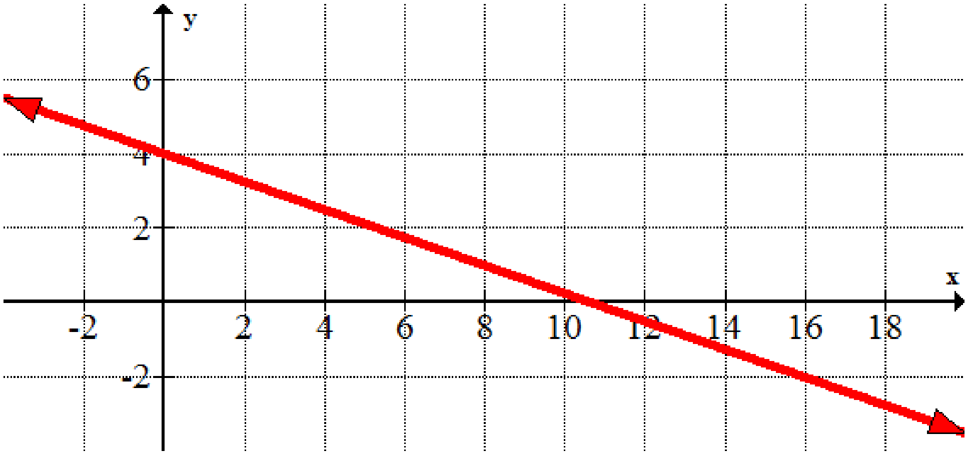
Vocabulary

Slope

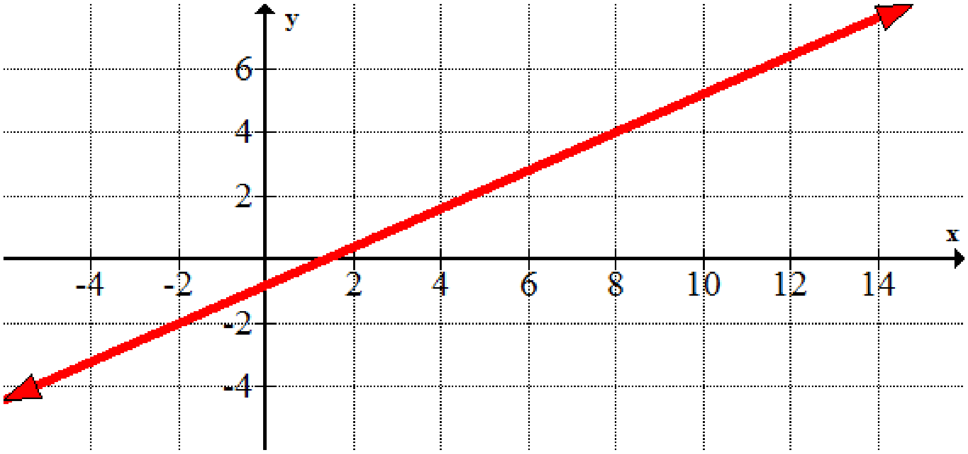
The ***slope*** of a line is the steepness of the line. One formula for slope is: $\frac{\text{rise}}{\text{run}}$.

Guided Practice

1. Identify the slope for the following graph.



2. Identify the slope for the following graph.

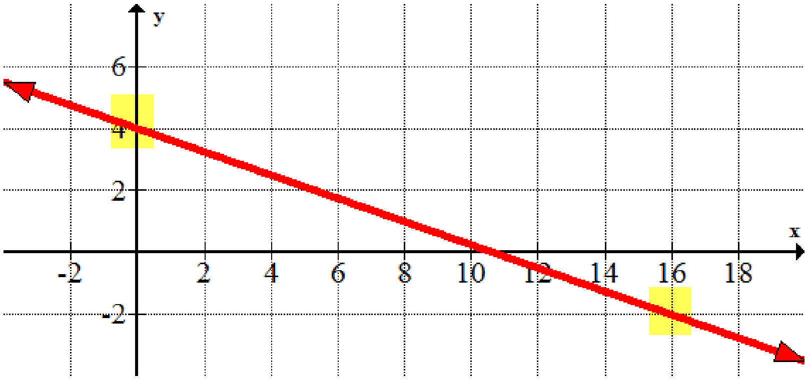


3. What is the slope of the line that passes through the point (2, 4) and is perpendicular to the x-axis?

4. What is the slope of the line that passes through the point (–6, 8) and is perpendicular to the y-axis?

Answers:

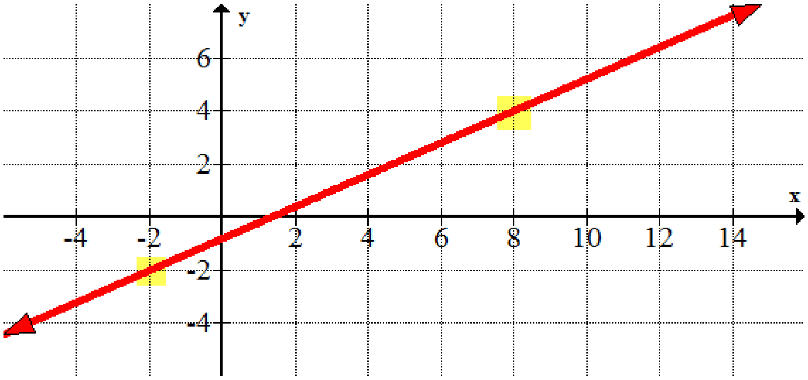
1.



Two exact points on the above graph are (0, 4) and (16, –2). From the point to the left, run sixteen spaces in a positive direction and move downward six spaces in a negative direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{-6}{16}$$
$$m = \frac{-3}{8}$$

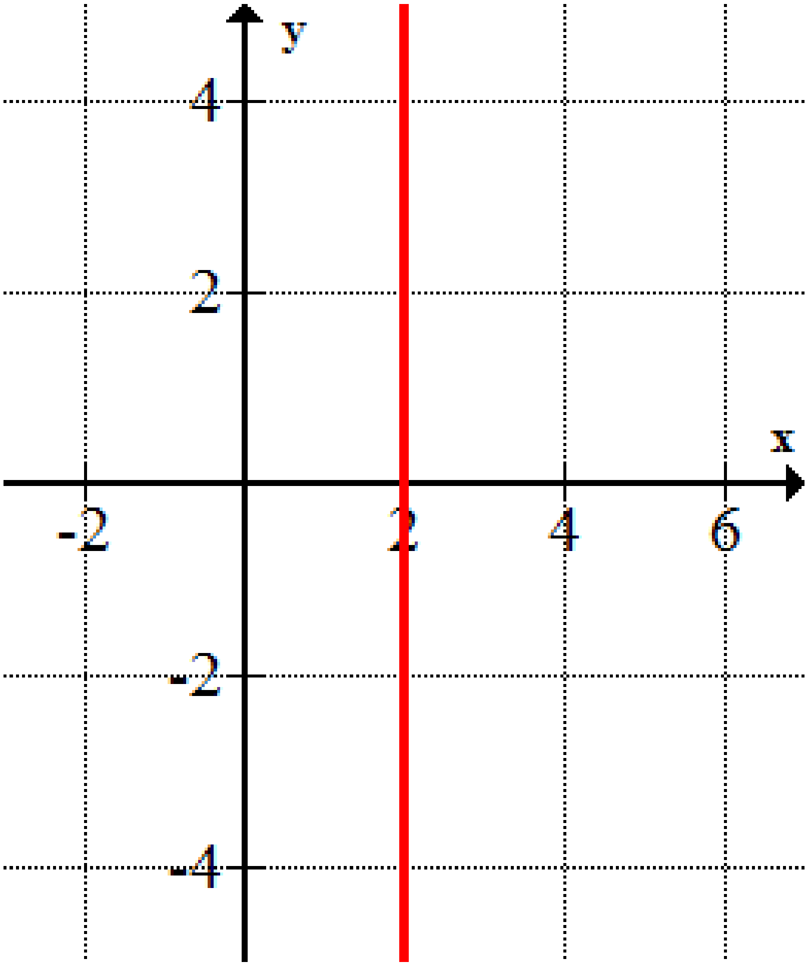
2.



Two exact points on the above graph are (–2, –2) and (8, 4). From the point to the left, run ten spaces in a positive direction and move upward six spaces in a positive direction.

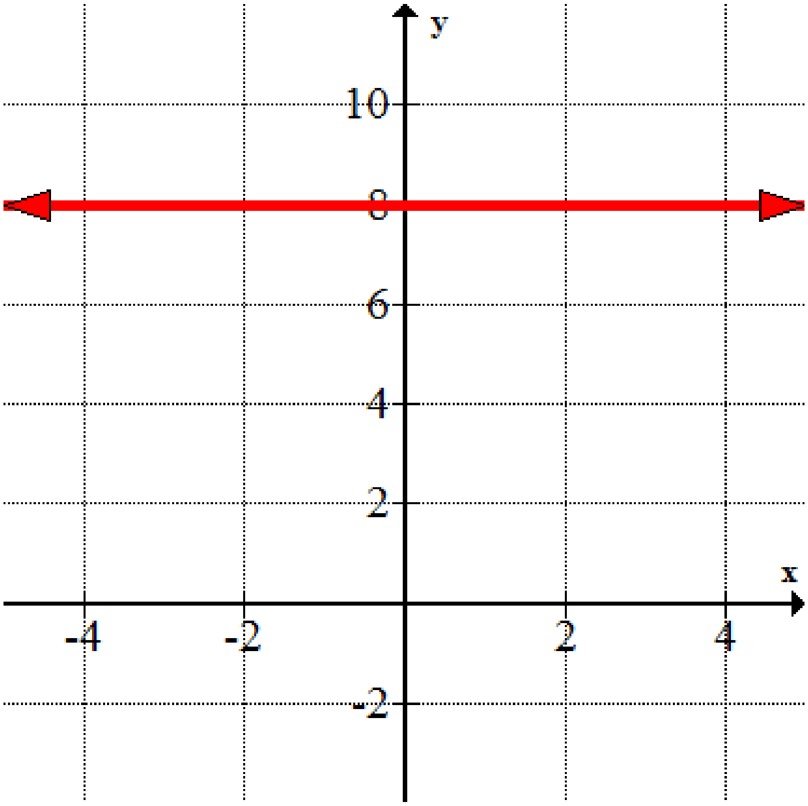
$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{6}{10}$$
$$m = \frac{3}{5}$$

3. You are not given the coordinates of two points. Sketch the graph according the information given.



A line that is perpendicular to the *x*-axis is parallel to the *y*-axis. The slope of a line that is parallel to the *y*-axis has a slope that is undefined.

4. You are not given the coordinates of two points. Sketch the graph according the information given.



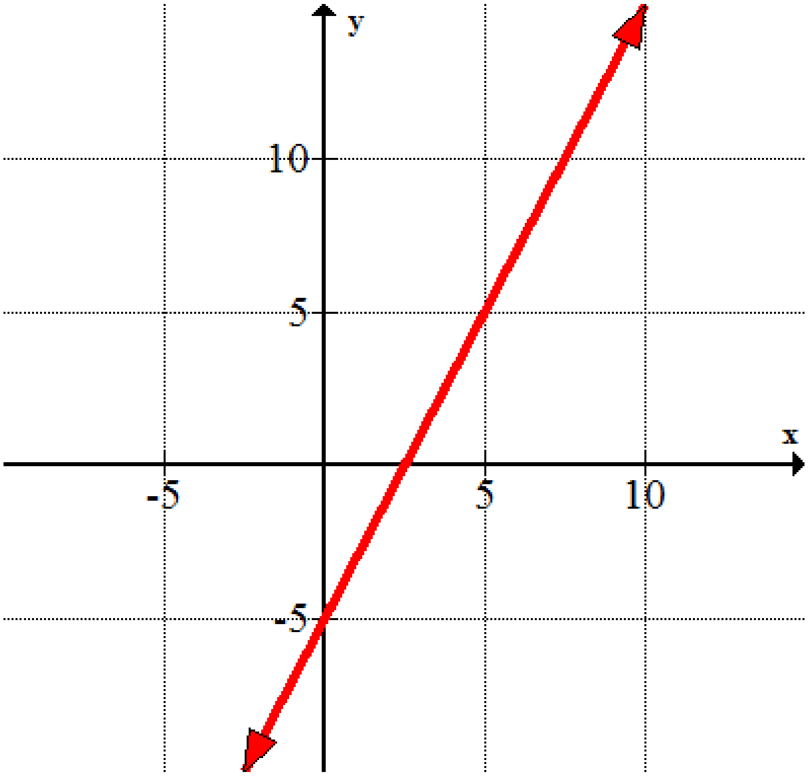
A line that is perpendicular to the *y*-axis is parallel to the *x*-axis. The slope of a line that is parallel to the *x*-axis has a slope that is zero.

Practice

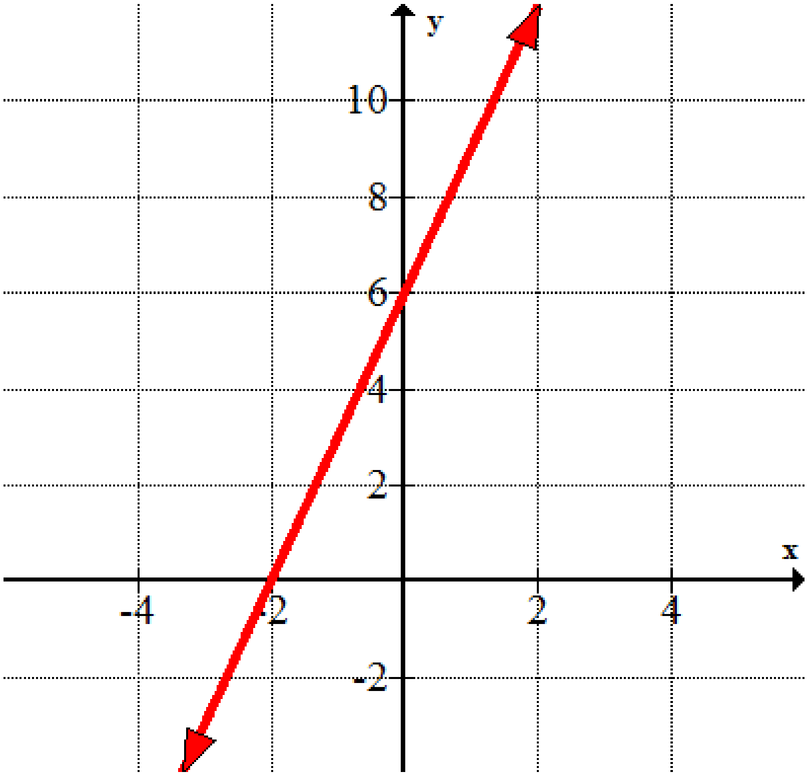
1. Explain how to find the slope of a line from its graph.
2. What does the slope of a line represent?
3. From left to right, a certain line points upwards. Is the slope of the line positive or negative?
4. How can you tell by looking at a graph if its slope is positive or negative?
5. What is the slope of a horizontal line?
6. What is the slope of a vertical line?

Find the slope of each of the following lines.

7.



8.



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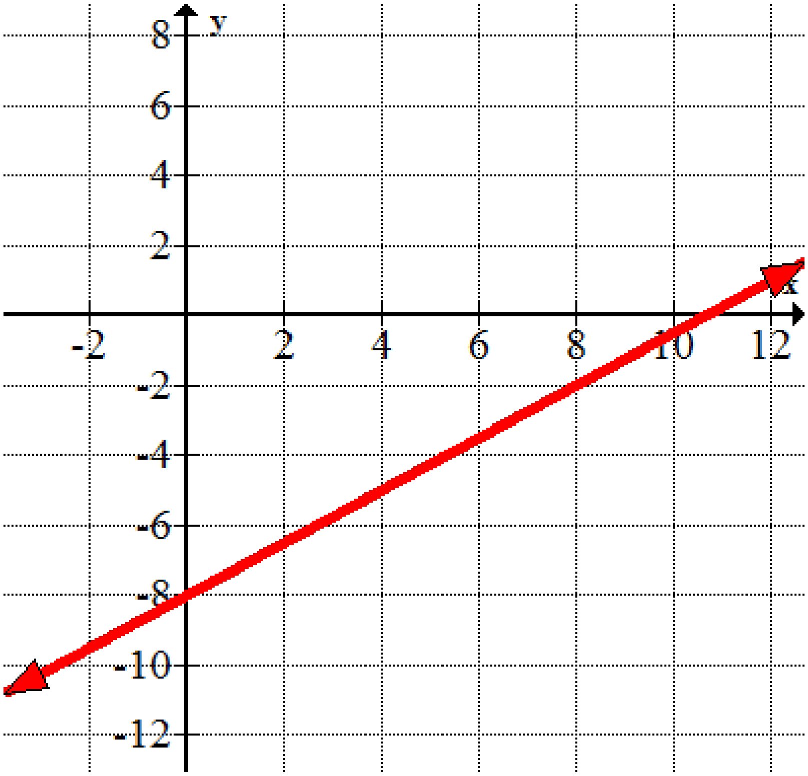

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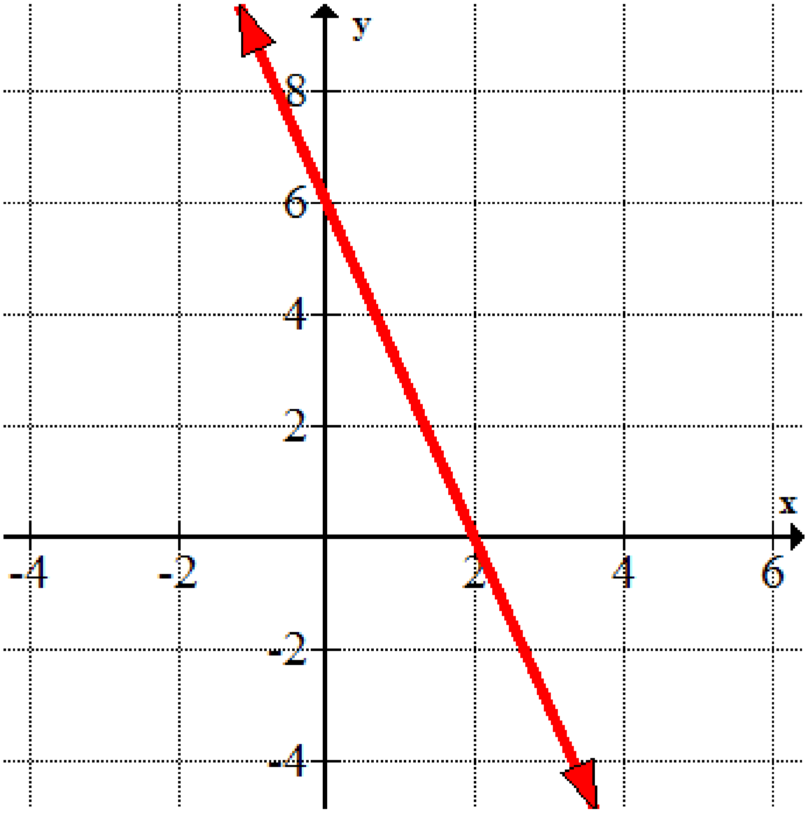

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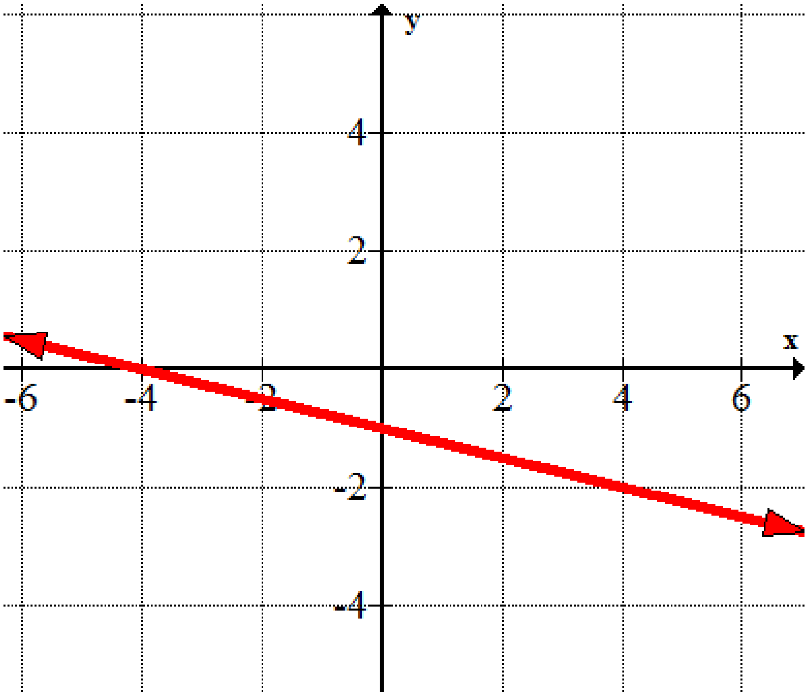

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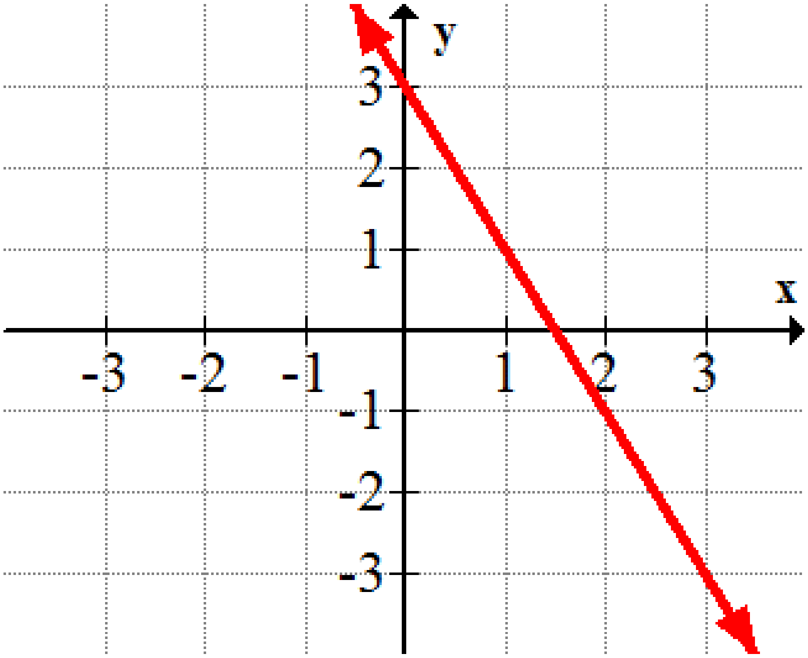
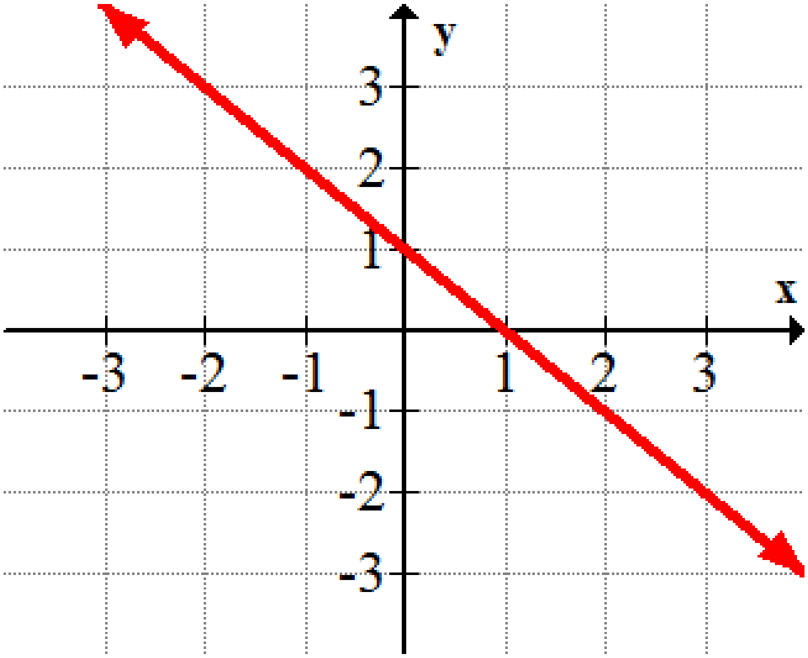
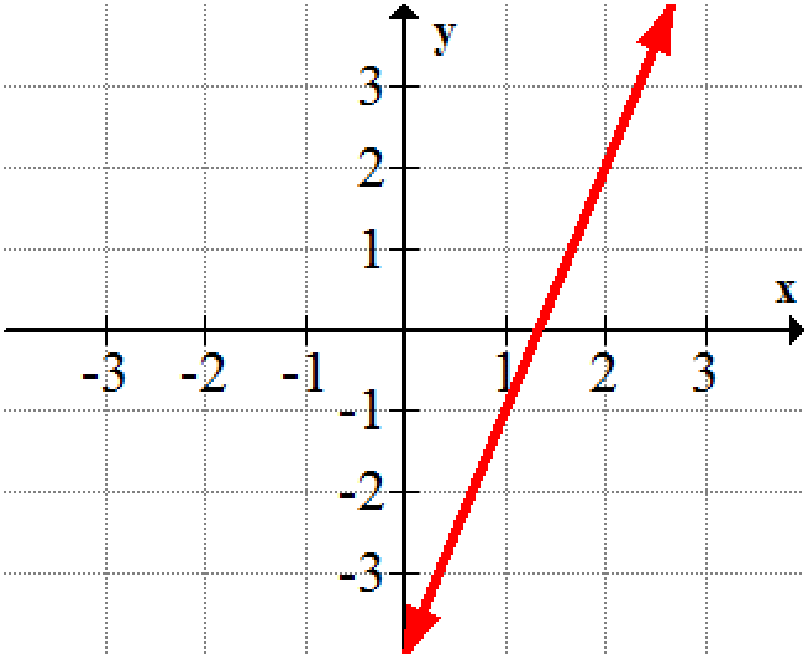


13.



14.



Plot the pairs of points and then find the slope of the line connecting the points. Can you come up with a way to find the slope without graphing?

15. $(2,4)$ and $(-1,3)$
16. $(-4,-2)$ and $(2,7)$

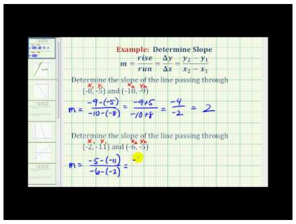
4.2 Slopes of Lines from Two Points

Here you'll learn how to find the slope of a line from two points on the line.

Can you determine the slope of the line with an x-intercept of 4 and y-intercept of –3?

Watch This

James Sousa: Ex. Determine the Slope of a Line Given Two Points on the Line]



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Guidance

The **slope** of a line is the steepness, slant or gradient of that line. Slope is defined as $\frac{\text{rise}}{\text{run}}$ (rise over run) or $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ (change in y over change in x). Whatever definition of slope is used, they all mean the same. The slope of a line is represented by the letter 'm' and its value is a real number.

You can calculate the slope of a line by using the coordinates of two points on the line. Consider a line that passes through the points $A(-6,-4)$ and $B(3,-8)$. The slope of this line can be determined by finding the change in y over the change in x.

The formula that is used is $m = \frac{y_2 - y_1}{x_2 - x_1}$ where 'm' is the slope, (x_1, y_1) are the coordinates of the first point and (x_2, y_2) are the coordinates of the second point. The choice of the first and second point will not affect the result.

$A\begin{pmatrix} x_1, & y_1 \\ -6, & -4 \end{pmatrix} \quad B\begin{pmatrix} x_2, & y_2 \\ 3, & -8 \end{pmatrix}$	Label the points to indicate the first and second points.
$m = \frac{y_2 - y_1}{x_2 - x_1}$	Substitute the values into the formula.
$m = \frac{-8 - -4}{3 - -6}$	Simplify the values (if possible)
$m = \frac{-8+4}{3+6}$	Evaluate the numerator and the denominator
$m = \frac{-4}{9}$	Reduce the fraction (if possible)

Example A

Determine the slope of the line passing through the pair of points(–3, –8) and (5, 8).

Solution: To determine the slope of a line from two given points, the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ can be used. Don't forget to designate your choice for the first and the second point. Designating the points will reduce the risk of entering the values in the wrong location of the formula.

$$\begin{pmatrix} x_1, & y_1 \\ -3, & -8 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 5, & 8 \end{pmatrix}$$

Substitute the values into the formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

$$m = \frac{8 - -8}{5 - -3}$$

Simplify $m = \dfrac{8+8}{5+3}$

Calculate $m = \dfrac{16}{8}$

Simplify $m = 2$

Example B

Determine the slope of the line passing through the pair of points $(9,5)$ and $(-1,6)$.

Solution:

$$\begin{pmatrix} x_1, & y_1 \\ 9, & 5 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -1, & 6 \end{pmatrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{6 - 5}{-1 - 9}$$

$$m = -\frac{1}{10}$$

Example C

Determine the slope of the line passing through the pair of points $(-2,7)$ and $(-3,-1)$.

Solution:

$$\begin{pmatrix} x_1, & y_1 \\ -2, & 7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -3, & -1 \end{pmatrix}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-1 - 7}{-3 - -2}$$

$$m = \frac{-1 - 7}{-3 + 2}$$

$$m = \frac{-8}{-1}$$

$$m = 8$$

Concept Problem Revisited

Determine the slope of the line with an x-intercept of 4 and y-intercept of –3.

$$\begin{pmatrix} x_1, & y_1 \\ 4, & 0 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 0, & -3 \end{pmatrix}$$

Express the $x-$ and y-intercepts as coordinates of a point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-3 - 0}{0 - 4}$$

$$m = \frac{-3}{-4}$$

$$m = \frac{3}{4}$$

Vocabulary

Slope

The ***slope*** of a line is the steepness, slant or gradient of that line. Slope is defined as $\frac{\text{rise}}{\text{run}}$ (rise over run) or $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ (change in y over change in x).

Guided Practice

Calculate the slope of the line that passes through the following pairs of points:

1. (5, –7) and (16, 3)

2. (–6, –7) and (–1, –4)

3. (5, –12) and (0, –6)

4. The local *Wine and Dine Restaurant* has a private room that can serve as a banquet facility for up to 200 guests. When the manager quotes a price for a banquet she includes the cost of the room rent in the price of the meal. The price of a banquet for 80 people is $900 while one for 120 people is $1300.

i) Plot a graph of cost versus the number of people.
ii) What is the slope of the line and what meaning does it have for this situation?

Answers:

1. The slope is $\frac{10}{11}$.

$$\begin{pmatrix} x_1, & y_1 \\ 5, & -7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 16, & 3 \end{pmatrix}$$

Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{3 - -7}{16 - 5}$$

Fill in the values

$$m = \frac{3 + 7}{16 - 5}$$

Simplify the numerator and denominator (if possible)

$$m = \frac{10}{11}$$

Calculate the value of the numerator and the denominator

2. The slope is $\frac{3}{5}$.

$$\begin{pmatrix} x_1, & y_1 \\ -6, & -7 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ -1, & -4 \end{pmatrix}$$

Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-4 - -7}{-1 - -6}$$

Fill in the values

$$m = \frac{-4 + 7}{-1 + 6}$$

Simplify the numerator and denominator (if possible)

$$m = \frac{3}{5}$$

Calculate the value of the numerator and the denominator

3. The slope is $-\frac{6}{5}$.

$$\begin{pmatrix} x_1, & y_1 \\ 5, & -12 \end{pmatrix} \quad \begin{pmatrix} x_2, & y_2 \\ 0, & -6 \end{pmatrix}$$

Designate the points as to the first point and the second point.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{-6 - -12}{0 - 5}$$

Fill in the values

$$m = \frac{-6 + 12}{0 - 5}$$

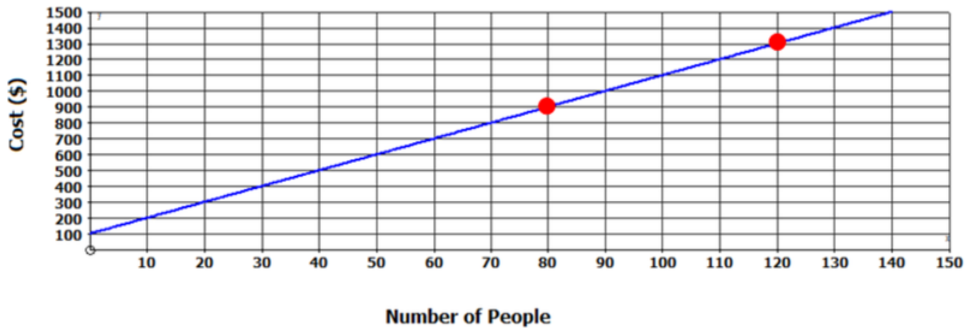
Simplify the numerator and denominator (if possible)

$$m = \frac{6}{-5}$$

Calculate the value of the numerator and the denominator

$$m = -\frac{6}{5}$$

4.



The domain for this situation is N. However, to demonstrate the slope and its meaning, it is more convenient to draw the graph as $x \varepsilon R$ instead of showing just the points on the Cartesian grid. The x-axis has a scale of 10 and the y-axis has a scale of 100. The slope can be calculated by counting to determine $\frac{\text{rise}}{\text{run}}$.

From the point to the left, run four spaces (40) in a positive direction and move upward four spaces (400) in a positive direction.

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{400}{40}$$
$$m = \frac{10}{1}$$
$$m = \frac{10\ dollars}{1\ person}$$

The slope represents the cost of the meal for each person. It will cost $10 per person for the meal.

Practice

Calculate the slope of the line that passes through the following pairs of points:

1. (3, 1) and (–3, 5)
2. (–5, –57) and (5, –5)
3. (–3, 2) and (7, –1)
4. (–4, 2) and (4, 4)
5. (–1, 5) and (4, 3)
6. (0, 2) and (4, 1)
7. (12, 15) and (17, 3)
8. (2, –43) and (2, –14)
9. (–16, 21) and (7, 2)

The cost of operating a car for one month depends upon the number of miles you drive. According to a recent survey completed by drivers of midsize cars, it costs $124/month if you drive 320 miles/month and $164/month if you drive 600 miles/month.

10. Plot a graph of distance/month versus cost/month.
11. What is the slope of the line and what does it represent?

A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for $1950 and 15 in another for $2850. The number of computers and the cost forms a linear relationship.

12. Plot a graph of number of computers sold versus cost.
13. What is the slope of the line and what does it represent?

Shop Rite sells one-quart cartons of milk for $1.65 and two-quart cartons for $2.95. Assume there is a linear relationship between the volume of milk and the price.

14. Plot a graph of volume of milk sold versus cost.
15. What is the slope of the line and what does it represent?

Some college students, who plan on becoming math teachers, decide to set up a tutoring service for high school math students. One student was charged $25 for 3 hours of tutoring. Another student was charged $55 for 7 hours of tutoring. The relationship between the cost and time is linear.

16. Plot a graph of time spent tutoring versus cost.
17. What is the slope of the line and what does it represent?

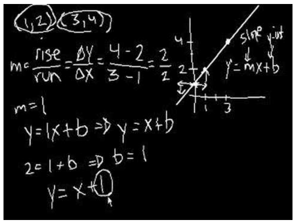
4.3 Equations of Lines from Two Points

Here you'll learn how to find the equation of a line in slope intercept form or standard form.

Write the equation of the line that has the same slope as $3x+2y-8=0$ and passes through the point (–6, 7).

Watch This

Khan Academy Equation of a Line



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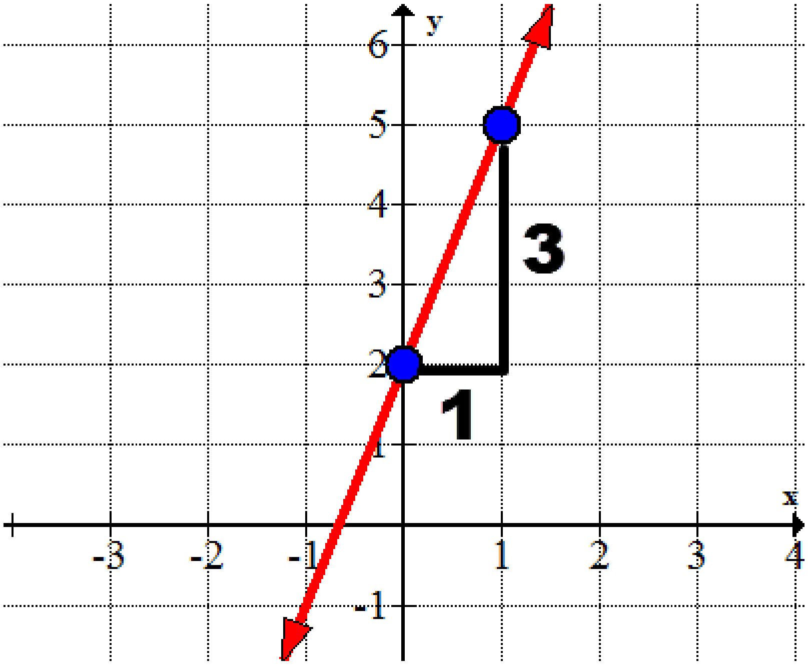
Guidance

Equations of Lines in Slope-Intercept Form

You can always find the **equation of a line** if you know its **slope** (m) and y**-intercept** (b). You can then write the equation of the line in slope-intercept form:

$$y = mx + b$$

Always remember that as long as you know the slope and the y-intercept, you can write the equation of the line. *Sometimes you will have to do some work to determine either the slope or the y-intercept, as will be shown in the examples.*



In the above graph, the line crosses the y-axis at the point (0, 2). This is the y-intercept of the line, so $b = 2$. The slope is $m = \frac{\text{rise}}{\text{run}} = \frac{3}{1} = 3$. Now that you know the values for m and b, you can write the equation of the line:

$$\boxed{y = 3x + 2.}$$

Equations of Horizontal and Vertical Lines

A line that is parallel to the x-axis (a horizontal line) will always have the equation $y = a$, where a is the $y-$coordinate of the point through which the line passes.

A line that is parallel to the y-axis (a vertical line) will always have the equation $x = c$, where c is the $x-$coordinate of the point through which the line passes.

Equations of Lines in Standard Form

The equation of a line can also be written in another form that is known as standard form. Standard form is $Ax + By + C = 0$. You can rewrite the equation of a line given in standard form as an equation in slope-intercept form by solving for y. This will allow you to determine the slope and y-intercept of the line. For example, can you rewrite the equation $3x + 2y - 8 = 0$ in slope-intercept form?

$$\begin{aligned}
3x + 2y - 8 &= 0 \\
3x - 3x + 2y - 8 &= 0 - 3x \\
2y - 8 &= -3x \\
2y - 8 + 8 &= -3x + 8 \\
2y &= -3x + 8 \\
\frac{2y}{2} &= \frac{-3x}{2} + \frac{8}{2} \\
&\boxed{y = \frac{-3}{2}x + 4}
\end{aligned}$$

The equation has been solved for y and is now in the form $y = mx + b$. You can now see that the slope of the line is $m = -\frac{3}{2}$.

Occasionally you will be given the equation of a line given in slope-intercept form and want to rewrite this equation in standard form. To do this, multiply all terms by the denominators of any fractions to get rid of the fractions and then move all variables and constants to one side of the equation in order to set it equal to 0.

Example A

Write the equation for the line that passes through the points $A(3,4)$ and $B(8,2)$.

Solution: Remember that to find the equation you need to figure out the slope (m) and the y-intercept (b). First, determine the slope of the line:

$$\begin{pmatrix} x_1, & y_1 \\ 3, & 4 \end{pmatrix} \qquad \begin{pmatrix} x_2, & y_2 \\ 8, & 2 \end{pmatrix}$$

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m &= \frac{2-4}{8-3} \\ m &= -\frac{2}{5} \end{aligned}$$

Next, determine the y-intercept of the line. This can be done using the slope-intercept form for the equation of the line by substituting one of the given points for x and y (it doesn't matter which point you use).

$$\begin{aligned} y &= mx + b \\ (4) &= \left(-\frac{2}{5}\right)(3) + b && \text{Use one of the given points for } (x,y) \text{ and } \left(-\frac{2}{5}\right) \text{ for } m. \\ 4 &= -\frac{6}{5} + b && \text{Solve for } b. \\ 4 + \frac{6}{5} &= -\frac{6}{5} + \frac{6}{5} + b \\ 4 + \frac{6}{5} &= b && \text{To add these numbers, a common denominator is necessary.} \\ \left(\frac{4}{1}\right)\left(\frac{5}{5}\right) + \frac{6}{5} &= b \\ \frac{20}{5} + \frac{6}{5} &= b \\ \frac{26}{5} &= b && \text{The } y\text{-intercept is } \left(0, \frac{26}{5}\right) \end{aligned}$$

The equation for the line is

$$\boxed{y = -\frac{2}{5}x + \frac{26}{5}}$$

Example B

(a) Write the equation of the line passing through the point (6, –4) and parallel to the x-axis.

(b) Write the equation of the line passing through the point (3, –2) and parallel to the y-axis.

Solution:

(a) A line that is parallel to the x-axis will always have the equation $y = a$, where a is the $y-$coordinate of the point through which the line passes. Therefore, the equation of this line is

$$\boxed{y = -4}$$

.

(b) A line that is parallel to the y-axis will always have the equation $x = c$, where c is the $x-$coordinate of the point through which the line passes. Therefore, the equation of this line is

$$\boxed{x = 3}$$

.

Example C

Write the equation of the line that passes through the point (–2, 5) and has the same y-intercept as: $-3x+6y+18=0$.

Solution: In order to find the equation of any line, you can figure out the slope and the intercept. First, find the y-intercept.

$$-3x+6y+18=0$$

is written in standard form. To determine the y-intercept, solve for y

$$\begin{aligned}
-3x+3x+6y+18-18 &= 3x-18\\
6y &= 3x-18\\
\frac{6y}{6} &= \frac{3}{6}x-\frac{18}{6}\\
y &= \frac{1}{2}x-3
\end{aligned}$$

You can see now that $b=-3$. The line also passes through the point (–2, 5). You can use this point along with the y-intercept to help find the slope.

$$\begin{aligned}
y &= mx+b && \text{Fill in } -3 \text{ for } b \text{ and } (x,y)=(-2,5)\\
(5) &= m(-2)+(-3)\\
5 &= -2m-3 && \text{Solve for } m.\\
5+3 &= -2m-3+3\\
8 &= -2m\\
\frac{8}{-2} &= \frac{-2m}{-2}\\
-4 &= m
\end{aligned}$$

$$\boxed{y = -4x - 3}$$

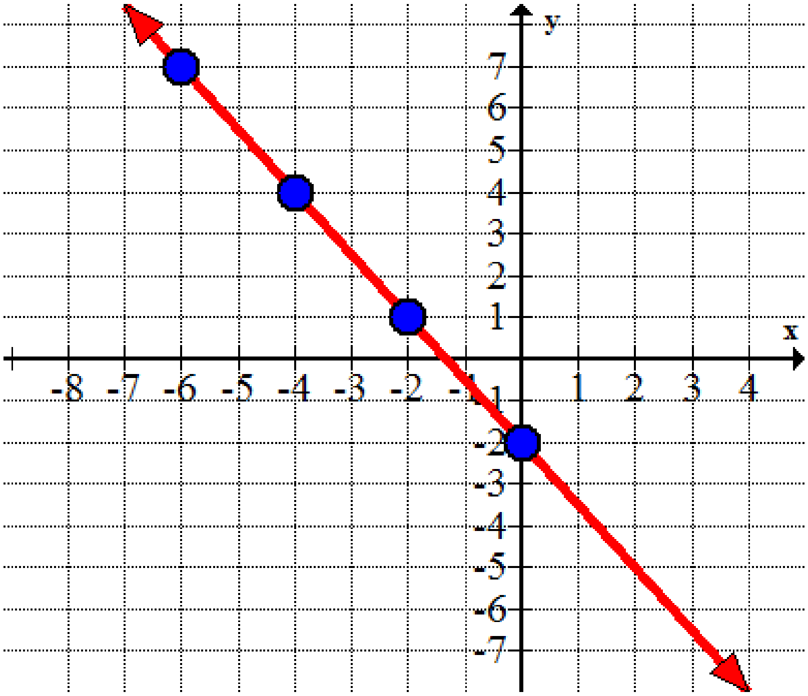
is the equation of the line.

Concept Problem Revisited

To determine the slope of $3x+2y-8=0$, solve the equation for 'y'.

$$\begin{aligned}
3x+2y-8&=0\\
3x-3x+2y-8&=0-3x\\
2y-8&=-3x\\
2y-8+8&=-3x+8\\
2y&=-3x+8\\
\frac{2y}{2}&=\frac{-3x}{2}+\frac{8}{2}\\
&\boxed{y=\frac{-3}{2}x+4}
\end{aligned}$$

The equation is now in the form $y = mx + b$. The slope of the line is $m = -\frac{3}{2}$. The line can now be sketched on the Cartesian grid.



The point (–6, 7) was plotted first and then the slope was applied – run two units to the right and then move downwards three units. The line crosses the y-axis at the point (0, –2). This is the y-intercept of the graph. The equation for the line has a slope of $-\frac{3}{2}$ and a y-intercept of –2.

The equation of the line is

$$\boxed{y=-\frac{3}{2}x-2}$$

.

Vocabulary

Slope – Intercept Form

The ***slope-intercept form*** is one method for writing the equation of a line. The slope-intercept form is $y = mx+b$ where m refers to the slope and b identifies the y-intercept.

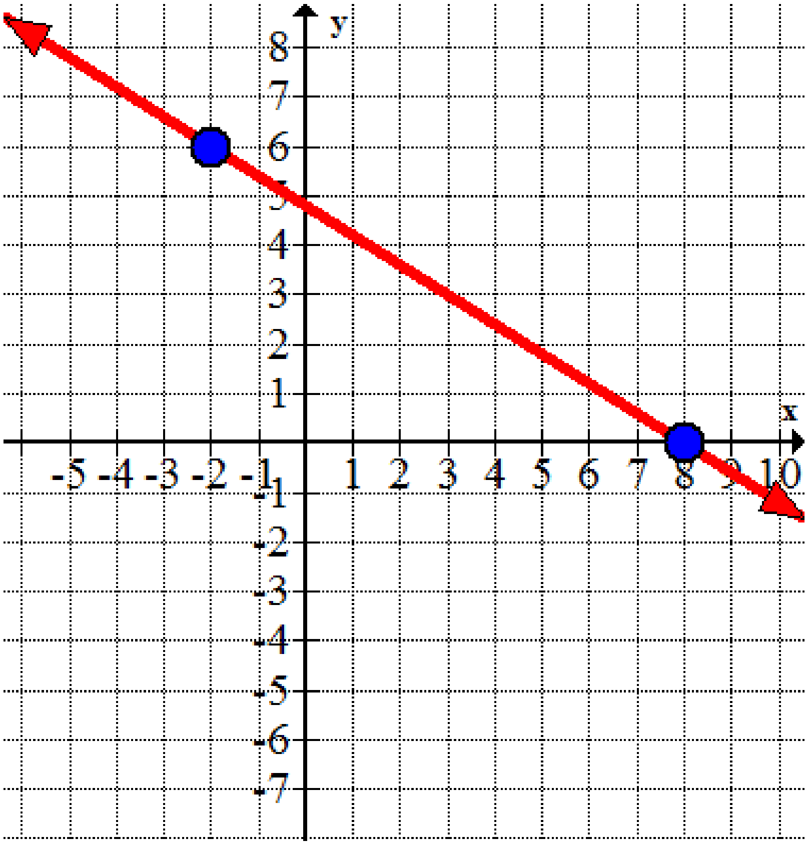
Standard Form

The ***standard form*** is another method for writing the equation of a line. The standard form is $Ax+By+C=0$ where A is the coefficient of x, B is the coefficient of y and C is a constant.

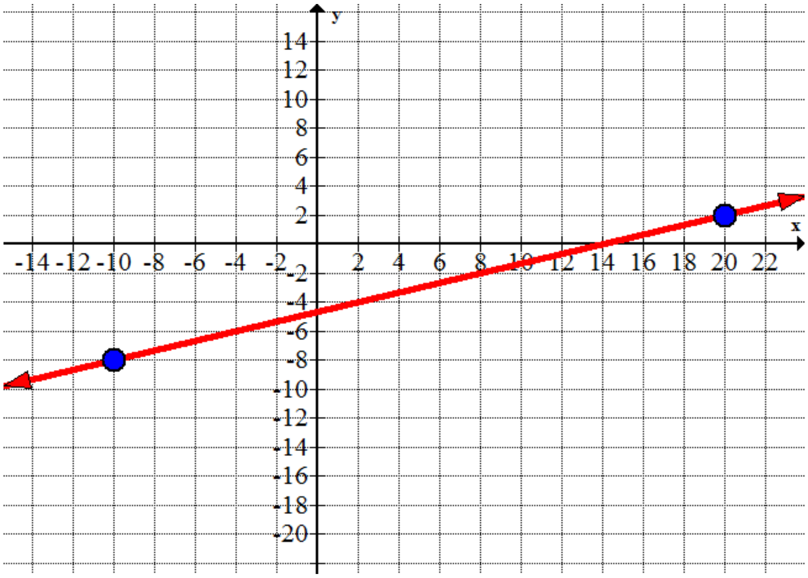
Guided Practice

Write the equation for each of the lines graphed below in slope-intercept form and in standard form.

1.



2.



3. Write the equation for the line that passes through the point (–4, 7) and is perpendicular of the *y*-axis.

Answers:

1. Two points on the graph are (–2, 6) and (8, 0). First use the formula to determine the slope of this line.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
m &= \frac{0-6}{8--2} \\
m &= \frac{0-6}{8+2} \\
m &= -\frac{6}{10} \\
m &= -\frac{3}{5}
\end{aligned}$$

Use the slope and one of the points to determine the *y*-intercept.

$$y = mx + b$$

$$(0) = \left(-\frac{3}{5}\right)(8) + b$$ Use one of the given points for (x,y) and $\left(-\frac{3}{5}\right)$ for m.

$$0 = -\frac{24}{5} + b$$ Solve for b.

$$0 + \frac{24}{5} = -\frac{24}{5} + \frac{24}{5} + b$$

$$\frac{24}{5} = b$$

The equation for the line in slope-intercept form is

$$\boxed{y = -\frac{3}{5}x + \frac{24}{5}}$$

. To express the equation in standard form, multiply each term by 5 and set the equation equal to zero.

$$\begin{aligned}
y &= -\frac{3}{5}x+\frac{24}{5}\\
5(y) &= 5\left(-\frac{3}{5}x\right)+5\left(\frac{24}{5}\right)\\
5(y) &= \cancel{5}\left(-\frac{3}{\cancel{5}}x\right)+\cancel{5}\left(\frac{24}{\cancel{5}}\right)\\
5y &= -3x+24
\end{aligned}$$

Apply the zero principle to move $-3x$ to the left side of the equation.

$$\begin{aligned}
5y+3x &= -3x+3x+24\\
5y+3x &= 24\\
5y+3x-24 &= 24-24\\
5y+3x-24 &= 0
\end{aligned}$$

$$\boxed{3x+5y-24=0}$$

2. Two exact points on the graph are (–10, –8) and (20, 2). The slope of the line can be calculated by counting to determine the value of $m=\frac{\text{rise}}{\text{run}}$.

$$\begin{aligned}
m &= \frac{\text{rise}}{\text{run}}\\
m &= \frac{10}{30}\\
m &= \frac{1}{3}
\end{aligned}$$

Now, use the slope and one of the points to calculate the y-intercept of the line.

$$y = mx+b \qquad m=\frac{1}{3} \text{ and } \begin{pmatrix} x, & y\\ 20, & 2\end{pmatrix}$$

$$\begin{aligned}
2 &= \frac{1}{3}(20)+b\\
2 &= \frac{20}{3}+b
\end{aligned}$$

Solve for b.

$$\begin{aligned}
2-\frac{20}{3} &= \frac{20}{3}-\frac{20}{3}+b\\
2-\frac{20}{3} &= b
\end{aligned}$$

A common denominator is needed.

$$\begin{aligned}
\left(\frac{3}{3}\right)\frac{2}{1}-\frac{20}{3} &= b\\
\frac{6}{3}-\frac{20}{3} &= b\\
-\frac{14}{3} &= b
\end{aligned}$$

The equation of the line in slope-intercept form is

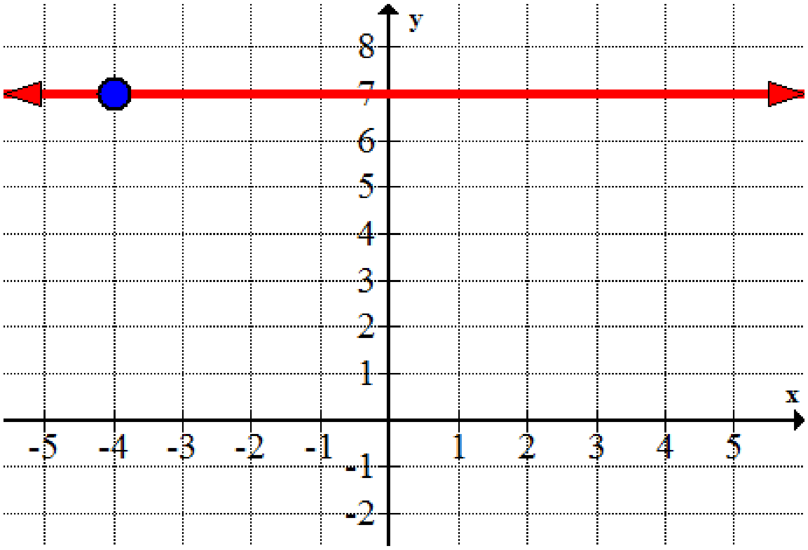
$$\boxed{y=\frac{1}{3}x-\frac{14}{3}}$$

. Multiply the equation by 3 and set the equation equal to zero to write the equation in standard form. The equation of the line in standard form is

$$\boxed{x-3y-14=0}$$

.

3. Begin by sketching the graph of the line.



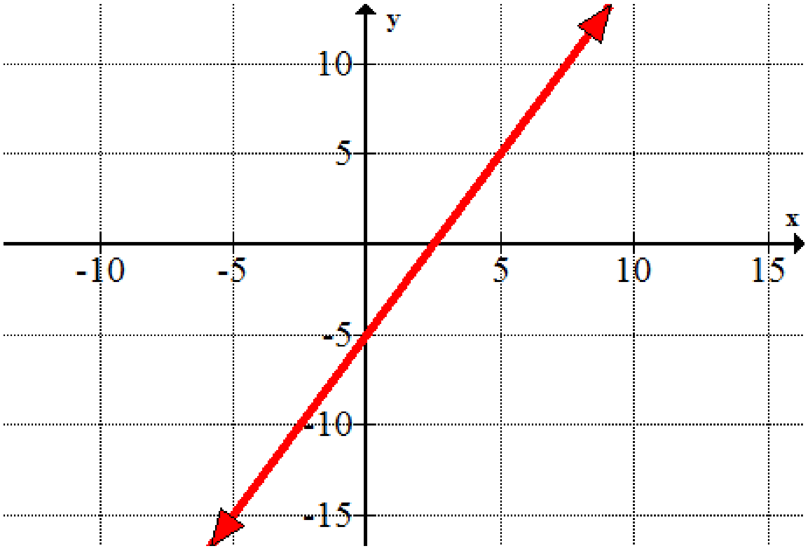
A line that is perpendicular to the *y*-axis is parallel to the *x*-axis. The slope of such a line is zero. The equation of this line is

$$\boxed{y=7}$$

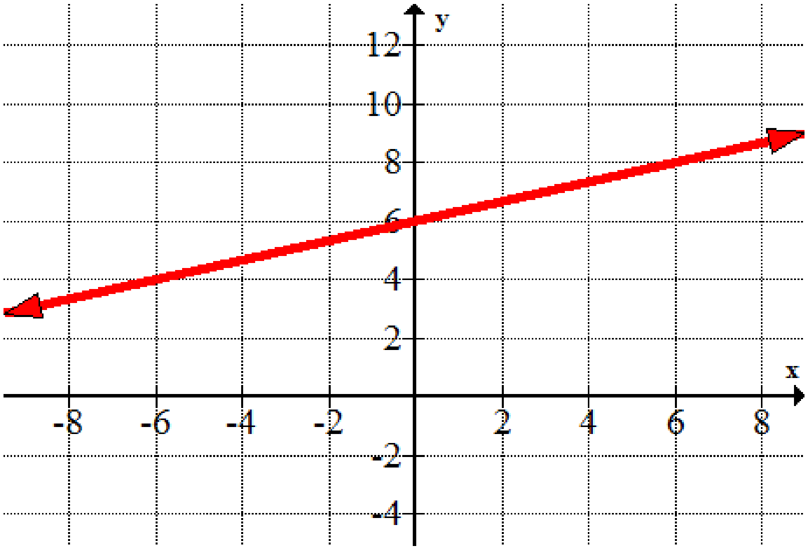
Practice

For each of the following graphs, write the equation of the line in slope-intercept form.

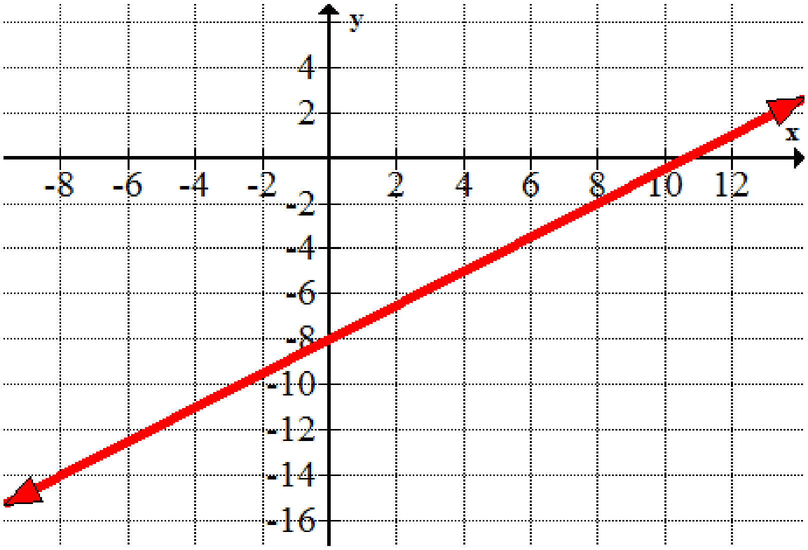
1.



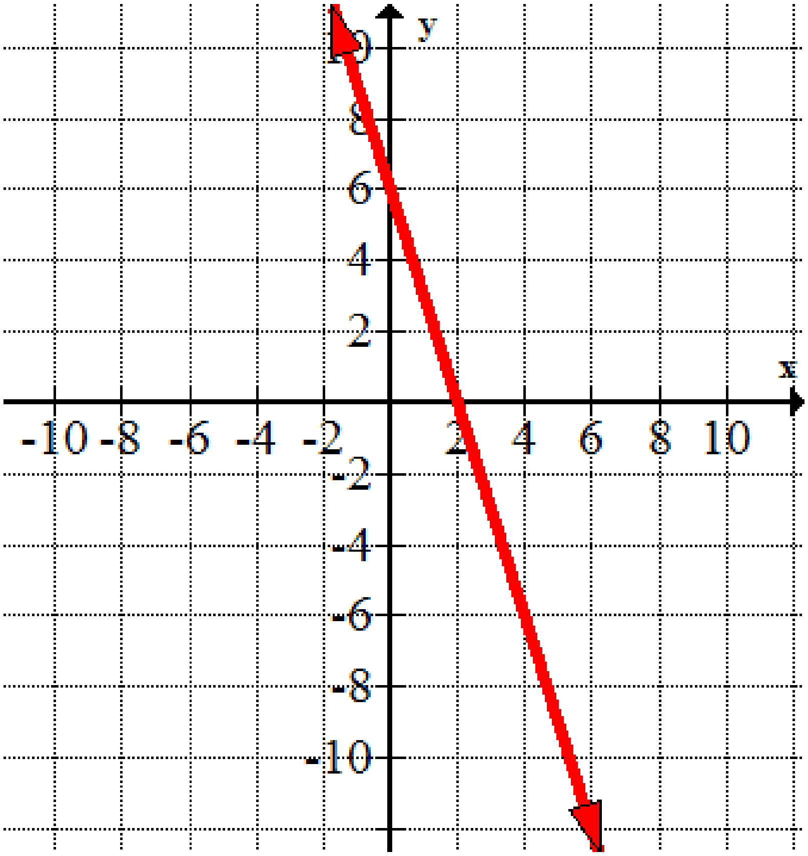
2.



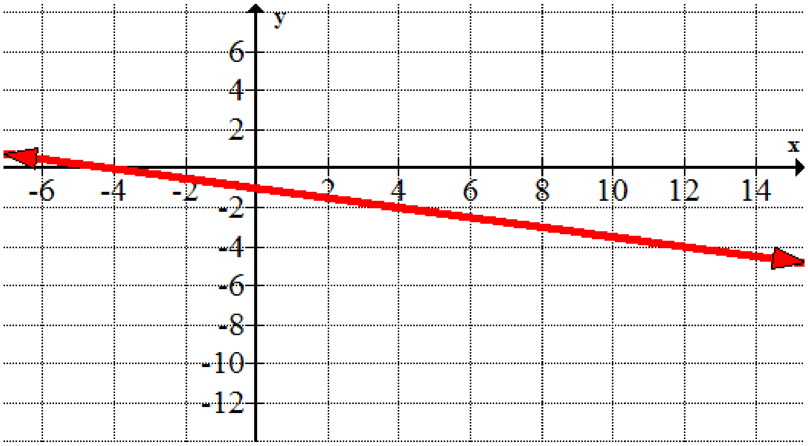
3.



4.



5.



Determine the equation of the line that passes through the following pairs of points:

6. (–3, 1) and (–3, –7)
7. (–5, –5) and (10, –5)
8. (–8, 4) and (2, –6)
9. (14, 8) and (4, 4)
10. (0, 5) and (4, –3)
11. (4, 7) and (2, –5)

For each of the following real world problems, write the linear equation in standard form that would best model the problem.

12. The cost of operating a car for one month depends upon the number of miles you drive. According to a recent survey completed by drivers of midsize cars, it costs \$124/month if you drive 320 miles/month and \$164/month if you drive 600 miles/month.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
13. A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for \$1950 and 15 in another for \$2850. The number of computers and the cost forms a linear relationship.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
14. Shop Rite sells a one-quart carton of milk for \$1.65 and a two-quart carton for \$2.95. Assume there is a linear relationship between the volume of milk and the price.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?
15. Some college students who plan on becoming math teachers decide to set up a tutoring service for high school math students. One student was charged \$25 for 3 hours of tutoring. Another student was charged \$55 for 7 hours of tutoring. The relationship between the cost and time is linear.
    a. Designate two data values for this problem. State the dependent and independent variables.
    b. Write an equation to model the situation. What do the numbers in the equation represent?

4.4 Graphs of Lines from Equations

Here you will learn how to graph a linear function from its equation without first making a table of values.

Can you graph the linear function $4y - 5x = 16$ on a Cartesian grid?

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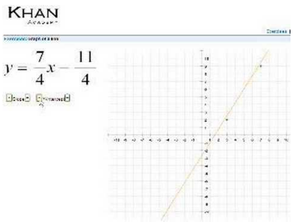
Khan Academy Slope and y-Intercept Intuition



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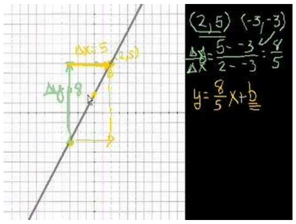
Khan Academy Slope 2



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Guidance

The graph of any linear function can be plotted using the slope-intercept form of the equation.

- Step 1: Solve the equation for y if it is not already in the form $y = mx + b$.
- Step 2: To graph the function, start by plotting the y-intercept.
- Step 3: Use the slope to find another point on the line. From the y-intercept, move to the right the number of units equal to the denominator of the slope and then up or down the number of units equal to the numerator of the slope. Plot the point.
- Step 4: Connect these two points to form a line and extend the line.

Note: You can repeat Step 3 multiple times in order to find more points on the line if you wish.

Because the equations of horizontal and vertical lines are special, these types of lines can be graphed differently:

- The graph of a horizontal line will have an equation of the form $y = a$ where a is the y-intercept of the line. You can simply draw a horizontal line through the y-intercept to sketch the graph.
- The graph of a vertical line will have an equation of the form $x = c$, where x is the x-intercept of the line. You can simply draw a vertical line through the x-intercept to sketch the graph.

Example A

For the following linear function, state the y-intercept and the slope: $4x-3y-9=0$.

Solution:

The first step is to rewrite the equation in the form $y=mx+b$. To do this, solve the equation for 'y'.

$$4x-3y-9=0$$

Apply the zero principle to move x to the right side of the equation.

$$4x-4x-3y-9=0-4x$$

$$-3y-9=-4x$$

Apply the zero principle to move -9 to the right of the equation.

$$-3y-9+9=-4x+9$$

$$-3y=-4x+9$$

Divide all terms by the coefficient of y. Divide by -3.

$$\frac{-3y}{-3}=\frac{-4x}{-3}+\frac{9}{-3}$$

$$\frac{\cancel{-3}y}{\cancel{-3}}=\frac{-4x}{-3}+\frac{9}{-3}$$

$$\boxed{y=\frac{4}{3}x-3}$$

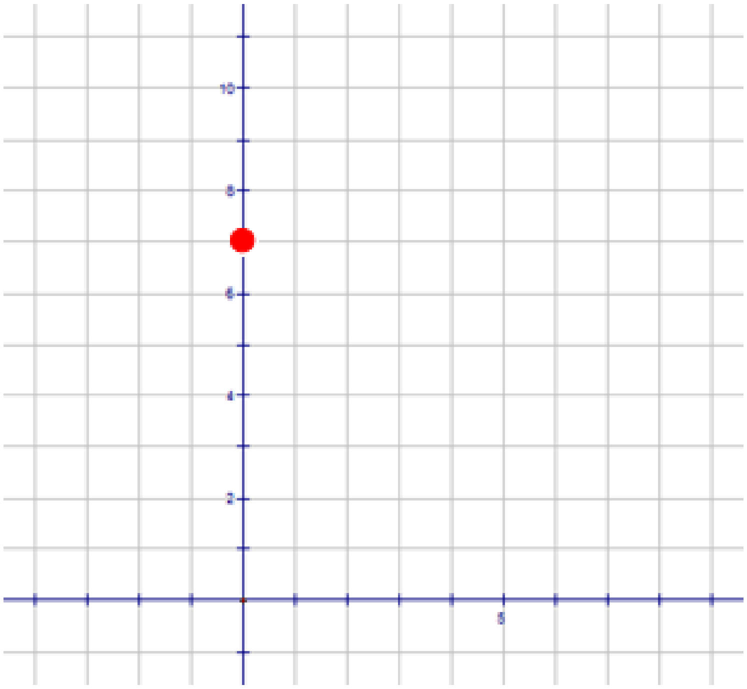
The y-intercept is (0, –3) and the slope is $\frac{4}{3}$.

Example B

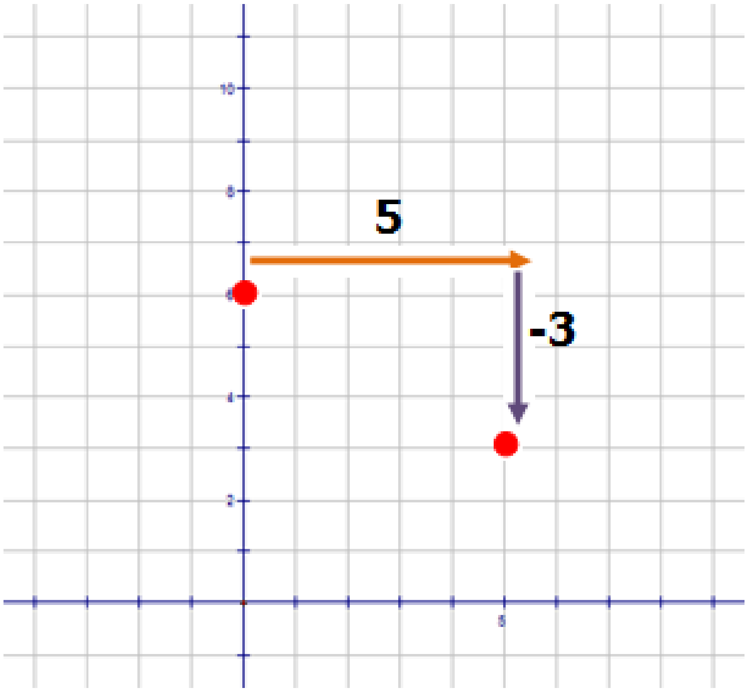
Graph the linear function $y=\frac{-3}{5}x+7$ on a Cartesian grid.

Solution:

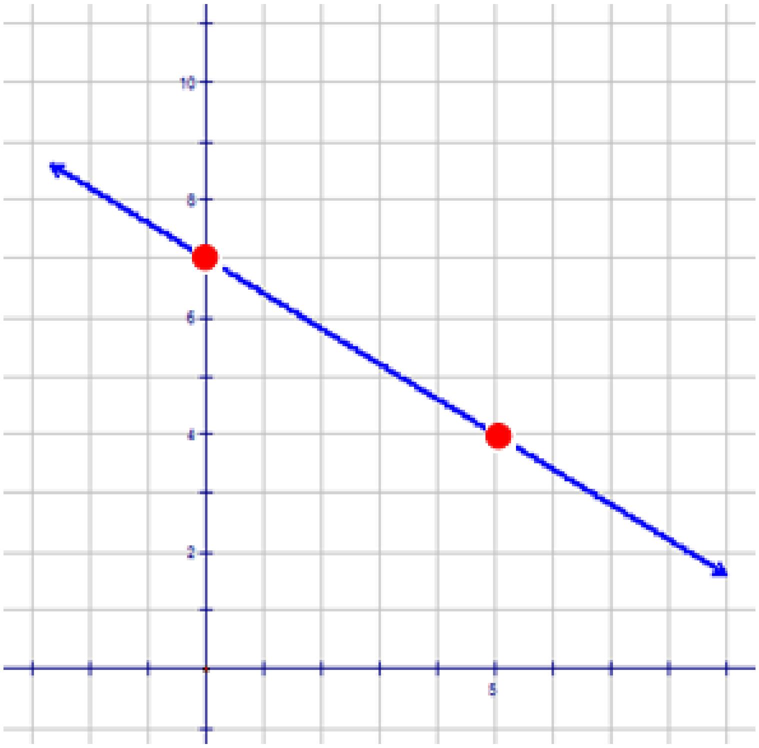
The y-intercept is (0, 7) and the slope is $\frac{-3}{5}$. Begin by plotting the y-intercept on the grid.



From the *y*-intercept, move to the right (run) 5 units and then move downward (rise) 3 units. Plot a point here.



Join the points with a straight line. Use a straight edge to draw the line.



Example C

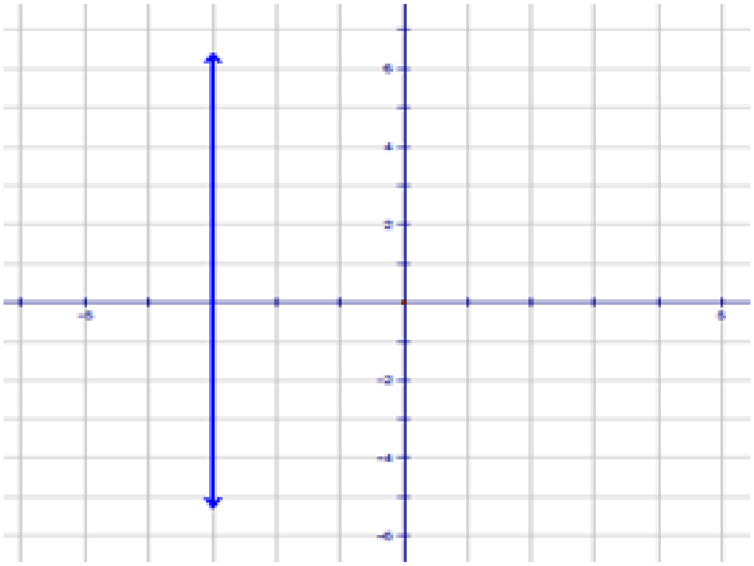
Plot the following linear equations on a Cartesian grid.

i) $x = -3$

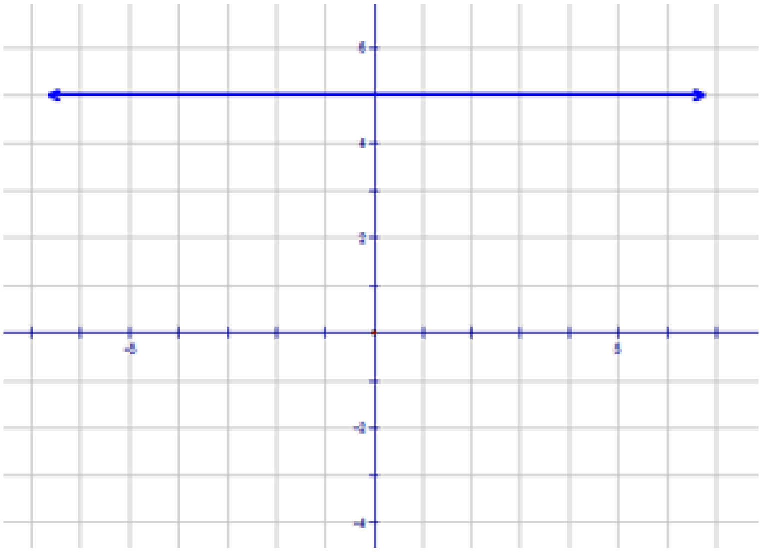
ii) $y = 5$

Solution:

i) A line that has $x = -3$ as its equation passes through all points that have –3 as the $x-$coordinate. The line also has a slope that is undefined. This line is parallel to the y-axis.



ii) A line that $y = 5$ has as its equation passes through all points that have 5 as the $y-$coordinate. The line also has a slope of zero. This line is parallel to the x-axis.



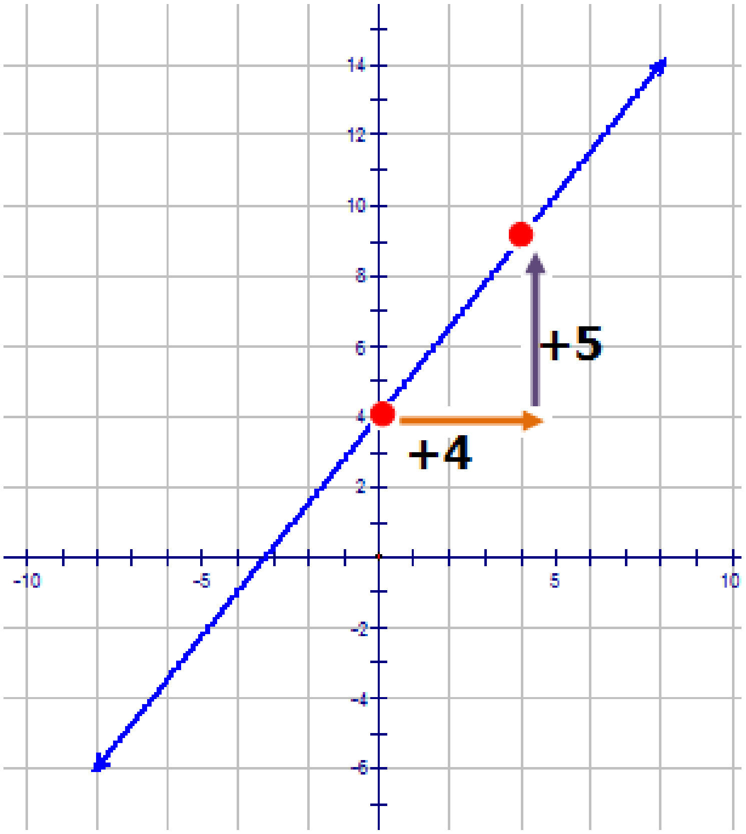
Concept Problem Revisited

Plot the linear function $4y-5x=16$ on a Cartesian grid.

The first step is to rewrite the function in slope-intercept form.

$$\begin{aligned}4y-5x&=16 && \text{Apply the zero principle to move } 5x \text{ to the right side of the equation.}\\ 4y-5x+5x&=16+5x\\ 4y&=16+5x && \text{Divide every term 4.}\\ \frac{4y}{4}&=\frac{16}{4}+\frac{5x}{4}\\ y&=4+\frac{5}{4}x && \text{Write the equation in the form } y=mx+b.\\ &\boxed{y=\frac{5}{4}x+4}\end{aligned}$$

The slope of the line is $\frac{5}{4}$ and the y-intercept is (0, 4)



Plot the y-intercept at (0, 4). From the y-intercept, move to the right 4 units and then move upward 5 units. Plot the point. Using a straight edge, join the points.

Vocabulary

Slope-Intercept Form

The ***slope-intercept form*** is one method for writing the equation of a line. The slope-intercept form is $y = mx + b$ where m refers to the slope and b identifies the y-intercept. This form is used to plot the graph of a linear function.

Guided Practice

1. Using the slope-intercept method, graph the linear function $y = -\frac{3}{2}x - 1$

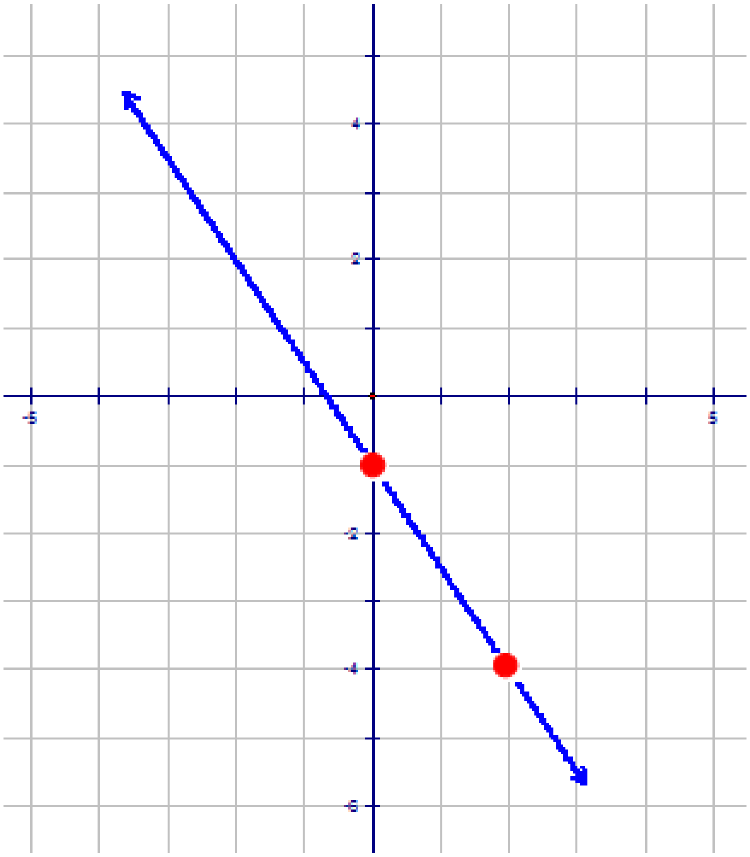
2. Using the slope-intercept method, graph the linear function $7x - 3y - 15 = 0$

3. Graph the following lines on the same Cartesian grid. What shape is formed by the graphs?

(a) $y = -3$
(b) $x = 4$
(c) $y = 2$
(d) $x = -6$

Answers:

1. The slope of the line is $-\frac{3}{2}$ and the y-intercept is (0, –1). Plot the y-intercept. Apply the slope to the y-intercept. Use a straight edge to join the two points.

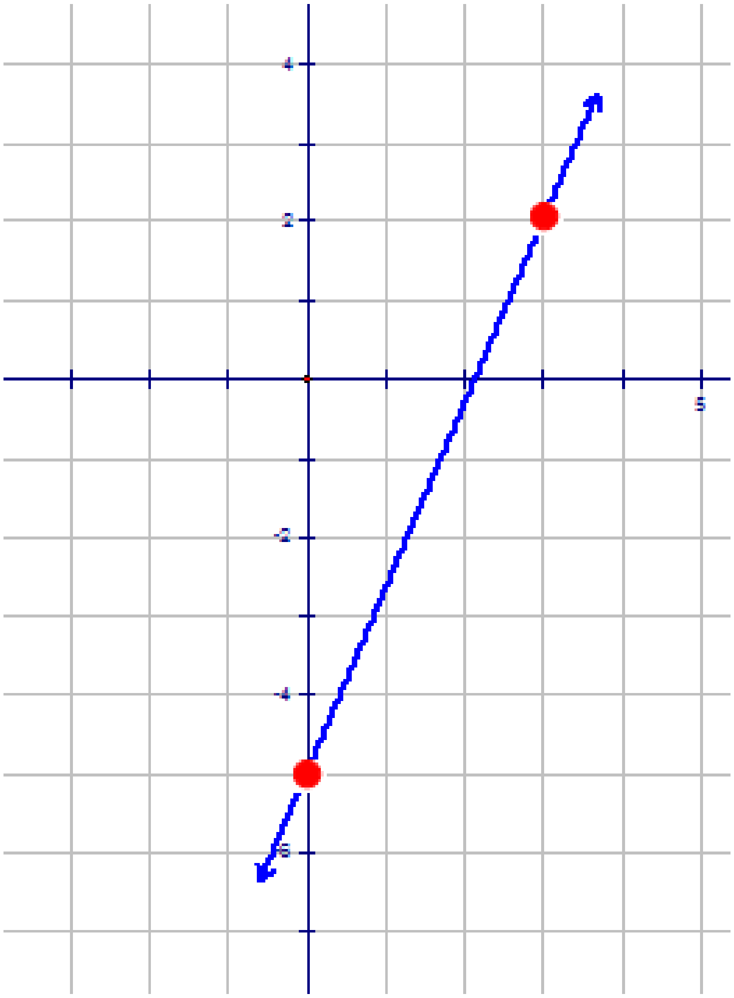


2. Write the equation in slope-intercept form.

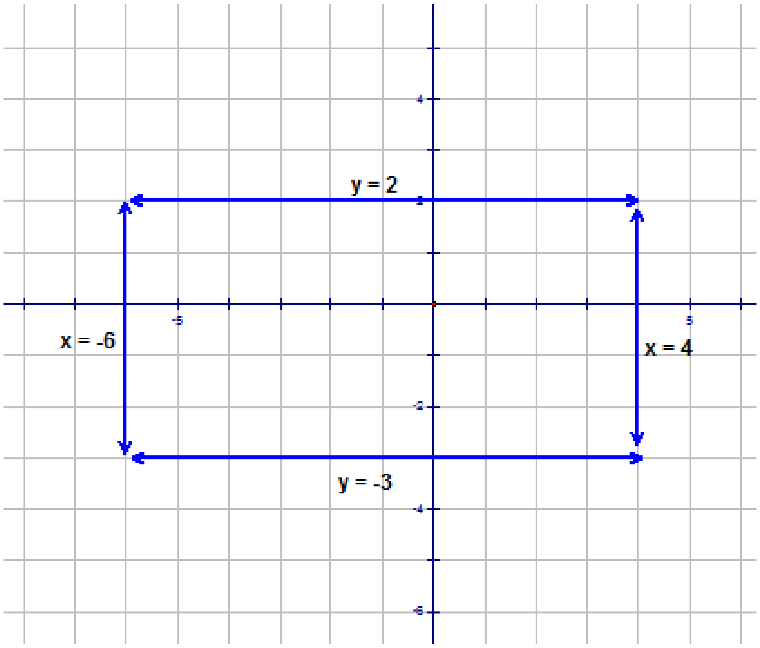
$$\begin{aligned}
7x-3y-15&=0\\
7x-7x-3y-15&=0-7x\\
-3y-15&=-7x\\
-3y-15+15&=-7x+15\\
-3y&=-7x+15\\
\frac{-3y}{-3}&=\frac{-7x}{-3}+\frac{15}{-3}\\
&\boxed{y=\frac{7}{3}x-5}
\end{aligned}$$

Solve the equation for the variable y.

The slope is $\frac{7}{3}$ and the y-intercept is (0, –5). Plot the y-intercept. Apply the slope to the y-intercept. Use a straight edge to join the two points.



3. There are four lines to be graphed. The lines a and c are lines with a slope of zero and are parallel to the x-axis. The lines b and d are lines that have a slope that is undefined and are parallel to the x-axis. The shape formed by the intersections of the lines is a rectangle.



Practice

For each of the following linear functions, state the slope and the y-intercept:

1. $y = \frac{5}{8}x + 3$
2. $4x + 5y - 3 = 0$
3. $4x - 3y + 21 = 0$
4. $y = -7$
5. $9y - 8x = 27$

Using the slope-intercept method, graph the following linear functions:

6. $3x + y = 4$
7. $3x - 2y = -4$
8. $2x + 6y + 18 = 0$
9. $3x + 7y = 0$
10. $4x - 5y = -30$
11. $6x - 2y = 8$

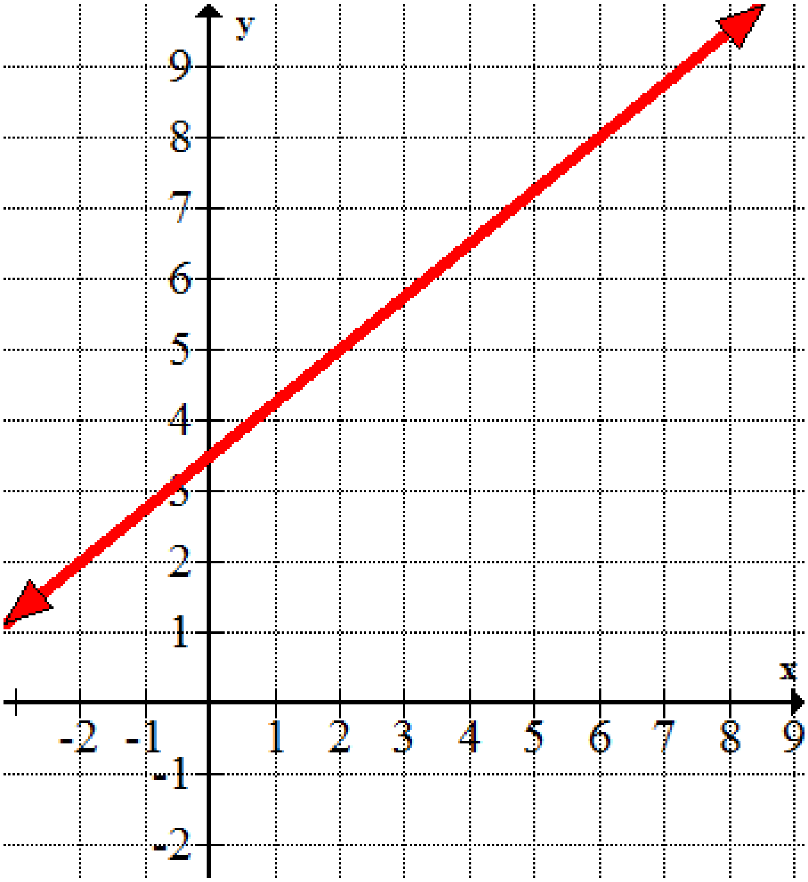
Graph the following linear equations and state the slope of the line:

12. $x = -5$
13. $y = 8$
14. $y = -4$
15. $x = 7$

4.5 Equations of Lines from Graphs

Here you'll learn how to find the equation of a line from its graph.

Write the equation, in standard form, of the following graph:



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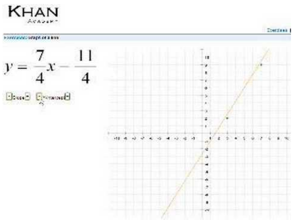

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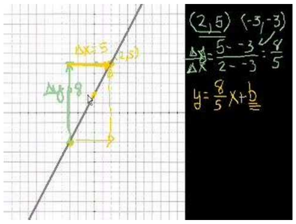
Khan Academy Slope 2



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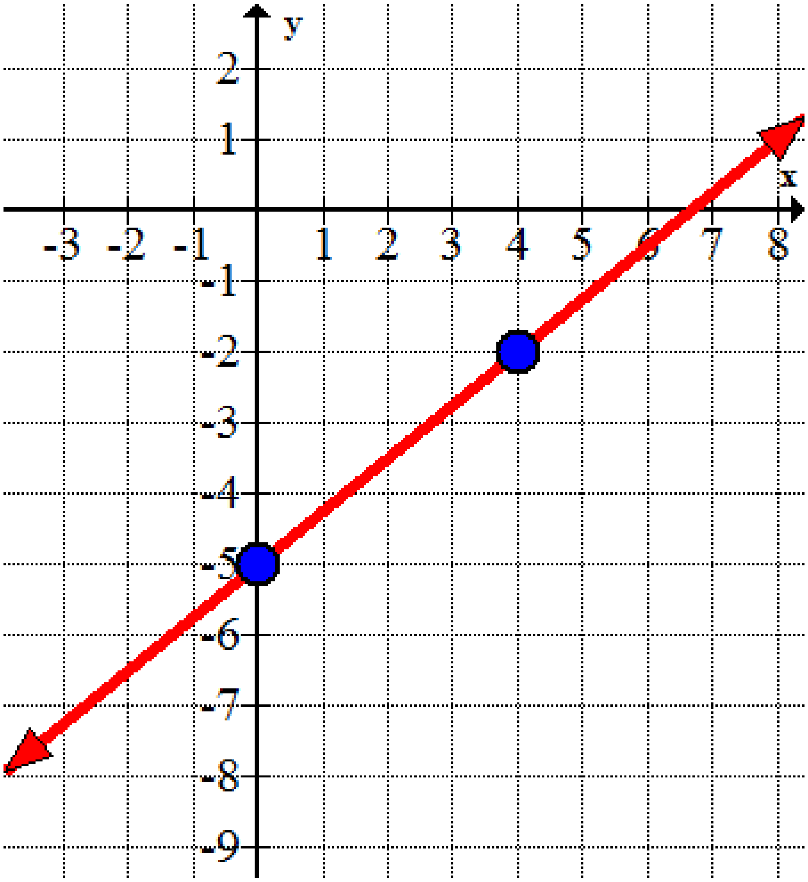
Guidance

You can determine the equation of a line from a graph by counting. Find the y-intercept (b) first and then a second point on the line. Use the y-intercept and second point to determine the slope (m). Then, write the equation in slope intercept form: $y = mx + b$.



The y-intercept of the graph is (0, –5). The slope of the line is $\frac{3}{4}$. The equation of the line in slope-intercept form is

$$\boxed{y = \frac{3}{4}x - 5}$$

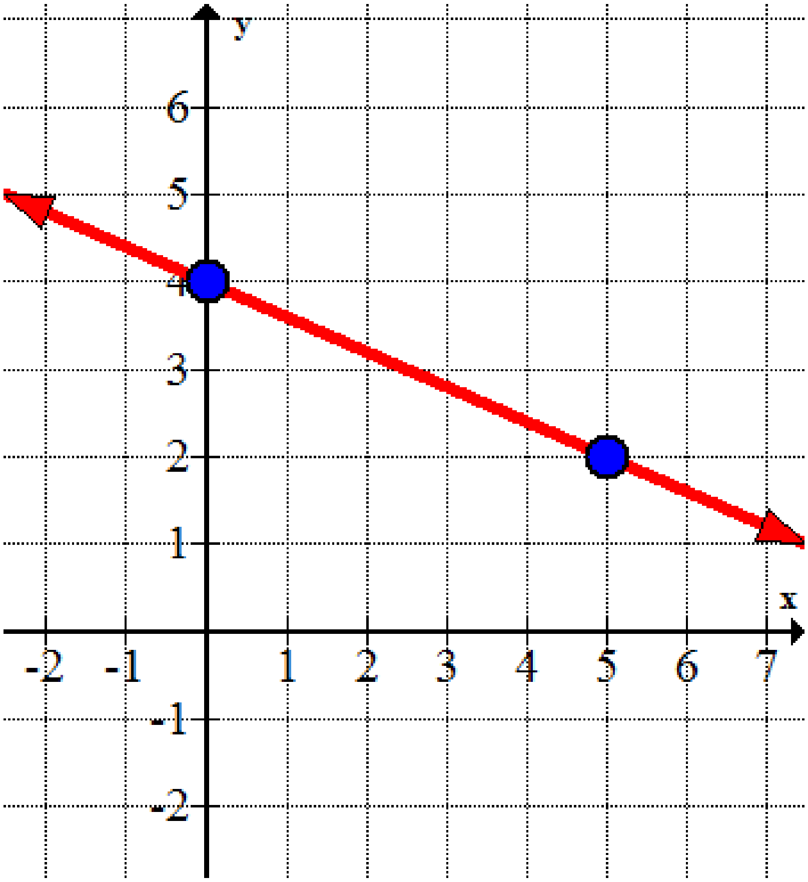
.

If you cannot determine the y-intercept, you can algebraically determine the equation of a line by using the coordinates of two points on the graph. These two points can be used to calculate the slope of the line by counting and then the y-intercept can then be determined algebraically.

To write the equation of a line in standard form, the value of the y-intercept is not needed. The slope can be determined by counting. The value of the slope and the coordinates of one other point on the line are used in the function $y - y_1 = m(x - x_1)$. This equation is then set equal to 0 to write the equation in standard form.

Example A

Determine the equation of the following graph. Write the equation in slope-intercept form.

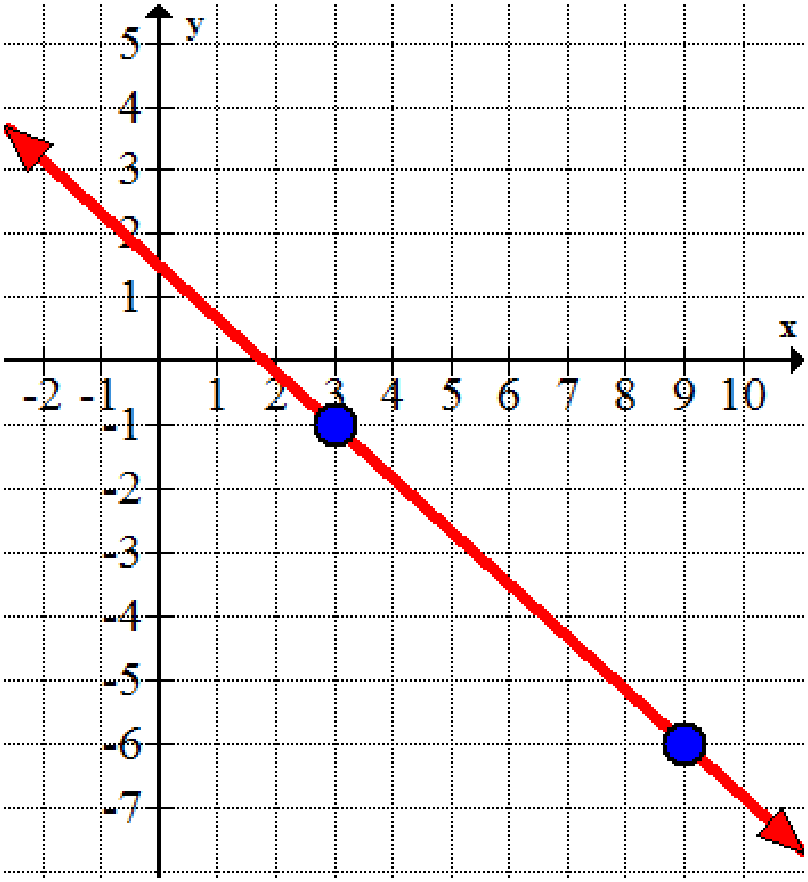


Solution:

The y-intercept is (0, 4) so $b = 4$. The slope has a run of five units to the right and a rise of 2 units downward. The slope of the line is $-\frac{2}{5}$. The equation of the line in slope-intercept form is $y = mx + b$ so $y = -\frac{2}{5}x + 4$.

Example B

Determine the equation in slope-intercept form of the line shown on the following graph:



Solution:

The y-intercept is not an exact point on this graph. The value of fractions on a Cartesian grid can only be estimated. Therefore, the points (3, –1) and (9, –6) will be used to determine the slope of the line. The slope is $-\frac{5}{6}$. The slope and one of the points will be used to algebraically calculate the y-intercept of the line.

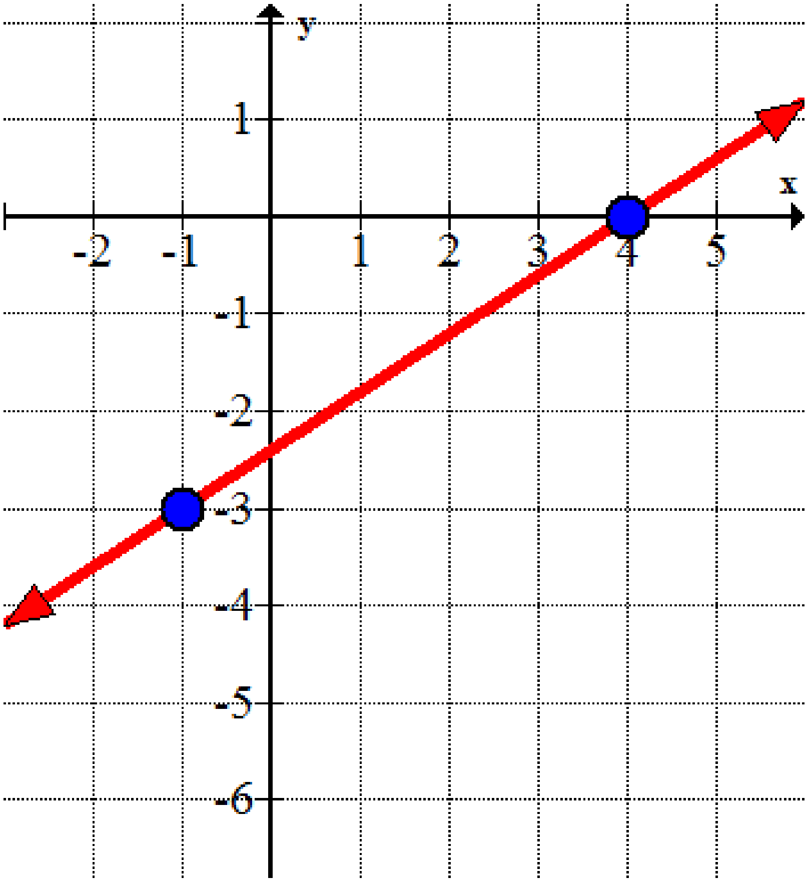
$$\begin{aligned}
y &= mx+b \\
-1 &= \left(\frac{-5}{6}\right)(3)+b \\
-1 &= \left(\frac{-5}{\cancel{6}_2}\right)(\cancel{3})+b \\
-1 &= \frac{-5}{2}+b \\
-1+\frac{5}{2} &= \frac{-5}{2}+\frac{5}{2}+b \\
-1+\frac{5}{2} &= b \\
\frac{-2}{2}+\frac{5}{2} &= b \\
\frac{3}{2} &= b
\end{aligned}$$

The equation in slope-intercept form is

$$\boxed{y = -\frac{5}{6}x+\frac{3}{2}}$$

Example C

Determine the equation, in standard form, for the line on the following graph:



Solution:

The y-intercept is not an exact point on this graph. Therefore, the points (4, 0) and (–1, –3) will be used to determine the slope of the line. The slope is $\frac{3}{5}$. The slope and one of the points will be used to algebraically calculate the equation of the line in standard form.

$$y - y_1 = m(x - x_1)$$

Use this formula to determine the equation in standard form.

$$y - 0 = \frac{3}{5}(x - 4)$$

Fill in the value for m of $\frac{3}{5}$ and $\begin{pmatrix} x_1, & y_1 \\ 4, & 0 \end{pmatrix}$

$$y = \frac{3}{5}x - \frac{12}{5}$$

$$5(y) = 5\left(\frac{3}{5}x\right) - 5\left(\frac{12}{5}\right)$$

Multiply every term by 5.

$$5(y) = \cancel{5}\left(\frac{3}{\cancel{5}}x\right) - \cancel{5}\left(\frac{12}{\cancel{5}}\right)$$

Simplify and set the equation equal to zero.

$$\begin{aligned}
5y &= 3x - 12 \\
5y - 3x &= 3x - 3x - 12 \\
5y - 3x &= -12 \\
5y - 3x + 12 &= -12 + 12 \\
5y - 3x + 12 &= 0 \\
-3x + 5y + 12 &= 0 \\
3x - 5y - 12 &= 0
\end{aligned}$$

The coefficient of x cannot be a negative value.

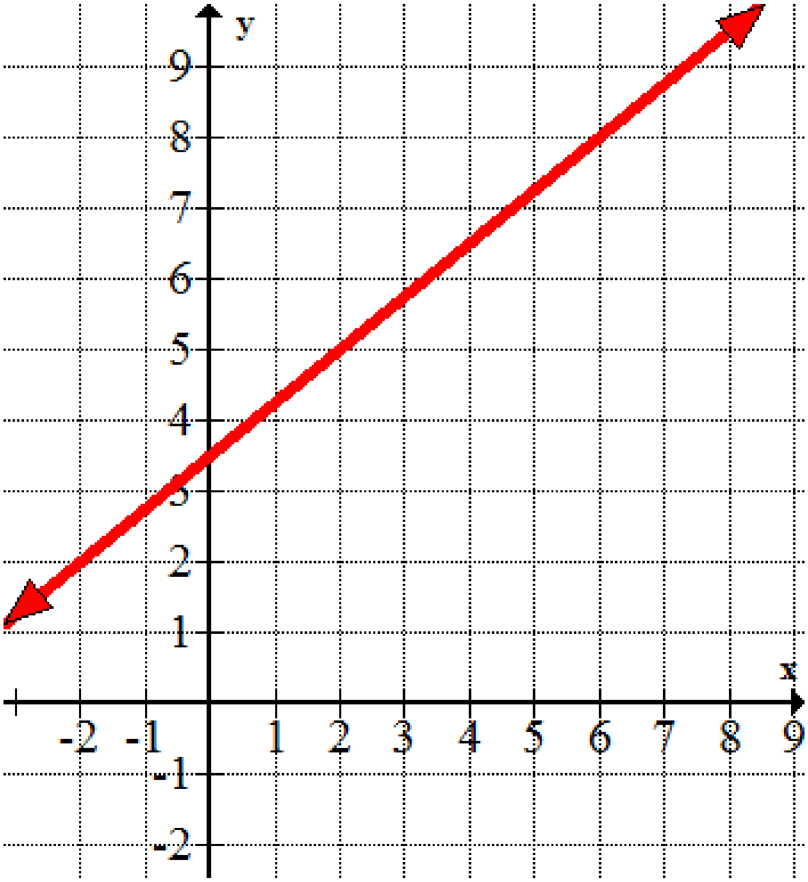
The equation of the line in standard form is

$$\boxed{3x - 5y - 12 = 0}$$

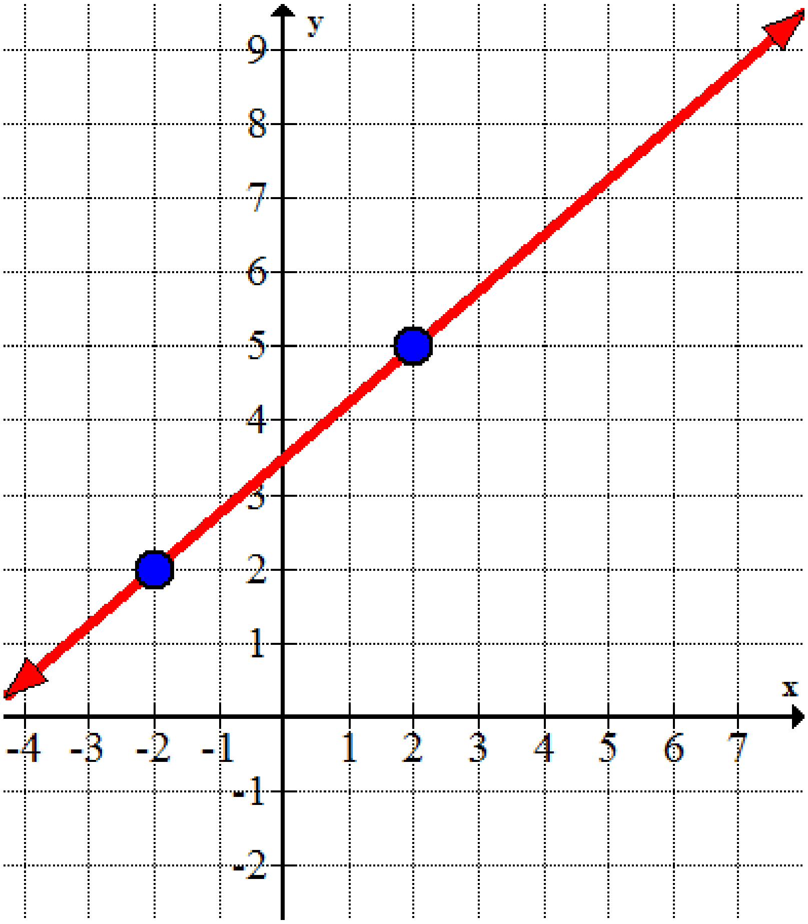
.

Concept Problem Revisited

Write the equation, in standard form, of the following graph:



The first step is to determine the slope of the line.



The slope of the line is $\frac{3}{4}$. The coordinates of one point on the line are (2, 5).

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y-5 &= \frac{3}{4}(x-2)\\
y-5 &= \frac{3}{4}x-\frac{6}{4}\\
4(y)-4(5) &= 4\left(\frac{3}{4}\right)x-4\left(\frac{6}{4}\right)\\
4(y)-4(5) &= \cancel{4}\left(\frac{3}{\cancel{4}}\right)x-\cancel{4}\left(\frac{6}{\cancel{4}}\right)\\
4y-20 &= 3x-6\\
-3x+4y-20 &= 3x-3x-6\\
-3x+4y-20 &= -6\\
-3x+4y-20+6 &= -6+6\\
-3x+4y-14 &= 0\\
3x-4y+14 &= 0
\end{aligned}$$

The equation of the line in standard form is

$$\boxed{3x-4y+14=0}$$

.

Vocabulary

Slope – Intercept Form

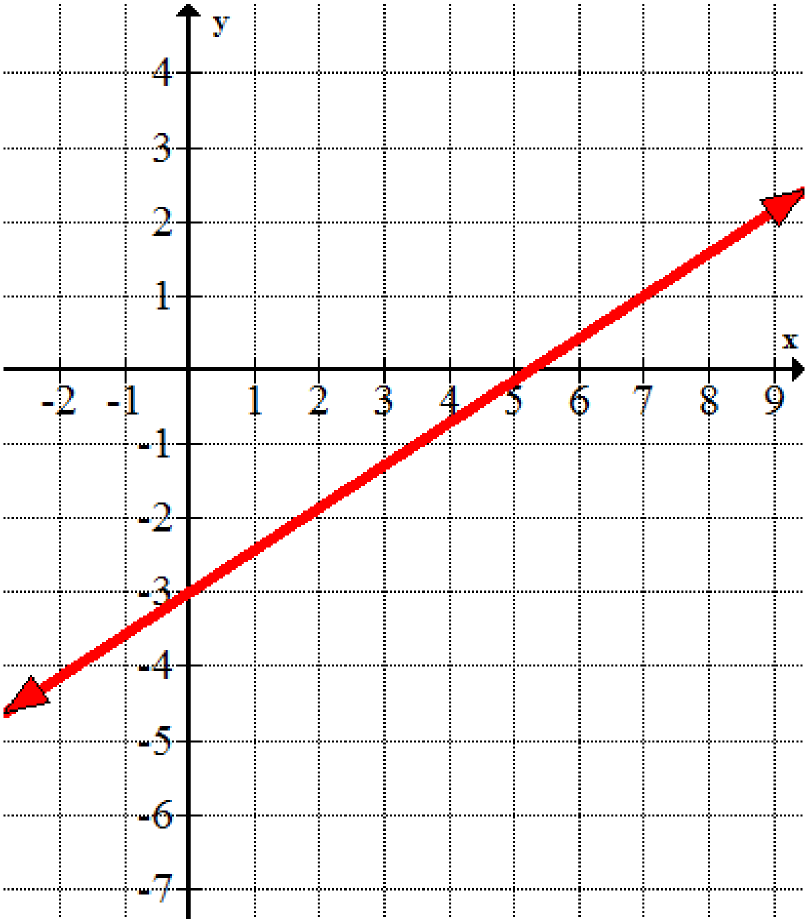
The ***slope-intercept form*** is one method for writing the equation of a line. The slope-intercept form is $y = mx + b$ where m refers to the slope and b identifies the y-intercept.

Standard Form

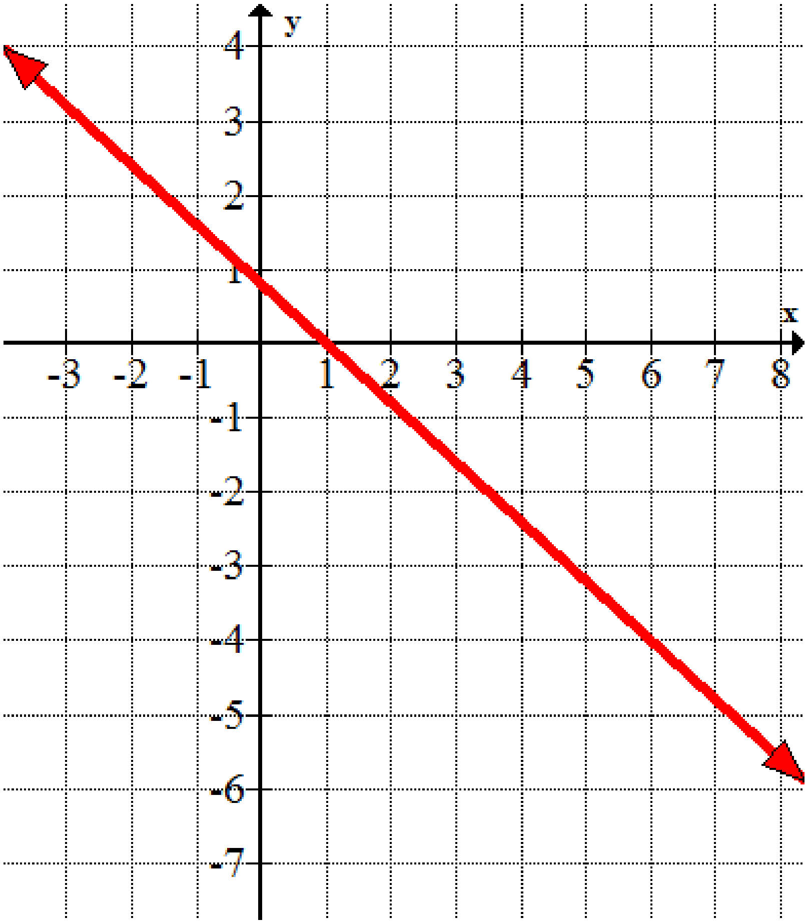
The ***standard form*** is another method for writing the equation of a line. The standard form is $Ax + By + C = 0$ where A is the coefficient of x, B is the coefficient of y and C is a constant.

Guided Practice

1. Write the equation, in slope-intercept form, of the following graph:



2. Write the equation, in slope-intercept form, of the following graph:



3. Rewrite the equation of the line from #2 in standard form.

Answers:

1. The first step is to determine the coordinates of the y-intercept. The y-intercept is (0, –3) so $b = -3$. The second step is to count to determine the value of the slope. Another point on the line is $(7, 1)$ so the slope is $\frac{4}{7}$. The equation of the line in slope-intercept form is

$$\boxed{y = \frac{4}{7}x - 3}$$

2. The y-intercept is not an exact point on the graph. Therefore begin by determining the slope of the line by counting between two points on the line. The coordinates of two points on the line are (1, 0) and (6, –4). The slope is is $-\frac{4}{5}$. The y-intercept of the line must be calculated by using the slope and one of the points on the line.

$$\begin{aligned} y &= mx + b \\ 0 &= \frac{-4}{5}(1) + b \\ 0 &= \frac{-4}{5} + b \\ 0 + \frac{4}{5} &= \frac{-4}{5} + \frac{4}{5} + b \\ \frac{4}{5} &= b \end{aligned}$$

The equation of the line in slope-intercept form is

$$\boxed{y=-\frac{4}{5}x+\frac{4}{5}}$$

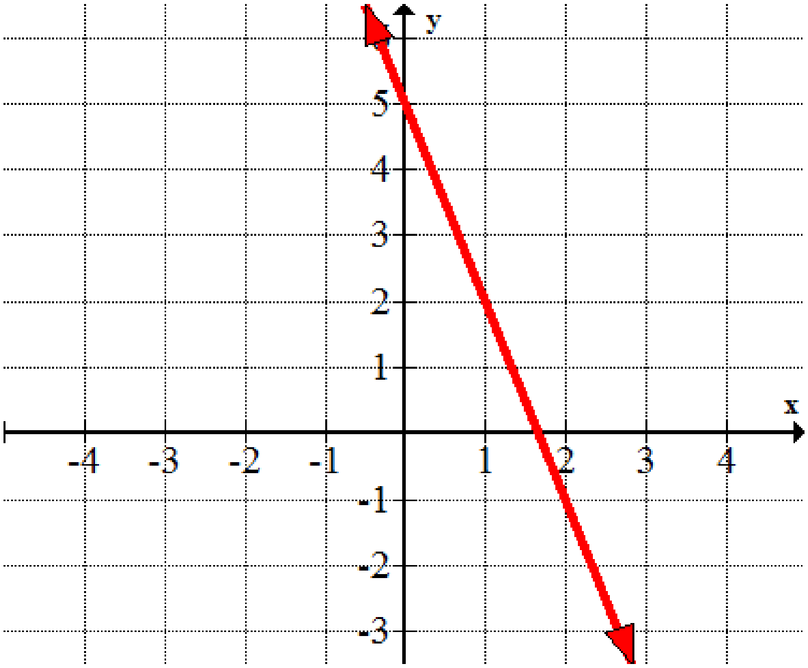
3. To rewrite the equation in standard form, first multiply the equation by 5 to get rid of the fractions. Then, set the equation equal to 0.

$$\begin{aligned} y &= -\frac{4}{5}x+\frac{4}{5} \\ 5y &= -4x+4 \\ 4x+5y-4 &= 0 \end{aligned}$$

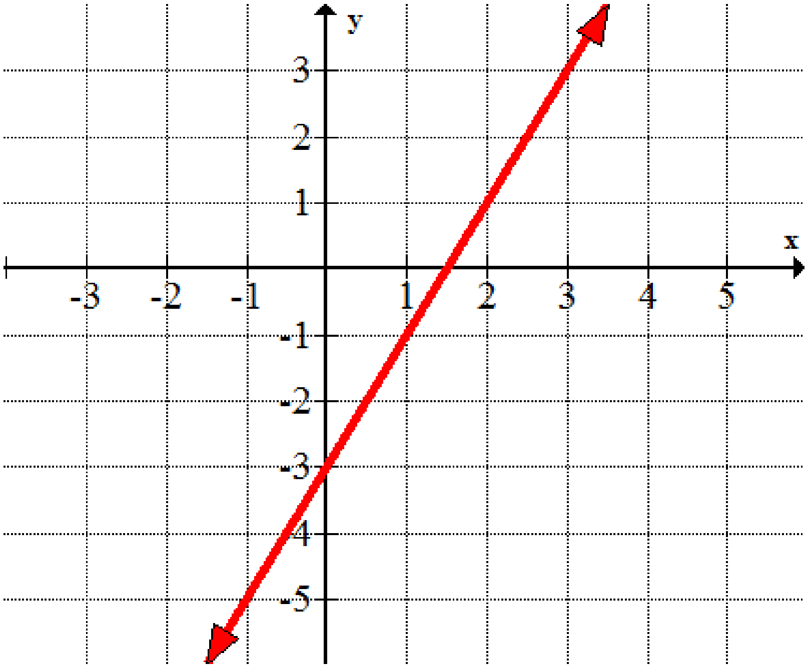
Practice

For each of the following graphs, write the equation in slope-intercept form:

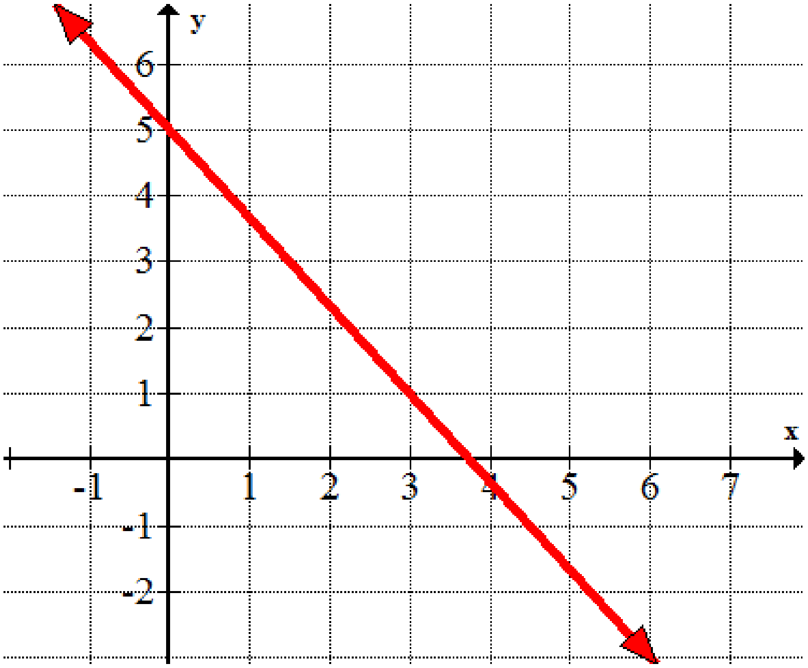
1.



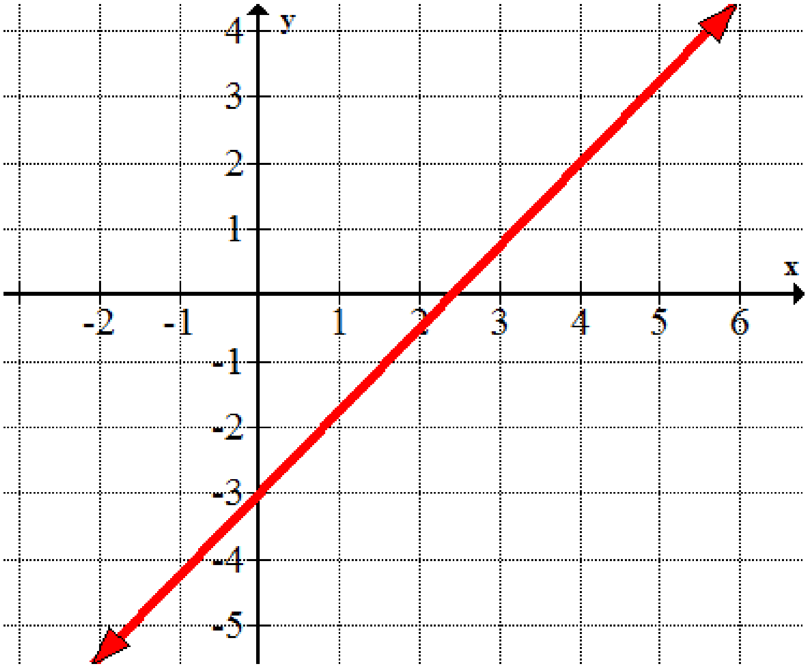
2.



3.

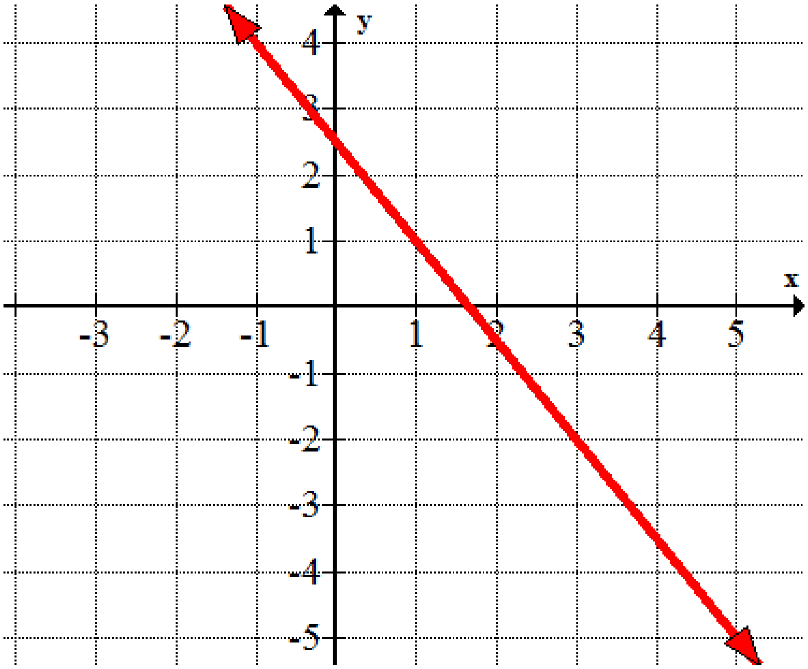


4.



For each of the following graphs, write the equation in slope-intercept form:

5.



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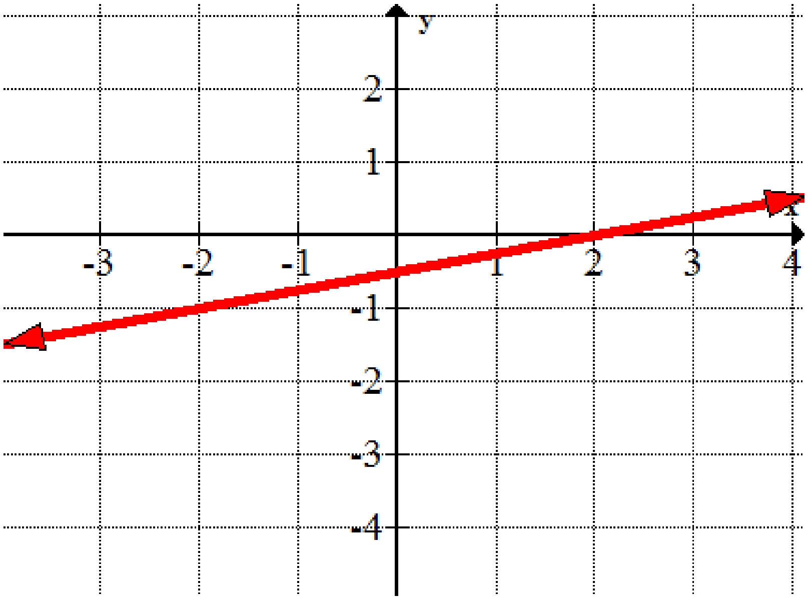

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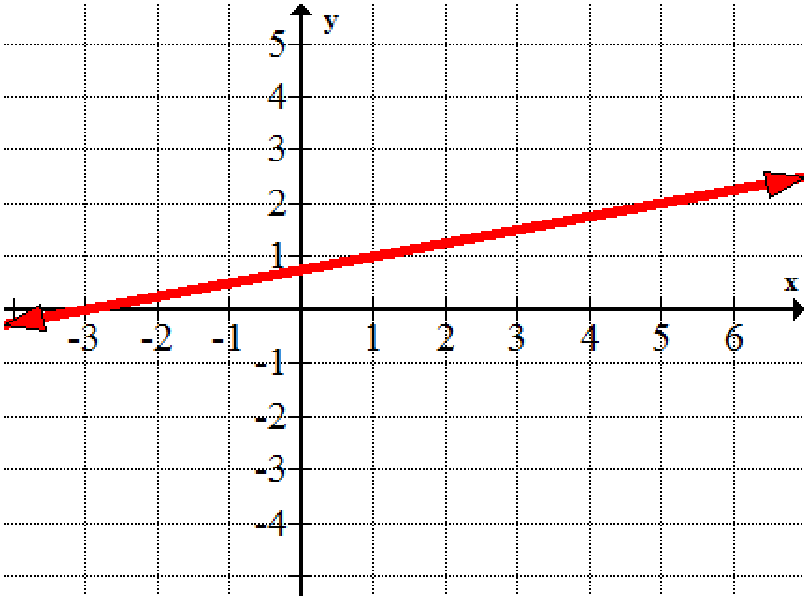

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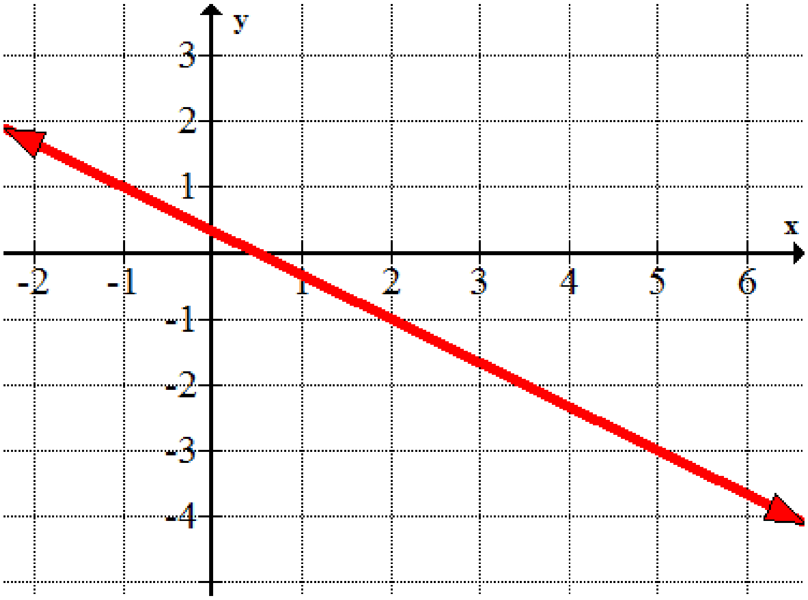
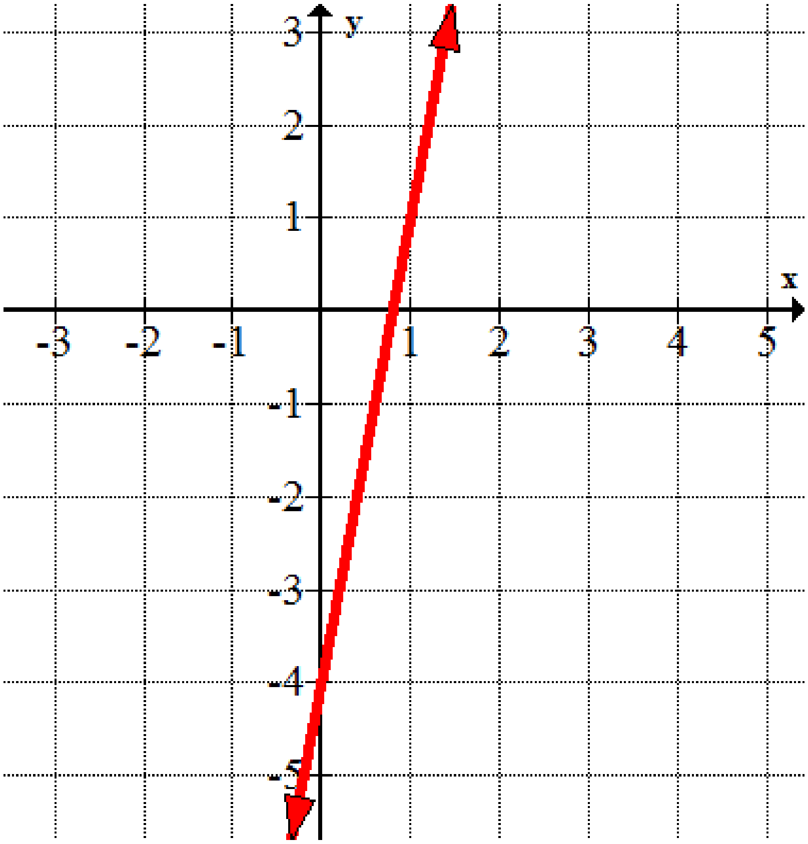
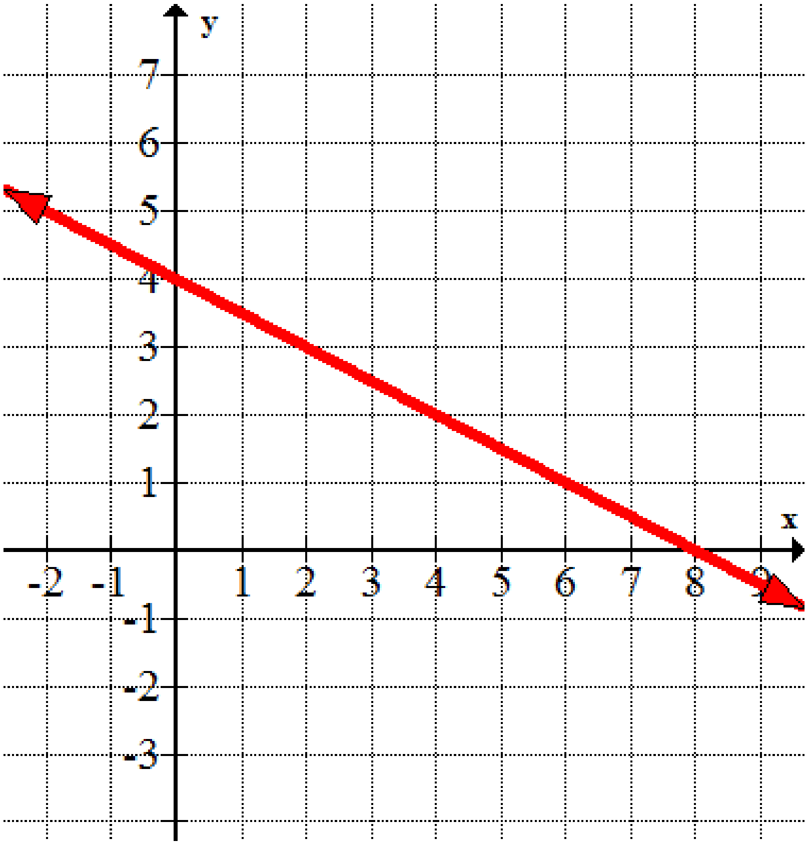


For each of the following graphs, write the equation standard form:

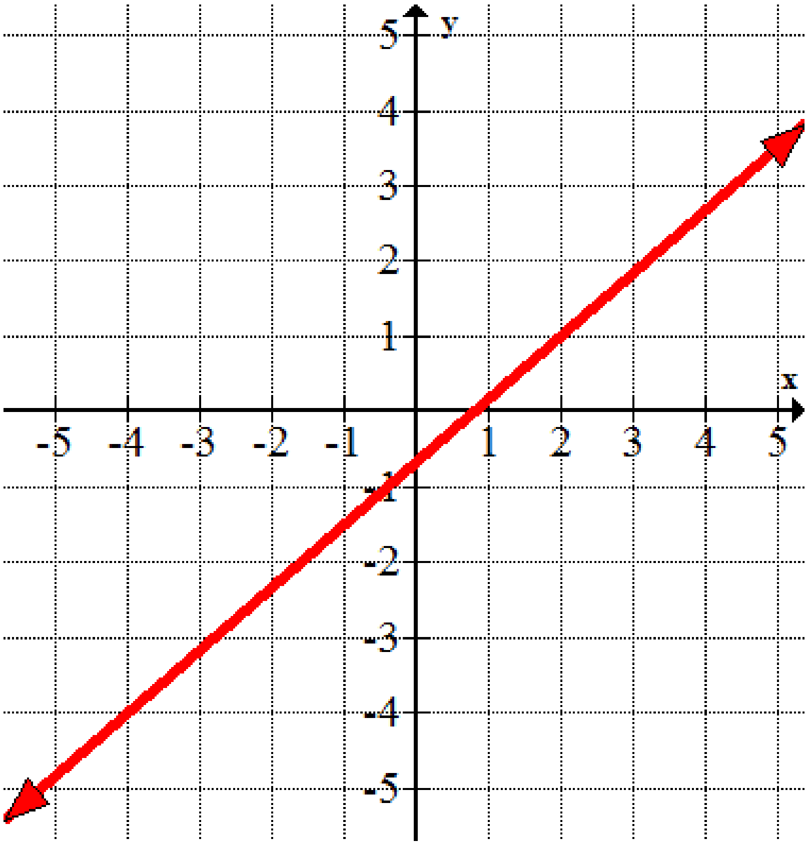
9.



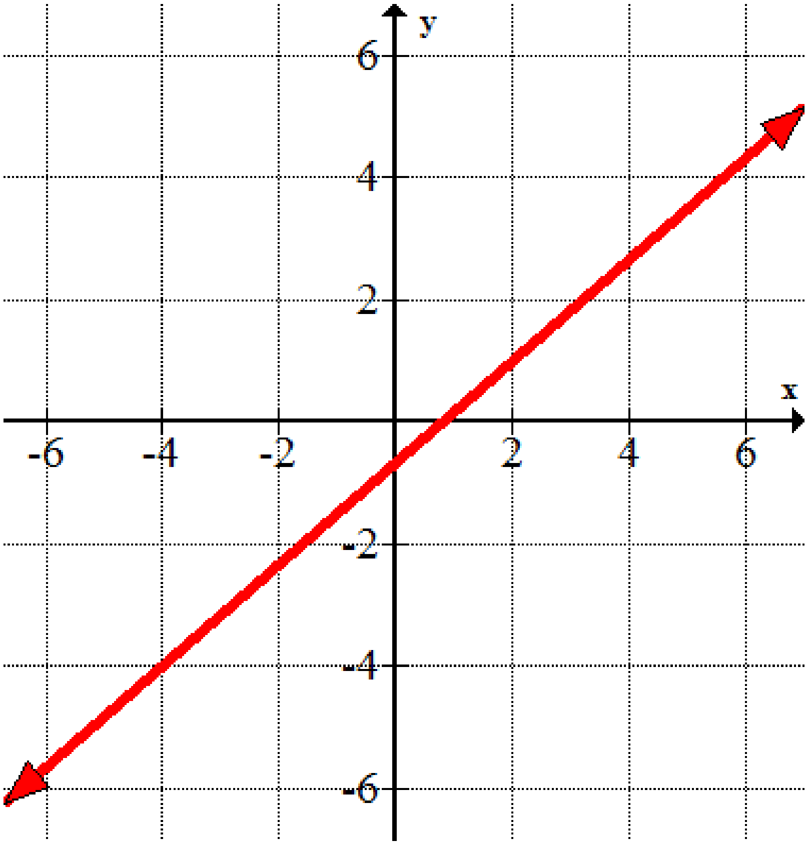
10.



11.



12.



13. Can you always find the equation of a line from its graph?
14. How do you find the equation of a vertical line? What about a horizontal line?
15. Rewrite the equation $y = \frac{1}{4}x - 5$ in standard form.
16. Rewrite the equation $y = \frac{2}{3}x + 1$ in standard form.
17. Rewrite the equation $y = \frac{1}{3}x - \frac{3}{7}$ in standard form.

4.6 Equations of Parallel and Perpendicular Lines

Here you will learn about parallel and perpendicular lines and how to determine whether or not two lines are parallel or perpendicular using slope.

Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

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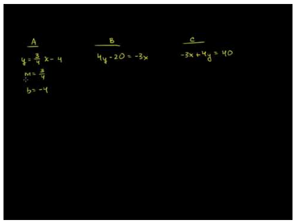
Khan Academy Parallel Lines



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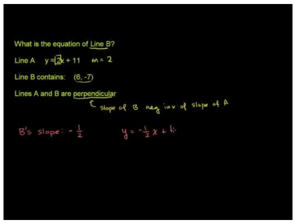
Khan Academy Perpendicular Lines



What is the equation of Line B?

Line A $y=\boxed{2}x+11$ $m=2$

Line B contains: (6, -7)

Lines A and B are perpendicular

slope of B neg inv of slope of A

B's slope: $-\frac{1}{2}$ $y=-\frac{1}{2}x+b$

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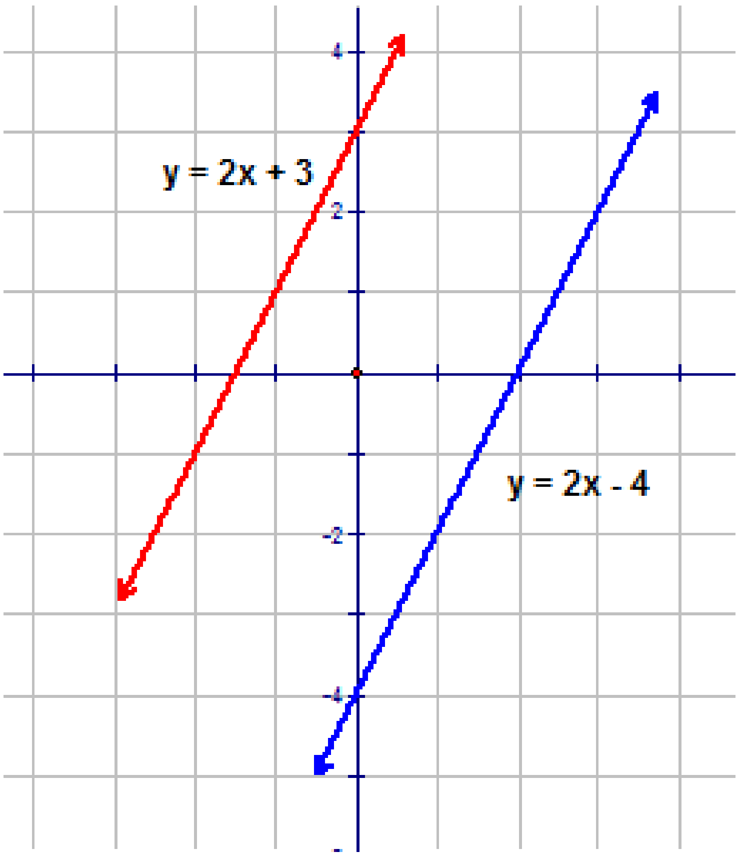
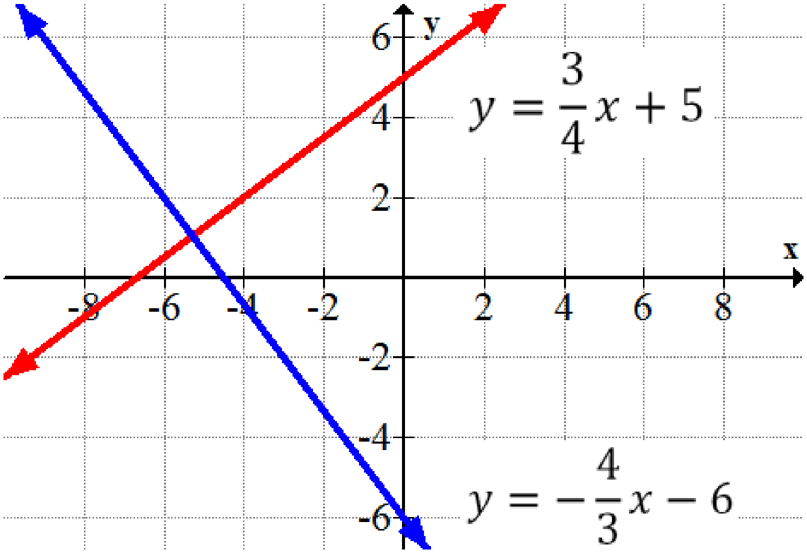
Guidance

Parallel lines are lines in the same plane that never intersect. Parallel lines maintain the **same slope**, or no slope (vertical lines) and the same distance from each other. The following graph shows two lines with the same slope. The slope of each line is 2. Notice that the lines are the same distance apart for the entire length of the lines. The lines will never intersect. The following lines are parallel.



Two lines in the same plane that intersect or cross each other at right angles are **perpendicular lines**. Perpendicular lines have **slopes that are opposite reciprocals**. The following graph shows two lines with slopes that are opposite reciprocals. The slope of one line is $\frac{3}{4}$ and the slope of the other line is $-\frac{4}{3}$. The product of the slopes is negative one. $\left(\frac{3}{4}\right)\left(\frac{-4}{3}\right) = \frac{-12}{12} = -1$. Notice that the lines intersect at a right angle. The lines are perpendicular lines.



You can use the relationship between the slopes of parallel lines and the slopes of perpendicular lines to write the equations of other lines.

Example A

Given the slopes of two lines, tell whether the lines are parallel, perpendicular or neither.

i) $m_1 = 4, m_2 = \frac{1}{4}$

ii) $m_1 = -3, m_2 = \frac{1}{3}$

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$

iv) $m_1 = -1, m_2 = 1$

v) $m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$

Solutions:

i) $m_1 = 4, m_2 = \frac{1}{4}$ The slopes are reciprocals but **not** opposite reciprocals. The lines are neither parallel nor perpendicular.

ii) $m_1 = -3, m_2 = \frac{1}{3}$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

iii) $m_1 = \frac{3}{12}, m_2 = \frac{1}{4}$ The slopes are the same. The fractions are equivalent. The lines are parallel.

iv) $m_1 = -1, m_2 = 1$ The slopes are reciprocals and are also opposite reciprocals. The lines are perpendicular.

v)$m_1 = -\frac{1}{3}, m_2 = \frac{1}{3}$ The slopes are not the same. The lines are neither parallel nor perpendicular.

Example B

Determine the equation of the line passing through the point (–4, 6) and parallel to the graph of $3x + 2y - 7 = 0$. Write the equation in standard form.

Solution:

If the equation of the line you are looking for is parallel to the given line, then the two lines have the same slope. Begin by expressing $3x + 2y - 7 = 0$ in slope-intercept form in order to find its slope.

$$\begin{aligned}
3x+2y-7&=0\\
3x-3x+2y-7&=0-3x\\
2y-7&=-3x\\
2y-7+7&=-3x+7\\
2y&=-3x+7\\
\frac{\cancel{2}y}{\cancel{2}}&=-\frac{3x}{2}+\frac{7}{2}\\
y&=-\frac{3}{2}x+\frac{7}{2}\\
&\ \ \updownarrow\\
y&=\ mx+b
\end{aligned}$$

The slope of the line is $-\frac{3}{2}$. The line passes through the point $(-4,6)$.

$$y-y_1=m(x-x_1)$$

Substitute the values into this equation.

$$\begin{aligned}
y-6&=\frac{-3}{2}(x--4)\\
y-6&=\frac{-3}{2}(x+4)\\
y-6&=\frac{-3x}{2}-\frac{12}{2}\\
2(y)-2(6)&=\cancel{2}\left(\frac{-3x}{\cancel{2}}\right)-\cancel{2}\left(\frac{12}{\cancel{2}}\right)\\
2y-12&=-3x-12\\
2y-12+12&=-3x-12+12\\
2y&=-3x\\
3x+2y&=-3x+3x\\
3x+2y&=0
\end{aligned}$$

The equation of the line is

$$\boxed{3x+2y=0}$$

Example C

Determine the equation of the line that passes through the point (6, –2) and is perpendicular to the graph of $3x = 2y-4$. Write the equation in standard form.

Solution: Begin by writing the equation $3x = 2y-4$ in slope-intercept form.

$$\begin{aligned}
3x &= 2y-4\\
2y-4 &= 3x\\
2y-4+4 &= 3x+4\\
2y &= 3x+4\\
\frac{\cancel{2}y}{\cancel{2}} &= \frac{3x}{2}+\frac{4}{2}\\
y &= \frac{3}{2}x+2\\
&\ \updownarrow\\
y &= mx+b
\end{aligned}$$

The slope of the given line is $\frac{3}{2}$. The slope of the perpendicular line is

$$\boxed{-\frac{2}{3}}$$

. The line passes through the point (6, –2).

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y--2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2}{3}(x-6)\\
y+2 &= -\frac{2x}{3}+\frac{12}{3}\\
3(y)+3(2) &= 3\left(-\frac{2x}{3}\right)+3\left(\frac{12}{3}\right)\\
3(y)+3(2) &= \cancel{3}\left(-\frac{2x}{\cancel{3}}\right)+\cancel{3}\left(\frac{12}{\cancel{3}}\right)\\
3y+6 &= -2x+12\\
3y+6-12 &= -2x+12-12\\
3y-6 &= -2x\\
2x+3y-6 &= -2x+2x\\
2x+3y-6 &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{2x+3y-6=0}$$

.

Concept Problem Revisited

Can you write the equation for the line that passes through the point (–2, –3) and is parallel to the graph of $y+2x=8$? Can you write the equation of the line in standard form?

Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the parallel line is the same as the slope of the given line.

$$\begin{aligned}
y+2x &= 8\\
y+2x-2x &= -2x+8\\
y &= -2x+8
\end{aligned}$$

The slope of the given line is –2. The slope of the parallel line is also

$$\boxed{-2}$$

.

$$\begin{aligned}
y-y_1 &= m(x-x_1)\\
y--3 &= -2(x--2)\\
y+3 &= -2(x+2)\\
y+3 &= -2x-4\\
y+3 &= -2x-4\\
2x+y+3 &= -2x+2x-4\\
2x+y+3 &= -4\\
2x+y+3+4 &= -4+4\\
2x+y+7 &= 0
\end{aligned}$$

The equation of the line is

$$\boxed{2x+y+7=0}$$

Vocabulary

Parallel Lines

Parallel lines are lines in the same plane that have the same slope. The lines never intersect and always maintain the same distance apart.

Perpendicular Lines

Perpendicular lines are lines in the same plane that intersect each other at right angles. The slopes of perpendicular lines are opposite reciprocals. The product of the slopes of two perpendicular lines is –1.

Guided Practice

Determine whether the lines that pass through the two pairs of points are parallel, perpendicular or neither parallel nor perpendicular.

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

4. Write the equation for the line that passes through the point (–3, 6) and is perpendicular to the graph of $3x = 5y + 6$. Write the equation of the line in slope-intercept form.

Answers:

1. (–2, 8), (3, 7) and (4, 3), (9, 2)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} & m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Use the formula to calculate the slopes}\\
m &= \frac{7-8}{3--2} & m &= \frac{2-3}{9-4} && \text{Calculate the slopes for each pair of points}\\
m &= \frac{7-8}{3+2} & &\boxed{m = \frac{-1}{5}} &&\\
&\boxed{m = \frac{-1}{5}} & & &&
\end{aligned}$$

The slopes of the lines are the same. The lines are parallel.

2. (2, 5), (8, 7) and (–3, 1), (–2, –2)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} & m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Use the formula to calculate the slopes}\\
m &= \frac{7-5}{8-2} & m &= \frac{-2-1}{-2--3} && \text{Calculate the slopes for each pair of points}\\
m &= \frac{2}{6} & m &= \frac{-2-1}{-2+3} &&\\
&\boxed{m = \frac{1}{3}} & &\boxed{m = \frac{-3}{1}} &&
\end{aligned}$$

The slopes of the lines are opposite reciprocals. The lines are perpendicular.

3. (4, 6), (–3, –1) and (6, –3), (4, 5)

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} & m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Use the formula to calculate the slopes}\\
m &= \frac{-1-6}{-3-4} & m &= \frac{5--3}{4-6} && \text{Calculate the slopes for each pair of points}\\
m &= \frac{-7}{-7} & m &= \frac{5+3}{4-6} &&\\
&\boxed{m = 1} & m &= \frac{8}{-2} &&\\
& & &\boxed{m = -4} &&
\end{aligned}$$

The lines are neither parallel nor perpendicular.

4. Begin by writing the given equation in slope-intercept form. This will give the slope of this line. The slope of the perpendicular line is the opposite reciprocal.

$$\begin{aligned}3x &= 5y+6\\5y+6 &= 3x\\5y+6-6 &= 3x-6\\5y &= 3x-6\\\frac{5y}{5} &= \frac{3x}{5}-\frac{6}{5}\\\frac{\cancel{5}y}{\cancel{5}} &= \frac{3x}{5}-\frac{6}{5}\\y &= \frac{3}{5}x-\frac{6}{5}\end{aligned}$$

The slope of the given line is $\frac{3}{5}$. The slope of the perpendicular line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the perpendicular line that passes through the point (–3, 6) is:

$$\begin{aligned}y &= mx+b\\6 &= -\frac{5}{3}(-3)+b\\6 &= -\frac{5}{\cancel{3}}\left(\overset{-1}{\cancel{-3}}\right)+b\\6 &= 5+b\\6-5 &= 5-5+b\\1 &= b\end{aligned}$$

The y-intercept is (0, 1) and the slope of the line is

$$\boxed{-\frac{5}{3}}$$

. The equation of the line is

$$\boxed{y=-\frac{5}{3}x+1}$$

.

Practice

For each pair of given equations, determine if the lines are parallel, perpendicular or neither.

1. $y=2x-5$ and $y=2x+3$
2. $y=\frac{1}{3}x+5$ and $y=-3x-5$
3. $x=8$ and $x=-2$
4. $y=4x+7$ and $y=-4x-7$

5. $y = -x - 3$ and $y = x + 6$
6. $3y = 9x + 8$ and $y = 3x - 4$

Determine the equation of the line satisfying the following conditions:

7. through the point (5, –6) and parallel to the line $y = 5x + 4$
8. through the point (–1, 7) and perpendicular to the line $y = -4x + 5$
9. containing the point (–1, –5) and parallel to $3x + 2y = 9$
10. containing the point (0, –6) and perpendicular to $6x - 3y + 8 = 0$
11. through the point (2, 4) and perpendicular to the line $y = -\frac{1}{2}x + 3$
12. containing the point (–1, 5) and parallel to $x + 5y = 3$
13. through the point (0, 4) and perpendicular to the line $2x - 5y + 1 = 0$

If $D(4, -1), E(-4, 5)$ and $F(3, 6)$ are the vertices of ΔDEF determine:

14. the equation of the line through D and parallel to EF.
15. the equation of the line containing the altitude from D to EF (the line perpendicular to EF that contains D).

4.7 Applications of Linear Functions

Here you will learn how to use what you know about the equations and graphs of lines to help you to solve real-life problems.

Joe's Warehouse has banquet facilities to accommodate a maximum of 250 people. When the manager quotes a price for a banquet she is including the cost of renting the room plus the cost of the meal. A banquet for 70 people costs $1300. For 120 people, the price is $2200.

(a) Plot a graph of cost versus the number of people.

(b) From the graph, estimate the cost of a banquet for 150 people.

(c) Determine the slope of the line. What quantity does the slope of the line represent?

(d) Write an equation to model this real-life situation.

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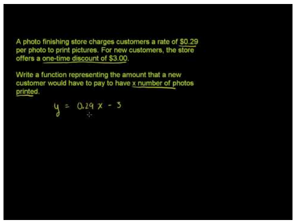
Khan Academy Basic Linear Function



A photo finishing store charges customers a rate of $0.29 per photo to print pictures. For new customers, the store offers a one-time discount of $3.00.

Write a function representing the amount that a new customer would have to pay to have x number of photos printed.

$y = 0.29x - 3$

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Guidance

Linear relationships are often used to model real-life situations. In order to create an equation and graph to model the real-life situation, you need at least two data values related to the real-life situation. When the data values have been represented graphically and the equation of the line has been determined, questions relating to the real-life situation can be presented and answered.

When equations and graphs are used to model real-life situations, the domain of the graph is sometimes $x \varepsilon N$. However, it is often more convenient to sketch the graph as though $x \varepsilon R$ instead of showing the function as a series of points in the plane.

Example A

A cab company charges $2.00 for the first 0.6 miles and $0.50 for each additional 0.2 miles.

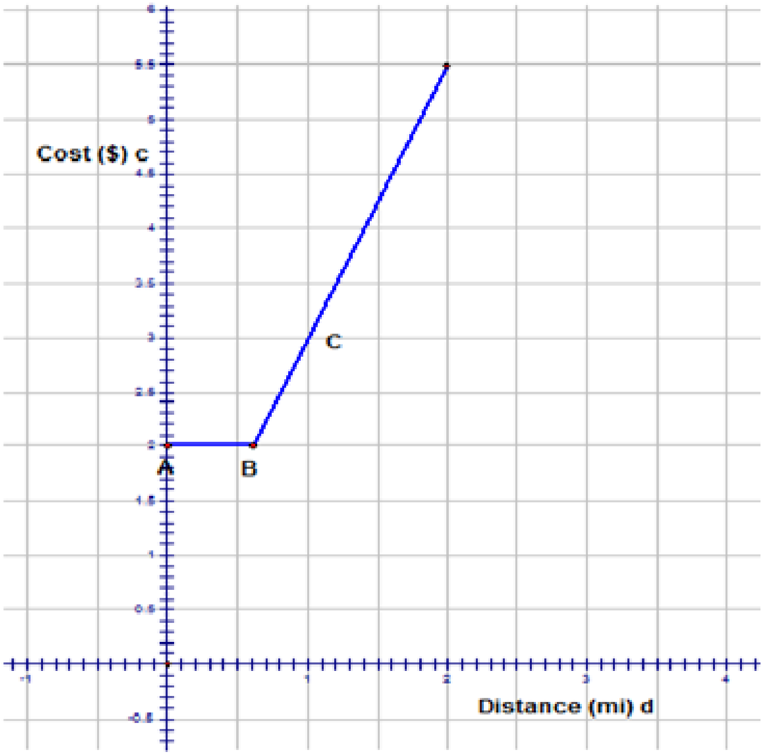
(a) Draw the graph of cost versus distance.

(b) Determine the equations that model this situation.

(c) What is the cost to travel 16 miles by cab?

Solution: This example demonstrates a real-life situation that cannot be modeled with just one equation.

(a) On the x-axis is the distance in miles and on the y-axis is the cost in dollars. The first line from A to B extends horizontally across the distance from 0 to 0.6 miles. The cost is constant at \$2.00. The equation for this constant function is $y = 2.00$ or $c = 2.00$. The second line from B to C and upward is not constant.



(b) The equation that models the second graph can be determined by using the data points (0.6, 2.00) and (1, 3.00)

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ m &= \frac{3.00 - 2.00}{1 - 0.6} \\ m &= \frac{3.00 - 2.00}{1 - 0.6} \\ m &= \frac{1.00}{0.4} \\ m &= 2.5 \end{aligned}$$

Use the data points to calculate the slope.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 2.5(x - 0.6) \\ y - 2 &= 2.5x - 1.5 \\ y - 2 + 2 &= 2.5x - 1.5 + 2 \\ y &= 2.5x + 0.5 \\ c &= 2.5d + 0.5 \end{aligned}$$

Use the slope and one point to determine the equation.

Therefore, the equations that model this situation are:

$$c = \begin{cases} 2.00 & 0 < d \leq 0.6 \\ 2.5d + 0.5 & d > 0.6 \end{cases}$$

(c) The cost to travel 16 miles in the cab is:

The distance is greater than 0.6 miles. The cost must be calculated using the equation $c = 2.5d + 0.5$. Substitute 16 in for 'd'.

$$\begin{aligned} c &= 2.5d + 0.5 \\ c &= 2.5(16) + 0.5 \\ c &= 40 + 0.5 \\ &\boxed{c = \$40.50} \end{aligned}$$

Example B

When a 40 gram mass was suspended from a coil spring, the length of the spring was 24 inches. When an 80 gram mass was suspended from the same coil spring, the length of the spring was 36 inches.

(a) Plot a graph of length versus mass.

(b) From the graph, estimate the length of the spring for a mass of 70 grams.

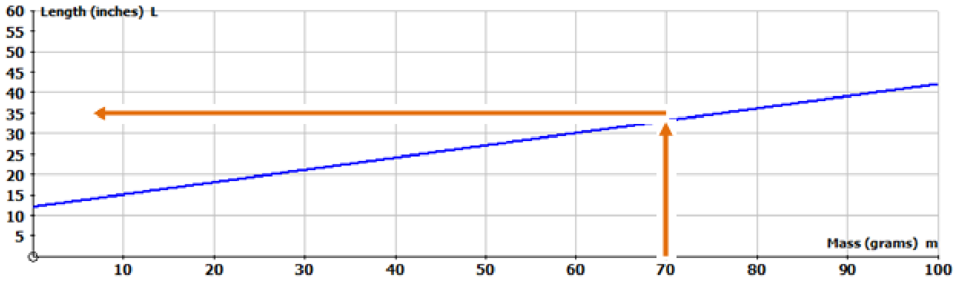
(c) Determine an equation that models this situation. Write the equation in slope-intercept form.

(d) Use the equation to determine the length of the spring for a mass of 60 grams.

(e) What is the y-intercept? What does the y-intercept represent?

Solution:

(a) On the x-axis is the mass in grams and on the y-axis is the length of the spring in inches.



(b) The length of the coil spring for a mass of 70 grams is approximately 33 inches.

(c) The equation of the line can be determined by using the two data values (40, 24) and (80, 36).

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
m &= \frac{36 - 24}{80 - 40} \\
m &= \frac{12}{40} \\
&= \frac{3}{10}
\end{aligned}$$

$$\begin{aligned}
y &= mx + b \\
24 &= \frac{3}{10}(40) + b \\
24 &= \frac{3}{\cancel{10}}\left(\overset{4}{\cancel{40}}\right) + b \\
24 &= 12 + b \\
24 - 12 &= 12 - 12 + b \\
12 &= b
\end{aligned}$$

The y-intercept is (0, 12). The equation that models the situation is

$y = \frac{3}{10}x + 12$

$$\boxed{l = \frac{3}{10}m + 12}$$

where 'l' is the length of the spring in inches and 'm' is the mass in grams.

(d)

$$\begin{aligned}
l &= \frac{3}{10}m + 12 && \text{Use the equation and substitute 60 in for } m. \\
l &= \frac{3}{10}(60) + 12 \\
l &= \frac{3}{\cancel{10}}\left(\overset{6}{\cancel{60}}\right) + 12 \\
l &= 18 + 12 \\
&\boxed{l = 30 \ inches}
\end{aligned}$$

(e) The y-intercept is (0, 12). The y-intercept represents the length of the coil spring before a mass was suspended from it. The length of the coil spring was 12 inches.

Example C

Juan drove from his mother's home to his sister's home. After driving for 20 minutes he was 62 miles away from his sister's home and after driving for 32 minutes he was only 38 miles away. The time driving and the distance away from his sister's home form a linear relationship.

(a) What is the independent variable? What is the dependent variable?

(b) What are the two data values?

(c) Draw a graph to represent this problem. Label the axis appropriately.

(d) Write an equation expressing distance in terms of time driving.

(e) What is the slope and what is its meaning in this problem?

(f) What is the time-intercept and what does it represent?

(g) What is the distance-intercept and what does it represent?

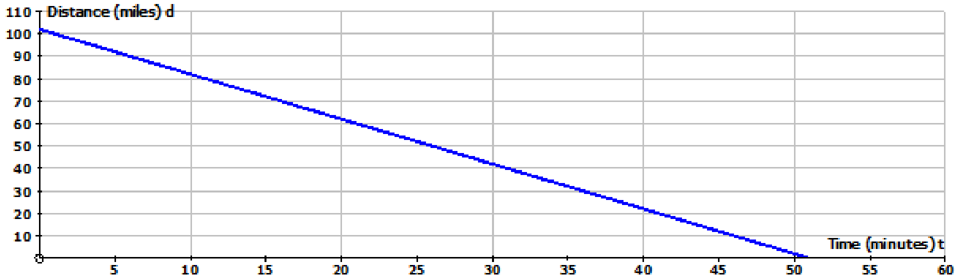
(h) How far is Juan from his sister's home after he had been driving for 35 minutes?

Solution:

(a) The independent variable is the time driving. The dependent variable is the distance.

(b) The two data values are (20, 62) and (32, 38).

(c) On the x-axis is the time in minutes and on the y-axis is the distance in miles.



(d) (20, 62) and (32, 38) are the coordinates that will be used to calculate the slope of the line.

$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1}\\
m &= \frac{38-62}{32-20}\\
m &= \frac{-24}{12}\\
m &= -2
\end{aligned}$$

$$\begin{aligned}
y &= mx+b\\
62 &= -2(20)+b\\
62 &= -40+b\\
62+40 &= -40+40+b\\
102 &= b \qquad \text{The } y\text{-intercept is } (0,102)
\end{aligned}$$

$$\begin{aligned}
y &= mx+b\\
y &= -2x+102\\
d &= -2t+102
\end{aligned}$$

(e) The slope is $-2 = \frac{-2}{1} = \frac{-2(miles)}{1(minute)}$. The slope means that for each minute of driving, the distance that Juan has to drive to his sister's home is reduced by 2 miles.

(f) The time-intercept is actually the x-intercept. This value is:

$$\begin{aligned}
d &= -2t+102 && \text{Set } d=0 \text{ and solve for } t.\\
0 &= -2t+102\\
0+2t &= -2t+2t+102\\
2t &= 102\\
\frac{2t}{2} &= \frac{102}{2}\\
\frac{\cancel{2}t}{\cancel{2}} &= \frac{102}{2}\\
t &= 51\ minutes
\end{aligned}$$

The time-intercept is 51 minutes and this represents the time it took Juan to drive from his mother's home to his sister's home.

(g) The distance-intercept is the y-intercept. This value has been calculated as (0, 102). The distance-intercept represents the distance between his mother's home and his sister's home. The distance is 102 miles.

(h)

$$\begin{aligned}
d &= -2t+102 && \text{Substitute 35 into the equation for } t \text{ and solve for } d.\\
d &= -2(35)+102\\
d &= -70+102\\
d &= 32\ miles
\end{aligned}$$

After driving for 35 minutes, Juan is 32 miles from his sister's home.

Concept Problem Revisited

Joe's Warehouse has banquet facilities to accommodate a maximum of 250 people. When the manager quotes a price for a banquet she is including the cost of renting the room plus the cost of the meal. A banquet for 70 people costs $1300. For 120 people, the price is $2200.

(a) Plot a graph of cost versus the number of people.

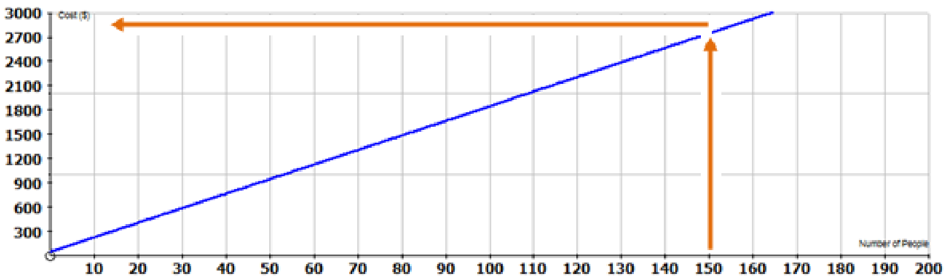
(b) From the graph, estimate the cost of a banquet for 150 people.

(c) Determine the slope of the line. What quantity does the slope of the line represent?

(d) Write an equation to model this real-life situation.

Solution:

(a) On the x-axis is the number of people and on the y-axis is the cost of the banquet.



(b) The approximate cost of a banquet for 150 people is $2700.

(c) The two data points (70, 1300) and (120, 2200) will be used to calculate the slope of the line.

$$\begin{aligned}m &= \frac{y_2 - y_1}{x_2 - x_1}\\ m &= \frac{2200 - 1300}{120 - 70}\\ m &= \frac{900}{50}\\ m &= \frac{18}{1}\end{aligned}$$

The slope represents the cost of the banquet for each person. The cost is $18 per person.

When a linear function is used to model the real life situation, the equation can be written in the form or in the form $y = mx + b$ or in the form $Ax + By + C = 0$.

(d)

$$\begin{aligned}y &= mx + b\\ 1300 &= 18(70) + b\\ 1300 &= 1260 + b\\ 1300 - 1260 &= 1260 - 1260 + b\\ 40 &= b\end{aligned}$$

The y-intercept is (0, 40)

The equation to model the real-life situation is $y = 18x + 40$. The variables should be changed to match the labels on the axes. The equation that best models the situation is $c = 18n + 40$ where 'c' represents the cost and 'n' represents the number of people.

Vocabulary

Slope – Intercept Form

The ***slope-intercept form*** is one method for writing the equation of a line. The slope-intercept form is $y = mx + b$ where m refers to the slope and b identifies the y-intercept.

Standard Form

The ***standard form*** is another method for writing the equation of a line. The standard form is $Ax + By + C = 0$ where A is the coefficient of x, B is the coefficient of y and C is a constant.

Guided Practice

1. Some college students who plan on becoming math teachers decide to set up a tutoring service for high school math students. One student was charged $25 for 3 hours of tutoring. Another student was charged $55 for 7 hours of tutoring. The relationship between the cost and time is linear.

(a) What is the independent variable?
(b) What is the dependent variable?
(c) What are two data values for this relationship?

(d) Draw a graph of cost versus time.
(e) Determine an equation to model the situation.
(f) What is the significance of the slope?
(g) What is the cost-intercept? What does the cost-intercept represent?

2. A Glace Bay developer has produced a new handheld computer called the ***Blueberry***. He sold 10 computers in one location for $1950 and 15 in another for $2850. The number of computers and cost forms a linear relationship

(a) State the dependent and independent variables.
(b) Sketch a graph.
(c) Find an equation expressing cost in terms of the number of computers.
(d) State the slope of the line and tell what the slope means to the problem.
(e) State the cost-intercept and tell what it means to this problem.
(f) Using your equation, calculate the number of computers you could get for $6000.

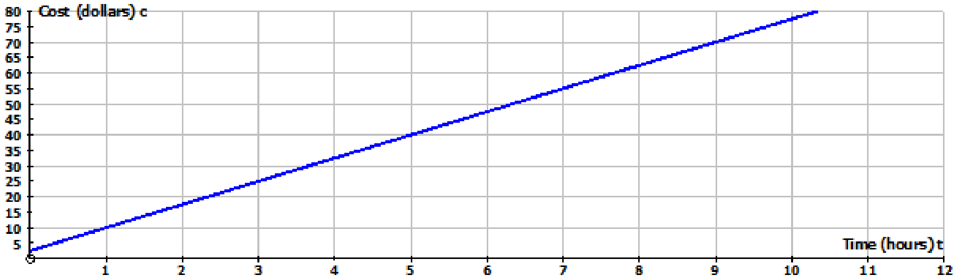
3. Handy Andy sells one quart can of paint thinner for $7.65 and a two quart can for $13.95. Assume there is a linear relationship between the volume of paint thinner and the price.

(a) What is the independent variable?
(b) What is the dependent variable?
(c) Write two data values for this relationship.
(d) Draw a graph to represent this relationship.
(e) What is the slope of the line?
(f) What does the slope represent in this problem?
(g) Write an equation to model this problem.
(h) What is the cost-intercept?
(i) What does the cost-intercept represent in this problem?
(j) How much would you pay for 6 quarts of paint thinner?

Answers:

1. (a) The cost for tutoring depends upon the amount of time. The independent variable is the time.

(b) The dependent variable is the cost.
(c) Two data values for this relationship are (3, 25) and (7, 55).
(d) On the x-axis is the time in hours and on the y-axis is the cost in dollars.



(e) Use the two data values (3, 25) and (7, 55) to calculate the slope of the line. $m = \frac{15}{2}$. Determine the y-intercept of the graph.

$$\begin{aligned} y &= mx+b \\ 25 &= \frac{15}{2}(3)+b \\ 25 &= \frac{45}{2}+b \\ 25-\frac{45}{2} &= \frac{45}{2}-\frac{45}{2}+b \\ \frac{50}{2}-\frac{45}{2} &= b \\ \frac{5}{2} &= b \end{aligned}$$

Use the slope and one of the data values to determine the value of b.

The equation to model the relationship is $y = \frac{15}{2}x + \frac{5}{2}$. To match the variables of the equation with the graph the equation is

$$\boxed{c = \frac{15}{2}t + \frac{5}{2}}$$

. The relationship is cost in dollars versus time in hours. The equation could also be written as

$$\boxed{c = 7.50t + 2.50}$$

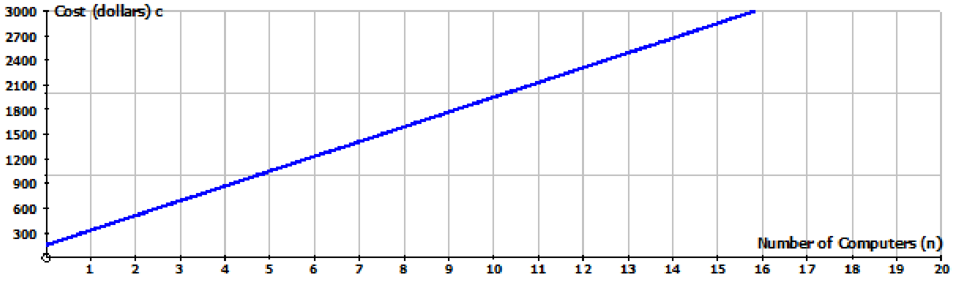
.

(f) The slope of $\frac{15}{2}$ means that it costs \$15.00 for 2 hours of tutoring. If the slope is expressed as a decimal, it means that it costs \$7.50 for 1 hour of tutoring.

(g) The cost-intercept is the y-intercept. The y-intercept is (0, 2.50). This value could represent the cost of having a scheduled time or the cost that must be paid for cancelling the appointment. In a problem like this, the y-intercept must represent a meaningful quantity for the problem.

2. (a) The number of dollars in sales from the computers depends upon the number of computers sold. The dependent variable is the dollars in sales and the independent variable is the number of computers sold.

(b) On the x-axis is the number of computers and on the y-axis is the cost of the computers.



(c) Use the data values (10, 1950) and (15, 2850) to calculate the slope of the line. $m = 180$. Next determine the y-intercept of the graph.

$$\begin{aligned} y &= mx+b \\ 1950 &= 180(10)+b \\ 1950 &= 1800+b \\ 1950-1800 &= 1800-1800+b \\ 150 &= b \end{aligned}$$

The equation of the line that models the relationship is

$$\boxed{y = 180x + 150}$$

. To make the equation match the variables of the graph the equation is

$$\boxed{c = 180n + 150}$$

.

(d) The slope is $\frac{180}{1}$. This means that the cost of one computer is \$180.00.

(e) The cost intercept is the y-intercept. The y-intercept is (0, 150). This could represent the cost of renting the location where the sales are being made or perhaps the salary for the sales person.

(f)

$$\begin{aligned}
c &= 180n + 150\\
6000 &= 180n + 150\\
6000 - 150 &= 180n + 150 - 150\\
5850 &= 180n\\
\frac{5850}{180} &= \frac{180n}{180}\\
\frac{5850}{180} &= \frac{\cancel{180}n}{\cancel{180}}\\
32.5 &= n
\end{aligned}$$

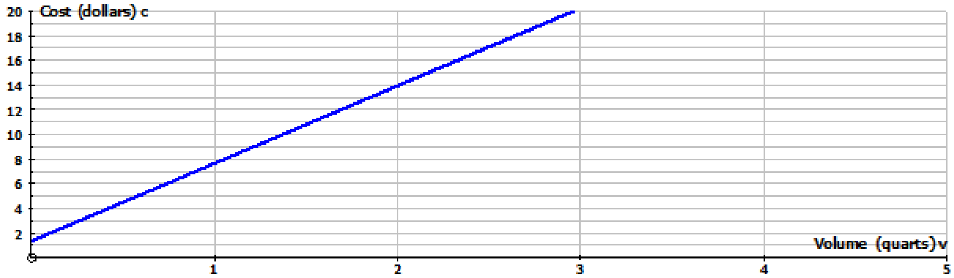
With \$6000 you could get **32** computers.

3. (a) The independent variable is the volume of paint thinner.

(b) The dependent variable is the cost of the paint thinner.

(c) Two data values are (1, 7.65) and (2, 13.95).

(d) On the x-axis is the volume in quarts and on the y-axis is the cost in dollars.



(e) Use the two data values (1, 7.65) and (2, 13.95) to calculate the slope of the line. The slope is $m = 6.30$.

(f) The slope represents the cost of one quart of paint thinner. The cost is \$6.30.

(g)

$$\begin{aligned}
y &= mx + b\\
7.65 &= 6.30(1) + b\\
7.65 &= 6.30 + b\\
7.65 - 6.30 &= 6.30 - 6.30 + b\\
1.35 &= b
\end{aligned}$$

The equation to model the relationship is $y = 6.30x + 1.35$. The equation that matches the variables of the graph is

$$\boxed{c = 6.30v + 1.35}$$

.

(h) The cost-intercept is (0, 1.35).
(i) This could represent the cost of the can that holds the paint thinner.
(j)

$$\begin{aligned}
c &= 6.30v + 1.35 \\
c &= 6.30(6) + 1.35 \\
c &= 37.80 + 1.35 \\
c &= \$39.15
\end{aligned}$$

The cost of 6 quarts of paint thinner is \$39.15.

Practice

Players on the school soccer team are selling candles to raise money for an upcoming trip. Each player has 24 candles to sell. If a player sells 4 candles a profit of \$30 is made. If he sells 12 candles a profit of \$70 is made. The profit and the number of candles sold form a linear relation.

1. State the dependent and the independent variables.
2. What are the two data values for this relation?
3. Draw a graph and label the axis.
4. Determine an equation to model this situation.
5. What is the slope and what does it mean in this problem?
6. Find the profit-intercept and explain what it represents.
7. Calculate the maximum profit that a player can make.
8. Write a suitable domain and range.
9. If a player makes a profit of \$90, how many candles did he sell?
10. Is this data continuous, discrete, or neither? Justify your answer.

Jacob leaves his summer cottage and drives home. After driving for 5 hours, he is 112 km from home, and after 7 hours, he is 15 km from home. Assume that the distance from home and the number of hours driving form a linear relationship.

11. State the dependent and the independent variables.
12. What are the two data values for this relationship?
13. Represent this linear relationship graphically.
14. Determine the equation to model this situation.
15. What is the slope and what does it represent?
16. Find the distance-intercept and its real-life meaning in this problem.
17. How long did it take Jacob to drive from his summer cottage to home?
18. Write a suitable domain and range.
19. How far was Jacob from home after driving 4 hours?
20. How long had Jacob been driving when he was 209 km from home?

Summary

You learned that slope is a measure of the steepness of a line. You learned how to find the slope of a line given two points on the line or the graph of the line. You also learned how to write the equation of a line in two forms: slope-intercept form and standard form. You learned how to graph a line directly from its equation without first making a table.

You also learned about parallel and perpendicular lines. You learned parallel lines always have the same slope while perpendicular lines always have slopes that are opposite reciprocals. Finally, you learned how to model real-world problems with linear functions.

CHAPTER 5 Systems of Equations and Inequalities

Chapter Outline

Introduction

Here you'll learn all about *systems* in algebra. You will learn about systems of equations and how to solve them graphically and algebraically. You will also learn about systems of inequalities and how to solve them graphically and use them to solve real-world problems involving maximizing and minimizing.

5.1 Graphical Solutions to Systems of Equations

Here you'll learn to solve a system of linear equations by graphing.

When you graph two linear functions on the same Cartesian plane, the resulting lines may intersect. Do the following two lines intersect? If so, where?

$$\left\{\begin{array}{c}2x+y=5\\ x-y=1\end{array}\right\}$$

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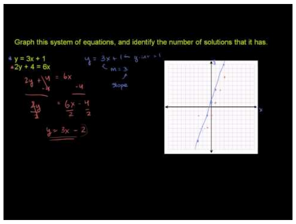
Khan Academy Solving Systems by Graphing3



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Guidance

A 2×2 system of linear equations consists of two equations with two variables, such as athe one below:

$$\left\{\begin{array}{c}2x+y=5\\ x-y=1\end{array}\right\}$$

When graphed, a system of linear equations is two lines. To solve a system of linear equations, figure out if the two lines intersect and if so, at what point. One way to solve a system of equations is by graphing. Graph both lines and look for the point where they intersect.

Keep in mind that even though most of the time when you graph two lines they will intersect in just one point, there are two other possibilities:

1. The lines might never intersect (they are parallel lines)
2. The lines might coincide (be exactly the same line)

A system that results in **one point of intersection** is **consistent and independent**. A system that results in lines that **coincide** is **consistent and dependent**. A system that results in **two parallel lines** is **inconsistent**.

Example A

Solve the following system of linear equations graphically:

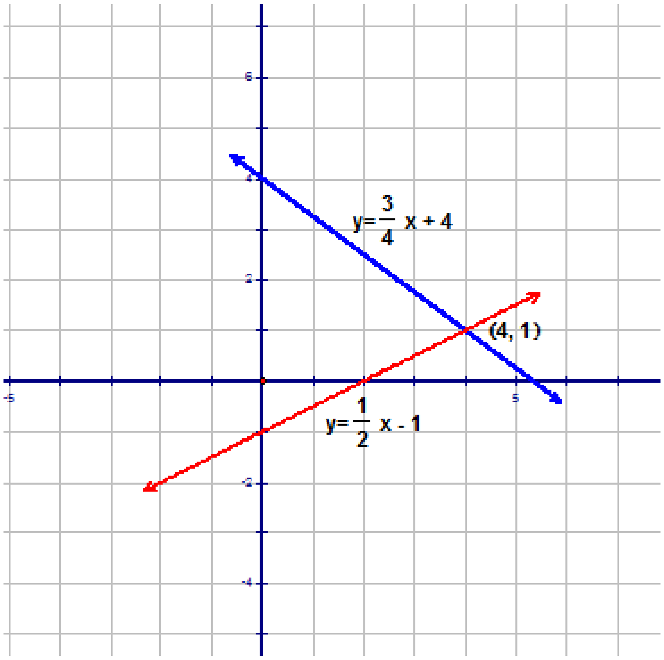
$$\left\{\begin{matrix} x-2y-2=0 \\ 3x+4y=16 \end{matrix}\right\}$$

Solution: Begin by writing each linear equation in slope-intercept form.

$$\begin{aligned}
&x-2y-2=0\\
&x-x-2y-2=0-x\\
&-2y-2=-x\\
&-2y-2+2=-x+2\\
&-2y=-x+2\\
&\frac{-2y}{-2}=\frac{-x}{-2}+\frac{2}{-2}\\
&\boxed{y=\frac{1}{2}x-1} \qquad \text{Equation One}
\end{aligned}$$

$$\begin{aligned}
&3x+4y=16\\
&3x-3x+4y=16-3x\\
&4y=16-3x\\
&\frac{4y}{4}=\frac{16}{4}-\frac{3x}{4}\\
&y=4-\frac{3}{4}x\\
&\boxed{y=-\frac{3}{4}x+4} \qquad \text{Equation Two}
\end{aligned}$$

Graph both equations on the same Cartesian plane.



The lines intersect at the point (4, 1). The solution is an ordered pair that should satisfy both of the equations in the system.

Test (4, 1) in equation one:

$$\begin{aligned}
x-2y-2&=0 && \text{Use the original equation}\\
(4)-2(1)-2&=0 && \text{Replace } x \text{ with 4 and replace } y \text{ with 1.}\\
4-2-2&=0 && \text{Perform the indicated operations and simplify the result.}\\
4-4&=0 &&\\
0&=0 && \text{Both sides of the equation are equal. The ordered pair } (4,1) \text{ satisfies the equation.}
\end{aligned}$$

Test (4, 1) in equation two:

$$\begin{aligned}
3x+4y&=16 && \text{Use the original equation}\\
3(4)+4(1)&=16 && \text{Replace } x \text{ with 4 and replace } y \text{ with 1.}\\
12+4&=16 && \text{Perform the indicated operations and simplify the result.}\\
16&=16 && \text{Both sides of the equation are equal. The ordered pair } (4,1) \text{ satisfies the equation.}
\end{aligned}$$

This system of equations has a solution and is therefore called a **consistent** system. Because it has only one ordered pair as a solution, it is called an **independent** system.

Example B

Solve the following system of linear equations graphically:

$$\begin{Bmatrix} 2y-3x=6 \\ 4y-6x=12 \end{Bmatrix}$$

Solution: Graph both equations on the same Cartesian plane using the intercept method. Let $x=0$. Solve for y

$$\begin{aligned}
&2y-3x=6 &&\\
&2y-3(0)=6 && \text{Replace } x \text{ with zero.}\\
&2y=6 && \text{Simplify}\\
&\frac{2y}{2}=\frac{6}{2} && \text{Solve for } y.\\
&\boxed{y=3} && \text{The } y\text{-intercept is } (0,3)
\end{aligned}$$

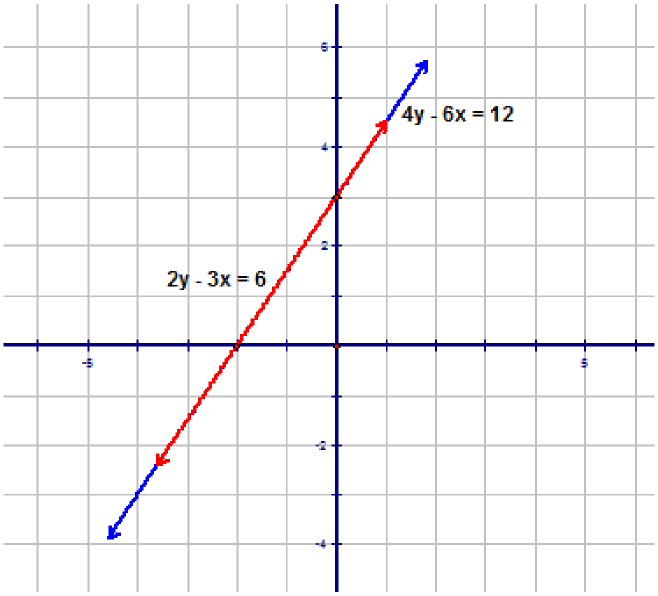
Let $y=0$. Solve for x.

$$\begin{aligned}
&2y-3x=6 &&\\
&2(0)-3x=6 && \text{Replace } y \text{ with zero.}\\
&-3x=6 && \text{Simplify}\\
&\frac{-3x}{-3}=\frac{6}{-3} && \text{Solve for } y.\\
&\boxed{x=-2} && \text{The } x\text{-intercept is } (-2,0)
\end{aligned}$$

$$\begin{array}{ll}
4y-6x=12 & \\
4y-6(0)=12 & \text{Replace } x \text{ with zero.} \\
4y=12 & \text{Simplify} \\
\frac{4y}{4}=\frac{12}{4} & \text{Solve for } y. \\
\boxed{y=3} & \text{The } y\text{-intercept is } (0,3)
\end{array}$$

Let $y=0$. Solve for x.

$$\begin{array}{ll}
4y-6x=12 & \\
4(0)-6x=12 & \text{Replace } y \text{ with zero.} \\
-6x=12 & \text{Simplify} \\
\frac{-6x}{-6}=\frac{12}{-6} & \text{Solve for } y. \\
\boxed{x=-2} & \text{The } x\text{-intercept is } (-2,0)
\end{array}$$



When the x and y-intercepts were calculated for each equation, they were the same for both lines. The graph resulted in the same line being graphed twice. The blue line is longer to show that the same line is graphed directly on top of the red line. The system does have solutions so it is also known as a **consistent** system. However, the system does not have one solution; it has an infinite number of solutions. This type of consistent system is called a **dependent** system. All the ordered pairs found on the line will satisfy both equations. If you look at the two given equations

$$\left\{\begin{array}{l} 2y-3x=6 \\ 4y-6x=12 \end{array}\right\}$$

, equation two is simply a multiple of equation one.

Example C

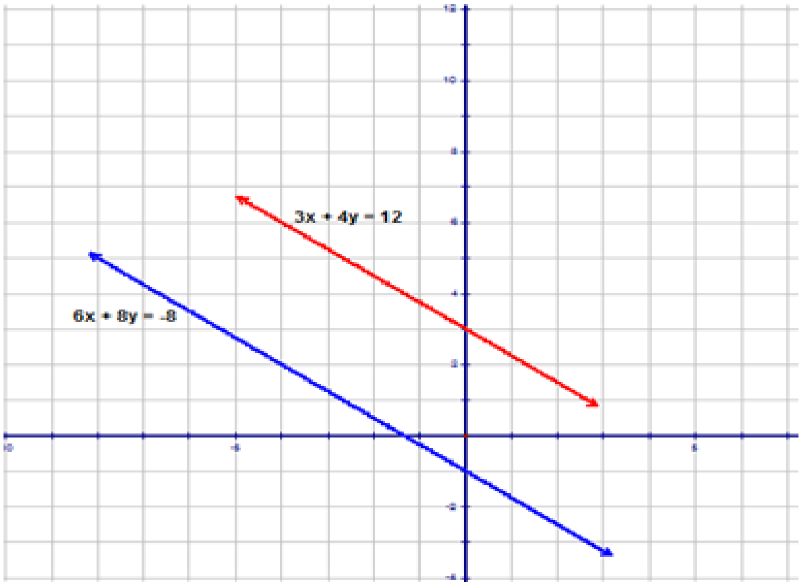
Solve the following system of linear equations graphically:

$$\left\{\begin{array}{l}3x+4y=12\\6x+8y=-8\end{array}\right\}$$

Solution: Graph both equations on the same Cartesian plane using the slope-intercept method. Begin by writing each linear equation in slope-intercept form.

$$\begin{aligned}&3x+4y=12\\&3x-3x+4y=12-3x\\&4y=12-3x\\&\frac{4y}{4}=\frac{12}{4}-\frac{3x}{4}\\&y=3-\frac{3}{4}x\\&\boxed{y=-\frac{3}{4}x+3}\end{aligned}$$

$$\begin{aligned}&6x+8y=-8\\&6x-6x+8y=-8-6x\\&8y=-8-6x\\&\frac{8y}{8}=\frac{-8}{8}-\frac{6x}{8}\\&y=-1-\frac{6}{8}x\\&\boxed{y=-\frac{6}{8}x-1}\end{aligned}$$



The lines do not intersect. This means that the system of equations has no solution. The lines are parallel and will never intersect. If you look at the equations that were written in slope-intercept form $y=-\frac{3}{4}x+3$**and** $y=-\frac{6}{8}x-1$, the slopes are the same $\left(-\frac{6}{8}=-\frac{3}{4}\right)$. A system of linear equations that has no solution is called an **inconsistent** system.

Example D

Before graphing calculators, graphing was not considered the best way to determine the solution for a system of linear equations, especially if the solutions were not integers. However, technology has changed this outlook. In this example, a graphing calculator will be used to determine the solution for

$$\left\{\begin{array}{c} x+4y=-14 \\ 2x-y=4 \end{array}\right\}$$

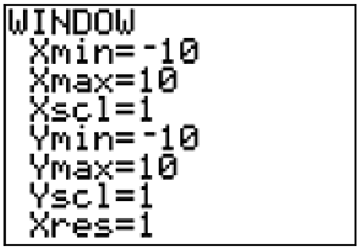
.

Solution: To use a graphing calculator, the equations must be written in slope-intercept form:

$$\begin{aligned}
&x+4y=-14\\
&x-x+4y=-x-14\\
&4y=-x-14\\
&\frac{4y}{4}=-\frac{x}{4}-\frac{14}{4}\\
&y=-\frac{1}{4}x-\frac{14}{4}\\
&\boxed{y=-\frac{1}{4}x-\frac{7}{2}}
\end{aligned}$$

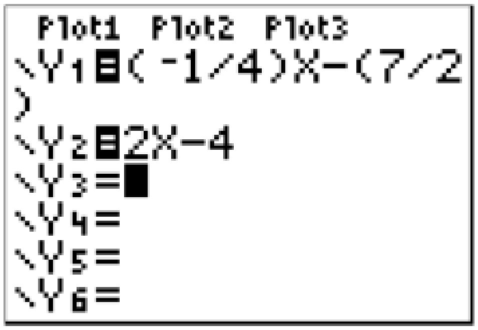
$$\begin{aligned}
&2x-y=4\\
&2x-2x-y=-2x+4\\
&-y=-2x+4\\
&\frac{-y}{-1}=\frac{-2x}{-1}+\frac{4}{-1}\\
&\boxed{y=2x-4}
\end{aligned}$$

The equations are both in slope-intercept form. Set the window on the calculator as shown below:



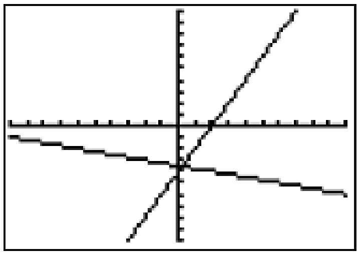
```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```

Enter the equations:

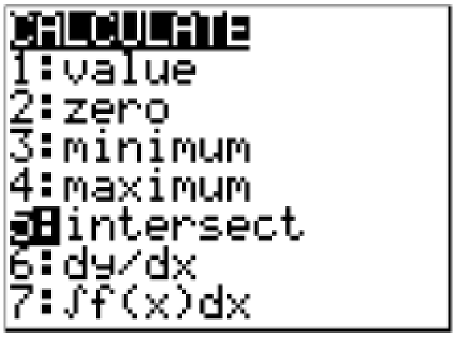


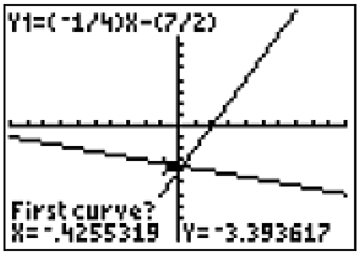
```
  Plot1  Plot2  Plot3
\Y1=(-1/4)X-(7/2
)
\Y2=2X-4
\Y3=
\Y4=
\Y5=
\Y6=
```

Graph the equations:

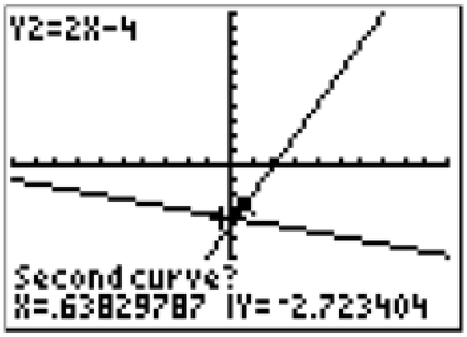


Calculate the intersection point:

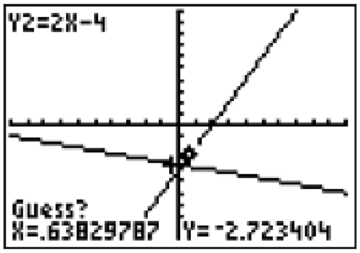




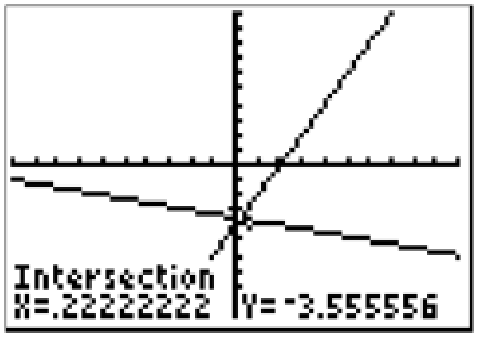
```
Y1=(-1/4)X-(7/2)
First curve?
X=-.4255319   Y=-3.393617
```



```
Y2=2X-4
Second curve?
X=.63829787   Y=-2.723404
```

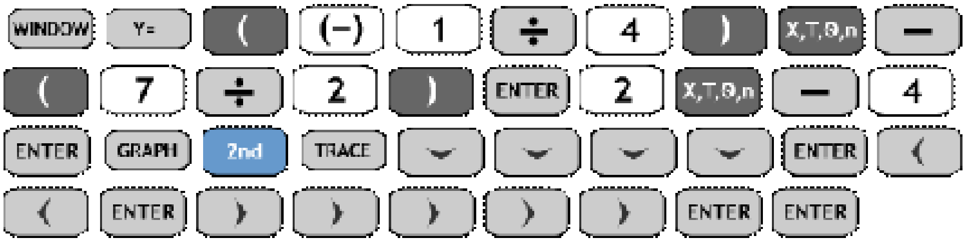


```
Y2=2X-4
Guess?
X=.63829787   Y=-2.723404
```



```
Intersection
X=.2222222   Y=-3.555556
```

The intersection point of the linear equations is (0.22, –3.56). The following represents the keys that were pressed on the calculator to obtain the above results:



Concept Problem Revisited

The following linear equations will be graphed by using the slope-intercept method.

$$\begin{Bmatrix} 2x+y=5 \\ x-y=1 \end{Bmatrix}$$

$$2x+y=5$$

$$2x-2x+y=5-2x$$

$$y=5-2x$$

$$\boxed{y=-2x+5}$$

$$x-y=1$$

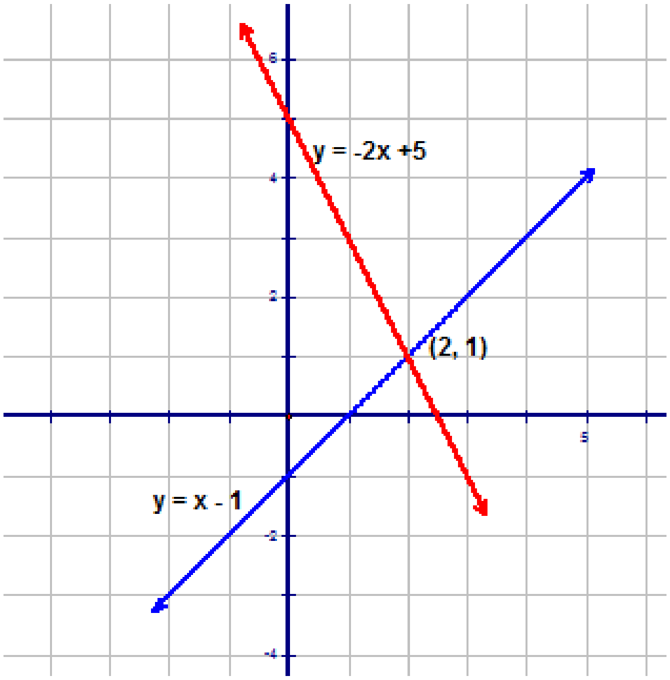
$$x-x-y=1-x$$

$$-y=1-x$$

$$\frac{-y}{-1}=\frac{1}{-1}-\frac{x}{-1}$$

$$y=-1+x$$

$$\boxed{y=x-1}$$



The two lines intersect at one point. The coordinates of the point of intersection are (2, 1).

Vocabulary

Consistent System of Linear Equations

A ***consistent system of linear equations*** is a system of linear equations that has a solution. The solution may be one solution or an infinite number of solutions.

Dependent System of Linear Equations

A ***dependent system of linear equations*** is a system of linear equations that has an infinite number of solutions. The equations are multiples and the graphs of each equation are the same. Therefore, the infinite number of ordered pairs satisfies both equations.

Equivalent Systems of Linear Equations

Equivalent systems of linear equations are systems of linear equations that have the same solution set.

Inconsistent System of Linear Equations

An ***inconsistent system of linear equations*** is a system of linear equations that has no solution. The graphs of an inconsistent system of linear equations are parallel lines. The lines never intersect so there is no common point of intersection.

Independent System of Linear Equations

An ***independent system of linear equations*** is a system of linear equations that has one solution. There is only one ordered pair that satisfies both equations.

System of Linear Equations

A ***system of linear equations*** is more than one linear equation. Two equations with two unknowns is called a 2×2 system of linear equations.

Guided Practice

1. Solve the following system of linear equations by graphing:

$$\begin{Bmatrix} -3x+4y=20 \\ x-2y=-8 \end{Bmatrix}$$

Is the system consistent and dependent, consistent and independent, or inconsistent?

For #2 and #3, use technology to determine whether the system is consistent and independent, consistent and dependent, or inconsistent.

2.

$$\begin{Bmatrix} 3x-2y=8 \\ 6x-4y=20 \end{Bmatrix}$$

3.

$$\begin{Bmatrix} x+3y=4 \\ 5x-y=4 \end{Bmatrix}$$

Answers:

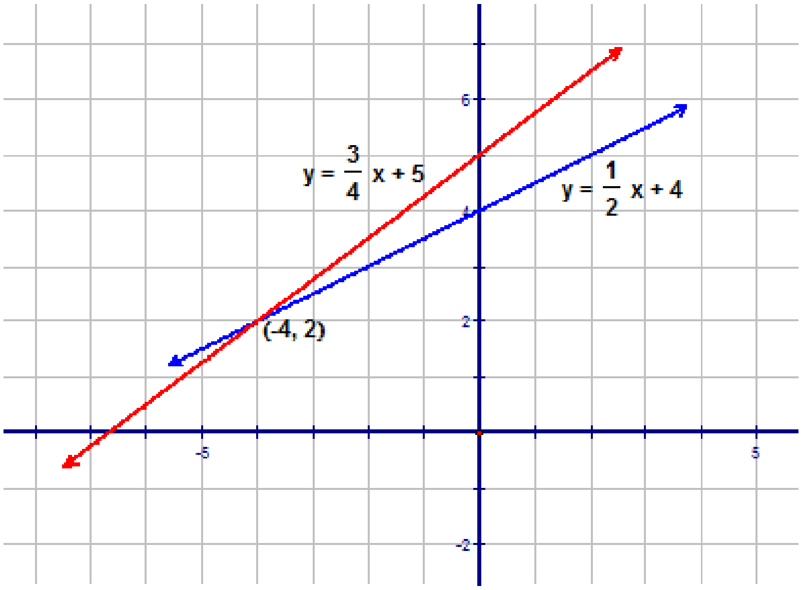
1.

$$\begin{Bmatrix} -3x+4y=20 \\ x-2y=-8 \end{Bmatrix}$$

Begin by writing the equations in slope-intercept form.

$$\begin{aligned}&-3x+4y=20\\&-3x+3x+4y=20+3x\\&4y=20+3x\\&\frac{4y}{4}=\frac{20}{4}+\frac{3x}{4}\\&y=5+\frac{3}{4}x\\&\boxed{y=\frac{3}{4}x+5}\end{aligned}$$

$$\begin{aligned}&x-2y=-8\\&x-x-2y=-8-x\\&-2y=-8-x\\&\frac{-2y}{-2}=\frac{-8}{-2}-\frac{x}{-2}\\&y=4+\frac{1}{2}x\\&\boxed{y=\frac{1}{2}x+4}\end{aligned}$$

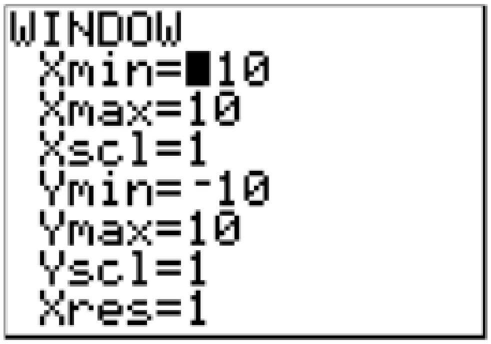


The lines intersect at the point (–4, 2). This ordered pair is the one solution for the system of linear equations. The system is **consistent** and **independent**.

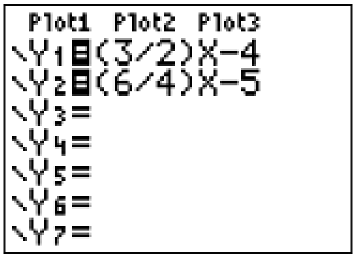
2.

$$\left\{\begin{aligned}3x-2y&=8\\6x-4y&=20\end{aligned}\right\}$$

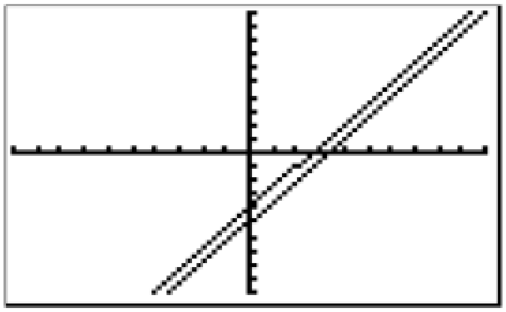
$$
\begin{aligned}
&3x-2y=8\\
&3x-3x-2y=-3x+8\\
&-2y=-3x+8\\
&\frac{-2y}{-2}=\frac{-3x}{-2}+\frac{8}{-2}\\
&\boxed{y=\frac{3}{2}x-4} \quad \text{Slope-intercept form}
\end{aligned}
\qquad
\begin{aligned}
&6x-4y=20\\
&6x-6x-4y=-6x+20\\
&-4y=-6x+20\\
&\frac{-4y}{-4}=\frac{-6x}{-4}+\frac{20}{-4}\\
&\boxed{y=\frac{6}{4}x-5}
\end{aligned}
$$



```
WINDOW
 Xmin=█10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



```
 Plot1  Plot2  Plot3
\Y1■(3/2)X-4
\Y2■(6/4)X-5
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

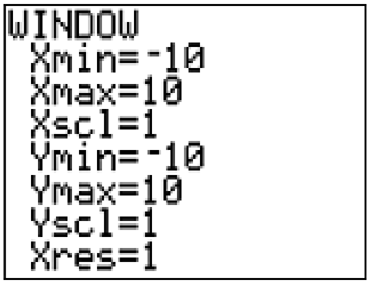


The lines are parallel. The lines will never intersect so there is no solution. The system is inconsistent.

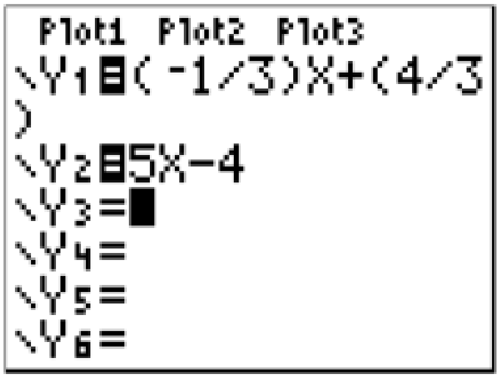
3.

$$
\begin{Bmatrix}
x+3y=4\\
5x-y=4
\end{Bmatrix}
$$

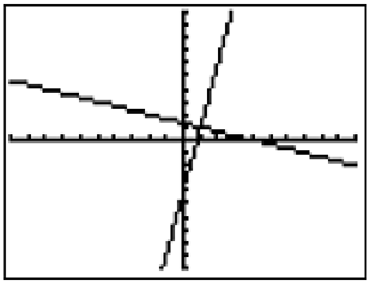
$$
\begin{aligned}
&x+3y=4\\
&x-x+3y=-x+4\\
&3y=-x+4\\
&\frac{3y}{3}=\frac{-x}{3}+\frac{4}{3}\\
&\boxed{y=\frac{-1}{3}x+\frac{4}{3}}
\end{aligned}
\qquad
\begin{aligned}
&5x-y=4\\
&5x-5x-y=-5x+4\\
&-y=-5x+4\\
&\frac{-y}{-1}=\frac{-5x}{-1}+\frac{4}{-1}\\
&\boxed{y=5x-4}
\end{aligned}
$$

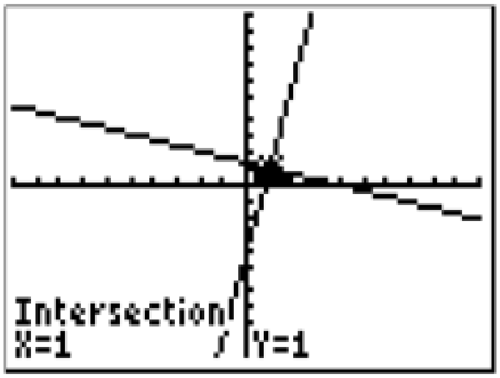


```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```



```
 Plot1  Plot2  Plot3
\Y1=(-1/3)X+(4/3
)
\Y2=5X-4
\Y3=
\Y4=
\Y5=
\Y6=
```





```
Intersection
X=1            Y=1
```

There is one point of intersection (1, 1). The system is consistent and independent.

Practice

Without graphing, determine whether the system is consistent and independent, consistent and dependent, or inconsistent.

1.

$$\left\{\begin{array}{c} 2x+3y=6 \\ 6x+9y=18 \end{array}\right\}$$

2.

$$\left\{\begin{array}{c} 2x-y=-14 \\ 12x-6y=-11 \end{array}\right\}$$

3.

$$\left\{\begin{array}{c} 3x+2y=14 \\ 5x-y=6 \end{array}\right\}$$

4.

$$\begin{Bmatrix} 2x+3y=-12 \\ 3x-y=3 \end{Bmatrix}$$

5.

$$\begin{Bmatrix} 20x+15y=-30 \\ 4x+3y=18 \end{Bmatrix}$$

Solve the following systems of linear equations by graphing.

6.

$$\begin{Bmatrix} x+2y=8 \\ 3x+6y=24 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 4x+2y=-2 \\ 2x-3y=9 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} 3x+5y=11 \\ 4x-2y=-20 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 2x+y=5 \\ 6x=15-3y \end{Bmatrix}$$

10.

$$\begin{Bmatrix} 2x-y=2 \\ 4x-3y=-2 \end{Bmatrix}$$

11.

$$\begin{Bmatrix} 2x-3y=15 \\ 4x+y=2 \end{Bmatrix}$$

12.

$$\left\{\begin{array}{c} 2x+3y=-6 \\ 9y+6x+18=0 \end{array}\right\}$$

13.

$$\left\{\begin{array}{c} 6x+12y=-24 \\ 5x+10y=30 \end{array}\right\}$$

14.

$$\left\{\begin{array}{c} x-3y=7 \\ 2x+5y=-19 \end{array}\right\}$$

15.

$$\left\{\begin{array}{c} x+3y=9 \\ x-y=-3 \end{array}\right\}$$

5.2 Substitution Method for Systems of Equations

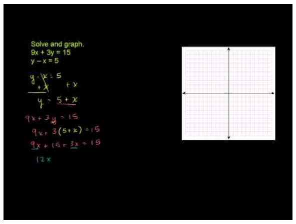
Here you'll learn how to solve systems of linear equations algebraically using the substitution method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. How can you algebraically solve a system of equations like the one below?

$$\begin{Bmatrix} 2x+3y=13 \\ y=4x-5 \end{Bmatrix}$$

Watch This

Khan Academy Slightly More Complicated Equations



MEDIA

Click image to the left for more content.

Guidance

A 2×2 system of linear equations can be solved algebraically by the substitution method. In order to use this method, follow these steps:

1. Solve one of the equations for one of the variables.
2. Substitute that expression into the remaining equation. The result will be a linear equation, with one variable, that can be solved.
3. Solve the remaining equation.
4. Substitute the solution into the other equation to determine the value of the other variable.
5. The solution to the system is the intersection point of the two equations and it represents the coordinates of the ordered pair.

Example A

Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} 3x+y=1 \\ 2x+5y=18 \end{Bmatrix}$$

Solution: To begin, solve one of the equations in terms of one of the variables. This step is simplified if one of the equations has one variable with a coefficient that is either +1 or –1. In the above system the first equation has 'y' with a coefficient of 1.

$$\begin{aligned}&3x+y=1\\&3x-3x+y=1-3x\\&\boxed{y=1-3x}\end{aligned}$$

Substitute $(1-3x)$ into the second equation for 'y'.

$$\begin{aligned}2x+5y&=18\\2x+5(1-3x)&=18\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}&2x+5-15x=18\\&-13x+5=18\\&-13x+5-5=18-5\\&-13x=13\\&\frac{-13x}{-13}=\frac{13}{-13}\\&\frac{\cancel{-13}x}{\cancel{-13}}=\frac{\overset{-1}{\cancel{-13}}}{\cancel{-13}}\\&\boxed{x=-1}\end{aligned}$$

Substitute –1 for x into the equation

$$\boxed{y=1-3x}$$

.

$$\begin{aligned}&y=1-3x\\&y=1-3(-1)\end{aligned}$$

$$\begin{aligned}&y=1+3\\&\boxed{y=4}\end{aligned}$$

The solution is (–1, 4). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid. Another way to write 'the lines intersect at (–1, 4)' is:

$$\boxed{\text{Line1}}:3x+y=1$$

$$\boxed{\text{Line2}} : 2x+5y=18$$

Line 1 intersects Line 2 at (–1, 4)

$$\begin{array}{c}\text{at}\\ \uparrow \\ \boxed{l_1 \cap l_2 @ (-1,4)} \\ \downarrow \\ \text{intersects}\end{array}$$

Example B

Solve the following system of linear equations by substitution:

$$\begin{Bmatrix} 8x-3y=6 \\ 6x+12y=-24 \end{Bmatrix}$$

Solution: There is no variable that has a coefficient of +1 or of –1. However, the second equation has coefficients and a constant that are multiples of 6. The second equation will be solved for the variable 'x'.

$$\begin{aligned}
&6x+12y=-24\\
&6x+12y-12y=-24-12y\\
&6x=-24-12y\\
&\frac{6x}{6}=\frac{-24}{6}-\frac{12y}{6}\\
&\frac{\cancel{6}x}{\cancel{6}}=\frac{\overset{-4}{\cancel{-24}}}{\cancel{6}}-\frac{\overset{2}{\cancel{12}}y}{\cancel{6}}\\
&\boxed{x=-4-2y}
\end{aligned}$$

Substitute $(-4-2y)$ into the first equation for 'x'.

$$\begin{aligned}
8x-3y&=6\\
8(-4-2y)-3y&=6
\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&-32-16y-3y=6\\
&-32-19y=6\\
&-32+32-19y=6+32\\
&-19y=38\\
&\frac{-19y}{-19}=\frac{38}{-19}\\
&\frac{\cancel{-19}y}{\cancel{-19}}=\frac{\overset{-2}{\cancel{38}}}{\cancel{-19}}\\
&\boxed{y=-2}
\end{aligned}$$

Substitute –2 for y into the equation

$$\boxed{x=-4-2y}$$

$$\begin{aligned}
&x=-4-2y\\
&x=-4-2(-2)
\end{aligned}$$

$$\begin{aligned}
&x=-4+4\\
&\boxed{x=0}\\
&\boxed{l_1 \cap l_2 @ (0,-2)}
\end{aligned}$$

Example C

Jason, who is a real computer whiz, decided to set up his own server and to sell space on his computer so students could have their own web pages on the Internet. He devised two plans. One plan charges a base fee of \$25.00 plus \$0.50 every month. His other plan has a base fee of \$5.00 plus \$1 per month.

i) Write an equation to represent each plan.

ii) Solve the system of equations.

Solution: Both plans deal with the cost of buying space from Jason's server. The cost involves a base fee and a monthly rate. The equations for the plans are:

- $y=0.50x+25$
- $y=1x+5$

where 'y' represents the **cost** and 'x' represents the **number of months.** Both equations are equal to 'y'. Therefore, the expression for y can be substituted for the y in the other equation.

$$\left\{\begin{array}{c} y=0.50x+25 \\ y=1x+5 \end{array}\right\}$$

$$\begin{aligned}
&0.50x+25=1x+5\\
&0.50x+25-25=1x+5-25\\
&0.50x=1x-20\\
&0.50x-1x=1x-1x-20\\
&-0.50x=-20\\
&\frac{-0.50x}{-0.50}=\frac{-20}{-0.50}\\
&\frac{\cancel{-0.50}x}{\cancel{-0.50}}=\frac{\overset{40}{\cancel{-20}}}{\cancel{-0.50}}\\
&\boxed{x=40\ months}
\end{aligned}$$

Since the equations were equal, the value for 'x' can be substituted into either of the original equations. The result will be the same.

$$\begin{aligned}
&y=1x+5\\
&y=1(40)+5
\end{aligned}$$

$$\begin{aligned}
&y=40+5\\
&\boxed{y=45\ dollars}\\
&\boxed{l_1\cap l_2@(40,45)}.
\end{aligned}$$

Concept Problem Revisited

When graphing is not a feasible method for solving a system, you can solve by substitution:

$$\left\{\begin{array}{c}2x+3y=13\\ y=4x-5\end{array}\right\}$$

The second equation is solved in terms of the variable 'y'. The expression $(4x+5)$ can be used to replace 'y' in the first equation.

$$\begin{aligned}
2x+3y&=13\\
2x+3(4x-5)&=13
\end{aligned}$$

The equation now has one variable. Apply the distributive property.

$2x+12x-15=13$

Combine like terms to simplify the equation.

$14x-15=13$

Solve the equation.

$$\begin{aligned}&14x-15+15=13+15\\&14x=28\\&\frac{14x}{14}=\frac{28}{14}\\&\frac{\cancel{14}x}{\cancel{14}}=\frac{\overset{2}{\cancel{28}}}{\cancel{14}}\\&\boxed{x=2}\end{aligned}$$

To determine the value of 'y', substitute this value into the equation $y=4x-5$.

$$\begin{aligned}&y=4x-5\\&y=4(2)-5\end{aligned}$$

$$\begin{aligned}&y=8-5\\&\boxed{y=3}\end{aligned}$$

The solution is (2, 3). This represents the intersection point of the lines if the equations were graphed on a Cartesian grid.

Vocabulary

Substitution Method

The ***substitution method*** is a way of solving a system of linear equations algebraically. The substitution method involves solving an equation for a variable and substituting that expression into the other equation.

Guided Practice

1. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix}x=2y+1\\x=4y-3\end{Bmatrix}$$

2. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix}2x+y=3\\3x+2y=12\end{Bmatrix}$$

.

3. Solve the following system of linear equations by substitution:

$$\begin{Bmatrix}\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}\\-\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4}\end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} x = 2y+1 \\ x = 4y-3 \end{Bmatrix}$$

Both equations are equal to the variable ’x’. Set $(2y+1) = (4y-3)$
$2y+1 = 4y-3$
Solve the equation.

$$\begin{aligned}
&2y+1-1 = 4y-3-1\\
&2y = 4y-4\\
&2y-4y = 4y-4y-4\\
&-2y = -4\\
&\frac{-2y}{-2} = \frac{-4}{-2}\\
&\frac{\cancel{-2}y}{\cancel{-2}} = \frac{\overset{2}{\cancel{-4}}}{\cancel{-2}}\\
&\boxed{y=2}
\end{aligned}$$

Substitute this value for ’y’ into one of the original equations.

$$\begin{aligned}
&x = 2y+1\\
&x = 2(2)+2
\end{aligned}$$

$$\begin{aligned}
&x = 4+1\\
&\boxed{x=5}\\
&\boxed{l_1 \cap l_2 @ (5,2)}
\end{aligned}$$

2.

$$\begin{Bmatrix} 2x+y = 3 \\ 3x+2y = 12 \end{Bmatrix}$$

The first equation has the variable ’y’ with a coefficient of 1. Solve the equation in terms of ’y’.

$$\begin{aligned}
&2x+y = 3\\
&2x-2x+y = 3-2x\\
&\boxed{y = 3-2x}
\end{aligned}$$

Substitute $(3-2x)$ into the second equation for ’y’.

$$\begin{aligned}
3x+2y &= 12\\
3x+2(3-2x) &= 12
\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&3x+6-4x=12\\
&6-x=12\\
&6-6-x=12-6\\
&-x=6\\
&\frac{-x}{-1}=\frac{6}{-1}\\
&\frac{\cancel{-1}x}{\cancel{-1}}=\frac{\overset{-6}{\cancel{6}}}{\cancel{-1}}\\
&\boxed{x=-6}
\end{aligned}$$

Substitute this value for 'x' into the equation $y=3-2x$.

$$\begin{aligned}
&y=3-2x\\
&y=3-2(-6)
\end{aligned}$$

$$\begin{aligned}
&y=3+12\\
&\boxed{y=15}\\
&\boxed{l_1 \cap l_2 @ (-6,15)}
\end{aligned}$$

3.

$$\left\{\begin{array}{c}\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}\\ -\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4}\end{array}\right\}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m+\frac{3}{4}n=\frac{5}{2}$ The LCM for the denominators is 20.

$$\begin{aligned}
20\left(\frac{2}{5}\right)m+20\left(\frac{3}{4}\right)n&=20\left(\frac{5}{2}\right)\\
\overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m+\overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n&=\overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right)\\
8m+15n&=50\\
8m+15n&=50
\end{aligned}$$

$-\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4}$ The LCM for the denominators is 12.

$$\begin{aligned}
-12\left(\frac{2}{3}\right)m+12\left(\frac{1}{2}\right)n&=12\left(\frac{3}{4}\right)\\
-\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m+\overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n&=\overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right)\\
-8m+6n&=9\\
-8m+6n&=9
\end{aligned}$$

The two equations that need to be solved by substitution are:

$$\left\{\begin{array}{l}8m+15n=50\\-8m+6n=9\end{array}\right\}$$

Neither of the equations have a variable with a coefficient of 1 nor does one equation have coefficients that are multiples of a given coefficient. Solve the first equation in terms of 'm'.

$$\begin{aligned}
&8m+15n=50\\
&8m+15n-15n=50-15n\\
&8m=50-15n\\
&\frac{8m}{8}=\frac{50}{8}-\frac{15n}{8}\\
&\frac{\cancel{8}m}{\cancel{8}}=\frac{50}{8}-\frac{15n}{8}\\
&m=\frac{25}{4}-\frac{15}{8}n\\
&\boxed{m=\frac{25}{4}-\frac{15}{8}n}
\end{aligned}$$

Substitute this value for 'm' into the second equation.

$$\begin{aligned}
-8m+6n&=9\\
-8\left(\frac{25}{4}-\frac{15}{8}n\right)+6n&=9
\end{aligned}$$

Apply the distributive property and solve the equation.

$$\begin{aligned}
&-\frac{200}{4}+\frac{120}{8}n+6n=9\\
&-\frac{\overset{50}{\cancel{200}}}{\cancel{4}}+\frac{\overset{15}{\cancel{120}}}{\cancel{8}}n+6n=9\\
&-50+15n+6n=9\\
&-50+21n=9\\
&-50+50+21n=9+50\\
&21n=59\\
&\frac{21n}{21}=\frac{59}{21}\\
&\frac{\cancel{21}n}{\cancel{21}}=\frac{59}{21}\\
&\boxed{n=\frac{59}{21}}
\end{aligned}$$

Substitute this value into the equation that has been solved in terms of 'm' or into one of the original equations or into one of the new equations that resulted from multiplying by the LCM.

Whichever substitution is performed, the same result will occur.

$$\begin{aligned}
m &= \frac{25}{4} - \frac{15}{8}n \\
m &= \frac{25}{4} - \frac{15}{8}\left(\frac{59}{21}\right) \\
m &= \frac{25}{4} - \frac{885}{168}
\end{aligned}$$

A common denominator is required to subtract the fractions.

$$\begin{array}{r}
42 \\
4\overline{)168} \\
\underline{-16\downarrow} \\
8 \\
\underline{-\ \ 8} \\
0
\end{array}$$

Multiply $\frac{25}{4}$ *by* $\frac{42}{42}$:

$$\begin{aligned}
m &= \frac{42}{42}\left(\frac{25}{4}\right) - \frac{885}{168} \\
m &= \frac{1050}{168} - \frac{885}{168} \\
m &= \frac{165}{168} \\
m &= \frac{55}{56}
\end{aligned}$$

$$\boxed{m = \frac{55}{56}}$$

$$\boxed{l_1 \cap l_2 @ \left(\frac{55}{56}, \frac{59}{21}\right)}$$

Practice

Solve the following systems of linear equations using the substitution method.

1.

$$\left\{\begin{array}{c} y = 3x \\ 5x - 2y = 1 \end{array}\right\}$$

2.

$$\left\{\begin{array}{c} y = 3x + 1 \\ 2x - y = 2 \end{array}\right\}$$

3.

$$\begin{Bmatrix} x = 2y \\ x = 3y - 3 \end{Bmatrix}$$

4.

$$\begin{Bmatrix} x - y = 6 \\ 6x - y = 40 \end{Bmatrix}$$

5.

$$\begin{Bmatrix} x + y = 6 \\ x + 3(y + 2) = 10 \end{Bmatrix}$$

6.

$$\begin{Bmatrix} 2x + y = 5 \\ 3x - 4y = 2 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 5x - 2y = -4 \\ 4x + y = -11 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} 3y - x = -10 \\ 3x + 4y = -22 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 4e + 2f = -2 \\ 2e - 3f = 1 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{1}{4}x + y = -\frac{7}{2} \\ \frac{1}{2}x - \frac{1}{4}y = 1 \end{Bmatrix}$$

11.

$$\left\{\begin{array}{c} x = -4 + y \\ x = 3y - 6 \end{array}\right\}$$

12.

$$\left\{\begin{array}{c} 3y - 2x = -3 \\ 3x - 3y = 6 \end{array}\right\}$$

13.

$$\left\{\begin{array}{c} 2x = 5y - 12 \\ 3x + 5y = 7 \end{array}\right\}$$

14.

$$\left\{\begin{array}{c} 3y = 2x - 5 \\ 2x = y + 3 \end{array}\right\}$$

15.

$$\left\{\begin{array}{c} \frac{x+y}{3} + \frac{x-y}{2} = \frac{25}{6} \\ \frac{x+y-9}{2} = \frac{y-x-6}{3} \end{array}\right\}$$

5.3 Elimination Method for Systems of Equations

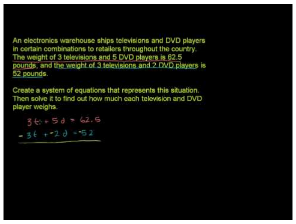
Here you'll learn how to solve a system of linear equations algebraically using the elimination method.

When you solve a system of consistent and independent equations by graphing, a single ordered pair is the solution. The ordered pair satisfies both equations and the point is the intersection of the graphs of the linear equations. The coordinates of this point of intersection are not always integers. Therefore, some method has to be used to determine the values of the coordinates. How can you algebraically solve the system of equations below without first rewriting the equations?

$$\left\{\begin{array}{c} 2x+3y=5 \\ 3x-3y=10 \end{array}\right\}$$

Watch This

Khan Academy Solving Systems by Elimination



An electronics warehouse ships televisions and DVD players in certain combinations to retailers throughout the country. The weight of 3 televisions and 5 DVD players is 62.5 pounds, and the weight of 3 televisions and 2 DVD players is 52 pounds.

Create a system of equations that represents this situation. Then solve it to find out how much each television and DVD player weighs.

$3t + 5d = 62.5$

$-3t + {}^-2d = {}^-52$

MEDIA

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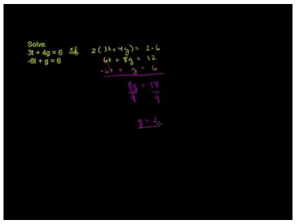
Khan Academy Solving Systems by Elimination 2



Solve.

3t + 4g = 6 $\xrightarrow{\times 2}$ $2(3t+4g) = 2 \cdot 6$

-6t + g = 6

$6t + 8g = 12$

$-6t + g = 6$

$\frac{9g}{9} = \frac{18}{9}$

$g = 2$

MEDIA

Click image to the left for more content.

Guidance

A 2×2 system of linear equations can be solved algebraically by the elimination method. To use this method you must write an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated. The solution is the intersection point of the two equations and it represents the coordinates of the ordered pair. This method is demonstrated in the examples.

Example A

Solve the following system of linear equations by elimination:

$$\left\{\begin{array}{c}3y=2x-5\\2x=y+3\end{array}\right\}$$

Solution: To begin, set up the equations so that they are in the format

$$\left\{\begin{array}{c}a_1x+b_1y=c_1\\a_2x+b_2y=c_2\end{array}\right\}$$

$$\begin{aligned}3y&=2x-5 & 2x&=y+3\\-2x+3y&=2x-2x-5 & 2x-y&=y-y+3\\-2x+3y&=-5 & 2x-y&=3\end{aligned}$$

Solve the formatted system of equations:

$$\left\{\begin{array}{c}-2x+3y=-5\\2x-y=3\end{array}\right\}$$

Both equations have a term that is $2x$. In the first equation the coefficient of x is a negative two and in the second equation the coefficient of x is a positive two. If the two equations are added, the x variable is eliminated.

$$\begin{aligned}-\cancel{2x}+3y&=-5\\\underline{\cancel{2x}-\ y}&\underline{=+3}\\2y&=-2 &&\text{Eliminate the variable } x.\\2y&=-2 &&\text{Solve the equation.}\\\frac{2y}{2}&=\frac{-2}{2}\\\frac{\cancel{2}y}{\cancel{2}}&=\frac{\overset{-1}{\cancel{-2}}}{\cancel{2}}\\\boxed{y=-1}\end{aligned}$$

The value of y is –1. This value can now be substituted into one of the original equations to determine the value of x. Remember x is the variable that was eliminated from the system of linear equations.

$$\begin{aligned}2x-y&=3\\2x-(-1)&=5 &&\text{Substitute in the value for } y.\\2x+1&=5 &&\text{Multiply the value of } x \text{ by the coefficient } (-1).\\2x+1-1&=3-1 &&\text{Isolate the variable } x.\\2x&=2 &&\text{Solve the equation.}\\\frac{2x}{2}&=\frac{2}{2}\\\frac{\cancel{2}x}{\cancel{2}}&=\frac{\overset{1}{\cancel{2}}}{\cancel{2}}\\\boxed{x=1}\\\boxed{l_1\cap l_2@(1,-1)}\end{aligned}$$

This means "Line 1 intersects Line 2 at the point (1, –1)".

Example B

Solve the following system of linear equations by elimination:

$$\left\{\begin{array}{c} 2x-3y=13 \\ 3x+4y=-6 \end{array}\right\}$$

Solution: The coefficients of 'x' are 2 and 3. The coefficients of 'y' are –3 and 4. To eliminate a variable the coefficients must be the same number but with opposite signs. This can be accomplished by multiplying one or both of the equations.

The first step is to choose a variable to eliminate. If the choice is 'x', the least common multiple of 2 and 3 is 6. This means that the equations must be multiplied by 3 and 2 respectively. One of the multipliers must be a negative number so that one of the coefficients of 'x' will be a negative 6. When this is done, the coefficients of 'x' will be +6 and –6. The variable will then be eliminated when the equations are added.

Multiply the first equation by negative three.

$$\begin{array}{c} -3(2x-3y=13) \\ -6x+9y=-39 \end{array}$$

Multiply the second equation by positive two.

$$\begin{array}{c} 2(3x+4y=-6) \\ 6x+8y=-12 \end{array}$$

Add the two equations.

$$\begin{array}{ll} -\cancel{6x}+9y=-39 & \\ \underline{\cancel{6x}+8y=-12} & \\ 17y=-51 & \text{Solve the equation.} \\ 17y=-51 & \\ \dfrac{17y}{17}=\dfrac{-51}{17} & \\ \dfrac{\cancel{17}y}{\cancel{17}}=\dfrac{\overset{-3}{\cancel{-51}}}{\cancel{17}} & \\ \boxed{y=-3} & \end{array}$$

Substitute the value for y into one of the original equations.

$$\begin{aligned}
&2x-3y=13 \\
&2x-3(-3)=13 && \text{Substitute in the value for } y. \\
&2x+9=13 && \text{Multiply the value of } y \text{ by the coefficient } (-3). \\
&2x+9-9=13-9 && \text{Isolate the variable } x. \\
&2x=4 && \text{Solve the equation.} \\
&\frac{2x}{2}=\frac{4}{2} \\
&\frac{\cancel{2}x}{\cancel{2}}=\frac{\overset{2}{\cancel{4}}}{\cancel{2}} \\
&\boxed{x=2} \\
&\boxed{l_1 \cap l_2 @ (2,-3)}
\end{aligned}$$

Example C

Solve the following system of linear equations by elimination:

$$\left\{\begin{array}{l} \frac{3}{4}x+\frac{5}{4}y=4 \\ \frac{1}{2}x+\frac{1}{3}y=\frac{5}{3} \end{array}\right\}$$

Solution: Begin by multiplying each equation by the LCD to create two equations with integers as the coefficients of the variables.

$$\begin{aligned}
\frac{3}{4}x+\frac{5}{4}y&=4 \\
4\left(\frac{3}{4}\right)x+4\left(\frac{5}{4}\right)y&=4(4) \\
\cancel{4}\left(\frac{3}{\cancel{4}}\right)x+\cancel{4}\left(\frac{5}{\cancel{4}}\right)y&=4(4) \\
3x+5y&=16
\end{aligned}
\qquad\qquad
\begin{aligned}
&\frac{1}{2}x+\frac{1}{3}y=\frac{5}{3} \\
&6\left(\frac{1}{2}\right)x+6\left(\frac{1}{3}\right)y=6\left(\frac{5}{3}\right) \\
&\overset{3}{\cancel{6}}\left(\frac{1}{\cancel{2}}\right)x+\overset{2}{\cancel{6}}\left(\frac{1}{\cancel{3}}\right)y=\overset{2}{\cancel{6}}\left(\frac{5}{\cancel{3}}\right) \\
&3x+2y=10
\end{aligned}$$

Now solve the following system of equations by elimination:

$$\left\{\begin{array}{l} 3x+5y=16 \\ 3x+2y=10 \end{array}\right\}$$

The coefficients of the ’x’ variable are the same – positive three. To change one of them to a negative three, multiply one of the equations by a negative one.

$$\begin{aligned}
-1(3x+5y&=16) \\
-3x-5y&=-16
\end{aligned}$$

The two equations can now be added.

$$\begin{array}{ll}
-\cancel{3x} - 5y = -16 & \\
\underline{\cancel{3x} + 2y = \quad 10} & \\
\qquad -3y = \quad -6 & \text{Solve the equation.} \\
-3y = -6 & \\
\frac{-3y}{-3} = \frac{-6}{-3} & \\
\frac{\cancel{-3}y}{\cancel{-3}} = \frac{\overset{2}{\cancel{-6}}}{\cancel{-3}} & \\
\boxed{y = 2} &
\end{array}$$

Substitute the value for ’*y*’ into one of the original equations.

$$\begin{array}{ll}
\frac{3}{4}x + \frac{5}{4}y = 4 & \\
\frac{3}{4}x + \frac{5}{4}(2) = 4 & \text{Substitute in the value for } y. \\
\frac{3}{4}x + \frac{10}{4} = 4 & \text{Multiply the value of } y \text{ by the coefficient } \left(\frac{5}{4}\right). \\
\frac{3}{4}x + \frac{10}{4} - \frac{10}{4} = 4 - \frac{10}{4} & \text{Isolate the variable } x. \\
\frac{3}{4}x = \frac{16}{4} - \frac{10}{4} & \\
\frac{3}{4}x = \frac{6}{4} & \text{Multiply both sides by 4.} \\
4\left(\frac{3}{4}x\right) = 4\left(\frac{6}{4}\right) & \\
\cancel{4}\left(\frac{3}{\cancel{4}}x\right) = \cancel{4}\left(\frac{6}{\cancel{4}}\right) & \\
3x = 6 & \text{Solve the equation.} \\
\frac{3x}{3} = \frac{6}{3} & \\
\frac{\cancel{3}x}{\cancel{3}} = \frac{\overset{2}{\cancel{6}}}{\cancel{3}} & \\
\boxed{x = 2} & \\
\boxed{l_1 \cap l_2 @ (2,2)} &
\end{array}$$

Concept Problem Revisited

Solve by elimination:

$$\left\{\begin{array}{r} 2x + 3y = 5 \\ 3x - 3y = 10 \end{array}\right\}$$

Both equations have a term that is $3y$. In the first equation the coefficient of 'y' is a positive three and in the second equation the coefficient of 'y' is a negative three. If the two equations are added, the 'y' variable is eliminated.

$$\begin{aligned}&2x+3y=5\\&\underline{3x-3y=10}\\&5x=15\end{aligned}$$

The resulting equation now has one variable. Solve this equation:

$$\begin{aligned}&5x=15\\&\frac{5x}{5}=\frac{15}{5}\\&\frac{\cancel{5}x}{\cancel{5}}=\frac{\overset{3}{\cancel{15}}}{\cancel{5}}\\&\boxed{x=3}\end{aligned}$$

The value of 'x' is 3. This value can now be substituted into one of the original equations to determine the value of 'y'.

$$\begin{aligned}&2x+3y=5\\&2(3)+3y=5 && \text{Substitute in the value for } x.\\&6+3y=5 && \text{Multiply the value of } x \text{ by the coefficient } (2).\\&6-6+3y=5-6 && \text{Isolate the variable } y.\\&3y=-1 && \text{Solve the equation.}\\&\frac{3y}{3}=\frac{-1}{3}\\&\frac{\cancel{3}y}{\cancel{3}}=-\frac{1}{3}\\&\boxed{y=-\frac{1}{3}}\end{aligned}$$

The solution to the system of linear equations is $x=3$ and $y=-\frac{1}{3}$. This solution means

$$\boxed{l_1 \cap l_2 @ \left(3,-\frac{1}{3}\right)}$$

Vocabulary

Elimination Method

The ***elimination method*** is a method used for solving a system of linear equations algebraically. This method involves obtaining an equivalent system of equations such that, when two of the equations are added or subtracted, one of the variables is eliminated.

Guided Practice

1. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 4x-15y=5 \\ 6x-5y=4 \end{Bmatrix}$$

2. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} 3x=7y+41 \\ 5x=3y+51 \end{Bmatrix}.$$

3. Solve the following system of linear equations by elimination:

$$\begin{Bmatrix} \frac{2}{5}m+\frac{3}{4}n=\frac{5}{2} \\ -\frac{2}{3}m+\frac{1}{2}n=\frac{3}{4} \end{Bmatrix}$$

Answers:

1.

$$\begin{Bmatrix} 4x-15y=5 \\ 6x-5y=4 \end{Bmatrix}$$

Multiply the second equation by (–3) to eliminate the variable ’y’.

$$\begin{aligned} &-3(6x-5y=4) \\ &-18x+15y=-12 \\ &-18x+15y=-12 \end{aligned}$$

Add the equations:

$$\begin{array}{r} 4x-\cancel{15y}=5 \\ -18x+\cancel{15y}=-12 \\ \hline -14x=-7 \end{array}$$

Solve the equation:

$$\begin{aligned} -14x&=-7 \\ \frac{-14x}{-14}&=\frac{-7}{-14} \\ \frac{\cancel{-14}x}{\cancel{-14}}&=\frac{-7}{-14} \\ \boxed{x=\frac{1}{2}} \end{aligned}$$

Substitute this value for 'x' into one of the original equations.

$4x-15y=5$

$4\left(\frac{1}{2}\right)-15y=5$ Substitute in the value for x.

$2-15y=5$ Multiply the value of x by the coefficient (4).

$2-2-15y=5-2$ Isolate the variable y.

$-15y=3$ Solve the equation.

$$\frac{-15y}{-15}=\frac{3}{-15}$$

$$\frac{\cancel{-15}y}{\cancel{-15}}=\frac{3}{-15}$$

$$\boxed{y=-\frac{1}{5}}$$

$$\boxed{l_1\cap l_2@\left(\frac{1}{2},-\frac{1}{5}\right)}$$

2.

$$\begin{Bmatrix}3x=7y+41\\5x=3y+51\end{Bmatrix}$$

Arrange the equations so that they are of the form

$$\begin{Bmatrix}a_1x+b_1y=c_1\\a_2x+b_2y=c_2\end{Bmatrix}$$

.

$$\begin{aligned}3x&=7y+41 & 5x&=3y+51\\3x-7y&=7y-7y+41 & 5x-3y&=3y-3y+51\\3x-7y&=41 & 5x-3y&=51\end{aligned}$$

Multiply the first equation by (–5) and the second equation by (3).

$$\begin{aligned}&-5(3x-7y=41) && 3(5x-3y=51)\\&-15x+35y=-205 && 15x-9y=153\\&-15x+35y=-205 && 15x-9y=153\end{aligned}$$

Add the equations to eliminate 'x'.

$$\begin{array}{r}-\cancel{15x}+35y=-205\\\cancel{15x}-9y=153\\\hline 26y=-52\end{array}$$

Solve the equation:

$$26y = 52$$

$$\frac{26y}{26} = \frac{-52}{26}$$

$$\frac{\cancel{26}y}{\cancel{26}} = \frac{\overset{-2}{\cancel{-52}}}{\cancel{26}}$$

$$\boxed{y = -2}$$

$$5x - 3y = 51$$

$5x - 3(-2) = 51$ Substitute in the value for y.

$5x + 6 = 51$ Multiply the value of y by the coefficient (-3).

$5x + 6 - 6 = 51 - 6$ Isolate the variable x.

$5x = 45$ Solve the equation.

$$\frac{5x}{5} = \frac{45}{5}$$

$$\frac{\cancel{5}x}{\cancel{5}} = \frac{\overset{9}{\cancel{45}}}{\cancel{5}}$$

$$\boxed{x = 9}$$

$$\boxed{l_1 \cap l_2 @ (9, -2)}$$

3.

$$\left\{\begin{array}{c} \frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\ -\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4} \end{array}\right\}$$

Begin by multiplying each equation by the LCM of the denominators to simplify the system.
$\frac{2}{5}m + \frac{3}{4}n = \frac{5}{2}$ The LCM for the denominators is 20.

$$20\left(\frac{2}{5}\right)m + 20\left(\frac{3}{4}\right)n = 20\left(\frac{5}{2}\right)$$

$$\overset{4}{\cancel{20}}\left(\frac{2}{\cancel{5}}\right)m + \overset{5}{\cancel{20}}\left(\frac{3}{\cancel{4}}\right)n = \overset{10}{\cancel{20}}\left(\frac{5}{\cancel{2}}\right)$$

$$8m + 15n = 50$$

$$\boxed{8m + 15n = 50}$$

$-\frac{2}{3}m + \frac{1}{2}n = \frac{3}{4}$ The LCM for the denominators is 12.

$$-12\left(\frac{2}{3}\right)m + 12\left(\frac{1}{2}\right)n = 12\left(\frac{3}{4}\right)$$

$$-\overset{4}{\cancel{12}}\left(\frac{2}{\cancel{3}}\right)m + \overset{6}{\cancel{12}}\left(\frac{1}{\cancel{2}}\right)n = \overset{3}{\cancel{12}}\left(\frac{3}{\cancel{4}}\right)$$

$$-8m + 6n = 9$$

$$\boxed{-8m + 6n = 9}$$

The two equations that need to be solved are:

$$\left\{\begin{array}{c} 8m + 15n = 50 \\ -8m + 6n = 9 \end{array}\right\}$$

The equations will be solved by using the elimination method. The variable 'm' has the same numerical coefficient with opposite signs. The variable will be eliminated when the equations are added.

$$\begin{aligned} \cancel{8m} + 15n &= 50 \\ \underline{-\cancel{8m} + 6n} &\underline{= 9} \\ 21n &= 59 \end{aligned}$$

Solve the equation:

$$\begin{aligned}
&21n = 59 \\
&\frac{21n}{21} = \frac{59}{21} \\
&\frac{\cancel{21}n}{\cancel{21}} = \frac{59}{21} \\
&\boxed{n = \frac{59}{21}} \\
&\frac{2}{5}m + \frac{3}{4}n = \frac{5}{2} \\
&\frac{2}{5}m + \frac{3}{4}\left(\frac{59}{21}\right) = \frac{5}{2} && \text{Substitute in the value for } n. \\
&\frac{2}{5}m + \frac{177}{84} = \frac{5}{2} && \text{Multiply the value of } y \text{ by the coefficient } \left(\frac{59}{21}\right). \\
&\frac{2}{5}m + \frac{177}{84} - \frac{177}{84} = \frac{5}{2} - \frac{177}{84} && \text{Isolate the variable } x. \\
&\frac{2}{5}m = \frac{210}{84} - \frac{177}{84} \\
&\frac{2}{5}m = \frac{33}{84} && \text{Solve the equation.} \\
&420\left(\frac{2}{5}\right)x = 420\left(\frac{33}{84}\right) \\
&\overset{84}{\cancel{420}}\left(\frac{2}{\cancel{5}}\right)x = \overset{5}{\cancel{420}}\left(\frac{33}{\cancel{84}}\right) \\
&168x = 165 \\
&\frac{168x}{168} = \frac{165}{168} \\
&\frac{\cancel{168}x}{\cancel{168}} = \frac{165}{168} \\
&\boxed{x = \frac{55}{56}} \\
&\boxed{l_1 \cap l_2 @ \left(\frac{55}{56}, \frac{59}{21}\right)}
\end{aligned}$$

Practice

Solve the following systems of linear equations using the elimination method.

1.

$$\begin{Bmatrix} 16x-y-181=0 \\ 19x-y=214 \end{Bmatrix}$$

2.

$$\begin{Bmatrix} 3x+2y+9=0 \\ 4x=3y+5 \end{Bmatrix}$$

3.

$$\begin{Bmatrix} x=7y+38 \\ 14y=-x-46 \end{Bmatrix}$$

4.

$$\begin{Bmatrix} 2x+9y=-1 \\ 4x+y=15 \end{Bmatrix}$$

5.

$$\begin{Bmatrix} x-\frac{3}{5}y=\frac{26}{5} \\ 4y=61-7x \end{Bmatrix}$$

6.

$$\begin{Bmatrix} 3x-5y=12 \\ 2x+10y=4 \end{Bmatrix}$$

7.

$$\begin{Bmatrix} 3x+2y+9=0 \\ 4x=3y+5 \end{Bmatrix}$$

8.

$$\begin{Bmatrix} x=69+6y \\ 3x=4y-45 \end{Bmatrix}$$

9.

$$\begin{Bmatrix} 3(x-1)-4(y+2)=-5 \\ 4(x+5)-(y-1)=16 \end{Bmatrix}$$

10.

$$\begin{Bmatrix} \frac{3}{4}x-\frac{2}{5}y=2 \\ \frac{1}{7}x+\frac{3}{2}y=\frac{113}{7} \end{Bmatrix}$$

11.

$$\begin{Bmatrix} 3x+5y=17 \\ 2x+3y=11 \end{Bmatrix}$$

12.

$$\begin{Bmatrix} 3x-5y=-29 \\ 2x-8y=-42 \end{Bmatrix}$$

13.

$$\begin{Bmatrix} 7x-8y=-26 \\ 5x-12y=-45 \end{Bmatrix}$$

14.

$$\begin{Bmatrix} 6x+5y=5.1 \\ 4x-2y=-1.8 \end{Bmatrix}$$

15. When does it make sense to use the elimination method to solve a system of equations?

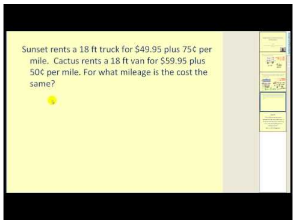
5.4 Applications of Systems of Equations

Here you'll learn how to use systems of linear equations to solve word problems and you will see some common applications of linear systems of equations.

Can you determine two numbers such that the sum of the numbers is 763 and the difference between the same two numbers is 179? How can a system of equations help you?

Watch This

James Sousa: Applications Involving Systems of Equations



Sunset rents a 18 ft truck for $49.95 plus 75¢ per mile. Cactus rents a 18 ft van for $59.95 plus 50¢ per mile. For what mileage is the cost the same?

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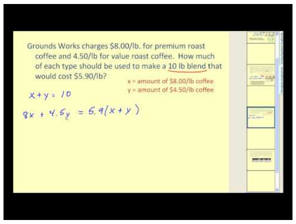
James Sousa: More Applications Involving Systems of Equations



Grounds Works charges $8.00/lb. for premium roast coffee and 4.50/lb for value roast coffee. How much of each type should be used to make a 10 lb blend that would cost $5.90/lb?

x = amount of $8.00/lb coffee

y = amount of $4.50/lb coffee

$x+y=10$

$8x+4.5y=5.9(x+y)$

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Guidance

A system of linear equations can be used to represent real-world problems. A system of equations is used when there are two variables and you are given two pieces of information about how those variables are related. The examples below will show you some common problems that can be solved using a system of equations.

Example A

The length of a rectangular plot of land is 255 yards longer than the width. If the perimeter is 1206 yards, find the dimensions of the rectangle.

Solution: The perimeter of a rectangle is found by the formula $P = 2l + 2w$ where P is the perimeter, l is the length and w is the width. The quantities are the length and the width of the rectangle.

- Let the length of the rectangle be represented by 'l'.
- Let the width of the rectangle be represented by 'w'.

The equations would be:

- *The length is 255 yards longer than the width*

$$\rightarrow \boxed{l = w + 255}$$

- *The perimeter is 1206 yards*

$$\rightarrow \boxed{2l + 2w = 1206}$$

The system of equations can now be solved to determine the dimensions of the rectangle. In the first equation the length is expressed in terms of the width. Substitution will be used to solve the system of equations.

$$\begin{aligned}
&\left\{\begin{array}{c} l = w + 255 \\ 2l + 2w = 1206 \end{array}\right\} && \\
&2l + 2w = 1206 && \text{Substitute } (w+255) \text{ for } l \text{ in the equation.} \\
&2(w+255) + 2w = 1206 && \text{Apply the distributive property.} \\
&2w + 510 + 2w = 1206 && \text{Simplify} \\
&4w + 510 = 1206 && \text{Solve the equation.} \\
&4w + 510 - 510 = 1206 - 510 && \\
&4w = 696 && \\
&\frac{4w}{4} = \frac{696}{4} && \\
&\frac{\cancel{4}w}{\cancel{4}} = \frac{\overset{174}{\cancel{696}}}{\cancel{4}} && \\
&\boxed{w = 174} &&
\end{aligned}$$

$$\begin{aligned}
&l = w + 255 && \text{Substitute the value for } w \text{ into the equation.} \\
&l = 174 + 255 && \text{Solve the equation.} \\
&\boxed{l = 429} &&
\end{aligned}$$

The length of the rectangular plot of land is 429 yards and the width is 174 yards.

Example B

Maria had \$12100 to invest. She decided to invest her money in bonds and mutual funds. She invested a portion of the money in bonds paying 8% interest per year and the remainder in a mutual fund paying 9% per year. After one year the total income she had earned from the investments was \$1043. How much had she invested at each rate?

Solution: The two quantities in this problem are the amount she had invested in bonds and the amount she had invested in mutual funds.

- Let the amount invested in bonds be represented by 'b'.
- Let the amount invested in mutual funds be represented by 'm'.

The equations would be:

- *The total amount of money she had to invest was*

$$\$12,100 \rightarrow \boxed{b+m=12,100}$$

- *The amount of money she earned from the investments was*

$$\$1043 \rightarrow \boxed{0.08b+0.09m=1043}$$

The system of equations will be solved using elimination.

$$\left\{\begin{array}{r} b+m=12,100 \\ 0.08b+0.09m=1043 \end{array}\right\}$$

The first equation will be multiplied by (–0.08)

$$\begin{aligned} &-0.08(b+m=12,100) \\ &-0.08b-0.08m=-968 \\ &-0.08b-0.08m=-968 \end{aligned}$$

The equations now have opposite, numerical coefficients for the variable *b*. Add the equations to eliminate the variable *b*.

$$\begin{aligned} &-\cancel{0.08b}-0.08m=-968 \\ &\underline{\cancel{0.08b}+0.09m=1043} && \text{Solve the equation.} \\ &0.01m=75 \\ &0.01m=75 \\ &\frac{0.01m}{0.01}=\frac{75}{0.01} \\ &\frac{\cancel{0.01}m}{\cancel{0.01}}=\frac{\overset{7500}{\cancel{75}}}{\cancel{0.01}} \\ &\boxed{m=7500} \end{aligned}$$

$$\begin{aligned} &b+m=12,100 && \text{Substitute the value for } m \text{ into the equation.} \\ &b+7500=12,100 && \text{Solve the equation.} \\ &b+7500-7500=12,100-7500 \\ &\boxed{b=4600} \end{aligned}$$

Maria invested $4600 in bonds and $7500 in mutual funds.

Example C

Pedro was saving quarters and dimes to buy a new skateboard. After months of saving his coins in a bottle, he emptied its contents and counted the money. The 561 coins in the bottle had a total value of $107.85. How many of each coin were in the bottle?

Solution: The two quantities in this problem are the number of quarters and the number of dimes.

- Let the number of quarters be represented by 'q'.
- Let the number of dimes be represented by 'd'.

The equations would be:

- *The total number of coins in the bottle was*

$$561 \rightarrow \boxed{q+d=561}$$

- *The amount of money in the bottle was*

$$\$107.85 \rightarrow \boxed{0.25q+0.10d=107.85}$$

This system will be solved using the substitution method. Solve the first equation in terms of quarters.

$$\begin{aligned}
&q+d=561\\
&q+d-d=561-d\\
&\boxed{q=561-d}
\end{aligned}$$

$$\begin{aligned}
&0.25q+0.10d=107.85 && \text{Substitute the value for } q \text{ into the equation.}\\
&0.25(561-d)+0.10d=107.85 && \text{Apply the distributive property.}\\
&140.25-0.25d+0.10d=107.85 && \text{Solve the equation.}\\
&140.25-0.15d=107.85\\
&140.25-140.25-0.15d=107.85-140.25\\
&-0.15d=-32.40\\
&\frac{-0.15d}{-0.15}=\frac{-32.40}{-0.15}\\
&\frac{\cancel{-0.15}d}{\cancel{-0.15}}=\frac{\overset{216}{\cancel{-32.40}}}{\cancel{-0.15}}\\
&\boxed{d=216}
\end{aligned}$$

$$\begin{aligned}
&q+d=561 && \text{Substitute the value for } d \text{ into the equation.}\\
&q+216=561 && \text{Solve the equation.}\\
&q+216-216=561-216\\
&q=345\\
&\boxed{q=345}
\end{aligned}$$

The number of quarters that Pedro had saved in the bottle was 345 and the number of dimes was 216.

Example D

The Bayplex and Centre 200 rent their ice out to the community whenever possible. The Bayplex charges a flat rate of \$20.00 plus \$15.00 for every hour rented. Centre 200 charges \$50.00 for a flat rate but only asks for \$10.00 for every hour rented.

a) Write an equation to model the cost of renting the ice surface for each arena.

b) Determine the intersection point of the equations. What does this intersection point represent?

c) Explain when it is best to use the Bayplex and when it is best to use Center 200.

Solution:

a) Begin by writing the equations to model the cost of renting the ice surface in each arena.

- *The cost of renting the ice at the Bayplex*

$$\rightarrow \boxed{c = 15h + 20}$$

- *The cost of renting the ice at Centre*

$$200 \rightarrow \boxed{c = 10h + 50}$$

b) The intersection point of the costs for the arenas can be determined by using the comparison method. Both equations are equal to the variable 'c'.

$$\begin{aligned} 15h + 20 &= 10h + 50 && \text{Solve the equation.} \\ 15h + 20 - 20 &= 10h + 50 - 20 \\ 15h &= 10h + 30 \\ 15h - 10h &= 10h - 10h + 30 \\ 5h &= 30 \\ \frac{5h}{5} &= \frac{30}{5} \\ \frac{\cancel{5}h}{\cancel{5}} &= \frac{\overset{6}{\cancel{30}}}{\cancel{5}} \\ h &= 6 \end{aligned}$$

Substitute the value for 'h' into one of the original equations.

$$\begin{aligned} c &= 15h + 20 \\ c &= 15(6) + 20 && \text{Evaluate the equation.} \\ c &= 90 + 20 \\ &\boxed{c = \$110} \end{aligned}$$

The intersection point of the system of linear equations is (6,110). This point is the time when the cost of renting the arena will be the same. Renting the Bayplex for six hours will cost \$110 and renting Centre 200 for six hours will cost \$110.

c) The flat rate to rent the Bayplex is only \$20.00. Therefore, it will cost less to rent the Bayplex for less than six hours but more to rent it after six hours. If you need to rent the ice surface for more than six hours, rent the Centre 200.

Concept Problem Revisited

Determine two numbers such that the sum of the numbers is 763 and the difference of the same two numbers is 179.

- Let ’x’ represent the larger number.
- Let ’y’ represent the smaller number.

Write an equation to model – *the sum of two numbers is 763.*

$$\rightarrow \boxed{x+y=763}$$

Write an equation to model – *the difference of two numbers is 179.*

$$\rightarrow \boxed{x-y=179}$$

The system of equations

$$\left\{\begin{matrix} x+y=763 \\ x-y=179 \end{matrix}\right\}$$

can be solved by a method of your choice. Choose one of the algebraic methods that you have learned to solve the system. Elimination would be a good choice since the variable ’y’ has opposite numerical coefficients.

$$\left\{\begin{matrix} x+y=763 \\ x-y=179 \end{matrix}\right\}$$

$$\begin{aligned} x+\cancel{y}&=763 \\ \underline{x-\cancel{y}}&\underline{=179} \\ 2x&=942 \end{aligned}$$

Add the equations to eliminate y.

$$\begin{aligned} 2x&=942 \\ \frac{2x}{2}&=\frac{942}{2} \\ \frac{\cancel{2}x}{\cancel{2}}&=\frac{\overset{471}{\cancel{942}}}{\cancel{2}} \\ \boxed{x=471} \end{aligned}$$

Solve the equation for the variable x.

$$\begin{aligned} x+y&=763 \\ (471)+y&=763 \\ 471-471+y&=763-471 \\ \boxed{y=292} \end{aligned}$$

Substitute the value for x into one of the original equations.

Solve the equation.

The larger number is 471 (x) and the smaller number is 292 (y).

Vocabulary

System of Linear Equations

A ***system of linear equations*** is two linear equations each having two variables. This type of system – two equations with two unknowns-is called a 2×2 system of linear equations.

Guided Practice

1. The sum of Henry's and his mother's age is 67. Three times Henry's age increased by 7 is his mother's age. How old is Henry?

2. The Sydney Schooners played a total of 41 games of hockey. The number of games lost was ten less than one-half the number of games won. How many games did the Schooners win?

3. Tim invested \$3000. A portion of the money was invested into a college fund that paid 8% interest per year and the remainder was invested into a retirement fund that paid 7 % interest per year. At the end of the first year, the interest from the college fund was \$60 more than the interest from the retirement fund. How much money did Tim invest in each fund?

Answers:

1. The two quantities in this problem are Henry's age and mother's age. Let Henry's age be represented by 'h'. Let mother's age be represented by 'm'. The equations would be:

The sum of Henry's age and his mother's age is 67.

$$\rightarrow \boxed{h+m=67}$$

Three times Henry's age increased by 7 is his mother's age

$$\rightarrow 3h+7=m$$

The system of equations will be solved by substitution.

$$\begin{array}{ll}
\left\{\begin{array}{l} h+m=67 \\ 3h+7=m \end{array}\right\} & \\
h+m=67 & \text{Substitute } 3h+7 \text{ into the equation for } m. \\
h+(3h+7)=67 & \text{Apply the distributive property.} \\
h+3h+7=67 & \text{Solve the equation.} \\
4h+7=67 & \\
4h+7-7=67-7 & \\
4h=60 & \\
\frac{4h}{4}=\frac{60}{4} & \\
\frac{\cancel{4}h}{\cancel{h}}=\frac{\overset{15}{\cancel{60}}}{\cancel{4}} & \\
\boxed{h=15} & \text{Henry is 15 years old. The problem asked for his mother's age.} \\
 & \\
3h+7=m & \text{Substitute 15 into the equation for } h. \\
3(15)+7=m & \text{Solve the equation.} \\
45+7=m & \\
\boxed{52=m} & \text{Henry's mother is 52 years of age.}
\end{array}$$

2. The two quantities in this problem are the games won and the games lost. Let games won be represented by 'w'. Let games lost be represented by 'l'. The equations would be:

The total number of games played (losses and wins) is 41.

$$\rightarrow \boxed{l+w=41}$$

The number of games lost was 10 less than one-half the number of games won.

$$\rightarrow \boxed{l = \frac{w}{2} - 10}$$

The system of equations will be solved by elimination.

$$l = \frac{w}{2} - 10$$

Multiply the equation by 2.

$$2(l) = 2\left[\frac{w}{2}\right] - 2(10)$$

$$2(l) = \cancel{2}\left[\frac{w}{\cancel{2}}\right] - 2(10)$$

$$2l = w - 20$$

Align the equations such that $\left\{\begin{matrix} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{matrix}\right\}$

$$2l - w = w - w - 20$$

$$\boxed{2l - w = -20}$$

The system of linear equations to solve by elimination is:

$$\left\{\begin{matrix} l + w = 41 \\ 2l - w = -20 \end{matrix}\right\}$$

$$l + \cancel{w} = 41$$

$$\underline{2l - \cancel{w} = -20}$$

Add the equations to eliminate w.

$$3l = 21$$

$$3l = 21$$

Solve the equation.

$$\frac{3l}{3} = \frac{21}{3}$$

$$\frac{\cancel{3}l}{\cancel{3}} = \frac{\overset{7}{\cancel{21}}}{\cancel{3}}$$

$$\boxed{l = 7}$$

The number of games the Schooners lost was seven.

$$l + w = 41$$

Substitute 7 into the equation for the variable l.

$$7 + w = 41$$

Solve the equation.

$$7 - 7 + w = 41 - 7$$

$$w = 34$$

$$\boxed{w = 34}$$

The number of games the Schooners won was 34.

3. The two quantities in this problem are the amount of money invested in the college fund and the amount of money invested in the retirement fund. Let the amount of money invested in the college fund be represented by 'c'. Let the amount of money invested in the retirement fund be represented by 'r'. The equations would be:

The total amount of money to be invested is $3000

$$\rightarrow \boxed{c + r = 3000}$$

The amount of interest from the 8% college fund is $60 more than the amount of interest from the 7% retirement fund.

$$\rightarrow \boxed{.08c = .07r + 60}$$

The system of equations will be solved by substitution.

$c + r = 3000$ Solve the equation for c.

$c + r - r = 3000 - r$

$\boxed{c = 3000 - r}$

$0.08c = 0.07r + 60$ Substitute $3000 - r$ into the equation for the variable c.

$0.08(3000 - r) = 0.07r + 60$ Apply the distributive property. Solve the equation.

$240 - 0.08r = 0.07r + 60$

$240 - 240 - 0.08r = 0.07r + 60 - 240$

$-0.08r - 0.07r = 0.07r - 0.07r - 180$

$-0.15r = -180$

$$\frac{-0.15r}{-0.15} = \frac{-180}{-0.15}$$

$$\frac{\cancel{-0.15}r}{\cancel{-0.15}} = \frac{\overset{1200}{\cancel{-180}}}{\cancel{-0.15}}$$

$\boxed{r = 1200}$ The amount of money invested in the retirement fund was $1200.

$c = 3000 - r$ Substitute 1200 into the equation for r.

$c = 3000 - 1200$ Solve the equation.

$c = 1800$

$\boxed{c = 1800}$ The amount of money invested in the college fund was $1800.

Practice

Write each of the following statements as a linear equation in two variables.

1. The sum of two numbers is 100.
2. When six times the larger of two numbers is added to 4 times the smaller, the result is 112.
3. The length of a rectangle is 8m more than 7 times its width.
4. Four times the number of nickels less three times the number of pennies is 56.
5. Jason has some $5 bills and some $1 bills which have a total of $91.

Solve each of the following problems using 2×2 systems of linear equations.

6. At a party, there were 72 people. The hostess counted the shoes and found that there were 32 more pairs of ladies' shoes compared to the number of pairs of males' shoes. How many males and females were at the party?
7. Two weight loss programs offer competitive services. Super Slim charges $33 to join and $1.50 per session whereas Think Thin charges $2.50 per session and $15 to join. Determine algebraically, under what circumstances you would choose each plan.
8. Today Sam is twice Jenny's age. Three years ago the sum of their ages was 45. How old are Sam and Jenny today?

9. The parking lot at a local amusement park contained 123 vehicles (cars and buses). Each car is charged \$3 to park for the day and each bus is charged \$10. If the total revenue for the day was \$481.00, how many cars were on the parking lot?
10. Seven times the larger of two numbers less three times the smaller is 351. Six times the larger less twice the smaller is 342. What are the numbers?
11. The debate team washed cars to raise money for a trip. They charged \$8 for a large car and \$5 for a small car. All together they raised \$550 and washed 80 cars. How many of each type of car did they wash?
12. Pencils cost \$0.10 each and notebooks cost \$2 each. You buy 15 items and spend \$9.10. How many pencils did you buy? How many notebooks did you buy?
13. The sum of two numbers is 15 and the product of the same two numbers is 36. What are the numbers?
14. Cereal costs \$3.50 a box and milk costs \$2.79 a gallon. Suppose you buy five items and spend \$16.08. How many boxes of cereal did you buy and how many gallons of milk did you buy?
15. Twice the sum of two number is 72 while the difference between the two numbers is 22. What are the numbers?

5.5 Graphs of Linear Inequalities

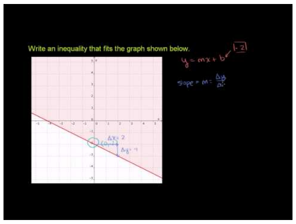
Here you'll learn how to graph a linear inequality in the Cartesian plane.

A linear equation is a line when graphed. What about a linear inequality? Can you represent the following linear inequality in the Cartesian plane?

$2x-3y<6$

Watch This

Khan Academy Graphing Linear Inequalities



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Guidance

When a linear inequality is graphed on a grid, the solution will appear as a shaded area, as opposed to simply a straight line. The general form of a linear inequality with two variables is $ax+by>c$ or $ax+by<c$, where 'a' and 'b' are the coefficients of the variables and are not both equal to zero. The constant is 'c'.

The inequality symbol $(>)$ is read "greater than" and the inequality symbol $(<)$ is read "less than". The inequality symbol $(\geq)$ is read "greater than or equal to" and the inequality symbol $(\leq)$ is read "less than or equal to". The inequality symbol determines whether the line is dashed or solid.

- All inequalities that have the symbol $(>)$ or $(<)$ are graphed with a dashed line.
- All inequalities that have the symbol $(\geq)$ or $(\leq)$ are graphed with a solid line.

Don't forget that when an inequality is divided or multiplied by a negative number, the direction of the inequality sign is reversed.

Example A

Graph the inequality $3x+4y\leq 12$.

Solution:

$3x+4y \leq 12$ Write the inequality in slope intercept form $(y = mx+b)$.

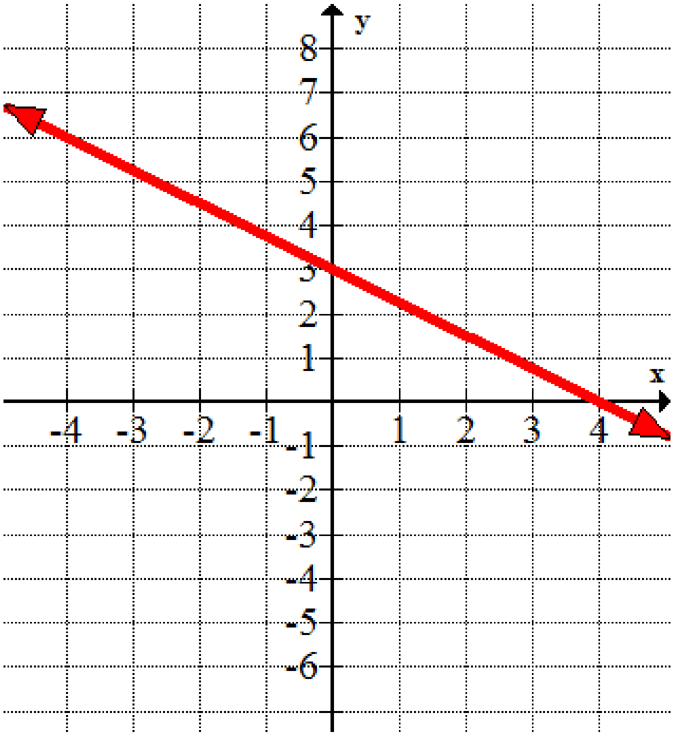
$3x-3x+4y \leq -3x+12$

$4y \leq -3x+12$

$\frac{4y}{4} \leq \frac{-3x}{4} + \frac{12}{4}$

$\frac{\cancel{4}y}{\cancel{4}} \leq \frac{-3x}{4} + \frac{\overset{3}{\cancel{12}}}{\cancel{4}}$

$\boxed{y \leq -\frac{3}{4}x+3}$



The above graph represents the graph of the inequality before the solution set region is shaded. The line is a solid line because the inequality symbol is $(\leq)$. To determine whether to shade above the line or below the line, choose a point that is not on the line and test its coordinates in the original inequality. If the coordinates of the point satisfy the inequality, then the area above or below the line, containing the point, will be shaded. If the coordinates of the point do not satisfy the inequality, then the area above or below the line, that does not contain the test point, will be shaded. The point (1, 1) is not on the graphed line. The point will be tested to determine if the coordinates satisfy the inequality.

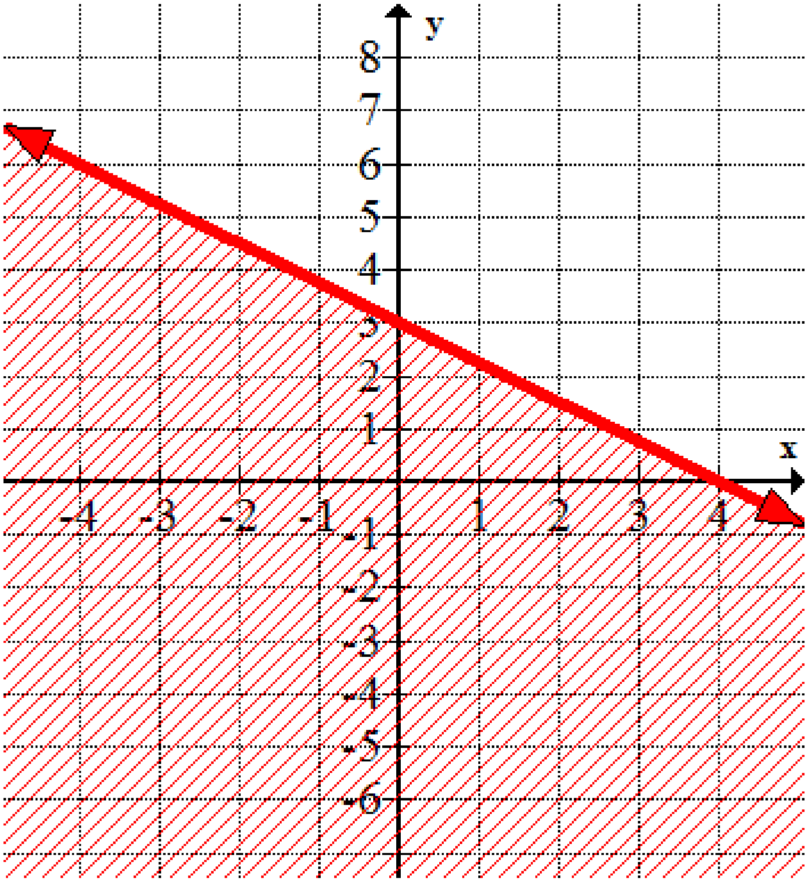
$3x+4y \leq 12$

$3(1)+4(1) \leq 12$ Substitute $(1,1)$ for x and y of the original inequality.

$3+4 \leq 12$ Evaluate the inequality.

$7 \leq 12$ Is it true?

Yes, seven is less than 12. The point (1, 1) satisfies the inequality. Therefore, the solution set is all of the area below the line that contains the point (1, 1). The ordered pair that satisfies the inequality will lie within the shaded region.



The solution set for the inequality is the entire shaded region shown in the graph. The solid line means that all of the points on the line will satisfy the inequality. In general, you will shade below the line if the inequality is of the form $y < mx + b$ or $y \leq mx + b$.

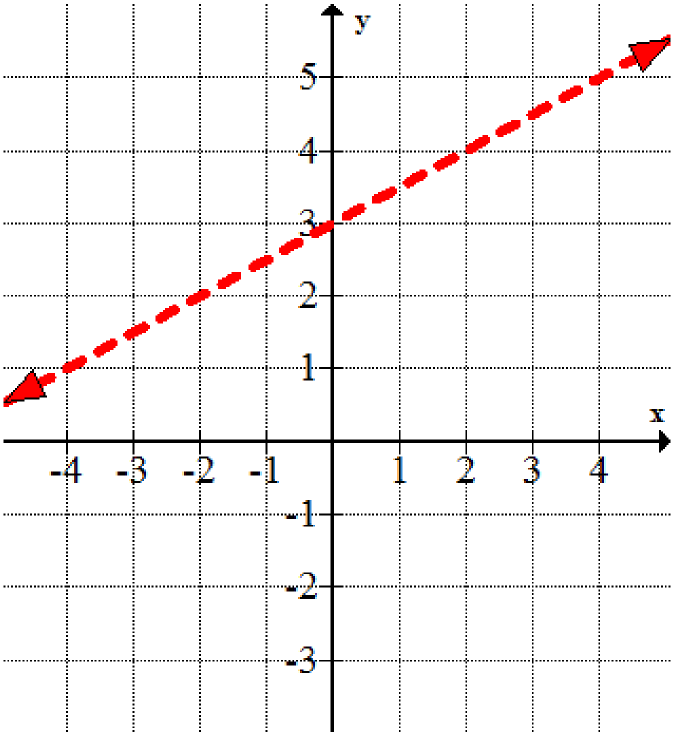
Example B

Graph the inequality $2x - 4y < -12$.

Solution:

$$\begin{aligned}
&2x - 4y < -12 && \text{Write the inequality in slope intercept form } (y = mx + b). \\
&2x {\color{red}-2x} - 4y < {\color{red}-2x} - 12 && \\
&-4y < {\color{red}-2x} - 12 && \\
&\frac{-4y}{\color{red}-4} < \frac{-2x}{\color{red}-4} - \frac{12}{\color{red}-4} && \\
&\frac{\cancel{-4}y}{\cancel{-4}} < \frac{-2x}{-4} - \frac{\overset{\color{red}-3}{\cancel{12}}}{\cancel{-4}} && \\
&\boxed{y {\color{red}>} \frac{1}{2}x + 3} &&
\end{aligned}$$

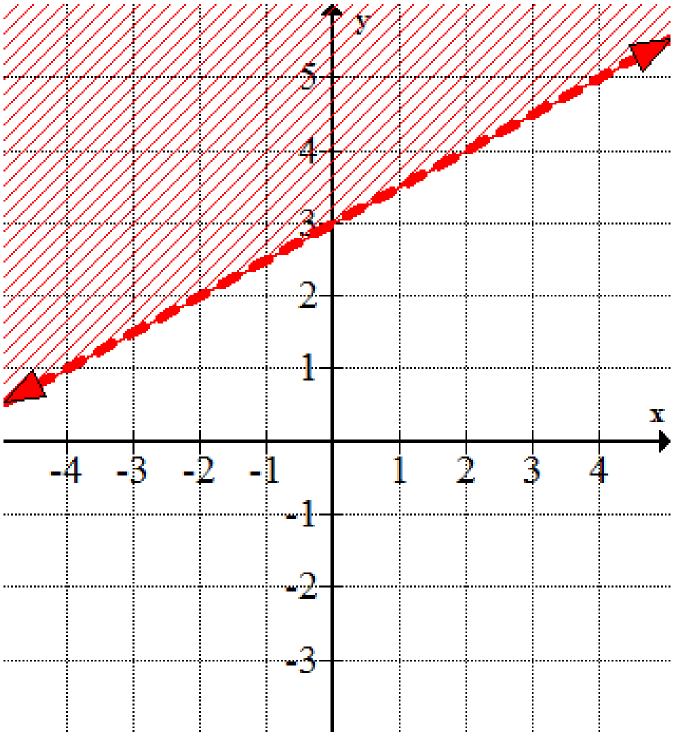
Note that the inequality was divided by negative 4 which caused the inequality sign to reverse its direction.



The above graph represents the graph of the inequality before the solution set region is shaded. The line is a dashed line because the inequality symbol is $(<)$. To determine whether to shade above the line or below the line, choose a point that is not on the line and test its coordinates in the original inequality. If the coordinates of the point satisfy the inequality, then the area above or below the line, containing the point, will be shaded. If the coordinates of the point do not satisfy the inequality, then the area above or below the line, that does not contain the test point, will be shaded. The point (1, 1) is not on the graphed line. The point will be tested to determine if the coordinates satisfy the inequality.

$$\begin{aligned}
2x-4y &< -12 && \\
2(1)-4(1) &< -12 && \text{Substitute } (1,1) \text{ for } x \text{ and } y \text{ of the original inequality.}\\
2-4 &< -12 && \text{Evaluate the inequality.}\\
-2 &< -12 && \text{Is it true?}
\end{aligned}$$

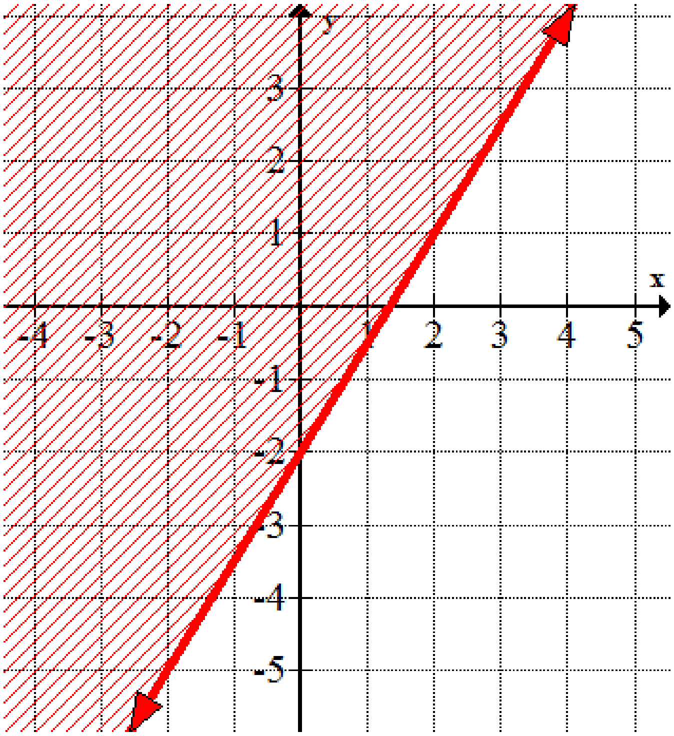
No, negative two is greater than negative twelve. The point (1, 1) does not satisfy the inequality. Therefore, the solution set is all of the area above the line that does not contain the point (1, 1). The ordered pair does not satisfy the inequality and will not lie within the shaded region.



The solution set for the inequality is the entire shaded region shown in the graph. The dashed or dotted line means that none of the points on the line will satisfy the inequality. In general, you will shade above the line if the inequality is of the form $y > mx + b$ or $y \geq mx + b$.

Example C

For the following, determine the inequality, in slope-intercept form, that is graphed.



Solution: Begin by determining the slope of the line. The slope of the line is determined by counting 2 units to the right and 3 units upward. The slope of the line for this graph is

$m = \frac{rise}{run} = \frac{3}{2}$

The y-intercept for the line is (0, –2). The equation of the line in slope-intercept form is

$y = \frac{3}{2}x - 2$

The solution set is found in the shaded region that is above the line. The line is a solid line. Therefore the inequality symbol that must be inserted is greater than or equal to. The inequality that is modeled by the above graph is:

$$\boxed{y \geq \frac{3}{2}x - 2}$$

Concept Problem Revisited

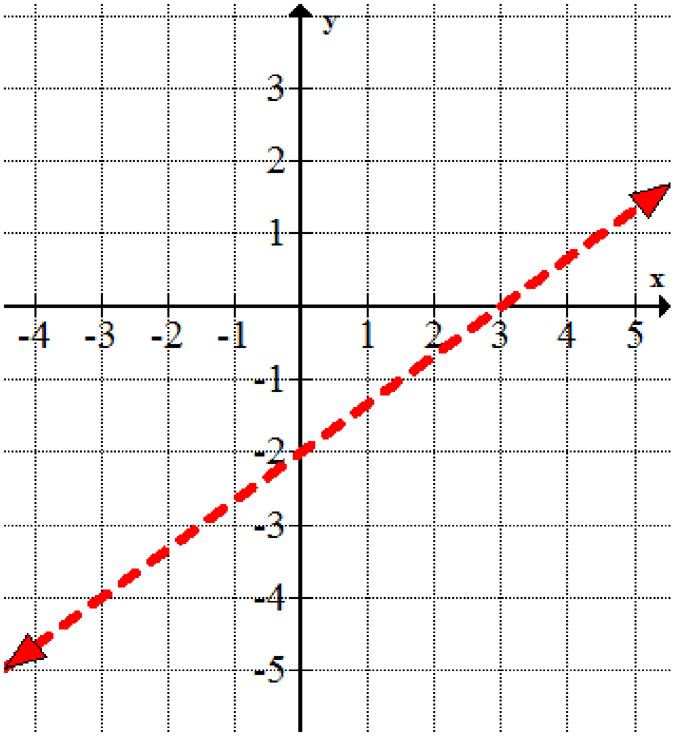
$2x - 3y < 6$

The first step is to rearrange the inequality in slope-intercept form. This process is the same as it is for linear equations.

$$\begin{aligned}
&2x - 3y < 6\\
&2x - 2x - 3y < -2x + 6\\
&-3y < -2x + 6\\
&\frac{-3y}{-3} < \frac{-2x}{-3} + \frac{6}{-3}\\
&\frac{\cancel{-3}y}{\cancel{-3}} < \frac{-2x}{-3} + \frac{\overset{-2}{\cancel{6}}}{\cancel{-3}}\\
&\boxed{y > \frac{2}{3}x - 2}
\end{aligned}$$

Note that the inequality was divided by negative 3 which caused the inequality sign to reverse its direction.

The graph of the inequality is done the same as it is for a linear equation. In this case the graph will be a dashed or dotted line because the sign is greater than $(>)$.

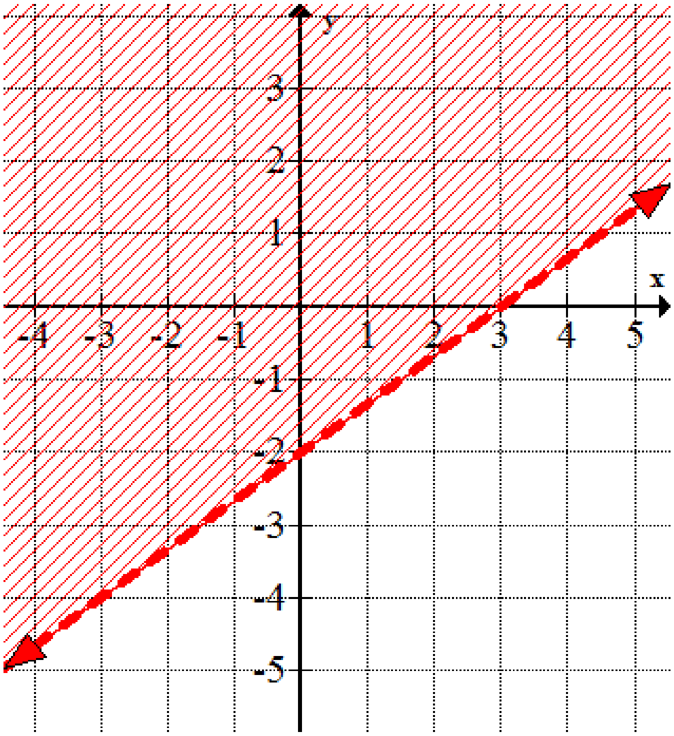


The above graph represents the graph of the inequality before the solution set region is shaded. To determine whether to shade above the line or below the line, choose a point that is not on the line and test its coordinates in the original inequality. If the coordinates of the point satisfy the inequality, then the area above or below the line, containing the point, will be shaded. If the coordinates of the point do not satisfy the inequality, then the area above or below the line, that does not contain the test point, will be shaded.

The point (1, 1) is not on the graphed line. The point will be tested to determine if the coordinates satisfy the inequality.

$$\begin{aligned}
2x-3y &< 6 \\
2(1)-3(1) &< 6 && \text{Substitute } (1,1) \text{ for } x \text{ and } y \text{ of the original inequality.} \\
2-3 &< 6 && \text{Evaluate the inequality.} \\
-1 &< 6 && \text{Is it true?}
\end{aligned}$$

Yes, negative one is less than six. The point (1, 1) satisfies the inequality. Therefore, the solution set is all of the area above the line that contains the point (1, 1). The ordered pair that satisfies the inequality will lie within the shaded region.



The solution set for the inequality is the entire shaded region shown in the graph. The dashed or dotted line means that none of the points on the line will satisfy the inequality.

Vocabulary

Inequality

An ***inequality*** is a statement that shows a relationship between two expressions that are not always equal. An inequality is written using one of the following inequality symbols: greater than $(>)$; less than $(<)$; greater than or equal to $(\geq)$; less than or equal to $(\leq)$. The solution to an inequality is indicated by a shaded region that contains all the ordered pairs that satisfy the inequality.

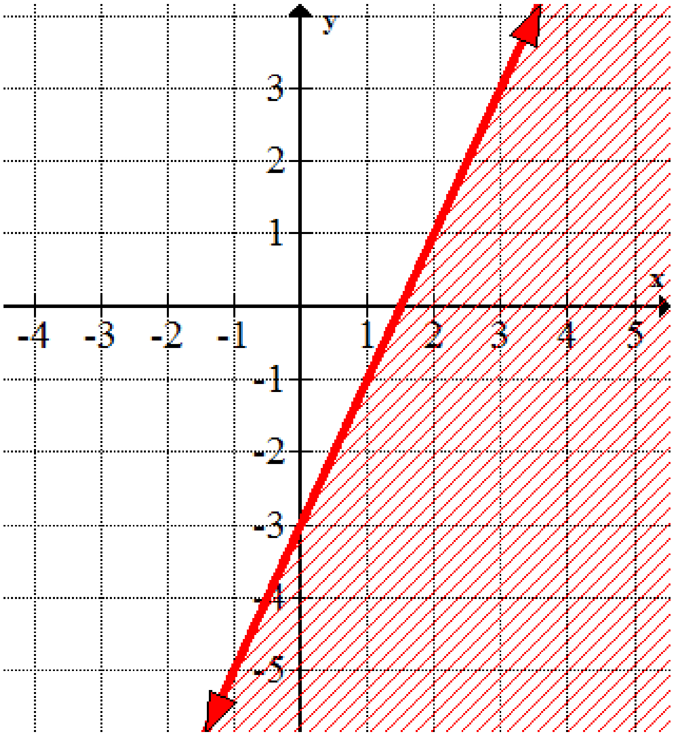
Guided Practice

1. Without graphing, determine if each point is in the shaded region for each inequality.

i) (2, –3) and $2y < -3x+1$
ii) (–3, 5) and $-3x > 2y+6$

2. Graph the following inequality: $5x-3y \geq 15$

3. Determine the inequality that models the following graph:



Answers:

1. If the point satisfies the inequality, then the point will lie within the shaded region. Substitute the coordinates of the point into the inequality and evaluate the inequality. If the solution is true, then the point is in the shaded area.

i)

$2y < -3x+1$	Substitute $(2,-3)$ for x and y in the inequality.
$2(-3) < -3(2)+1$	Evaluate the inequality.
$-6 < -6+1$	
$-6 < -5$	Is it true?

Yes, negative six is less than negative five. The point (2, –3) satisfies the inequality. The ordered pair will lie within the shaded region.

ii)

$-3x > 2y+6$	Substitute $(-3,5)$ for x and y in the inequality.
$-3(-3) > 2(5)+6$	Evaluate the inequality.
$9 > 10+6$	
$9 > 16$	Is it true?

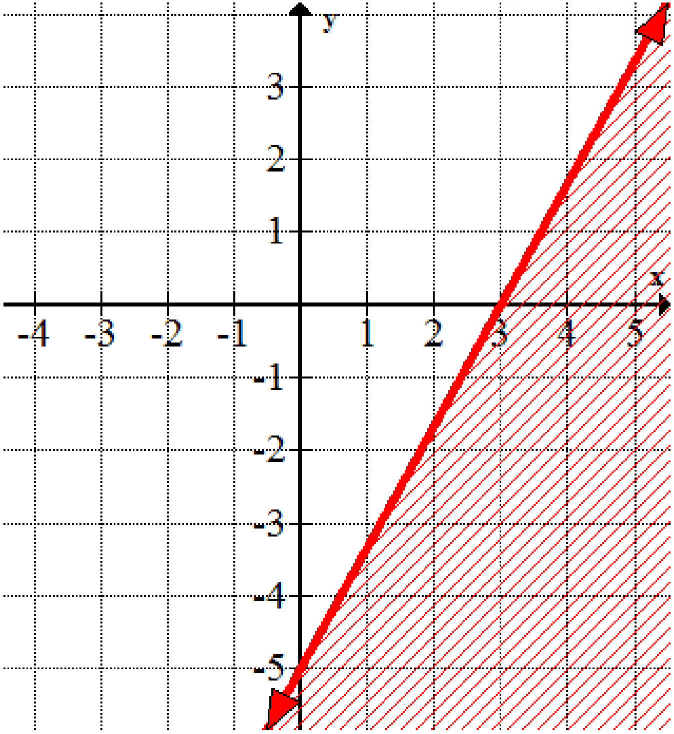
No, nine is not greater than sixteen. The point (–3, 5) does not satisfy the inequality. The ordered pair does not satisfy the inequality and will not lie within the shaded region.

2.

$5x-3y \geq 15$ Write the inequality in slope intercept form $(y = mx+b)$.

$$\begin{aligned}
&5x-5x-3y \geq -5x+15\\
&-3y \geq -5x+15\\
&\frac{-3y}{-3} \geq \frac{-5x}{-3}+\frac{15}{-3}\\
&\frac{\cancel{-3}y}{\cancel{-3}} \geq \frac{5}{3}x+\frac{\overset{-5}{\cancel{15}}}{\cancel{-3}}\\
&\boxed{y \leq \frac{5}{3}x-5}
\end{aligned}$$

Note that the inequality was divided by negative 3 which caused the inequality sign to reverse its direction.



Is the graph of the inequality shaded correctly?

The point (1, 1) is not on the graphed line. The point will be tested to determine if the coordinates satisfy the inequality.

$5x-3y \geq 15$

$5(1)-3(1) \geq 15$ Substitute $(1,1)$ for x and y of the original inequality.

$5-3 \geq 15$ Evaluate the inequality.

$2 \geq 15$ Is it true?

No, two is not greater than or equal to fifteen. The point (1, 1) does not satisfy the inequality. The ordered pair does not satisfy the inequality and will not lie within the shaded region.

3. Begin by determining the slope of the line. The slope of the line is determined by counting 2 units to the left and 4 units downward. The slope of the line for this graph is

$m = \frac{rise}{run} = \frac{-4}{-2} = 2$

The y-intercept for the line is (0, –3). The equation of the line in slope-intercept form is $y = 2x - 3$
The solution set is found in the shaded region that is below the line. The line is a solid line. Therefore the inequality symbol that must be inserted is less than or equal to. The inequality that is modeled by the above graph is:

$$\boxed{y \leq 2x - 3}$$

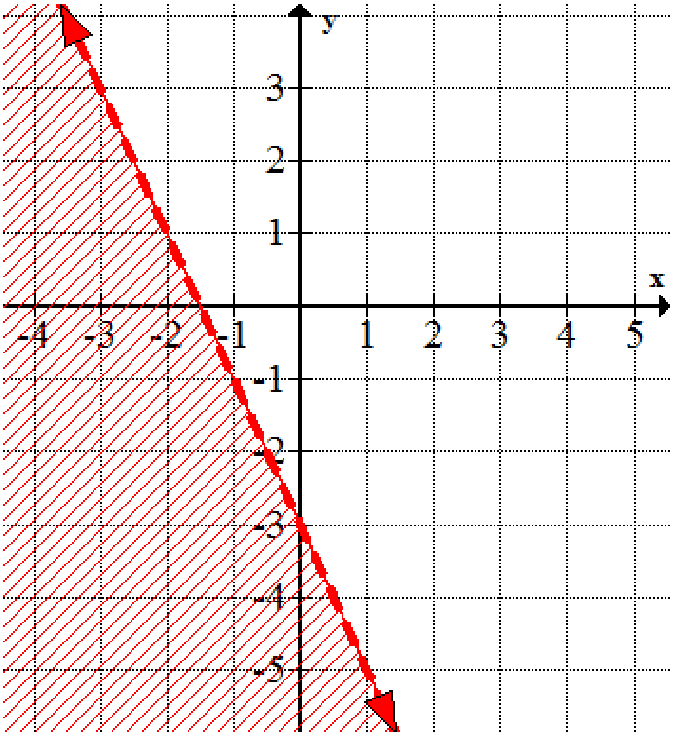
Practice

Without graphing, determine if each point is in the shaded region for each inequality.

1. (2, 1) and $2x + y > 5$
2. (–1, 3) and $2x - 4y \leq -10$
3. (–5, –1) and $y > -2x + 8$
4. (6, 2) and $2x + 3y \geq -2$
5. (5, –6) and $2y < 3x + 3$

Determine the inequality that is modeled by each of the following graphs.

6.



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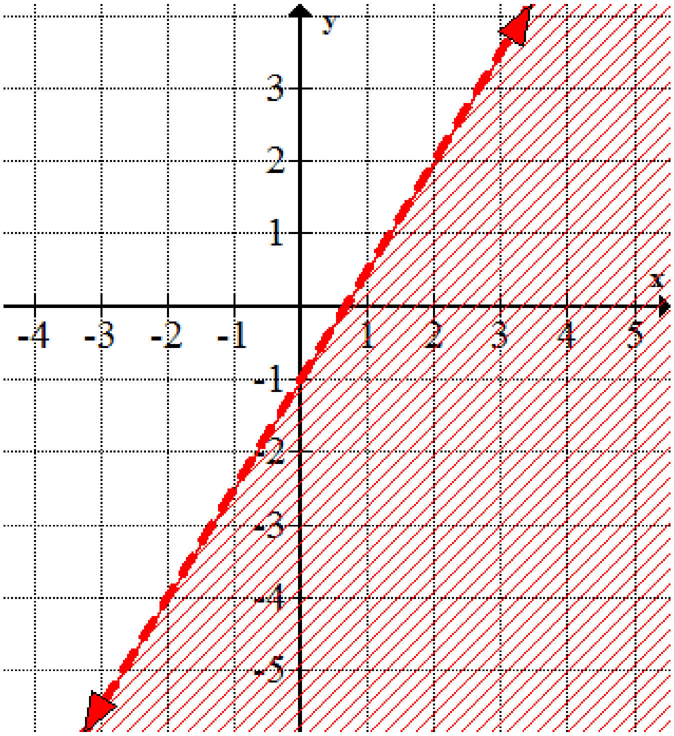

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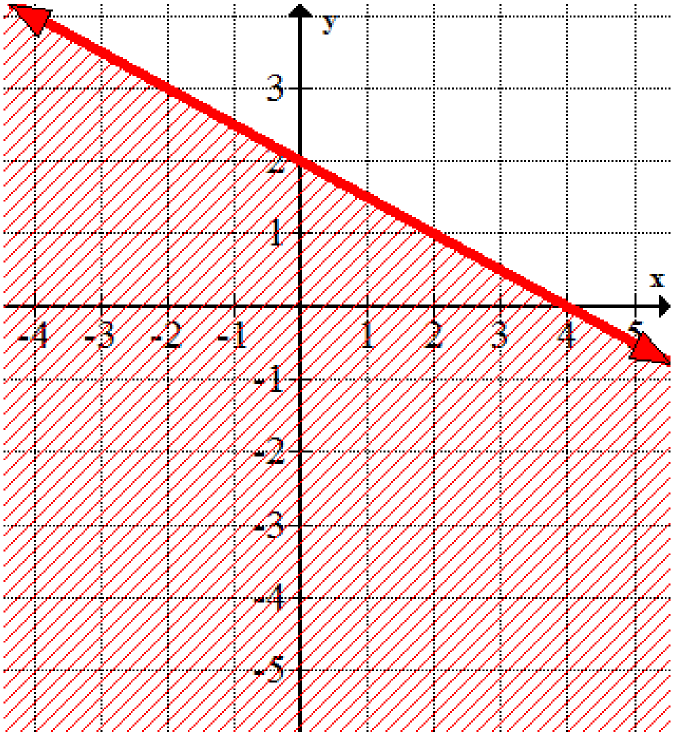

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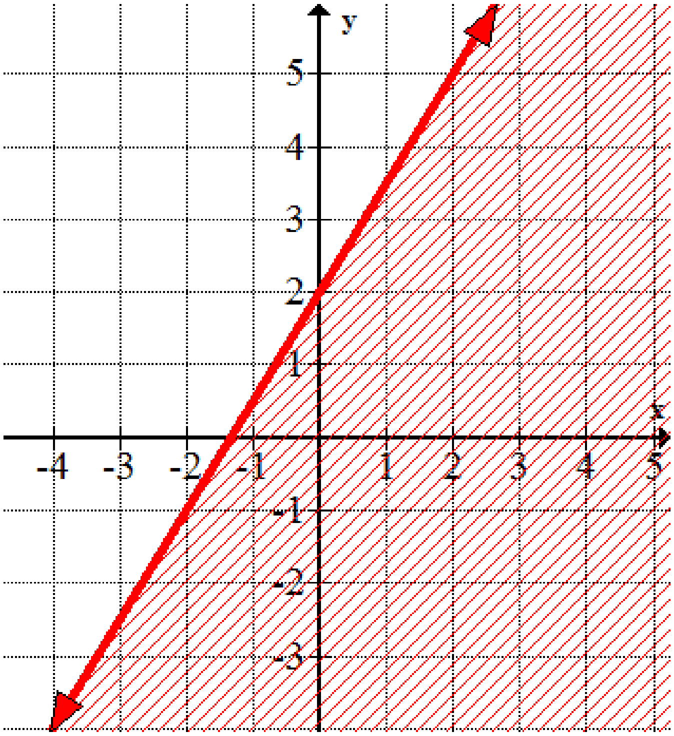
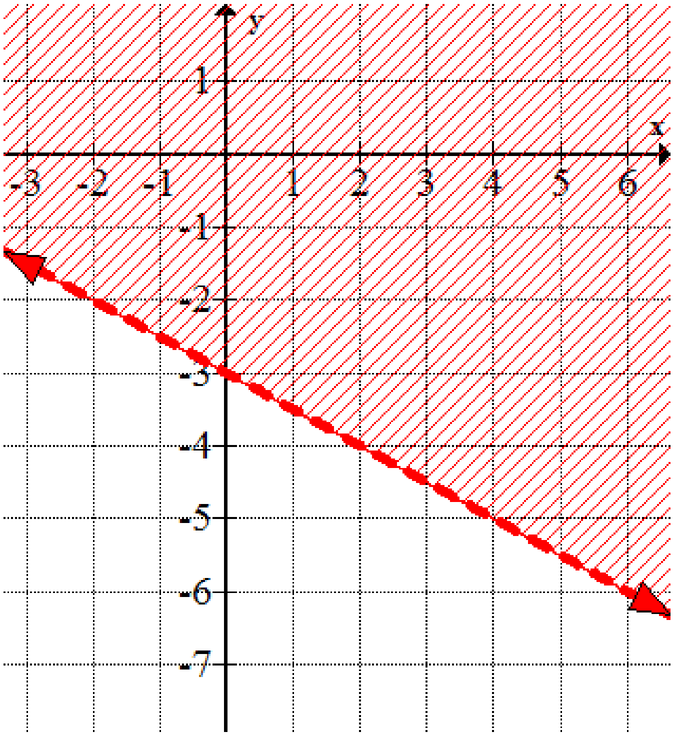


10.



Graph the following linear inequalities on a Cartesian plane.

11. $4x - 2y > 8$
12. $4y - 3x \leq -8$
13. $x - y < -3$
14. $3x + y > -1$
15. $x + 3y \geq 9$

5.6 Graphical Solutions to Systems of Inequalities

Here you'll learn to solve a system of linear inequalities by graphing.

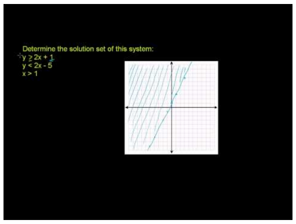
Graph the following system of linear inequalities on the same Cartesian grid.

$$\begin{Bmatrix} y > -\frac{1}{2}x + 2 \\ y \leq 2x - 3 \end{Bmatrix}$$

What is the **solution** to this system?

Watch This

Khan Academy Systems of Linear Inequalities



Determine the solution set of this system:
$y \geq 2x + 1$
$y < 2x - 5$
$x > 1$

MEDIA

Click image to the left for more content.

Guidance

When a linear inequality is graphed it appears as a shaded region on the Cartesian plane. Therefore, when a system of linear inequalities is graphed, you will see two shaded regions that most likely overlap in some places. The solution to a system of linear inequalities is this ***region*** of intersection. The solution to a system of linear inequalities is also known as the **feasible region**.

Example A

Solve the following system of linear inequalities by graphing:

$$\begin{Bmatrix} -2x - 6y \leq 12 \\ -x + 2y > -4 \end{Bmatrix}$$

Solution:

Write each inequality in slope-intercept form.

$$\begin{aligned}
&-2x-6y \le 12\\
&-2x+2x-6y \le 2x+12\\
&-6y \le 2x+12\\
&\frac{-6y}{-6} \le \frac{2x}{-6}+\frac{12}{-6}\\
&\frac{\cancel{-6}y}{\cancel{-6}} \le -\frac{2}{6}x+\frac{\overset{-2}{\cancel{12}}}{\cancel{-6}}
\end{aligned}$$

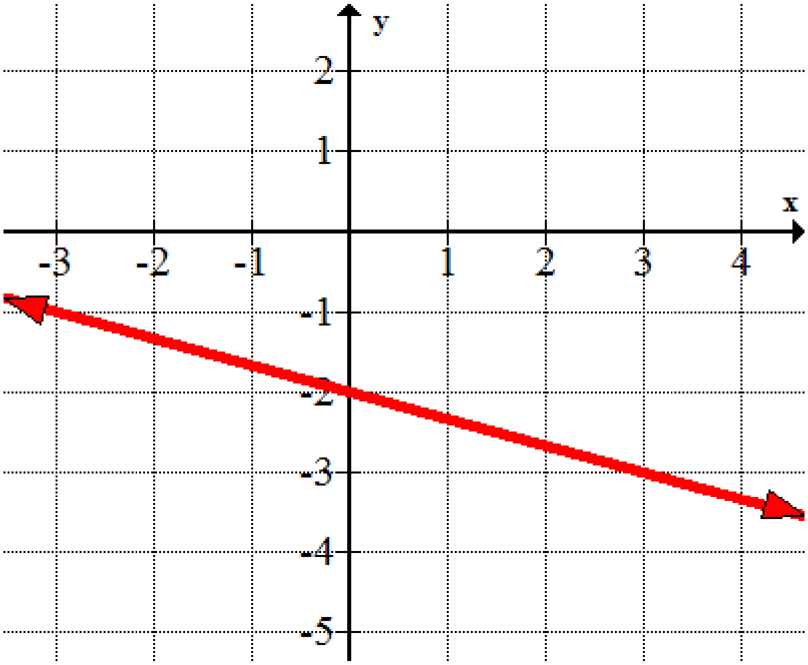
Simplify the slope to lowest terms. $\boxed{-\frac{2}{6}=-\frac{1}{3}}$

$$\boxed{y \ge -\frac{1}{3}x-2}$$

$$\begin{aligned}
&-x+2y > -4\\
&-x+x+2y > x-4\\
&2y > x-4\\
&\frac{2y}{2} > \frac{x}{2}-\frac{4}{2}\\
&\frac{\cancel{2}y}{\cancel{2}} > \frac{1}{2}x-\frac{\overset{2}{\cancel{4}}}{\cancel{2}}
\end{aligned}$$

$$\boxed{y > \frac{1}{2}x-2}$$

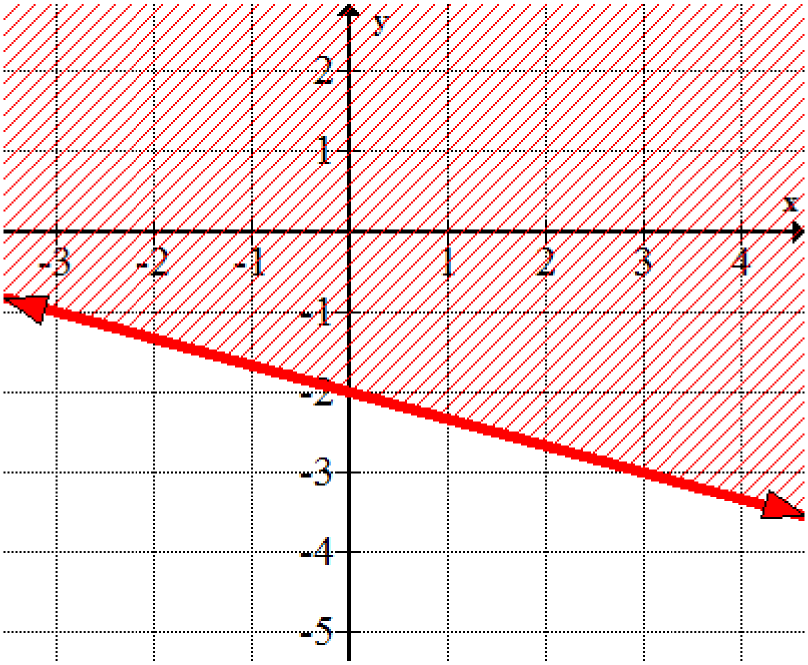
Graph: $y \ge -\frac{1}{3}x-2$.



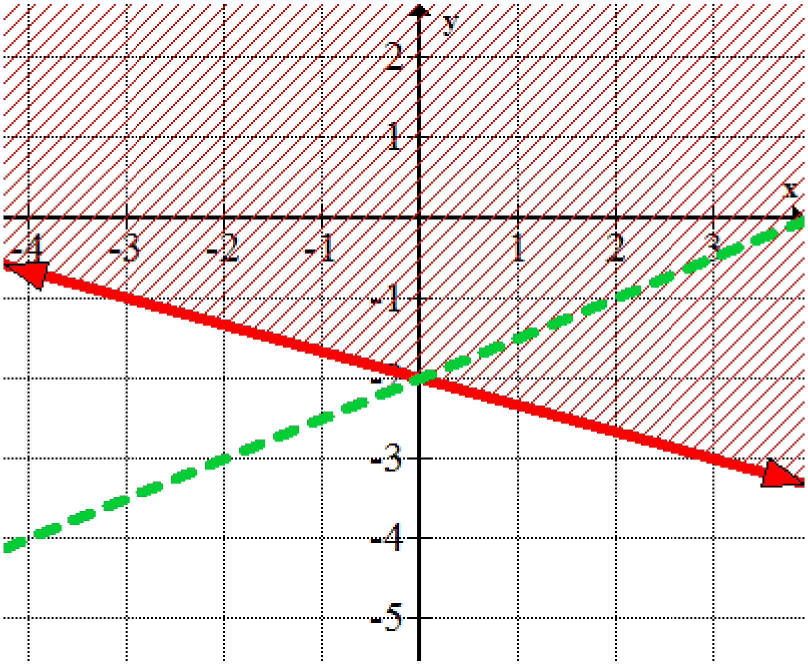
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} -2x-6y &\le 12 \\ -2(1)-6(1) &\le 12 \\ -2-6 &\le 12 \\ -8 &\le 12 \quad \text{Is it true?} \end{aligned}$$

Yes, negative eight is less than or equal to twelve. The point (1, 1) satisfies the inequality and will lie within the shaded region.



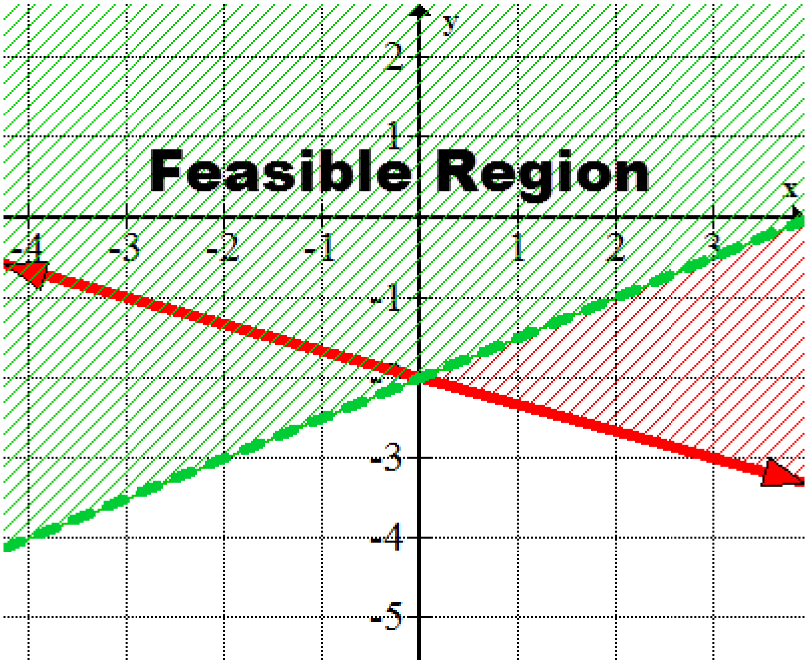
Now graph $y > \frac{1}{2}x - 2$ on the same Cartesian grid.



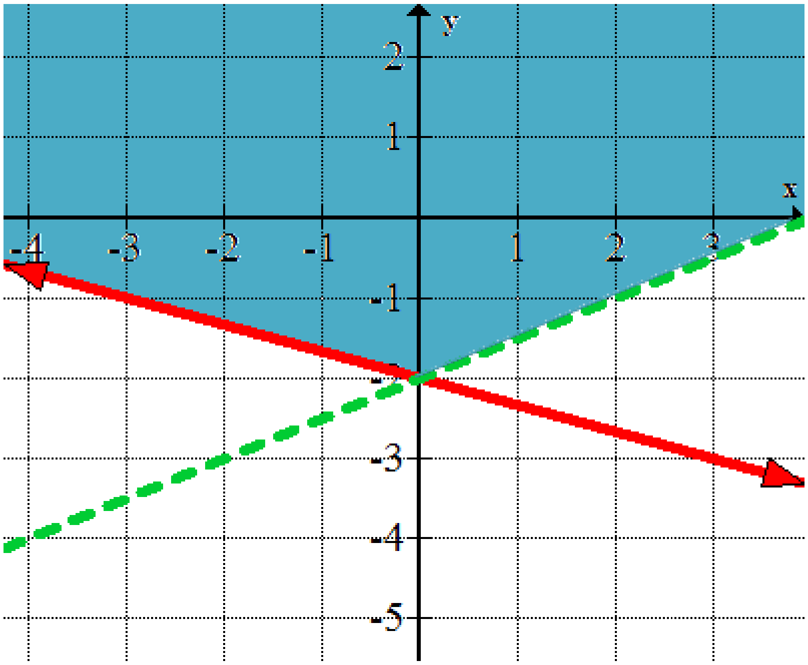
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} -x+2y &> -4 \\ -(1)+2(1) &> -4 \\ -1+2 &> -4 \\ 1 &> -4 \quad \text{Is it true?} \end{aligned}$$

Yes, one is greater than negative four. The point (1, 1) satisfies the inequality and will lie within the shaded region.



The region that is indicated as the **feasible region** is the area on the graph where the shading from each line overlaps. This region contains all the points that will satisfy both inequalities.



The **feasible region** is the area shaded in blue.

Example B

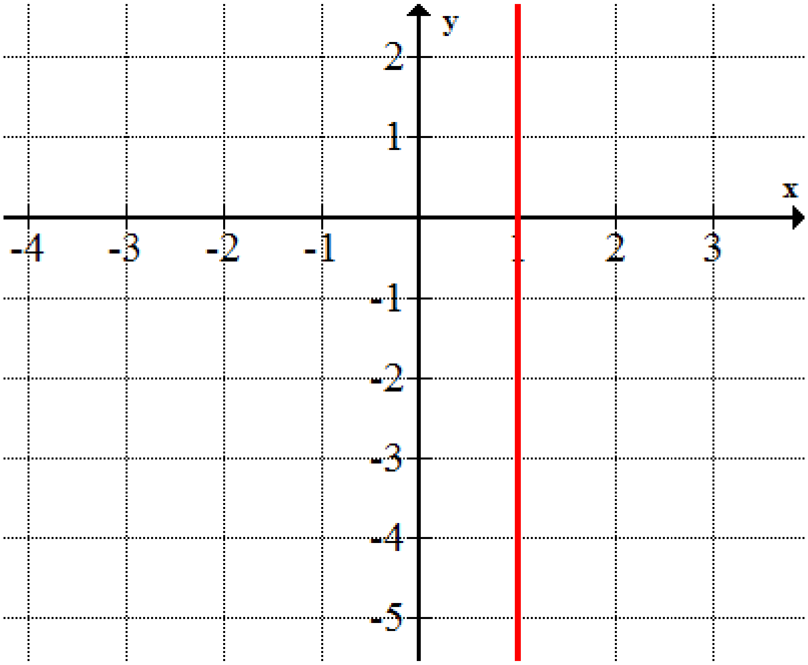
Solve the following system of linear inequalities by graphing:

$$\begin{Bmatrix} x \geq 1 \\ y > 2 \end{Bmatrix}$$

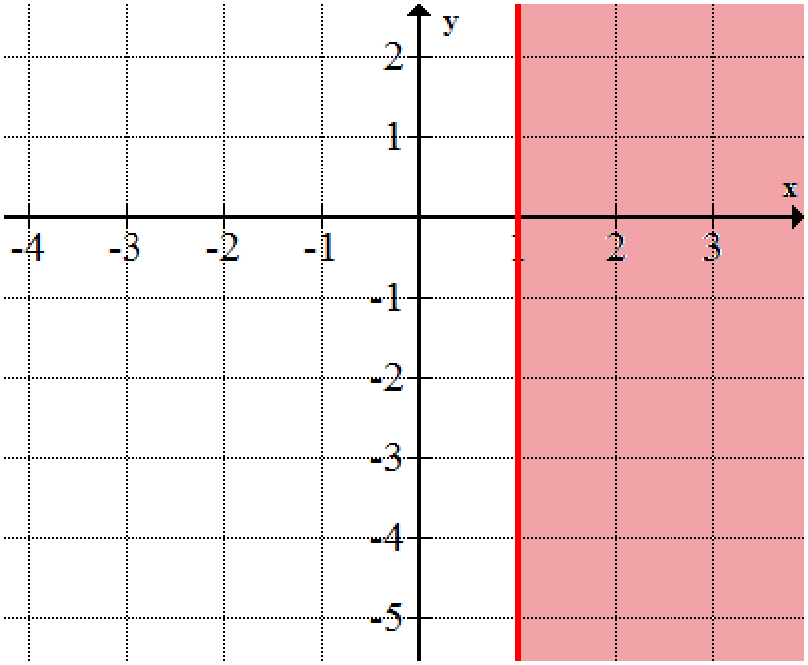
Solution:

These are the special lines that need to be graphed. The first line is a line that has an undefined slope. The graph is a vertical line parallel to the y-axis. The second line is a line that has a slope of zero. The graph is a line parallel to the x-axis.

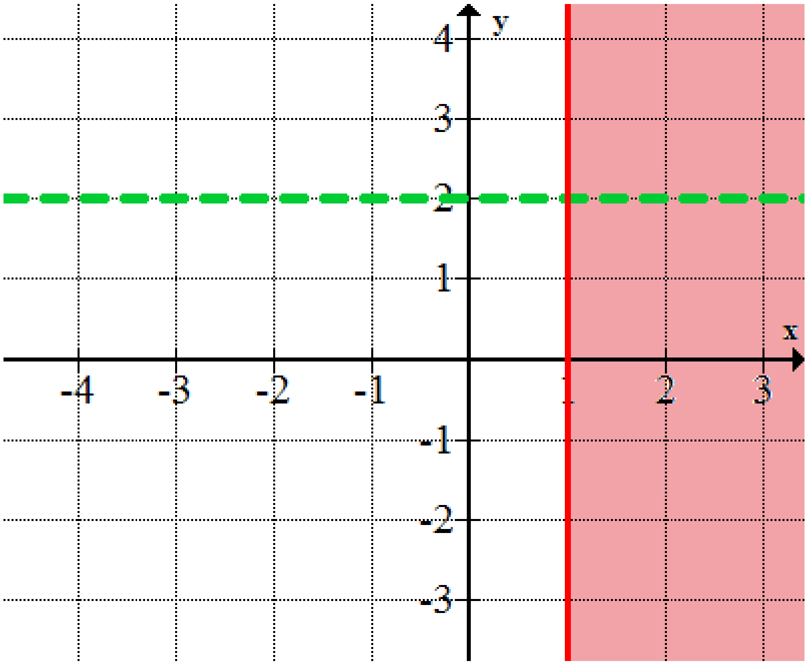
Graph: $x \geq 1$.



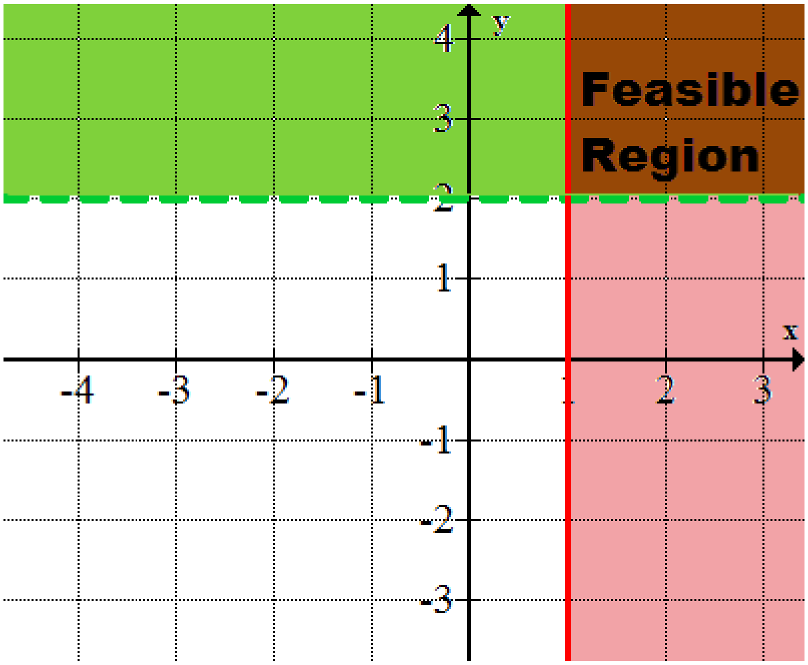
On the graph, every point to the right of the vertical line has an x-value that is greater than one. Therefore, the graph must be shaded to the right of the vertical line.



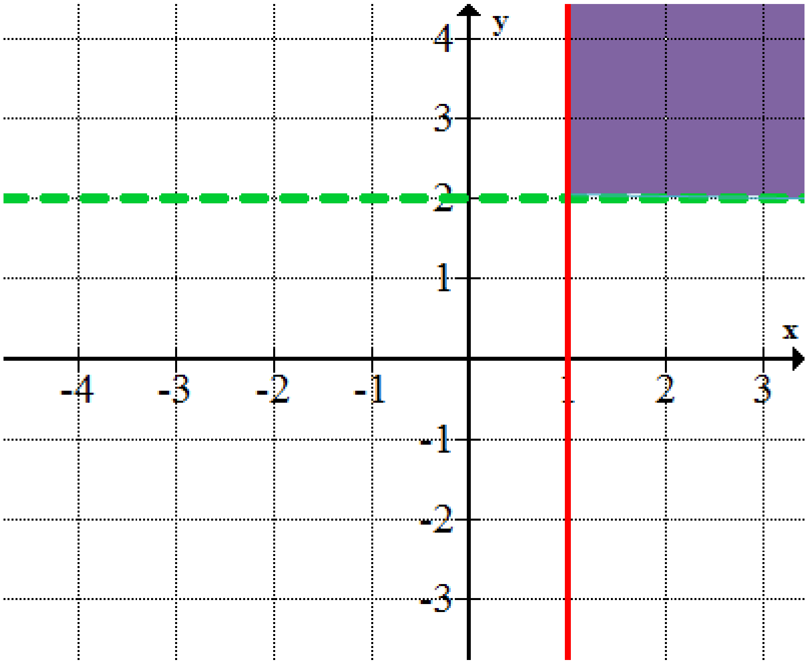
Now graph $y > 2$ on the same Cartesian grid.



On the graph, every point above the horizontal line has a y-value that is greater than two. Therefore, the graph must be shaded above the horizontal line.



The region that is indicated as the **feasible region** is the area on the graph where the shading from each line overlaps. This region contains all the points that will satisfy both inequalities.



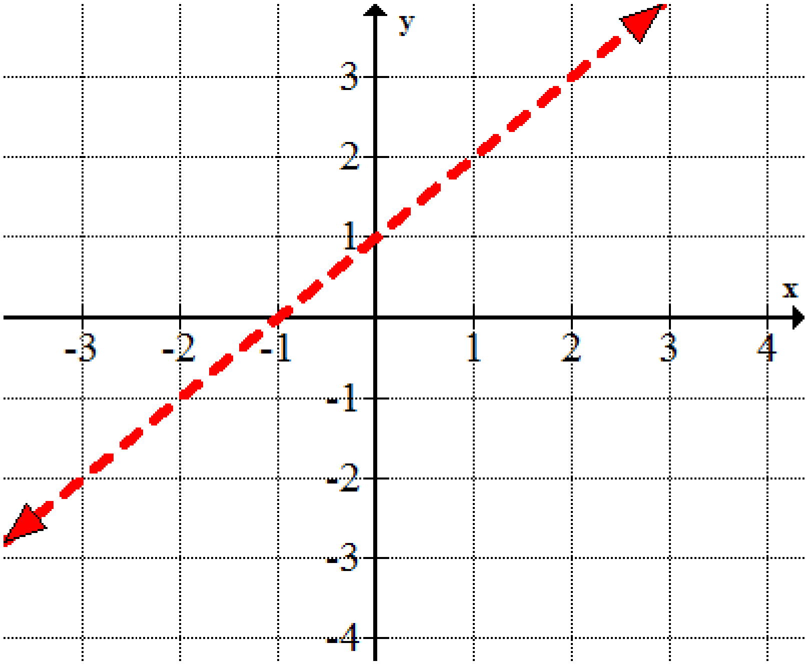
The **feasible region** is the area shaded in purple.

Example C

More than two inequalities can be shaded on the same Cartesian plane. The solution set is all of the coordinates that lie within the shaded regions that overlap. When more than two inequalities are being shaded on the same grid, the shading must be done accurately and neatly. Solve the following system of linear inequalities by graphing:

$$\left\{\begin{array}{c} y < x+1 \\ y \geq -2x+4 \\ y > 0 \end{array}\right\}$$

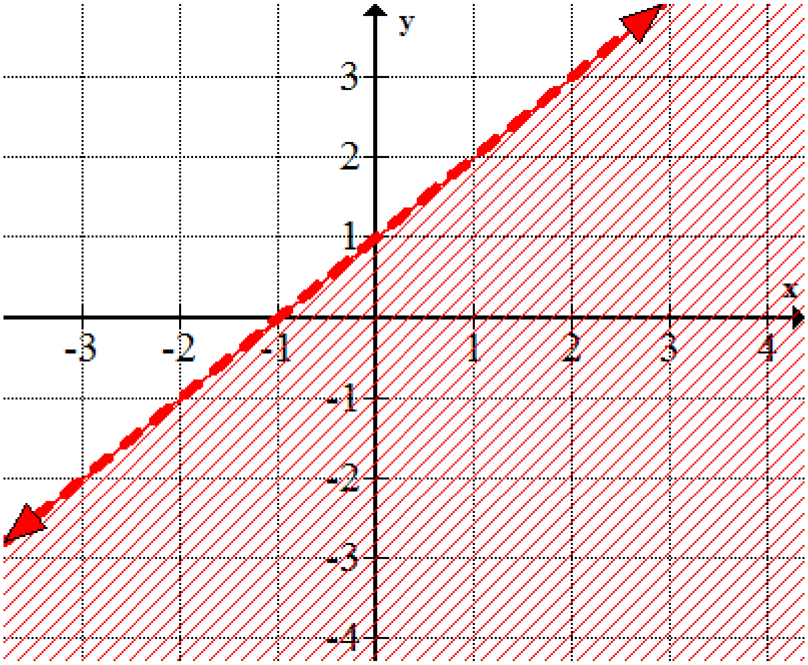
Solution: Graph $y < x+1$:



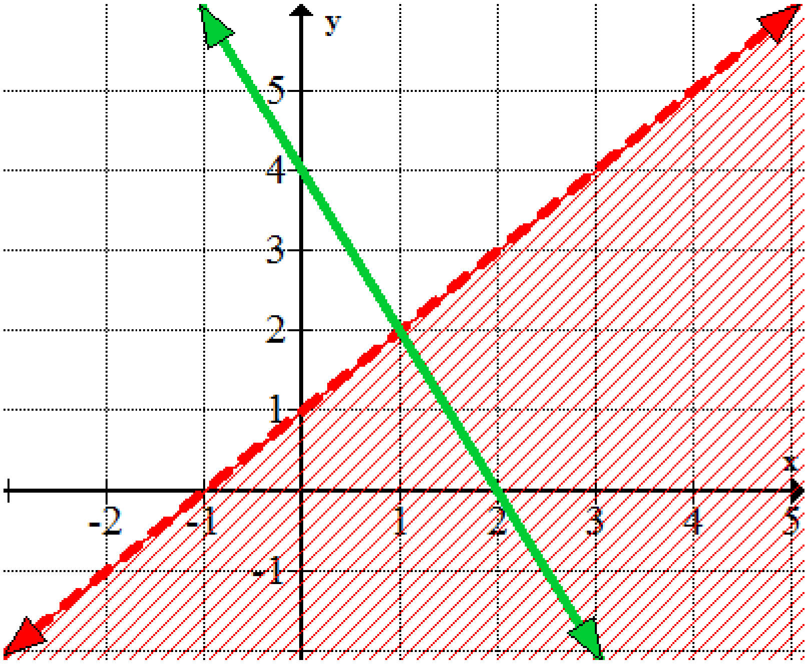
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} y &< x+1 \\ (\color{red}{1}) &< (\color{red}{1})+1 \\ \color{red}{1} &< \color{red}{1}+1 \\ 1 &< \color{red}{2} \quad \text{Is it true?} \end{aligned}$$

Yes, one is less than two. The point (1, 1) satisfies the inequality and will lie within the shaded region.



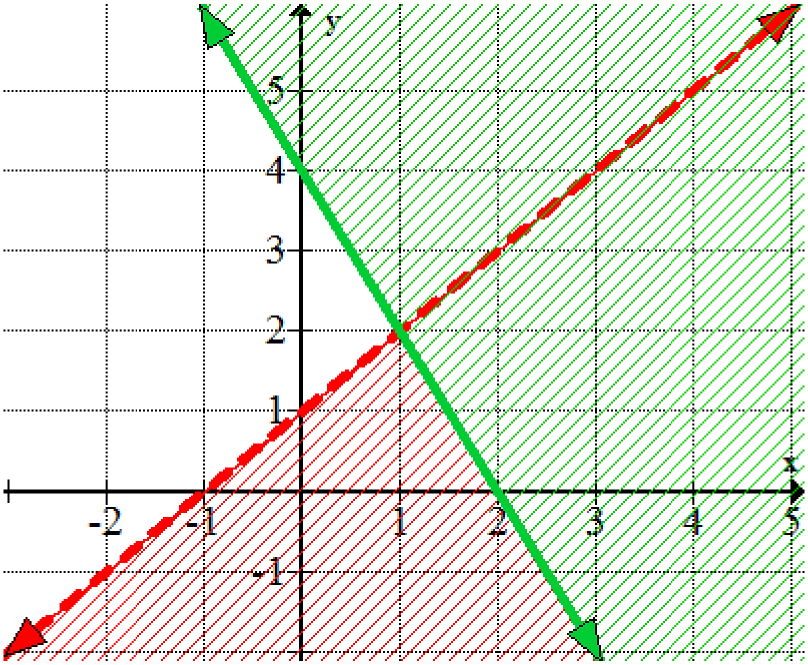
$y \geq -2x+4$



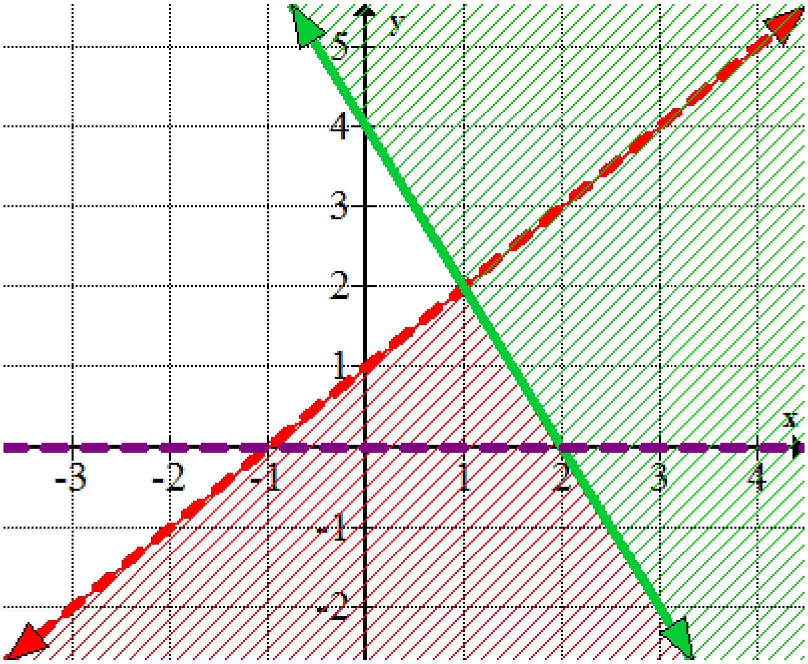
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} y &\geq -2x+4 \\ (1) &\geq -2(1)+4 \\ 1 &\geq -2+4 \\ 1 &\geq 2 \quad \text{Is it true?} \end{aligned}$$

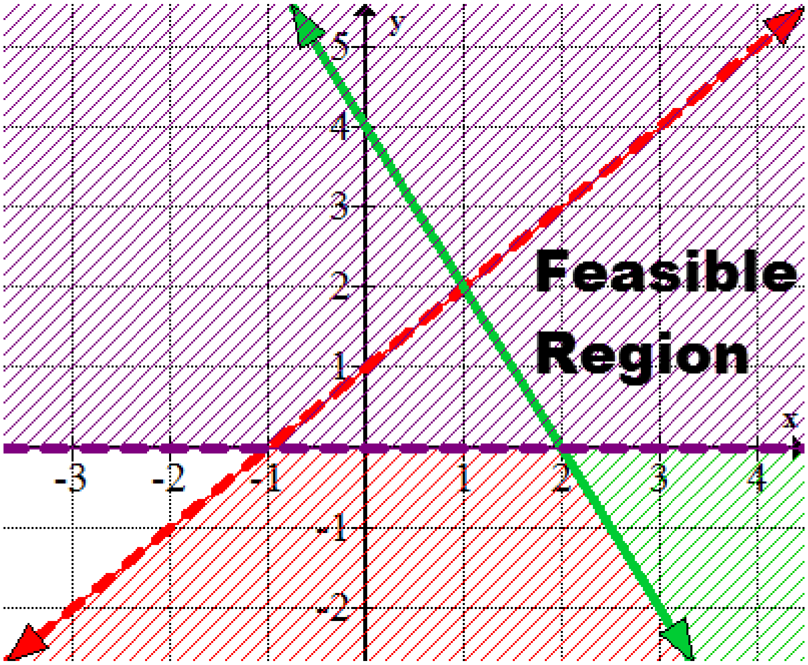
No, one is not greater than or equal to two. The point (1, 1) does not satisfy the inequality and will not lie within the shaded region.



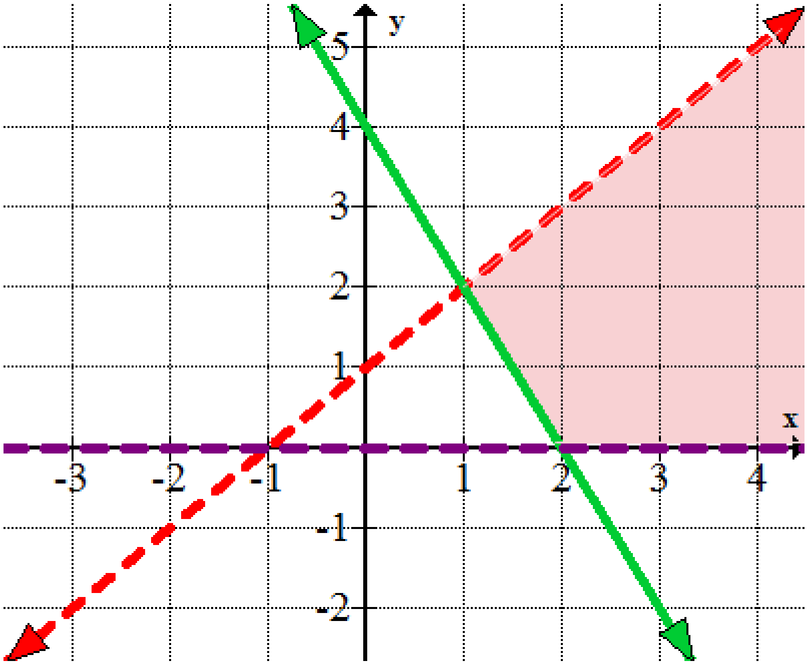
$y > 0$



The graph of $y > 0$ is a horizontal line along the x-axis. Every point above the horizontal line has a y-value greater than zero. Therefore, the shaded area will be above the graphed line.



The region that is indicated as the **feasible region** is the area on the graph where the shading from each line overlaps. This region contains all the points that will satisfy both inequalities.



The **feasible region** is the area on the right shaded in pink.

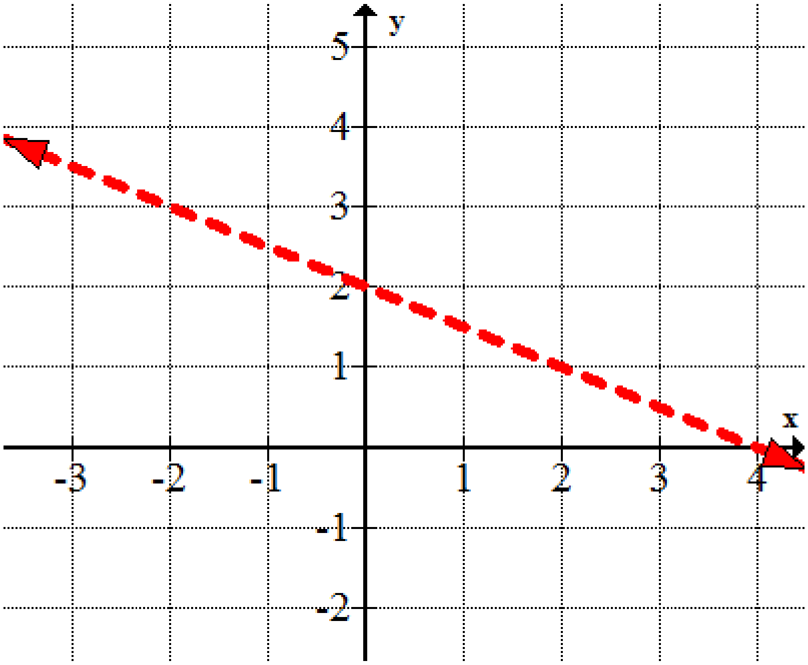
Concept Problem Revisited

$$\begin{Bmatrix} y > -\frac{1}{2}x + 2 \\ y \leq 2x - 3 \end{Bmatrix}$$

Both inequalities are in slope-intercept form. Begin by graphing

$$\boxed{y > -\frac{1}{2}x+2}$$

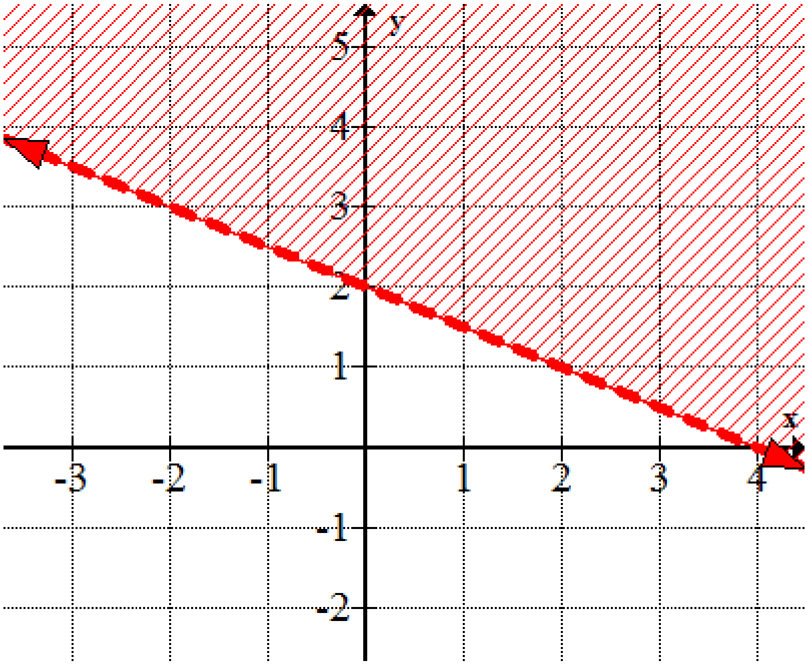
.



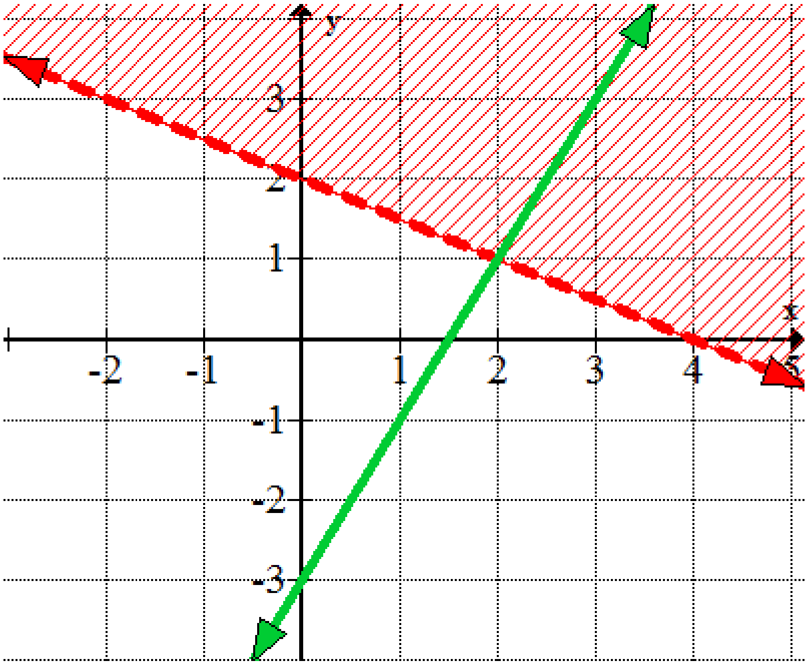
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$y > -\frac{1}{2}x+2$$
$$(1) > -\frac{1}{2}(1)+2$$
$$1 > -\frac{1}{2}+2$$
$$\boxed{1 > 1\frac{1}{2}} \quad \text{Is it true?}$$

No, one is not greater than one and one-half. Therefore, the point (1, 1) does not satisfy the inequality and will not lie in the shaded area. The shaded area is above the dashed line.



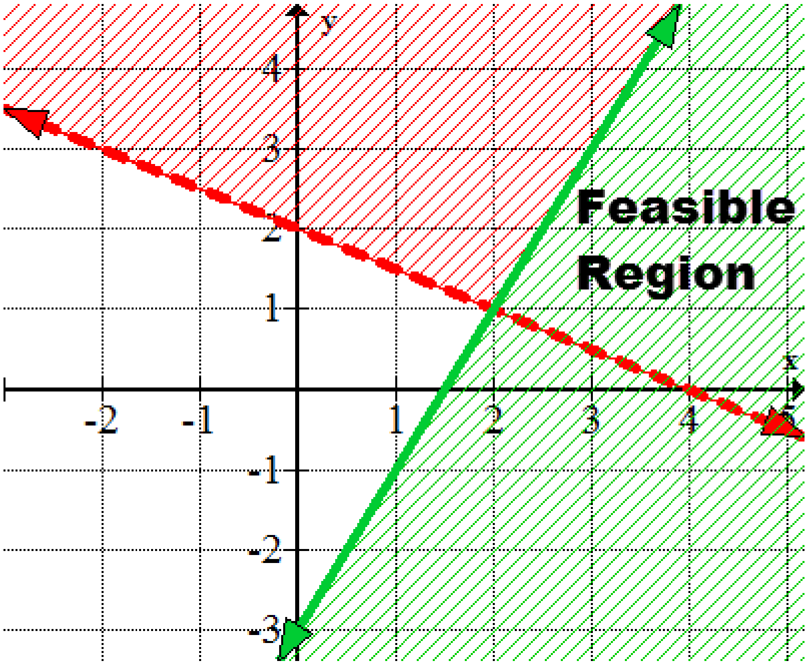
Now, the inequality $y \leq 2x-3$ will be graphed on the same Cartesian grid.



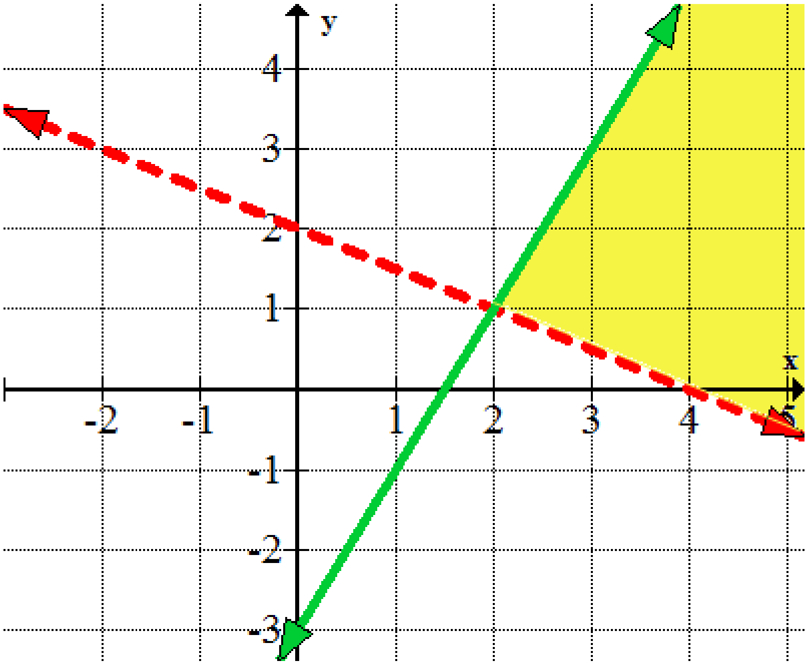
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$
\begin{aligned}
&y \leq 2x-3\\
&(1) \leq 2(1)-3\\
&1 \leq 2-3\\
&1 \leq -1\\
&\boxed{1 \leq -1} \quad \text{Is it true?}
\end{aligned}
$$

No, one is not less than or equal to negative one. Therefore, the point (1, 1) does not satisfy the inequality and will not lie in the shaded area. The shaded area is below the solid line.



The region that is indicated as the **feasible region** is the area on the graph where the shading from each line overlaps. This region contains all the points that will satisfy both inequalities. Another way to indicate the feasible region is to shade the entire region a different color. If you were to do this exercise in your notebook using colored pencils, the feasible region would be very obvious.



The **feasible region** is the area shaded in yellow.

Vocabulary

Feasible region

The ***feasible region*** is the part on the graph where the shaded areas of the inequalities overlap. This area contains all the solutions for the inequalities.

Guided Practice

1. Solve the following system of linear inequalities by graphing:

$$\left\{\begin{matrix} 4x+5y \le 20 \\ 3x+y \le 6 \end{matrix}\right\}$$

2. Solve the system of linear inequalities by graphing:

$$\left\{\begin{matrix} 2x+y \le 8 \\ 2x+3y < 12 \\ x \ge 0 \\ y \ge 0 \end{matrix}\right\}$$

3. Determine and prove three points that satisfy the following system of linear inequalities:

$$\left\{\begin{matrix} y < 2x+7 \\ y \ge -3x-4 \end{matrix}\right\}$$

Answers:

1.

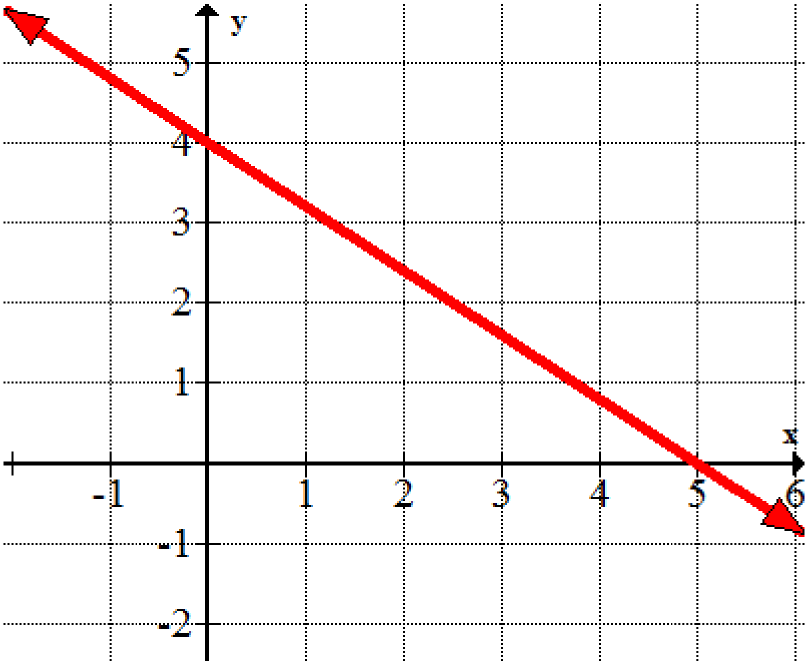
$$\left\{\begin{matrix} 4x+5y \le 20 \\ 3x+y \le 6 \end{matrix}\right\}$$

Write each inequality in slope-intercept form.

$$\begin{aligned} &4x+5y \le 20 \\ &4x-4x+5y \le -4x+20 \\ &5y \le -4x+20 \\ &\frac{5y}{5} \le \frac{-4x}{5}+\frac{20}{5} \\ &\frac{\cancel{5}y}{\cancel{5}} \le -\frac{4}{5}x+\frac{\overset{4}{\cancel{20}}}{\cancel{5}} \\ &\boxed{y \le -\frac{4}{5}x+4} \end{aligned}$$

$$\begin{aligned} &3x+y \le 6 \\ &3x-3x+y \le -3x+6 \\ &y \le -3x+6 \\ &\boxed{y \le -3x+6} \end{aligned}$$

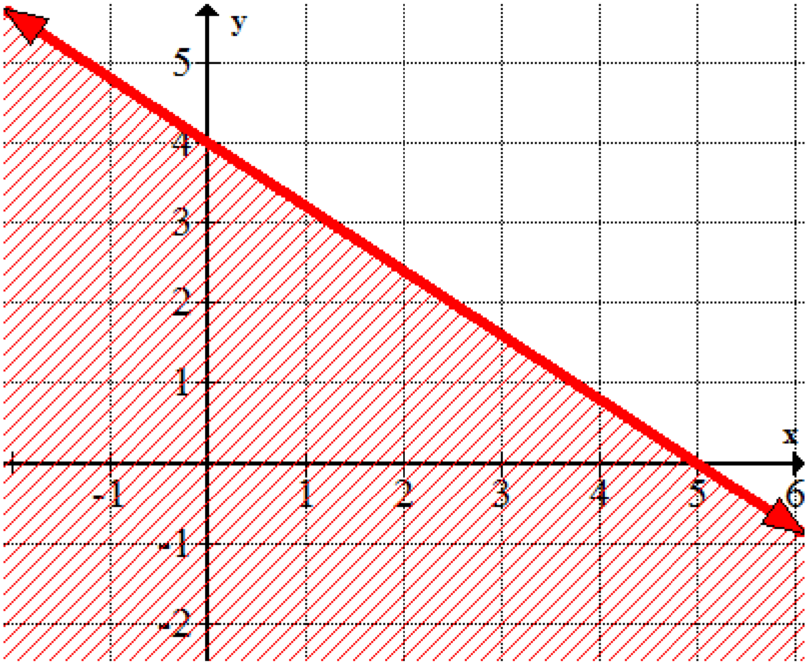
Graph: $y \le -\frac{4}{5}x+4$



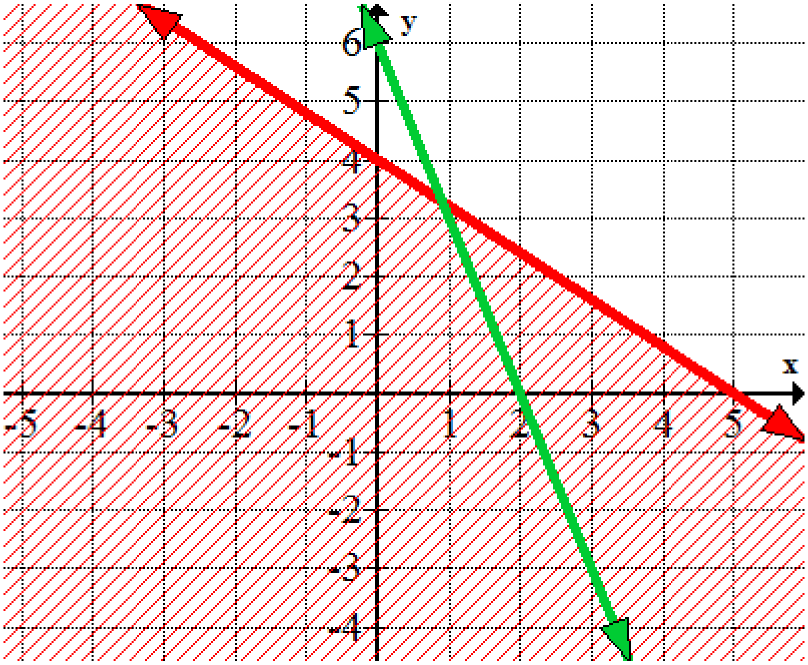
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} 4x+5y &\le 20 \\ 4(1)+5(1) &\le 20 \\ 4+5 &\le 20 \\ 9 &\le 20 \quad \text{Is it true?} \end{aligned}$$

Yes, nine is less than or equal to twenty. The point (1, 1) satisfies the inequality and will lie within the shaded region.



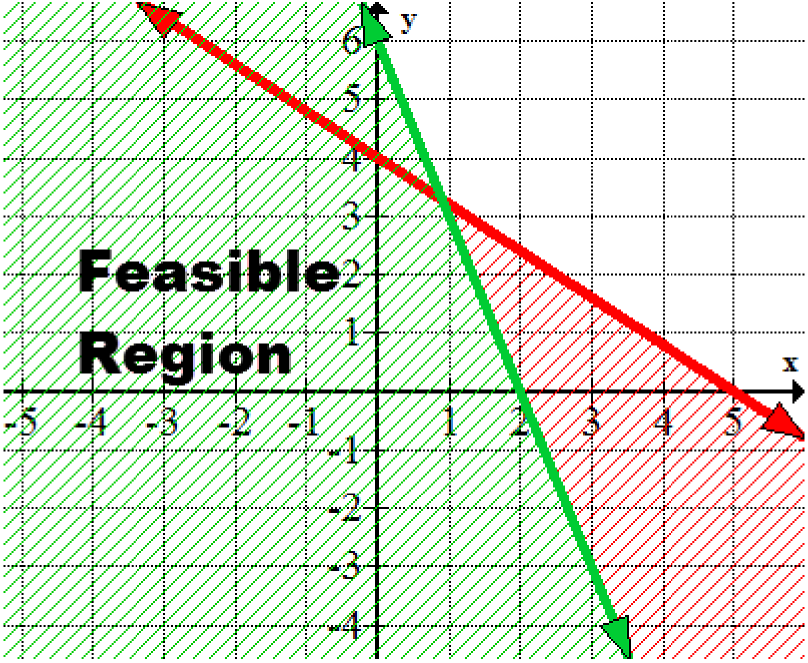
Graph: $y \le -3x+6$

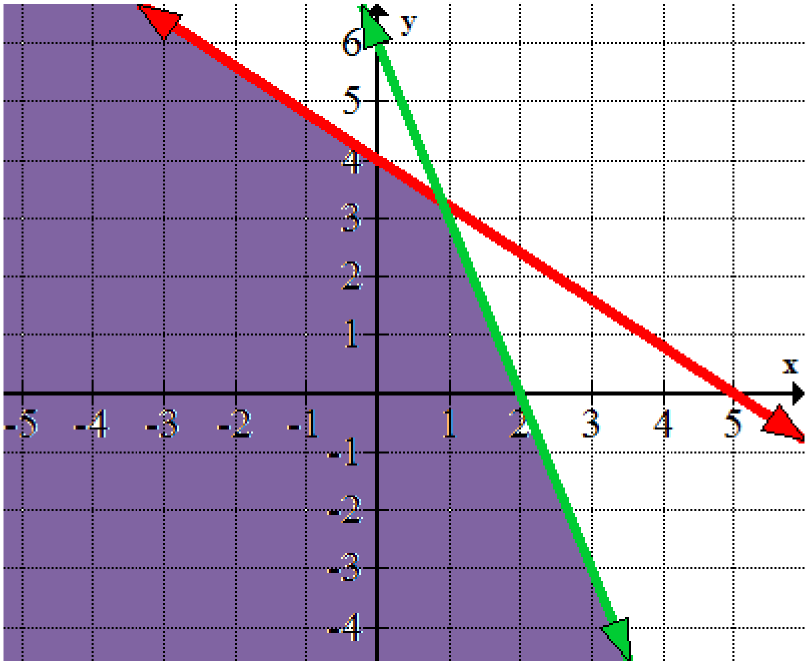


The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} 3x+y &\leq 6 \\ 3(\textcolor{red}{1})+(\textcolor{red}{1}) &\leq 6 \\ \textcolor{red}{3}+\textcolor{red}{1} &\leq 6 \\ \textcolor{red}{4} &\leq 6 \quad \text{Is it true?} \end{aligned}$$

Yes, four is less than or equal to six. The point (1, 1) satisfies the inequality and will lie within the shaded region.





The **feasible region** is the area shaded in purple.

2.

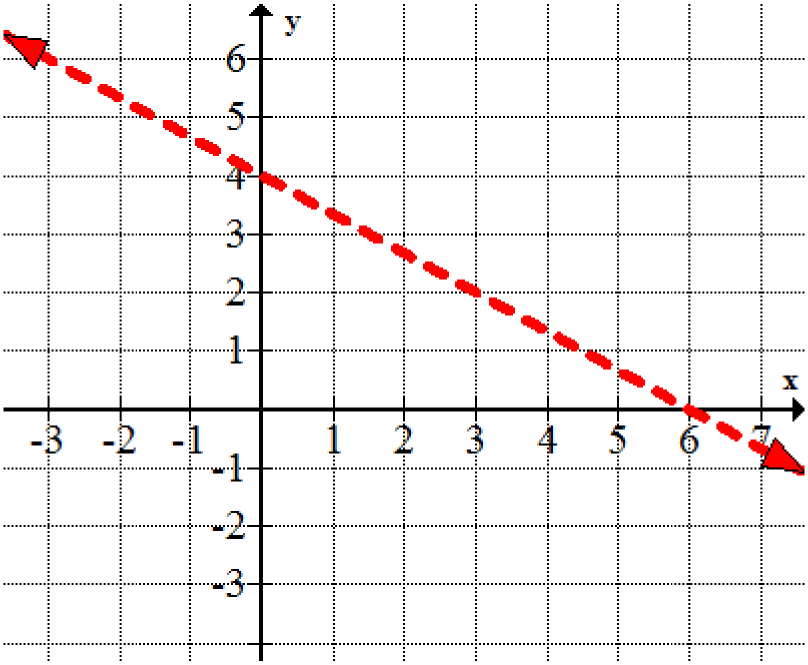
$$\begin{cases} 2x+y \le 8 \\ 2x+3y < 12 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

Write the first two inequalities in slope-intercept form.

$$\begin{aligned}
&2x+3y<12\\
&2x-2x+3y<-2x+12\\
&3y<-2x+12\\
&\frac{3y}{3}<\frac{-2x}{3}+\frac{12}{3}\\
&\frac{\cancel{3}y}{\cancel{3}}<-\frac{2}{3}x+\frac{\overset{4}{\cancel{12}}}{\cancel{3}}\\
&\boxed{y<-\frac{2}{3}x+4}
\end{aligned}$$

$$\begin{aligned}
&2x+y\le 8\\
&2x-2x+y\le -2x+8\\
&y\le -2x+8\\
&\boxed{y\le -2x+8}
\end{aligned}$$

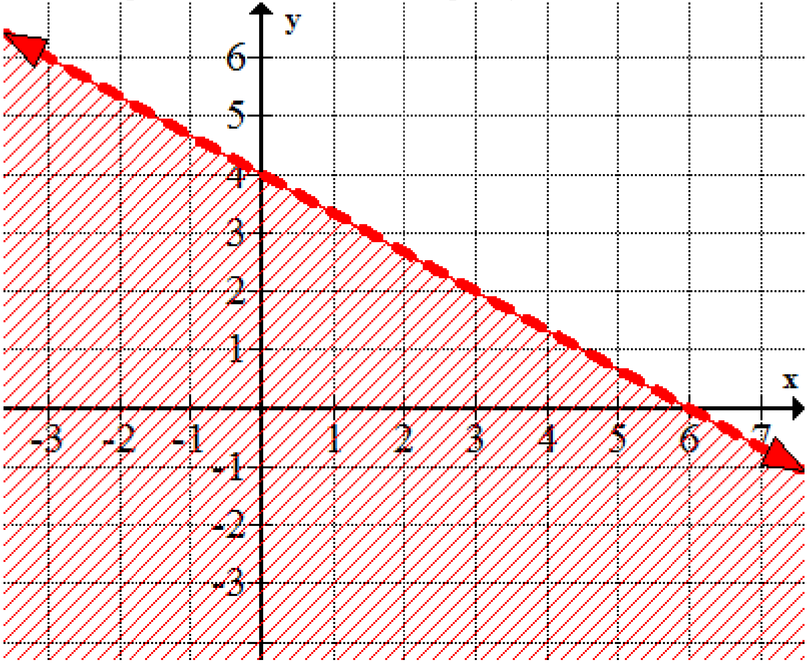
Graph: $y<-\frac{2}{3}x+4$



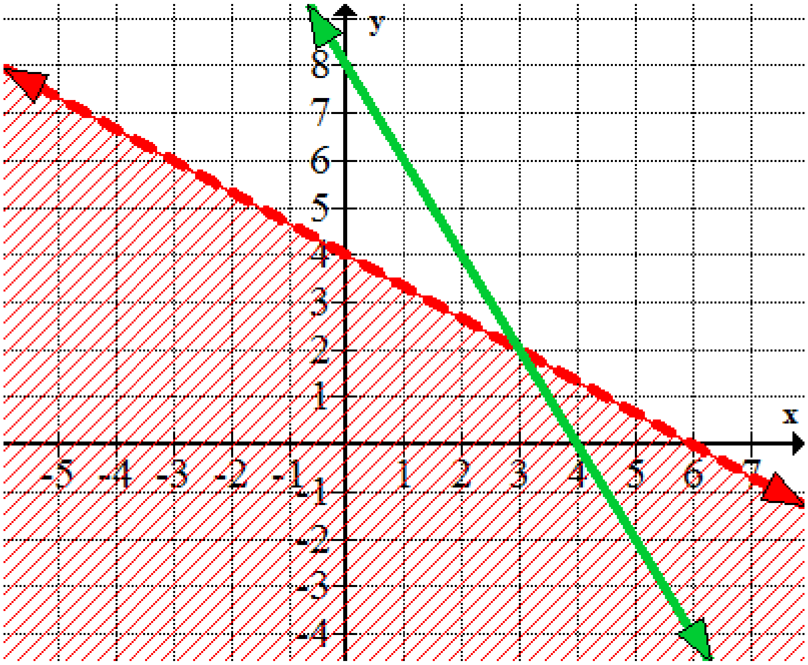
The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

$$\begin{aligned} 2x+3y &< 12 \\ 2(1)+3(1) &< 12 \\ 2+3 &< 12 \\ 5 &< 12 \quad \text{Is it true?} \end{aligned}$$

Yes, five is less than twelve. The point (1, 1) satisfies the inequality and will lie within the shaded region.



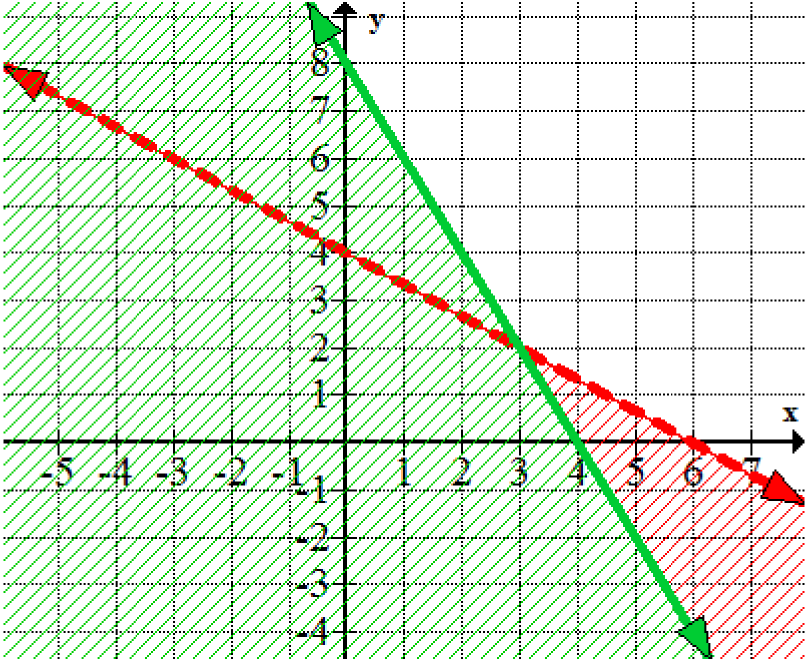
Graph: $y \leq -2x+8$



The point (1, 1) is not on the graphed line. Test the point (1, 1) to determine the location of the shading.

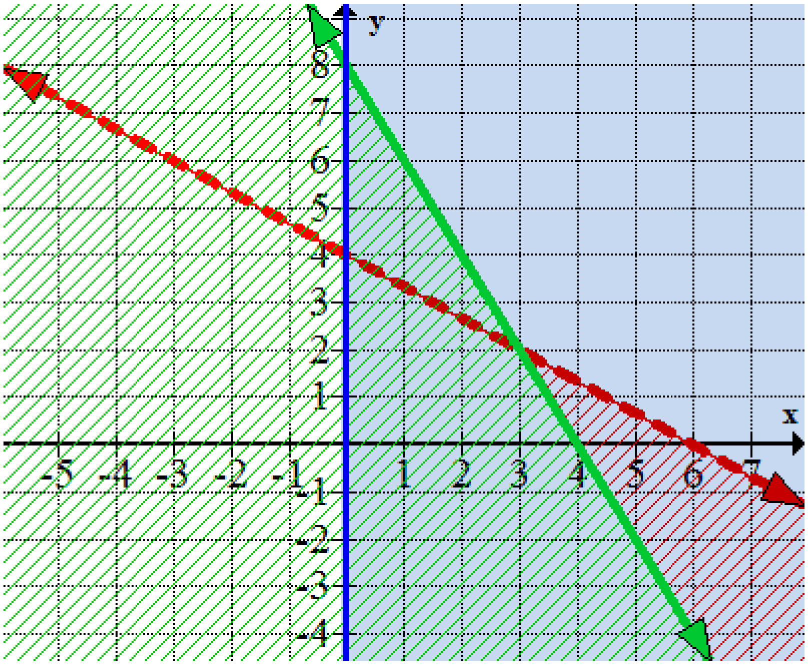
$$\begin{aligned} 2x+y &\leq 8 \\ 2({\color{red}1})+({\color{red}1}) &\leq 8 \\ {\color{red}2+1} &\leq 8 \\ {\color{red}3} &\leq 8 \quad \text{Is it true?} \end{aligned}$$

Yes, three is less than or equal to eight. The point (1, 1) satisfies the inequality and will lie within the shaded region.



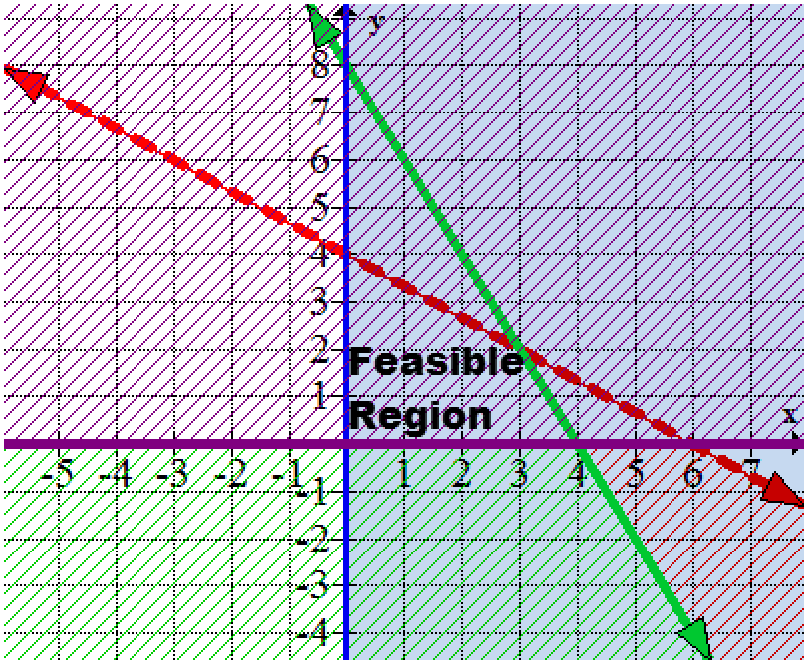
Graph: $x \geq 0$

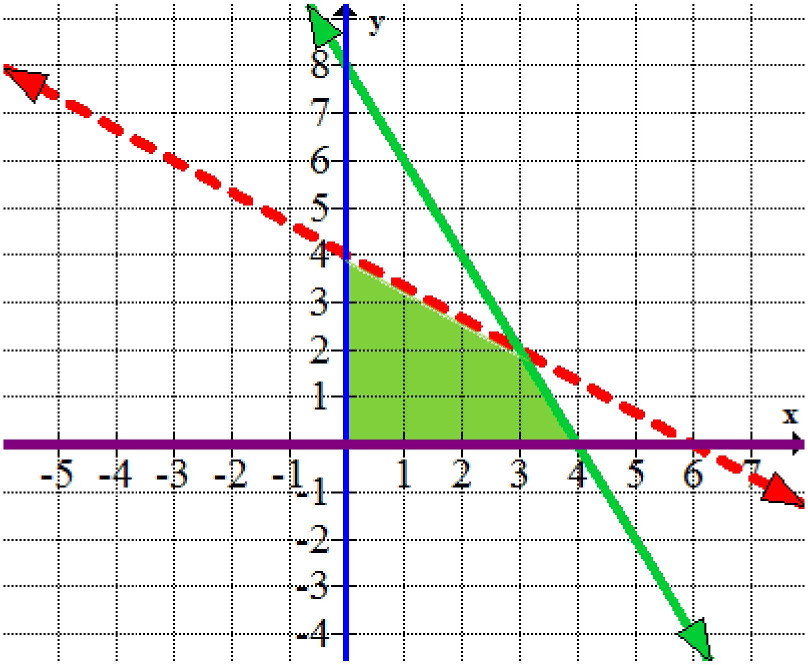
The graph will be a vertical line that will coincide with the y**-axis.** All x-values to the right of the line are greater than or equal to zero. The shaded area will be to the right of the vertical line.



Graph: $y \geq 0$

The graph will be a horizontal line that will coincide with the x**-axis**. All y-values above the line are greater than or equal to zero. The shaded area will be above the horizontal line.



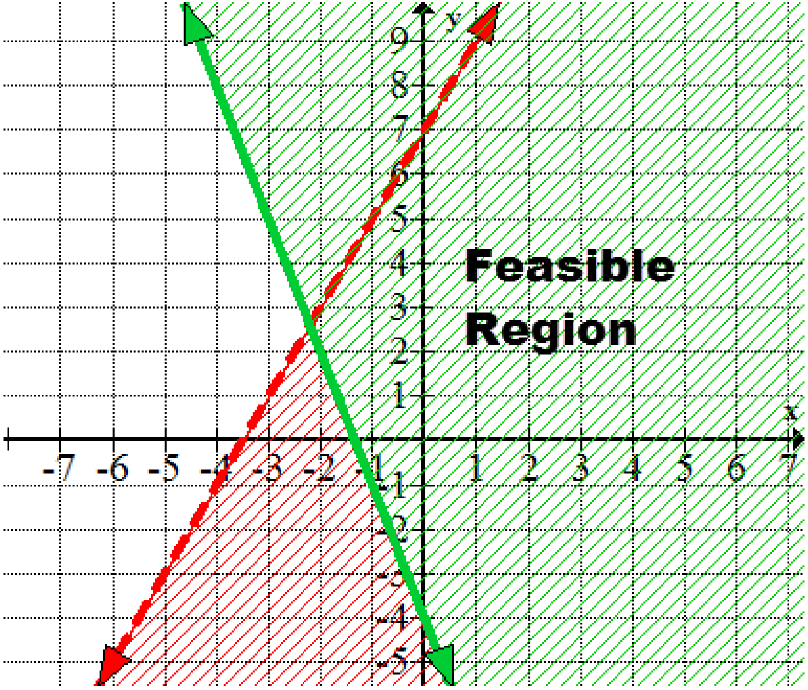


The **feasible region** is the area shaded in green.

3.

$$\begin{cases} y < 2x+7 \\ y \geq -3x-4 \end{cases}$$

Graph the system of inequalities to determine the feasible region.



Three points in the feasible region are (–1, 3); (4, –2); and (6, 5). These points will be tested in each of the linear inequalities. All of these points should satisfy both inequalities.

Test (–1, 3)

$$\begin{array}{lcl} y < 2x+7 & \textit{and} & y \geq -3x-4 \\ y < 2x+7 & & y \geq -3x-4 \\ (3) < 2(-1)+7 & & (3) \geq -3(-1)-4 \\ 3 < -2+7 & & 3 \geq 3-4 \\ 3 < 5 & & 3 \geq -1 \end{array}$$

The point (–1, 3) satisfies both inequalities. In the first inequality, three is less than five. In the second inequality three is greater than or equal to negative one. Therefore, the point lies within the feasible region and is a solution for the system of linear inequalities.

Test (4, –2)

$$\begin{array}{ll} y < 2x+7 & y \geq -3x-4 \\ (-2) < 2(4)+7 & (-2) \geq -3(4)-4 \\ -2 < 8+7 & -2 \geq -12-4 \\ -2 < 15 & -2 \geq -16 \end{array}$$

The point (4, –2) satisfies both inequalities. In the first inequality, negative two is less than fifteen. In the second inequality negative two is greater than or equal to negative sixteen. Therefore, the point lies within the feasible region and is a solution for the system of linear inequalities.

Test (6, 5)

$$\begin{array}{ll} y < 2x+7 & y \geq -3x-4 \\ (5) < 2(6)+7 & (5) \geq -3(6)-4 \\ 5 < 12+7 & 5 \geq -18-4 \\ 5 < 19 & 5 \geq -22 \end{array}$$

The point (6, 5) satisfies both inequalities. In the first inequality, five is less than nineteen. In the second inequality five is greater than or equal to negative twenty-two. Therefore, the point lies within the feasible region and is a solution for the system of linear inequalities.

Practice

$$\begin{Bmatrix} 3x+5y > 15 \\ 2x-7y \leq 14 \end{Bmatrix}$$

1. Solve the above system of linear inequalities by graphing.
2. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} 3x+2y \geq 10 \\ x-y < -1 \end{Bmatrix}$$

3. Solve the above system of linear inequalities by graphing.
4. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} x-y>4 \\ x+y>6 \end{Bmatrix}$$

5. Solve the above system of linear inequalities by graphing.
6. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} y>3x-2 \\ y<-2x+5 \end{Bmatrix}$$

7. Solve the above system of linear inequalities by graphing.
8. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} 3x-6y>-6 \\ 5x+9y\geq -18 \end{Bmatrix}$$

9. Solve the above system of linear inequalities by graphing.
10. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} 2x-y<4 \\ x\geq -1 \\ y\geq -2 \end{Bmatrix}$$

11. Solve the above system of linear inequalities by graphing.
12. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} 2x+y>6 \\ x+2y\geq 6 \\ x\geq 0 \\ y\geq 0 \end{Bmatrix}$$

13. Solve the above system of linear inequalities by graphing.
14. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} x\leq 3 \\ x\geq -2 \\ y\leq 4 \\ y\geq -1 \end{Bmatrix}$$

15. Solve the above system of linear inequalities by graphing.
16. Determine three points that satisfy the system of linear inequalities.

$$\begin{Bmatrix} x+y>-1 \\ 3x-2y\geq 2 \\ x<3 \\ y\geq 0 \end{Bmatrix}$$

17. Solve the above system of linear inequalities by graphing.
18. Determine three points that satisfy the system of linear inequalities.

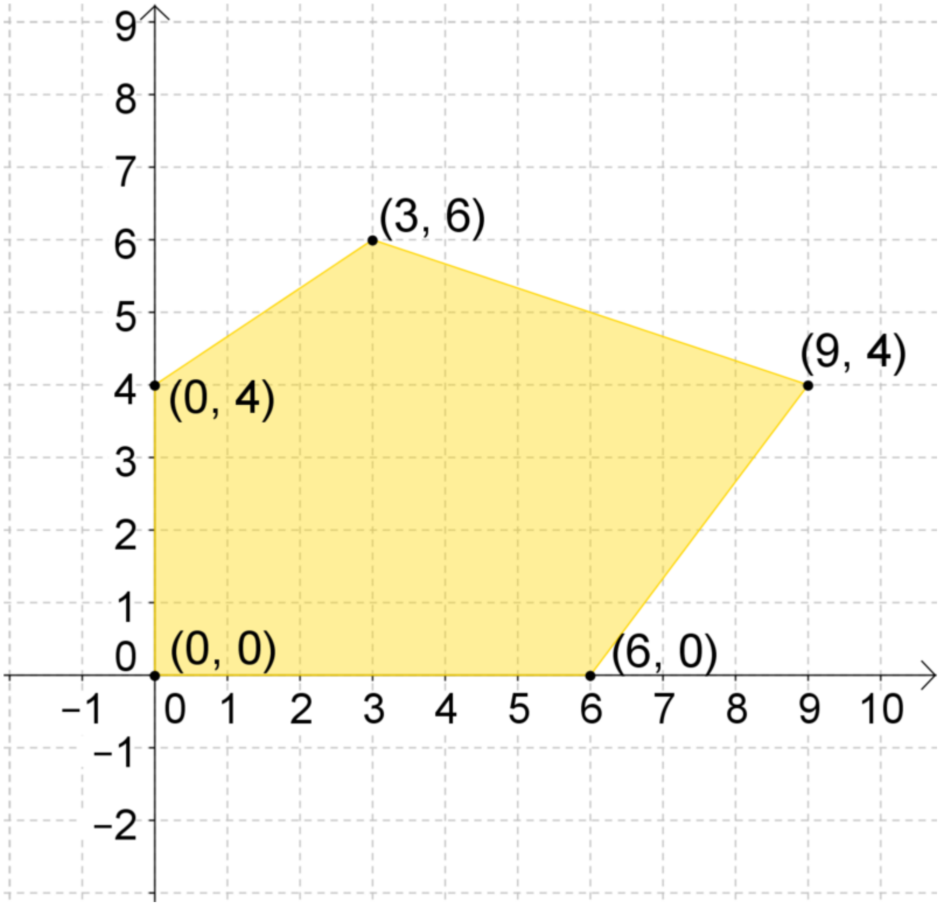
$$\begin{Bmatrix} y<x+1 \\ y\geq -2x+3 \\ y>0 \end{Bmatrix}$$

19. Solve the above system of linear inequalities by graphing.
20. Determine three points that satisfy the system of linear inequalities.

5.7 Applications of Systems of Inequalities

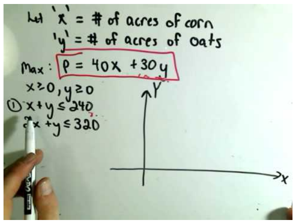
Here you'll learn how to use systems of linear inequalities to solve real-world problems.

The following diagram shows a feasible region solution for a system of linear inequalities.



If $z = 2x + 3y$, at what point on the graph is the value of z the largest?

Watch This



MEDIA

Click image to the left for more content.

Guidance

A system of linear inequalities is often used to determine the maximum or minimum values of a situation with multiple constraints. For example, you might be determining how many of a product should be produced to maximize a profit.

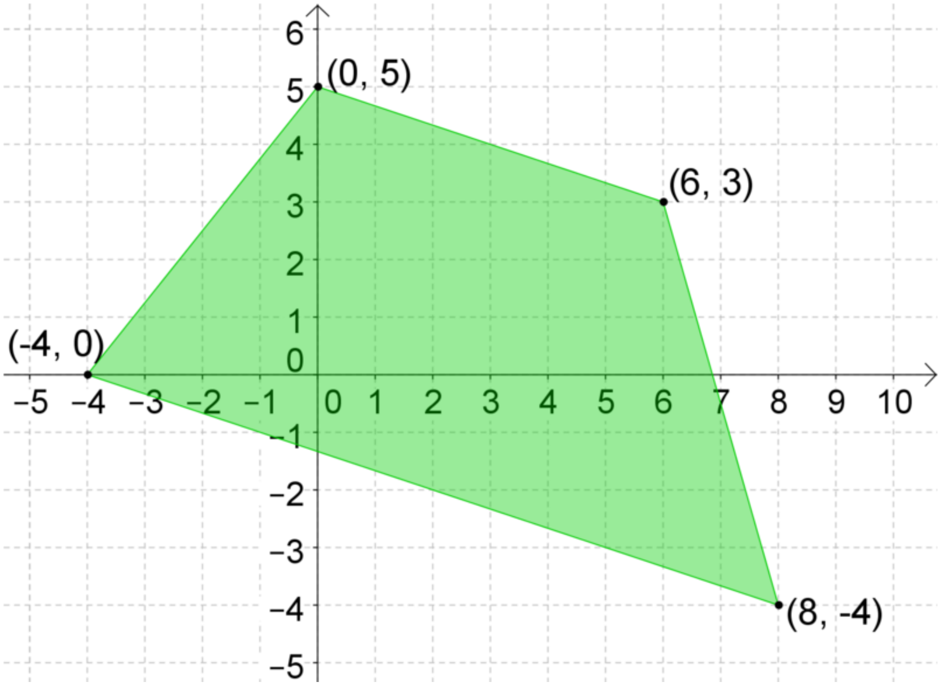
In order to solve this type of problem using linear inequalities, follow these steps:

- Step 1: Make a table to organize the given information.
- Step 2: List the constraints of the situation. Write an inequality for each constraint.
- Step 3: Write an equation for the quantity you are trying to maximize (like profit) or minimize (like cost).
- Step 4: Graph the constraints as a system of inequalities.
- Step 5: Find the exact coordinates for each vertex from the graph or algebraically.
- Step 6: Use the Vertex Theorem. Test all vertices of the feasible region in the equation and see which point is the maximum or minimum.

Look at the examples below to see how this process works. Example A allows you to see Steps 5 and 6. Examples B and C allow you to see all the steps in the process.

Example A

Evaluate the expression $z = 3x + 4y$ for the given feasible region to determine the point at which z has a maximum value and the point at which z has a minimum value.



Solution:

$(-4,0)$	$z=3x+4y \rightarrow z=3(-4)+4(0) \rightarrow z=-12+0 \rightarrow z=-12$ Therefore $3x+4y=-12$
$(0,5)$	$z=3x+4y \rightarrow z=3(0)+4(5) \rightarrow z=0+20 \rightarrow z=20$ Therefore $3x+4y=20$
$(6,3)$	$z=3x+4y \rightarrow z=3(6)+4(3) \rightarrow z=18+12 \rightarrow z=30$ Therefore $3x+4y=30$
$(8,-4)$	$z=3x+4y \rightarrow z=3(8)+4(-4) \rightarrow z=24-16 \rightarrow z=8$ Therefore $3x+4y=8$

The maximum value of z occurred at the vertex (6, 3). The minimum value of z occurred at the vertex (–4, 0).

Note: Using the vertices of the feasible region to determine the maximum or the minimum value is a branch of mathematics known as ***linear programming***. Linear programming is a technique used by businesses to solve problems. The types of problems that usually employ linear programming are those where the profit is to be maximized and those where the expenses are to be minimized. However, linear programming can also be used to solve other types of problems. The solution provides the business with a program to follow to obtain the best results for the company.

Example B

A company that produces flags makes two flags for Nova Scotia-the traditional blue flag and the green flag for Cape Breton. To produce each flag, two types of material, nylon and cotton, are used. The company has 450 units of nylon in stock and 300 units of cotton. The traditional blue flag requires 6 units of nylon and 3 units of cotton. The Cape Breton flag requires 5 units of nylon and 5 units of cotton. Each blue flag that is made realizes a profit of \$12 for the company, whereas each Cape Breton flag realizes a profit of \$15. For the nylon and cotton that the company currently has in stock, how many of each flag should the company make to maximize their profit?

Solution: Let 'x' represent the number of blue flags. Let 'y' represent the number of green flags.

Step 1: Transfer the information presented in the problem to a table.

TABLE 5.1:

	Units Required per Blue Flag	Units Required Per Green Flag	Units Available
Nylon	6	5	450
Cotton	3	5	300
Profit(per flag)	\$12	\$15	

The information presented in the problem identifies the restrictions or conditions on the production of the flags. These restrictions are known as ***constraints*** and are written as inequalities to represent the information presented in the problem.

Step 2: From the information (now in the table), list the constraints.

- The number of blue flags that are produced must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of green flags that are produced must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total number of units of nylon required to make both types of flags cannot exceed 450. Therefore, the constraint is

$$\boxed{6x + 5y \leq 450}$$

.

- The total number of units of cotton required to make both types of flags cannot exceed 300. Therefore, the constraint is

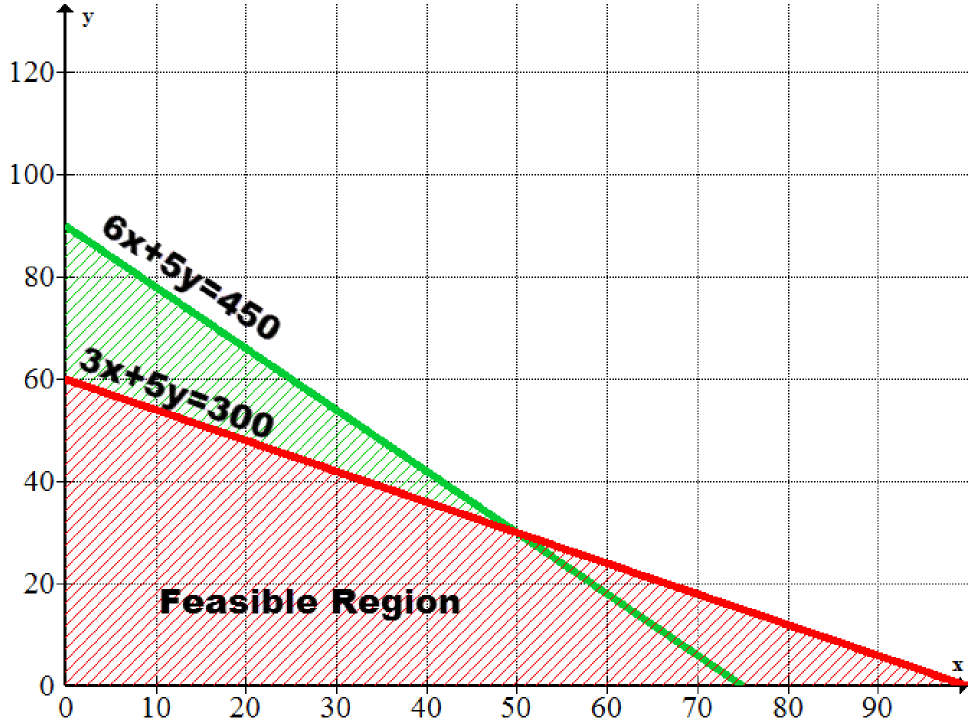
$$\boxed{3x + 5y \leq 300}$$

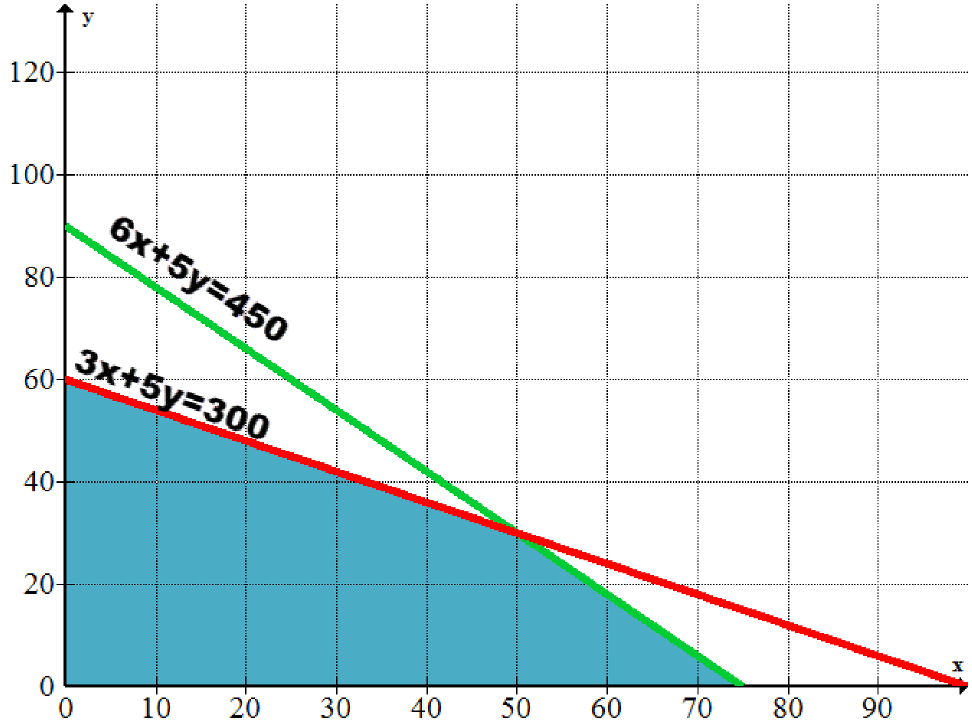
.

Step 3: Write an equation to identify the profit.

$$\boxed{P = 12x + 15y}$$

Step 4: Graph the listed constraints to identify the feasible region.





The feasible region is the area shaded in teal blue.

Step 5: Algebraically, determine the exact point of intersection between the constraints. Also, the x-intercept of the feasible region must be calculated. Write the constraints as linear equations and solve the system by elimination.

$$\begin{array}{llllllll}
6x+5y=450 & \rightarrow & 6x+5y=450 & \rightarrow & 6x+\cancel{5y}=450 & & 6x+5y=450 \\
3x+5y=300 & & -1(3x+5y=300) & \rightarrow & \underline{-3x-\cancel{5y}=-300} & & 6({\color{red}50})+5y=450 \\
 & & & & 3x=150 & \rightarrow & {\color{red}300}+5y=450 \\
 & & & & \dfrac{\cancel{3}x}{\cancel{3}}=\dfrac{\overset{\color{red}50}{\cancel{150}}}{\cancel{3}} & & 300{\color{red}-300}+5y=450{\color{red}-300} \\
 & & & & x=50 & & 5y=150 \\
 & & & & & & \dfrac{\cancel{5}y}{\cancel{5}}=\dfrac{\overset{\color{red}30}{\cancel{150}}}{\cancel{5}} \\
 & & & & & & y=30
\end{array}$$

$$\boxed{l_1 \cap l_2 @ (50,30)}$$

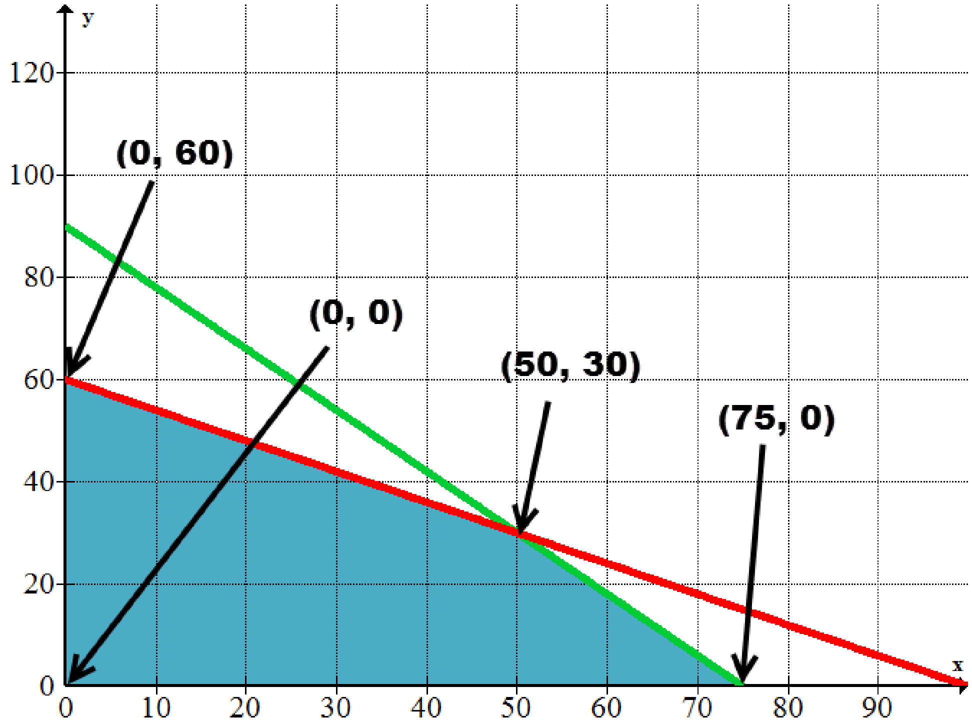
The x-intercept for the inequality $6x+5y \leq 450$ must be calculated. Write the inequality as a linear equation. Set 'y' equal to zero and solve the equation for 'x'.

$$\begin{aligned}6x+5y&=450\\6x+5(0)&=450\\6x&=450\\\frac{\cancel{6}x}{\cancel{6}}&=\frac{\overset{75}{\cancel{450}}}{\cancel{6}}\\x&=75\end{aligned}$$

The *x*-intercept of the feasible region is (75, 0).

The *y*-intercept is (0, 60). This point was plotted when the inequalities were put into slope-intercept form for graphing.

The following graph shows the vertices of the polygon that encloses the feasible region.



Step 6: Calculate the profit, using the profit equation, for each vertex of the feasible region:

$(0,0)$	$P = 12x + 15y \rightarrow P = 12(0) + 15(0) \rightarrow P = 0 + 0 \rightarrow P = 0$ Therefore $12x + 15y = \$0$
$(0,60)$	$P = 12x + 15y \rightarrow P = 12(0) + 15(60) \rightarrow P = 0 + 900 \rightarrow P = 900$ Therefore $12x + 15y = \$900$
$(50,30)$	$P = 12x + 15y \rightarrow P = 12(50) + 15(30) \rightarrow P = 600 + 450 \rightarrow P = 1050$ Therefore $12x + 15y = \$1050$
$(75,0)$	$P = 12x + 15y \rightarrow P = 12(75) + 15(0) \rightarrow P = 900 + 0 \rightarrow P = 900$ Therefore $12x + 15y = \$900$

The maximum profit occurred at the vertex (50, 30). This means, with the supplies in stock, the company should make 50 blue flags and 30 green flags to maximize their profit.

Example C

A local smelting company is able to provide its customers with iron, lead and copper by melting down either of two ores, A or B. The ores arrive at the company in railroad cars. Each railroad car of ore A contains 3 tons of iron, 3 tons of lead and I ton of copper. Each railroad car of ore B contains 1 ton of iron, 4 tons of lead and 3 tons of copper. The smelting receives an order for 7 tons of iron, 19 tons of lead and 8 tons of copper. The cost to purchase and process a carload of ore A is $7000 while the cost for ore B is $6000. If the company wants to fill the order at a minimum cost, how many carloads of each ore must be bought?

Solution: Let 'x' represent the number of carloads of ore A to purchase. Let 'y' represent the number of carloads of ore B to purchase.

Step 1: Transfer the information presented in the problem to a table.

TABLE 5.2:

	One Carload of ore A	**One Carload of ore B**	**Number of tons to fill the order**
Tons of Iron	**3**	**1**	**7**
Tons of Lead	**3**	**4**	**19**
Tons of Copper	**1**	**3**	**8**

Step 2: From the information, list the constraints.

- The number of carloads of ore A that must be bought is either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of carloads of ore B that must be bought is either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total number of tons of iron from ore A and ore B must be greater than or equal to the 7 tons needed to fill the order. Therefore, the constraint is

$$\boxed{3x+y \geq 7}$$

.

- The total number of tons of lead from ore A and ore B must be greater than or equal to the 20 tons needed to fill the order. Therefore, the constraint is

$$\boxed{3x+4y \geq 19}$$

.

- The total number of tons of copper from ore A and ore B must be greater than or equal to the 8 tons needed to fill the order. Therefore, the constraint is

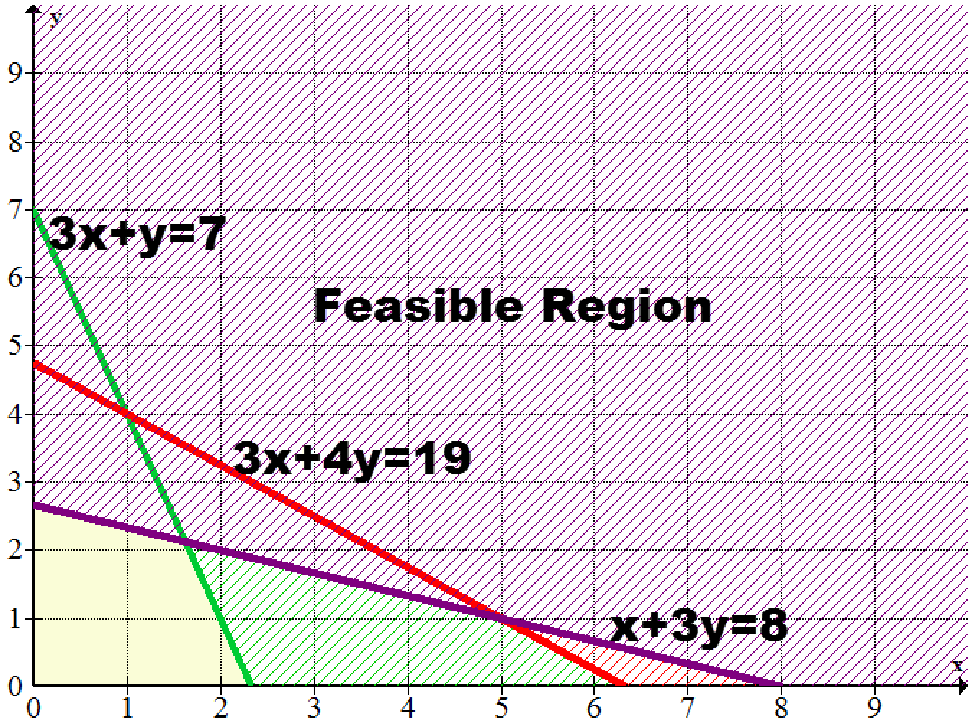
$$\boxed{x+3y \geq 8}$$

.

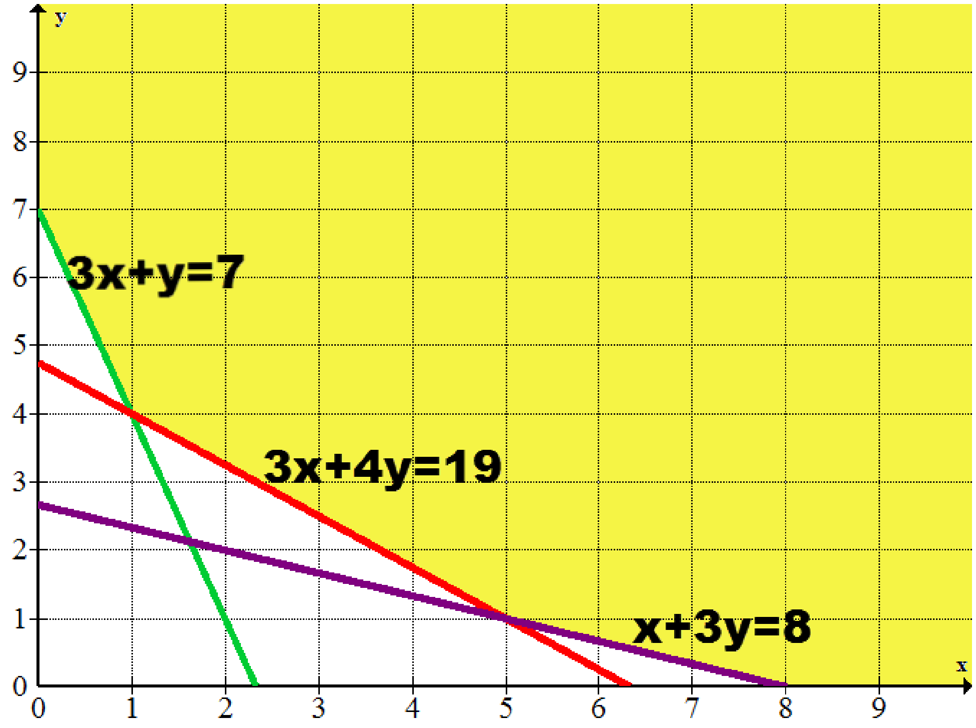
Step 3: Write an equation to represent the cost in dollars of x carloads of ore A and y carloads of ore B.

$$\boxed{c = 7000x+6000y}$$

Step 4: Graph the listed constraints to identify the feasible region.



The feasible region shows that there are an infinite number of ways to fill the order. The feasible region is the large shaded area that is sitting above the graphed lines.



Step 5: Algebraically, determine the exact point of intersection between the constraints. Also, the *x*-intercept of the feasible region must be calculated. Write the constraints as linear equations and solve the system by elimination.

$$\begin{array}{llllll} 3x+y=7 & \rightarrow & -1(3x+y=7) & \rightarrow & -\cancel{3x}-y=-7 & 3x+4y=19 \\ 3x+4y=19 & & 3x+4y=19 & \rightarrow & \underline{\cancel{3x}+4y=19} & 3x+4(4)=19 \\ & & & & 3y=12 \quad \rightarrow & 3x+16=19 \\ & & & & \frac{\cancel{3}y}{\cancel{3}}=\frac{\overset{4}{\cancel{12}}}{\cancel{3}} & 3x+16-16=19-16 \\ & & & & y=4 & 3x=3 \\ & & & & & \frac{\cancel{3}x}{\cancel{3}}=\frac{\overset{1}{\cancel{3}}}{\cancel{3}} \\ & & & & & x=1 \end{array}$$

$$\boxed{l_1 \cap l_2 @ (1,4)}$$

$$\begin{array}{lllll}
3x+4y=19 \rightarrow & 3x+4y=19 & \rightarrow & \cancel{3x}+4y=19 & 3x+4y=19 \\
x+3y=8 & -3(x+3y=8) & \rightarrow & \underline{\cancel{-3x}-9y=-24} & 3x+4(1)=19 \\
 & & & -5y=-5 \quad \rightarrow & 3x+4=19 \\
 & & & \dfrac{\cancel{-5}y}{\cancel{-5}}=\dfrac{\overset{1}{\cancel{-5}}}{\cancel{-5}} & 3x+4-4=19-4 \\
 & & & y=1 & 3x=15 \\
 & & & & \dfrac{\cancel{3}x}{\cancel{3}}=\dfrac{\overset{5}{\cancel{15}}}{\cancel{3}} \\
 & & & & x=5
\end{array}$$

$$\boxed{l_2 \cap l_3 @ (5,1)}$$

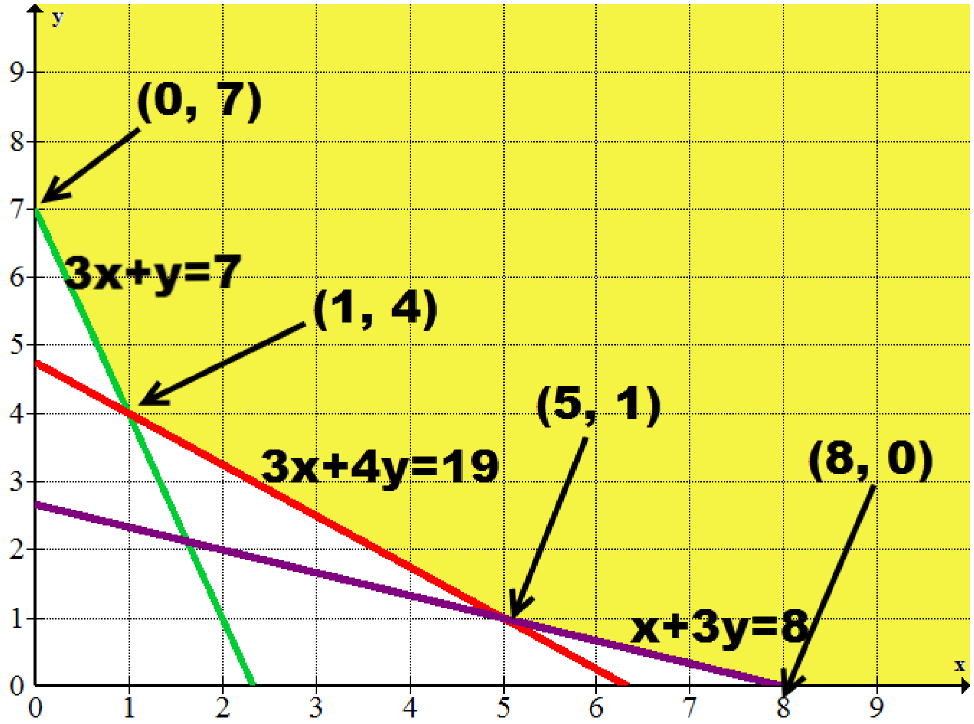
The x-intercept for the inequality $x+3y \geq 8$ must be calculated. Write the inequality as a linear equation. Set 'y' equal to zero and solve the equation for 'x'.

$$\begin{aligned}
x+3y&=8\\
x+3(0)&=8\\
x&=8
\end{aligned}$$

The x-intercept of the feasible region is (8, 0).

The y-intercept is (0, 7). This point was plotted when the inequalities were put into slope-intercept form for graphing.

The following graph shows the vertices of the region borders the feasible region.



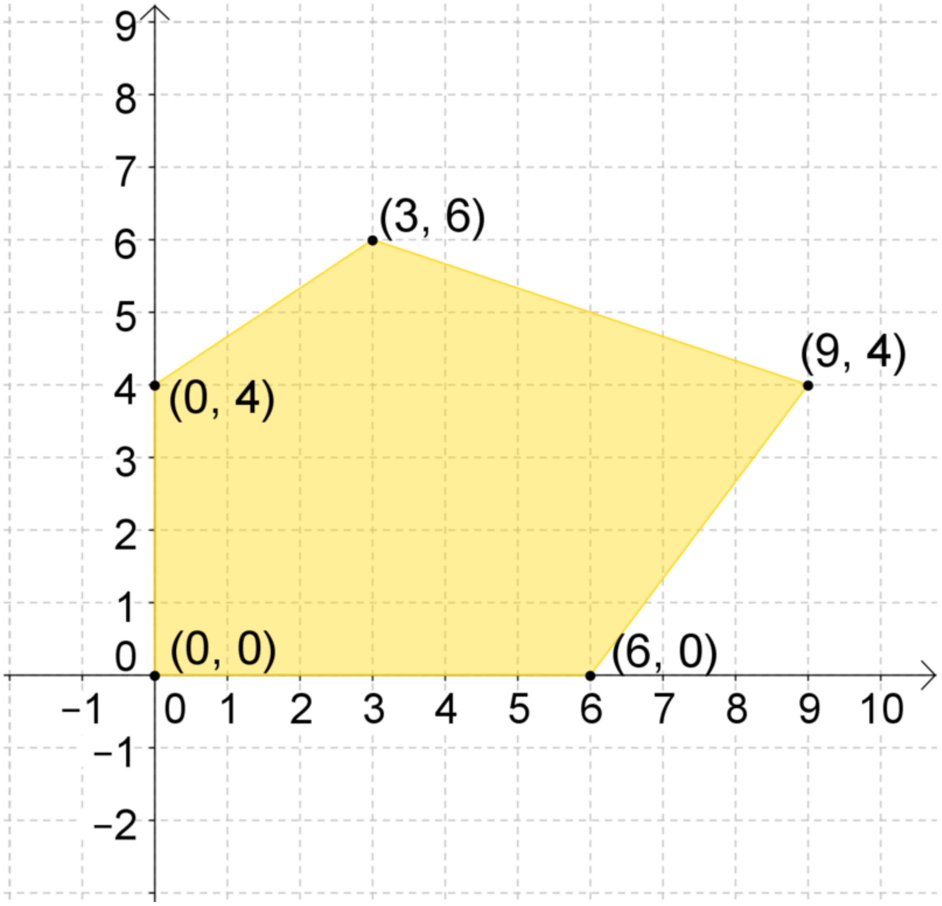
Step 6: Calculate the cost, using the cost equation, for each vertex of the feasible region:

$(0,7)$	$c = 7000x + 6000y \rightarrow c = 7000(0) + 6000(7) \rightarrow c = 0 + 42,000 \rightarrow P = 42,000$ Therefore $7000x + 6000y = \$42,000$
$(1,4)$	$c = 7000x + 6000y \rightarrow c = 7000(1) + 6000(4) \rightarrow c = 7000 + 24,000 \rightarrow P = 31,000$ Therefore $7000x + 6000y = \$31,000$
$(5,1)$	$c = 7000x + 6000y \rightarrow c = 7000(5) + 6000(1) \rightarrow c = 35,000 + 6000 \rightarrow P = 41,000$ Therefore $7000x + 6000y = \$41,000$
$(8,0)$	$c = 7000x + 6000y \rightarrow c = 7000(8) + 6000(0) \rightarrow c = 56,000 + 0 \rightarrow P = 56,000$ Therefore $7000x + 6000y = \$56,000$

The minimum cost is located at the vertex (1, 4). Therefore the company should buy one carload of ore A and four carloads of ore B.

Concept Problem Revisited

The following diagram shows a feasible region that is within a polygonal region.

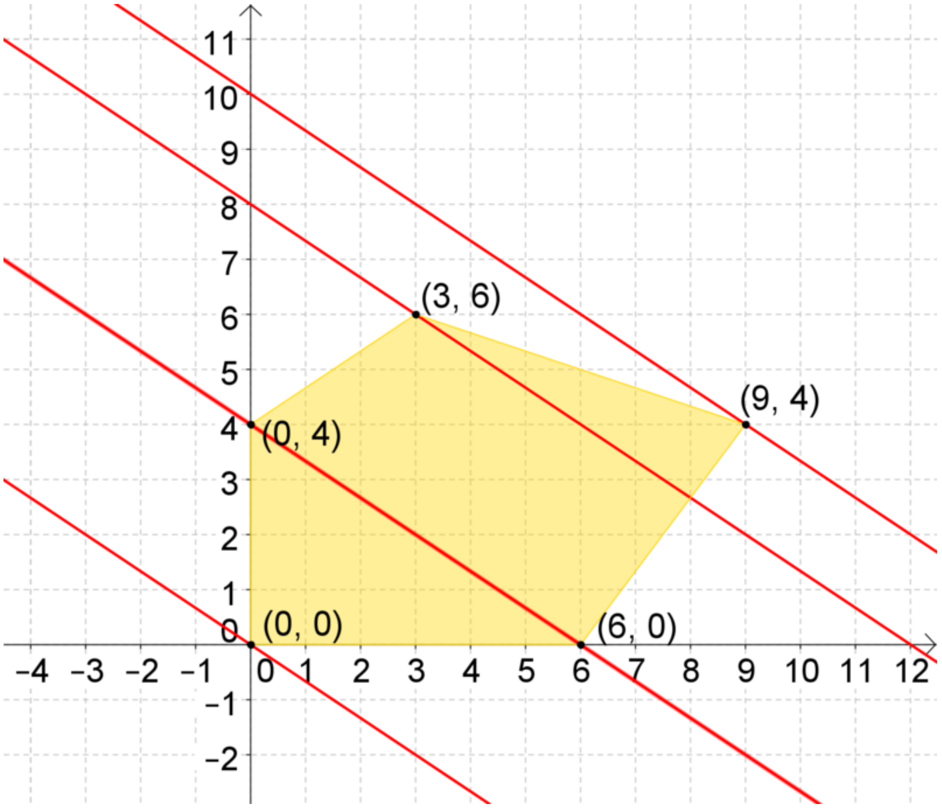


The linear function $z = 2x + 3y$ will now be evaluated for each of the vertices of the polygon.

To evaluate the value of ’z’ substitute the coordinates of the point into the expression for ’x’ and ’y’.

$(0,0)$	$z = 2x+3y \rightarrow z = 2(0)+3(0) \rightarrow z = 0+0 \rightarrow z = 0$ Therefore $2x+3y = 0$
$(0,4)$	$z = 2x+3y \rightarrow z = 2(0)+3(4) \rightarrow z = 0+12 \rightarrow z = 12$ Therefore $2x+3y = 12$
$(6,0)$	$z = 2x+3y \rightarrow z = 2(6)+3(0) \rightarrow z = 12+0 \rightarrow z = 12$ Therefore $2x+3y = 12$
$(3,6)$	$z = 2x+3y \rightarrow z = 2(3)+3(6) \rightarrow z = 6+18 \rightarrow z = 24$ Therefore $2x+3y = 24$
$(9,4)$	$z = 2x+3y \rightarrow z = 2(9)+3(4) \rightarrow z = 18+12 \rightarrow z = 30$ Therefore $2x+3y = 30$

The value of $z = 2x+3y$, for each of the vertices, remains constant along any line with a slope of $-\frac{2}{3}$. This is obvious on the following graph.



As the line moved away from the origin, the value of $z = 2x+3y$ increased. The maximum value for the shaded region occurred at the vertex (9, 4) while the minimum value occurred at the vertex (0, 0). These statements confirm the vertex theorem for a feasible region:

If a linear expression

$$\boxed{z = ax + by + c}$$

is to be evaluated for all points of a convex, polygonal region, then the maximum value of z, if one exists, will occur at one of the vertices of the feasible region. Also, the minimum value of z, if one exists, will occur at one of the vertices of the feasible region.

Vocabulary

Constraint

A ***constraint*** is a restriction or condition presented in a real-world problem. The constraints are written as inequalities and are used to solve the problem.

Linear Programming

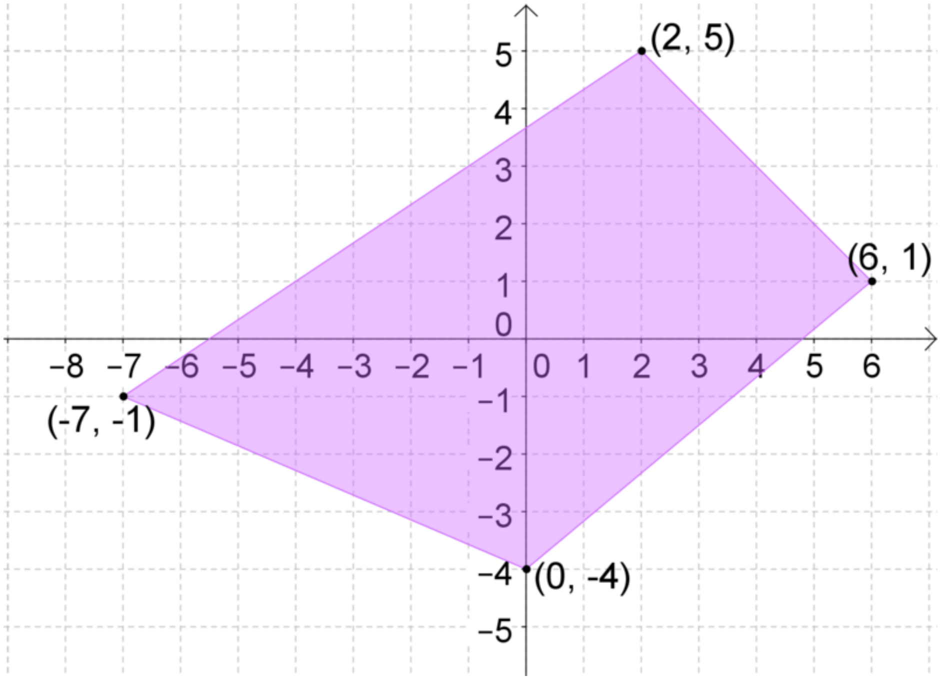
Linear programming is a branch of mathematics that uses systems of linear inequalities to solve real-world problems. The vertex theorem of regions is applied to the vertices to determine the best solution to the problem.

Vertex Theorem for Regions

The ***vertex theorem for regions*** states that if a linear expression $z = ax + by + c$ is to be evaluated for all points of a convex, polygonal region, then the maximum value of z, if one exists, will occur at one of the vertices of the feasible region. Also, the minimum value of z, if one exists, will occur at one of the vertices of the feasible region.

Guided Practice

1. For the following graphed region and the expression $z = 5x + 7y - 1$, find a point where 'z' has a maximum value and a point where 'z' has a minimum value.



2. The following table shows the time required on three machines for a company to produce Super 1 and Super 2 coffee percolators. The table also shows the amount of time that each machine is available during a one hour period.

The company is trying to determine how many of each must be made to maximize a profit if they make $30 on each Super 1 model and $35 on each Super 2 model. List the constraints and write a profit statement to represent the information.

TABLE 5.3:

	Super 1	**Super 2**	**Time Available**
Machine A	**1 minute**	**3minutes**	**24 minutes**
Machine B	**3 minutes**	**2minutes**	**36 minutes**
Machine C	**3 minutes**	**4 minutes**	**44 minutes**

3. A local paint company has created two new paint colors. The company has 28 units of yellow tint and 22 units of red tint and intends to mix as many quarts as possible of color X and color Y. Each quart of color X requires 4 units of yellow tint and 1 unit of red tint. Each quart of color Y requires 1 unit of yellow tint and 4 units of red tint. How many quarts of each color can be mixed with the units of tint that the company has available? List the constraints, complete the graph and determine the solution using linear programming.

Answers:

1. The vertices of the polygonal region are (–7, –1); (2, 5); (6, 1); and (0, –4).

$$\begin{array}{ll} (-7,-1) & z=5x+7y-1 \rightarrow z=5(-7)+7(-1)-1 \rightarrow z=-35-7-1 \rightarrow z=-43 \\ & \text{Therefore } 5x+7y-1=-43 \\ \\ (2,5) & z=5x+7y-1 \rightarrow z=5(2)+7(5)-1 \rightarrow z=10+35-1 \rightarrow z=44 \\ & \text{Therefore } 5x+7y-1=44 \\ \\ (6,1) & z=5x+7y-1 \rightarrow z=5(6)+7(1)-1 \rightarrow z=30+7-1 \rightarrow z=36 \\ & \text{Therefore } 5x+7y-1=36 \\ \\ (0,-4) & z=5x+7y-1 \rightarrow z=5(0)+7(-4)-1 \rightarrow z=0-28-1 \rightarrow z=-29 \\ & \text{Therefore } 5x+7y-1=-29 \end{array}$$

The maximum value of 'z' occurred at the vertex (2, 5). The minimum value of 'z' occurred at the vertex (–7, –1).

2. Let 'x' represent the number of Super 1 coffee percolators. Let 'y' represent the number of Super 2 coffee percolators.

- The number of Super 1 coffee percolators that are made must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{x \geq 0}$$

.

- The number of Super 2 coffee percolators that are made must be either zero or greater than zero. Therefore, the constraint is

$$\boxed{y \geq 0}$$

.

- The total amount of time that both a Super 1 and a Super 2 model can be processed on Machine A is less than or equal to 24 minutes. Therefore, the constraint is

  $$\boxed{x+3y \le 24}$$

  .

- The total amount of time that both a Super 1 and a Super 2 model can be processed on Machine B is less than or equal to 36 minutes. Therefore, the constraint is

  $$\boxed{3x+2y \le 36}$$

  .

- The total amount of time that both a Super1 and a Super 2 model can be processed on Machine C is less than or equal to 44 minutes. Therefore, the constraint is

  $$\boxed{3x+4y \le 44}$$

  .

- The profit equation is

  $$\boxed{P = 30x+35y}$$

3. **Table:**

TABLE 5.4:

	Color X	**Color Y**	**Units Available**
Yellow Tint	**4 units**	**1 unit**	**28**
Red Tint	**1 unit**	**4 units**	**22**

Constraints: Let '*x*' represent the number of quarts of Color X paint to be made. Let '*y*' represent the number of quarts of Color Y paint to be made.

- The number of quarts of Color X paint that are mixed must be either zero or greater than zero. Therefore, the constraint is

  $$\boxed{x \ge 0}$$

  .

- The number of quarts of Color Y paint that are mixed must be either zero or greater than zero. Therefore, the constraint is

  $$\boxed{y \ge 0}$$

  .

- The total amount of yellow tint that is used to mix Color X and Color Y must be less than or equal to 28. Therefore, the constraint is

  $$\boxed{4x+y \le 28}$$

  .

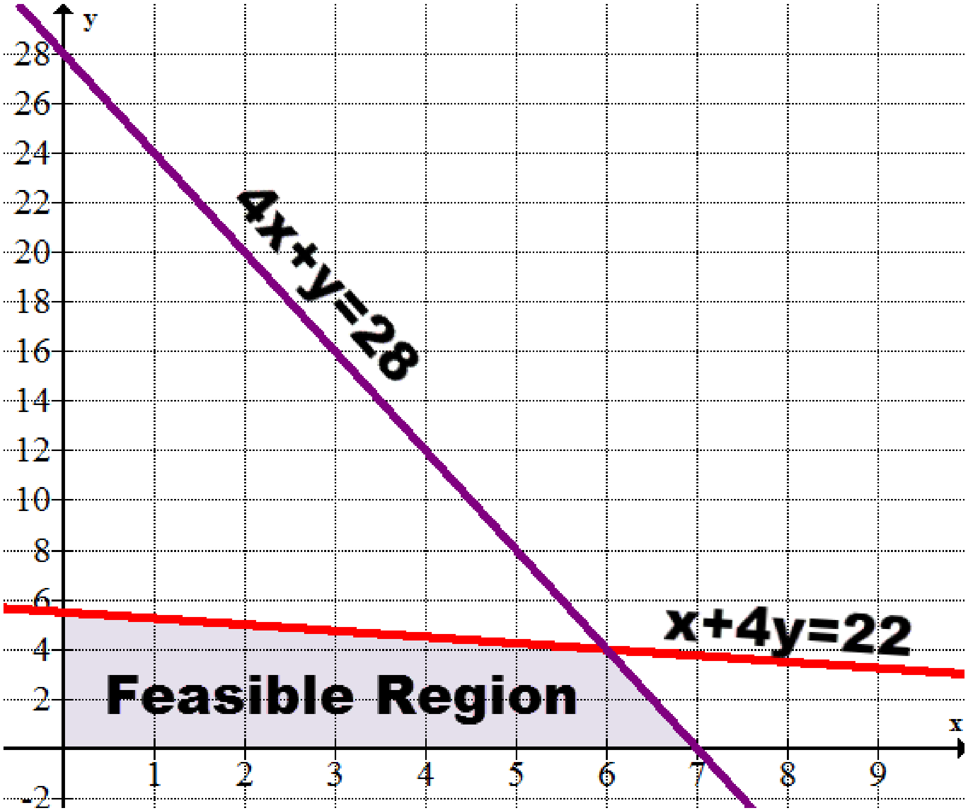
- The total amount of red tint that is used to mix Color X and Color Y must be less than or equal to 22. Therefore, the constraint is

$$\boxed{x+4y \leq 22}$$

.

Equation: The company wants to mix as many quarts as possible. They want to maximize the value of Q given by $Q = x+y$.

Graph:



Vertices:

$$\begin{array}{llllll}
4x+y=28 & \rightarrow & 4x+y=28 & \rightarrow & \cancel{4x}+y=28 & 4x+y=28 \\
x+4y=22 & & -4(x+4y=22) & \rightarrow & \underline{-\cancel{4x}-16y=-88} & 4x+(4)=28 \\
 & & & & -15y=-60 \quad \rightarrow & 4x=24 \\
 & & & & \frac{\cancel{-15}y}{\cancel{-15}} = \frac{\overset{4}{\cancel{-60}}}{\cancel{-15}} & \frac{\cancel{4}x}{\cancel{4}} = \frac{\overset{6}{\cancel{24}}}{\cancel{4}} \\
 & & & & y=4 & x=6
\end{array}$$

$$\boxed{l_1 \cap l_2 @ (6,4)}$$

The three points in the feasible region are $(6,4),(7,0),(0,5.5)$. The company wants to maximize $Q = x+y$. The point that produces the maximum value is $(6,4)$.

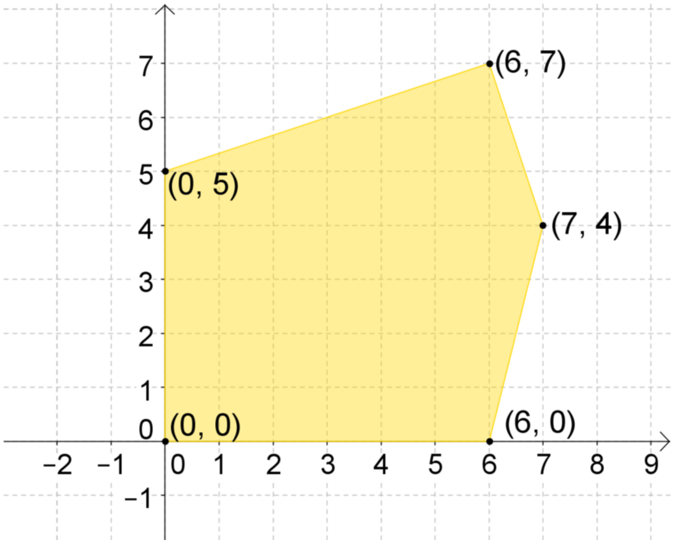
The company should mix **6** quarts of Color X paint and **4** quarts of Color Y paint.

Practice

For each graphed region and corresponding equation, find a point at which ’z’ has a maximum value and a point at which ’z’ has a minimum value.

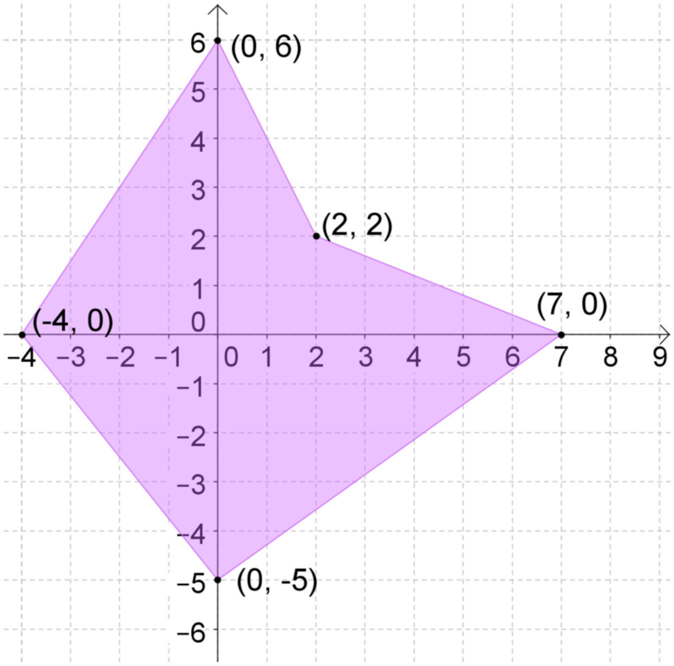
1.

$\boxed{z = 7x - 2y}$



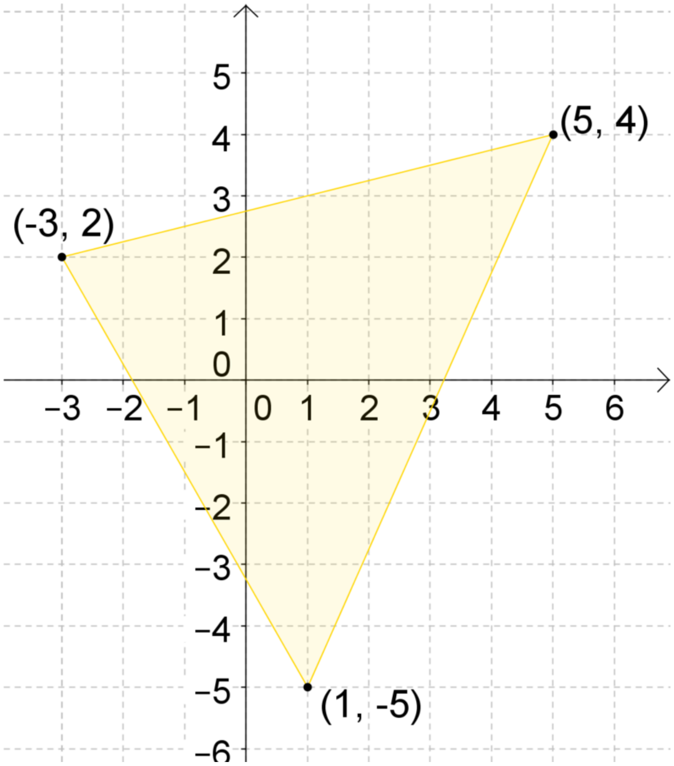
2.

$\boxed{z = 3y - 4x}$



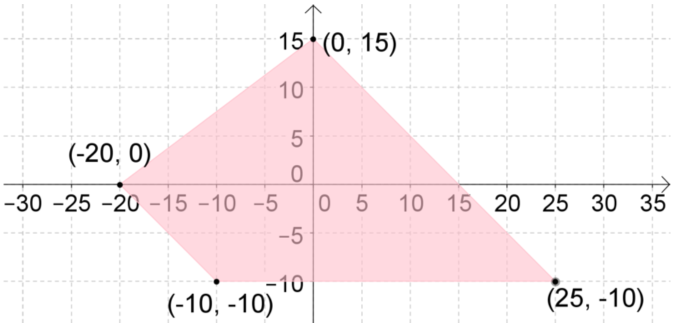
3.

$\boxed{z = 4x - 2y}$



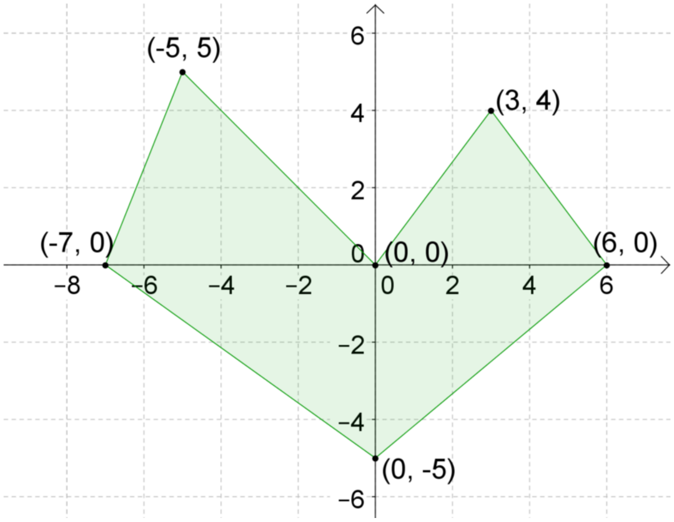
4.

$\boxed{z = 10x + 20y}$



5.

$\boxed{z = 20x - 15y + 4}$



A small manufacturing company makes \$125 on each DVD player it produces and \$100 profit on each color TV set it makes. Each DVD player and each TV must be processed by a cutting machine (A), a fitting machine (B) and a polishing machine (C). Each DVD player must be processed on Machine A for one hour, on Machine B for one hour and on Machine C for four hours. Each TV set must be processed on Machine A for two hours, on Machine B for one hour and on Machine C for one hour. Machines A, B, and C are available for 16, 9, and 24 hours per day respectively. How many DVD players and TV sets must be made each day to maximize the profit?

6. List the constraints and state the profit equation.
7. Create a graph and identify the feasible region.
8. Determine what the company must do to maximize their profit.

April has a small business during the winter months making hats and scarves. A hat requires 2 hours on Machine A, 4 hours on Machine B and 2 hours on Machine C. A scarf requires 3 hours on Machine A, 3 hours on Machine B and 1 hour on Machine C. Machine A is available 36 hours each week, Machine B is available 42 hours each week and Machine C is available 20 hours each week. The profit on a hat is \$7.00 and the profit on a scarf is \$4.00. How many of each should be made each week to maximize the profit?

9. List the constraints and state the profit equation.
10. Create a graph and identify the feasible region.
11. Determine what the April must do to maximize her profit.

Beth is knitting mittens and gloves. Each pair must be processed on three machines. Each pair of mittens requires 2 hours on Machine A, 2 hours on Machine B and 4 hours on Machine C. Each pair of gloves requires 4 hours on Machine A, 2 hours on Machine B and 1 hour on Machine C. Machine A, B, and C are available 32, 18 and 24 minutes each day respectively. The profit on a pair of mittens is \$8.00 and on a pair of gloves is \$10.00. How many pairs of each should be made each day to maximize the profit?

12. List the constraints and state the profit equation.
13. Create a graph and identify the feasible region.
14. Determine what the Beth must do to maximize her profit.

A patient is prescribed a pill that contains vitamins A, B and C. These vitamins are available in two different brands of pills. The first type is called Brand X and the second type is called Brand Y. The following table shows the amount of each vitamin that a Brand X and a Brand Y pill contain. The table also shows the minimum daily requirement needed by the patient. Each Brand X pill costs \$0.32 and each Brand Y pill costs \$0.29. How many pills of each brand should the patient take each day to minimize the cost?

TABLE 5.5:

	Brand X	**Brand Y**	**Minimum Daily Requirement**
Vitamin A	**2mg**	**1mg**	**5mg**
Vitamin B	**3mg**	**3mg**	**12mg**
Vitamin C	**25mg**	**50mg**	**125mg**

15. List the constraints and state the cost equation.
16. Create a graph and identify the feasible region.
17. Determine what the patient must do to minimize his/her cost.

A local smelting company is able to provide its customers with lead, copper and iron by melting down either of two ores, X or Y. The ores arrive at the company in railroad cars. Each railroad car of ore X contains 5 tons of lead, 1 ton of copper and 1 ton of iron. Each railroad car of ore Y contains 1 ton of lead, 1 ton of copper and 2 tons of iron. The smelting company receives an order and must make at least 20 tons of lead, 12 tons of copper and 20 tons of iron. The cost to purchase and process a carload of ore X is $6000 while the cost for ore Y is $5000. If the company wants to fill the order at a minimum cost, how many carloads of each ore must be bought?

18. List the constraints and state the cost equation.
19. Create a graph and identify the feasible region.
20. Determine what the company must do to minimize their cost.

Summary

You learned that a *system* of equations or inequalities means more than one equation or inequality.

To solve a system of equations you can graph the system and look for the point of intersection or use one of two algebraic methods (substitution or elimination). Sometimes a system of equations has no solution because the two lines are parallel. Other times the system has an infinite number of solutions because the lines coincide.

A linear inequality appears as a region in the Cartesian plane. To solve a system of linear inequalities, graph both and look for where their solution regions overlap. This region is often called the feasible region. Systems of linear inequalities can help you solve problems where you have multiple constraints on different variables and you are trying to figure out how to maximize or minimize something (like profit or cost). The maximum or minimum values will occur at one of the vertices of the feasible region according to the Vertex Theorem. These types of problems are sometimes referred to as linear programming problems.

CHAPTER 6

Exponents and Exponential Functions

Chapter Outline

Introduction

Here you'll learn all about *exponents* in algebra. You will learn the properties of exponents and how to simplify exponential expressions. You will learn how exponents can help you write very large or very small numbers with scientific notation. You will also learn how to solve different types of exponential equations where the variable appears as the exponent or the base. Finally, you will explore different types of exponential functions of the form $y = b^x, y = ab^x, y = ab^{\frac{x}{c}}$, and $y = ab^{\frac{x}{c}} + d$ as well as applications of exponential functions.

6.1 Product Rules for Exponents

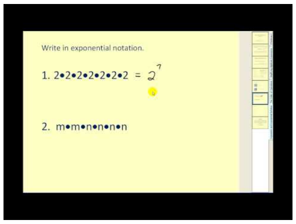
Here you'll learn how to multiply two terms with the same base and how to find the power of a product.

Suppose you have the expression:

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$

How could you write this expression in a more concise way?

Watch This



Write in exponential notation.

1. 2•2•2•2•2•2•2 = 2^7

2. m•m•n•n•n•n

MEDIA

Click image to the left for more content.

James Sousa: Exponential Notation

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and products.

RULE: To multiply two terms with the same base, add the exponents.

$$a^m \times a^n = \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ m \text{ factors}}} \underbrace{(a \times a \times \ldots \times a)}_{\substack{\downarrow \\ n \text{ factors}}}$$

$$a^m \times a^n = \underbrace{(a \times a \times a \ldots \times a)}_{\substack{\downarrow \\ m+n \text{ factors}}}$$

$$a^m \times a^n = a^{m+n}$$

RULE: To raise a product to a power, raise each of the factors to the power.

$$
\begin{aligned}
(ab)^n &= \underbrace{(ab)\times(ab)\times\ldots\times(ab)}_{\substack{\downarrow \\ n \text{ factors}}} \\
(ab)^n &= \underbrace{(a\times a\times\ldots\times a)}_{\substack{\downarrow \\ n \text{ factors}}}\times\underbrace{(b\times b\times\ldots\times b)}_{\substack{\downarrow \\ n \text{ factors}}} \\
(ab)^n &= a^n b^n
\end{aligned}
$$

Example A

Simplify $3^2 \times 3^3$.

Solution:

$3^2 \times 3^3$	The base is 3.
3^{2+3}	Keep the base of 3 and add the exponents.
3^5	This answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$
\begin{aligned}
&3^5 = 3\times3\times3\times3\times3 \\
&3^5 = 243 \\
&\boxed{3^2\times3^3 = 3^5 = 243}
\end{aligned}
$$

Example B

Simplify $(x^3)(x^6)$.

Solution:

$(x^3)(x^6)$	The base is x.
x^{3+6}	Keep the base of x and add the exponents.
x^9	The answer is in exponential form.
$\boxed{(x^3)(x^6) = x^9}$	

Example C

Simplify $y^5 \cdot y^2$.

Solution:

$y^5 \cdot y^2$	The base is y.
y^{5+2}	Keep the base of y and add the exponents.
y^7	The answer is in exponential form.
$\boxed{y^5 \cdot y^2 = y^7}$	

Example D

Simplify $5x^2y^3 \cdot 3xy^2$.

Solution:

$5x^2y^3 \cdot 3xy^2$	The bases are x and y.
$15(x^2y^3)(xy^2)$	Multiply the coefficients - $5 \times 3 = 15$. Keep the base of x and y and add the exponents of the same base. If a base does not have a written exponent, it is understood as 1.
$15x^{2+1}y^{3+2}$	
$15x^3y^5$	The answer is in exponential form.
$\boxed{5x^2y^3 \cdot 3xy^2 = 15x^3y^5}$	

Concept Problem Revisited

$x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x \cdot x \cdot x$ can be rewritten as $x^9y^5x^4$. Then, you can use the rules of exponents to simplify the expression to $x^{13}y^5$. This is certainly much quicker to write!

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $(-3x)^2$

2. $(5xy)^3$

3. $(2^3 \cdot 3^2)^2$

Answers:

1. $9x^2$. Here are the steps:

$$\begin{aligned}(-3x)^2 &= (-3)^2 \cdot (x)^2\\ &= 9x^2\end{aligned}$$

2. $125x^3y^3$. Here are the steps:

$$\begin{aligned}(5x^2y^4)^3 &= (5)^3 \cdot (x)^3 \cdot (y)^3\\ &= 125x^3y^3\end{aligned}$$

3. 5184. Here are the steps:

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2\\ &= (72)^2\\ &= 5184\end{aligned}$$

OR

$$\begin{aligned}(2^3 \cdot 3^2)^2 &= (8 \cdot 9)^2\\ &= 8^2 \cdot 9^2\\ &= 64 \cdot 81\\ &= 5184\end{aligned}$$

Practice

Simplify each of the following expressions, if possible.

1. $4^2 \times 4^4$
2. $x^4 \cdot x^{12}$
3. $(3x^2y^4)(9xy^5z)$
4. $(2xy)^2(4x^2y^3)$
5. $(3x)^5(2x)^2(3x^4)$
6. $x^3y^2z \cdot 4xy^2z^7$
7. $x^2y^3 + xy^2$
8. $(0.1xy)^4$

9. $(xyz)^6$
10. $2x^4(x^2 - y^2)$
11. $3x^5 - x^2$
12. $3x^8(x^2 - y^4)$

Expand and then simplify each of the following expressions.

13. $(x^5)^3$
14. $(x^6)^8$
15. $(x^a)^b$ *Hint: Look for a pattern in the previous two problems.*

6.2 Quotient Rules for Exponents

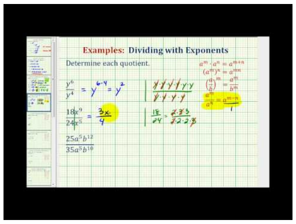
Here you'll learn how to divide two terms with the same base and find the power of a quotient.

Suppose you have the expression:

$$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$$

How could you write this expression in a more concise way?

Watch This


Examples: Dividing with Exponents

Determine each quotient.

MEDIA

Click image to the left for more content.

James Sousa: Simplify Exponential Expressions- Quotient Rule

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn two rules that have to do with exponents and quotients.

RULE: To divide two powers with the same base, subtract the exponents.

$$\frac{a^m}{a^n} = \frac{\overbrace{(a\times a\times \ldots \times a)}^{\overset{m\text{ factors}}{\uparrow}}}{\underbrace{(a\times a\times \ldots \times a)}_{\underset{n\text{ factors}}{\downarrow}}} \quad m>n; a\neq 0$$

$$\frac{a^m}{a^n} = \underbrace{(a\times a\times \ldots \times a)}_{\underset{m-n\text{ factors}}{\downarrow}}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

RULE: To raise a quotient to a power, raise both the numerator and the denominator to the power.

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b}\times\frac{a}{b}\times\ldots\times\frac{a}{b}}_{\underset{n\text{ factors}}{\downarrow}}$$

$$\left(\frac{a}{b}\right)^n = \frac{\overbrace{(a\times a\times \ldots \times a)}^{\overset{n\text{ factors}}{\uparrow}}}{\underbrace{(b\times b\times \ldots \times b)}_{\underset{n\text{ factors}}{\downarrow}}}$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \ (b\neq 0)$$

Example A

Simplify $2^7 \div 2^3$.

Solution:

$2^7 \div 2^3$	The base is 2.
2^{7-3}	Keep the base of 2 and subtract the exponents.
2^4	The answer is in exponential form.

The answer can be taken one step further. The base is numerical so the term can be evaluated.

$$2^4 = 2 \times 2 \times 2 \times 2$$
$$2^4 = 16$$
$$\boxed{2^7 \div 2^3 = 2^4 = 16}$$

Example B

Simplify $\frac{x^8}{x^2}$.

Solution:

$\frac{x^8}{x^2}$ The base is x.

x^{8-2} Keep the base of x and subtract the exponents.

x^6 The answer is in exponential form.

$$\boxed{\frac{x^8}{x^2} = x^6}$$

Example C

Simplify $\frac{16x^5y^5}{4x^2y^3}$.

Solution:

$\frac{16x^5y^5}{4x^2y^3}$ The bases are x and y.

$4\left(\frac{x^5y^5}{x^2y^3}\right)$ Divide the coefficients - $16 \div 4 = 4$. Keep the base of x and y and subtract the exponents of the same base.

$4x^{5-2}y^{5-3}$

$4x^3y^2$

Concept Problem Revisited

$\frac{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y}{x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y}$ can be rewritten as $\frac{x^9y^5}{x^6y^3}$ and then simplified to x^3y^2.

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Simplify each of the following expressions.

1. $\left(\frac{2}{3}\right)^2$
2. $\left(\frac{x}{6}\right)^3$
3. $\left(\frac{3x}{4y}\right)^2$

Answers:

1. $\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$
2. $\left(\frac{x}{6}\right)^3 = \frac{x^3}{6^3} = \frac{x^3}{216}$
3. $\left(\frac{3x}{4y}\right)^2 = \frac{3^2x^2}{4^2y^2} = \frac{9x^2}{16y^2}$

Practice

Simplify each of the following expressions, if possible.

1. $\left(\frac{2}{5}\right)^6$
2. $\left(\frac{4}{7}\right)^3$
3. $\left(\frac{x}{y}\right)^4$
4. $\frac{20x^4y^5}{5x^2y^4}$
5. $\frac{42x^2y^8z^2}{6xy^4z}$
6. $\left(\frac{3x}{4y}\right)^3$
7. $\frac{72x^2y^4}{8x^2y^3}$
8. $\left(\frac{x}{4}\right)^5$
9. $\frac{24x^{14}y^8}{3x^5y^7}$
10. $\frac{72x^3y^9}{24xy^6}$
11. $\left(\frac{7}{y}\right)^3$
12. $\frac{20x^{12}}{-5x^8}$

13. Simplify using the laws of exponents: $\frac{2^3}{2^5}$
14. Evaluate the numerator and denominator separately and then simplify the fraction: $\frac{2^3}{2^5}$
15. Use your result from the previous problem to determine the value of a: $\frac{2^3}{2^5} = \frac{1}{2^a}$

16. Use your results from the previous three problems to help you evaluate 2^{-4}.

6.3 Power Rule for Exponents

Here you'll learn how to find the power of a power.

Can you simplify an expression where an exponent has an exponent? For example, how would you simplify $[(2^3)^2]^4$?

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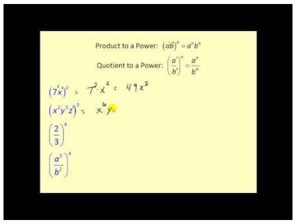

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James Sousa: Properties of Exponents

Guidance

In the expression x^3, the x is called the **base** and the 3 is called the **exponent**. **Exponents** are often referred to as **powers**. When an exponent is a positive whole number, it tells you how many times to multiply the base by itself. For example:

- $x^3 = x \cdot x \cdot x$
- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$.

There are many rules that have to do with exponents (often called the **Laws of Exponents**) that are helpful to know so that you can work with expressions and equations that involve exponents more easily. Here you will learn a rule that has to do with raising a power to another power.

RULE: To raise a power to a new power, multiply the exponents.

$$(a^m)^n = (\underbrace{a \times a \times \ldots \times a}_{\substack{\downarrow \\ m \text{ factors}}})^n$$

$$(a^m)^n = \underbrace{(\underbrace{a \times a \times \ldots \times a}_{\substack{\downarrow \\ m \text{ factors}}}) \times (\underbrace{a \times a \times \ldots \times a}_{\substack{\downarrow \\ m \text{ factors}}}) \; (\underbrace{a \times a \times \ldots \times a}_{\substack{\downarrow \\ m \text{ factors}}})}_{n \text{ times}}$$

$$(a^m)^n = \underbrace{a \times a \times a \ldots \times a}_{mn \text{ factors}}$$

$$(a^m)^n = a^{mn}$$

Example A

Evaluate $(2^3)^2$.

Solution: $(2^3)^2 = 2^6 = 64$.

Example B

Simplify $(x^7)^4$.

Solution: $(x^7)^4 = x^{28}$.

Example C

Evaluate $(3^2)^3$.

Solution: $(3^2)^3 = 3^6 = 729$.

Example D

Simplify $(x^2y^4)^2 \cdot (xy^4)^3$.

Solution: $(x^2y^4)^2 \cdot (xy^4)^3 = x^4y^8 \cdot x^3y^{12} = x^7y^{20}$.

Concept Problem Revisited

$[(2^3)^2]^4 = [2^6]^4 = 2^{24}$. Notice that the power rule applies even when a number has been raised to more than one power. The overall exponent is 24 which is $3 \cdot 2 \cdot 4$.

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

You know you can rewrite 2^4 as $2 \times 2 \times 2 \times 2$ and then calculate in order to find that

$$\boxed{2^4 = 16}$$

. This concept can also be reversed. To write 32 as a power of 2, $32 = 2 \times 2 \times 2 \times 2 \times 2$. There are 5 twos; therefore,

$$\boxed{32 = 2^5}$$

. Use this idea to complete the following problems.

1. Write 81 as a power of 3.

2. Write $(9)^3$ as a power of 3.

3. Write $(4^3)^2$ as a power of 2.

Answers:

1. $81 = 3 \times 3 = 9 \times 3 = 27 \times 3 = 81$

There are 4 threes. Therefore

$$\boxed{81 = 3^4}$$

2. $9 = 3 \times 3 = 9$

There are 2 threes. Therefore

$$\boxed{9 = 3^2}$$

.

$(3^2)^3$ Apply the law of exponents for power to a power-multiply the exponents.
$3^{2 \times 3} = 3^6$
Therefore

$$\boxed{(9)^3 = 3^6}$$

3. $4 = 2 \times 2 = 4$

There are 2 twos. Therefore

$$\boxed{4 = 2^2}$$

$\left((2^2)^3\right)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{2 \times 3} = 2^6}$$

$(2^6)^2$ Apply the law of exponents for power to a power-multiply the exponents.

$$\boxed{2^{6 \times 2} = 2^{12}}$$

Therefore

$$\boxed{(4^3)^2 = 2^{12}}$$

Practice

Simplify each of the following expressions.

1. $\left(\frac{x^4}{y^3}\right)^5$
2. $\frac{(5x^2y^4)^5}{(5xy^2)^3}$
3. $\frac{x^8y^9}{(x^2y)^3}$
4. $(x^2y^4)^3$
5. $(3x^2)^2 \cdot (4xy^4)^2$
6. $(2x^3y^5)(5x^2y)^3$
7. $(x^4y^6z^2)^2(3xyz)^3$
8. $\left(\frac{x^2}{2y^3}\right)^4$
9. $\frac{(4xy^3)^4}{(2xy^2)^3}$
10. True or false: $(x^2+y^3)^2 = x^4+y^6$
11. True or false: $(x^2y^3)^2 = x^4y^6$
12. Write 64 as a power of 4.
13. Write $(16)^3$ as a power of 2.
14. Write $(9^4)^2$ as a power of 3.
15. Write $(81)^2$ as a power of 3.
16. Write $(25^3)^4$ as a power of 5.

6.4 Zero and Negative Exponents

Here you'll learn how to work with zero and negative exponents.

How can you use the quotient rules for exponents to understand the meaning of a zero or negative exponent?

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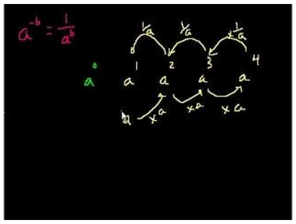
Khan Academy Negative Exponent Intuition



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Guidance

Zero Exponent

Recall that

$$\boxed{\frac{a^m}{a^n} = a^{m-n}}$$

. If $m = n$, then the following would be true:

$$\frac{a^m}{a^n} = a^{m-n} = a^0$$
$$\frac{3^3}{3^3} = 3^{3-3} = 3^0$$

However, any quantity divided by itself is equal to one. Therefore, $\frac{3^3}{3^3} = 1$ which means $3^0 = 1$. This is true in general:

$$\boxed{a^0 = 1 \text{ if } a \neq 0.}$$

Note that if $a = 0$, 0^0 is not defined.

Negative Exponents

$4^2 \times 4^{-2} = 4^{2+(-2)} = 4^0 = 1$

Therefore:

$$4^2 \times 4^{-2} = 1$$

$$\frac{4^2 \times 4^{-2}}{4^2} = \frac{1}{4^2} \qquad \text{Divide both sides by } 4^2.$$

$$\frac{\cancel{4^2} \times 4^{-2}}{\cancel{4^2}} = \frac{1}{4^2} \qquad \text{Simplify the equation.}$$

$$\boxed{4^{-2} = \frac{1}{4^2}}$$

This is true in general and creates the following laws for negative exponents:

- $$\boxed{a^{-m} = \frac{1}{a^m}}$$
- $$\boxed{\frac{1}{a^{-m}} = a^m}$$

These laws for negative exponents can be expressed in many ways:

- If a term has a negative exponent, write it as 1 over the term with a positive exponent. For example: $a^{-m} = \frac{1}{a^m}$ and $\frac{1}{a^{-m}} = a^m$
- If a term has a negative exponent, write the reciprocal with a positive exponent. For example: $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2$ and $a^{-m} = \frac{a^{-m}}{1} = \frac{1}{a^m}$
- If the term is a factor in the numerator with a negative exponent, write it in the denominator with a positive exponent. For example: $3x^{-3}y = \frac{3y}{x^3}$ and $a^{-m}b^n = \frac{1}{a^m}(b^n) = \frac{b^n}{a^m}$
- If the term is a factor in the denominator with a negative exponent, write it in the numerator with a positive exponent. For example: $\frac{2x^3}{x^{-2}} = 2x^3(x^2)$ and $\frac{b^n}{a^{-m}} = b^n\left(\frac{a^m}{1}\right) = b^n a^m$

These ways for understanding negative exponents provide shortcuts for arriving at solutions without doing tedious calculations. The results will be the same.

Example A

Evaluate the following using the laws of exponents.

$\left(\frac{3}{4}\right)^{-2}$

Solution:

There are two methods that can be used to evaluate the expression.

Method 1: Apply the negative exponent rule

$$\boxed{a^{-m} = \frac{1}{a^m}}$$

$\left(\frac{3}{4}\right)^{-2} = \frac{1}{\left(\frac{3}{4}\right)^2}$ Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$\frac{1}{\left(\frac{3}{4}\right)^2} = \frac{1}{\frac{3^2}{4^2}}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

$\frac{1}{\frac{3^2}{4^2}} = \frac{1}{\frac{9}{16}}$ Evaluate the powers.

$\frac{1}{\frac{9}{16}} = 1 \div \frac{9}{16}$ Divide

$1 \div \frac{9}{16} = 1 \times \frac{16}{9} = \frac{16}{9}$

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Method 2: Apply the shortcut and write the reciprocal with a positive exponent.

$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$ Write the reciprocal with a positive exponent.

$\left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}}$

$\frac{4^2}{3^2} = \frac{16}{9}$ Simplify.

$$\boxed{\left(\frac{3}{4}\right)^{-2} = \frac{16}{9}}$$

Applying the shortcut facilitates the process for obtaining the solution.

Example B

State the following using only positive exponents: (If possible, use shortcuts)

i) y^{-6}

ii) $\left(\frac{a}{b}\right)^{-3}$

iii) $\frac{x^5}{y^{-4}}$

iv) $a^2 \times a^{-5}$

Solutions:

i)

y^{-6} Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$$\boxed{y^{-6} = \frac{1}{y^6}}$$

ii)

$\left(\frac{a}{b}\right)^{-3}$ Write the reciprocal with a positive exponent.

$\left(\frac{a}{b}\right)^{-3} = \left(\frac{b}{a}\right)^{3}$ Apply the law of exponents for raising a quotient to a power. $\boxed{\left(\frac{a}{b}\right)^{n} = \frac{a^n}{b^n}}$

$\left(\frac{b}{a}\right)^{3} = \frac{b^3}{a^3}$

$\boxed{\left(\frac{a}{b}\right)^{-3} = \frac{b^3}{a^3}}$

iii)

$\frac{x^5}{y^{-4}}$ Apply the negative exponent rule. $\boxed{\frac{1}{a^{-m}} = a^{m}}$

$\frac{x^5}{y^{-4}} = x^5\left(\frac{y^4}{1}\right)$ Simplify.

$\boxed{\frac{x^5}{y^{-4}} = x^5y^4}$

iv)

$a^2 \times a^{-5}$ Apply the product rule for exponents $\boxed{a^m \times a^n = a^{m+n}}$.

$a^2 \times a^{-5} = a^{2+(-5)}$ Simplify.

$a^{2+(-5)} = a^{-3}$ Write the expression with a positive exponent by applying $\boxed{a^{-m} = \frac{1}{a^m}}$.

$a^{-3} = \frac{1}{a^3}$

$\boxed{a^2 \times a^{-5} = \frac{1}{a^3}}$

Example C

Evaluate the following: $\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$

Solution:

There are two methods that can be used to evaluate the problem.

Method 1: Work with the terms in the problem in exponential form.

Numerator:

$$7^{-2}=\frac{1}{7^2} \text{ and } 7^{-1}=\frac{1}{7}$$ Apply the definition $a^{-m}=\frac{1}{a^m}$

$$\frac{1}{7^2}+\frac{1}{7}$$ A common denominator is needed to add the fractions.

$$\frac{1}{7^2}+\frac{1}{7}\left(\frac{7}{7}\right)$$ Multiply $\frac{1}{7}$ by $\frac{7}{7}$ to obtain the common denominator of 7^2

$$\frac{1}{7^2}+\frac{7}{7^2}=\frac{1+7}{7^2}=\frac{8}{7^2}$$ Add the fractions.

Denominator:

$$7^{-3}=\frac{1}{7^3} \text{ and } 7^{-4}=\frac{1}{7^4}$$ Apply the definition $a^{-m}=\frac{1}{a^m}$

$$\frac{1}{7^3}+\frac{1}{7^4}$$ A common denominator is needed to add the fractions.

$$\left(\frac{7}{7}\right)\frac{1}{7^3}+\frac{1}{7^4}$$ Multiply $\frac{1}{7^3}$ by $\frac{7}{7}$ to obtain the common denominator of 7^4

$$\frac{7}{7^4}+\frac{1}{7^4}=\frac{1+7}{7^4}=\frac{8}{7^4}$$ Add the fractions.

Numerator and Denominator:

$$\frac{8}{7^2}\div\frac{8}{7^4}$$ Divide the numerator by the denominator.

$$\frac{8}{7^2}\times\frac{7^4}{8}$$ Multiply by the reciprocal.

$$\frac{\cancel{8}}{7^2}\times\frac{7^4}{\cancel{8}}=\frac{7^4}{7^2}=7^2=49$$ Simplify.

$$\boxed{\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}=49}$$

Method 2: Multiply the numerator and the denominator by 7^4. This will change all negative exponents to positive exponents. Apply the product rule for exponents and work with the terms in exponential form.

$$\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$$

$$\left(\frac{7^4}{7^4}\right)\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}$$ Apply the distributive property with the product rule for exponents.

$$\frac{7^2+7^3}{7^1+7^0}$$ Evaluate the numerator and the denominator.

$$\frac{49+343}{7+1}=\frac{392}{8}=49$$

$$\boxed{\frac{7^{-2}+7^{-1}}{7^{-3}+7^{-4}}=49}$$

Whichever method is used, the result is the same.

Concept Problem Revisited

By the quotient rule for exponents, $\frac{x^m}{x^m} = x^{m-m} = x^0$. Since anything divided by itself is equal to 1 (besides 0), $\frac{x^m}{x^m} = 1$. Therefore, $x^0 = 1$ as long as $x \neq 0$.

Also by the quotient rule for exponents, $\frac{x^2}{x^5} = x^{2-5} = x^{-3}$. If you were to expand and reduce the original expression you would have $\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x^3}$. Therefore, $x^{-3} = \frac{1}{x^3}$. This generalizes to $x^{-a} = \frac{1}{x^a}$.

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

1. Use the laws of exponents to simplify the following: $(-3x^2)^3(9x^4y)^{-2}$

2. Rewrite the following using only positive exponents. $(x^2y^{-1})^2$

3. Use the laws of exponents to evaluate the following: $[5^{-4} \times (25)^3]^2$

Answers:

1.

$$(-3x^2)^3(9x^4y)^{-2}$$

Apply the laws of exponents $\boxed{(a^m)^n = a^{mn}}$ and $\boxed{a^{-m} = \frac{1}{a^m}}$

$$(-3x^2)^3(9x^4y)^{-2} = (-3^3x^6)\left(\frac{1}{(9x^4y)^2}\right)$$

Simplify and apply $\boxed{(ab)^n = a^nb^n}$

$$(-3^3x^6)\left(\frac{1}{(9x^4y)^2}\right) = -27x^6\left(\frac{1}{(9^2x^8y^2)}\right)$$

Simplify.

$$-27x^6\left(\frac{1}{(9^2x^8y^2)}\right) = \frac{-27x^6}{81x^8y^2}$$

Simplify and apply the quotient rule for exponents $\boxed{\frac{a^m}{a^n} = a^{m-n}}$.

$$\frac{-27x^6}{81x^8y^2} = -\frac{1x^{-2}}{3y^2}$$

Apply the negative exponent rule $\boxed{a^{-m} = \frac{1}{a^m}}$

$$\boxed{(-3x^2)^3(9x^4y)^{-2} = -\frac{1}{3x^2y^2}}$$

2.

$$\begin{aligned}(x^2y^{-1})^2 &= x^4y^{-2}\\ &= \frac{x^4}{y^2}\end{aligned}$$

3.

$$[5^{-4}\times(25)^3]^2$$

Try to do this one by applying the laws of exponents.

$$[5^{-4}\times(25)^3]^2 = [5^{-4}\times(5^2)^3]^2$$

$$[5^{-4}\times(5^2)^3]^2 = [5^{-4}\times 5^6]^2$$

$$[5^{-4}\times 5^6]^2 = (5^2)^2$$

$$(5^2)^2 = 5^4$$

$$5^4 = 625$$

$$\boxed{[5^{-4}\times(25)^3]^2 = 5^4 = 625}$$

Practice

Evaluate each of the following expressions:

1. $-\left(\frac{2}{3}\right)^0$
2. $\left(-\frac{2}{5}\right)^{-2}$
3. $(-3)^{-3}$
4. $6\times\left(\frac{1}{2}\right)^{-2}$
5. $7^{-4}\times 7^4$

Rewrite the following using positive exponents only. Simplify where possible.

6. $(4wx^{-2}y^3z^{-4})^3$

7. $\frac{a^2b^3c^{-2}}{d^{-2}bc^{-6}}$
8. $x^{-2}(x^2-1)$
9. $m^4(m^2+m-5m^{-2})$
10. $\frac{x^{-2}y^{-2}}{x^{-1}y^{-1}}$
11. $\left(\frac{x^{-2}}{y^4}\right)^3\left(\frac{y^{-4}}{x^6}\right)^{-7}$
12. $\frac{(x^{-2}y^4)^2}{(x^5y^{-3})^4}$
13. $\frac{(3xy^2)^3}{(3x^2y)^4}$
14. $\left(\frac{x^2y^{-25}z^5}{-12.4x^3y}\right)^0$
15. $\left(\frac{x^{-2}}{y^3}\right)^5\left(\frac{y^{-2}}{x^4}\right)^{-3}$

6.5 Fractional Exponents

Here you'll learn how to work with exponents that are fractions.

If an exponent usually tells you the number of times to multiply the base by itself, what does it mean if the exponent is a fraction? How can you think about and calculate $4^{\frac{3}{2}}$?

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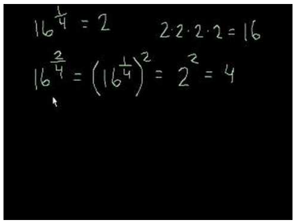
Khan Academy Level 3 Exponents



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Guidance

A fraction exponent is related to a root. Raising a number to the power of $\frac{1}{2}$ is the same as taking the square root of the number. If you have $a^{\frac{m}{n}}$, you can think about this expression in multiple ways:

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} \quad \text{or} \quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} \quad \text{or} \quad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

All of these ideas can be summarized as the following rule for fractional exponents:

$$\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m \ m,n \in N}$$

Example A

Simplify the following:

$(125)^{-\frac{2}{3}}$

Solution:

$(125)^{-\frac{2}{3}}$ Apply the law of exponents for negative exponents $\boxed{a^{-m} = \frac{1}{a^m}}$.

$\frac{1}{125^{\frac{2}{3}}}$ Apply the law of exponents for rational exponents $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \quad m,n \in N.}$

$\frac{1}{\left(\sqrt[3]{125}\right)^2}$

The cube root of 125 is ’**5**’.

$\frac{1}{5^2}$

Evaluate the denominator.

$$\frac{1}{25}$$

$$\boxed{(125)^{-\frac{2}{3}} = \frac{1}{25}}$$

Example B

Simplify the following:

$(2a^2b^4)^{\frac{3}{2}}$

Solution:

$(2a^2b^4)^{\frac{3}{2}}$ Apply the law of exponents for raising a power to a power $\boxed{(a^m)^n = a^{mn}}$.

$(2a^2b^4)^{\frac{3}{2}} = 2^{1\times\frac{3}{2}}(a^2)^{\frac{3}{2}}(b^4)^{\frac{3}{2}}$ Simplify the expression.

$2^{1\times\frac{3}{2}}(a^2)^{\frac{3}{2}}(b^4)^{\frac{3}{2}} = 2^{\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{4\times\frac{3}{2}}$ Simplify. Apply the rule for rational exponents $\boxed{a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \quad m,n \in N}$.

$2^{\frac{3}{2}}(a)^{\cancel{2}\times\frac{3}{\cancel{2}}}(b)^{\overset{2}{\cancel{4}}\times\frac{3}{\cancel{2}}} = \sqrt{2^3}(a)^3(b)^6$ Simplify

$\sqrt{2^3}(a)^3(b)^6 = \sqrt{8}a^3b^6$

$\sqrt{8}a^3b^6 = 2\sqrt{2}a^3b^6$

$\boxed{(2a^2b^4)^{\frac{3}{2}} = 2\sqrt{2}a^3b^6}$

Example C

State the following using radicals:

i) $2^{\frac{3}{8}}$

ii) $7^{-\frac{1}{5}}$

iii) $3^{\frac{3}{4}}$

Solutions:

i) $2^{\frac{3}{8}} = \sqrt[8]{2^3} = \sqrt[8]{8}$

ii) $7^{-\frac{1}{5}} = \frac{1}{\sqrt[5]{7}}$

iii) $3^{\frac{3}{4}} = \sqrt[4]{3^3} = \sqrt[4]{27}$

Example D

State the following using exponents:

i) $\sqrt[3]{7^2}$

ii) $\frac{1}{\left(\sqrt[4]{5}\right)^3}$

iii) $\left(\sqrt[5]{a}\right)^2$

Solutions:

i) $\sqrt[3]{7^2} = 7^{\frac{2}{3}}$

ii)

$$\frac{1}{\left(\sqrt[4]{5}\right)^3}$$

$$\frac{1}{\left(\sqrt[4]{5}\right)^3} = \frac{1}{5^{\frac{3}{4}}}$$

$$\frac{1}{5^{\frac{3}{4}}} = 5^{-\frac{3}{4}}$$

iii) $\left(\sqrt[5]{a}\right)^2 = a^{\frac{2}{5}}$

Concept Problem Revisited

$4^{\frac{3}{2}} = (\sqrt{4})^3 = 2^3 = 8$

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2 \times 2 \times 2 \times 2 \times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y \times -3y \times -3y \times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

1. Use the laws of exponents to evaluate the following: $9^{\frac{3}{2}} \div 36^{-\frac{1}{2}}$

2. Simplify the following using the laws of exponents. $(20a^2b^3c^{-1})^{\frac{3}{2}}$

3. Use the laws of exponents to evaluate the following: $\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}}$

Answers:

1. $9^{\frac{3}{2}} \div 36^{-\frac{1}{2}}$

$$9^{\frac{3}{2}} \div 36^{-\frac{1}{2}} = \left(\sqrt{9}\right)^{3} \div \frac{1}{36^{\frac{1}{2}}}$$

$$\left(\sqrt{9}\right)^{3} \div \frac{1}{36^{\frac{1}{2}}} = (3)^{3} \div \frac{1}{\sqrt{36}}$$

Simplify

$$(3)^{3} \div \frac{1}{\sqrt{36}} = 27 \div \frac{1}{6}$$

Perform the indicated operation of division.

$$27 \div \frac{1}{6} = 27 \times \frac{6}{1} = 162$$

$$\boxed{9^{\frac{3}{2}} \div 36^{-\frac{1}{2}} = 162}$$

2.

$$(20a^2b^3c^{-1})^{\frac{3}{2}}$$

Apply the law of exponents $\boxed{(ab)^n = a^nb^n}$.

$$(20a^2b^3c^{-1})^{\frac{3}{2}} = 20^{1\times\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{3\times\frac{3}{2}}(c)^{-1\times\frac{3}{2}}$$

Simplify the exponents.

$$20^{1\times\frac{3}{2}}(a)^{2\times\frac{3}{2}}(b)^{3\times\frac{3}{2}}(c)^{-1\times\frac{3}{2}} = 20^{\frac{3}{2}}(a)^{3}(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}}$$

$$20^{\frac{3}{2}}(a)^{3}(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}} = \left(\sqrt{20}\right)^{3}(a)^{3}\sqrt{b^9}\left(\frac{1}{c^{\frac{3}{2}}}\right)$$

Simplify

$$20^{\frac{3}{2}}(a)^{3}(b)^{\frac{9}{2}}(c)^{-\frac{3}{2}} = \left(2\sqrt{5}\right)^{3}\left(a^{3}\right)\left(b^{4}\sqrt{b}\right)\left(\frac{1}{\sqrt{c^{3}}}\right)$$

Simplify

$$\left(2\sqrt{5}\right)^{3}\left(a^{3}\right)\left(b^{4}\sqrt{b}\right)\left(\frac{1}{\sqrt{c^{3}}}\right) = 8\sqrt{125}a^{3}b^{4}\sqrt{b}\frac{1}{c\sqrt{c}}$$

$$8\sqrt{125}a^{3}b^{4}\sqrt{b}\frac{1}{c\sqrt{c}} = 40\sqrt{5}a^{3}b^{4}\sqrt{b}\left(c\sqrt{c}\right)^{-1}$$

Simplify

$$\boxed{(20a^2b^3c^{-1})^{\frac{3}{2}} = 40\sqrt{5}a^{3}b^{4}\sqrt{b}\left(c\sqrt{c}\right)^{-1}}$$

3. $\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}}$.

Numerator	**Denominator**
$64^{\frac{2}{3}}$	$216^{-\frac{1}{3}}$
$64^{\frac{2}{3}} = \left(\sqrt[3]{64}\right)^2$	$216^{-\frac{1}{3}} = \frac{1}{216^{\frac{1}{3}}}$
$\left(\sqrt[3]{64}\right)^2 = (4)^2$	$216^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{216}}$
$(4)^2 = 16$	$\frac{1}{\sqrt[3]{216}} = \frac{1}{6}$

Numerator divided by denominator:

$$16 \div \frac{1}{6}$$

$$16 \times \frac{6}{1} = 96$$

$$\boxed{\frac{64^{\frac{2}{3}}}{216^{-\frac{1}{3}}} = 96}$$

Practice

Express each of the following as a radical and if possible, simplify.

1. $x^{\frac{1}{2}}$
2. $5^{\frac{3}{4}}$
3. $2^{\frac{3}{2}}$
4. $2^{-\frac{1}{2}}$
5. $9^{-\frac{1}{5}}$

Express each of the following using exponents:

6. $\sqrt{26}$
7. $\sqrt[3]{5^2}$
8. $\left(\sqrt[6]{a}\right)^5$
9. $\sqrt[4]{m}$
10. $\left(\sqrt[3]{7}\right)^2$

Evaluate each of the following using the laws of exponents:

11. $3^{\frac{2}{5}} \times 3^{\frac{3}{5}}$
12. $(6^{0.4})^5$
13. $2^{\frac{1}{7}} \times 4^{\frac{3}{7}}$
14. $\left(\frac{64}{125}\right)^{-\frac{1}{2}}$
15. $(81^{-1})^{-\frac{1}{4}}$

6.6 Exponential Expressions

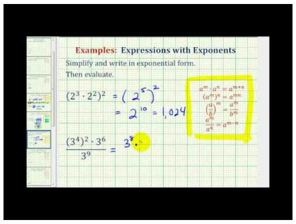
Here you'll learn how to use all of the laws of exponents to simplify and evaluate exponential expressions.

Can you simplify the following expression so that it has only positive exponents?

$$\frac{8x^3y^{-2}}{(-4a^2b^4)^{-2}}$$

Watch This

James Sousa: Simplify Exponential Expressions



Examples: Expressions with Exponents

Simplify and write in exponential form. Then evaluate.

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Guidance

The following table summarizes all of the rules for exponents.

Laws of Exponents

If $a \in R, a \geq 0$ and $m,n \in Q$, then

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$ (if $m > n, a \neq 0$)
3. $(a^m)^n = a^{mn}$
4. $(ab)^n = a^n b^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$
6. $a^0 = 1$ $(a \neq 0)$
7. $a^{-m} = \frac{1}{a^m}$
8. $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Example A

Evaluate $81^{-\frac{1}{4}}$.

Solution: First, rewrite with a positive exponent:

$81^{-\frac{1}{4}} = \frac{1}{81^{\frac{1}{4}}} = \left(\frac{1}{81}\right)^{\frac{1}{4}}$.

Next, evaluate the fractional exponent:

$\left(\frac{1}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{1}{81}} = \frac{1}{3}$

Example B

Simplify $(4x^3y)(3x^5y^2)^4$.

Solution:

$$\begin{aligned}(4x^3y)(3x^5y^2)^4 &= (4x^3y)(81x^{20}y^8)\\ &= 324x^{23}y^9\end{aligned}$$

Example C

Simplify $\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2}$.

Solution:

$$\begin{aligned}\left(\frac{x^{-2}y}{x^4y^3}\right)^{-2} &= \left(\frac{x^4y^3}{x^{-2}y}\right)^2\\ &= (x^6y^2)^2\\ &= x^{12}y^4\end{aligned}$$

Concept Problem Revisited

$$\begin{aligned}\frac{8x^3y^{-2}}{(-4x^2y^4)^{-2}} &= (8x^3y^{-2})(-4x^2y^4)^2\\ &= (8x^3y^{-2})(16x^4y^8)\\ &= 8\cdot 16\cdot x^3\cdot x^4\cdot y^{-2}\cdot y^8\\ &= 128x^7y^6\end{aligned}$$

Vocabulary

Base

In an algebraic expression, the ***base*** is the variable, number, product or quotient, to which the exponent refers. Some examples are: In the expression 2^5, '2' is the base. In the expression $(-3y)^4$, '$-3y$' is the base.

Exponent

In an algebraic expression, the ***exponent*** is the number to the upper right of the base that tells how many times to multiply the base times itself. Some examples are:

In the expression 2^5, '5' is the exponent. It means to multiply 2 times itself 5 times as shown here: $2^5 = 2\times 2\times 2\times 2\times 2$.

In the expression $(-3y)^4$, '4' is the exponent. It means to multiply $-3y$ times itself 4 times as shown here: $(-3y)^4 = -3y\times -3y\times -3y\times -3y$.

Laws of Exponents

The ***laws of exponents*** are the algebra rules and formulas that tell us the operation to perform on the exponents when dealing with exponential expressions.

Guided Practice

Use the laws of exponents to simplify each of the following:

1. $(-2x)^5(2x^2)$
2. $(16x^{10})\left(\frac{3}{4}x^5\right)$
3. $\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8}$

Answers:

1. $(-2x)^5(2x^2) = (-32x^5)(2x^2) = -64x^7$
2. $(16x^{10})\left(\frac{3}{4}x^5\right) = 12x^{15}$
3. $\frac{(x^{15})(x^{24})(x^{25})}{(x^7)^8} = \frac{x^{64}}{x^{56}} = x^8$

Practice

Simplify each expression.

1. $(x^{10})(x^{10})$
2. $(7x^3)(3x^7)$
3. $(x^3y^2)(xy^3)(x^5y)$
4. $\frac{(x^3)(x^2)}{(x^4)}$
5. $\frac{x^2}{x^{-3}}$
6. $\frac{x^6y^8}{x^4y^{-2}}$
7. $(2x^{12})^3$
8. $(x^5y^{10})^7$
9. $\left(\frac{2x^{10}}{3y^{20}}\right)^3$

Express each of the following as a power of 3. Do not evaluate.

10. $(3^3)^5$
11. $(3^9)(3^3)$
12. $(9)(3^7)$
13. 9^4
14. $(9)(27^2)$

Apply the laws of exponents to evaluate each of the following without using a calculator.

15. $(2^3)(2^2)$
16. $6^6 \div 6^5$
17. $-(3^2)^3$
18. $(1^2)^3 + (1^3)^2$
19. $\left(\frac{1}{3}\right)^6 \div \left(\frac{1}{3}\right)^8$

Use the laws of exponents to simplify each of the following.

20. $(4x)^2$
21. $(-3x)^3$

22. $(x^3)^4$
23. $(3x)(x^7)$
24. $(5x)(4x^4)$
25. $(-3x^2)(-6x^3)$
26. $(10x^8) \div (2x^4)$

Simplify each of the following using the laws of exponents.

27. $5^{\frac{1}{2}} \times 5^{\frac{1}{3}}$
28. $(d^4e^8f^{12})^{\frac{1}{4}}$
29. $\sqrt[4]{\frac{y^{\frac{1}{2}}\sqrt{xy}}{x^{\frac{2}{3}}}}$
30. $(32a^{20}b^{-15})^{\frac{1}{5}}$
31. $(729x^{12}y^{-6})^{\frac{2}{3}}$

6.7 Scientific Notation

Here you'll learn about scientific notation.

Very large and very small quantities and measures are often used to provide information in magazines, textbooks, television, newspapers and on the Internet. Some examples are:

- The distance between the sun and Neptune is 4,500,000,000 km.
- The diameter of an electron is approximately 0.00000000000022 inches.

Scientific notation is a convenient way to represent such numbers. How could you write the numbers above using scientific notation?

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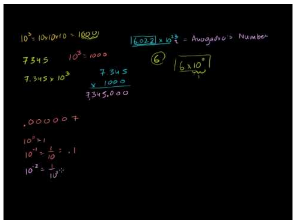
Khan Academy Scientific Notation



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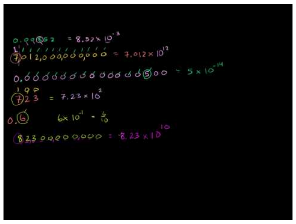
Khan Academy Scientific Notation Examples



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Guidance

To represent a number in scientific notation means to express the number as a product of two factors: a number between 1 and 10 (including 1) and a power of 10. A positive real number 'x' is said to be written in **scientific notation** if it is expressed as

$$x = a \times 10^n$$

where

$$1 \leq a < 10 \text{ and } n \in Z.$$

In other words, a number in scientific notation is a single nonzero digit followed by a decimal point and other digits, all multiplied by a power of 10.

When working with numbers written in scientific notation, you can use the following rules. These rules are proved by example in Example B and Example C.

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

$$\boxed{(A \times 10^m) \times (B \times 10^n) = (A \times B) \times (10^{m+n})}$$

$$\boxed{(A \times 10^m) \div (B \times 10^n) = (A \div B) \times (10^{m-n})}$$

Example A

Write the following numbers using scientific notation:

i) 2,679,000

ii) 0.00005728

Solutions:

i)

$$\begin{aligned} 2,679,000 &= 2.679 \times 1,000,000 \\ 2.679 \times 1,000,000 &= 2.679 \times 10^{6} \end{aligned}$$

The exponent, $n = 6$, represents the decimal point that is 6 places to the right of the **standard position of the decimal point.**

ii)

$$\begin{aligned} 0.00005728 &= 5.728 \times 0.00001 \\ 5.728 \times 0.00001 &= 5.728 \times \frac{1}{100,000} \\ 5.728 \times \frac{1}{100,000} &= 5.728 \times \frac{1}{10^{5}} \\ 5.728 \times \frac{1}{100,000} &= 5.728 \times 10^{-5} \end{aligned}$$

The exponent, $n = -5$, represents the decimal point that is 5 places to the left of the **standard position of the decimal point.**

One advantage of scientific notation is that calculations with large or small numbers can be done by **applying the laws of exponents.**

Example B

Complete the following table.

TABLE 6.1:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$			
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$			
$4.6 \times 10^4 - 2.2 \times 10^4$			
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$			

Solution:

TABLE 6.2:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$1.3 \times 10^5 + 2.5 \times 10^5$	$130,000 + 250,000$	380,000	3.8×10^5
$3.7 \times 10^{-2} + 5.1 \times 10^{-2}$	$0.037 + 0.051$	0.088	8.8×10^{-2}
$4.6 \times 10^4 - 2.2 \times 10^4$	$46,000 - 22,000$	24,000	2.4×10^4
$7.9 \times 10^{-2} - 5.4 \times 10^{-2}$	$0.079 - 0.054$	0.025	2.5×10^{-2}

Note that the numbers in the last column have the same power of 10 as those in the first column.

Example C

Complete the following table.

TABLE 6.3:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$			
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$			
$(4.4 \times 10^4) \div (2.2 \times 10^2)$			
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$			

Solution:

TABLE 6.4:

Expression in Scientific Notation	Expression in Standard Form	Result in Standard Form	Result in Scientific Notation
$(3.6 \times 10^2) \times (1.4 \times 10^3)$	360×1400	504,000	5.04×10^5
$(2.5 \times 10^3) \times (1.1 \times 10^{-6})$	2500×0.0000011	0.00275	2.75×10^{-3}
$(4.4 \times 10^4) \div (2.2 \times 10^2)$	$44,000 \div 220$	200	2.0×10^2
$(6.8 \times 10^{-4}) \div (3.2 \times 10^{-2})$	$0.00068 \div 0.032$	0.02125	2.125×10^{-2}

Note that for multiplication, the power of 10 is the result of adding the exponents of the powers in the first column. For division, the power of 10 is the result of subtracting the exponents of the powers in the first column.

Example D

Calculate each of the following:

i) $4.6 \times 10^4 + 5.3 \times 10^5$

ii) $4.7 \times 10^{-3} - 2.4 \times 10^{-4}$

iii) $(7.3 \times 10^5) \times (6.8 \times 10^4)$

iv) $(4.8 \times 10^9) \div (5.79 \times 10^7)$

Solution:

i) Before the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.6×10^4

$4.6 \times 10^4 = (0.46 \times 10^1) \times 10^4$ The power 10^1 indicates the number of places to the right that the decimal point must be moved to return 0.46 to the original number of 4.6.

$(0.46 \times 10^1) \times 10^4 = 0.46 \times 10^5$ Add the exponents of the power.

Rewrite the question and substitute 4.6×10^4 with 0.46×10^5.

$0.46 \times 10^5 + 5.3 \times 10^5$

Apply the rule

$$\boxed{(A \times 10^n) + (B \times 10^n) = (A + B) \times 10^n}$$

.

$$(0.46 \times 10^5) + (5.3 \times 10^5) = (0.46 + 5.3) \times 10^5$$
$$(0.46 + 5.3) \times 10^5 = 5.76 \times 10^5$$
$$\boxed{4.6 \times 10^4 + 5.3 \times 10^5 = 5.76 \times 10^5}$$

ii) Before the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

can be used, one of the numbers must be rewritten so that the powers of 10 are the same.

Rewrite 4.7×10^{-3}

$4.7 \times 10^{-3} = (47 \times 10^{-1}) \times 10^{-3}$ The power 10^{-1} indicates the number of places to the left that the decimal point must be moved to return 47 to the original number of 4.7.

$(47 \times 10^{-1}) \times 10^{-3} = 47 \times 10^{-4}$ Add the exponents of the power.

Rewrite the question and substitute 4.7×10^{-3} with 47×10^{-4}.

$47 \times 10^{-4} - 2.4 \times 10^{-4}$

Apply the rule

$$\boxed{(A \times 10^n) - (B \times 10^n) = (A - B) \times 10^n}$$

.

$$(47\times10^{-4})-(2.4\times10^{-4})=(47-2.4)\times10^{-4}$$
$$(47\times10^{-4})-(2.4\times10^{-4})=44.6\times10^{-4}$$

The answer must be written in scientific notation.

$44.6\times10^{-4}=(4.46\times10^{1})\times10^{-4}$ Apply the law of exponents – add the exponents of the power.

$4.46\times10\times10^{-4}=4.46\times10^{-3}$

$\boxed{4.7\times10^{-3}-2.4\times10^{-4}=4.46\times10^{-3}}$

iii) $(7.3\times10^{5})\times(6.8\times10^{4})$

$7.3\times10^{5}\times6.8\times10^{4}$ Apply the rule $\boxed{(A\times10^{m})\times(B\times10^{n})=(A\times B)\times(10^{m+n})}$.

$(7.3\times10^{5})\times(6.8\times10^{4})=(7.3\times6.8)\times(10^{5+4})$

$(7.3\times6.8)\times(10^{5+4})=(49.64)\times(10^{9})$

$(49.64)\times(10^{9})=49.64\times10^{9}$ Write the answer in scientific notation.

$49.64\times10^{9}=(4.964\times10^{1})\times10^{9}$ Apply the law of exponents – add the exponents of the power.

$49.64\times10^{9}=4.964\times10^{10}$

$\boxed{(7.3\times10^{5})\times(6.8\times10^{4})=4.964\times10^{10}}$

iv) $(4.8\times10^{9})\div(5.79\times10^{7})$

$(4.8\times10^{9})\div(5.79\times10^{7})$ Apply the rule $\boxed{(A\times10^{m})\div(B\times10^{n})=(A\div B)\times(10^{m-n})}$.

$(4.8\times10^{9})\div(5.79\times10^{7})=(4.8\div5.79)\times10^{9-7}$ Apply the law of exponents –subtract the exponents of the power.

$(4.8\div5.79)\times10^{9-7}=(0.829)\times10^{2}$ Write the answer in scientific notation.

$(0.829)\times10^{2}=(8.29\times10^{-1})\times10^{2}$ Apply the law of exponents – add the exponents of the power.

$\boxed{(8.29\times10^{-1})\times10^{2}=8.29\times10^{1}}$

Concept Problem Revisited

The distance between the sun and Neptune would be written as 4.5×10^{9} *km* and the diameter of an electron would be written as 2.2×10^{-13} *in*.

Vocabulary

Scientific Notation

Scientific notation is a way of writing numbers in the form of a number between 1 and 10 multiplied by a power of 10. The number 196.5 written in scientific notation is 1.965×10^{2} and the number 0.0760 written in scientific notation is 7.60×10^{-2}.

Guided Practice

1. Express the following product in scientific notation: $(4 \times 10^{12})(9.2 \times 10^{7})$

2. Express the following quotient in scientific notation: $\frac{6{,}400{,}000}{0.008}$

3. If $a = 0.000415, b = 521$, and $c = 71,640$, find an approximate value for $\frac{ab}{c}$. Express the answer in scientific notation.

Answers:

1. Apply the rule

$$\boxed{(A \times 10^{m}) \times (B \times 10^{n}) = (A \times B) \times (10^{m+n})}$$

$$(4 \times 10^{12}) \times (9.2 \times 10^{7}) = (4 \times 9.2) \times (10^{12+7})$$
$$(4 \times 9.2) \times (10^{12+7}) = 36.8 \times 10^{19}$$

Express the answer in scientific notation.

$$36.8 \times 10^{19} = (3.68 \times 10^{1}) \times 10^{19}$$
$$(3.68 \times 10^{1}) \times 10^{19} = 3.68 \times 10^{20}$$
$$\boxed{(4 \times 10^{12})(9.2 \times 10^{7}) = 3.68 \times 10^{20}}$$

2. Begin by expressing the numerator and the denominator in scientific notation.

$\frac{6.4 \times 10^{6}}{8.0 \times 10^{-3}}$

Apply the rule

$$\boxed{(A \times 10^{m}) \div (B \times 10^{n}) = (A \div B) \times (10^{m+n})}$$

.

$(6.4 \times 10^{6}) \div (8.0 \times 10^{-3}) = (6.4 \div 8.0) \times (10^{6--3})$ Apply the law of exponents – subtract the exponents of the powers.

$(6.4 \div 8.0) \times (10^{6--3}) = (0.8) \times (10^{9})$

$(0.8) \times (10^{9}) = 0.8 \times 10^{9}$ Express the answer in scientific notation.

$0.8 \times 10^{9} = (8.0 \times 10^{-1}) \times 10^{9}$

$0.8 \times 10^{9} = 8.0 \times 10^{-1} \times 10^{9}$ Apply the law of exponents – add the exponents of the powers.

$8.0 \times 10^{-1} \times 10^{9} = 8.0 \times 10^{8}$

$\boxed{\frac{6,400,000}{0.008} = 8.0 \times 10^{8}}$ Express the answer in scientific notation.

3. Express all values in scientific notation.

$$0.000415 = 4.15 \times 10^{-4}$$
$$521 = 5.21 \times 10^{2}$$
$$71,640 = 7.1640 \times 10^{4}$$

Use the values in scientific notation to determine an approximate value for $\frac{ab}{c}$.

$\frac{ab}{c} = \frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}}$

In the numerator, apply the rule

$$\boxed{(A \times 10^{m}) \times (B \times 10^{n}) = (A \times B) \times (10^{m+n})}$$

$$\frac{(4.15 \times 10^{-4})(5.21 \times 10^{2})}{7.1640 \times 10^{4}} = \frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}}$$

$$\frac{(4.15 \times 5.21) \times (10^{-4} \times 10^{2})}{7.1640 \times 10^{4}} = \frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}}$$

Apply the rule $\boxed{(A \times 10^{m}) \div (B \times 10^{n}) = (A \div B) \times (10^{m-n})}$.

$$\frac{21.6215 \times 10^{-2}}{7.1640 \times 10^{4}} = (21.6215 \div 7.1640) \times (10^{-2-4})$$

$$\boxed{(21.6215 \div 7.1640) \times (10^{-2} \times 10^{4}) = 3.018 \times 10^{-6}}$$

Practice

Express each of the following in scientific notation:

1. 42,000
2. 0.00087
3. 150.64
4. 56,789
5. 0.00947

Express each of the following in standard form:

6. 4.26×10^{5}
7. 8×10^{4}
8. 5.967×10^{10}
9. 1.482×10^{-6}
10. 7.64×10^{-3}

Perform the indicated operations and express the answer in scientific notation

11. $8.9 \times 10^{4} + 4.3 \times 10^{5}$
12. $8.7 \times 10^{-4} - 6.5 \times 10^{-5}$
13. $(5.3 \times 10^{6}) \times (7.9 \times 10^{5})$
14. $(3.9 \times 10^{8}) \div (2.8 \times 10^{6})$

For the given values, perform the indicated operations for $\frac{ab}{c}$ and express the answer in scientific notation and standard form.

15.

$$\begin{aligned} a &= 76.1 \\ b &= 818,000,000 \\ c &= 0.000016 \end{aligned}$$

16.

$$\begin{aligned} a &= 9.13 \times 10^{9} \\ b &= 5.45 \times 10^{-23} \\ c &= 1.62 \end{aligned}$$

6.8 Exponential Equations

Here you'll learn how to solve basic equations that contain exponents.

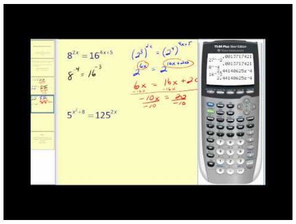
The following exponential equation is one in which the variable appears in the exponent.

$9^{x+1} = \sqrt{27}$

How can you solve this type of equation where you can't isolate the variable?

Watch This

James Sousa: Solving Exponential Equations



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Guidance

When an equation has exponents, sometimes the variable will be in the exponent and sometimes it won't. There are different strategies for solving each type of equation.

- ***When the variable is in the exponent:*** Rewrite each side of the equation so that the bases of the exponent are the same. Then, create a new equation where you set the exponents equal to each other and solve (see Example A).
- ***When the variable is not in the exponent:*** Manipulate the equation so the exponent is no longer there (see Example B). Or, rewrite each side of the equation so that both sides have the same exponent. Then, create a new equation where you set the bases equal to each other and solve (see Example C).

Example A

Solve the following exponential equation:

$25^{x-3} = \left(\frac{1}{5}\right)^{3x+18}$

Solution: The variable appears in the exponent. Write both sides of the equation as a power of 5.

$(5^2)^{x-3} = (5^{-1})^{3x+18}$

Apply the law of exponents for raising a power to a power

$$\boxed{(a^m)^n = a^{mn}}$$

.

$$5^{2(x-3)} = 5^{-1(3x+18)}$$

Simplify the exponents.

$$5^{2x-6} = 5^{-3x-18}$$

The bases are the same so the exponents are equal quantities.

$$2x-6 = -3x-18$$

Set the exponents equal to each other and solve the equation.

$$2x-6+6 = -3x-18+6$$
$$2x = -3x-12$$
$$2x+3x = -3x+3x-12$$
$$5x = -12$$
$$\frac{5x}{5} = \frac{-12}{5}$$
$$\frac{\cancel{5}x}{\cancel{5}} = \frac{-12}{5}$$
$$\boxed{x = \frac{-12}{5}}$$

Example B

Solve the following exponential equation:

$4(x-2)^{\frac{1}{2}} = 16$

Solution: The variable appears in the base.

$$4(x-2)^{\frac{1}{2}} = 16$$

Divide both sides of the equation by 4.

$$\frac{4(x-2)^{\frac{1}{2}}}{4} = \frac{16}{4}$$
$$\frac{\cancel{4}(x-2)^{\frac{1}{2}}}{\cancel{4}} = \frac{\overset{4}{\cancel{16}}}{\cancel{4}}$$
$$(x-2)^{\frac{1}{2}} = 4$$

Multiply the exponents on each side of the equation by the reciprocal of $\frac{1}{2}$.

$$\left[(x-2)^{\frac{1}{2}}\right]^2$$

Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$.

$$(x-2)^{\frac{1}{2}\times 2} = (4)^{1\times 2}$$

Simplify the exponents.

$$(x-2)^1 = 4^2$$
$$x-2 = 16$$

Solve the equation.

$$x-2+2 = 16+2$$
$$x = 18$$
$$\boxed{x = 18}$$

Example C

Solve the following exponential equation:

$(2x-4)^{\frac{2}{3}} = \sqrt[3]{9}$

Solution: The variable appears in the base.

$(2x-4)^{\frac{2}{3}}=\sqrt[3]{9}$ Apply $\boxed{a^{\frac{m}{n}}=\sqrt[n]{a^m}=\left(\sqrt[n]{a}\right)^m \quad m,n\in N}$ to the right side of the equation.

$(2x-4)^{\frac{2}{3}}=(9)^{\frac{1}{3}}$ Write 9 as a power of 3.

$(2x-4)^{\frac{2}{3}}=(3^2)^{\frac{1}{3}}$ Apply the law of exponents $\boxed{(a^m)^n=a^{mn}}$ to the right side of the equation.

$(2x-4)^{\frac{2}{3}}=(3)^{2\times\frac{1}{3}}$ Simplify the exponents.

$(2x-4)^{\frac{2}{3}}=(3)^{\frac{2}{3}}$ The exponents are equal so the bases are equal quantities.

$2x-4=3$ Solve the equation.

$$2x-4+4=3+4$$

$$2x=7$$

$$\frac{2x}{2}=\frac{7}{2}$$

$$\frac{\cancel{2}x}{\cancel{2}}=\frac{7}{2}$$

$$\boxed{x=\frac{7}{2}}$$

Concept Problem Revisited

$9^{x+1}=\sqrt{27}$

To begin, write each side of the equation with a common base. Both 9 and 27 can be written as a power of '3'. Therefore, $(3^2)^{x+1}=\sqrt{3^3}$.

Apply

$$\boxed{(a^m)^n=a^{mn}}$$

to the left side of the equation. $3^{2x+2}=\sqrt{3^3}$

Express the right side of the equation in exponential form and apply

$$\boxed{(a^m)^n=a^{mn}}$$

.

$$3^{2x+2}=(3^3)^{\frac{1}{2}}$$
$$3^{2x+2}=3^{\frac{3}{2}}$$

Now that the bases are the same, then the exponents are equal quantities.

$2x+2=\frac{3}{2}$

Solve the equation.

$2(2x+2)=2\left(\frac{3}{2}\right)$ Multiply both sides of the equation by '2'. Simplify and solve.

$$4x+4=\cancel{2}\left(\frac{3}{\cancel{2}}\right)$$
$$4x+4=3$$
$$4x+4-4=3-4$$
$$\frac{\cancel{4}x}{\cancel{4}}=\frac{-1}{4}$$
$$\boxed{x=-\frac{1}{4}}$$

Vocabulary

Exponential Equation

An ***exponential equation*** is an equation in which the variable appears in either the exponent or in the base. The equation is solved by applying the laws of exponents.

Guided Practice

1. Use the laws of exponents to solve the following exponential equation: $27^{1-x}=\left(\frac{1}{9}\right)^{2-x}$

2. Use the laws of exponents to solve the following exponential equation: $(x-3)^{\frac{1}{2}}=(25)^{\frac{1}{4}}$

3. Use the laws of exponents to solve $\frac{(8^{x-4})(2^x)(4^{2x+3})}{32^x}=16$.

Answers:

1.

$27^{1-x}=\left(\frac{1}{9}\right)^{2-x}$	The variable appears in the exponent.
$(3^3)^{1-x}=(3^{-2})^{2-x}$	Write each side of the equation as a power of 3.
$(3^3)^{1-x}=(3^{-2})^{2-x}$	Apply the law of exponents $\boxed{(a^m)^n=a^{mn}}$.
$(3)^{3(1-x)}=(3)^{-2(2-x)}$	Simplify the exponents.
$3^{3-3x}=3^{-4+2x}$	The bases are the same so the exponents are equal quantities.
$3-3x=-4+2x$	Solve the equation.
$3-3-3x=-4-3+2x$	
$-3x=-7+2x$	
$-3x-2x=-7x+2x-2x$	
$-5x=-7$	
$\frac{-5x}{5}=\frac{-7}{5}$	
$\frac{\cancel{-5}x}{\cancel{-5}}=\frac{7}{5}$	
$\boxed{x=\frac{7}{5}}$	

2.

$(x-3)^{\frac{1}{2}} = (25)^{\frac{1}{4}}$	The variable appears in the base.
$(x-3)^{\frac{1}{2}} = (5^2)^{\frac{1}{4}}$	Write 25 as a power of 2.
$(x-3)^{\frac{1}{2}} = (5^2)^{\frac{1}{4}}$	Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$ to the right side of the equation.
$(x-3)^{\frac{1}{2}} = (5)^{2\times\frac{1}{4}}$	Simplify the exponents.
$(x-3)^{\frac{1}{2}} = (5)^{\frac{2}{4}}$	
$(x-3)^{\frac{1}{2}} = (5)^{\frac{1}{2}}$	The exponents are equal so the bases are equal quantities.
$x-3=5$	Solve the equation.
$x-3+3=5+3$	
$x=8$	
$\boxed{x=8}$	

3.

$\dfrac{(8^{x-4})(2^x)(4^{2x+3})}{32^x} = 16$	The variable appears in the exponent.
$\dfrac{[(2^3)^{x-4}](2^x)[(2^2)^{2x+3}]}{(2^5)^x} = 2^4$	Write all bases as a power of 2. Write 16 as a power of 2.
$\dfrac{[(2^3)^{x-4}](2^x)[(2^2)^{2x+3}]}{(2^5)^x} = 2^4$	Apply the law of exponents $\boxed{(a^m)^n = a^{mn}}$.
$\dfrac{[(2)^{3(x-4)}](2^x)[(2)^{2(2x+3)}]}{(2)^{5(x)}} = 2^4$	
$\dfrac{[(2)^{3x-12}](2^x)[(2)^{4x+6}]}{2^{5x}} = 2^4$	Simplify the exponents.
$\dfrac{[2^{3x-12}](2^x)[2^{4x+6}]}{2^{5x}} = 2^4$	Apply the law of exponents $\boxed{a^m \times a^n = a^{m+n}}$.
$\dfrac{[2^{3x+x+4x-12+6}]}{2^{5x}} = 2^4$	Simplify the exponents.
$\dfrac{2^{8x-6}}{2^{5x}} = 2^4$	Apply the laws of exponents $\boxed{\dfrac{a^m}{a^n} = a^{m-n}}$.
$2^{8x-6-5x} = 2^4$	Simplify the exponents.
$2^{3x-6} = 2^4$	The bases are the same so the exponents are equal quantities.
$3x-6=4$	Solve the equation.
$3x-6+6=4+6$	
$3x=10$	
$\dfrac{3x}{3} = \dfrac{10}{3}$	
$\dfrac{\cancel{3}x}{\cancel{3}} = \dfrac{10}{3}$	
$\boxed{x = \dfrac{10}{3}}$	

Practice

Use the laws of exponents to solve the following exponential equations:

1. $2^{3x-1} = \sqrt[3]{16}$
2. $36^{x-2} = \left(\frac{1}{6}\right)^{2x+5}$
3. $6(x-4)^{\frac{1}{3}} = 18$
4. $(3x-2)^{\frac{2}{5}} = 4$
5. $36^{x+1} = \sqrt{6}$
6. $3^{5x-1} = \sqrt[3]{9}$
7. $9^{2x-1} = \left(\sqrt[4]{27}\right)^{x}$
8. $(3x-2)^{\frac{3}{2}} = 8$
9. $(x+1)^{-\frac{5}{2}} = 32$
10. $\left(\sqrt{3}\right)^{4x} = 27^{x-3}$
11. $4^{3x-1} = \sqrt[3]{32}$
12. $(x+2)^{\frac{2}{3}} = (27)^{\frac{2}{9}}$
13. $(2^{x-3})(8^{x}) = 32$
14. $(x-2)^{\frac{1}{2}} = 9^{\frac{1}{4}}$
15. $8^{x+12} = \left(\frac{1}{16}\right)^{2x-7}$

6.9 Exponential Functions

Here you'll learn to sketch and recognize basic exponential functions. You will also learn a real world application of exponential functions.

Roberta invested \$600 into a mutual fund that paid 4% interest each year compounded annually.

i) Complete a table showing the value of the mutual fund for the first five years.

ii) Write an exponential function of the form $y = a \cdot b^x$ to describe the value of the mutual fund.

iii) Use the exponential function to determine the value of the mutual fund in 15 years.

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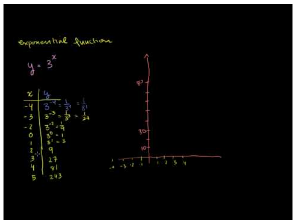
Khan Academy Exponential Growth Functions



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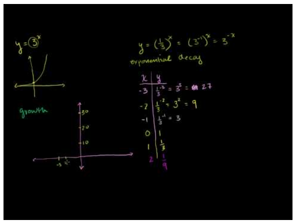
Khan Academy Exponential Decay Functions



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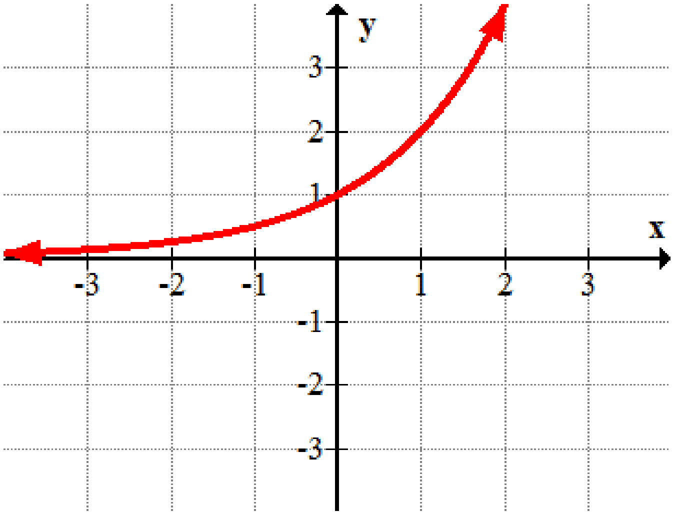
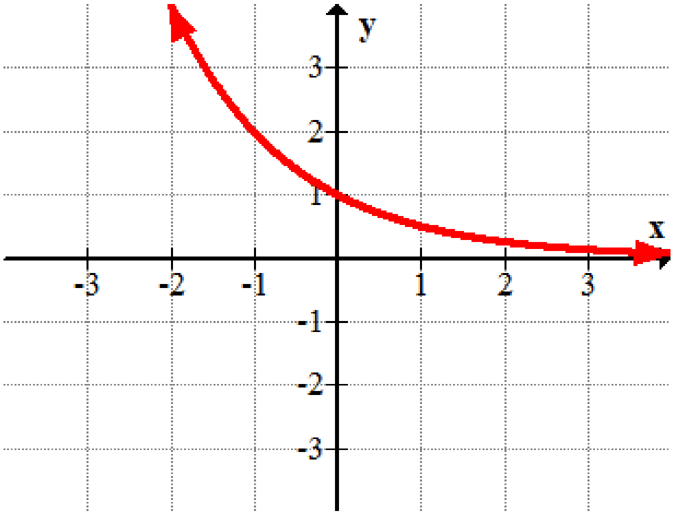
Guidance

An exponential function is a function with a variable in the exponent. Two examples of exponential functions are shown below:

$$\boxed{y = 2^x}$$



$$\boxed{y = \left(\frac{1}{2}\right)^x}$$



Here are some facts to notice about the functions and their graphs:

- The graph of $y = 2^x$ is an increasing curve. It shows growth.
- Each y-value of $y = 2^x$ is 2 times the previous y-value (for the integer values of x). For example, the points on the graph go from (0, 1) to (1, 2) to (2, 4). The next point would be (3, 8). The y-values keep being multiplied by 2.
- The graph of $y = \left(\frac{1}{2}\right)^x$ is a decreasing curve. It shows decay.
- Each y-value of $y = \left(\frac{1}{2}\right)^x$ is $\frac{1}{2}$ the value of the previous y-value (for the integer values of x). For example, the points on the graph go from (0, 1) to $(1, \frac{1}{2})$ to $(2, \frac{1}{4})$. The y-values keep being multiplied by $\frac{1}{2}$.
- Both graphs have a y-intercept of 1. This is because anything to the zero power is equal to 1.
- The domain of each function is $D = \{x|x \in R\}$.
- The range for each function is $R = \{y|y > 0,\ y \in R\}$.

Based on the above observations, you can deduce that an exponential function of the form $y = ab^x$ where $b > 0$ has the following properties:

Properties of an Exponential Function of the form

- 'b' is the value of the common ratio. Within the function, as the x-value increases by 1, the y-value is multiplied by the common ratio.
- If $b > 1$ then the curve will represent exponential growth.
- If $0 < b < 1$ then the curve will represent exponential decay.
- Every exponential function of the form $y = ab^x$ will pass through the point $(0, a)$. a will always be the y-intercept of the function, or its value at time 0.
- Every exponential function of the form $y = ab^x$ will have the domain and range:

$$D = \{x|x \in R\} \ and \ R = \{y|y > 0, \ y \in R\}$$

Example A

For the following tables of values that represent exponential functions, determine the common ratio:

i)

x	0	1	2	3	4	$\cdots$
y	1	2	4	8	16	$\cdots$

ii)

x	0	1	2	3	4	$\cdots$
y	100	50	25	12.5	6.25	$\cdots$

Solutions:

i) The common ratio is a constant that is determined by $r = \frac{t_{n+1}}{t_n}$.

$$\begin{aligned}
r &= \frac{t_{n+1}}{t_n} = \frac{2}{1} = 2\\
r &= \frac{t_{n+1}}{t_n} = \frac{4}{2} = 2\\
r &= \frac{t_{n+1}}{t_n} = \frac{8}{4} = 2\\
r &= \frac{t_{n+1}}{t_n} = \frac{16}{8} = 2\\
&\boxed{\text{The common ratio is 2.}}
\end{aligned}$$

ii) The common ratio is a constant that is determined by $r = \frac{t_{n+1}}{t_n}$.

$$r = \frac{t_{n+1}}{t_n}$$
$$r = \frac{50}{100} = \frac{1}{2}$$
$$r = \frac{25}{50} = \frac{1}{2}$$
$$r = \frac{12.5}{25} = \frac{1}{2}$$
$$r = \frac{6.25}{12} = \frac{1}{2}$$
$$\boxed{\text{The common ratio is } \frac{1}{2}.}$$

Example B

Using the exponential function

$$\boxed{f(x) = 3^x}$$

, determine the value of each of the following:

i) $f(2)$

ii) $f(3)$

iii) $f(0)$

iv) $f(4)$

v) $f(-2)$

Solutions: $f(x) = 3^x$ is another way to express $y = 3^x$. To determine the value of the function for the given values, replace the exponent with that value and evaluate the expression.

i)

$$f(x) = 3^x$$
$$f(2) = 3^2$$
$$f(2) = 9$$
$$\boxed{f(2) = 9}$$

ii)

$$f(x) = 3^x$$
$$f(3) = 3^3$$
$$f(3) = 27$$
$$\boxed{f(3) = 27}$$

iii)

$$f(x) = 3^x$$
$$f(0) = 3^0$$
$$f(0) = 1$$
$$\boxed{f(0) = 1}$$

iv)

$$f(x) = 3^x$$
$$f(4) = 3^4$$
$$f(4) = 81$$
$$\boxed{f(4) = 81}$$

v)

$$f(x) = 3^x$$
$$f(-2) = 3^{-2}$$
$$f(-2) = \frac{1}{3^2}$$
$$\boxed{f(-2) = \frac{1}{9}}$$

Example C

On January 1, Juan invested \$1.00 at his bank at a rate of 10% interest compounded daily.

i) Create a table of values for the first 8 days of the investment.

ii) What is the common ratio?

iii) Determine the equation of the function that would best represent Juan's investment.

iv) How much money will Juan have in his account on January 31?

v) If Juan had originally invested \$100 instead of \$1.00 at 10%, what exponential equation would describe the investment. How much money would he have in his account on January 31?

Solution:

i)

$$\begin{array}{llll} 1.00(.10) = 0.10 & 1.10(.10) = 0.11 & 1.21(.10) = 0.12 & 1.33(.10) = 0.13 \\ 1.00 + 0.10 = 1.10 & 1.10 + 0.11 = 1.21 & 1.21 + 0.12 = 1.33 & 1.33 + 0.13 = 1.46 \\ \\ 1.46(.10) = 0.15 & 1.61(.10) = 0.16 & 1.77(.10) = 0.18 & 1.95(.10) = 0.20 \\ 1.46 + 0.15 = 1.61 & 1.61 + 0.16 = 1.77 & 1.77 + 0.18 = 1.95 & 1.95 + 0.20 = 2.15 \end{array}$$

of days	0	1	2	3	4	5	6	7	8
Money ($)	1	1.10	1.21	1.33	1.46	1.61	1.77	1.95	2.15

ii) The common ratio is a constant that is determined by $\frac{t_{n+1}}{t_n}$. Therefore, the common ratio for this problem is $r = \frac{t_{n+1}}{t_n} \rightarrow \frac{1.10}{1} = 1.10 \rightarrow \frac{1.21}{1.10} = 1.10 \rightarrow \frac{1.33}{1.21} = 1.10$.

The common ratio is

$$\boxed{1.10}$$

.

iii) The equation of the function to model Juan's investment is $y = 1.10^x$

iv)

$$y = 1.10^x \rightarrow y = 1.10^{31} \rightarrow \boxed{y = \$19.19}$$

.

On January 31, Juan will have \$19.19 in his account.

v)

$$\begin{aligned} y &= 100(1.10)^x \\ y &= 100(1.10)^{31} \rightarrow y = \$1919.43 \rightarrow \boxed{y = \$1919.43}. \end{aligned}$$

On January 31, Juan would have \$1919.43 in his account if he had invested \$100 instead of \$1.00.

Concept Problem Revisited

Roberta invested \$600 into a mutual fund that paid 4% interest each year compounded annually.

i)

$600(.04) = 24$
$600 + 24 = 624$

$624(.04) = 24.96$
$624 + 24.96 = 648.96$

$648.96(.04) = 25.96$
$648.96 + 25.96 = 674.92$

$674.92(.04) = 27.00$
$674.92 + 27.00 = 701.92$

$701.92(.04) = 28.08$
$701.92 + 28.08 = 730.00$

Time $(years)$	0	1	2	3	4	5
Value ($)	600	624	648.96	674.92	701.92	730.00

ii) The initial value is \$600. The common ratio is 1.04 which represents the initial investment and the interest rate of 4%. The exponent is the time in years. The exponential function is $y = 600(1.04)^x$ or $v = 600(1.04)^t$.

iii)

$$v = 600(1.04)^t$$
$$v = 600(1.04)^{15}$$
$$v = \$1080.57$$
$$\boxed{v = \$1080.57}$$

The value of the mutual fund in fifteen years will be $1080.57.

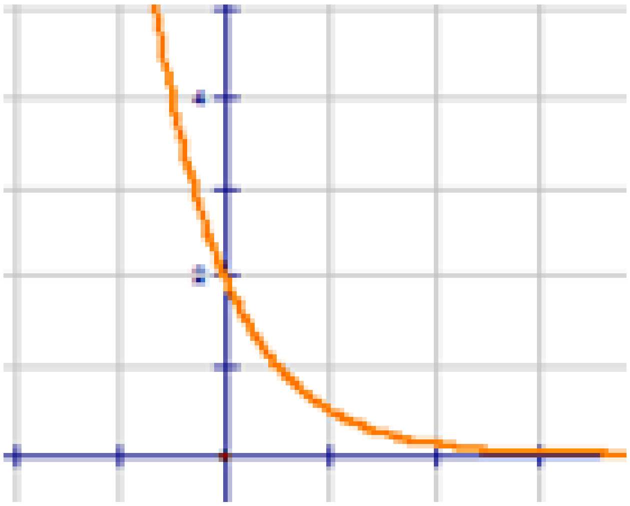
Vocabulary

Common Ratio

The ***common ratio*** is the constant that exists between successive terms and is determined by applying the formula $r = \frac{t_{n+1}}{t_n}$. In an exponential function of the form $y = b^x$, 'b' represents the common ratio.

Decay Curve

A ***decay curve*** is the name given to the graph of an exponential function in which the common ratio is such that $0 < b < 1$. The graph is decreasing since the value of the function falls as the value of 'x' increases. The following shows a decay curve:

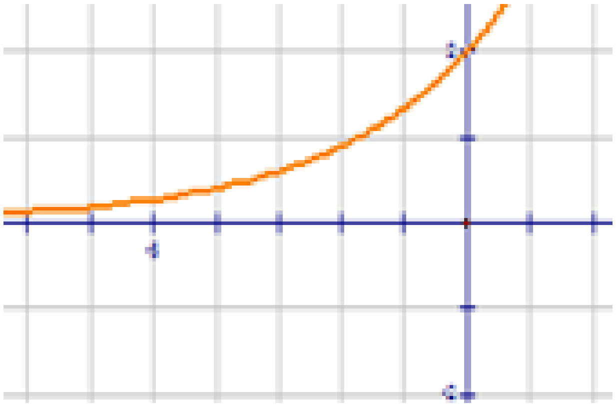


Exponential Function

An ***exponential function*** is a function (y) of the form $y = b^x$, where 'b' is the common ratio and 'x' is an exponent that represents the variable.

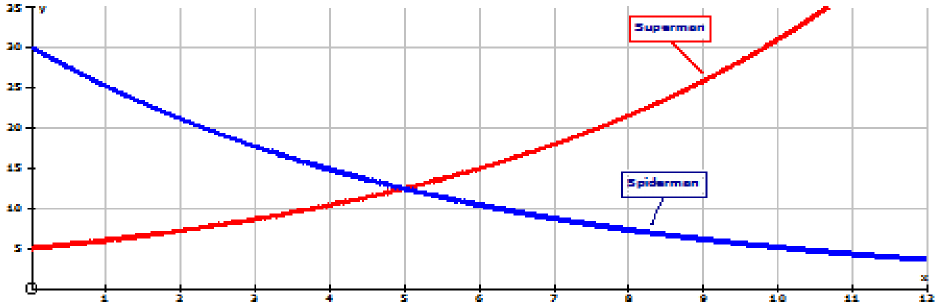
Growth Curve

A ***growth curve*** is the name given to the graph of an exponential function in which the common ratio is such that $b > 1$. The graph is increasing since the value of the function rises as the value of 'x' increases. The following shows a growth curve:



Guided Practice

1. The graph below shows the change in value of two comic books purchases in the year 2000. Both comics were expected to be good investments, but one of them did not perform as expected. Use the graphs to answer the questions.



a) What was the purchase price of each comic book?
b) Which comic book shows exponential growth?
c) Which comic book shows exponential decay?
d) In what year were both comic books equal in value?
e) State the domain and range for each comic.

2. Paulette bought a Bobby Orr rookie card for \$300. The value of the card appreciates (increases) by 30% each year.

a) Complete a table of values to show the first five years of the investment.
b) Determine the common ratio for the successive terms.
c) Determine the equation of the exponential function that models this investment.

3. Due to the closure of the pulp and paper mill, the population of the small town is decreasing at a rate of 12% annually. If there are now 2400 people living in the town, what will the town's projected population be in eight years?

Answers:

1. a) The purchase price of each comic book is the y-intercept. The y-intercept is the initial value of the books. The Spiderman comic book cost \$30.00 and the Superman comic book cost \$5.00.

b) The Superman comic book shows exponential growth.

c) The Spiderman comic book shows exponential decay.

d) In 2005 both comic books were equal in value. The graphs intersect at approximately (5, \$12.50), where 5 represents five years after the books were purchased.

e) The domain and range for each comic is $D = \{x|x \in R\}$ and $R = \{y|y > 0,\ y \in R\}$

2.

$$300(.30) = 90 \qquad 390(.30) = 117 \qquad 507(.30) = 152.10$$
$$300 + 90 = 390 \qquad 390 + 117 = 507 \qquad 507 + 152.10 = 659.10$$

$$659.10(.30) = 197.73 \qquad 856.83(.30) = 257.05$$
$$659.10 + 197.73 = 856.83 \qquad 856.83 + 257.05 = 1113.88$$

a)

Time $(years)$	0	1	2	3	4	5
Value (\$)	300	390	507	659.10	856.83	1113.88

b)

$$\begin{aligned}
r &= \frac{t_{n+1}}{t_n}\\
r &= \frac{390}{300} = 1.3\\
r &= \frac{507}{390} = 1.3\\
r &= \frac{659.10}{507} = 1.3\\
r &= \frac{856.83}{659.10} = 1.3\\
r &= \frac{1113.88}{856.83} = 1.3
\end{aligned}$$

c) The exponential function that would model Paulette's investment is

$$\boxed{y = 300(1.3)^x \textit{ or } v = 300(1.3)^t}$$

3. The town's population is decreasing by 12% annually. The simplest way to use this in an exponential function is to use the percent of the population that still exists each year – 88%.

Therefore, the exponential function would consist of the present population (a), the common ratio is 0.88 (b) and the time in years would be the exponent (x). The function is $p = 2400(0.88)^t$

The population in eight years would be

$$\begin{aligned}
p &= 2400(0.88)^t\\
p &= 2400(0.88)^8\\
p &= 863.123\\
p &\approx 863 \textit{ people}
\end{aligned}$$

Practice

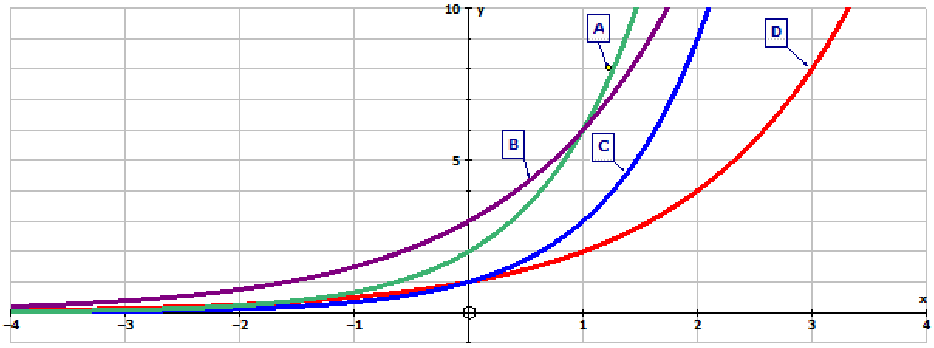
Brandon bought a car for \$13,000. The value of the car depreciates by 20% each year.

1. Complete a table of values to show the car's values for the first five years.
2. Determine the exponential function that would model the depreciation of Brandon's car.

For each of the following exponential functions, identify the common ratio and the y-intercept, and tell if the function represents a growth or decay curve.

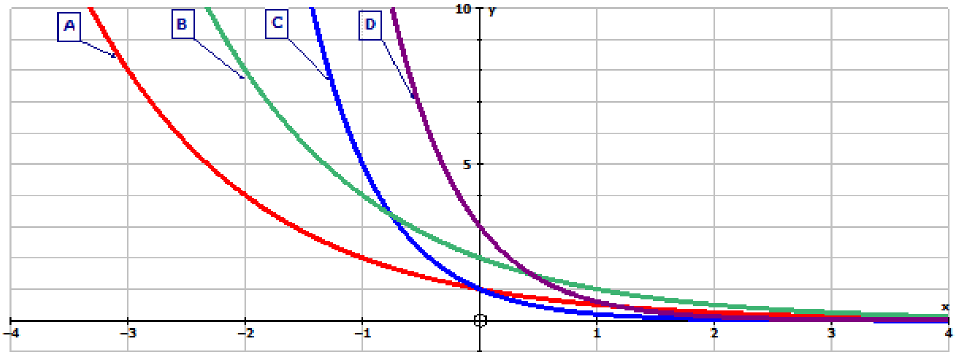
3. $y = 4(5)^x$
4. $y = 13(2.3)^x$
5. $y = 0.85(0.16)^x$
6. $y = 1.6(0.5)^x$
7. $y = 0.4(2.1)^x$

Match each graph below with its corresponding equation:



8. $y = 2^x$
9. $y = 3^x$
10. $y = 2(3)^x$
11. $y = 3(2)^x$
12. Do these graphs represent growth or decay?

Match each graph below with its corresponding equation:



13. $y = 0.5^x$

14. $y = 0.2^x$
15. $y = 2(0.5)^x$
16. $y = 3(0.2)^x$
17. Do these graphs represent growth or decay?

18. Jolene purchased a summer home for $120,000 in 2002. If the property has consistently increased in value by 11% each year, what will be the value of her summer home in 2012?

6.10 Advanced Exponential Functions

Here you'll learn more about exponential functions that are of the form $y = a(b)^{\left(\frac{x}{c}\right)}$ and $y = a(b)^{\left(\frac{x}{c}\right)} + d$.

Andrea invested money into a mutual fund. Every three months she received a bank statement indicating the growth of her investment. Andrea recorded the following data from her bank statements:

Time $(months)$	0	3	6	9	12	15
Value ($)	1200	1224	1248.48	1273.45	1298.92	1324.90

i) How much was the initial investment?

ii) What was the rate of interest the bank applied to her investment? Express this as a percent.

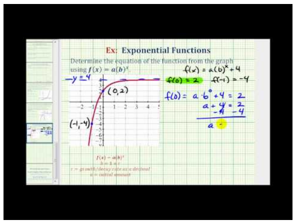
iii) What is the exponential function that best models the value of Andrea's investment?

iv) What will be the mutual fund's value in two years?

v) Can you use technology to determine the length of time it will take for the mutual fund to double in value?

Watch This

James Sousa: Find the Equation of a Transformed Exponential Function From a Graph



MEDIA

Click image to the left for more content.

Guidance

You have learned about exponential functions of the form $y = ab^x$ in which the x-value increased by increments of one. However, data is often collected using other intervals such as every three months, every two hours, every fifteen minutes, and so forth. You still want to obtain the exponential function to represent such a situation, but the function must be modified to accommodate this interval change.

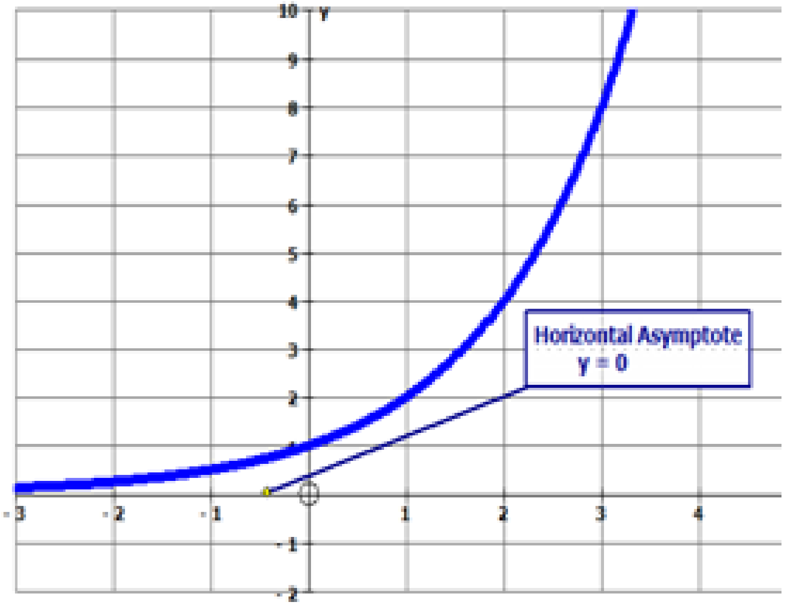
The exponential function will now be expressed in the form $y = a(b)^{\frac{x}{c}}$ where:

- 'a' represents the initial value (y-intercept)
- 'b' represents the common ratio (the rate of growth or decay)
- 'c' represents the increment in value of x.

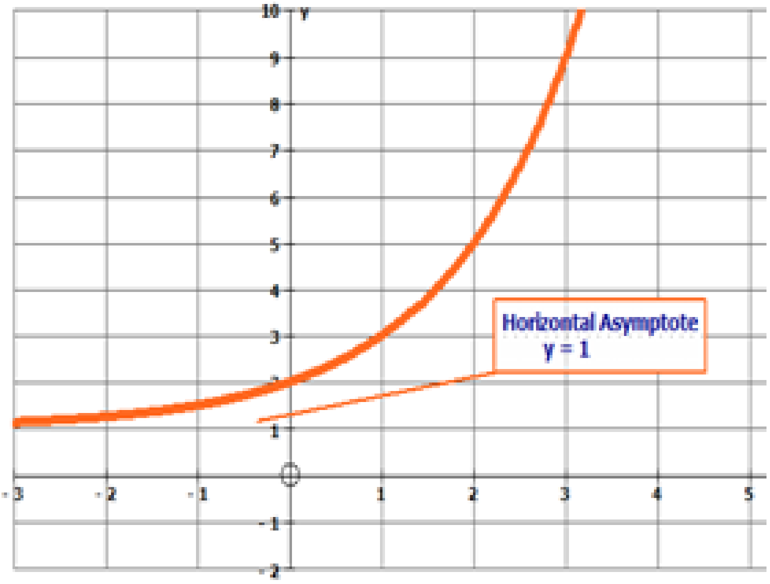
With exponential functions of the form $y = ab^x$, the **horizontal asymptote** is the x-axis. A horizontal asymptote is a horizontal line that a function keeps getting closer and closer to. Look at the graph of the function $y = 2^x$. The growth curve gets closer and closer to the x-axis, so this is the horizontal asymptote. The equation of the horizontal asymptote is

$$\boxed{y = 0}$$

.



Now look at the graph of the function $y = 2^x + 1$. The growth curve gets closer and closer to the line $y = 1$, so this is the horizontal asymptote.



For exponential functions of the form $y = a(b)^{\frac{x}{c}} + d$:

- '$a + d$' represents the initial value (y-intercept)
- 'b' represents the common ratio (the rate of growth or decay)
- 'c' represents the increment in value of x.
- '$y = d$' represents the horizontal asymptote.

Example A

Write an exponential function of the form $y = a(b)^{\frac{x}{c}}$ to describe the table of values.

X	0	2	4	6	8	10
Y	3	6	12	24	48	96

Solution:

The initial value '*a*' is 3. The common ratio is:

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{6}{3} = 2} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{12}{6} = 2} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{24}{12} = 2}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{48}{24} = 2} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{96}{48} = 2}$$

The increment in the x-value is 2. The exponential function to describe the table of values is

$$\boxed{y = 3(2)^{\frac{x}{2}}}$$

.

Example B

A radioactive isotope decays exponentially over time according to the equation $A(t) = 42\left(\frac{1}{2}\right)^{\frac{t}{20}}$, where $A(t)$ is the amount of the isotope present at time t, in days.

i) What is the half-life of the isotope?

ii) Explain in words what the equation represents.

iii) How much isotope will remain after 35 days?

Solutions:

i) Half life is the length of time it takes for only half of the original amount of isotope to be present. Since $b = \frac{1}{2}$, the half life of the isotope is the increment in the x-value which is 'c'. The half life is 20 days.

ii) The equation represents the amount (A) of a decaying isotope that remains at any time x, in days. There are 42 units initially present and its half-life is 20 days.

iii)

$$A(t) = 42\left(\frac{1}{2}\right)^{\frac{t}{20}}$$
$$A(t) = 42\left(\frac{1}{2}\right)^{\frac{35}{20}} = 12.5\ units$$

Example C

Brian bought a cup of coffee from the cafeteria and placed it on the ground while recording its change in temperature (the temperature that day was $0°C$). The temperature of the coffee was recorded every 3 minutes, and the results are shown in the following table:

Time (min)	0	3	6	9	12	15
Temp $(0°C)$	90.0	81.0	72.9	65.6	59.0	53.1

i) Write an exponential function to describe the temperature of the coffee after 't' minutes.

ii) Determine the temperature of the coffee after 30 minutes.

iii) Determine the temperature of the coffee after I hour.

Solutions:

i) The initial value 'a' is 90.

The common ratio is

$$\boxed{r=\frac{81}{90}=0.9}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{72.9}{81}=0.9}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{65.6}{72.9}=0.899=0.9}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{59.0}{65.6}=0.899=0.9}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{53.1}{59}=0.9}$$

The increment in the x-value is 3.

The exponential function to describe the table of values is

$$\boxed{y=90(0.9)^{\frac{x}{3}}}$$

.

ii) Replace 'x' in the exponent with '30' and calculate the value of the function. $y=90(0.9)^{\frac{30}{3}}=31.4°C$

iii) One hour equals 60 minutes. Replace 'x' in the exponent with '60' and calculate the value of the function. $y=90(0.9)^{\frac{60}{3}}=10.9°C$

Example D

In the example above, the outdoor temperature was $0°C$. This temperature was chosen to make the value of the horizontal asymptote zero. More importantly, this enabled us to determine the value of the common ratio.

Now you will investigate the same problem again with the outdoor temperature now being $15°C$. What effect will this have on the value of the common ratio? How can the value of 'b' be determined so that an exponential function can be created to model the data?

Brian bought a cup of coffee from the cafeteria and placed it on the ground while recording its change in temperature (the temperature that day was $15°C$). The temperature of the coffee was recorded every 3 minutes, and the results are shown in the following table:

Time (min)	0	3	6	9	12	15
Temp $(0°C)$	105.0	96.0	87.9	80.6	74.0	68.1

i) Determine the rate at which the coffee is cooling.

ii) Write an exponential function to describe the temperature of the coffee after 't' minutes.

iii) Determine the temperature of the coffee after I hour.

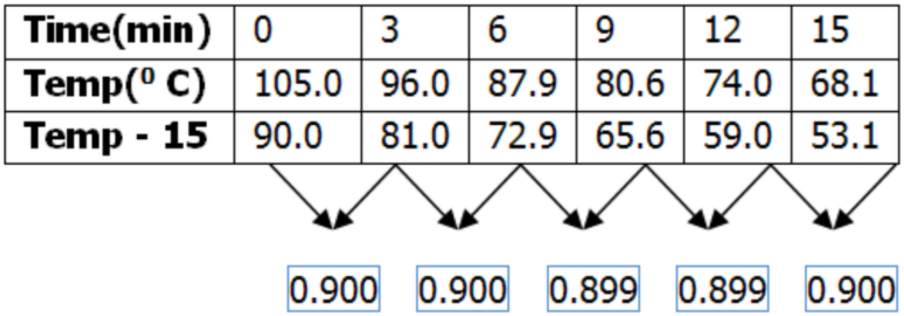
Solutions:

i) You can first try to determine the common ratio by using the values in the given table.

$$\boxed{r=\frac{96}{105}=0.914} \quad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{87.9}{96}=0.916} \quad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{80.6}{87.9}=0.917}$$

$$\boxed{r=\frac{t_{n+1}}{t_n}=\frac{74.0}{80.6}=0.918} \quad \boxed{r=\frac{t_{n+1}}{t_n}=\frac{68.1}{74.0}=0.920}$$

Notice that this doesn't work! The common ratio must be consistent. To eliminate the difficulty with identifying the common ratio, simply subtract the value of the horizontal asymptote (15) from each of the y-values. Then, look for the common ratio.



Time(min)	0	3	6	9	12	15
Temp(^{0} C)	105.0	96.0	87.9	80.6	74.0	68.1
Temp - 15	90.0	81.0	72.9	65.6	59.0	53.1

The common ratio is 0.9. Therefore, the temperature of the coffee is decreasing by 10% every 3 minutes.

ii) The value of the 'a' is the initial temperature less the outdoor temperature. $a = 105.0°C - 15.0°C = 90°C$

The initial value 'a' is 90. The common ratio 'b' is 0.9. The increment in the x-values is 3. Therefore, $c = 3$. The coolest temperature that will be reached by the cooling coffee is that of the outdoor temperature. Therefore $d = 15$. The exponential function that models the problem is

$$\boxed{y=90(0.9)^{\frac{x}{3}}+15}$$

iii) One hour equals 60 minutes. Replace 'x' in the exponent with '60' and calculate the value of the function.

$$\boxed{y=90(0.9)^{\frac{60}{3}}+15=25.9°C}$$

Concept Problem Revisited

Andrea invested money into a mutual fund. Every three months she received a bank statement indicating the growth of her investment. Andrea recorded the following data from her bank statements:

Time $(months)$	0	3	6	9	12	15
Value ($)	1200	1224	1248.48	1273.45	1298.92	1324.90

i) The initial investment was \$1200.

ii) The rate of interest that the bank applied to her investment was:

$$\begin{array}{ccccc}
r = \frac{t_{n+1}}{t_n} & r = \frac{t_{n+1}}{t_n} & r = \frac{t_{n+1}}{t_n} & r = \frac{t_{n+1}}{t_n} & r = \frac{t_{n+1}}{t_n} \\
r = \frac{1224}{1200} & r = \frac{1248.48}{1224} & r = \frac{1273.45}{1248.48} & r = \frac{1298.92}{1273.45} & r = \frac{1324.90}{1298.92} \\
\boxed{r = 1.02} & \boxed{r = 1.02} & \boxed{r = 1.02} & \boxed{r = 1.02} & \boxed{r = 1.02}
\end{array}$$

The common ratio is 1.02. Therefore the interest rate was 2%.

iii) The increment of the x-value is 3. Therefore 'c' $= 3$. The exponential function to model Andrea's investment is

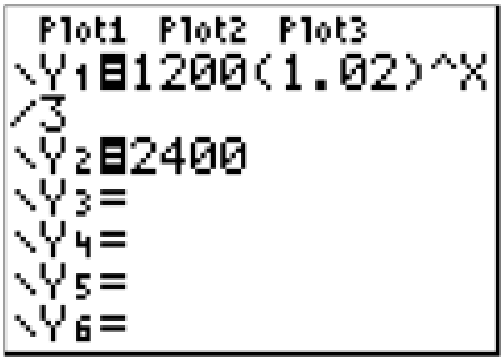
$$\boxed{y = 1200(1.02)^{\frac{x}{3}}}$$

.

iv) The value of 'c' is 3. This represents the every 3 months that the bank applies the interest to the investment. Therefore, the value of 'x' must be in months. There are 24 months in two years. The value of Andrea's investment in two years time will be:

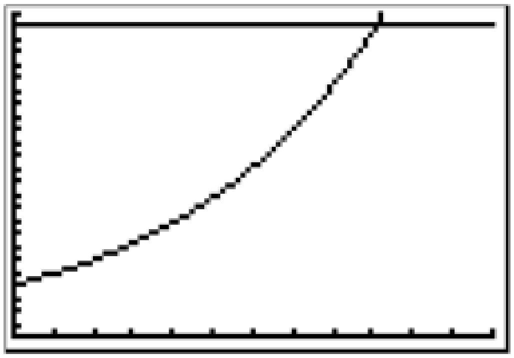
$$\boxed{y = 1200(1.02)^{\frac{24}{3}} = \$1405.99}$$

v) Press $y =$ and enter the equations $y = 1200(1.02)^{\frac{x}{3}}$ and $y = 2400$ as shown below.



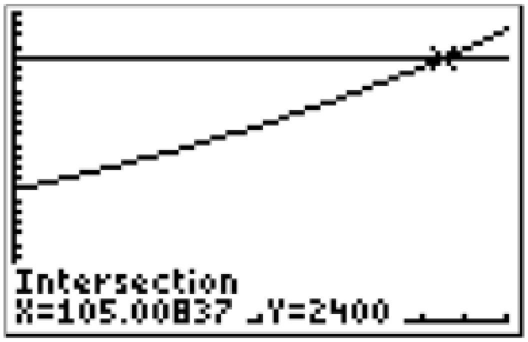
```
Plot1 Plot2 Plot3
\Y1■1200(1.02)^X
/3
\Y2■2400
\Y3=
\Y4=
\Y5=
\Y6=
```

Graph the equations:

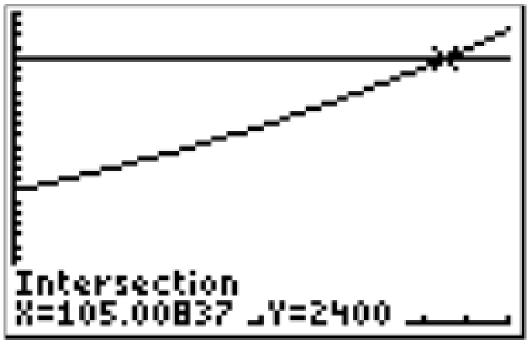


Determine the intersection point of the growth curve and the horizontal line by using

2^{nd} TRACE 5 ENTER ENTER ENTER



```
Intersection
X=105.00837  Y=2400
```



```
Intersection
X=105.00837  Y=2400
```

The intersection point is $(105.00837, 2400)$. It will take 105 months (8 years and 9 months) for Andrea's investment to double in value.

Vocabulary

Horizontal Asymptote

A ***horizontal asymptote*** is a horizontal line that a curve gets closer and closer to.

Common Ratio

The ***common ratio*** is the constant that exists between successive terms and is determined by applying the formula $r = \frac{t_{n+1}}{t_n}$. In an exponential function of the form $y = b^x$, 'b' represents the common ratio.

Decay Curve

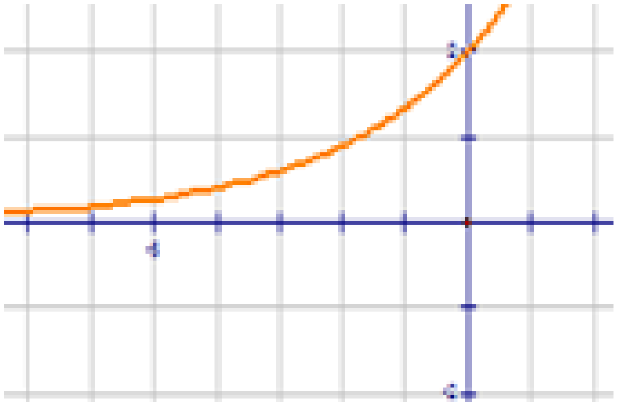
A ***decay curve*** is the name given to the graph of an exponential function in which the common ratio is such that $0 < b < 1$. The graph is decreasing since the value of the function falls as the value of 'x' increases. The following shows a decay curve:

Exponential Function

An ***exponential function*** is a function (y) of the form $y = b^x$, where 'b' is the common ratio and 'x' is an exponent that represents the variable.

Growth Curve

A ***growth curve*** is the name given to the graph of an exponential function in which the common ratio is such that $b > 1$. The graph is increasing since the value of the function rises as the value of 'x' increases. The following shows a growth curve:



Guided Practice

1. A bucket of tar is heated outside to repair a roof that is leaking. Once the leak has been repaired, the remaining tar is left in the bucket to cool down to use again at a later time. Its change in temperature, T, measured in $^\circ C$, with respect to time, t, in minutes, can be modeled with the function

$$\boxed{T = 95(0.72)^{\frac{t}{15}} + 30}$$

.

i) What is the initial temperature of the tar?
ii) What is the outdoor temperature?

iii) At what rate is the temperature of the tar changing over time?
iv) Determine the temperature of the tar after 30 minutes.
v) Determine the temperature of the tar after 2 hours.
vi) Draw a sketch to show the change in temperature of the tar over time.

2. Roberto's car radiator is always overheating. He decided to monitor its change in temperature. The outdoor temperature was $73.4^\circ F$, so he did not mind doing the task at hand. Roberto recorded the temperature that was determined every five minutes. The following table is the recorded temperatures.

Time (min)	0	5	10	15	20	25
Temp $(0^\circ F)$	224.6	197.42	175.1	156.7	141.8	129.4

i) Determine the rate at which the radiator is cooling.
ii) Determine the value of 'a' in $y = a(b)^{\frac{x}{c}} + d$ for this problem.
iii) What is the equation of the horizontal asymptote?
iv) Write an exponential function to indicate the temperature of the radiator after 't' minutes.
v) What was the temperature of the radiator after one hour?
vi) Draw a sketch of the radiator's change in temperature over time.

3. A deadly bacteria is threatening the small town of Norman. The bacteria are doubling every 3 hours. If there were initially 250 spores, how many will be present in 12 hours?

Answers:

1. i) The initial temperature is the temperature 'a' plus the outdoor temperature of $30^\circ C$.

The exponential function is of the form $y = a(b)^{\frac{x}{c}} + d$. The initial temperature is

$$\boxed{a + d = 95^\circ C + 30^\circ C = 125^\circ C}$$

.

ii) The outdoor temperature is the horizontal asymptote ('d') of the function. The outdoor temperature is

$$\boxed{30^\circ C}$$

.

iii) The common ratio 'b' represents the rate expressed as a decimal that the tar is maintaining. Therefore the rate at which it is cooling down is

$$100 - 0.72 = 0.28$$
$$0.28 \times 100\% = 28\%$$

. The temperature of the tar is dropping 28% every 15 minutes.

iv) The temperature of the tar after 30 minutes is determined by replacing 't' in the exponent with 30 and then performing the calculation.

$$T = 95(0.72)^{\frac{t}{15}} + 30$$
$$T = 95(0.72)^{\frac{30}{15}} + 30$$
$$T = 79.2^\circ C$$

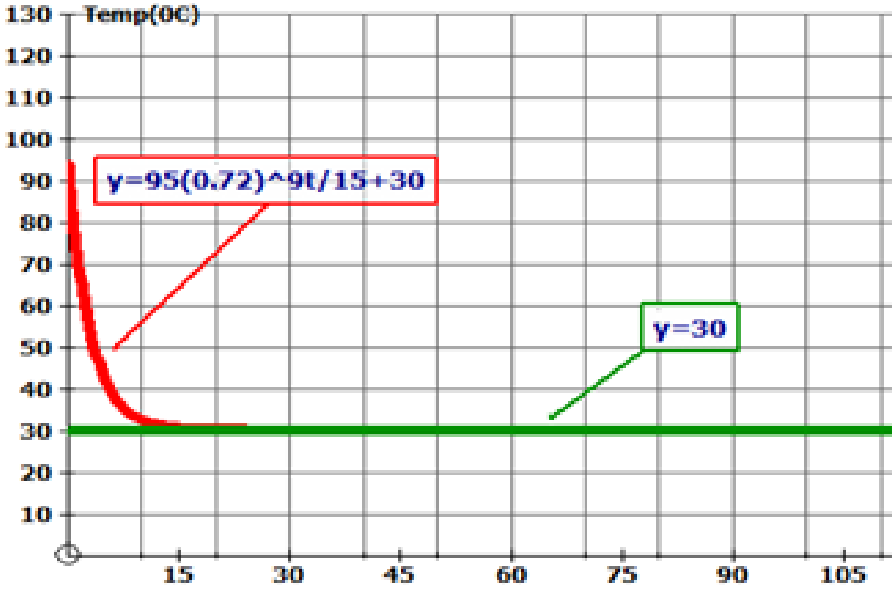
v) There are 60 minutes in one hour and 120 minutes in two hours. The temperature of the tar after 2 hours is determined by replacing 't' in the exponent with 120 and then performing the calculation.

$$\begin{aligned}T &= 95(0.72)^{\frac{t}{15}}+30\\ T &= 95(0.72)^{\frac{120}{15}}+30\\ T &= 36.9^\circ C\end{aligned}$$

vi) A sketch to represent

$$\boxed{T = 95(0.72)^{\frac{t}{15}}+30}$$

is shown below:



2. i) The rate by which the radiator is cooling must be determined by first subtracting the outdoor temperature of $73.4^\circ F$ from each of the temperature values.

Time (min)	0	5	10	15	20	25
Temp$(^\circ F)$	224.6	197.4	175.1	156.7	141.8	129.4
Temp $-$ 73.4	151.2	124.0	101.7	83.3	68.4	56.0

Now, determine the common ratio using the final row of the table.

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{124.0}{151.2} = 0.820} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{101.7}{124.0} = 0.820} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{83.3}{101.7} = 0.819}$$

$$\boxed{r = \frac{t_{n+1}}{t_n} = \frac{68.4}{83.3} = 0.821} \quad \boxed{r = \frac{t_{n+1}}{t_n} = \frac{56.0}{68.4} = 0.818}$$

Therefore the common ratio is 0.82. The rate at which the radiator is cooling is

$$\boxed{100\% - 82\% = 18\%}$$

every five minutes.

ii) The value of 'a' in $y = a(b)^{\frac{x}{c}} + d$ is the initial temperature given in the table less the outdoor temperature.

$$\boxed{a = 224.6^\circ F - 73.4^\circ F = 151.2^\circ F}$$

iii) The equation of the horizontal asymptote is $y =$ the outdoor temperature or

$$\boxed{y = 73.4}$$

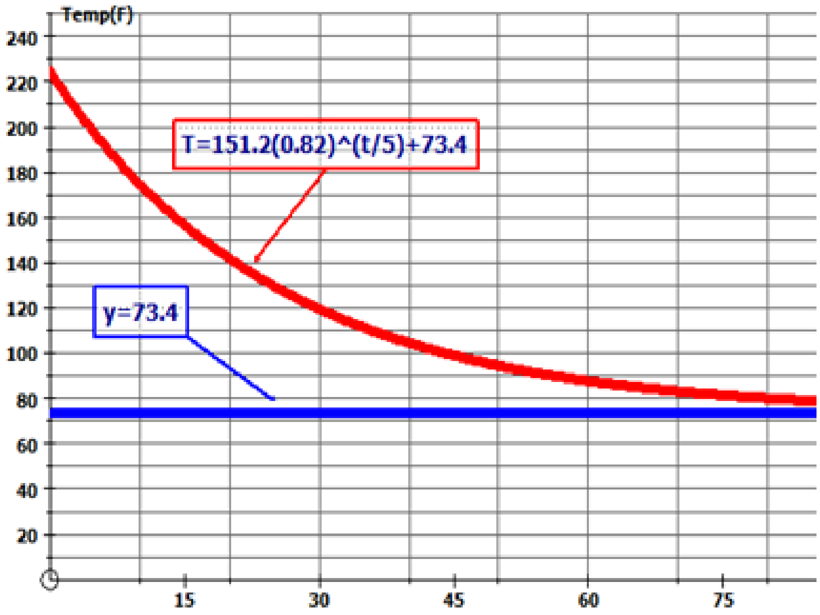
iv) The exponential function that models the change in temperature of the radiator is:

$$\begin{aligned}a &= 151.2\\ b &= 0.82\\ c &= 5\\ d &= 73.4\\ T &= 151.2(0.82)^{\frac{t}{5}} + 73.4\end{aligned}$$

v) There are 60 minutes in one hour. Therefore

$$\boxed{T = 151.2(0.82)^{\frac{60}{5}} + 73.4 = 87.4^\circ F}$$

vi)



3. Write the exponential function to model the number of spores produced over time.

$$\begin{aligned}a &= 250\\ b &= 3\\ x &= 12\\ y &= a(b)^x\\ y &= 250(3)^{12}\end{aligned}$$

Use the exponential function to determine the number of spores present after 12 hours.

$$\boxed{y = 250(3)^{12} = 132,860,250 \; spores}$$

Practice

Write an exponential function in the form $y = a(b)^{\frac{x}{c}}$ to model each table of values.

1.

X	0	5	10	15	20	25
Y	8	4	2	1	0.5	0.25

2.

X	0	3	6	9	12	15
Y	2	10	50	250	1250	6250

3.

X	-4	0	4	8	12	16
Y	6	2.4	0.96	0.384	0.1536	0.06144

4.

X	-0.6	-0.3	0	0.3	0.6	0.9
Y	10	12	14.4	17.28	20.736	24.8832

5.

X	0	0.1	0.2	0.3	0.4	0.5
Y	5	15	45	135	405	1215

A radioactive isotope decays at the rate indicated by the exponential function $A(t) = 800\left(\frac{1}{2}\right)^{\frac{t}{1500}}$, where '$t$' is the time in years and $A(t)$ is the amount of the isotope, in grams, remaining.

6. What is the initial mass of the isotope?

7. How long will it take for the isotope to be reduced to half of its original amount?
8. What will the mass of the isotope be after 4500 years?

For each of the following exponential functions state the equation of the horizontal asymptote, the y-intercept, the range, and whether it is a growth or decay function.

9. $y = 2^x + 5$
10. $y = 2(3)^x$
11. $y = 6\left(\frac{1}{3}\right)^x + 5$
12. $y = 4(0.4)^x + 1.8$
13. $y = 12(1.25)^x$

A hot cup of coffee cools exponentially with time as it sits on the teacher's desk. Its change in temperature, T, measured in $^\circ C$, with respect to time, t, in minutes, is modeled with the following function:

$$T = 82(0.6)^{\frac{t}{12}} + 20$$

14. What is the initial temperature of the coffee?
15. What is the room temperature of the classroom?
16. At what rate is the temperature of the coffee changing over time?
17. What is the coffee's temperature after 30 minutes?
18. What is the temperature of the coffee after 1 hour?

Summary

You learned that in an expression like 2^x, the "2" is the base and the "x" is the exponent. You learned the following laws of exponents that helped you to simplify expressions with exponents:

- $a^m \times a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$ (if $m > n, a \neq 0$)
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ $(b \neq 0)$
- $a^0 = 1$ $(a \neq 0)$
- $a^{-m} = \frac{1}{a^m}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

You learned that scientific notation is a way to express large or small numbers in the form

$$x = a \times 10^n$$

where

$$1 \leq a < 10 \text{ and } n \in Z.$$

You learned that to solve exponential equations with variables in the exponent you should try to rewrite the equations so the bases are the same. Then, set the exponents equal to each other and solve. If the equation has a variable in the base you can try to get rid of the exponent or, make the exponents on each side of the equation the same and then set the bases equal to each other and solve.

Finally, you learned all about exponential functions. You learned that for exponential functions of the form $y = ab^{\frac{x}{c}} + d$, if $0 < b < 1$ then the function is decreasing and represents exponential decay. If $b > 1$ then the function is increasing and represents exponential growth. Exponential functions are used in many real-life situations such as with the decay of radioactive isotopes and with interest that compounds.

CHAPTER 7

Polynomials

Chapter Outline

Introduction

Here you'll learn all about polynomials. You'll start by learning how to add, subtract, and multiply polynomials. Then you will learn how to factor polynomials, which can be thought of as the opposite of multiplying. Next you'll learn how to divide polynomials and how this connects to factoring. Finally, you'll learn how to use your graphing calculator to graph polynomials in order to determine the factors and roots of polynomials.

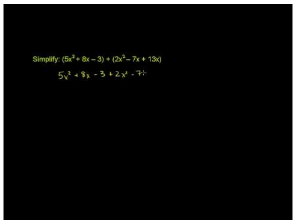
7.1 Addition and Subtraction of Polynomials

Here you'll learn how to add and subtract polynomials.

You are going to build a rectangular garden in your back yard. The garden is 2 m more than 1.5 times as long as it is wide. Write an expression to show the area of the garden.

Watch This

Khan Academy Adding and Subtracting Polynomials 1



Simplify: $(5x^2+8x-3)+(2x^2-7x+13x)$

$5x^2+8x-3+2x^2-7$

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Guidance

The word polynomial comes from the Greek word *poly* meaning "many". Polynomials are made up of one or more terms and each term must have an exponent that is 0 or a whole number. This means that $3x^2+2x+1$ is a polynomial, but $3x^{0.5}+2x^{-2}+1$ is not a polynomial. Some common polynomials have special names based on how many terms they have:

- A monomial is a polynomial with just one term. Examples of monomials are $3x$, $2x^2$ and 7.
- A binomial is a polynomial with two terms. Examples of binomials are $2x+1$, $3x^2-5x$ and $x-5$.
- A trinomial is a polynomial with three terms. An example of a trinomial is $2x^2+3x-4$.

To add and subtract polynomials you will go through two steps.

1. Use the distributive property to remove parentheses. Remember that when there is no number in front of the parentheses, it is like there is a 1 in front of the parentheses. Pay attention to whether or not the sign in front of the parentheses is $+$ or $-$, because this will tell you if the number you need to distribute is $+1$ or -1.
2. Combine similar terms. This means, combine the x^2 terms with the x^2 terms, the x terms with the x terms, etc.

Example A

Find the sum: $(3x^2+2x-7)+(5x^2-3x+3)$.

Solution: First you want to remove the parentheses. Because this is an addition problem, it is like there is a $+1$ in front of each set of parentheses. When you distribute a $+1$, none of the terms will change.

$1(3x^2+2x-7)+1(5x^2-3x+3) = 3x^2+2x-7+5x^2-3x+3$

Next, combine the similar terms. Sometimes it can help to first reorder the expression to put the similar terms next to one another. Remember to keep the signs with the correct terms. For example, in this problem the 7 is negative and the 3x is negative.

$$\begin{aligned} 3x^2+2x-7+5x^2-3x+3 &= 3x^2+5x^2+2x-3x-7+3 \\ &= 8x^2-x-4 \end{aligned}$$

This is your final answer.

Example B

Find the difference: $(5x^2+8x+6)-(4x^2+5x+4)$.

Solution: First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign.

$1(5x^2+8x+6)-1(4x^2+5x+4) = 5x^2+8x+6-4x^2-5x-4$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned} 5x^2+8x+6-4x^2-5x-4 &= 5x^2-4x^2+8x-5x+6-4 \\ &= x^2+3x+2 \end{aligned}$$

This is your final answer.

Example C

Find the difference: $(3x^3+6x^2-7x+5)-(4x^2+3x-8)$

Solution: First you want to remove the parentheses. Because this is a subtraction problem, it is like there is a -1 in front of the second set of parentheses. When you distribute a -1, each term inside that set of parentheses will change its sign..

$1(3x^3+6x^2-7x+5)-1(4x^2+3x-8) = 3x^3+6x^2-7x+5-4x^2-3x+8$

Next, combine the similar terms. Remember to keep the signs with the correct terms.

$$\begin{aligned} 3x^3+6x^2-7x+5-4x^2-3x+8 &= 3x^3+6x^2-4x^2-7x-3x+5+8 \\ &= 3x^3+2x^2-10x+13 \end{aligned}$$

This is your final answer.

Concept Problem Revisited

Remember that the area of a rectangle is length times width.

$$Area = l \times w$$
$$Area = (1.5x+2)x$$
$$Area = 1.5x^2+2x$$

Vocabulary

Binomial

A ***binomial*** has two terms that are added or subtracted from each other. Each of the terms of a binomial is a variable (x), a product of a number and a variable $(4x)$, or the product of multiple variables with or without a number $(4x^2y+3)$. One of the terms in the binomial can be a number.

Monomial

A ***monomial*** can be a number or a variable (like x) or can be the product of a number and a variable (like $3x$ or $3x^2$). A monomial has only one term.

Polynomial

A ***polynomial***, by definition, is also a monomial or the sum of a number of monomials. So $3x^2$ can be considered a polynomial, $2x+3$ can be considered a polynomial, and $2x^2+3x-4$ can be considered a polynomial.

Trinomial

A ***trinomial*** has three terms $(4x^2+3x-7)$. The terms of a trinomial can be a variable (x), a product of a number and a variable $(3x)$, or the product of multiple variables with or without a number $(4x^2)$. One of the terms in the trinomial can be a number (-7).

Variable

A ***variable*** is an unknown quantity in a mathematical expression. It is represented by a letter. It is often referred to as the literal coefficient.

Guided Practice

1. Find the sum: $(2x^2+4x+3)+(x^2-3x-2)$.

2. Find the difference: $(5x^2-9x+7)-(3x^2-5x+6)$.

3. Find the sum: $(8x^3+5x^2-4x+2)+(4x^3+7x-5)$.

Answers:

1. $(2x^2+4x+3)+(x^2-3x-2)=2x^2+4x+3+x^2-3x-2=3x^2+x+1$

2. $(5x^2-9x+7)-(3x^2-5x+6)=5x^2-9x+7-3x^2+5x-6=2x^2-4x+1$

3. $(8x^3+5x^2-4x+2)+(4x^3+7x-5)=8x^3+5x^2-4x+2+4x^3+7x-5=12x^3+5x^2+3x-3$

Practice

For each problem, find the sum or difference.

1. $(x^2+4x+5)+(2x^2+3x+7)$
2. $(2r^2+6r+7)-(3r^2+5r+8)$
3. $(3t^2-2t+4)+(2t^2+5t-3)$
4. $(4s^2-2s-3)-(5s^2+7s-6)$
5. $(5y^2+7y-3)+(-2y^2-5y+6)$
6. $(6x^2+36x+13)-(4x^2+13x+33)$
7. $(12a^2+13a+7)+(9a^2+15a+8)$
8. $(9y^2-17y-12)+(5y^2+12y+4)$
9. $(11b^2+7b-12)-(15b^2-19b-21)$
10. $(25x^2+17x-23)-(-14x^3-14x-11)$
11. $(-3y^2+10y-5)-(5y^2+5y+8)$
12. $(-7x^2-5x+11)+(5x^2+4x-9)$
13. $(9a^3-2a^2+7)+(3a^2+8a-4)$
14. $(3x^2-2x+4)-(x^2+x-6)$
15. $(4s^3+4s^2-5s-2)-(-2s^2-5s+6)$

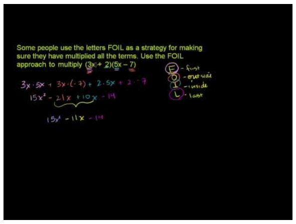
7.2 Multiplication of Polynomials

Here you will learn how to multiply polynomials using the distributive property.

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

Watch This

Khan Academy Multiplying Polynomials



Some people use the letters FOIL as a strategy for making sure they have multiplied all the terms. Use the FOIL approach to multiply (3x + 2)(5x – 7)

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Guidance

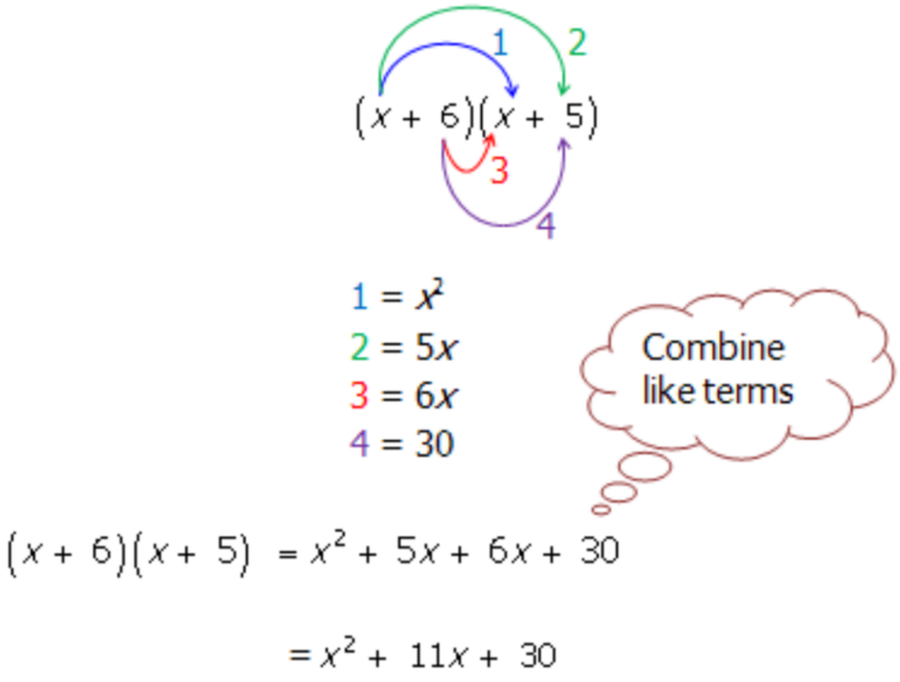
To multiply polynomials you will need to use the distributive property. Recall that the distributive property says that if you start with an expression like $3(5x+2)$, you can simplify it by multiplying both terms inside the parentheses by 3 to get a final answer of $15x+6$.

When multiplying polynomials, you will need to use the distributive property more than once for each problem.

Example A

Find the product: $(x+6)(x+5)$

Solution: To answer this question you will use the distributive property. The distributive property would tell you to multiply x in the first set of parentheses by everything inside the second set of parentheses , then multiply 6 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



$$(x+6)(x+5)$$

$1 = x^2$
$2 = 5x$
$3 = 6x$
$4 = 30$

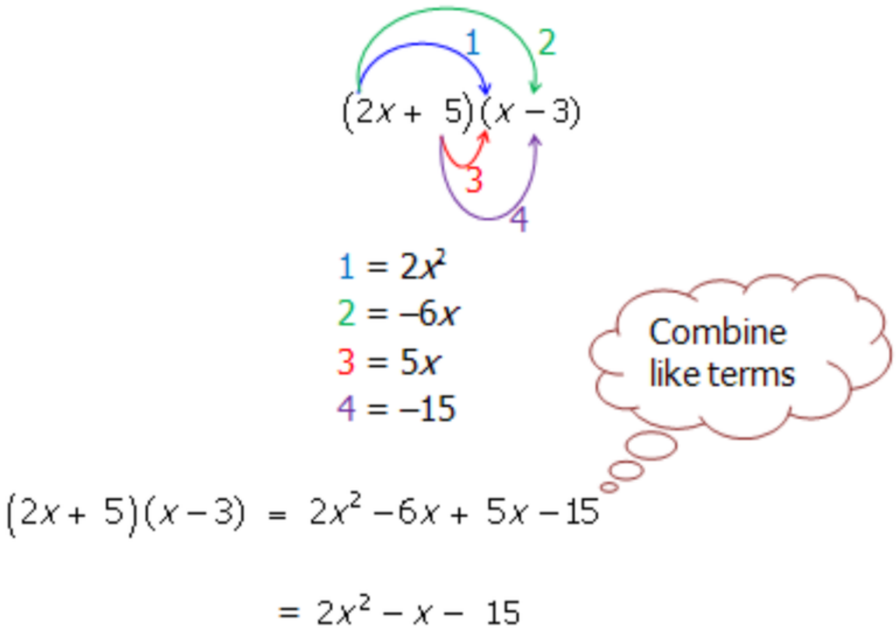
$$(x+6)(x+5) = x^2 + 5x + 6x + 30$$

$$= x^2 + 11x + 30$$

Example B

Find the product: $(2x+5)(x-3)$

Solution: Again, use the distributive property. The distributive property tells you to multiply $2x$ in the first set of parentheses by everything inside the second set of parentheses , then multiply 5 in the first set of parentheses by everything in the second set of parentheses . Here is what that looks like:



$$(2x+5)(x-3)$$

$1 = 2x^2$
$2 = -6x$
$3 = 5x$
$4 = -15$

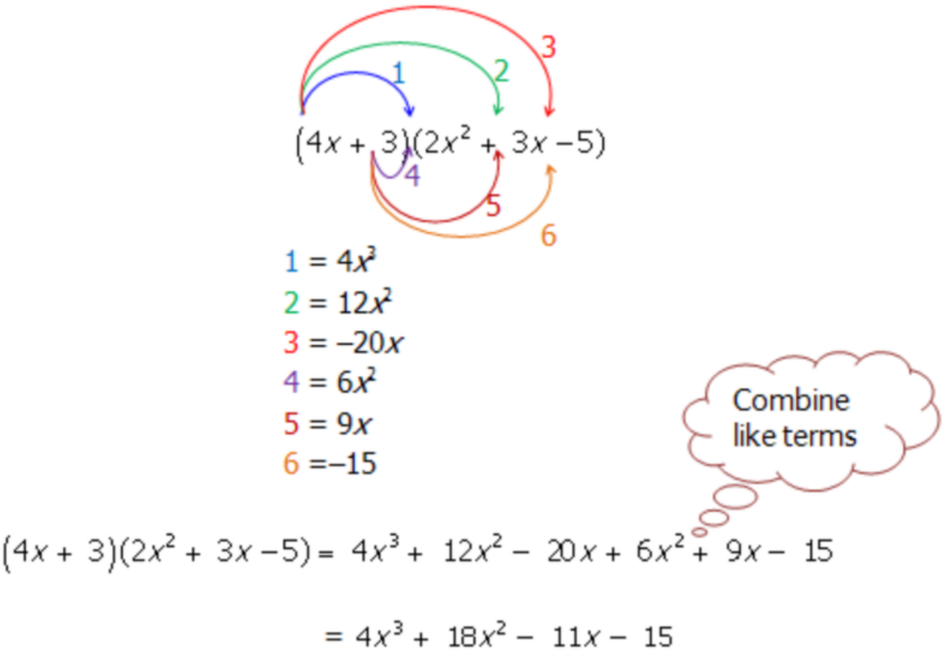
$$(2x+5)(x-3) = 2x^2 - 6x + 5x - 15$$

$$= 2x^2 - x - 15$$

Example C

Find the product: $(4x+3)(2x^2+3x-5)$

Soltuion: Even though at first this question may seem different, you can still use the distributive property to find the product. The distributive property tells you to multiply $4x$ in the first set of parentheses by everything inside the second set of parentheses, then multiply 3 in the first set of parentheses by everything in the second set of parentheses. Here is what that looks like:



$(4x+3)(2x^2+3x-5)$

$1 = 4x^3$
$2 = 12x^2$
$3 = -20x$
$4 = 6x^2$
$5 = 9x$
$6 = -15$

$$(4x+3)(2x^2+3x-5) = 4x^3+12x^2-20x+6x^2+9x-15$$

$$= 4x^3+18x^2-11x-15$$

Concept Problem Revisited

Jack was asked to frame a picture. He was told that the width of the frame was to be 5 inches longer than the glass width and the height of the frame was to be 7 inches longer than the glass height. Jack measures the glass and finds the height to width ratio is 4:3. Write the expression to determine the area of the picture frame.

What is known?

- The width is 5 inches longer than the glass
- The height is 7 inches longer than the glass
- The glass has a height to width ratio of 4:3

The equations:

- The height of the picture frame is $4x+7$
- The width of the picture frame is $3x+5$

The formula:

$$\begin{aligned}
\text{Area} &= w \times h \\
\text{Area} &= (3x+5)(4x+7) \\
\text{Area} &= 12x^2+21x+20x+35 \\
\text{Area} &= 12x^2+41x+35
\end{aligned}$$

Vocabulary

Distributive Property

The ***distributive property*** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $\frac{2}{3}(x+5)$, the distributive property states that the product of a number $\left(\frac{2}{3}\right)$ and a sum $(x+5)$ is equal to the sum of the individual products of the number $\left(\frac{2}{3}\right)$ and the addends (x and 5).

Like Terms

Like terms refers to terms in which the degrees match and the variables match. For example $3x$ and $4x$ are like terms. Like terms are also known as **similar terms**.

Guided Practice

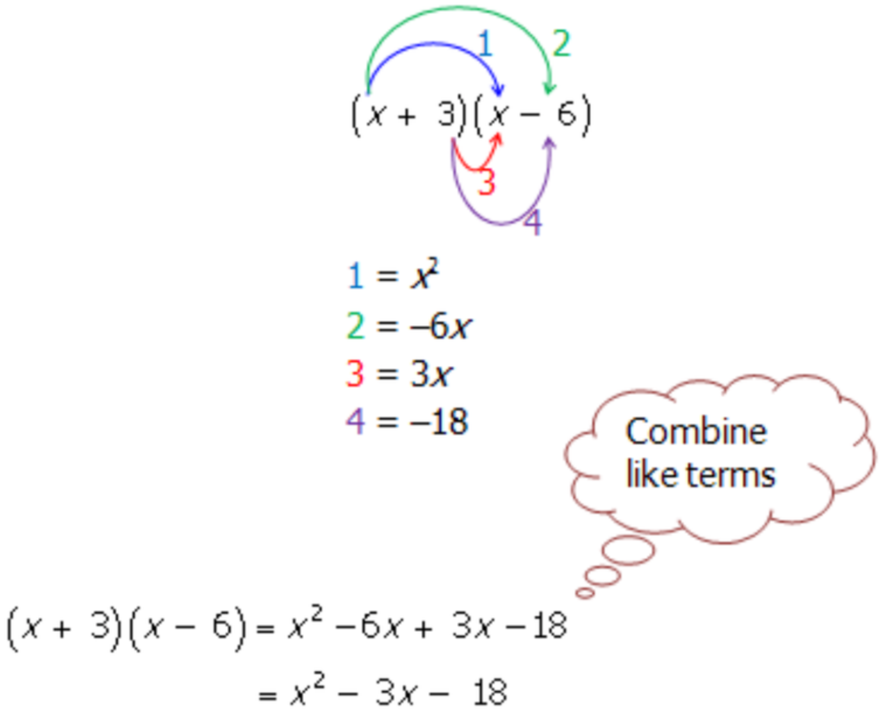
1. Find the product: $(x+3)(x-6)$

2. Find the product: $(2x+5)(3x^2-2x-7)$

3. An average football field has the dimensions of 160 ft by 360 ft. If the expressions to find these dimensions were $(3x+7)$ and $(7x+3)$, what value of x would give the dimensions of the football field?

Answers:

1. $(x+3)(x-6)$



$1 = x^2$

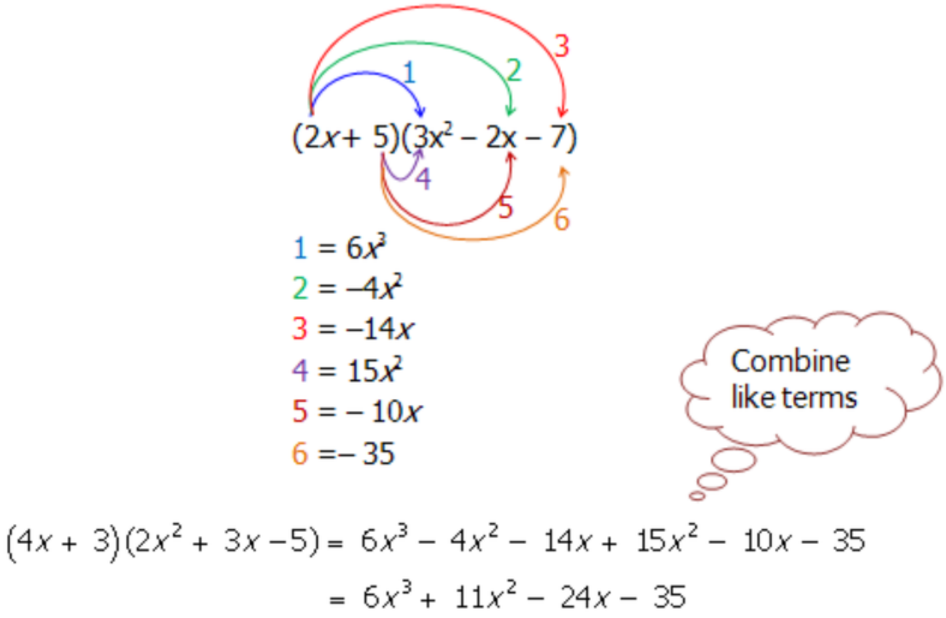
$2 = -6x$

$3 = 3x$

$4 = -18$

$$(x+3)(x-6) = x^2 - 6x + 3x - 18$$
$$= x^2 - 3x - 18$$

2. $(2x+5)(3x^2-2x-7)$



$1 = 6x^3$

$2 = -4x^2$

$3 = -14x$

$4 = 15x^2$

$5 = -10x$

$6 = -35$

$$(4x+3)(2x^2+3x-5) = 6x^3 - 4x^2 - 14x + 15x^2 - 10x - 35$$
$$= 6x^3 + 11x^2 - 24x - 35$$

3. Area $= l \times w$

$$\begin{aligned}
\text{Area} &= 360 \times 160 \\
(7x+3) &= 360 \\
7x &= 360-3 \\
7x &= 357 \\
x &= 51
\end{aligned}$$

$$\begin{aligned}
(3x+7) &= 160 \\
3x &= 160-7 \\
3x &= 153 \\
x &= 51
\end{aligned}$$

The value of x that satisfies these expressions is 51.

Practice

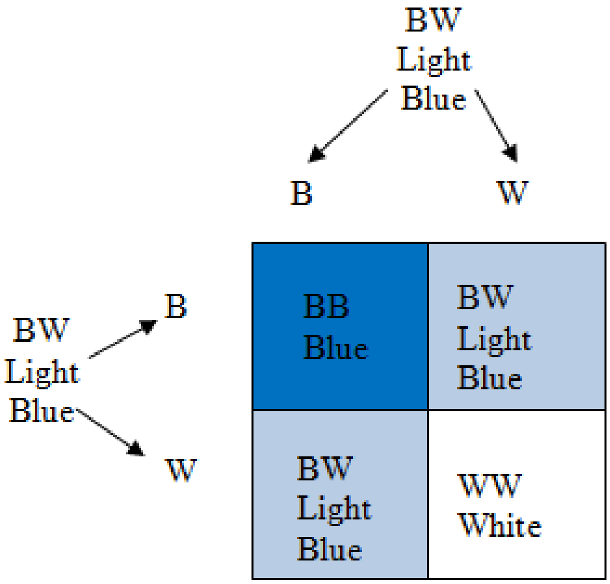
Use the distributive property to find the product of each of the following polynomials:

1. $(x+4)(x+6)$
2. $(x+3)(x+5)$
3. $(x+7)(x-8)$
4. $(x-9)(x-5)$
5. $(x-4)(x-7)$
6. $(x+3)(x^2+x+5)$
7. $(x+7)(x^2-3x+6)$
8. $(2x+5)(x^2-8x+3)$
9. $(2x-3)(3x^2+7x+6)$
10. $(5x-4)(4x^2-8x+5)$
11. $9a^2(6a^3+3a+7)$
12. $-4s^2(3s^3+7s^2+11)$
13. $(x+5)(5x^3+2x^2+3x+9)$
14. $(t-3)(6t^3+11t^2+22)$
15. $(2g-5)(3g^3+9g^2+7g+12)$

7.3 Special Products of Polynomials

Here you will learn about special cases of binomial multiplication.

A flower is homozygous blue (RR) and another flower is homozygous white (rr). Use a Punnett square to show that a mixture of the two can produce a white flower.



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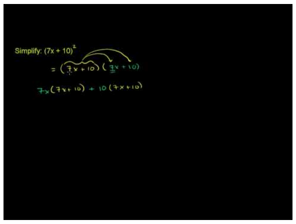
Khan Academy Special Products of Binomials



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Guidance

There are two special cases of multiplying binomials. If you can learn to recognize them, you can multiply these binomials more quickly.

Here are the two special products that you should learn to recognize:

Special Case 1 (Binomial Squared): $(x \pm y)^2 = x^2 \pm 2xy + y^2$

- Example: $(x+5)^2 = x^2 + 10x + 25$

- Example: $(2x-8)^2 = 4x^2 - 32x + 64$

Special Case 2 (Difference of Perfect Squares): $(x+y)(x-y) = x^2 - y^2$

- Example: $(5x+10)(5x-10) = 25x^2 - 100$
- Example: $(2x-4)(2x+4) = 4x^2 - 16$

Keep in mind that you can always use the distributive property to do the multiplications if you don't notice that the problem is a special case.

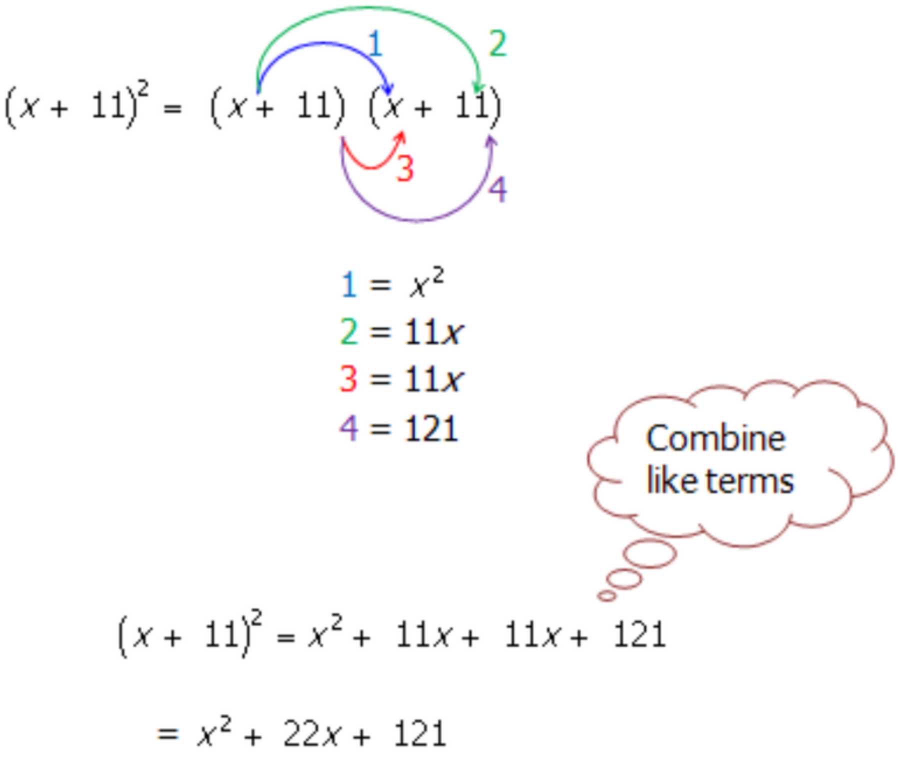
Example A

Find the product: $(x+11)^2$

Solution: These is an example of Special Case 1. You can use that pattern to quickly multiply.

$$\begin{aligned}(x+11)^2 &= x^2 + 2\cdot x \cdot 11 + 11^2 \\ &= x^2 + 22x + 121\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:



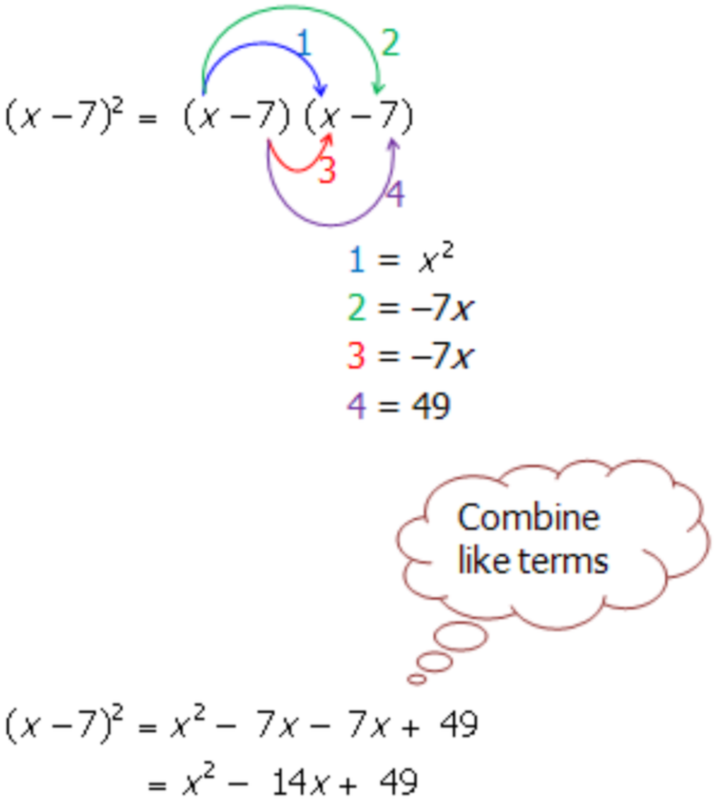
Example B

Find the product: $(x-7)^2$

Solution: These is another example of Special Case 1. You can use that pattern to quickly multiply.

$$\begin{aligned}(x-7)^2 &= x^2-2\cdot x\cdot 7+7^2\\ &= x^2-14x+49\end{aligned}$$

You can verify that this is the correct answer by using the distributive property:



Example C

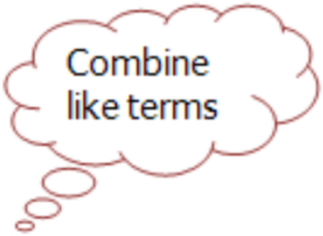
Find the product: $(x+9)(x-9)$

Solution: These is an example of Special Case 2. You can use that pattern to quickly multiply.

$$\begin{aligned}(x+9)(x-9) &= x^2-9^2\\ &= x^2-81\end{aligned}$$

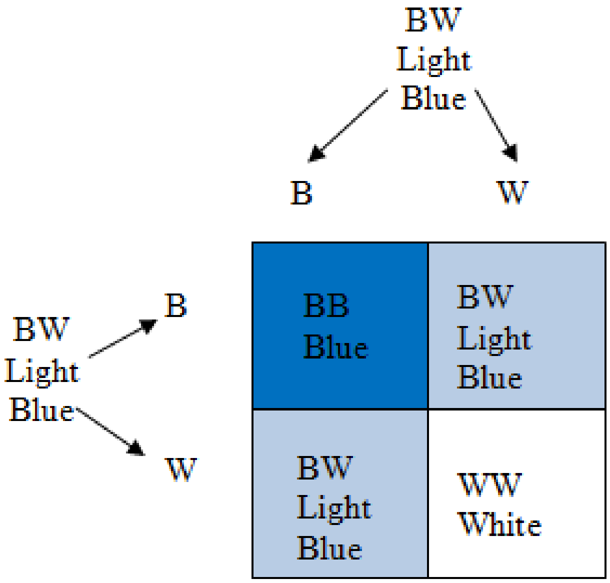
You can verify that this is the correct answer by using the distributive property:

$1 = x^2$
$2 = -9x$
$3 = 9x$
$4 = -81$



$$\begin{aligned}(x+9)(x-9) &= x^2+9x-9x-81\\ &= x^2-81\end{aligned}$$

Concept Problem Revisited



BB Blue	BW Light Blue
BW Light Blue	WW White

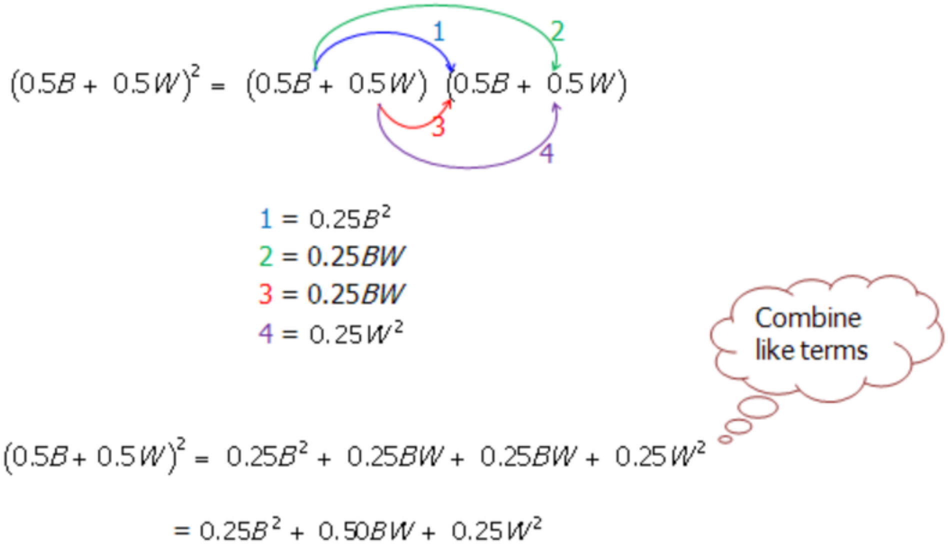
Each flower will have one-half of the blue genes and one-half of the white genes. Therefore the equation formed will be:

$0.5B+0.5W$

The offspring will have the genetic makeup (the mixture produced) using the equation:

$(0.5B+0.5W)^2$

Notice that this is an example of Special Case 1. You can expand the offspring genetic makeup equation to find out the percentage of offspring (or flowers) that will be blue, white, or light blue.



Therefore 25% of the offspring flowers will be blue, 50% will be light blue, and 25% will be white.

Vocabulary

Distributive Property

The ***distributive property*** is a mathematical way of grouping terms. It states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, in the expression: $3(x+5)$, the distributive property states that the product of a number (3) and a sum $(x+5)$ is equal to the sum of the individual products of the number (3) and the addends (x and 5).

Guided Practice

1. Expand the following binomial: $(x+4)^2$.

2. Expand the following binomial: $(5x-3)^2$.

3. Determine whether or not each of the following is a difference of two perfect squares:

a) a^2-16

b) $9b^2-49$

c) c^2-60

Answers:

1. $(x+4)^2 = x^2+4x+4x+16 = x^2+8x+16$.

2. $(5x-3)^2 = 25x^2-15x-15x+9 = 25x^2-30x+9$

3. a)Yes, $a^2-16 = (a+4)(a-4)$

b) Yes, $9b^2-49 = (3b+7)(3b-7)$
c) No, 60 is not a perfect square.

Practice

Expand the following binomials:

1. $(t+12)^2$
2. $(w+15)^2$
3. $(2e+7)^2$
4. $(3z+2)^2$
5. $(7m+6)^2$
6. $(g-6)^2$
7. $(d-15)^2$
8. $(4x-3)^2$
9. $(2p-5)^2$
10. $(6t-7)^2$

Find the product of the following binomials:

11. $(x+13)(x-13)$
12. $(x+6)(x-6)$
13. $(2x+5)(2x-5)$
14. $(3x+4)(3x-4)$
15. $(6x+7)(6x-7)$

7.4 Monomial Factors of Polynomials

Here you will learn to find a common factor in a polynomial and factor it out of the polynomial.

Can you write the following polynomial as a product of a monomial and a polynomial?

$12x^4 + 6x^3 + 3x^2$

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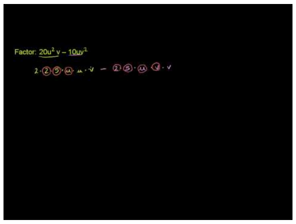
Khan Academy Factoring and the Distributive Property



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Guidance

In the past you have studied common factors of two numbers. Consider the numbers 25 and 35. A common factor of 25 and 35 is 5 because 5 goes into both 25 and 35 evenly.

This idea can be extended to polynomials. A common factor of a polynomial is a number and/or variable that are a factor in all terms of the polynomial. The Greatest Common Factor (or GCF) is the largest monomial that is a factor of each of the terms of the polynomial.

To factor a polynomial means to write the polynomial as a product of other polynomials. One way to factor a polynomial is:

1. Look for the greatest common factor.
2. Write the polynomial as a product of the **greatest common factor** and **the polynomial that results when you divide all the terms of the original polynomial by the greatest common factor.**

One way to think about this type of factoring is that you are essentially doing the distributive property in reverse.

Example A

Factor the following binomial: $5a+15$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5 and 15 can both be divided by 5. The GCF for this binomial is 5.

Step 2: Divide the GCF out of each term of the binomial:

$5a+15 = 5(a+3)$

Example B

Factor the following polynomial: $4x^2+8x-2$

Solution: *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 4, 8 and 2 can all be divided by 2. The GCF for this polynomial is 2.

Step 2: Divide the GCF out of each term of the polynomial:

$4x^2+8x-2=2(2x^2+4x-1)$

Example C

Factor the following polynomial: $3x^5-9x^3-6x^2$

Solution: *Step 1:* Identify the GCF. Looking at each of the terms, you can see that 3, 9 and 6 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$3x^5-9x^3-6x^2=3x^2(x^3-3x-2)$

Concept Problem Revisited

To write as a product you want to try to factor the polynomial: $12x^4+6x^3+3x^2$.

Step 1: Identify the GCF of the polynomial. Looking at each of the numbers, you can see that 12, 6, and 3 can all be divided by 3. Also notice that each of the terms has an x^2 in common. The GCF for this polynomial is $3x^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$12x^4+6x^3+3x^2=3x^2(4x^2+2x+1)$$

Vocabulary

Common Factor

Common factors are numbers (numerical coefficients) or letters (literal coefficients) that are a factor in all parts of the polynomials.

Greatest Common Factor

The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

1. Find the common factors of the following: $a^2(b+7)-6(b+7)$
2. Factor the following polynomial: $5k^6+15k^4+10k^3+25k^2$
3. Factor the following polynomial: $27x^3y+18x^2y^2+9xy^3$

Answers:

1. *Step 1:* Identify the GCF

This problem is a little different in that if you look at the expression you notice that $(b+7)$ is common in both terms. Therefore $(b+7)$ is the common factor. The GCF for this expression is $(b+7)$.
Step 2: Divide the GCF out of each term of the expression:

$$a^2(b+7)-6(b+7)=(a^2-6)(b+7)$$

2. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 5, 15, 10, and 25 can all be divided by 5. Also notice that each of the terms has an k^2 in common. The GCF for this polynomial is $5k^2$.

Step 2: Divide the GCF out of each term of the polynomial:

$$5k^6+15k^4+10k^3+25k^2=5k^2(k^4+3k^2+2k+5)$$

3. *Step 1:* Identify the GCF. Looking at each of the numbers, you can see that 27, 18 and 9 can all be divided by 9. Also notice that each of the terms has an xy in common. The GCF for this polynomial is $9xy$.

Step 2: Divide the GCF out of each term of the polynomial:
$27x^3y+18x^2y^2+9xy^3=9xy(3x^2+2xy+y^2)$

Practice

Factor the following polynomials by looking for a common factor:

1. $7x^2+14$
2. $9c^2+3$
3. $8a^2+4a$
4. $16x^2+24y^2$
5. $2x^2-12x+8$
6. $32w^2x+16xy+8x^2$
7. $12abc+6bcd+24acd$
8. $15x^2y-10x^2y^2+25x^2y$
9. $12a^2b-18ab^2-24a^2b^2$
10. $4s^3t^2-16s^2t^3+12st^2-24st^3$

Find the common factors of the following expressions and then factor:

1. $2x(x-5)+7(x-5)$
2. $4x(x-3)+5(x-3)$
3. $3x^2(e+4)-5(e+4)$
4. $8x^2(c-3)-7(c-3)$
5. $ax(x-b)+c(x-b)$

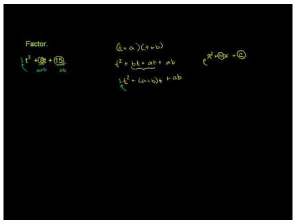
7.5 Factorization of Quadratic Expressions

Here you'll learn how to factor quadratic expressions.

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?

Watch This

Khan Academy Factoring trinomials with a leading 1 coefficient



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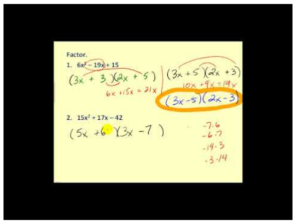
James Sousa: Factoring Trinomials using Trial and Error and Grouping



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Guidance

To factor a polynomial means to write the polynomial as a product of other polynomials. Here, you'll focus on factoring quadratic expressions. Quadratic expressions are polynomials of degree 2, of the form ax^2+bx+c. Consider the steps for finding the product of the following binomials:

$$\begin{aligned}(2x+3)(3x-5) &= 6x^2-10x+9x-15\\ &= 6x^2-x-15\end{aligned}$$

When factoring a quadratic expression, your job will be to take an expression like $6x^2-x-15$ and write it as $(2x+3)(3x-5)$. You can think of factoring as the reverse of multiplying. Notice that when factored, the $6x^2$ factors to $2x$ and $3x$. The -15 factors to -5 and 3. You can say then, in general, that with the trinomial ax^2+bx+c, you have to factor both "a" and "c".

-

$$ax^2+bx+c=(dx+e)(fx+g) \text{ where } a=d\times f \text{ and } c=e\times g$$

- The middle term (b) is

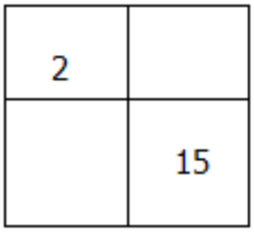
$$b = dg + ef$$

Here you will work through a number of examples to develop mastery at factoring trinomials using a box method.

Example A

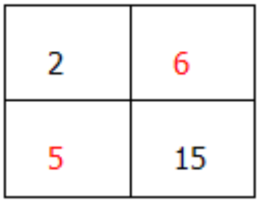
Factor: $2x^2 + 11x + 15$

Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 2 and the 'c' value is 15. Start by making a box and placing these values in the box as shown.



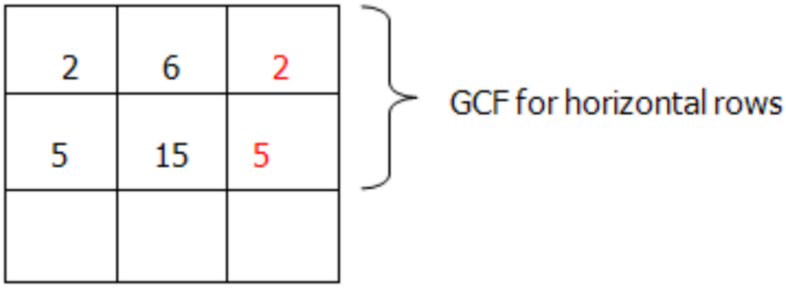
2	
	15

The product of 2 and 15 is 30. To continue filling in the box, you need to find two numbers that multiply to 30, but add up to +11 (the value of b in the original equation). The two numbers that work are 5 and 6: $5 + 6 = 11$ and $5 \cdot 6 = 30$. Put 5 and 6 in the box.



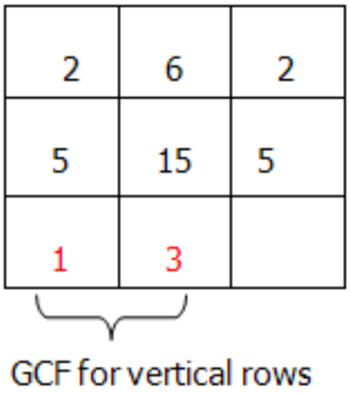
2	6
5	15

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 2 and 6, has a GCF of 2. The second row, 5 and 15, has a GCF of 5.



2	6	2
5	15	5

The first column, 2 and 5, has a GCF of 1. The second column, 6 and 15, has a GCF of 3.



2	6	2
5	15	5
1	3	

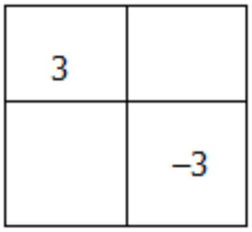
Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(1x+3)$ and $(2x+5)$. You can verify that those binomials multiply to create the original trinomial: $(x+3)(2x+5) = 2x^2+5x+6x+15 = 2x^2+11x+15$.

The factored form of $2x^2+11x+15$ is $(x+3)(2x+5)$.

Example B

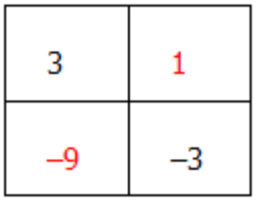
Factor: $3x^2-8x-3$

Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 3 and the 'c' value is –3. Start by making a box and placing these values in the box as shown.



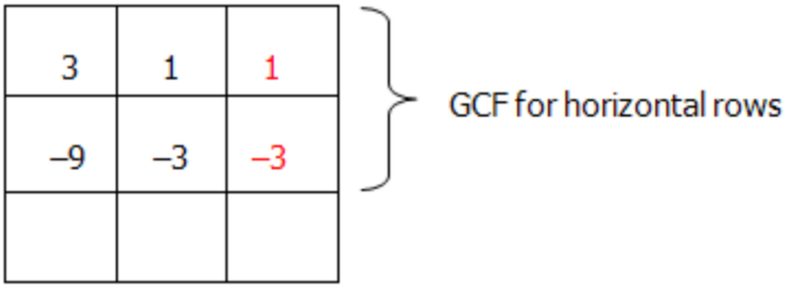
3	
	–3

The product of 3 and –3 is –9. To continue filling in the box, you need to find two numbers that multiply to –9, but add up to –8 (the value of b in the original equation). The two numbers that work are –9 and 1. $-9+1=-8$ and $-9 \cdot 1 = -9$. Put –9 and 1 in the box.



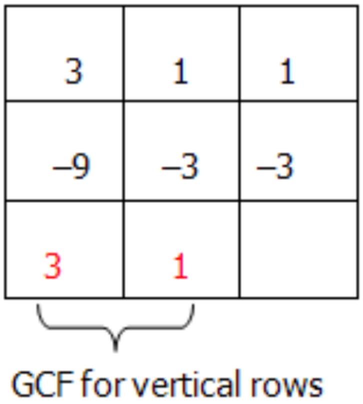
3	1
–9	–3

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 3 and 1, has a GCF of 1. The second row, –9 and –3, has a GCF of –3.



3	1	1
–9	–3	–3

The first column, 3 and –9, has a GCF of 3. The second column, 1 and –3, has a GCF of 1.



3	1	1
–9	–3	–3
3	1	

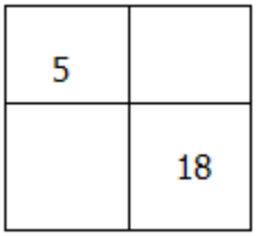
Notice that the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(3x+1)$ and $(1x-3)$. You can verify that those binomials multiply to create the original trinomial: $(3x+1)(x-3) = 3x^2-9x+1x-3 = 3x^2-8x-3$.

The factored form of $3x^2-8x-3$ is $(3x+1)(x-3)$.

Example C

Factor: $5w^2-21w+18$

Solution: First note that there is not a common factor in this trinomial. If there was, you would want to start by factoring out the common factor. In this trinomial, the 'a' value is 5 and the 'c' value is 18. Start by making a box and placing these values in the box as shown.



5	
	18

The product of 5 and 18 is 90. To continue filling in the box, you need to find two numbers that multiply to 90, but add up to –21 (the value of b is the original equation). The two numbers that work are –6 and –15. $-6+(-15) = -21$ and $-6 \cdot -15 = 90$. Put –6 and –15 in the box.

5	–6
–15	18

Next, find the GCF of the numbers in each row and each column and put these new numbers in the box. The first row, 5 and –6, has a GCF of 1. The second row, –15 and 18, has a GCF of 3.

5	–6	1
–15	18	–3

Note: When b is – and c is +
you need to use negative factors

The first column, 5 and –15, has a GCF of 5. The second column, –6 and 18, has a GCF of 6.

5	–6	1
–15	18	−3
5	–6	

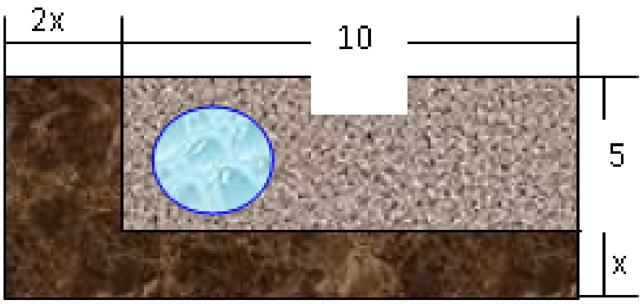
Note: When b is – and c is +
you need to use negative factors

Notice that you need to make two of the GCFs negative in order to make the products of each row GCF with each column GCF create the original 4 numbers in the box. The GCFs represent the coefficients of your factors. Your factors are $(5w-6)$ and $(w-3)$. You can verify that those binomials multiply to create the original trinomial: $(5w-6)(w-3) = 5w^2-15w-6w+18 = 5w^2-21w+18$.

The factored form of $5w^2-21w+18$ is $(5w-6)(w-3)$.

Concept Problem Revisited

Jack wants to construct a border around two sides of his garden. The garden measures 5 yards by 18 yards. He has enough stone to build a border with a total area of 30 square yards. The border will be twice as wide on the shorter end. What are the dimensions of the border?



$$\begin{aligned}
\text{Area of Garden} &= 18 \times 5 = 90\ yd^2\\
\text{Area of border} &= 30\ yd^2\\
\text{Area of Garden} + \text{border} &= (18+2x)(5+x)\\
\text{Area of border} &= (\text{Area of garden} + \text{border}) - \text{Area of garden}\\
30 &= (18+2x)(5+x) - 90\\
30 &= 90 + 18x + 10x + 2x^2 - 90\\
30 &= 28x + 2x^2\\
0 &= 2x^2 + 28x - 30
\end{aligned}$$

This trinomial has a common factor of 2. First, factor out this common factor:

$2x^2 + 28x - 30 = 2(x^2 + 14x - 15)$

Now, you can use the box method to factor the remaining trinomial. After using the box method, your result should be:

$2(x^2 + 14x - 15) = 2(x+15)(x-1)$

To find the dimensions of the border you need to solve a quadratic equation. This is explored in further detail in another concept:

$$\begin{array}{cc}
\multicolumn{2}{c}{2(x+15)(x-1) = 0}\\
\swarrow & \searrow\\
x+15 = 0 & x-1 = 0\\
x = -15 & x = 1
\end{array}$$

Since x cannot be negative, x must equal 1.

Width of Border: $2x = 2(1) = 2\ yd$

Length of Border: $x = 1\ yd$

Vocabulary

Greatest Common Factor

The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Quadratic Expression

A ***quadratic expression*** is a polynomial of degree 2. The general form of a quadratic expression is $ax^2 + bx + c$.

Guided Practice

1. Factor the following trinomial: $8c^2-2c-3$
2. Factor the following trinomial: $3m^2+3m-60$
3. Factor the following trinomial: $5e^3+30e^2+40e$

Answers:

1. Use the box method and you find that $8c^2-2c-3=(2c+1)(4c-3)$

2. First you can factor out the 3 from the polynomial. Then, use the box method. The final answer is $3m^2+3m-60=3(m-4)(m+5)$.

3. First you can factor out the $5e$ from the polynomial. Then, use the box method. The final answer is $5e^3+30e^2+40e=5e(e+2)(e+4)$.

Practice

Factor the following trinomials.

1. x^2+5x+4
2. $x^2+12x+20$
3. $a^2+13a+12$
4. $z^2+7z+10$
5. $w^2+8w+15$
6. $x^2-7x+10$
7. $x^2-10x+24$
8. m^2-4m+3
9. s^2-6s+7
10. $y^2-8y+12$
11. x^2-x-12
12. x^2+x-12
13. $x^2-5x-14$
14. $x^2-7x-44$
15. y^2+y-20
16. $3x^2+5x+2$
17. $5x^2+9x-2$
18. $4x^2+x-3$
19. $2x^2+7x+3$
20. $2y^2-15y-8$
21. $2x^2-5x-12$
22. $2x^2+11x+12$
23. $6w^2-7w-20$
24. $12w^2+13w-35$
25. $3w^2+16w+21$
26. $16a^2-18a-9$
27. $36a^2-7a-15$
28. $15a^2+26a+8$
29. $20m^2+11m-4$
30. $3p^2+17p-20$

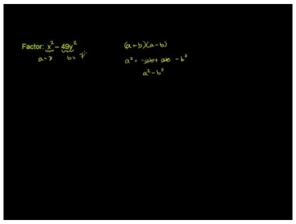
7.6 Special Cases of Quadratic Factorization

Here you'll learn to recognize two special kinds of quadratics and how to factor them quickly.

A box is to be designed for packaging with a side length represented by the quadratic $9b^2 - 64$. If this is the most economical box, what are the dimensions?

Watch This

Khan Academy Factoring the Sum and Difference of Squares



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Guidance

When factoring quadratics, there are special cases that can be factored more quickly. There are two special quadratics that you should learn to recognize:

Special Case 1 (Perfect Square Trinomial): $x^2 \pm 2xy + y^2 = (x \pm y)^2$

- Example: $x^2 + 10x + 25 = (x+5)^2$
- Example: $4x^2 - 32x + 64 = (2x-8)^2$

Special Case 2 (Difference of Perfect Squares): $x^2 - y^2 = (x+y)(x-y)$

- Example: $25x^2 - 100 = (5x+10)(5x-10)$
- Example: $4x^2 - 25 = (2x-5)(2x+5)$

Keep in mind that you can always use the box method to do the factoring if you don't notice the problem as a special case.

Example A

Factor $2x^2 + 28x + 98$.

Solution: First, notice that there is a common factor of 2. Factor out the common factor:

$2x^2 + 28x + 98 = 2(x^2 + 14x + 49)$

Next, notice that the first and last terms are both perfect squares ($x^2 = x \cdot x$ and $49 = 7 \cdot 7$, and the middle term is 2 times the product of the roots of the other terms ($14x = 2 \cdot x \cdot 7$). This means $x^2 + 14x + 49$ is a perfect square trinomial (Special Case 1). Using the pattern:

$x^2 + 14x + 49 = (x+7)^2$

Therefore, the complete factorization is $2x^2 + 28x + 98 = 2(x+7)^2$.

Example B

Factor $8a^2-24a+18$.

Solution: First, notice that there is a common factor of 2. Factor out the common factor:

$8a^2-24a+18=2(4a^2-12a+9)$

Next, notice that the first and last terms are both perfect squares and the middle term is 2 times the product of the roots of the other terms ($12a=2\cdot 2a\cdot 3$). This means $4a^2-12a+9$ is a perfect square trinomial (Special Case 1). Because the middle term is negative, there will be a negative in the binomial. Using the pattern:

$4a^2-12a+9=(2a-3)^2$

Therefore, the complete factorization is $8a^2-24a+18=2(2a-3)^2$.

Example C

Factor x^2-16.

Solution: Notice that there are no common factors. The typical middle term of the quadratic is missing and each of the terms present are perfect squares and being subtracted. This means x^2-16 is a difference of perfect squares (Special Case 2). Using the pattern:

$x^2-16=(x-4)(x+4)$

Note that it would also be correct to say $x^2-16=(x+4)(x-4)$. It does not matter whether you put the + version of the binomial first or the – version of the binomial first.

Concept Problem Revisited

A box is to be designed for packaging with a side length represented by the quadratic $9b^2-64$. If this is the most economical box, what are the dimensions?

First: factor the quadratic to find the value for *b*.

$9b^2-64$

This is a difference of perfect squares (Special Case 2). Use that pattern:

$9b^2-64=(3b-8)(3b+8)$

To finish this problem we need to **solve** a quadratic equation. This idea is explored in further detail in another concept.

$$\begin{array}{cc} \multicolumn{2}{c}{9b^2-64=(3b+8)(3b-8)} \\ \swarrow & \searrow \\ 3b+8=0 & 3b-8=0 \\ 3b=-8 & 3b=8 \\ b=\dfrac{-8}{3} & b=\dfrac{8}{3} \end{array}$$

The most economical box is a cube. Therefore the dimensions are $\frac{8}{3}\times\frac{8}{3}\times\frac{8}{3}$

Vocabulary

Difference of Perfect Squares

The ***difference of perfect squares*** is a special case of a quadratic expression where there is no middle term and the two terms present are both perfect squares. The general equation for the difference of two squares is:

$$x^2 - y^2 = (x+y)(x-y)$$

Perfect Square Trinomial

The ***perfect square trinomials*** are the result of a binomial being multiplied by itself. The two variations of the perfect square trinomial are:

1. $(x+y)^2 = x^2 + 2xy + y^2$
2. $(x-y)^2 = x^2 - 2xy + y^2$

Guided Practice

1. Factor completely $s^2 - 18s + 81$

2. Factor completely $50 - 98x^2$

3. Factor completely $4x^2 + 48x + 144$

Answers:

1. This is Special Case 1. $s^2 - 18s + 81 = (s-9)^2$

2. First factor out the common factor of 2. Then, it is Special Case 2. $50 - 98x^2 = 2(5-7x)(5+7x)$

3. First factor out the common factor of 4. Then, it is Special Case 1. $4x^2 + 48x + 144 = 4(x+6)^2$

Practice

Factor each of the following:

1. $s^2 + 18s + 81$
2. $x^2 + 12x + 36$
3. $y^2 - 14y + 49$
4. $4a^2 + 20a + 25$
5. $9s^2 - 48s + 64$
6. $s^2 - 81$
7. $x^2 - 49$
8. $4t^2 - 25$
9. $25w^2 - 36$
10. $64 - 81a^2$
11. $y^2 - 22y + 121$
12. $16t^2 - 49$
13. $9a^2 + 30a + 25$
14. $100 - 25b^2$
15. $4s^2 - 28s + 49$

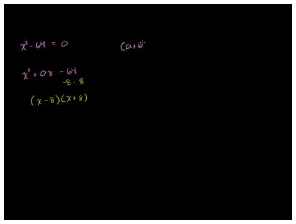
7.7 Zero Product Property for Quadratic Equations

Here you'll learn how to solve a quadratic equation by factoring and using the zero product property.

The area of a particular rectangle was found to be $A(w) = w^2 - 8w - 58$. Determine the dimensions of the rectangle if the area was known to be 7 units.

Watch This

Khan Academy Factoring Special Products



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Guidance

Recall that when solving an equation, you are trying to determine the values of the variable that make the equation true. For the equation $2x^2 + 10x + 8 = 0$, $x = -1$ and $x = -4$ are both solutions. You can check this:

- $2(-1)^2 + 10(-1) + 8 = 2(1) - 10 + 8 = 0$
- $2(-4)^2 + 10(-4) + 8 = 2(16) - 40 + 8 = 32 - 40 + 8 = 0$

Here you will focus on solving quadratic equations. One of the methods for quadratic equations utilizes your factoring skills and a property called the **zero product property**.

If $a \cdot b = 0$, what can you say about a or b? What you should realize is that either a or b have to be equal to 0, because that is the only way that their product will be 0. If both a and b were non-zero, then their product would have to be non-zero. This is the idea of the zero product property. The zero product property states that if the product of two quantities is zero, then one or both of the quantities must be zero.

The zero product property has to do with products being equal to zero. When you factor, you turn a quadratic expression into a product. If you have a quadratic expression equal to zero, you can factor it and then use the zero product property to solve. So, if you were given the equation $2x^2 + 5x - 3 = 0$, first you would want to turn the quadratic expression into a product by factoring it:

$2x^2 + 5x - 3 = (x+3)(2x-1)$

You can rewrite the equation you are trying to solve as $(x+3)(2x-1) = 0$.

Now, you have the product of two binomials equal to zero. This means at least one of those binomials must be equal to zero. So, you have two mini-equations that you can solve to find the values of x that cause each binomial to be equal to zero.

- $x + 3 = 0$, which means $x = -3$ OR
- $2x - 1 = 0$, which means $x = \frac{1}{2}$

The two solutions to the equation $2x^2+5x-3=0$ are $x=-3$ and $x=\frac{1}{2}$.

Keep in mind that you can only use the zero product property if your equation is set equal to zero! If you have an equation not set equal to zero, first rewrite it so that it is set equal to zero. Then factor and use the zero product property.

Example A

Solve for x: $x^2+5x+6=0$.

Solution: First, change x^2+5x+6 into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+5x+6=(x+3)(x+2)$

Next, rewrite the equation you are trying to solve:

$x^2+5x+6=0$ becomes $(x+3)(x+2)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+3=0$, which means that $x=-3$
- $x+2=0$, which means that $x=-2$

The solutions are $x=-3$ or $x=-2$.

Example B

Solve for x: $6x^2+x-15=0$.

In order to solve for x you need to factor the polynomial.

Solution: First, change $6x^2+x-15$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$6x^2+x-15=(3x+5)(2x-3)$

Next, rewrite the equation you are trying to solve:

$6x^2+x-15=0$ becomes $(3x+5)(2x-3)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $3x+5=0$, which means that $x=-\frac{5}{3}$
- $2x-3=0$, which means that $x=\frac{3}{2}$

The solutions are $x=-\frac{5}{3}$ or $x=\frac{3}{2}$.

Example C

Solve for x: $x^2+2x-35=0$.

Solution: First, change $x^2+2x-35$ into a product so that you can use the zero product property. Change the expression into a product by factoring:

$x^2+2x-35=(x+7)(x-5)$

Next, rewrite the equation you are trying to solve:

$x^2+2x-35=0$ becomes $(x+7)(x-5)=0$.

Finally, set up two mini-equations to solve in order to find the values of x that cause each binomial to be equal to zero.

- $x+7=0$, which means that $x=-7$
- $x-5=0$, which means that $x=5$

The solutions are $x=-7$ or $x=5$.

Concept Problem Revisited

The area of a particular rectangle was found to be $A(w)=w^2-8w-58$. Determine the dimensions of the rectangle if the area was known to be 7 units.

In other words, you are being asked to solve the problem:

$$\begin{aligned}&w^2-8w-58=7\\&OR\\&w^2-8w-65=0\end{aligned}$$

You can solve this problem by factoring and using the zero product property.

$w^2-8w-65=0$ becomes $(w+5)(w-13)=0$

$$\begin{array}{ccc} & (w+5)(w-13)=0 & \\ \swarrow & & \searrow \\ w+5=0 & & w-13=0 \\ w=-5 & or & w=13 \end{array}$$

Since you are asked for dimensions, a width of –5 units does not make sense. Therefore for the rectangle, the width would be 13 units.

Vocabulary

Quadratic Equation

A ***quadratic equation*** is an equation in which the highest power of a variable is 2. Standard form for a quadratic equation is $ax^2+bx+c=0$.

Zero-product property

The ***zero-product property*** states that if two factors are multiplied together and their product is zero, then one of the factors must equal zero

Guided Practice

1. Solve for the variable in the polynomial: $x^2+4x-21=0$

2. Solve for the variable in the polynomial: $20m^2+11m-4=0$

3. Solve for the variable in the polynomial: $2e^2+7e+6=0$

Answers:

1. $x^2+4x-21=(x-3)(x+7)$

$$\begin{array}{ccc} & (x-3)(x+7)=0 & \\ & \swarrow \quad \searrow & \\ (x-3)=0 & & (x+7)=0 \\ x=3 & & x=-7 \end{array}$$

2. $20m^2+11m-4=(4m-1)(5m+4)$

$$\begin{array}{ccc} & (4m-1)(5m+4)=0 & \\ & \swarrow \quad \searrow & \\ 4m-1=0 & & 5m+4=0 \\ 4m=1 & & 5m=-4 \\ m=\frac{1}{4} & & m=\frac{-4}{5} \end{array}$$

3. $2e^2+7e+6=(2e+3)(e+2)$

$$\begin{array}{ccc} & (2e+3)(e+2)=0 & \\ & \swarrow \quad \searrow & \\ 2e+3=0 & & e+2=0 \\ 2e=-3 & & e=-2 \\ e=\frac{-3}{2} & & \end{array}$$

Practice

Solve for the variable in each of the following equations.

1. $(x+1)(x-3)=0$
2. $(a+3)(a+5)=0$
3. $(x-5)(x+4)=0$
4. $(2t-4)(t+3)=0$
5. $(x-8)(3x-7)=0$
6. $x^2+x-12=0$
7. $b^2+2b-24=0$
8. $t^2+3t-18=0$
9. $w^2+3w-108=0$
10. $e^2-2e-99=0$

11. $6x^2 - x - 2 = 0$
12. $2d^2 + 14d - 16 = 0$
13. $3s^2 + 20s + 12 = 0$
14. $18x^2 + 12x + 2 = 0$
15. $3j^2 - 17j + 10 = 0$

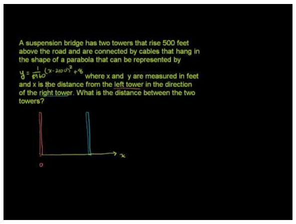
7.8 Applications of Quadratic Equations

Here you'll learn how to apply your knowledge of factoring quadratic expressions to solve real world application problems.

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles, the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

Watch This

Khan Academy Applying Quadratic Equations



A suspension bridge has two towers that rise 500 feet above the road and are connected by cables that hang in the shape of a parabola that can be represented by $y = \frac{1}{8960}(x-2100)^2+8$ where x and y are measured in feet and x is the distance from the left tower in the direction of the right tower. What is the distance between the two towers?

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Click image to the left for more content.

Guidance

Quadratic functions can be used to help solve many different real world problems. Here are two hints for solving quadratic word problems:

1. It is often helpful to start by drawing a picture in order to visualize what you are asked to solve.
2. Once you have solved the problem, it is important to make sure that your answers are realistic given the context of the problem. For example, if you are solving for the age of a person and one of your answers is a negative number, that answer does not make sense in the context of the problem and is not actually a solution.

Example A

The number of softball games that must be scheduled in a league with n teams is given by $G(n) = \frac{n^2-n}{2}$. Each team can only play every other team exactly once. A league schedules 21 games. How many softball teams are in the league?

Solution: You are given the function $G(n) = \frac{n^2-n}{2}$ and you are asked to find n when $G(n) = 21$. This means, you have to solve the equation:

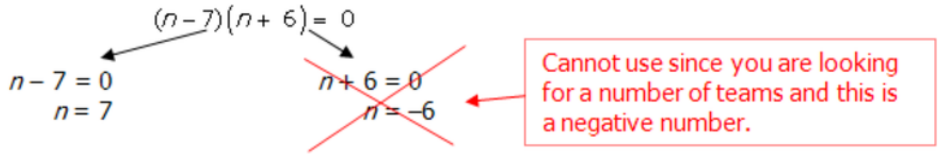
$21 = \frac{n^2-n}{2}$

Start by setting the equation equal to zero:

$$42 = n^2 - n$$
$$n^2 - n - 42 = 0$$

Now solve for n to find the number of teams (n) in the league. Start by factoring the left side of the equation and rewriting the equation:

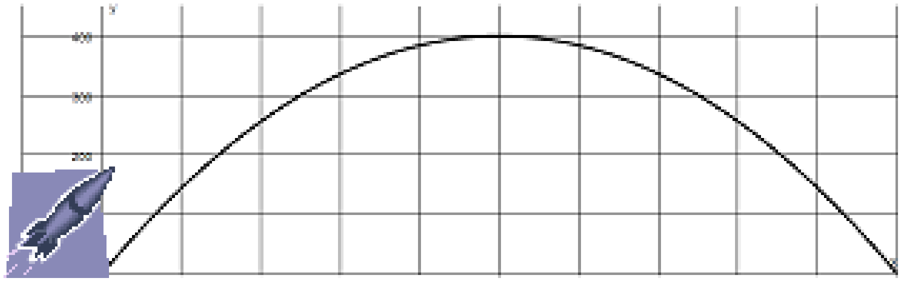
$n^2 - n - 42 = 0$ becomes $(n-7)(n+6) = 0$



There are 7 teams in the softball league.

Example B

When a home-made rocket is launched from the ground, it goes up and falls in the pattern of a parabola. The height, in feet, of a home-made rocket is given by the equation $h(t) = 160t - 16t^2$ where t is the time in seconds. How long will it take for the rocket to return to the ground?



Solution: The formula for the path of the rocket is $h(t) = 160t - 16t^2$. You are asked to find t when $h(t) = 0$, or when the rocket hits the ground and no longer has height. Start by factoring:

$160t - 16t^2 = 0$ becomes $16t(10-t) = 0$

This means $16t = 0$ (so $t = 0$) or $10 - t = 0$ (so $t = 10$). $t = 0$ represents the rocket being on the ground when it starts, so it is not the answer you are looking for. $t = 10$ represents the rocket landing back on the ground.

The rocket will hit the ground after 10 seconds.

Example C

Using the information in **Example B**, what is the height of the rocket after 2 seconds?

Solution: To solve this problem, you need to replace t with 2 in the quadratic function.

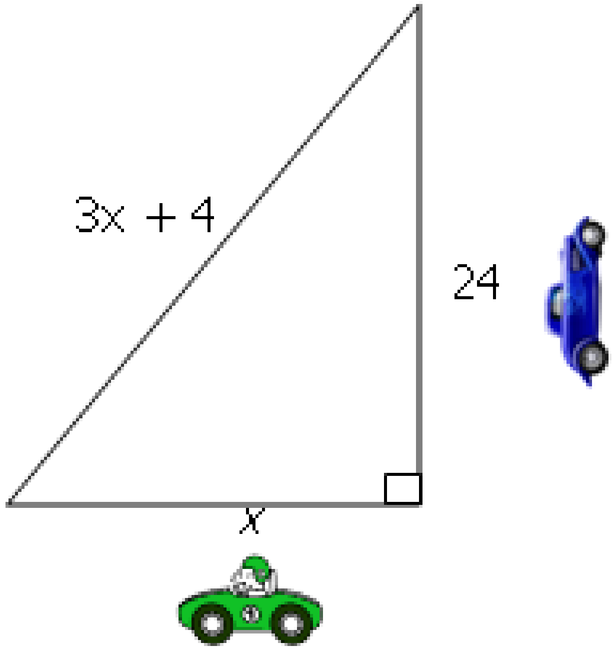
$$\begin{aligned} h(t) &= 160t - 16t^2 \\ h(2) &= 160(2) - 16(2)^2 \\ h(2) &= 320 - 64 \\ h(2) &= 256. \end{aligned}$$

Therefore, after 2 seconds, the height of the rocket is 256 feet.

Concept Problem Revisited

Two cars leave an intersection at the same time. One car travels north and the other car travels west. When the car traveling north had gone 24 miles the distance between the cars was four miles more than three times the distance traveled by the car heading west. Find the distance between the cars at that time.

First draw a diagram. Since the cars are traveling north and west from the same starting position, the triangle made to connect the distance between them is a right triangle. Since you have a right triangle, you can use the Pythagorean Theorem to set up an equation relating the lengths of the sides of the triangle.

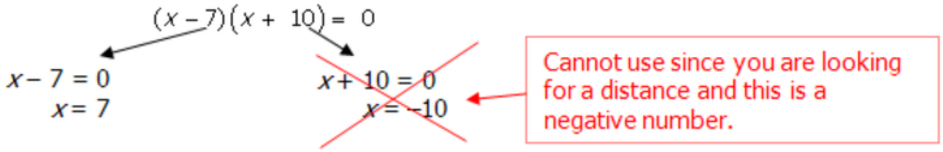


The Pythagorean Theorem is a geometry theorem that says that for all right triangles, $a^2+b^2=c^2$ where a and b are legs of the triangle and c is the longest side of the triangle, the hypotenuse. The equation for this problem is:

$$\begin{aligned}x^2+24^2&=(3x+4)^2\\x^2+576&=(3x+4)(3x+4)\\x^2+576&=9x^2+24x+16\end{aligned}$$

Now set the equation equal to zero and factor the quadratic expression so that you can use the zero product property.

$$\begin{aligned}x^2+576&=9x^2+24x+16\\0&=8x^2+24x-560\\0&=8(x^2+3x-70)\\0&=8(x-7)(x+10)\end{aligned}$$



So you now know that $x = 7$. Since the distance between the cars is represented by the expression $3x+4$, the actual distance between the two cars after the car going north has traveled 24 miles is:

$$\begin{aligned}3x+4 &= 3(7)+4\\ &= 21+4\\ &= 25\ miles\end{aligned}$$

Vocabulary

Factor

To ***factor*** means to rewrite an expression as a product.

Pythagorean Theorem

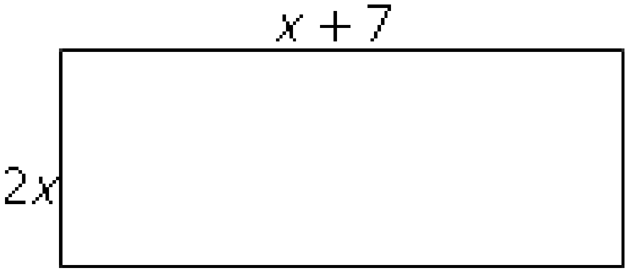
The ***Pythagorean Theorem*** is a right triangle theorem that relates all three sides of a right triangle according to the equation $a^2+b^2=c^2$ where a and b are legs of the triangle and c is the hypotenuse.

Quadratic Expression

A ***quadratic expression*** is a polynomial of degree 2. The general form of a quadratic expression is ax^2+bx+c.

Guided Practice

1. A rectangle is known to have an area of 520 square inches. The lengths of the sides are shown in the diagram below. Solve for both the length and the width.



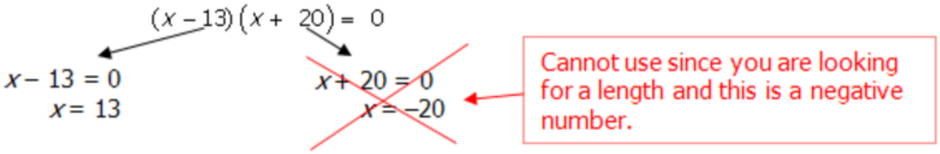
2. The height of a ball in feet can be found by the quadratic function $h(t) = -16t^2+80t+5$ where t is the time in seconds that the ball is in the air. Determine the time(s) at which the ball is 69 feet high.

3. A manufacturer measures the number of cell phones sold using the binomial $0.015c+2.81$. She also measures the wholesale price on these phones using the binomial $0.011c+3.52$. Calculate her revenue if she sells 100,000 cell phones.

Answers:

1. The rectangle has an area of 520 square inches and you know that the area of a rectangle has the formula: $A = l \times w$. Therefore:

$$\begin{aligned}520 &= (x+7)(2x)\\520 &= 2x^2+14x\\0 &= 2x^2+14x-520\\0 &= 2(x^2+7x-260)\\0 &= 2(x-13)(x+20)\end{aligned}$$



$(x-13)(x+20)=0$

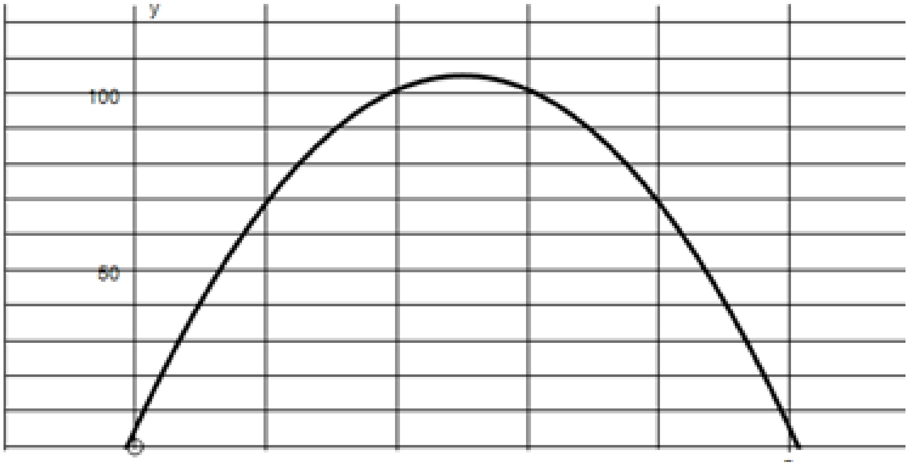
$x-13=0$ $\quad$ $x=13$

$\cancel{x+20=0}$ $\quad$ $\cancel{x=-20}$

Cannot use since you are looking for a length and this is a negative number.

Therefore the value of x is 13. This means that the width is $2x$ or $2(13)=26$ *inches*. The length is $x+7=13+7=20$ *inches*.

2. The equation for the ball being thrown is $h(t)=-16t^2+80t+5$. If you drew the path of the thrown ball, you would see something like that shown below.



You are asked to find the time(s) when the ball hits a height of 69 feet. In other words, solve for:

$$69=-16t^2+80t+5$$

To solve for t, you have to factor the quadratic and then solve for the value(s) of t.

$$\begin{aligned}0 &= -16t^2+80t-64\\0 &= -16(t^2-5t+4)\\0 &= -16(t-1)(t-4)\end{aligned}$$

$$\swarrow \qquad \searrow$$

$$\begin{array}{cc}t-1=0 & t-4=0\\t=1 & t=4\end{array}$$

Since both values are positive, you can conclude that there are two times when the ball hits a height of 69 feet. These times are at 1 second and at 4 seconds.

3. The number of cell phones sold is the binomial $0.015c+2.81$. The wholesale price on these phones is the binomial $0.011c+3.52$. The revenue she takes in is the wholesale price times the number that she sells. Therefore:

$R(c) = (0.015c+2.81)(0.011c+3.52)$
First, let's expand the expression for R to get the quadratic expression. Therefore:

$$\begin{aligned} R(c) &= (0.015c+2.81)(0.011c+3.52) \\ R(c) &= 0.000165c^2+0.08371c+9.8912 \end{aligned}$$

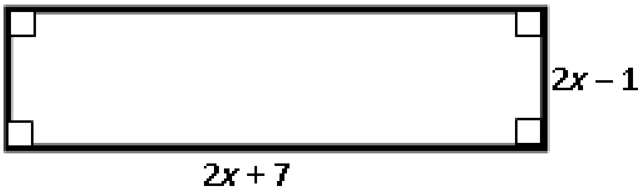
The question then asks if she sold 100,000 cell phones, what would her revenue be. Therefore what is $R(c)$ when $c = 100,000$.

$$\begin{aligned} R(c) &= 0.000165c^2+0.08371c+9.8912 \\ R(c) &= 0.000165(100,000)^2+0.08371(100,000)+9.8912 \\ R(c) &= 1,658,380.89 \end{aligned}$$

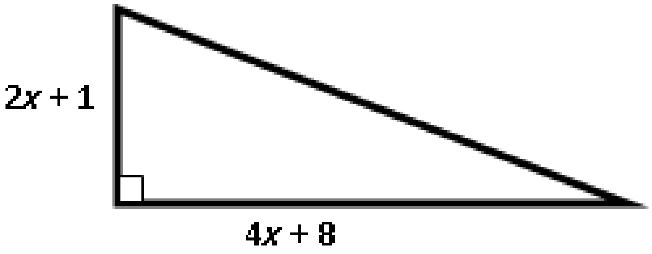
Therefore she would make $1,658,380.89 in revenue.

Practice

1. A rectangle is known to have an area of 234 square feet. The length of the rectangle is given by $x+3$ and the width of the rectangle is given by $x+8$. What is the value of x?
2. Solve for x in the rectangle below given that the area is 9 units.



3. Solve for x in the triangle below given that the area is 10 units.



A pool is treated with a chemical to reduce the amount of algae. The amount of algae in the pool t days after the treatment can be approximated by the function $A(t) = 40t^2 - 300t + 500$.

4. How many days after treatment will the pool have the no algae?
5. How much algae is in the pool before treatments are started?
6. How much less algae is in the pool after 1 day?

A football is kicked into the air. The height of the football in meters can be found by the quadratic function $h(t) = -5t^2 + 25t$ where t is the time in seconds since the ball has been kicked.

7. How high is the ball after 3 seconds? At what other time is the ball the same height?
8. When will the ball be 20 meters above the ground?
9. After how many seconds will the ball hit the ground?

A ball is thrown into the air. The height of the ball in meters can be found by the quadratic function $h(t) = -5t^2 + 30t$ where t is the time in seconds since the ball has been thrown.

10. How high is the ball after 3 seconds?
11. When will the ball be 25 meters above the ground?
12. After how many seconds will the ball hit the ground?

Kim is drafting the windows for a new building. Their shape can be modeled by the function $h(w) = -w^2 + 4$, where h is the height and w is the width of points on the window frame, measured in meters.

13. Find the width of each window at its base.
14. Find the width of each window when the height is 3 meters.
15. What is the height of the window when the width is 1 meter?

7.9 Complete Factorization of Polynomials

Here you will learn how to factor a polynomial completely by first looking for common factors and then factoring the resulting expression.

Can you factor the following polynomial completely?

$8x^3+24x^2-32x$

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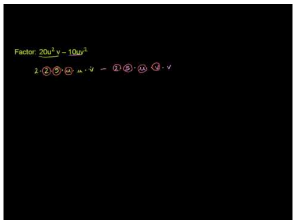
Khan Academy Factoring and the Distributive Property



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Guidance

A cubic polynomial is a polynomial of degree equal to 3. Examples of cubics are:

- $9x^3+10x-5$
- $8x^3+2x^2-5x-7$

Recall that to factor a polynomial means to rewrite the polynomial as a product of other polynomials . You will not be able to factor all cubics at this point, but you will be able to factor some using your knowledge of common factors and factoring quadratics. In order to attempt to factor a cubic, you should:

1. Check to see if the cubic has any common factors. If it does, factor them out.
2. Check to see if the resulting expression can be factored, especially if the resulting expression is a quadratic. To factor the quadratic expression you could use the box method, or any method you prefer.

Anytime you are asked to **factor completely**, you should make sure that none of the pieces (factors) of your final answer can be factored any further. If you follow the steps above of first checking for common factors and then checking to see if the resulting expressions can be factored, you can be confident that you have factored completely.

Example A

Factor the following polynomial completely: $3x^3-15x$.

Solution: Look for the common factors in each of the terms. The common factor is $3x$. Therefore:

$3x^3-15x=3x(x^2-5)$

The resulting quadratic, x^2-5, cannot be factored any further (it is NOT a difference of perfect squares). Your answer is $3x(x^2-5)$.

Example B

Factor the following polynomial completely: $2a^3+16a^2+30a$.

Solution: Look for the common factors in each of the terms. The common factor is $2a$. Therefore:

$2a^3+16a^2+30a = 2a(a^2+8a+15)$

The resulting quadratic, $a^2+8a+15$ can be factored further into $(a+5)(a+3)$. Your final answer is $2a(a+5)(a+3)$.

Example C

Factor the following polynomial completely: $6s^3+36s^2-18s-42$.

Solution: Look for the common factors in each of the terms. The common factor is 6. Therefore:

$6s^3+36s^2-18s-42 = 6(s^3+6s^2-3s-7)$

The resulting expression is a cubic, and you don't know techniques for factoring cubics without common factors at this point. Therefore, your final answer is $6(s^3+6s^2-3s-7)$.

Note: It turns out that the resulting cubic cannot be factored, even with more advanced techniques. Remember that not all expressions can be factored. In fact, in general most expressions ***cannot*** *be factored.*

Concept Problem Revisited

Factor the following polynomial completely: $8x^3+24x^2-32x$.

Look for the common factors in each of the terms. The common factor is $8x$. Therefore:

$8x^3+24x^2+32x = 8x(x^2+3x-4)$

The resulting quadratic can be factored further into $(x+4)(x-1)$. Your final answer is $8x(x+4)(x-1)$.

Vocabulary

Cubic Polynomial

A ***cubic polynomial*** is a polynomial of degree equal to 3. For example $8x^3+2x^2-5x-7$ is a cubic polynomial.

Distributive Property

The ***distributive property*** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, the distributive property says that $2(x+5) = 2x+10$.

Factor

To ***factor*** means to rewrite an expression as a product of other expressions. These resulting expressions are called the ***factors*** of the original expression.

Factor Completely

To ***factor completely*** means to factor an expression until none of its factors can be factored any further.

Greatest Common Factor

The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

Factor each of the following polynomials completely.

1. $9w^3+12w$.

2. y^3+4y^2+4y.

3. $2t^3-10t^2+8t$.

Answers:

1. The common factor is $3w$. Therefore, $9w^3+12w=3w(3w^2+4)$. The resulting quadratic cannot be factored any further, so your answer is $3w(3w^2+4)$.

2. The common factor is y. Therefore, $y^3+4y^2+4y=y(y^2+4y+4)$. The resulting quadratic can be factored into $(y+2)(y+2)$ or $(y+2)^2$. Your answer is $y(y+2)^2$.

3. The common factor is $2t$. Therefore, $2t^3-10t^2+8t=2t(t^2-5t+4)$. The resulting quadratic can be factored into $(t-4)(t-1)$. Your answer is $2t(t-4)(t-1)$.

Practice

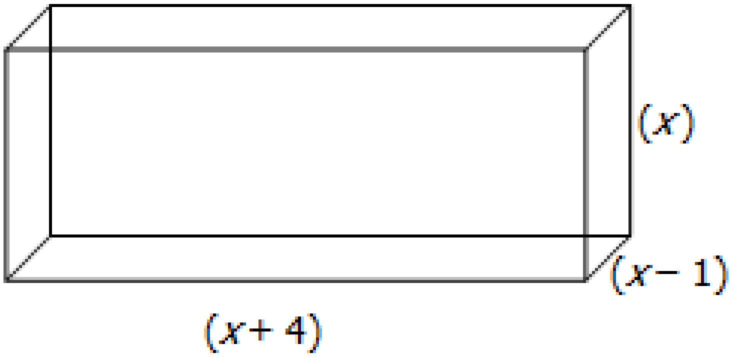
Factor each of the following polynomials completely.

1. $6x^3-12$
2. $4x^3-12x^2$
3. $8y^3+32y$
4. $15a^3+30a^2$
5. $21q^3+63q$
6. $4x^3-12x^2-8$
7. $12e^3+6e^2-6e$
8. $15s^3-30s+45$
9. $22r^3+66r^2+44r$
10. $32d^3-16d^2+12d$
11. $5x^3+15x^2+25x-30$
12. $3y^3-18y^2+27y$
13. $12s^3-24s^2+36s-48$
14. $8x^3+24x^2-80x$
15. $5x^3-25x^2-70x$

7.10 Factorization by Grouping

Here you will learn to factor polynomials by grouping.

A tank is bought at the pet store and is known to have a volume of 12 cubic feet. The dimensions are shown in the diagram below. If your new pet requires the tank to be at least 3 feet high, did you buy a big enough tank?



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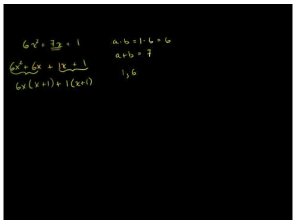
Khan Academy Factoring by Grouping



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Note: The above video shows factoring by grouping of quadratic (trinomial) expressions. The same problem-solving concept will be developed in this lesson for cubic polynomials.

Guidance

Recall that to factor means to rewrite an expression as a product. In general, quadratic expressions are the easiest to factor and cubic expressions are much more difficult to factor.

One method that can be used to factor *some* cubics is the factoring by grouping method. To factor cubic polynomials by grouping there are four steps:

- **Step 1:** Separate the terms into two groups.
- **Step 2:** Factor out the common terms in each of the two groups.
- **Step 3:** Factor out the common binomial.
- **Step 4:** If possible, factor the remaining quadratic expression.

Take a look at the examples to see what factoring by grouping looks like.

Example A

Factor the following polynomial by grouping: $w^3 - 2w^2 - 9w + 18$.

Solution: Step 1: Separate the terms into two groups. Notice the sign change on the second group because of the negative sign.

$w^3 - 2w^2 - 9w + 18 = (w^3 - 2w^2) - (9w - 18)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(w^3 - 2w^2) - (9w - 18) = w^2(w - 2) - 9(w - 2)$

Step 3: Factor out the common binomial $(w - 2)$.

$w^2(w - 2) - 9(w - 2) = (w - 2)(w^2 - 9)$

Step 4: Factor the remaining quadratic expression $(w^2 - 9)$.

$(w - 2)(w^2 - 9) = (w - 2)(w + 3)(w - 3)$

Therefore, your answer is: $w^3 - 2w^2 - 9w + 18 = (w - 2)(w + 3)(w - 3)$

Example B

Factor the following polynomial by grouping: $2s^3 - 8s^2 + 3s - 12$.

Solution: Step 1: Separate the terms into two groups.

$2s^3 - 8s^2 + 3s - 12 = (2s^3 - 8s^2) + (3s - 12)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(2s^3 - 8s^2) + (3s - 12) = 2s^2(s - 4) + 3(s - 4)$

Step 3: Factor out the common binomial $(s - 4)$.

$2s^2(s - 4) + 3(s - 4) = (s - 4)(2s^2 + 3)$

Step 4: Check to see if the remaining quadratic can be factored. In this case, the expression $(2s^3 + 3)$ cannot be factored.

Therefore, your final answer is $2s^3 - 8s^2 + 3s - 12 = (s - 4)(2s^2 + 3)$

Example C

Factor the following polynomial by grouping: $y^3 + 5y^2 - 4y - 20$.

Solution: Step 1: Separate the terms into two groups. Notice the sign change on the second group because of the negative sign.

$y^3 + 5y^2 - 4y - 20 = (y^3 + 5y^2) - (4y + 20)$

Step 2: Factor out the common terms in each of the sets of parentheses.

$(y^3 + 5y^2) - (4y + 20) = y^2(y + 5) - 4(y + 5)$

Step 3: Factor out the common binomial $(y + 5)$.

$y^2(y + 5) - 4(y + 5) = (y + 5)(y^2 - 4)$

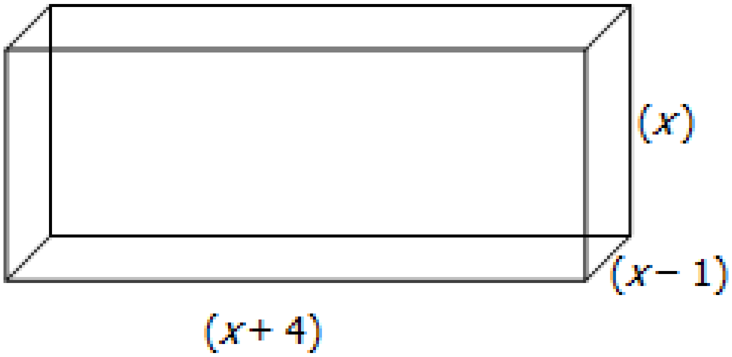
Step 4: Factor the remaining quadratic expression $(y^2 - 4)$.

$(y + 5)(y^2 - 4) = (y + 5)(y + 2)(y - 2)$

Therefore, your answer is $y^3 + 5y^2 - 4y - 20 = (y + 5)(y + 2)(y - 2)$.

Concept Problem Revisited

A tank is bought at the pet store and is known to have a volume of 12 cubic feet. The dimensions are shown in the diagram below. If your new pet requires the tank to be at least 3 feet high, did you buy a big enough tank?



To solve this problem, you need to calculate the volume of the tank.

$$\begin{aligned} V &= l \times w \times h \\ 12 &= (x+4) \times (x-1) \times (x) \\ 12 &= (x^2+3x-4) \times (x) \\ 12 &= x^3+3x^2-4x \\ 0 &= x^3+3x^2-4x-12 \end{aligned}$$

Now you start to solve by factoring by grouping.

$0 = (x^3+3x^2) - (4x+12)$

Factor out the common terms in each of the sets of parentheses.

$0 = x^2(x+3) - 4(x+3)$

Factor out the group of terms $(x+3)$ from the expression.

$0 = (x+3)(x^2-4)$

Completely factor the remaining quadratic expression.

$0 = (x+3)(x-2)(x+2)$

Now solve for the variable x.

$$\begin{array}{ccc} & 0 = (x+3)\ (x-2)\ (x+2) & \\ \swarrow & \downarrow & \searrow \\ x+3=0 & x-2=0 & x+2=0 \\ x=-3 & x=2 & x=-2 \end{array}$$

Since you are looking for a length, only $x = 2$ is a good solution (you can't have a negative length!). But since you need a tank 3 feet high and this one is only 2 feet high, you need to go back to the pet shop and buy a bigger one.

Vocabulary

Cubic Polynomial

A ***cubic polynomial*** is a polynomial of degree equal to 3. For example $8x^3+2x^2-5x-7$ is a cubic polynomial.

Distributive Property

The ***distributive property*** states that the product of a number and a sum is equal to the sum of the individual products of the number and the addends. For example, the distributive property says that $2(x+5)=2x+10$.

Factor

To ***factor*** means to rewrite an expression as a product of other expressions. These resulting expressions are called the ***factors*** of the original expression.

Factor Completely

To ***factor completely*** means to factor an expression until none of its factors can be factored any further.

Greatest Common Factor

The ***Greatest Common Factor*** (or ***GCF***) is the largest monomial that is a factor of (or divides into evenly) each of the terms of the polynomial.

Guided Practice

Factor each of the following polynomials by grouping.

1. Factor the following polynomial by grouping: $y^3-4y^2-4y+16$.
2. Factor the following polynomial by grouping: $3x^3-4x^2-3x+4$.
3. Factor the following polynomial by grouping: $e^3+3e^2-4e-12$.

Answers

1. Here are the steps:

$$\begin{aligned} y^3-4y^2-4y+16 &= y^2(y-4)-4(y-4)\\ &= (y^2-4)(y-4)\\ &= (y-2)(y+2)(y-4) \end{aligned}$$

2. Here are the steps:

$$\begin{aligned} 3x^3-4x^2-3x+4 &= x^2(3x-4)-1(3x-4)\\ &= (x^2-1)(3x-4)\\ &= (x-1)(x+1)(3x-4) \end{aligned}$$

3. Here are the steps:

$$\begin{aligned} e^3+3e^2-4e-12 &= e^2(e+3)-4(e+3)\\ &= (e^2-4)(e+3)\\ &= (e+2)(e-2)(e+3) \end{aligned}$$

Practice

Factor the following cubic polynomials by grouping.

1. $x^3 - 3x^2 - 36x + 108$
2. $e^3 - 3e^2 - 81e + 243$
3. $x^3 - 10x^2 - 49x + 490$
4. $y^3 - 7y^2 - 5y + 35$
5. $x^3 + 9x^2 + 3x + 27$
6. $3x^3 + x^2 - 3x - 1$
7. $5s^3 - 6s^2 - 45s + 54$
8. $4a^3 - 7a^2 + 4a - 7$
9. $5y^3 + 15y^2 - 45y - 135$
10. $3x^3 + 15x^2 - 12x - 60$
11. $2e^3 + 14e^2 + 7e + 49$
12. $2k^3 + 16k^2 + 38k + 24$
13. $-6x^3 + 3x^2 + 54x - 27$
14. $-5m^3 - 6m^2 + 20m + 24$
15. $-2x^3 - 8x^2 + 14x + 56$

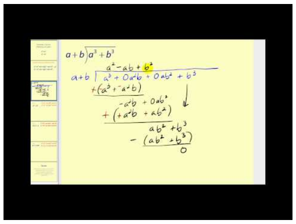
7.11 Factorization of Special Cubics

Here you'll learn to factor the sum and difference of perfect cubes.

Factor the following cubic polynomial: $375x^3 + 648$.

Watch This

James Sousa: Factoring Sum and Difference of Cubes



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Guidance

While many cubics cannot easily be factored, there are two special cases that can be factored quickly. These special cases are the sum of perfect cubes and the difference of perfect cubes.

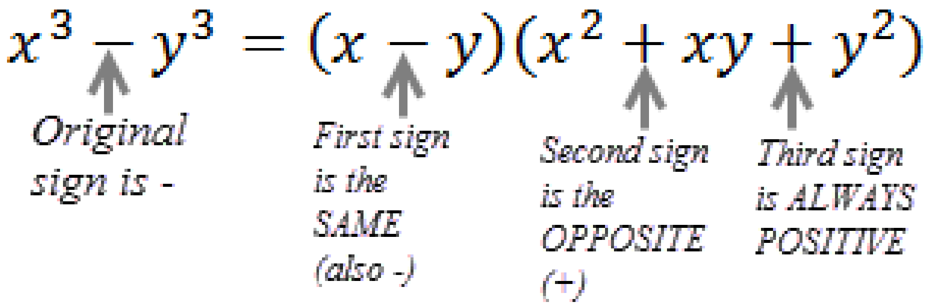
- Factoring the sum of two cubes follows this pattern:

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

- Factoring the difference of two cubes follows this pattern:

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$$

The acronym SOAP can be used to help you remember the positive and negative signs when factoring the sum and difference of cubes. SOAP stands for "**S**ame", "**O**pposite", "**A**lways **P**ositive". "Same" refers to the first sign in the factored form of the cubic being the same as the sign in the original cubic. "Opposite" refers to the second sign in the factored cubic being the opposite of the sign in the original cubic. "Always Positive" refers to the last sign in the factored form of the cubic being always positive. See below:



Example A

Factor: x^3+27.

Solution: This is the sum of two cubes and uses the factoring pattern: $x^3+y^3=(x+y)(x^2-xy+y^2)$.

$x^3+3^3=(x+3)(x^2-3x+9)$.

Example B

Factor: x^3-343.

Solution: This is the difference of two cubes and uses the factoring pattern: $x^3-y^3=(x-y)(x^2+xy+y^2)$.

$x^3-7^3=(x-7)(x^2+7x+49)$.

Example C

Factor: $64x^3-1$.

Solution: This is the difference of two cubes and uses the factoring pattern: $x^3-y^3=(x-y)(x^2+xy+y^2)$.

$(4x)^3-1^3=(4x-1)(16x^2+4x+1)$.

Concept Problem Revisited

Factor the following cubic polynomial: $375x^3+648$.

First you need to recognize that there is a common factor of 3. $375x^3+648=3(125x^3+216)$

Notice that the result is the sum of two cubes. Therefore, the factoring pattern is $x^3+y^3=(x+y)(x^2-xy+y^2)$.

$375x^3+648=3(5x+6)(25x^2-30x+36)$

Vocabulary

Difference of Two Cubes

The ***difference of two cubes*** is a special polynomial in the form of x^3-y^3. This type of polynomial can be quickly factored using the pattern:

$$(x^3-y^3)=(x-y)(x^2+xy+y^2)$$

Sum of Two Cubes

The ***sum of two cubes*** is a special polynomial in the form of x^3+y^3. This type of polynomial can be quickly factored using the pattern:

$$(x^3+y^3)=(x+y)(x^2-xy+y^2)$$

Guided Practice

Factor each of the following cubics.

1. x^3+512

2. $8x^3+125$

3. $x^3 - 216$

Answers:

1. $x^3 + 8^3 = (x+8)(x^2 - 8x + 64)$.

2. $(2x)^3 + 5^3 = (2x+5)(4x^2 - 10x + 25)$.

3. $x^3 - 6^3 = (x-6)(x^2 + 6x + 36)$.

Practice

Factor each of the following cubics.

1. $x^3 + h^3$
2. $a^3 + 125$
3. $8x^3 + 64$
4. $x^3 + 1728$
5. $2x^3 + 6750$
6. $h^3 - 64$
7. $s^3 - 216$
8. $p^3 - 512$
9. $4e^3 - 32$
10. $2w^3 - 250$
11. $x^3 + 8$
12. $y^3 - 1$
13. $125e^3 - 8$
14. $64a^3 + 2197$
15. $54z^3 + 3456$

7.12 Division of a Polynomial by a Monomial

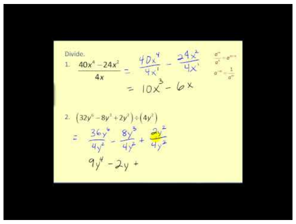
Here you'll learn how to divide a polynomial by a monomial.

Can you divide the polynomial by the monomial? How does this relate to factoring?

$4e^4 + 6e^3 - 10e^2 \div 2e$

Watch This

James Sousa: Dividing Polynomials by Monomials



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Guidance

Recall that a monomial is an algebraic expression that has only one term. So, for example, x, 8, –2, or $3ac$ are all monomials because they have only one term. The term can be a number, a variable, or a combination of a number and a variable. A polynomial is an algebraic expression that has more than one term.

When dividing polynomials by monomials, it is often easiest to separately divide each term in the polynomial by the monomial. When simplifying each mini-division problem, don't forget to use exponent rules for the variables. For example,

$$\frac{8x^5}{2x^3} = 4x^2$$

.

Remember that a fraction is just a division problem!

Example A

What is $(14s^2 - 21s + 42) \div (7)$?

Solution: This is the same as $\frac{14s^2-21s+42}{7}$. Divide each term of the polynomial numerator by the monomial denominator and simplify.

- $\frac{14s^4}{7} = 2s^4$
- $\frac{-21s}{7} = -3s$
- $\frac{42}{7} = 6$

Therefore, $(14s^2 - 21s + 42) \div (7) = 2s^4 - 3s + 6$.

Example B

What is $\frac{3w^3-18w^2-24w}{6w}$?

Solution: Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{3w^3}{6w}=\frac{w^2}{2}$
- $\frac{-18w^2}{6w}=-3w$
- $\frac{-24w}{6w}=-4$

Therefore, $\frac{3w^3-18w^2-24w}{6w}=\frac{w^2}{2}-3w-4$.

Example C

What is $(-27a^4b^5+81a^3b^4-18a^2b^3)\div(-9a^2b)$?

Solution: This is the same as $\frac{-27a^4b^5+81a^3b^4-18a^2b^3}{-9a^2b}$. Divide each term of the polynomial numerator by the monomial denominator and simplify. Remember to use exponent rules when dividing the variables.

- $\frac{-27a^4b^5}{-9a^2b}=3a^2b^4$
- $\frac{81a^3b^4}{-9a^2b}=-9ab^3$
- $\frac{-18a^2b^3}{-9a^2b}=2b^2$

Therefore, $(-27a^4b^5+81a^3b^4-18a^2b^3)\div(-9a^2b)=3a^2b^4-9ab^3+2b^2$.

Concept Problem Revisited

Can you divide the polynomial by the monomial? How does this relate to factoring?

$4e^4+6e^3-10e^2\div 2e$

This process is the same as factoring out a $2e$ from the expression $4e^4+6e^3-10e^2$.

- $\frac{4e^4}{2e}=2e^3$
- $\frac{6e^3}{2e}=3e^2$
- $\frac{-10e^2}{2e}=-5e$

Therefore, $4e^4+6e^3-10e^2\div 2e=2e^3+3e^2-5e$.

Vocabulary

Divisor

A ***divisor*** is the expression in the denominator of a fraction.

Monomial

A ***monomial*** is an algebraic expression that has only one term. x, 8, –2, or $3ac$ are all monomials because they have only one term.

Polynomial

A polynomial is an algebraic expression that has more than one term.

Guided Practice

Complete the following division problems.

1. $(3a^5-5a^4+17a^3-9a^2)\div(a)$
2. $(-40n^3-32n^7+88n^{11}+8n^2)\div(8n^2)$
3. $\frac{16m^6-12m^4+4m^2}{4m^2}$

Answers:

1. $(3a^5-5a^4+17a^3-9a^2)\div(a)=3a^4-5a^3+17a^2-9a$
2. $(-40n^3-32n^7+88n^{11}+8n^2)\div(8n^2)=-5n-4n^5+11n^9+1$
3. $\frac{(16m^6-12m^4+4m^2)}{(4m^2)}=4m^4-3m^2+1$

Practice

Complete the following division problems.

1. $(6a^3+30a^2+24a)\div 6$
2. $(15b^3+20b^2+5b)\div 5$
3. $(12c^4+18c^2+6c)\div 6c$
4. $(60d^{12}+90d^{11}+30d^8)\div 30d$
5. $(33e^7+99e^3+22e^2)\div 11e$
6. $(-8a^4+8a^2)\div(-4a)$
7. $(-3b^4+6b^3-30b^2+15b)\div(-3b)$
8. $(-40c^{12}-20c^{11}-25c^9-30c^3)\div 5c^2$
9. $(32d^{11}+16d^7+24d^4-64d^2)\div 8d^2$
10. $(14e^{12}-18e^{11}-12e^{10}-18e^7)\div -2e^5$
11. $(18a^{10}-9a^8+72a^7+9a^5+3a^2)\div 3a^2$
12. $(-24b^9+42b^7+42b^6)\div -6b^3$
13. $(24c^{12}-42c^7-18c^6)\div -2c^5$
14. $(14d^{12}+21d^9+42d^7)\div -7d^4$
15. $(-40e^{12}+30e^{10}-10e^4+30e^3+80e)\div -10e^2$

7.13 Long Division and Synthetic Division

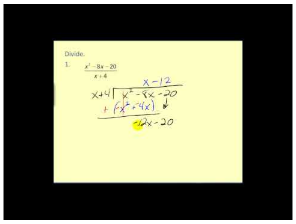
Here you will learn to divide polynomials using polynomial long division and synthetic division.

Can you divide the following polynomials?

$$\frac{x^2-5x+6}{x-2}$$

Watch This

James Sousa: Dividing Polynomials- Long Division



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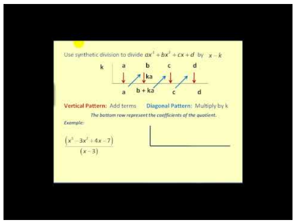
James Sousa: Dividing Polynomials- Synthetic Division



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Guidance

Polynomial Long Division

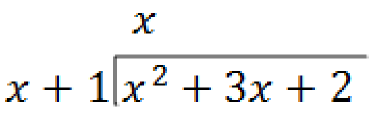
Whenever you want to divide a polynomial by a polynomial, you can use a process called polynomial long division. This process is similar to long division for regular numbers. Look at the example below:

$$\frac{(x^2+3x+2)}{(x+1)}$$

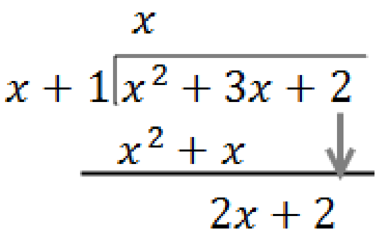
This is the same as the division problem below:

$$x+1\overline{\big)x^2+3x+2}$$

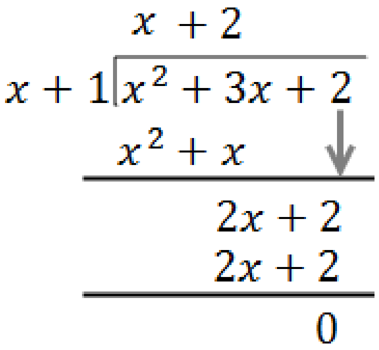
Step 1: Divide the first term in the numerator (x^2) by the first term in the denominator (x). Put this result above the division bar in your answer. In this case, $\frac{x^2}{x} = x$.



Step 2: Multiply the denominator ($x+1$) by the result from Step 1 (x), and put the new result below your numerator. Then, ***subtract*** to get your new polynomial. *This is the same process as in regular number long division!*



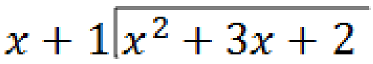
Step 3: Divide the first term in the new polynomial ($2x$) by the first term in the denominator (x). Put this result above the division bar in your answer. Multiply, subtract, and repeat this process until you cannot repeat it anymore.



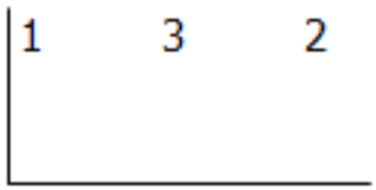
Therefore: $\frac{(x^2+3x+2)}{(x+1)} = (x+2)$

Synthetic Division

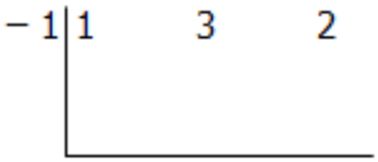
Synthetic division is another method of dividing polynomials. It is a shorthand of long division that only works when you are dividing by a polynomial of degree 1. Usually the divisor is in the form $(x \pm a)$. In synthetic division, unlike long division, you are only concerned with the coefficients in the polynomials. Consider the same example as above:



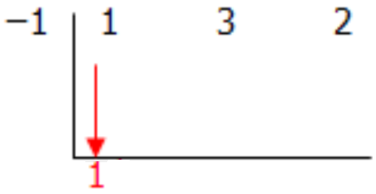
Step 1: Write the coefficients in an upside down division sign.



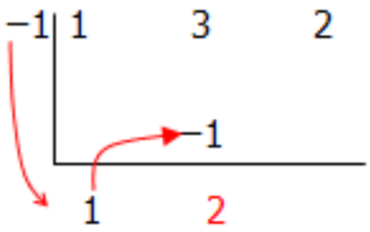
Step 2: Put the opposite of the number from the divisor to the left of the division symbol. In this case, the divisor is $x+1$, so you will use a -1.



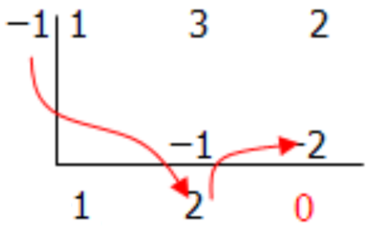
Step 3: Take your leading coefficient and bring it down below the division symbol.



Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



The numbers below the division sign represent your coefficients. Therefore, $\frac{(x^2+3x+2)}{(x+1)} = (x+2)$.

Example A

Use long division to divide:

$$\frac{x^2+6x-7}{x-1}$$

Solution: Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x}=x$.

$$\begin{array}{rl} & x \\ (x-1) & \overline{|x^2+6x-7} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{rl} & x \\ (x-1) & \overline{|x^2+6x-7} \\ & \underline{x^2-x\qquad} \\ & \qquad 7x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{rl} & x+7 \\ (x-1) & \overline{|x^2+6x-7} \\ & \underline{x^2-x \quad \downarrow} \\ & \qquad 7x-7 \\ & \underline{\qquad 7x-7} \\ & \qquad\qquad 0 \end{array}$$

Therefore: $\frac{x^2+6x-7}{x-1}=(x+7)$

Example B

Use long division to divide:

$$\frac{2x^2+7x+5}{2x+5}$$

Solution: Step 1: Divide the first term in the numerator by the first term in the denominator; put this in your answer. Therefore $\frac{2x^2}{2x}=x$.

$$\begin{array}{rl} & x \\ (2x+5) & \overline{|2x^2+7x+5} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} x \\ (2x+5)|\overline{2x^2+7x+5} \\ \underline{2x^2+5x} \\ 2x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} x+1 \\ (2x+5)|\overline{2x^2+7x+5} \\ \underline{2x^2+5x \quad \downarrow} \\ 2x+5 \\ \underline{2x+5} \\ 0 \end{array}$$

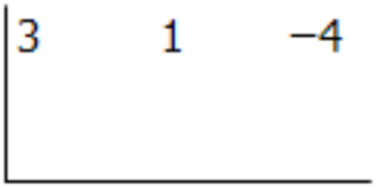
Therefore: $\frac{2x^2+7x+5}{2x+5} = (x+1)$

Example C

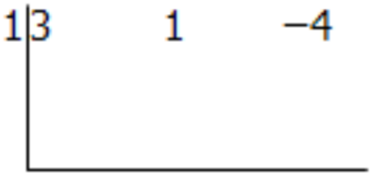
Use synthetic division to divide:

$$(x-1)|\overline{3x^2+x-4}$$

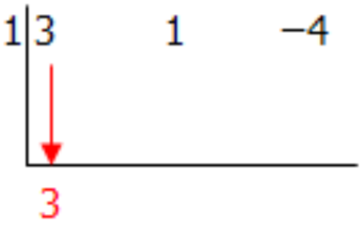
Solution: Step 1: Write the coefficients in an upside down division sign.



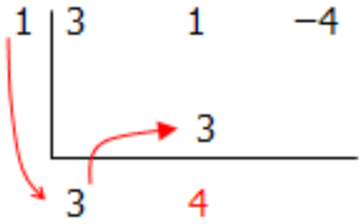
Step 2: Put the opposite of the number from the divisor to the left of the division symbol.



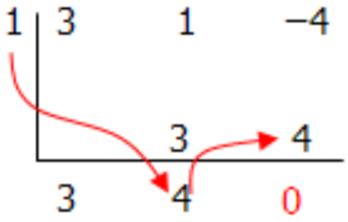
Step 3: Take your leading coefficient and bring it down below the division symbol.



Step 4: Multiply this number by the number to the left of the division symbol and place it in the next column. Add the two numbers together and place this new number below the division sign.



Step 5: Multiply this second number by the number to the left of the division symbol and place it into the third column. Add the two numbers together and place this new number below the division sign.



Therefore: $\frac{3x^2+x-4}{x-1}=(3x+4)$

Concept Problem Revisited

You can divide using long division or synthetic division.

$$\frac{x^2-5x+6}{x-2}$$

Long Division:

Step 1: Divide the first term in the numerator by the first term in the denominator, put this in your answer. Therefore $\frac{x^2}{x}=x$.

$$\begin{array}{rl} & \color{red}{x} \\ (x-2)| & \overline{x^2-5x+6} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{rl} & \color{red}{x} \\ (x-2)| & \overline{x^2-5x+6} \\ & \underline{x^2-2x} \\ & 3x \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{rl} & \overline{x-3} \\ (x-2) & |x^2-5x+6 \\ & \underline{x^2-2x\ \downarrow} \\ & \quad -3x+6 \\ & \underline{\ \ -3x+6} \\ & \qquad\ 0 \end{array}$$

Therefore: $\frac{x^2-5x+6}{x-2}=(x-3)$

Vocabulary

Dividend

The ***dividend*** is the number, variable, or expression you are dividing in a mathematical expression. In the expression $\frac{15}{4x}$, 15 is the dividend.

Divisor

The ***divisor*** is the number, variable, or expression you are dividing by in a mathematical expression. In the expression $\frac{15}{4x}$, $4x$ is the divisor.

Polynomial Long Division

Polynomial long division is a mathematical process similar to the long division process for numbers that allows you to divide polynomials.

Synthetic Division

Synthetic division is a mathematical process of dividing polynomials that works when dividing by a polynomial of degree 1.

Guided Practice

1. Use long division to divide $5x^2+6x+1$ by $x+1$.
2. Use synthetic division to divide $3x^2-2x-1$ by $x-1$.
3. Use synthetic division to divide $3x^3+11x^2+4x-4$ by $x+1$.

Answers:

1. $\frac{5x^2+6x+1}{x+1}=(5x+1)$
2. $\frac{3x^2-2x-1}{x-1}=(3x+1)$
3. $\frac{3x^3+11x^2+4x-4}{x+1}=(3x^2+8x-4)$

Practice

Use long division to divide each of the following:

1. $(x^2+7x+12)\div(x+3)$

2. $(x^2+4x+3) \div (x+3)$
3. $(a^2-4a-45) \div (a-9)$
4. $(3x^2+5x-2) \div (3x-1)$
5. $(2x^2-5x+2) \div (2x-1)$

Use synthetic division to divide each of the following:

6. $(b^2-5b+6) \div (b-3)$
7. $(x^2-6x+8) \div (x-4)$
8. $(a^2-1) \div (a+1)$
9. $(c^2-9) \div (c-3)$
10. $(5r^2+2r-3) \div (r+1)$

Divide each of the following:

11. $\frac{2x^3-7x^2-14x-5}{x-5}$
12. $\frac{9x^4-15x^3+12x^2-11x-15}{3x^3+4x+3}$
13. $\frac{6x^4+4x^3+9x^2+2x+3}{2x^2+1}$
14. $\frac{x^4+4x^3+3x^2+x+1}{x+1}$
15. $\frac{2x^3+7x^2-27x+18}{x+6}$
16. $\frac{8x^3-2x^2+7x+5}{2x+1}$
17. $\frac{3x^3-15x^2+4x-20}{x-5}$
18. $\frac{9x^3+26x^2-48x+5}{x^2+3x-5}$
19. $\frac{-x^3+13x+12}{x+3}$
20. $\frac{x^3-2x^2-5x+10}{x-2}$

7.14 The Factor Theorem

Here you will learn how to factor a polynomial using the factor theorem.

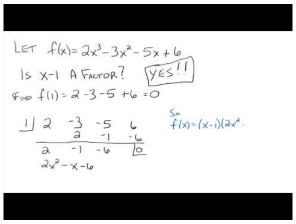
A rectangular shaped container is built in such a way that its volume can be represented by the polynomial $V(w) = w^3 + 7w^2 + 16w + 12$, where w is the width of the container.

a) Factor the polynomial.

b) If $w = 2\ ft$, what are the dimensions of the container?

Watch This

The Factor Theorem and the Remainder Theorem



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Guidance

You know techniques for factoring quadratics and special cases of cubics, but what about other cubics or higher degree polynomials? With the factor theorem, you can attempt to factor these types of polynomials. The factor theorem states that if $(x - a)$ is a factor of $p(x)$, then $p(a) = 0$. To use the factor theorem:

1. Guess factors of the given polynomial $p(x)$. Factors should be of the form $(x - a)$ where a is a factor of the constant term of the polynomial divided by a factor of the first coefficient of the polynomial.
2. Test potential factors by checking $p(a)$. If $p(a) = 0$, then $x - a$ is a factor of the polynomial.
3. Divide the polynomial by one of its factors.
4. Repeat Steps 2 and 3 until the result is a quadratic expression that you can factor using other methods.

Example A

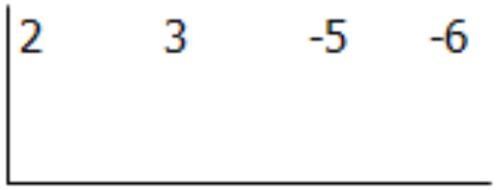
Use the factor theorem to determine if $x + 1$ is a factor of $p(x) = 2x^3 + 3x^2 - 5x - 6$. If so, find the other factors.

Solution: If $x + 1$ is a factor, then $p(-1) = 0$. Test this:

$$\begin{aligned} p(x) &= 2x^3 + 3x^2 - 5x - 6 \\ x = -1 : p(-1) &= 2(-1)^3 + 3(-1)^2 - 5(-1) - 6 \\ p(-1) &= -2 + 3 + 5 - 6 \\ p(-1) &= 0 \quad \text{(IS a factor)} \end{aligned}$$

Now that you have one of the factors, use division to find the others.

Step 1:



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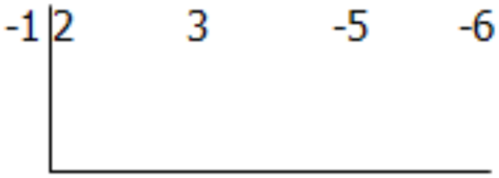

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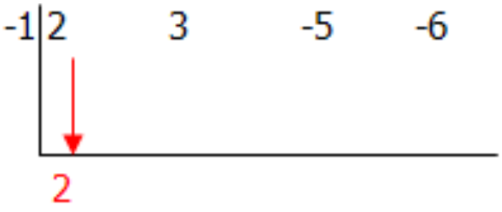

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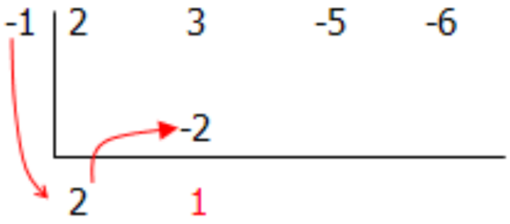

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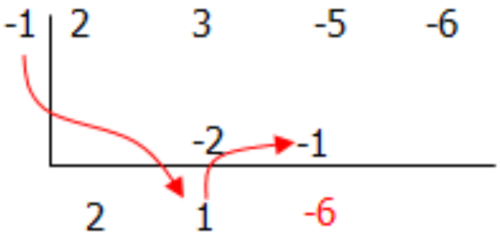

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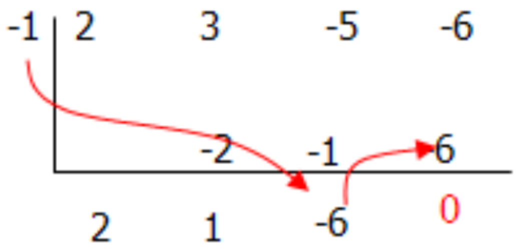


So:

$$\begin{aligned} p(x) &= 2x^3+3x^2-5x-6 \\ p(x) &= (x+1)(2x^2+x-6) \\ p(x) &= (x+1)(2x-3)(x+2) \end{aligned}$$

Example B

Use the factor theorem to determine if $x+3$ is a factor of $s(x)=5x^2-13x-84$. If so, find the other factors.

Solution: If $x+3$ is a factor, then $p(-3)=0$. Test this:

$$s(x)=5x^2-13x-84$$

$$\begin{aligned} x=-3: s(-3) &= 5(-3)^2-13(-3)-84 \\ s(-3) &= 45+39-84 \\ s(-3) &= 0 \quad (\text{IS a factor}) \end{aligned}$$

Now that you have one of the factors, use division to find the other factor.

Step 1: Divide the first term in the numerator by the first term in the denominator; put this in your answer. Therefore $\frac{5x^2}{x}=5x$.

$$\begin{array}{r} 5x \qquad\quad \\ (x+3)\overline{)5x^2-13x-84} \end{array}$$

Step 2: Multiply the denominator by this number (variable) and put it below your numerator, subtract and get your new polynomial.

$$\begin{array}{r} 5x \qquad\quad \\ (x+3)\overline{)5x^2-13x-84} \\ \underline{5x^2+15x \quad} \\ -28x \quad \end{array}$$

Step 3: Repeat the process until you cannot repeat it anymore.

$$\begin{array}{r} 5x-28 \\ (x+3)\overline{)5x^2-13x-84} \\ 5x^2+13x \quad \downarrow \\ \hline -28x-84 \\ -28x-84 \\ \hline 0 \end{array}$$

Therefore: $\frac{(5x^2-13x-84)}{(x+3)} = (5x-28)$

So:

$$s(x) = 5x^3 - 13x - 84$$
$$s(x) = (x+3)(5x-28)$$

Example C

Factor $f(t) = t^3 - 8t^2 + 17t - 10$.

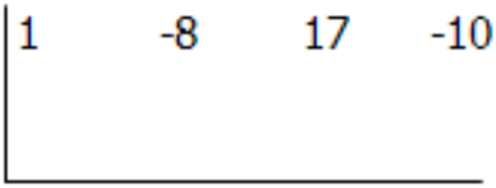
Solution: In order to begin to find the factors, look at the number –10 and find the factors of this number. The factors of –10 are –1, 1, –2, 2, –5, 5, –10, 10. Next, start testing the factors to see if you get a remainder of zero.

$$\begin{aligned} f(t) &= t^3 - 8t^2 + 17t - 10 \\ t = -1 : f(-1) &= (-1)^3 - 8(-1)^2 + 17(-1) - 10 \\ f(-1) &= -36 \quad \text{(NOT a factor)} \end{aligned}$$

$$\begin{aligned} t = 1 : f(1) &= (1)^3 - 8(1)^2 + 17(1) - 10 \\ f(1) &= 0 \quad \text{(IS a factor)} \end{aligned}$$

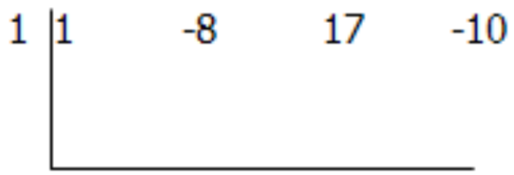
Now that you have one of the factors, use division to find the others.

Step 1:



1	-8	17	-10

Step 2:



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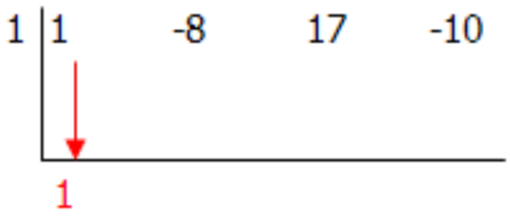

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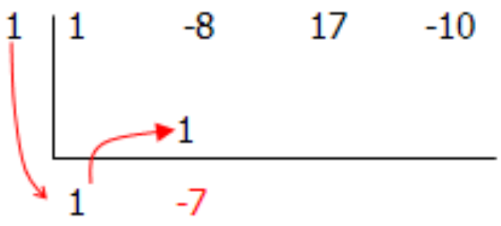

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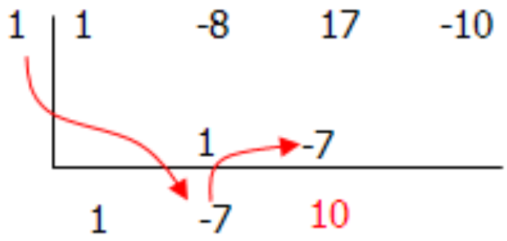

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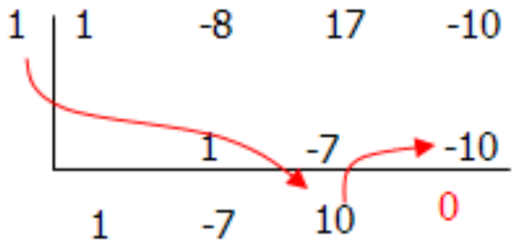


So :

$$f(t) = t^3 - 8t^2 + 17t - 10$$
$$f(t) = (t-1)(t^2 - 7t + 10)$$
$$f(t) = (t-1)(t-5)(t-2)$$

Therefore: $f(t) = (t-1)(t-5)(t-2)$

Concept Problem Revisited

A rectangular shaped container is built in such a way that its volume can be represented by the polynomial $V(w) = w^3 + 7w^2 + 16w + 12$, where w is the width of the container.

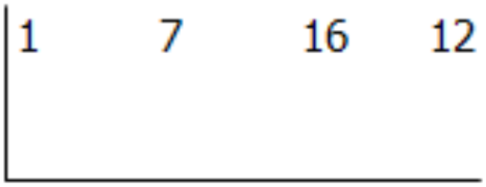
a) In order to begin to find the factors, look at the number 12 and find the factors of this number. The factors of 12 are $\pm1, \pm2, \pm3, \pm4, \pm6$, and ±12. Next, start testing the factors to see if you get a remainder of zero.

$$V(w) = w^3 + 7w^2 + 16w + 12$$
$$w = -1 : V(-1) = (-1)^3 + 7(-1)^2 + 16(-1) + 12$$
$$V(-1) = 2 \quad \text{(NOT a factor)}$$

$$w = -2 : V(-2) = (-2)^3 + 7(-2)^2 + 16(-2) + 12$$
$$V(-2) = 0 \quad \text{(IS a factor)}$$

Now that you have one of the factors, use division to find the others.

Step 1:



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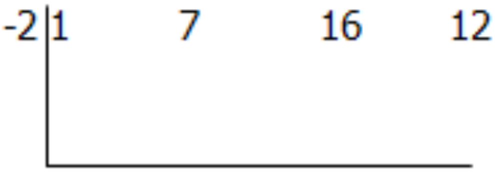

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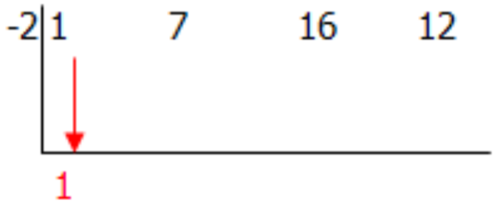

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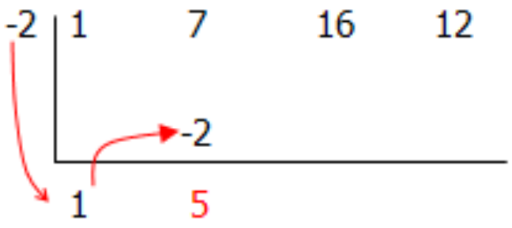

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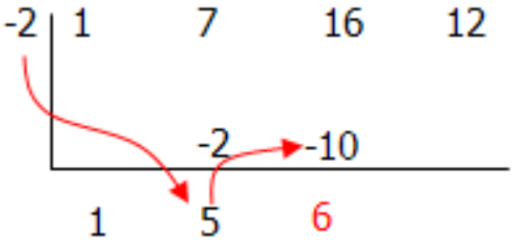

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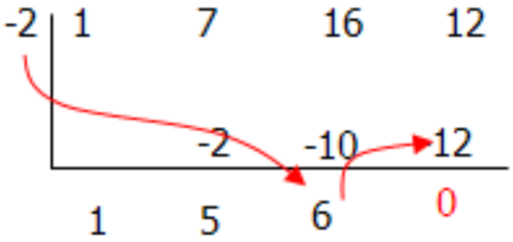


So:

$$V(w) = w^3 + 7w^2 + 16w + 12$$
$$V(w) = (w+2)(w^2+5w+6)$$
$$V(w) = (w+2)(w+2)(w+3)$$

b) If $w = 2$, what are the dimensions of the container?

$$
\begin{aligned}
(w+2) &= 2+2=4\\
(w+2) &= 2+2=4\\
(w+3) &= 2+3=5
\end{aligned}
$$

Therefore. the dimensions of the container are $4\ ft \times 4\ ft \times 5\ ft$.

Vocabulary

The Factor theorem

The ***factor theorem*** states that if $p(a)=0$, then $x-a$ is a factor of $p(x)$.

Guided Practice

1. Determine if $e+3$ is a factor of $2e^3-e^2+e-1$.

2. Factor: x^3+4x^2+x-6.

3. A tennis court is being built where the volume is represented by the polynomial $p(L)=3L^3+8L^2+3L-2$, where L represents the length of the court. Determine if $L+1$ is a factor and if so, find the other factors. If $L=5ft$, what are the dimensions of the court.

Answers:

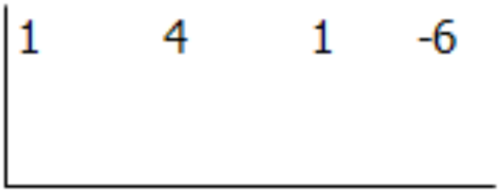
1. $e=-3:2(-3)^3-(-3)^2+(-3)-1=-67$. Therefore $(e+3)$ is not a factor of $2e^3-e^2+e-1$.

2. In order to begin to find the factors, look at the number –6 and find the factors of this number. The factors of –6 are $\pm1, \pm2, \pm3$, and ±6. Next, start testing the factors to see if you get a remainder of zero.

$$
\begin{aligned}
&x^3+4x^2+x-6\\
&x=-1:(-1)^3+4(-1)^2+(-1)-6=-4 \quad \text{(NOT a factor)}\\
&x=1:(1)^3+4(1)^2+(1)-6=0 \quad \text{(IS a factor)}
\end{aligned}
$$

Now that you have one of the factors, use division to find the others.

Step 1:



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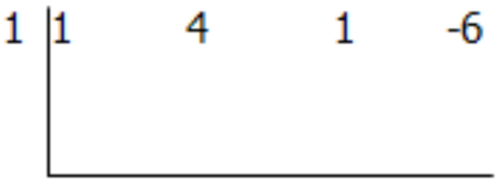

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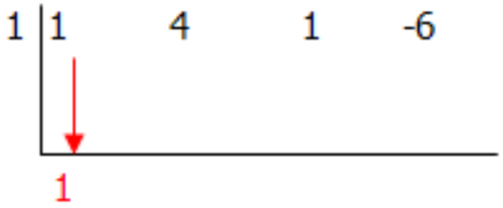

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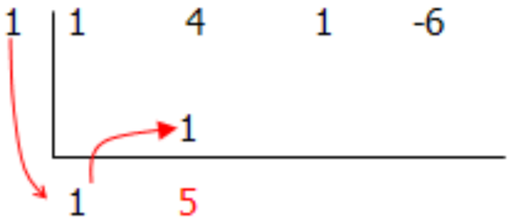

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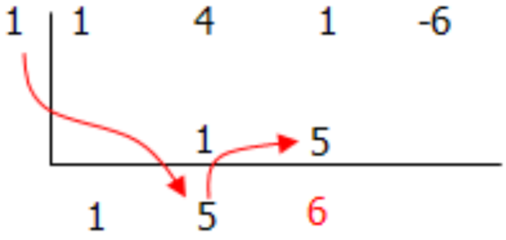

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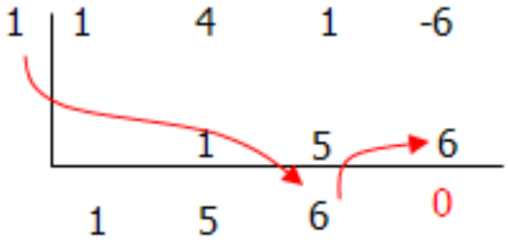


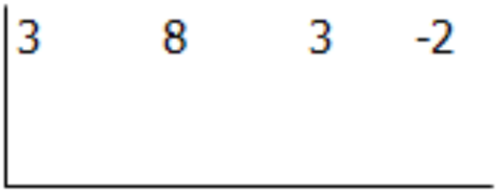
Therefore: $x^3+4x^2+x-6=(x-1)(x^2+5x+6)=(x-1)(x+2)(x+3)$

3. Start by testing the factor $L+1$ to see if you get a remainder of zero.

$$p(L)=3L^3+8L^2+3L-2L=-1:$$
$$p(L)=3(-1)^3+8(-1)^2+3(-1)-2$$
$$p(1)=0 \quad \text{(IS a factor)}$$

Now that you have one of the factors, use division to find the others.

Step 1:



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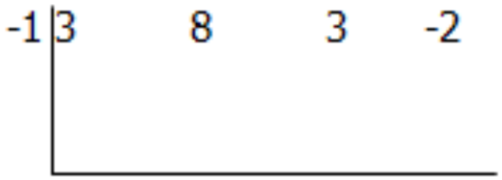

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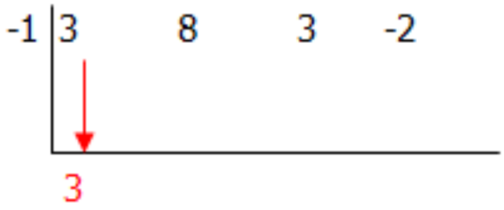

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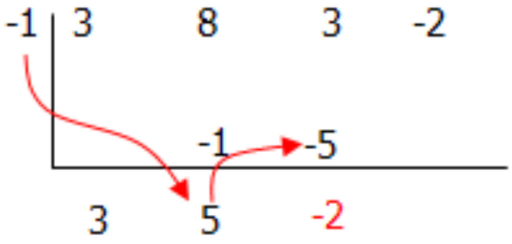

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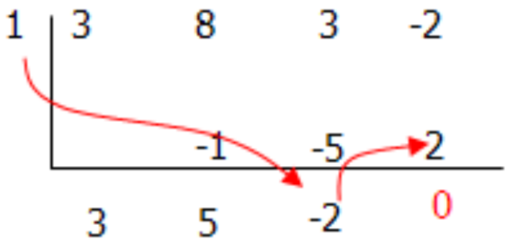


So:

$$p(L) = 3L^3 + 8L^2 + 3L - 2$$
$$p(L) = (L+1)(3L^2 + 5L - 2)$$
$$p(L) = (L+1)(3L-1)(L+2)$$

If $L = 5\ ft$, what are the dimensions of the container?

$$\begin{aligned}(L+1) &= 5+1=6\\(3L-1) &= 3(5)-1=14\\(L+2) &= 5+2=7\end{aligned}$$

Therefore the dimensions of the container are $6\ ft \times 14\ ft \times 7\ ft$.

Practice

Determine if $a-4$ is a factor of each of the following.

1. a^3-5a^2+3a+4
2. $3a^2-7a-20$
3. $-a^4+3a^3+5a^2-16$
4. $a^4-2a^3-8a^2+3a-4$
5. $2a^4-5a^3-7a^2-21a+4$

Factor each of the following:

6. x^3+2x^2+2x+1
7. x^3+x^2-x-1
8. $2x^3-5x^2+2x+1$
9. $2b^3+4b^2-3b-6$
10. $3c^3-4c^2-c+2$
11. $2x^3-13x^2+17x+12$
12. x^3+2x^2-x-2
13. $3x^3+2x^2-53x+60$
14. $x^3-7x^2+7x+15$
15. $x^4+4x^3-7x^2-34x-24$

7.15 Graphs of Polynomial Functions

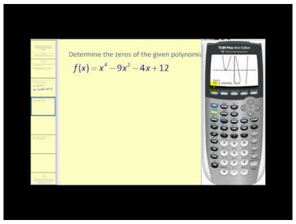
Here you will look at graphs of polynomial functions and identify real roots of polynomial functions from their graphs.

Use your graphing calculator to graph the functions below. What are the real roots of the functions?

1. $f(x) = x^3 - 6x^2 + 11x - 6$
2. $g(x) = 2x^4 - 4x^3 - 3x^2 + 12x - 8$

Watch This

James Sousa: Determing the Zeros or Roots of a Polynomial Function on the TI83/84



Determine the zeros of the given polynomial

$f(x) = x^4 - 9x^2 - 4x + 12$

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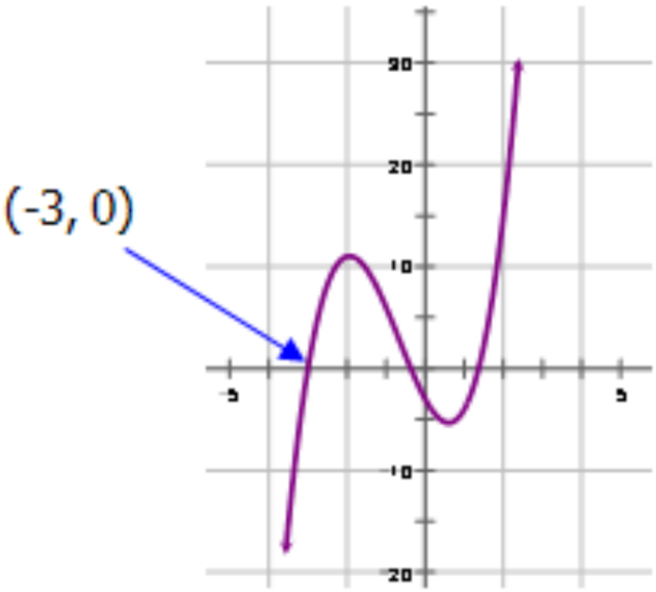
Click image to the left for more content.

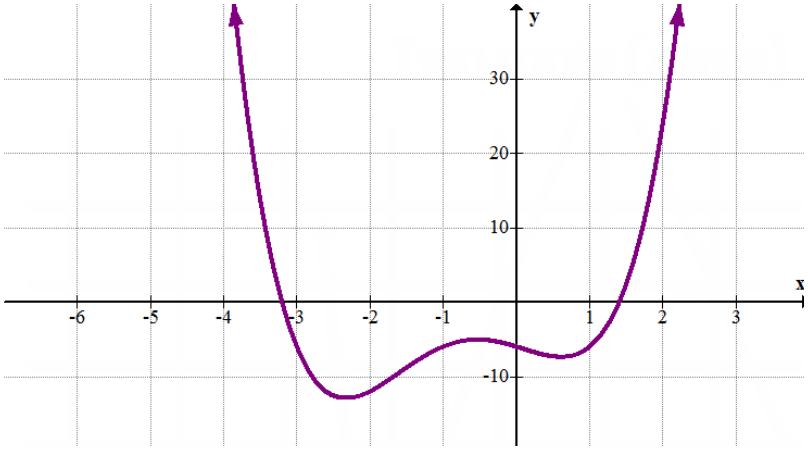
Guidance

Recall that $x - a$ is a factor of polynomial $p(x)$ if $p(a) = 0$. This means that on a graph, factors will appears as x-intercepts of a polynomial because they will occur at points with a y-coordinate equal to zero. For the function graphed below, you can see one of the x-intercepts is the point (–3, 0).

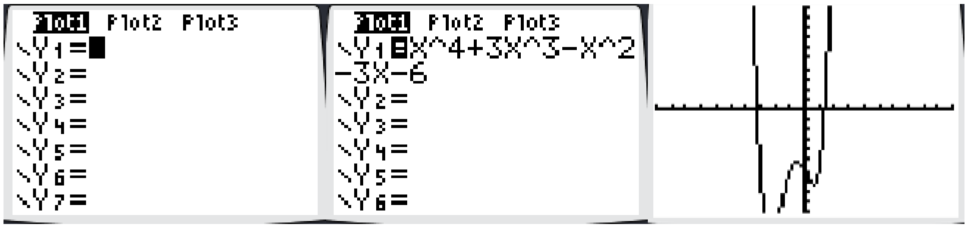


This means that one of the factors of the polynomial must be $(x+3)$. For the polynomial above, -3 is not only known as an x-intercept. It is also known as a **real root** of the polynomial. *Whenever a root (x-intercept) of a polynomial is an integer, it corresponds to a factor of the function.*

Cubic polynomials are degree three and are of the form $y = ax^3 + bx^2 + cx + d$. Graphs of cubics are like the graph above where overall one end of the graph points up and one end of the graph points down. Quartic polynomials are degree four and are of the form $y = ax^4 + bx^3 + cx^2 + dx + e$. Graphs of quartics are like the graph below where overall both ends of the graph point either up or down.



You can use a graphing calculator to graph cubics and quartics. To graph with your graphing calculator, push [Y=], enter your polynomial, push [GRAPH] to see the graph. Then, look at the graph for information about the factors of the polynomial. You can push [TABLE] ([2nd], [GRAPH]) to see the points on the graph more clearly.



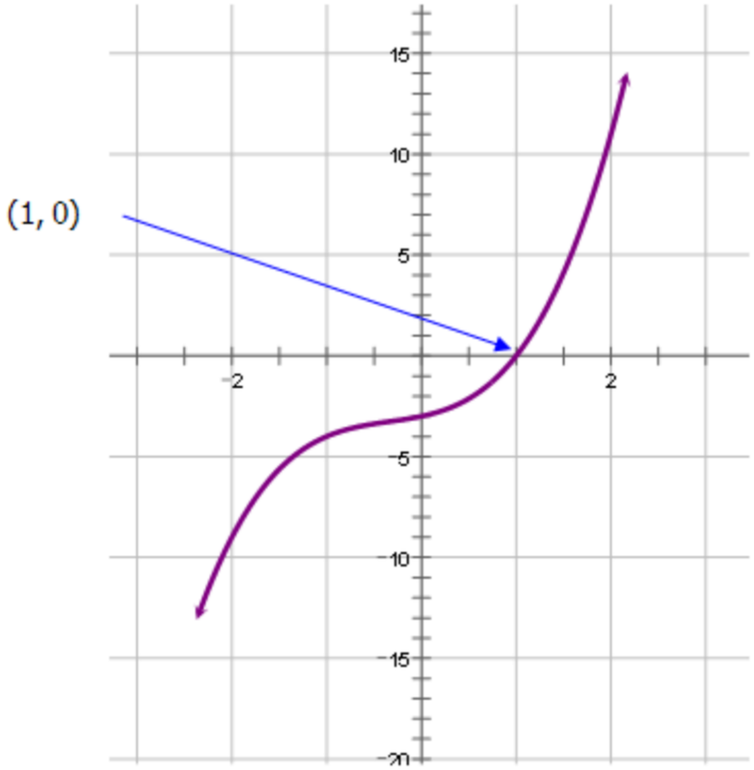
```
Plot1 Plot2 Plot3
\Y1=
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
Plot1 Plot2 Plot3
\Y1=X^4+3X^3-X^2
-3X-6
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

Example A

Graph the function $f(x) = x^3 + x^2 + x - 3$ to determine the number of real roots (x-intercepts).

Solution: Once you graph the function, this is what you should see:

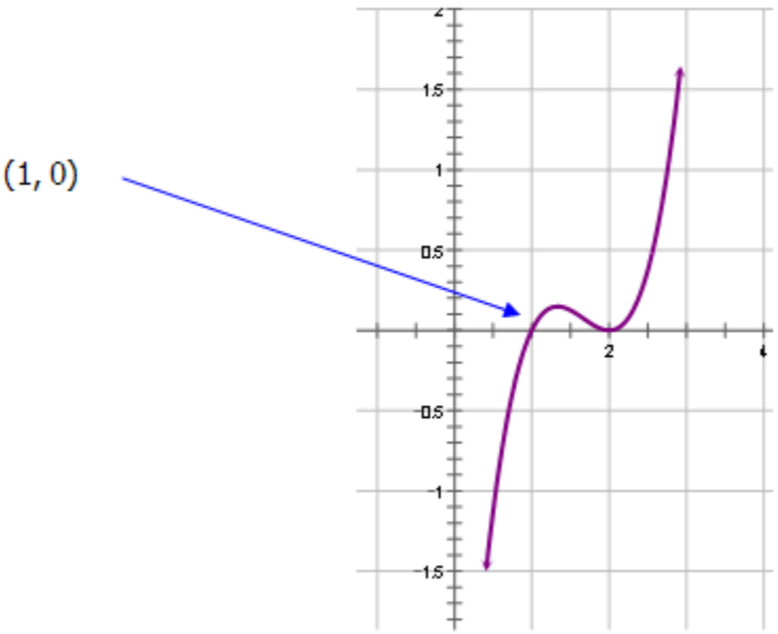


The polynomial $f(x) = x^3 + x^2 + x - 3$ has only one real root (x-intercept) at (1, 0).

Example B

Graph the function $g(x) = x^3 - 5x^2 + 8x - 4$ to determine if $x - 1$ is a factor of the polynomial.

Solution: Once you graph the function, this is what you should see:

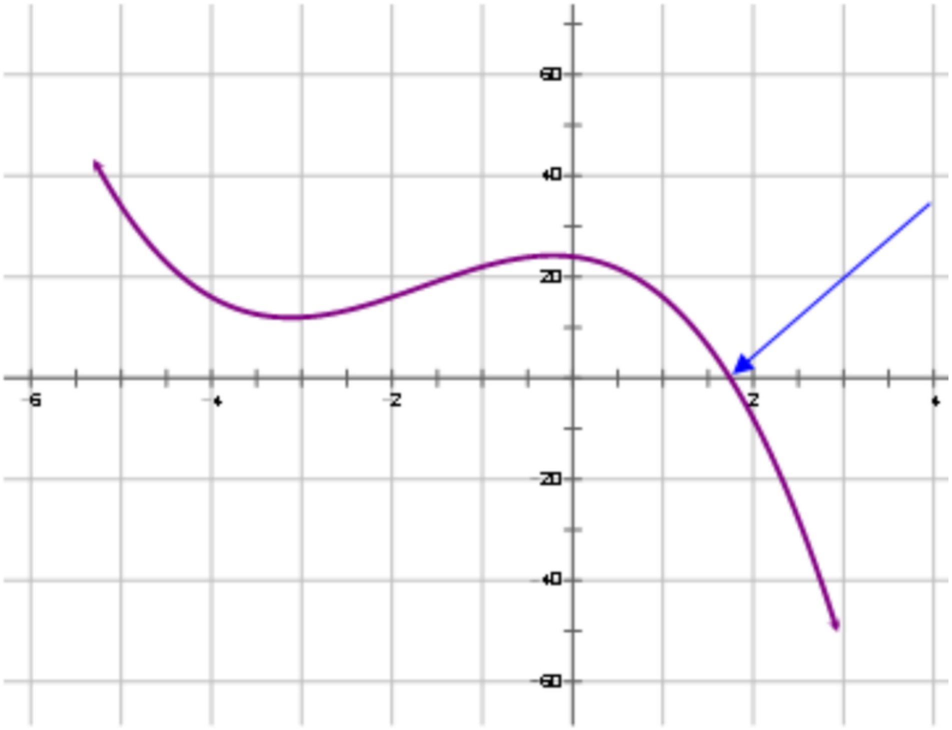


Since (1, 0) is an x-intercept of the polynomial $g(x) = x^3 - 5x^2 + 8x - 4$, $(x - 1)$ is a factor of this cubic.

Example C

How many real roots (x-intercepts) are there for the polynomial $h(x) = -x^3 - 5x^2 - 2x + 24$?

Solution: Once you graph the function, this is what you should see:



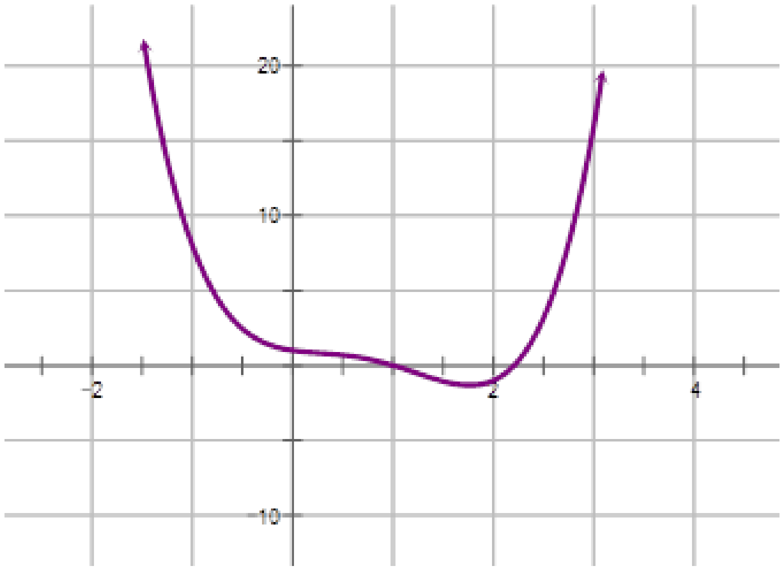
There is one x-intercept so there is one real root.

Example D

Find the real root(s) for the following quartic.

$$k(x) = x^4 - 3x^3 + 2x^2 - x + 1$$

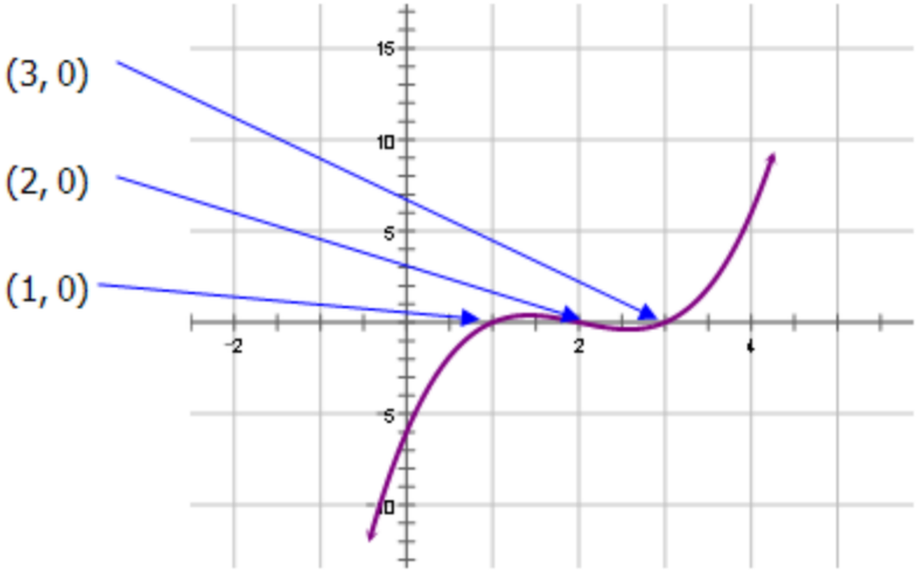
Solution: This is the graph of the quartic:



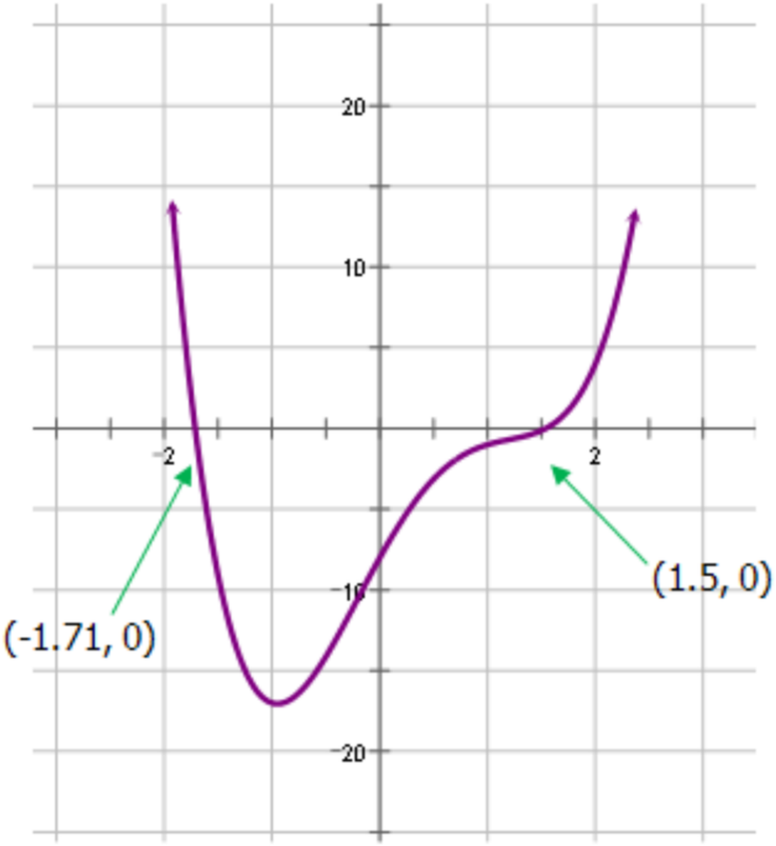
There are two real roots for this quartic. One is (1, 0) and the other occurs around (2.25,0).

Concept Problem Revisited

Here is the graph of the function $f(x) = x^3 - 6x^2 + 11x - 6$:



Here is the graph of the function $g(x) = 2x^4 - 4x^3 - 3x^2 + 12x - 8$. It has two real roots as indicated.



Vocabulary

Cubic Polynomial

A ***cubic polynomial*** is a polynomial where the largest degree is 3. So, for example, $2x^3+13x^2-8x+5$ is a cubic polynomial.

Real Root

A ***real root*** is a point where the graph of a function crosses the x-axis.

Quartic Polynomials

Quartic polynomials have a degree of 4. So for example $x^4-2x^3-13x^2-14x+24$ is a quartic because it has a degree of 4.

Guided Practice

1. Find the real roots for the cubic $y=x^3+3x-4$ using a graph.

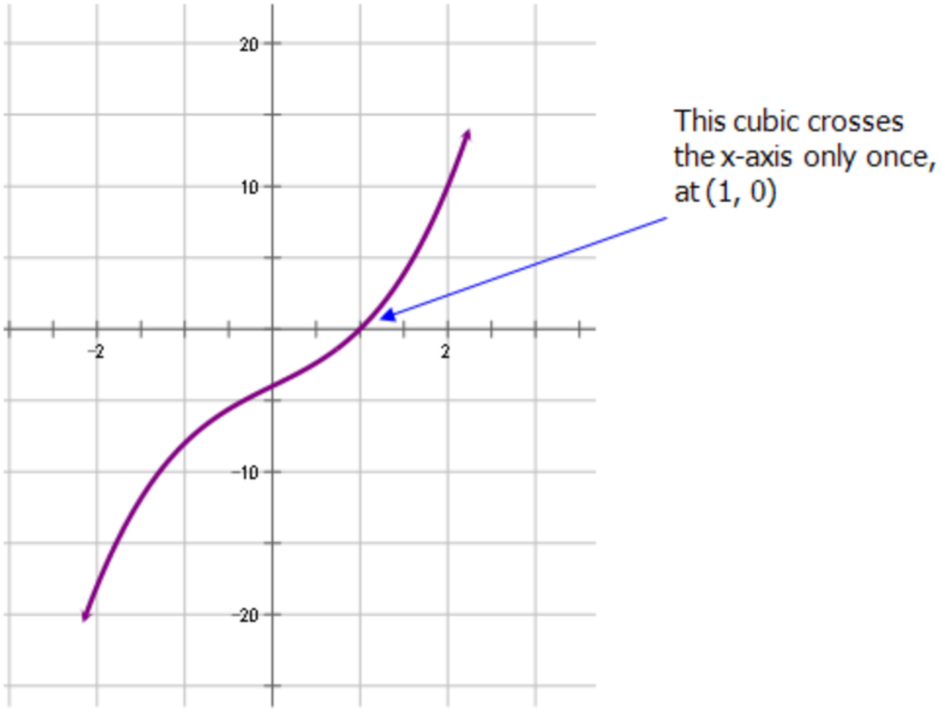
2. Graph the function $g(x)=3x^3+8x^2+3x-2$ and determine the number of real roots. Is $(x-2)$ one of the factors of this polynomial?

3. Graph the function $m(x)=-2x^3+10x^2+8x-1$ to determine if $x-1$ is a factor of the polynomial.

4. Describe the graph of the following quartic: $j(x)=-x^4-3x^3+2x^2+x-6$.

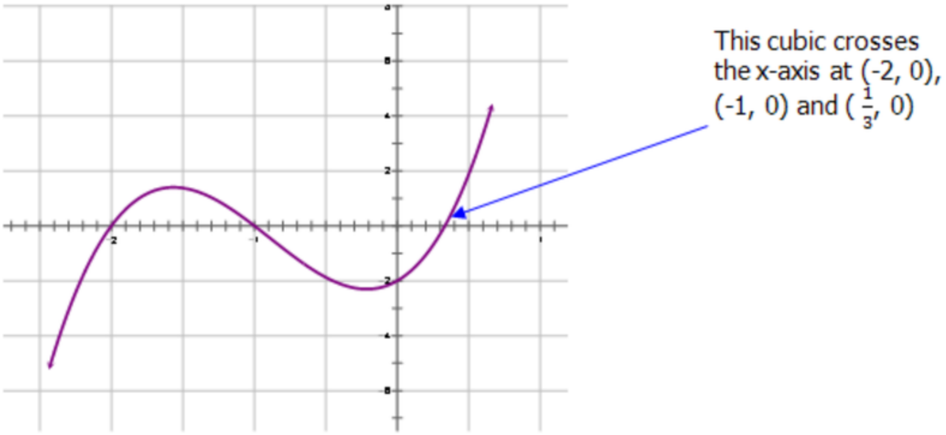
Answers

1.



This cubic crosses the x-axis only once, at (1, 0)

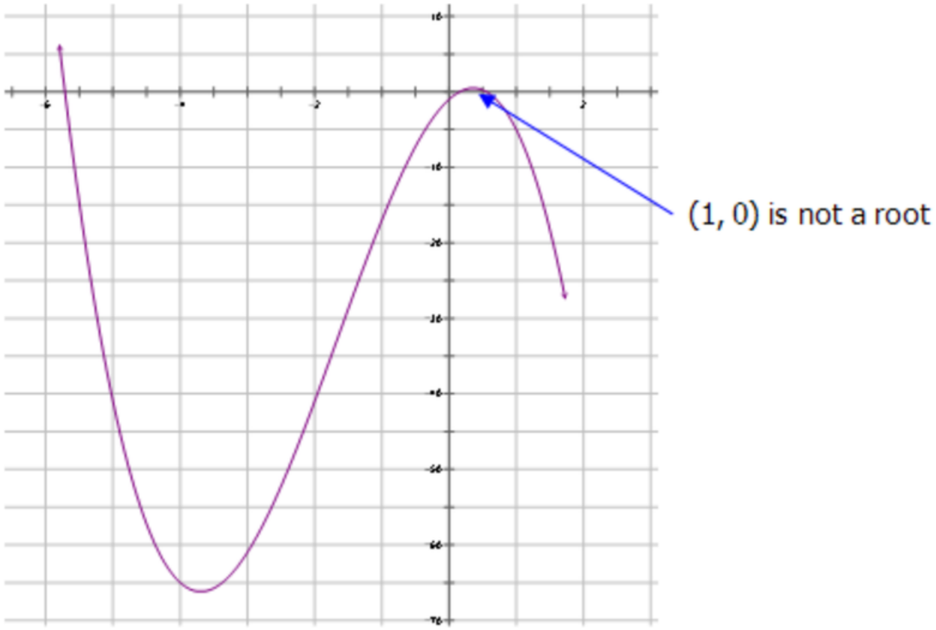
2.



This cubic crosses the x-axis at (-2, 0), (-1, 0) and ($\frac{1}{3}$, 0)

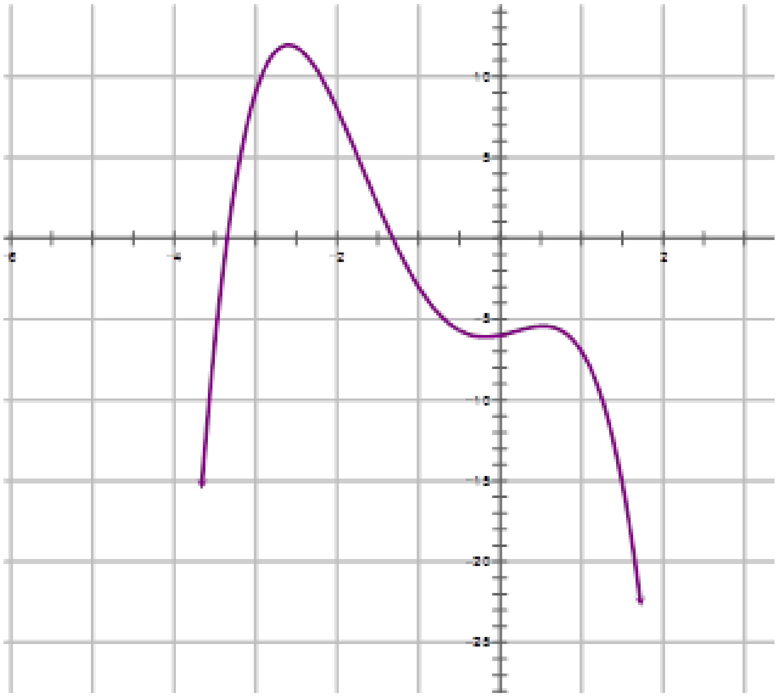
Since one of these root values is (–2, 0), the factor for the polynomial would be $(x+2)$ and not $(x-2)$.

3. If $(x-1)$ was one of the factors then one of the roots would have to be (1, 0). This is not the case.



(1, 0) is not a root

4. The graph has an M shape. It looks like an M because of the –1 coefficient before x^4. There are two real roots located at (–3.35, 0) and (–1.32, 0).



Practice

Find the real roots for the following cubic polynomials using a graph.

1. $y = x^3 - 2x^2 - 9x + 18$
2. $y = x^3 + 5x^2 - 4x - 20$
3. $y = 3x^3 - 6x^2 + 12x - 5$

4. $y = 2x^3 - 8x^2 + 3x - 12$
5. $y = -2x^3 - 3x^2 - 5x + 10$

Graph the functions below and determine the number of real roots. Give at least one factor of each polynomial from the graphed solution.

6. $y = x^3 - 3x^2 - 2x + 6$
7. $y = x^3 + x^2 - 3x - 3$
8. $y = x^3 + 2x^2 - 16x - 32$
9. $y = 2x^3 + 13x^2 + 9x + 6$
10. $y = 2x^3 + 15x^2 + 4x - 21$

Graph the functions below to determine if $x - 1$ is a factor of the polynomial.

11. $y = x^3 - 2x^2 + 3x - 6$
12. $y = x^3 + 3x^2 - 2x - 2$
13. $y = 3x^3 + 8x^2 - 5x - 6$
14. $y = x^3 + x^2 - 10x + 8$
15. $y = 2x^3 - x^2 - 3x + 2$

Indicate the real root(s) on the following quartic graphs:

16. $y = x^4 - 3x^3 - 6x^2 - 3$
17. $y = x^4 - 8x^2 - 8$
18. $y = 2x^4 + 2x^3 + x^2 - x - 8$
19. $y = x^4 - 6x^2 - x + 3$
20. $y = x^4 + x^3 - 7x^2 - x + 6$

Describe the following graphs:

21. $y = x^4 - 5x^2 + 2x + 2$
22. $y = x^4 + 3x^3 - x - 3$
23. $y = -x^4 + x^3 + 4x^2 - x + 6$
24. $y = -x^4 - 5x^3 - 5x^2 + 5x + 6$
25. $y = -2x^4 - 4x^3 - 5x^2 - 4x - 4$

Summary

You learned that adding, subtracting, and multiplying polynomials all rely on the distributive property. You also learned that factoring is the reverse of multiplying because when you factor a polynomial you are trying to rewrite the polynomial as a product of other polynomials. You learned how to factor completely by first looking for common factors and then using other methods to factor the remaining expression. You learned special cases of factoring to watch out for including the difference of perfect squares, perfect square trinomials, and the sum and difference of cubes. You learned how factoring can allow you to solve a quadratic equation with the help of the zero product property.

You learned how to divide polynomials and how dividing polynomials can connect to factoring. You learned about the factor theorem and that polynomial long division can actually help you to factor higher degree polynomials. Finally, you learned that factors of polynomials connect to their x-intercepts, and that you can use a graphing calculator to graph polynomial functions in order to find out more information about the factors of the polynomial.

CHAPTER 8

Rational Expressions and Rational Functions

Chapter Outline

Introduction

Here you'll learn all about rational expressions. You'll start by learning how to apply your factoring skills to simplifying rational expressions. You'll then learn how operations with fractions generalize to operations with rational expressions. Finally, you will learn about rational functions. You will learn what the graphs of rational functions look like and how to find the asymptotes for rational functions.

8.1 Rational Expression Simplification

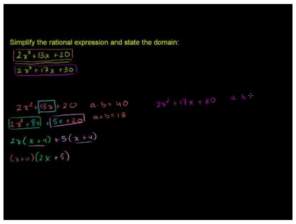
Here you'll learn how to simplify rational expressions.

How could you use factoring to help simplify the following rational expression?

$\frac{3x^2-27}{x^2+7x+12}$

Watch This

Khan Academy Simplifying Rationals Expressions



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Guidance

A rational number is any number of the form $\frac{a}{b}$, where $b \neq 0$. A ***rational expression*** is any algebraic expression of the form $\frac{a(x)}{b(x)}$, where $b \neq 0$. An example of a rational expression is: $\frac{4x^2+20x+24}{2x^2+8x+8}$.

Consider that any number or expression divided by itself is equal to 1. For example, $\frac{2}{2} = 1$ and $\frac{(x+2)}{(x+2)} = 1$. This fact allows you to simplify rational expressions that are in factored form by looking for "1's". Consider the following rational expression:

$\frac{4x^2+20x+24}{2x^2+8x+8}$

Factor both the numerator and denominator completely:

$\frac{4(x+2)(x+3)}{2(x+2)(x+2)}$

Notice that there is one factor of $x+2$ in both the numerator and denominator. These factors divide to make 1, so they "cancel out" (the second factor of $(x+2)$ in the denominator will remain there).

$\frac{4\cancel{(x+2)}(x+3)}{2\cancel{(x+2)}(x+2)}$

Also, the $\frac{4}{2}$ reduces to just 2. The simplified expression is:

$\frac{2(x+3)}{x+2}$

Keep in mind that you cannot "cancel out" common factors until both the numerator and denominator have been factored.

A rational expression is like any other fraction in that it is said to be undefined if the denominator is equal to zero. Values of the variable that cause the denominator of a rational expression to be zero are referred to as ***restrictions*** and must be excluded from the set of possible values for the variable. For the original expression above, the restriction is $x \neq -2$ because if $x = -2$ then the denominator would be equal to zero. Note that to determine the restrictions you must look at the **original** expression before any common factors have been cancelled.

Example A

Simplify the following and state any restrictions on the denominator.

$\frac{x-2}{x^2-10x+16}$

Solution: To begin, factor both the numerator and the denominator:

$\frac{x-2}{(x-8)(x-2)}$

Cancel out the common factor of $x-2$ to create the simplified expression:

$\frac{\cancel{(x-2)}}{(x-8)\cancel{(x-2)}}$

$\frac{1}{x-8}$

The restrictions are $x \neq 2$ and $x \neq 8$ because both of those values for x would have made the denominator of the original expression equal to zero.

Example B

Simplify the following and state any restrictions on the denominator.

$\frac{x^2+7x+12}{x^2-16}$

Solution: To begin, factor both the numerator and the denominator:

$\frac{(x+4)(x+3)}{(x-4)(x+4)}$

Cancel out the common factor of $x+4$ to create the simplified expression:

$\frac{\cancel{(x+4)}(x+3)}{(x-4)\cancel{(x+4)}}$

$\frac{x+3}{x-4}$

The restrictions are $x \neq 4$ and $x \neq -4$ because both of those values for x would have made the denominator of the original expression equal to zero.

Example C

Simplify the following and state any restrictions on the denominator.

$\frac{3x^2-7x-6}{4x^2-13x+3}$

Solution: To begin, factor both the numerator and the denominator:

$\frac{(x-3)(3x+2)}{(x-3)(4x-1)}$

Cancel out the common factor of $x-3$ to create the simplified expression:

$\frac{\cancel{(x-3)}(3x+2)}{\cancel{(x-3)}(4x-1)}$

$\frac{3x+2}{4x-1}$

The restrictions are $x \neq 3$ and $x \neq \frac{1}{4}$ because both of those values for x would have made the denominator of the original expression equal to zero.

Concept Problem Revisited

$$\begin{aligned}&\frac{3x^2-27}{x^2+7x+12}\\=&\frac{3(x^2-9)}{(x+3)(x+4)}\\=&\frac{3(x+3)(x-3)}{(x+3)(x+4)}\\=&\frac{3\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+4)}\\=&\frac{3(x-3)}{x+4}\end{aligned}$$

where $x \neq -3$ and $x \neq -4$

Vocabulary

Rational Expression

A ***rational expression*** is an algebraic expression that can be written in the form $\frac{a(x)}{b(x)}$ where $b \neq 0$.

Restriction

Any value of the variable in a rational expression that would result in a zero denominator is called a ***restriction*** on the denominator.

Guided Practice

Simplify each of the following and state the restrictions.

1. $\frac{m^2-9m+18}{4m^2-24m}$
2. $\frac{2x^2-8}{4x+8}$
3. $\frac{c^2+4c-5}{c^2-2c-35}$

Answers:

1. $\frac{m^2-9m+18}{4m^2-24m} = \frac{(m-6)(m-3)}{(4m)(m-6)} = \frac{(m-3)}{4m}, m \neq 0; m \neq 6$
2. $\frac{2x^2-8}{4x+8} = \frac{(2)(x-2)(x+2)}{(4)(x+2)} = \frac{(x-2)}{2}, x \neq -2$
3. $\frac{c^2+4c-5}{c^2-2c-35} = \frac{(c+5)(c-1)}{(c-7)(c+5)} = \frac{(c-1)}{(c-7)}, c \neq -5; c \neq 7$

Practice

For each of the following rational expressions, state the restrictions.

1. $\frac{7}{x+4}$
2. $\frac{-3}{x-5}$
3. $\frac{5x+1}{5x-1}$
4. $\frac{6}{4x-3}$

5. $\frac{(x+1)}{x^2-4}$
6. $\frac{x-8}{x^2+3x+2}$
7. $\frac{x+6}{x^2-5x-24}$
8. $\frac{5x+2}{2x^2+5x+2}$

Simplify each of the following rational expressions and state the restrictions.

9. $\frac{4}{4x+12}$
10. $\frac{4c^2}{8c^2-4c}$
11. $\frac{10x+5}{2x+1}$
12. $\frac{x-4}{x^2-16}$
13. $\frac{y+1}{y^2+5y+4}$
14. $\frac{c+2}{c^2-5c-14}$
15. $\frac{(b-3)^2}{b^2-6b+9}$
16. $\frac{3n^2-27}{6n+18}$
17. $\frac{6k^2+7k-20}{12k^2-19k+4}$
18. $\frac{4x^2-4x-3}{2x^2+3x-9}$

8.2 Rational Expression Multiplication and Division

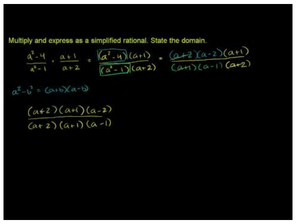
Here you'll learn how to multiply and divide rational expressions.

How can you use your knowledge of multiplying fractions to multiply the following rational expressions?

$$\frac{10y+20}{5y-15}\cdot\frac{y-3}{y^2+10y+16}$$

Watch This

Khan Academy Multiplying and Dividing Rational Expressions



Multiply and express as a simplified rational. State the domain.

$$\frac{a^2-4}{a^2-1}\cdot\frac{a+1}{a+2}=\frac{\boxed{(a^2-4)}(a+1)}{\boxed{(a^2-1)}(a+2)}=\frac{(a+2)(a-2)(a+1)}{(a+1)(a-1)(a+2)}$$

$$a^2-b^2=(a+b)(a-b)$$

$$\frac{(a+2)(a+1)(a-2)}{(a+2)(a+1)(a-1)}$$

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Guidance

Rational expressions are examples of fractions, so you multiply and divide rational expressions in the same ways that you multiply and divide fractions. As with fractions, after multiplying or dividing you will want to simplify your result. You will also want to state any restrictions that would cause the denominator of either original rational expression to be equal to zero. To multiply:

$$\frac{x^2+2x}{x+3}\cdot\frac{x^2+4x+3}{x}$$

First, factor all expressions that can be factored:

$$\frac{x(x+2)}{x+3}\cdot\frac{(x+3)(x+1)}{x}$$

Next, multiply the numerators and multiply the denominators to create one big rational expression. Leave in factored form:

$$\frac{x(x+2)(x+3)(x+1)}{x(x+3)}$$

Simplify:

$$\frac{\cancel{x}(x+2)\cancel{(x+3)}(x+1)}{\cancel{x}\cancel{(x+3)}}$$

$$=(x+2)(x+1)$$

$$=x^2+3x+2$$

Finally, state the restrictions based on the original rational expressions:

$x\neq-3$ and $x\neq0$.

To divide rational expressions, recall that dividing one fraction by another is the same as multiplying the first fraction by the reciprocal of the second fraction. For example, $\frac{x^2+2x}{x+3}\div\frac{x}{x^2+4x+3}$ is equivalent to, and can be rewritten as, $\frac{x^2+2x}{x+3}\cdot\frac{x^2+4x+3}{x}$, which can then be solved using the same steps as above.

Example A

Multiply the following rational expressions and state the restrictions.

$$\frac{4x-8}{x^2-7x+10}\cdot\frac{x^2-3x-10}{x^2-4}$$

Solution: Begin by factoring the numerator and denominator of each expression:

$$\frac{4(x-2)}{(x-5)(x-2)}\cdot\frac{(x-5)(x+2)}{(x-2)(x+2)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$$\frac{4(x-2)(x-5)(x+2)}{(x-5)(x-2)(x-2)(x+2)}$$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$$\frac{4\cancel{(x-2)}\cancel{(x-5)}\cancel{(x+2)}}{\cancel{(x-5)}\cancel{(x-2)}\cancel{(x+2)}(x-2)}$$

The final answer is: $\frac{4}{(x-2)}$ with restrictions: $x \neq 5$, $x \neq 2$, and $x \neq -2$.

Example B

Divide the following rational expressions and state the restrictions.

$$\frac{m^2-4}{m^2+9m+14}\div\frac{3m^2-6m}{m^2-49}$$

Solution: To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$$\frac{m^2-4}{m^2+9m+14}\cdot\frac{m^2-49}{3m^2-6m}$$

Begin by factoring the numerator and denominator of each expression.

$$\frac{(m-2)(m+2)}{(m+7)(m+2)}\cdot\frac{(m-7)(m+7)}{3m(m-2)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$$\frac{(m-2)(m+2)(m-7)(m+7)}{3m(m+7)(m+2)(m-2)}$$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$$\frac{\cancel{(m+2)}\cancel{(m-2)}\cancel{(m+7)}(m-7)}{3m\cancel{(m-2)}\cancel{(m+7)}\cancel{(m+2)}}$$

The final answer is: $\frac{(m-7)}{3m}$ with restrictions: $x \neq -7$, $x \neq 2$, $x \neq 0$, $x \neq 7$ and $x \neq -2$. Note that when dividing rational expressions, for restrictions you must consider ALL factors that ever appear in a denominator. This means that both the numerator and denominator of the second rational expression must be considered for restrictions.

Example C

Simplify the following rational expressions and state the restrictions.

$$\frac{12x^2+13x-35}{5x^2-21x+18}\div\frac{3x^2+16x+21}{5x^2+9x-18}$$

Solution: To divide rational expressions, multiply by the reciprocal of the divisor. Then, follow the process for multiplying rational expressions.

$$\frac{12x^2+13x-35}{5x^2-21x+18}\times\frac{5x^2+9x-18}{3x^2+16x+21}$$

Begin by factoring the numerator and denominator of each expression.

$$\frac{(4x-5)(3x+7)}{(5x-6)(x-3)}\times\frac{(5x-6)(x+3)}{(3x+7)(x+3)}$$

Next, multiply the numerators and the denominators to create one rational expression:

$\frac{(4x-5)(3x+7)(5x-6)(x+3)}{(5x-6)(x-3)(3x+7)(x+3)}$

Simplify by removing common factors from the numerator and denominator that divide to make 1:

$\frac{(4x-5)\cancel{(3x+7)}\cancel{(5x-6)}\cancel{(x+3)}}{\cancel{(5x-6)}(x-3)\cancel{(3x+7)}\cancel{(x+3)}}$

The final answer is: $\frac{(4x-5)}{(x-3)}$ with restrictions: $x \neq \frac{6}{5}$, $x \neq 3$, $x \neq -\frac{7}{3}$, and $x \neq -3$.

Concept Problem Revisited

$\frac{10y+20}{5y-15} \cdot \frac{y-3}{y^2+10y+16}$

Begin by factoring the numerator and denominator of each expression.

$\frac{10(y+2)}{5(y-3)} \cdot \frac{y-3}{(y+8)(y+2)}$

Express the factored rational expressions as a single rational expression.

$\frac{10(y+2)(y-3)}{5(y-3)(y+8)(y+2)}$

Cancel the common factors that exist in the numerator and the denominator.

$\frac{\overset{2}{\cancel{10}}\cancel{(y+2)}\cancel{(y-3)}}{\cancel{5}\cancel{(y-3)}(y+8)\cancel{(y+2)}}$

The result of cancelling the common factors is the answer. Don't forget to include the restrictions.

$$\boxed{\frac{2}{y+8}; y \neq 3; y \neq -8; y \neq -2}$$

Vocabulary

Rational Expression

A ***rational expression*** is an algebraic expression that can be written in the form $\frac{a(x)}{b(x)}$ where $b \neq 0$.

Restriction

Any value of the variable in a rational expression that would result in a zero denominator is called a ***restriction*** on the denominator.

Guided Practice

Multiply or divide each of the following and state the restrictions.

1. $\frac{x+7}{x^2-5x-36} \div \frac{x^2-2x-63}{x+4} \cdot \frac{x^2-15x+54}{x^2-36}$
2. $\frac{y^2-25}{y^2-6y} \cdot \frac{y^2-12y+36}{y^2+2y-15} \div \frac{y^2-11y+30}{y^2+4y-21}$
3. $\frac{2x^2+7x-4}{6x^2+x-2} \cdot \frac{15x^2+7x-2}{5x^2+19x-4}$

Answers:

1. Write the term after the division sign as a reciprocal and multiply.

$\frac{x+7}{x^2-5x-36} \cdot \frac{x+4}{x^2-2x-63} \cdot \frac{x^2-15x+54}{x^2-36}$

Factor the numerator and denominator of each expression.

$\frac{x+7}{(x-9)(x+4)} \cdot \frac{x+4}{(x-9)(x+7)} \cdot \frac{(x-9)(x-6)}{(x+6)(x-6)}$

Express the factored rational expressions as a single rational expression.

$\frac{(x+7)(x+4)(x-9)(x-6)}{(x-9)(x+4)(x-9)(x+7)(x+6)(x-6)}$

Cancel the common factors that exist in the numerator and the denominator.

$$\frac{\overbrace{\cancel{(x+7)}\cancel{(x+4)}\cancel{(x-9)}\cancel{(x-6)}}^{1}}{\cancel{(x-9)}\cancel{(x+4)}(x-9)\cancel{(x+7)}(x+6)\cancel{(x-6)}}$$

The result of cancelling the common factors is the answer.

$$\boxed{=\frac{1}{(x-9)(x-6)};x \neq 9;x \neq -4;x \neq -7;x \neq -6;x \neq 6;}$$

2. Write the term after the division sign as a reciprocal and multiply.

$\frac{y^2-25}{y^2-6y} \cdot \frac{y^2-12y+36}{y^2+2y-15} \cdot \frac{y^2+4y-21}{y^2-11y+30}$

Factor the numerator and denominator of each expression.

$\frac{(y+5)(y-5)}{y(y-6)} \cdot \frac{(y-6)(y-6)}{(y+5)(y-3)} \cdot \frac{(y+7)(y-3)}{(y-6)(y-5)}$

Express the factored rational expressions as a single rational expression.

$\frac{(y+5)(y-5)(y-6)(y-6)(y+7)(y-3)}{y(y-6)(y+5)(y-3)(y-6)(y-5)}$

Cancel the common factors that exist in the numerator and the denominator.

$$\frac{\cancel{(y+5)}\cancel{(y-5)}\cancel{(y-6)}\cancel{(y-6)}(y+7)\cancel{(y-3)}}{y\cancel{(y-6)}\cancel{(y+5)}\cancel{(y-3)}\cancel{(y-6)}\cancel{(y-5)}}$$

The result of cancelling the common factors is the answer.

$$\boxed{=\frac{y+7}{y};y \neq 0;y \neq 6;y \neq -5;y \neq 3;y \neq 5;y \neq -7}$$

3. Factor the numerator and denominator of each expression.

$\frac{(2x-1)(x+4)}{(2x-1)(3x+2)} \cdot \frac{(5x-1)(3x+2)}{(5x-1)(x+4)}$

Express the factored rational expressions as a single rational expression.

$\frac{(2x-1)(x+4)(5x-1)(3x+2)}{(2x-1)(3x+2)(5x-1)(x+4)}$

Cancel the common factors that exist in the numerator and the denominator.

$$\frac{\overbrace{\cancel{(2x-1)}\cancel{(x+4)}\cancel{(5x-1)}\cancel{(3x+2)}}^{1}}{\cancel{(2x-1)}\cancel{(3x+2)}\cancel{(5x-1)}\cancel{(x+4)}}$$

The result of cancelling the common factors is the answer.

$$\boxed{=1;x \neq \frac{1}{2};x \neq -\frac{2}{3};x \neq \frac{1}{5};x \neq -4}$$

Practice

Multiply or divide each of the following and state the restrictions for each.

1. $\frac{3x+9}{6x} \cdot \frac{x^2}{x^2-9}$
2. $\frac{c^2+5c+6}{c-1} \cdot \frac{c^2-1}{c+3}$
3. $\frac{a^2+3a}{3a-9} \cdot \frac{a^2-a-6}{2a^2+6a}$
4. $\frac{y-3}{y+3} \cdot \frac{y^2-9}{y+3} \cdot \frac{y^2+6y+9}{y^2-6y+9}$
5. $\frac{m^2-4m-5}{m^2-5m} \cdot \frac{m^2-6m+5}{m^2-1} \cdot \frac{m}{m-5}$
6. $\frac{x^2-x-20}{x^2-25} \div \frac{3x+12}{x+5}$
7. $\frac{d^2-9}{3-3d} \div \frac{d^2+5d+6}{d^2+3d-4}$
8. $\frac{4x^2-20x}{3x+6} \div \frac{x-5}{x^2-x-6}$
9. $\frac{4n^2-9}{2n^3+2n^2-4n} \div \frac{2n^2-n-3}{3n^2-6n+3}$
10. $\frac{e^2+10e+21}{2e^2+7e-15} \div \frac{e^2+8e+15}{e^2+10e+25}$
11. $\frac{x^2+2x-15}{x^2-6x+8} \cdot \frac{x^2+2x-8}{x^2-6x+9} \cdot \frac{x^2-7x+12}{x^2-x-30}$
12. $\frac{2x^2+5x-3}{4x^2-12x+5} \div \frac{3x^2+13x+12}{6x^2-7x-20}$
13. $\frac{5m^2-20}{m^2+14m+33} \cdot \frac{m^2+10m-11}{m^2-8m+12} \cdot \frac{m^2-3m-18}{m^2+m-2}$
14. $\frac{2y^2+5y-12}{y^2+9y+14} \div \frac{6y^2-7y-3}{3y^2+25y+8} \cdot \frac{y^2+3y-28}{y^2-16}$
15. $\frac{x^2-49}{x^2+3x-88} \cdot \frac{x^2+6x-55}{x^2-11x+28} \div \frac{x^2+2x-35}{x^2-12x+32}$

8.3 Rational Expression Addition and Subtraction

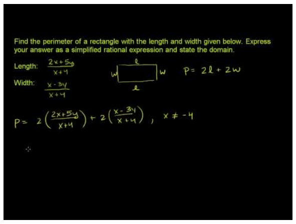
Here you'll learn how to add and subtract rational expressions.

Can you use your knowledge of rational expressions and adding fractions to add the following rational expressions?

$\frac{3x}{x^2+6x-16}+\frac{2x}{x-2}$

Watch This

Khan Academy Adding and Subtracting Rational Expressions


Find the perimeter of a rectangle with the length and width given below. Express your answer as a simplified rational expression and state the domain.

Length: $\frac{2x+5y}{x+4}$

Width: $\frac{x-3y}{x+4}$

$P = 2l + 2w$

$P = 2\left(\frac{2x+5y}{x+4}\right) + 2\left(\frac{x-3y}{x+4}\right), \quad x \neq -4$

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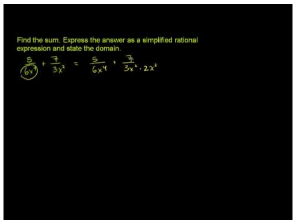
Khan Academy Adding and Subtracting Rational Expressions


Find the sum. Express the answer as a simplified rational expression and state the domain.

$\frac{5}{6x^4}+\frac{7}{3x^2} = \frac{5}{6x^4}+\frac{7}{3x^2\cdot 2x^2}$

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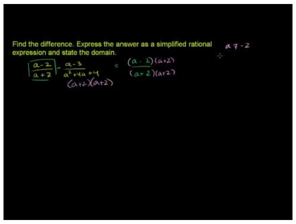
Khan Academy Adding and Subtracting Rational Expressions


Find the difference. Express the answer as a simplified rational expression and state the domain.

$\frac{a-2}{a+2}-\frac{a-3}{a^2+4a+4} = \frac{(a-2)(a+2)}{(a+2)(a+2)}$

$(a+2)(a+2)$

$a \neq -2$

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Guidance

Rational expressions are examples of fractions, so you add and subtract rational expressions in the same way that you add and subtract fractions. As with fractions, you will need a common denominator, ideally the lowest common denominator (LCD), in order to add or subtract the expressions.

For example, to add:

$\frac{x^2+2x}{x+3}+\frac{x}{x^2+4x+3}$

First, factor the denominators to get:

$\frac{x^2+2x}{x+3} + \frac{x}{(x+3)(x+1)}$.

Next, find the lowest common denominator (LCD). This will be the product of each unique factor in the denominators. In this case, the LCD is $(x+3)(x+1)$. Multiply the numerator and denominator of each fraction by the factors necessary to create the common denominator. In this case, you only need to multiply the fraction on the left by $\left(\frac{x+1}{x+1}\right)$. The expression becomes:

$\frac{x^2+2x}{x+3}\left(\frac{x+1}{x+1}\right) + \frac{x}{(x+3)(x+1)}$

$= \frac{(x^2+2x)(x+1)}{(x+3)(x+1)} + \frac{x}{(x+3)(x+1)}$

Now add the numerators and write as one rational expression:

$= \frac{(x^2+2x)(x+1)+x}{(x+3)(x+1)}$

Simplify the numerator by multiplying, combining like terms, and factoring if possible (the denominator is left in factored form):

$\frac{x^3+3x^2+3x}{(x+3)(x+1)}$

$= \frac{x(x^2+3x+3)}{(x+3)(x+1)}$

The rational expression cannot be simplified any further so this is your answer. The restrictions are $x \neq -3$ and $x \neq -1$ because those values would cause one or both of the original denominators to be equal to zero.

Example A

Identify the lowest common denominator (LCD) in factored form.

i) $\frac{2x-3}{x^2-7x+10} - \frac{x-5}{x^2-2x-15}$

ii) $\frac{2x+1}{x^2+6x+9} + \frac{3x-2}{x^2+x-6}$

Solution: To determine the LCD, begin by factoring the denominators.

i) $\frac{2x-3}{x^2-7x+10} - \frac{x-5}{x^2-2x-15} = \frac{2x-3}{(x-5)(x-2)} - \frac{x-5}{(x-5)(x+3)}$

The LCD is

$$\boxed{(x-5)(x-2)(x+3)}$$

ii) $\frac{2x+1}{x^2+6x+9} + \frac{3x-2}{x^2+x-6} = \frac{2x+1}{(x+3)(x+3)} + \frac{3x-2}{(x+3)(x-2)}$

The LCD is

$$\boxed{(x+3)(x+3)(x-2)}$$

Example B

Add the following rational expressions and state the restrictions.

$\frac{3x+1}{x^2+8x+16} + \frac{2x-3}{x^2+x-12}$

Solution: Begin by determining the LCD. Factor the denominators of each expression.

$\frac{3x+1}{x^2+8x+16} + \frac{2x-3}{x^2+x-12} = \frac{3x+1}{(x+4)(x+4)} + \frac{2x-3}{(x+4)(x-3)}$

The LCD is

$$\boxed{(x+4)(x+4)(x-3)}$$

Multiply the numerators and denominators of each expression by the necessary factors to create the LCD.

$\frac{3x+1}{(x+4)(x+4)}\left(\frac{x-3}{x-3}\right)+\frac{2x-3}{(x+4)(x-3)}\left(\frac{x+4}{x+4}\right)$

Multiply the numerators. Keep the denominators in factored form.

$\frac{3x^2-8x-3}{(x+4)(x+4)(x-3)}+\frac{2x^2+5x-12}{(x+4)(x+4)(x-3)}$

Write the two expressions as one rational expression.

$\frac{3x^2-8x-3+2x^2+5x-12}{(x+4)(x+4)(x-3)}$

Simplify the numerator by combining like terms.

$\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)}$

The numerator cannot be factored so the expression cannot be further simplified. The answer in lowest terms is:

$$\boxed{\frac{5x^2-3x-15}{(x+4)(x+4)(x-3)};x\neq -4;x\neq 3}$$

Example C

Subtract the following rational expressions and state the restrictions.

$\frac{x}{x^2-9x+18}-\frac{x-2}{x^2-10x+24}$

Solution: Begin by determining the LCD. Factor the denominators of each expression.

$\frac{x}{(x-6)(x-3)}-\frac{x-2}{(x-6)(x-4)}$

The LCD is

$$\boxed{(x-6)(x-3)(x-4)}$$

Multiply the numerators and denominators of each expression to get the LCD.

$\frac{x}{(x-6)(x-3)}\left(\frac{x-4}{x-4}\right)-\frac{x-2}{(x-6)(x-4)}\left(\frac{x-3}{x-3}\right)$

Multiply the numerators.

$\frac{x^2-4x}{(x-6)(x-3)(x-4)}-\frac{x^2-5x+6}{(x-6)(x-3)(x-4)}$

Write the expressions as one rational expression.

$\frac{x^2-4x-(x^2-5x+6)}{(x-6)(x-3)(x-4)}$

$=\frac{x^2-4x-x^2+5x-6}{(x-6)(x-3)(x-4)}$

Simplify the numerator by combining like terms.

$\frac{x-6}{(x-6)(x-3)(x-4)}$

The term $(x-6)$ is common to both the numerator and the denominator. This term can be "cancelled." The solution is:

$$\boxed{\frac{1}{(x-3)(x-4)};x\neq 3;x\neq 4;x\neq 6}$$

Concept Problem Revisited

$\frac{3x}{x^2+6x-16}+\frac{2x}{x-2}$

Factor the denominator of the first fraction and rewrite the problem:

$\frac{3x}{(x+8)(x-2)}+\frac{2x}{x-2}$

The LCD is $(x+8)(x-2)$.

$\frac{3x}{(x+8)(x-2)}+\frac{2x}{x-2}\left(\frac{x+8}{x+8}\right)$

Multiply the numerators.

$\frac{3x}{(x+8)(x-2)}+\frac{2x^2+16x}{(x-2)(x+8)}$

Write the two expressions a one rational expression.

$\frac{3x+2x^2+16x}{(x+8)(x-2)}$

Simplify the numerator by combining like terms. Your final answer is:

$$\boxed{\frac{2x^2+19x}{(x+8)(x-2)};x\neq-8;x\neq 2}$$

Vocabulary

Rational Expression

A ***rational expression*** is an algebraic expression that can be written in the form $\frac{a(x)}{b(x)}$ where $b\neq 0$.

Restriction

Any value of the variable in a rational expression that would result in a zero denominator is called a ***restriction*** on the denominator.

Guided Practice

Add or subtract the following and state the restrictions.

1. $\frac{2x}{x^2-4}-\frac{1}{x-2}$
2. $\frac{-2}{3y^2+5y+2}+\frac{3}{y^2-7y-8}$
3. $\frac{3m-1}{9m^3-36m^2}+\frac{2m+1}{2m^2-5m-12}$

Answers:

1.

$$\begin{aligned}\frac{2x}{x^2-4}-\frac{1}{x-2}&=\frac{2x}{(x-2)(x+2)}-\frac{x+2}{(x-2)(x+2)}\\&=\frac{2x-(x+2)}{(x-2)(x+2)}\\&=\frac{(x-2)}{(x-2)(x+2)}\\&=\frac{1}{(x+2)};x\neq-2;x\neq 2\end{aligned}$$

2.

$$\begin{aligned}\frac{-2}{3y^2+5y+2}+\frac{3}{y^2-7y-8}&=\frac{-2}{(3y+2)(y+1)}+\frac{3}{(y-8)(y+1)}\\&=\frac{-2(y-8)}{(3y+2)(y+1)(y-8)}+\frac{3(3y+2)}{(3y+2)(y-8)(y+1)}\\&=\frac{-2y+16+9y+6}{(3y+2)(y-8)(y+1)}\\&=\frac{7y+22}{(3y+2)(y-8)(y+1)};y\neq-\frac{2}{3};y\neq 8;y\neq -1\end{aligned}$$

3.

$$\begin{aligned}\frac{3m-1}{9m^3-36m^2}+\frac{2m+1}{2m^2-5m-12}&=\frac{3m-1}{9m^2(m-4)}+\frac{2m+1}{(2m+3)(m-4)}\\&=\frac{(3m-1)(2m+3)}{9m^2(m-4)(2m+3)}+\frac{9m^2(2m+1)}{9m^2(2m+3)(m-4)}\\&=\frac{6m^2-2m+9m-3+18m^3+9m^2}{9m^2(m-4)(2m+3)}\\&=\frac{18m^3+15m^2+7m-3}{9m^2(m-4)(2m+3)};m\neq-\frac{3}{2};m\neq 4;m\neq 0\end{aligned}$$

Practice

For each of the following rational expressions, determine the LCD.

1. $\frac{2a-3}{4}+\frac{3a-1}{5}-\frac{a-5}{2}$
2. $\frac{5}{3x^2}-\frac{1}{2x}+\frac{3}{5x^3}$
3. $\frac{x}{a^2b}-\frac{y}{ab^2}+\frac{z}{3a^3b^2}$
4. $\frac{2w}{w^2-6w+5}-\frac{3w}{w^2-11w+30}$
5. $\frac{1}{y^2+5y}-\frac{2}{y^2+12y+35}-\frac{3}{y^3+7y^2}$

For each of the following rational expressions, state the restrictions.

6. $\frac{3}{x^2-5x+4}+\frac{4}{x^2-16}$
7. $\frac{5}{a^2+a}-\frac{2}{a^2+3a+2}$
8. $\frac{6}{m^2-5m}+\frac{7}{m^2-4m-5}$
9. $\frac{3n}{n^2+2n-3}-\frac{4n}{n^2+n-6}$
10. $\frac{6}{y^2-4}+\frac{4}{y^2+4y+4}$

Add or subtract each of the following rational expressions and state the restrictions.

11. $\frac{2a-3}{4}+\frac{3a-1}{5}-\frac{a-5}{2}$
12. $\frac{5}{3x^2}-\frac{1}{2x}+\frac{3}{5x^3}$
13. $\frac{x}{a^2b}-\frac{y}{ab^2}+\frac{z}{3a^3b^2}$
14. $\frac{2w}{w^2-6w+5}-\frac{3w}{w^2-11w+30}$
15. $\frac{1}{y^2+5y}-\frac{2}{y^2+12y+35}-\frac{3}{y^3+7y^2}$

8.4 Graphs of Rational Functions

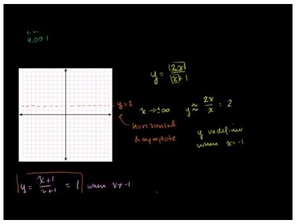
Here you'll investigate the graphs of rational functions using a graphing calculator.

Can you use your graphing calculator to help you sketch the graph of the rational function? Does this function have any zeros or asymptotes?

$y = \frac{4}{x^2+1}$

Watch This

Khan Academy Another Rational Function Graph Example



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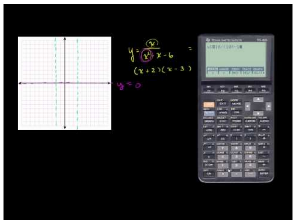
Khan Academy A Third Example of Graphing A Rational Function



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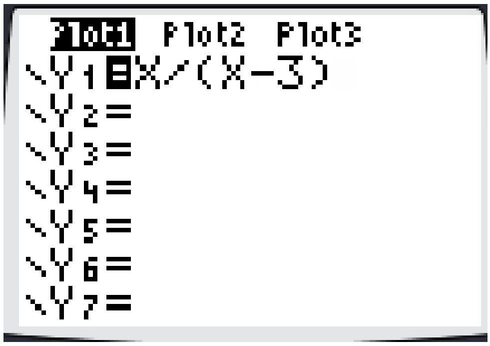
Guidance

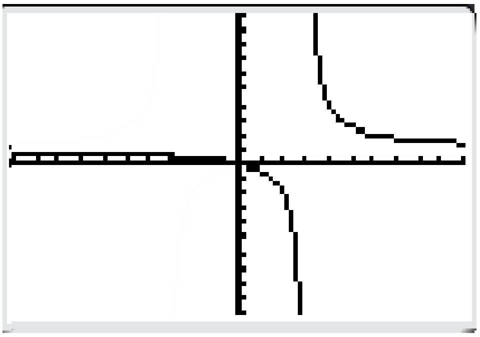
A rational function is the quotient of two polynomial functions. In general,

$f(x) = \frac{A(x)}{B(x)}$

where A and B are polynomials and $B \neq 0$. You can use your graphing calculator to graph a rational function and look for important features. Consider the function $y = \frac{x}{x-3}$.



```
Plot1 Plot2 Plot3
\Y1■X/(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



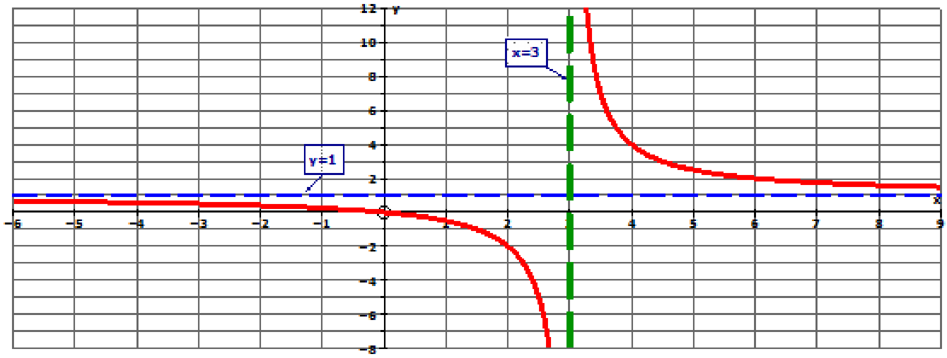
Notice that the function has two pieces. In between those two pieces are the asymptotes.

- A vertical asymptote occurs at the x value(s) that cause the denominator of the function to be equal to zero (which is undefined). This function has a vertical asymptote at $x = 3$.
- A horizontal asymptote occurs at the y value(s) that cause the denominator of the function to be zero if the function is rewritten and solved for x instead of y. This is what it looks like to solve for x:

$$\begin{aligned}
&y = \frac{x}{x-3}\\
&(x-3)(y) = (x-3)\left(\frac{x}{x-3}\right)\\
&xy - 3y = \cancel{(x-3)}\left(\frac{x}{\cancel{x-3}}\right)\\
&xy - 3y = x\\
&xy - x = 3y\\
&x(y-1) = 3y\\
&\boxed{x = \frac{3y}{y-1}}
\end{aligned}$$

Thus, $y \neq 1$ and this function has a horizontal asymptote at $y = 1$.

The image below shows the graph with the asymptotes drawn in and labelled. For rational functions, the asymptotes represent the lines that the function will approach but never touch.



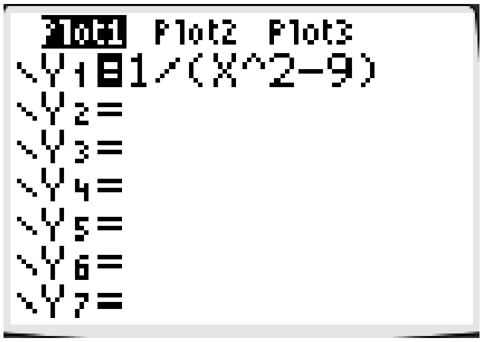
It is also important to note the x-intercepts (zeros) of the function. The zeros of the function will be the values for x that cause the numerator, but not also the denominator, to be equal to zero.

Example A

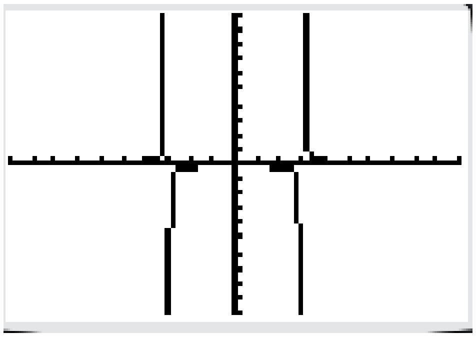
Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = \frac{1}{x^2-9} = \frac{1}{(x+3)(x-3)}$

Solution: Here is the sketch from the calculator:



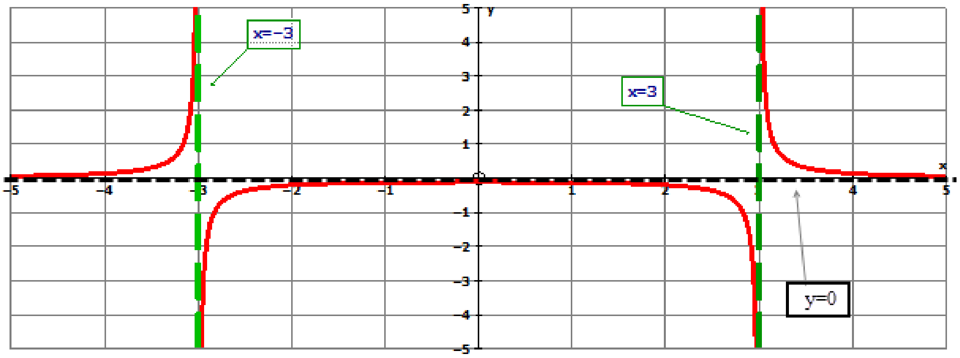
```
Plot1 Plot2 Plot3
\Y1=1/(X^2-9)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```



It can sometimes be hard to interpret what you see on the graphing calculator screen. Use algebra to find the asymptotes and zeros and sketch those first.

- There are no zeros for the numerator. Therefore, there are no x-intercepts for the function.
- The zeros of the denominator are 3 and –3. This means that there are two vertical asymptotes. One vertical asymptote is the line $x = 3$ and the other is the line $x = -3$.
- Another way to determine a horizontal asymptote besides solving the equation for x is to look at the degrees of the numerators and denominators. The degree is the highest exponent. The degree of the numerator is 0 and the degree of the denominator is 2. In general, if the degree of the numerator is less than the degree of the denominator, there will be a horizontal asymptote at $y = 0$.

Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.

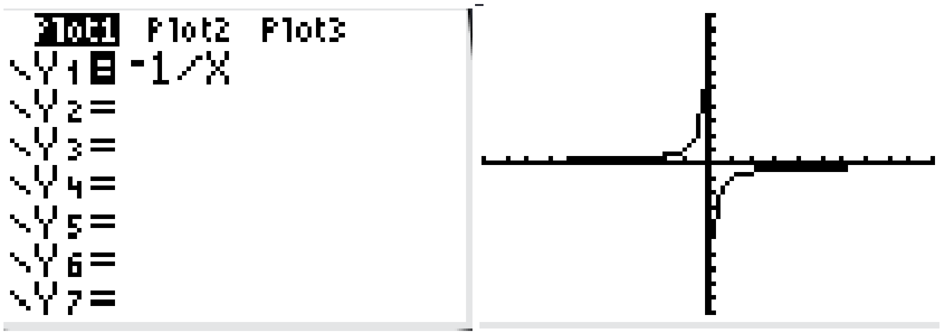


Example B

Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = -\frac{1}{x}$

Solution: Here is the sketch from the calculator:



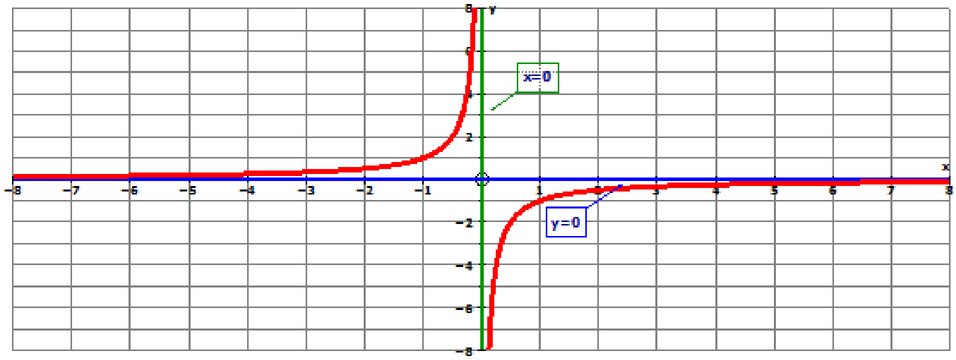
Plot1 Plot2 Plot3
\Y1 = -1/X
\Y2 =
\Y3 =
\Y4 =
\Y5 =
\Y6 =
\Y7 =

Use algebra to find the asymptotes and zeros and sketch those first.

- There are no zeros for the numerator. Therefore, there are no x-intercepts for the function.
- The zero of the denominator is 0. This means that there is one vertical asymptote, the line $x = 0$.
- The horizontal asymptote is $y = 0$. You can use algebra to solve the equation for x and look for the values of y that will cause the denominator to be equal to zero:

$$\begin{aligned} y &= -\frac{1}{x} \\ xy &= -1 \\ \frac{xy}{y} &= \frac{-1}{y} \\ &\boxed{x = -\frac{1}{y}} \end{aligned}$$

Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.

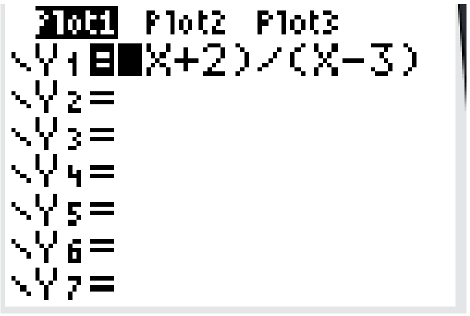


Example C

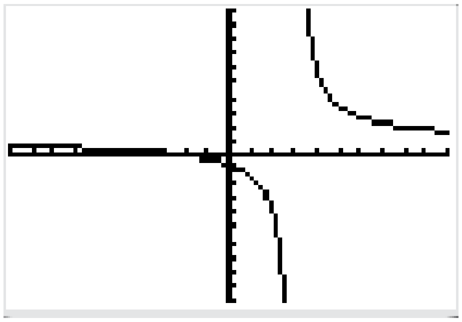
Use technology to sketch the graph of the rational function. Find all zeros and asymptotes and label those on your sketch.

$y = \frac{x+2}{x-3}$

Solution: Here is the sketch from the calculator:



```
Plot1 Plot2 Plot3
\Y1=█X+2)/(X-3)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

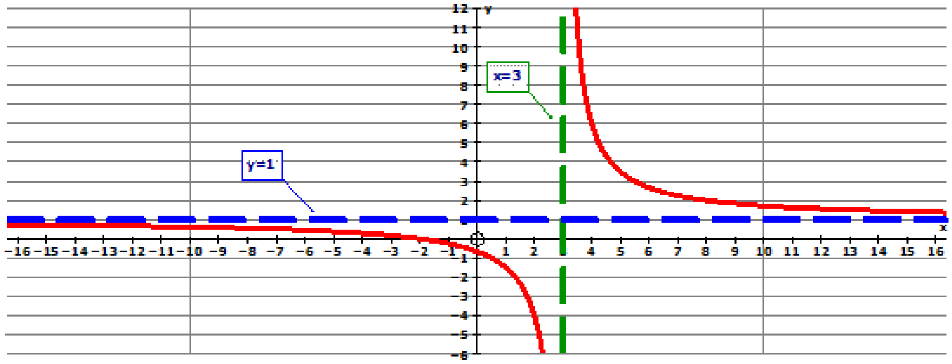


Use algebra to find the asymptotes and zeros and sketch those first.

- The zero of the numerator is –2 so the zero (x-intercept) of the function is (–2, 0).
- The zero of the denominator is 3. This means that there is one vertical asymptote, the line $x = 3$.
- The horizontal asymptote is $y = 1$. You can use algebra to solve the equation for x and look for the values of y that will cause the denominator to be equal to zero:

$$\begin{aligned}
&y = \frac{x+2}{x-3}\\
&(x-3)(y) = \left(\frac{x+2}{x-3}\right)(x-3)\\
&xy - 3y = x + 2\\
&xy - x = 2 + 3y\\
&x(y-1) = 2 + 3y\\
&\boxed{x = \frac{3y+2}{y-1}}
\end{aligned}$$

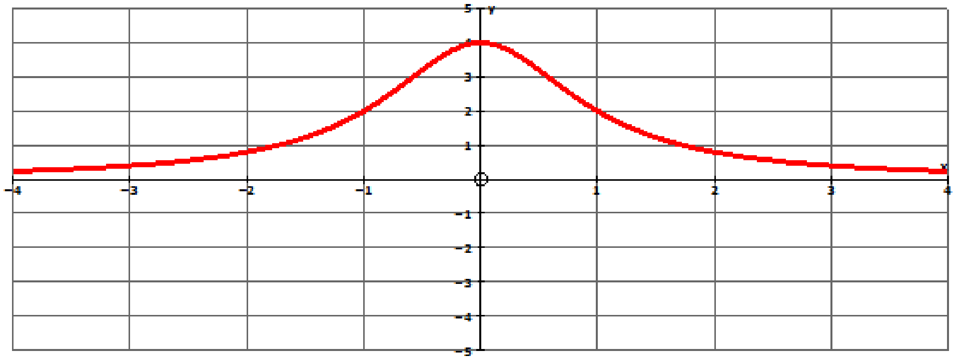
Once you have sketched the asymptotes, use the table and/or graph from the graphing calculator to decide what the rest of the graph looks like. Here is the graph with the asymptotes labelled.



Concept Problem Revisited

$y = \frac{4}{x^2+1}$

Here is a sketch of the function:



- There are no zeros for this function since there are no zeros for the numerator. The graph does not cross the x-axis.
- There are no zeros for the denominator. Therefore, there are no vertical asymptotes.
- Because the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y = 0$.

Vocabulary

Rational Function

A ***rational function*** is the quotient of two polynomial functions. In general,

$f(x) = \frac{A(x)}{B(x)}$

where A and B are polynomials and $B \neq 0$.

Guided Practice

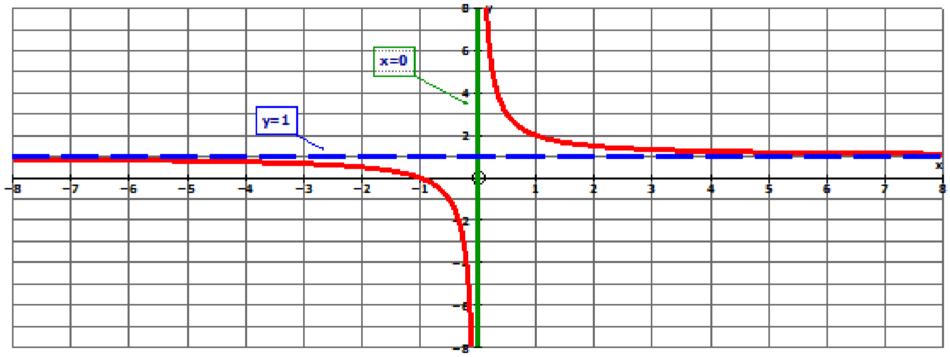
1. Sketch the graph of the rational function: $y = \frac{x+1}{x}$

2. Sketch the graph of the rational function: $y = \frac{x^2}{x-3}$

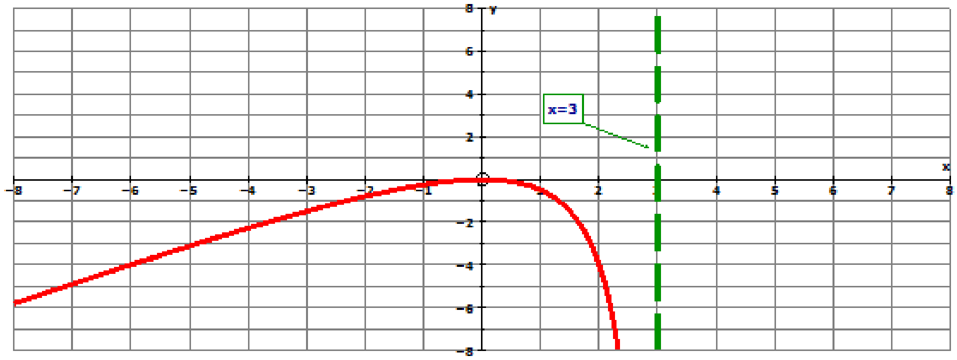
3. Find the asymptotes of the function: $y = \frac{1}{x^2-16}$

Answers:

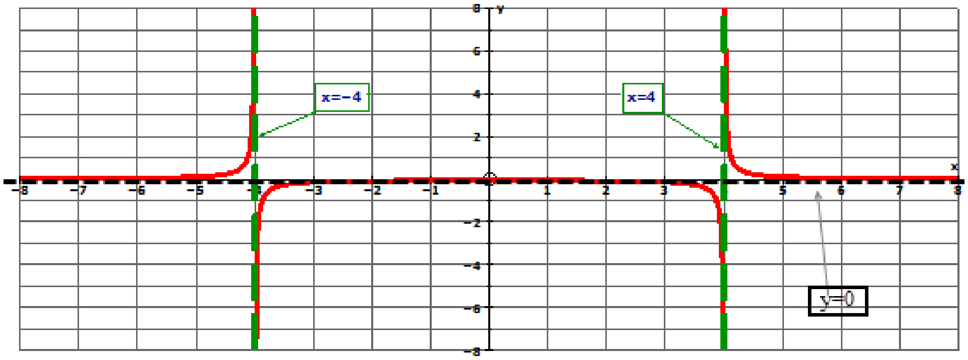
1. The zero of the denominator is 0. This means that there will be a vertical asymptote at $x = 0$ (the y-axis). Because the degree of the denominator and numerator are the same, you can solve the equation for x and get $x = \frac{1}{y-1}$. The zero of the denominator is now 1. This means there is a horizontal asymptote at $y = 1$. The numerator has a zero at -1, so there is an x-intercept at -1.



2. The zero of the denominator is 3. This means that there will be a vertical asymptote at $x = 3$. Because the degree of the numerator is greater than the degree of the denominator, there are no horizontal asymptotes. The zero of the numerator is 0, so there is an x-intercept at 0.



3. The zeros of the denominator are 4 and –4. This means that there will be two vertical asymptotes. One vertical asymptote will be the line $x = 4$ and the other will be the line $x = -4$. Because the degree of the numerator is less than the degree of the denominator, there is a horizontal asymptote at $y = 0$. Though you did not need to sketch the graph, here is the graph of the function:



Practice

Sketch the graph of each of the following rational functions.

1. $y = \frac{2}{x+3}$
2. $y = \frac{x}{x-1}$
3. $y = \frac{1}{x^2-4}$
4. $y = \frac{x+2}{x}$
5. $y = \frac{1}{x^2+2}$
6. $y = \frac{x}{x+2}$
7. $y = \frac{1}{x^2-x-12}$
8. $y = \frac{x-1}{x+3}$
9. $y = \frac{x-1}{x+4}$
10. $y = \frac{5}{x^2+1}$

Without graphing the following rational functions, state what you know about their asymptotes and zeros.

11. $y = \frac{1}{x^2-x-2}$

12. $y = -\frac{2}{x-4}$
13. $y = -\frac{2}{x^2+1}$
14. $y = \frac{6}{x^2+1}$
15. $y = \frac{x-1}{x+3}$

Summary

You learned that operations with rational expressions rely on factoring and operations with fractions. To multiply rational expressions, multiply across and simplify. To divide rational expressions, change the problem to a multiplication problem by multiplying the first fraction by the reciprocal of the second fraction. To add or subtract, find the lowest common denominator in order to combine the expressions.

Rational functions can be graphed on the graphing calculator as an aid for making a sketch. You can algebraically find both the vertical and horizontal asymptotes of a rational function. To find the vertical asymptotes, consider the values of x that cause the denominator to be equal to zero and thus the function to be undefined. One method for finding horizontal asymptotes is to solve the equation of the function for x, and then look for the values of y that cause the denominator to be equal to zero. In the case where the degree of the numerator is less than the degree of the denominator, the horizontal asymptote will automatically be at $y = 0$. Rational functions will be explored in further detail in future courses like Algebra II and PreCalculus.

CHAPTER 9 Quadratic Equations and Quadratic Functions

Chapter Outline

Introduction

Here you'll learn more about quadratic equations and quadratic functions. You will learn three new methods for solving quadratic equations and discover the connections between a quadratic equation and its corresponding quadratic function. You will discover a new set of numbers called complex numbers and see how complex numbers are related to quadratic functions with no x-intercepts. Finally, you will learn how to solve radical equations.

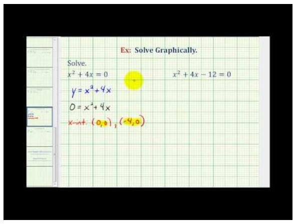
9.1 Graphs to Solve Quadratic Equations

Here you will learn how to solve a quadratic equation by graphing.

One way to solve the equation $x^2-2x-3=0$ is to use factoring and the zero product property. How could you use a graph to solve $x^2-2x-3=0$?

Watch This

James Sousa: Solve a Quadratic Equation Graphically on the Calculator



Ex: Solve Graphically.

Solve.

$x^2+4x=0$

$y=x^2+4x$

$0=x^2+4x$

x-int: $(0,0)$, $(-4,0)$

$x^2+4x-12=0$

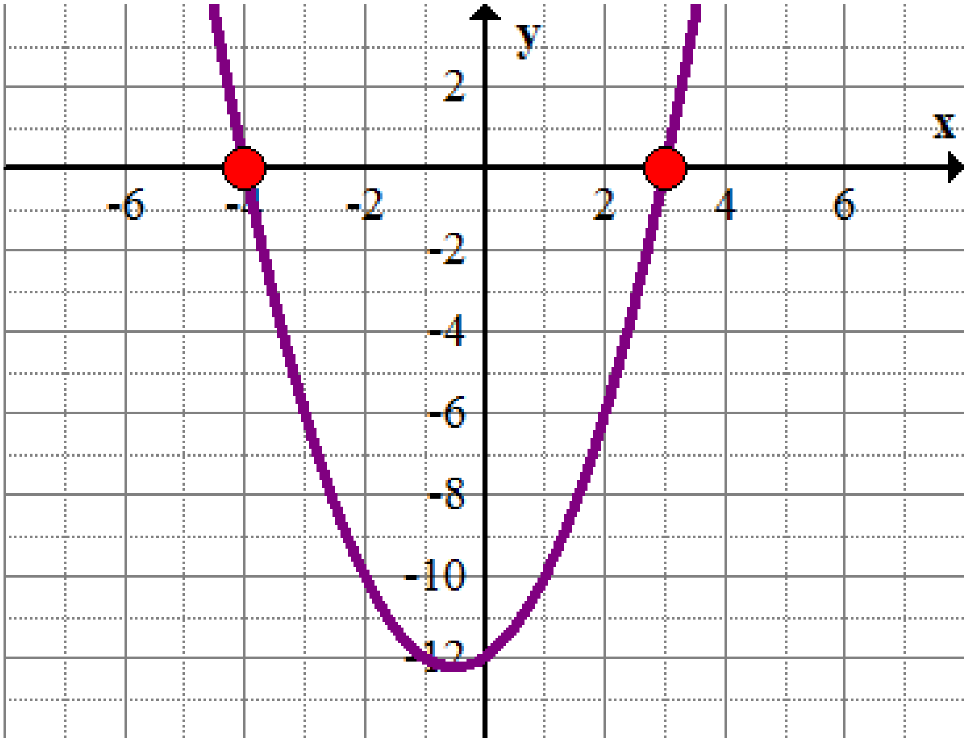
MEDIA

Click image to the left for more content.

Guidance

Recall that a quadratic equation is a degree 2 equation that can be written in the form $ax^2+bx+c=0$. Every quadratic equation has a corresponding quadratic function that you get by changing the "0" to a "y". Standard form for a quadratic function is $y=ax^2+bx+c$. Quadratic functions can be graphed by hand, or with a graphing calculator.

How do the solutions to the equation $x^2+x-12=0$ show up on the graph of $y=x^2+x-12$? On the graph you are looking for the points that have a y-coordinate that is equal to 0. Therefore, the solutions to the equation will show up as the x-intercepts on the graph of the function. These are also known as the ***roots*** or ***zeros*** of the function. Here is the graph of $y=x^2+x-12$:



You can see the x-intercepts are at $(-4,0)$ and $(3,0)$. This means that the solutions to the equation $x^2+x-12=0$ are $x=-4$ and $x=3$. You can verify these solutions by substituting them back into the equation:

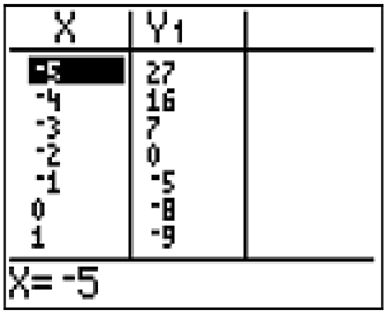
- $(-4)^2+(-4)-12=16-4-12=0$
- $(3)^2+(3)-12=9+3-12=0$

Graphing is a great way to solve quadratic equations. Keep in mind that you can also solve many quadratic equations by factoring or using other algebraic methods such as the quadratic formula or completing the square.

Example A

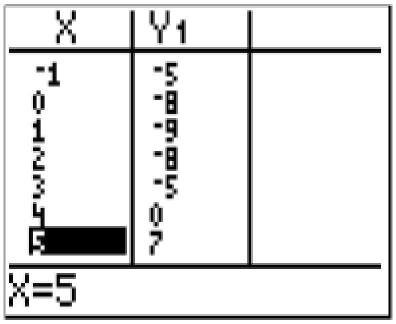
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2-2x-8=0$

Solution: The corresponding function is $y=x^2-2x-8$. Use your graphing calculator to make a table and a graph for this function.



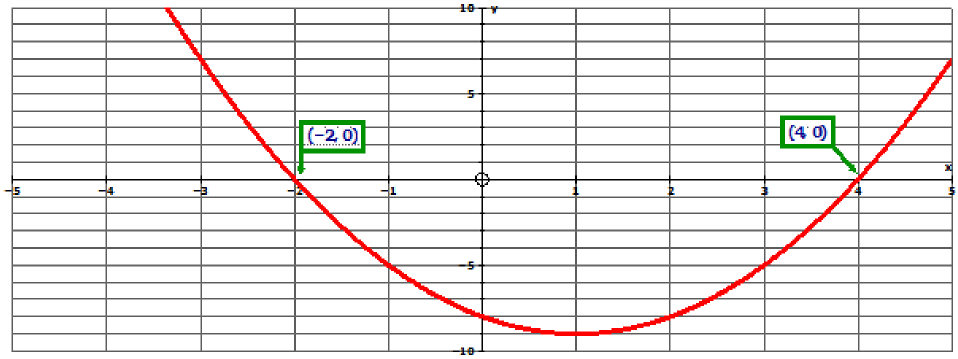
X	Y1	
-5	27	
-4	16	
-3	7	
-2	0	
-1	-5	
0	-8	
1	-9	

X= -5



X	Y_1	
-1	-5	
0	-8	
1	-9	
2	-8	
3	-5	
4	0	
5	7	

X=5

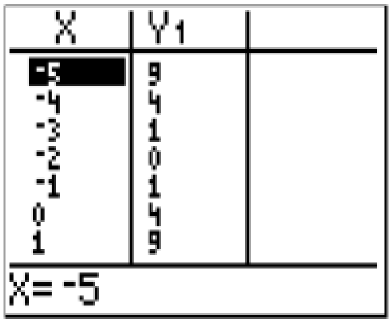


The *x*-intercepts are (–2, 0) and (4, 0). The *x*-intercepts are the values for '*x*' that result in $y = 0$ and are therefore the solutions to your equation. The solutions for the quadratic are $x = -2$ and $x = 4$.

Example B

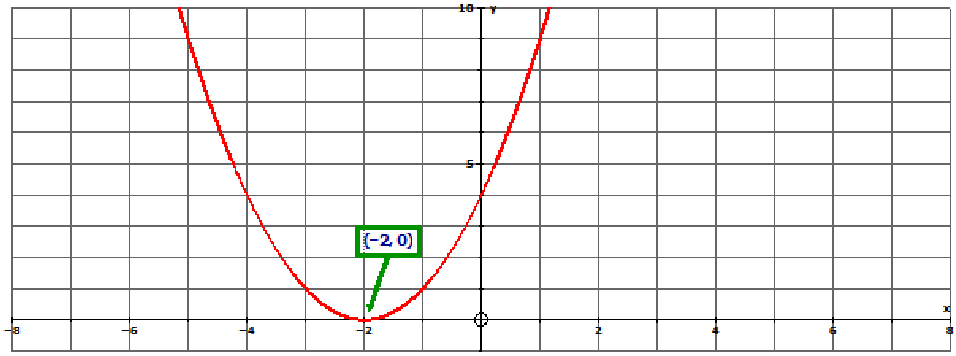
Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2 + 4x + 4 = 0$

Solution: The corresponding function is $y = x^2 + 4x + 4$. Use your graphing calculator to make a table and a graph for this function.



X	Y_1	
-5	9	
-4	4	
-3	1	
-2	0	
-1	1	
0	4	
1	9	

X= -5

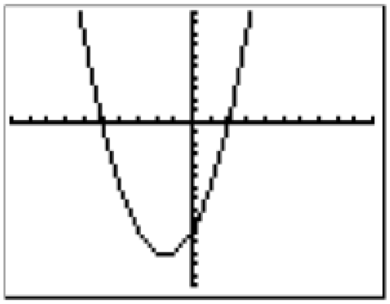


The only x-intercept is (–2, 0). There is only one solution to the equation: $x = -2$. Keep in mind that quadratic equations can have 0, 1, or 2 real solutions. If you were to factor the quadratic x^2+4x+4, you would get $(x+2)(x+2)$–two of the same factors. The root of -2 for this function is said to have a ***multiplicity*** of 2, because 2 factors produce the same solution. You will learn more about ***multiplicity*** when you study polynomials in future courses.

Example C

Solve the following quadratic equation by finding the x-intercepts of the corresponding quadratic function: $x^2+3x=10$

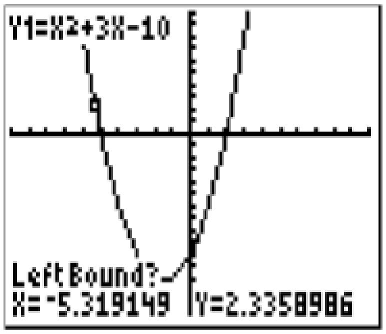
Solution: First rewrite the equation so it is set equal to zero to get $x^2+3x-10=0$. Now, the corresponding function is $y=x^2+3x-10$. Use your graphing calculator to make a graph for this function. You will see that there are two x-intercepts.



For this example you will see how the calculator can calculate the zeros of a function on a graph. *This technique is particularly useful when the intercepts are not at whole numbers.* Have the calculator find the x-intercept on the left first. Press

2nd TRACE 2

The calculator will display “Left Bound?” Use the arrow to position the cursor so that it is to the left and above the x-axis.

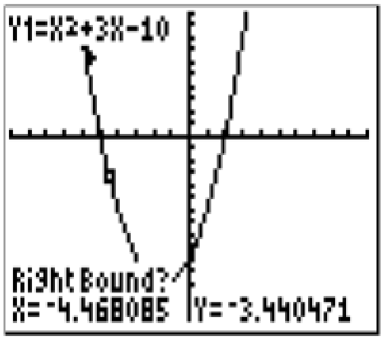


```
Y1=X²+3X-10
Left Bound?
X=⁻5.319149  Y=2.3358986
```

When the cursor has been positioned, press

ENTER

The calculator will now display “Right Bound?” Use the arrows to position the cursor so that it is to the right and below the *x*-axis.

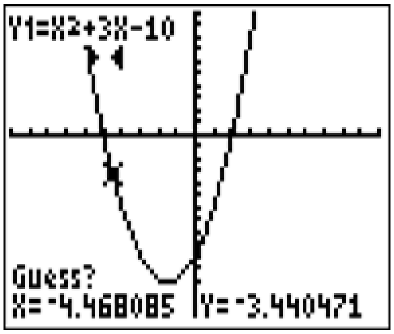


```
Y1=X²+3X-10
Right Bound?
X=⁻4.468085  Y=⁻3.440471
```

When the cursor has been positioned, press

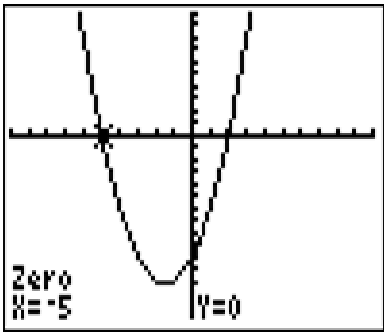
ENTER

The calculator will now display “Guess?”



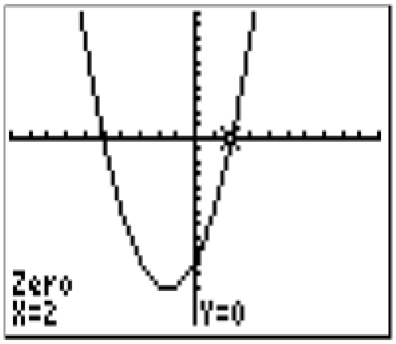
```
Y1=X²+3X-10
Guess?
X=⁻4.468085  Y=⁻3.440471
```

Press



At the bottom of the screen you can see it says "Zero" and the x and y coordinates. You are interested in the x-coordinate because that is one of the solutions to the original equation. The x-intercept is (–5, 0) which means that one of the solutions is $x = -5$.

Repeat this same process to determine the value of the x-intercept on the right.



The x-intercept is (2, 0) which means that the second solution is $x = 2$.

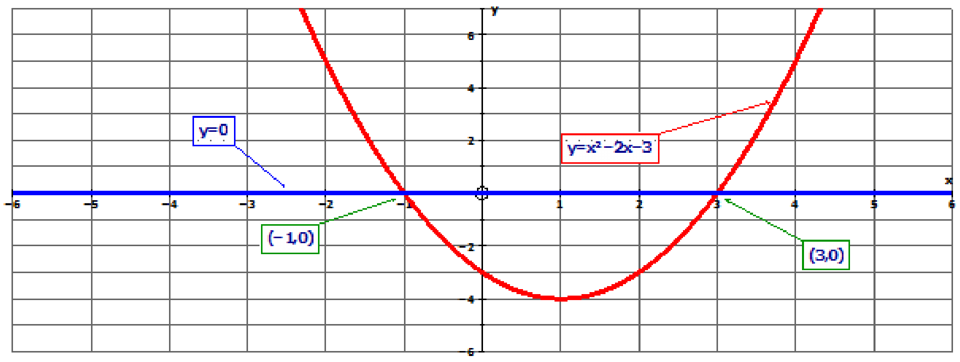
Concept Problem Revisited

To solve the equation $x^2 - 2x - 3 = 0$ using a graph, use a calculator to graph the corresponding function $y = x^2 - 2x - 3$. Then, look for the values on the graph where $y = 0$, which will be the x-intercepts.

Another way to think about this problem is to solve the system:

$$\left\{\begin{matrix} y = x^2 - 2x - 3 \\ y = 0 \end{matrix}\right\}$$

You are looking for where the parabola $y = x^2 - 2x - 3$ intersects with the line $y = 0$.



The points of intersection are (–1, 0) and (3, 0). The solutions to the original equation are $x=-1$ and $x=3$.

Vocabulary

Quadratic Equation

A ***quadratic equation*** is an equation of degree 2. The standard form of a quadratic equation is $ax^2+bx+c=0$ where $a \neq 0$.

Quadratic Function

A ***quadratic function*** is a function that can be written in the form $f(x)=ax^2+bx+c$ with $a \neq 0$. The graph of a quadratic function is a ***parabola***.

Zeros of a Quadratic Function

The ***zeros of a quadratic function*** are the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Roots of a Quadratic Function

The ***roots of a quadratic function*** are also the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Guided Practice

Solve each quadratic equation using a graph.

1.

$$x^2-3x-10=0$$

2.

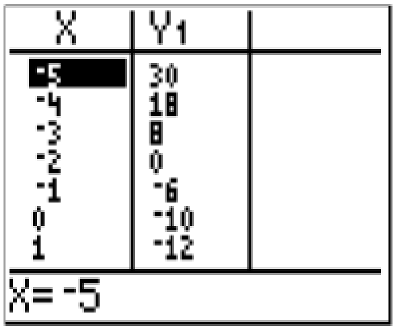
$$2x^2-5x+2=0$$

3.

$$2x^2-5x=3$$

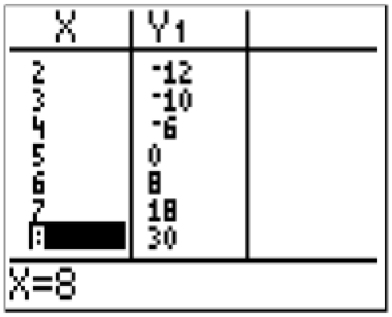
Answers:

1. To begin, create a table of values for the corresponding function $y=x^2-3x-10$ by using your graphing calculator:



X	Y1
-5	30
-4	18
-3	8
-2	0
-1	-6
0	-10
1	-12

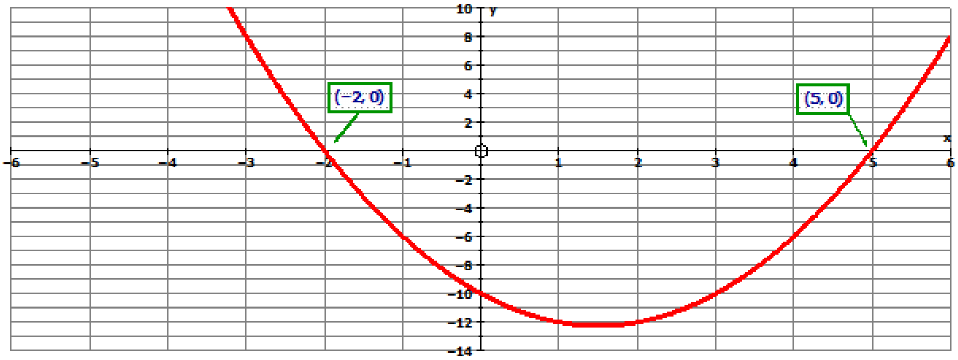
X= -5



X	Y1
2	-12
3	-10
4	-6
5	0
6	8
7	18
8	30

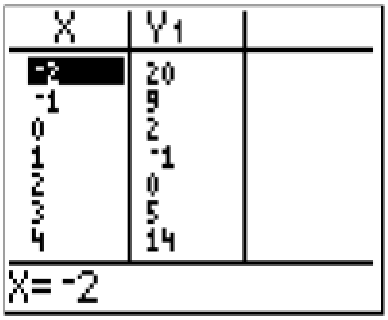
X=8

From the table, the *x*-intercepts are (–2, 0) and (5, 0). The *x*-intercepts are the values for '*x*' that result in $y = 0$ and are therefore the solutions to the equation.



The solutions to the equation are are $x = -2$ and $x = 5$.

2. To begin, create a table of values for the corresponding function $y = 2x^2 - 5x + 2$ by using your graphing calculator:

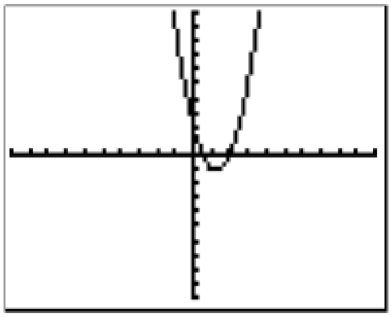


X	Y1
-2	20
-1	9
0	2
1	-1
2	0
3	5
4	14

X= -2

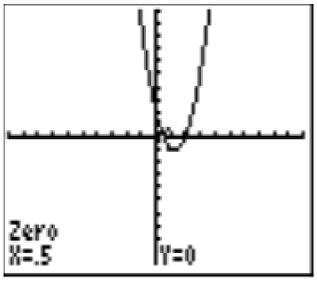
Press

GRAPH



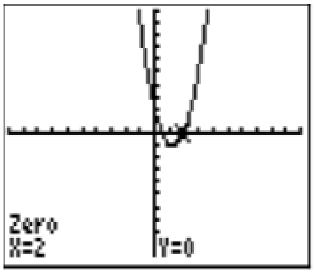
Press the following keys to determine the x-intercept to the left:

2nd TRACE 2 ❯ ENTER ❯ ❯ ❯ ENTER ENTER



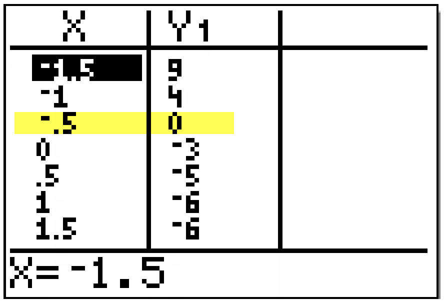
Press the following keys to determine the x-intercept to the right:

2nd TRACE 2 ❯ ENTER ❯ ❯ ❯ ENTER ENTER



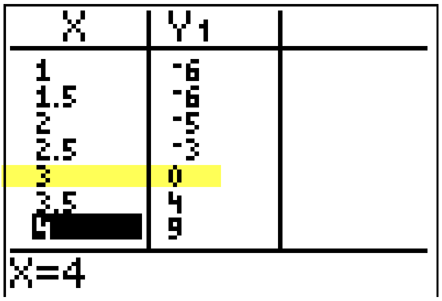
The x-intercepts of the function are (0.5, 0) and (2, 0). The solutions to the equation are, therefore, $x = 0.5$ and $x = 2$.

3. First rewrite the equation so it is set equal to zero: $2x^2 - 5x - 3 = 0$. Next, create a table of values for the corresponding function $y = 2x^2 - 5x - 3$ by using your graphing calculator:



X	Y1	
-1.5	9	
-1	4	
-.5	0	
0	-3	
.5	-5	
1	-6	
1.5	-6	

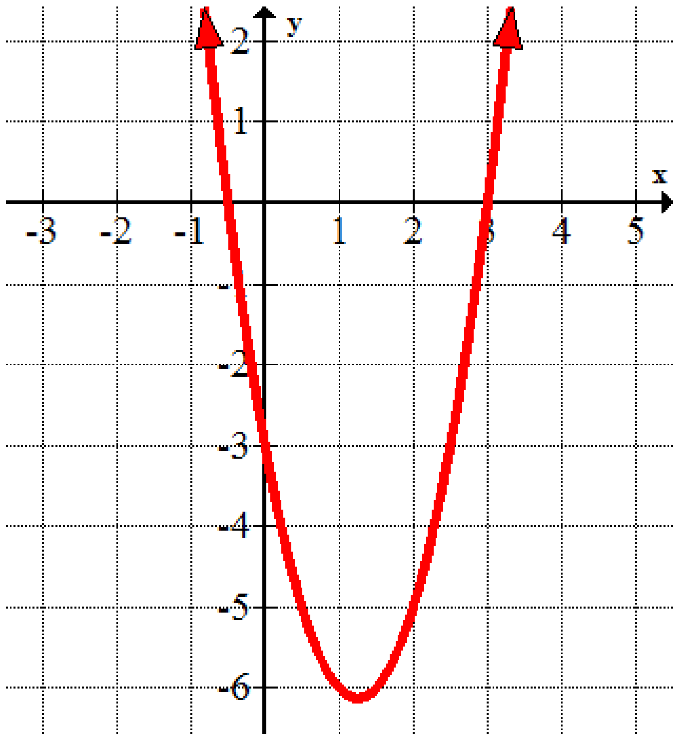
X= -1.5



X	Y1	
1	-6	
1.5	-6	
2	-5	
2.5	-3	
3	0	
3.5	4	
4	9	

X=4

Now sketch the graph of the function.



The zeros of the function are (–0.5, 0) and (3, 0). Therefore, the solutions to the equation are $x = -0.5$ and $x = 3$.

Practice

Use your graphing calculator to solve each of the following quadratic equations by graphing:

1. $2x^2 + 9x - 18 = 0$
2. $3x^2 + 8x - 3 = 0$
3. $-5x^2 + 13x + 6 = 0$
4. $2x^2 - 11x + 5 = 0$
5. $3x^2 + 8x - 3 = 0$
6. $x^2 - x - 20 = 0$
7. $2x^2 - 7x + 5 = 0$
8. $3x^2 + 7x = -2$
9. $2x^2 - 15 = -x$
10. $3x^2 - 10x = 8$
11. How could you use the graphs of a system of equations to solve $3x^2 - 10x = 8$?
12. What's the difference between a quadratic equation and a quadratic function?
13. Will a quadratic equation always have 2 solutions? Explain.
14. The quadratic equation$x^2 + 4 = 0$ has no real solutions. How does the graph of $y = x^2 + 4$ verify this fact?
15. When does it make sense to use the graphing method for solving a quadratic equation?

9.2 Completing the Square

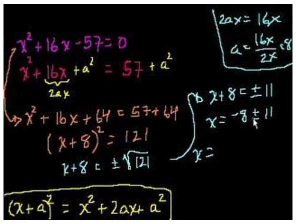
Here you will learn how to solve a quadratic equation by completing the square.

How can you use square roots to solve the quadratic equation $(x-3)^2 = 15$?

The equation $x^2 - 6x - 6 = 0$ is equivalent to $(x-3)^2 = 15$. How can you algebraically change $x^2 - 6x - 6 = 0$ into $(x-3)^2 = 15$ in order to solve it?

Watch This

Khan Academy Completing the Square



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Guidance

Recall that some quadratics are known as perfect square trinomials because they can be represented as a binomial squared. For example:

- $x^2 + 10x + 25$ can be written as $(x+5)^2$
- $x^2 + 6x + 9$ can be written as $(x+3)^2$
- $x^2 + 2x + 1$ can be written as $(x+1)^2$

You can solve a quadratic such as $(x+5)^2 = 0$ by taking the square root of both sides as shown:

$$\begin{aligned}
(x+5)^2 &= 0\\
\sqrt{(x+5)^2} &= \pm\sqrt{0}\\
x+5 = +0 \text{ OR } x+5 &= -0\\
x &= -5
\end{aligned}$$

You can also solve such a quadratic even if it is not set equal to zero. For example, you can solve $(x+5)^2 = 9$ in a similar way:

$$\begin{aligned}
(x+5)^2 &= 9\\
\sqrt{(x+5)^2} &= \pm\sqrt{9}\\
x+5 = +3 \text{ OR } x+5 &= -3\\
x = -2 \quad x &= -8
\end{aligned}$$

Completing the square is a technique for solving quadratic equations that turns a given equation into a perfect square trinomial set equal to a number so that it can be solved using the method above. Consider the equation:

$x^2-12x+20=0$

Step 1: Move 20 to the right side of the equation.

$$x^2-12x=-20$$

Step 2: *Complete the square* on the left side of the equation. This means, figure out what could be added to the left side to turn that equation into a perfect square trinomial. One method for figuring this out is to take the value of '*b*' (which is –12), divide it by 2, and square the result:

$$\begin{aligned}&b=-12\\&\frac{b}{2}=\frac{-12}{2}=-6\\&\left(\frac{b}{2}\right)^2=(-6)^2=36\\&x^2-12x+36=-20+36\end{aligned}$$

Remember that what is added to one side of the equation must be added to the other side of the equation. Once you simplify you get:

$x^2-12x+36=16$

Step 3: Rewrite the left side of the equation as a binomial squared.

$$(x-6)^2=16$$

Step 4: Take the square root of both sides of the equation.

$$\begin{aligned}\sqrt{(x-6)^2}&=\sqrt{16}\\x-6&=\pm4\end{aligned}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x-6=4$ and $x-6=-4$

$x-6+6=4+6$ and $x-6+6=-4+6$

$x=10$ or $x=2$

The solutions to the equation are:

$$\boxed{x=10\ or\ x=2}$$

Example A

What would you need to add to the following expression to turn it into a perfect square trinomial?

x^2+16x

Solution: To determine the value to add, you can divide 16 by 2 and square the result.

$$\begin{aligned}&b=16\\&\frac{b}{2}=\frac{16}{2}=8\\&\left(\frac{b}{2}\right)^2=(8)^2=64\end{aligned}$$

The expression is now $x^2+16x+64$, which can be rewritten as $(x+8)^2$.

Example B

Solve the following quadratic equation by completing the square: $x^2+2x-35=0$

Solution:

Step 1: Move –35 to the right side of the equation.

$$x^2+2x=35$$

Step 2: Complete the square on the left side of the equation. The value of b is 2.

$$\begin{aligned}&b=2\\&\frac{b}{2}=\frac{2}{2}=1\\&\left(\frac{b}{2}\right)^2=(1)^2=1\\&x^2+2x+1=35+1\end{aligned}$$

$x^2+2x+1=36$

Step 3: Rewrite the left side of the equation as a binomial squared.

$$(x+1)^2=36$$

Step 4: Take the square root of both sides of the equation.

$$\begin{aligned}\sqrt{(x+1)^2}&=\sqrt{36}\\x+1&=\pm 6\end{aligned}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x+1=6$ and $x+1=-6$

$x+1-1=6-1$ and $x+1-1=-6-1$

$x=5$ or $x=-7$

The solutions to the equation are:

$$\boxed{x=5\ or\ x=-7}$$

Example C

Solve the following quadratic equation by completing the square:

$$\boxed{2x^2-5x+2=0}$$

Solution: In this quadratic equation, the value of 'a' is not one. The first step will be to factor out the value of a.

Step 1: Factor out the '2'. Then, because the equation is set equal to zero, divide both sides by 2 in order to have a simpler equation.

$$\begin{aligned}2(x^2-\frac{5}{2}x+1)&=0\\x^2-\frac{5}{2}x+1&=0\end{aligned}$$

Step 2: Move 1 to the right side of the equation.

$$x^2-\frac{5}{2}x=-1$$

Step 3: Complete the square on the left side of the equation.

$$\begin{aligned}b&=-\frac{5}{2}\\\frac{b}{2}&=-\frac{5}{2}\div 2=-\frac{5}{2}\left(\frac{1}{2}\right)=-\frac{5}{4}\\\left(\frac{b}{2}\right)^2&=\left(-\frac{5}{4}\right)^2=\frac{25}{16}\\x^2-\frac{5}{2}x+\frac{25}{16}&=-1+\frac{25}{16}\end{aligned}$$

$x^2-\frac{5}{2}x+\frac{25}{16}=\frac{9}{16}$

Step 4: Rewrite the perfect square trinomial on the left side of the equation as a binomial squared.

$$\left(x-\frac{5}{4}\right)^2=\frac{9}{16}$$

Step 4: Take the square root of both sides of the equation.

$$\sqrt{\left(x-\frac{5}{4}\right)^2} = \sqrt{\frac{9}{16}}$$
$$x-\frac{5}{4} = \pm\frac{3}{4}$$

Step 5: Set the left side of the equation equal to each of the roots on the right side and solve each linear equation.

$x-\frac{5}{4}=\frac{3}{4}$ and $x-\frac{5}{4}=-\frac{3}{4}$

$x-\frac{5}{4}+\frac{5}{4}=\frac{3}{4}+\frac{5}{4}$ and $x-\frac{5}{4}+\frac{5}{4}=-\frac{3}{4}+\frac{5}{4}$

$x=\frac{8}{4}$ or $x=\frac{2}{4}$

$x=2$ and $x=\frac{1}{2}$

The solutions to the equation are:

$$\boxed{x=2 \; or \; x=\frac{1}{2}}$$

Concept Problem Revisited

To solve $(x-3)^2=15$, take the square root of both sides. You will get that $x-3=\sqrt{15}$ or $x-3=-\sqrt{15}$. The two solutions are therefore $x=3+\sqrt{15}, 3-\sqrt{15}$.

You can turn $x^2-6x-6=0$ into $(x-3)^2=15$ by completing the square.

Vocabulary

Completing the Square

Completing the square is a method used for solving quadratic equations. This method uses the principle of ***completing the square*** of an algebraic expression to make it a perfect square trinomial.

Perfect Square Trinomial

A ***perfect square trinomial*** is one of the form $(ax)^2+2abx+b^2$ where the first and last terms are perfect squares and the middle term is twice the product of the square root of the first term and the square root of the third term.

Roots of a Quadratic Function

The ***roots of a quadratic function*** are the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Zeros of a Quadratic Function

The ***zeros of a quadratic function*** are also the x-intercepts of the function. These are the values for the variable 'x' that will result in $y=0$.

Guided Practice

1. What would you need to add to the following expression to turn it into a perfect square trinomial?

$$x^2 - 7x$$

2. Solve the following quadratic equation by completing the square.

$$x^2 - 4x + 1 = 0$$

3. Solve the following quadratic equation by completing the square.

$$7x^2 - 2x - 2 = 0$$

Answers:

1. Divide b by $\frac{1}{2}$ and square the result.
$(\frac{-7}{2})^2 = \frac{49}{4}$
2. $x^2 - 4x + 1 = 0$

$$\begin{aligned} x^2 - 4x &= -1 \\ x^2 - 4x {\color{red}+4} &= -1 {\color{red}+4} \\ x^2 - 4x + 4 &= 3 \\ (x-2)^2 &= 3 \\ \sqrt{(x-2)^2} &= \sqrt{3} \\ x - 2 &= \pm\sqrt{3} \end{aligned}$$

The solutions to the equation are:

$$\boxed{x = 2 + \sqrt{3} \; or \; x = 2 - \sqrt{3}}$$

$$\boxed{x = 2 \pm \sqrt{3}}$$

3. $7x^2 - 2x - 2 = 0$

$$
\begin{aligned}
7(x^2 - \frac{2}{7}x - \frac{2}{7}) &= 0 \\
x^2 - \frac{2}{7}x - \frac{2}{7} &= 0 \\
x^2 - \frac{2}{7}x &= \frac{2}{7} \\
x^2 - \frac{2}{7}x + \frac{1}{49} &= \frac{2}{7} + \frac{1}{49} \\
\left(x - \frac{1}{7}\right)^2 &= \frac{15}{49}
\end{aligned}
$$

The exact solutions to the equation are:

$$\boxed{x = \frac{\sqrt{15}+1}{7} \ or\ x = \frac{-\sqrt{15}+1}{7}}$$

The approximate solutions (which have been rounded) are:

$$\boxed{x = 0.70\ or\ x = -0.41}$$

Practice

State the value of m that makes each trinomial a perfect square:

1. $x^2 - 10x + m$
2. $x^2 - 22x + m$
3. $x^2 + \frac{1}{2}x + m$
4. $x^2 + 9x + m$
5. $x^2 + x + m$

Solve each of the following quadratic equations by completing the square:

6. $x^2 + 18x = 85$
7. $x^2 - \frac{2}{3}x = 1$
8. $x^2 - 7x = 3$
9. $x^2 + \frac{1}{5}x = 2$
10. $x^2 - \frac{2}{3}x = 5$

Solve each of the following quadratic equations by completing the square:

11. $x^2 - 2x - 8 = 0$
12. $2x^2 + 8x + 5 = 0$
13. $3x^2 - 6x - 2 = 0$
14. $2x^2 - 3x - 5 = 0$
15. $3x^2 + 4x - 2 = 0$

9.3 The Quadratic Formula

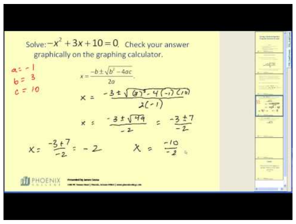
Here you will learn the quadratic formula and how to use it.

Solve the following quadratic equation algebraically:

$$3x^2 - 5x + 1 = 0$$

Watch This

James Sousa: Using the Quadratic Formula



Solve: $-x^2+3x+10=0$. Check your answer graphically on the graphing calculator.

$a=-1$
$b=3$
$c=10$

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$x=\frac{-3\pm\sqrt{(3)^2-4(-1)(10)}}{2(-1)}$

$x=\frac{-3\pm\sqrt{49}}{-2}=\frac{-3\pm7}{-2}$

$x=\frac{-3+7}{-2}=-2$ $x=\frac{-10}{-2}$

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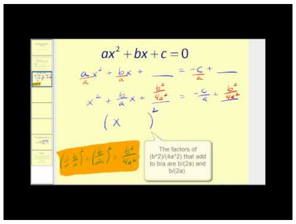
James Sousa: Proof of the Quadratic Formula



$ax^2+bx+c=0$

The factors of (b^2)/(4a^2) that add to b/a are b/(2a) and b/(2a).

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Guidance

You can use the method of completing the square to solve the general quadratic equation $ax^2 + bx + c = 0$. The result will be a formula that you can use to solve any quadratic equation given the values for a, b, and c. The following is a derivation of the quadratic formula:

Step 1: Divide the general equation by a. Then, move the third term on the left side to the right side of the equation.

$$ax^2 + bx + c = 0$$
$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 2: Complete the square. Note that your "b" value in this case is actually $\frac{b}{a}$.

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

Step 3: Simplify.

$$\begin{aligned}
x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2}{4a^2}-\frac{c}{a}\left(\frac{4a}{4a}\right)\\
x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}\\
x^2+\frac{b}{a}x+\frac{b^2}{4a^2}&=\frac{b^2-4ac}{4a^2}
\end{aligned}$$

Step 4: Rewrite the left side of the equation as a binominal squared. Then, take the square root of both sides and solve for x.

$$\begin{aligned}
\left(x+\frac{b}{2a}\right)^2&=\frac{b^2-4ac}{4a^2}\\
\sqrt{\left(x+\frac{b}{2a}\right)^2}&=\sqrt{\frac{b^2-4ac}{4a^2}}\\
x+\frac{b}{2a}&=\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x&=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}\\
x&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

This is known as the quadratic formula. You can use the quadratic formula to solve ANY quadratic equation. All you need to know are the values of a, b, and c. Keep in mind that while the factoring method for solving a quadratic equation will only sometimes work, the quadratic formula will ALWAYS work. You should memorize the quadratic formula because you will use it in algebra and future math courses.

QUADRATIC FORMULA: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Example A

Find the exact solutions of the following quadratic equation using the quadratic formula:

$$5x^2+2x-2=0$$

Solution: For this quadratic equation,

$$a=5, b=2, c=-2$$

. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
x &= \frac{-(2) \pm \sqrt{(2)^2-4(5)(-2)}}{2(5)}\\
x &= \frac{-2 \pm \sqrt{4+40}}{10}\\
x &= \frac{-2 \pm \sqrt{44}}{10}\\
x &= \frac{-2 \pm 2\sqrt{11}}{10}\\
x &= \frac{-1 \pm \sqrt{11}}{5}
\end{aligned}$$

The exact solutions to the quadratic equation are $\frac{-1+\sqrt{11}}{5}$ *or* $\frac{-1-\sqrt{11}}{5}$.

Example B

Use the quadratic formula to determine the approximate solutions of the equation:

$2x^2 - 3x = 3$

Solution: Start by rewriting the equation in standard form so that it is set equal to zero. $2x^2 - 3x = 3$ becomes $2x^2 - 3x - 3 = 0$. For this quadratic equation, $a = 2, b = -3, c = -3$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
m &= \frac{-(-3) \pm \sqrt{(-3)^2-4(2)(-3)}}{2(2)}\\
x &= \frac{3 \pm \sqrt{9+24}}{4}\\
x &= \frac{3 \pm \sqrt{33}}{4} \qquad \sqrt{33} = 5.74\\
x &= \frac{3 \pm 5.74}{4}\\
x &= \frac{3+5.74}{4} \text{ or } x = \frac{3-5.74}{4}\\
x &= \frac{8.74}{4} \text{ or } x = \frac{-2.74}{4}\\
x &= 2.2 \text{ or } x = -0.7
\end{aligned}$$

The approximate solutions to the quadratic equation to the nearest tenth are $x = 2.2$ or $x = -0.7$.

Example C

Solve the following equation using the quadratic formula:

$$\boxed{\frac{2}{y}-\frac{3}{y+1}=1}$$

Solution: While this does not look like a quadratic equation (it is actually a rational equation because it contains rational expressions), you can rewrite it as a quadratic equation by multiplying by $(y)(y+1)$ to get rid of the fractions. *Note that y and $y+1$ are the denominators you want to eliminate. This is why you want to multiply by* $(y)(y+1)$. After multiplying, simplify and put the equation in standard quadratic form set equal to 0.

$$\begin{aligned}
&\frac{2}{y}-\frac{3}{y+1}=1\\
&\frac{2}{y}(y)(y+1)-\frac{3}{y+1}(y)(y+1)=1(y)(y+1)\\
&\frac{2}{\cancel{y}}(\cancel{y})(y+1)-\frac{3}{\cancel{y+1}}(y)(\cancel{y+1})=1(y)(y+1)\\
&2(y+1)-3(y)=1(y^2+y)\\
&2y+2-3y=y^2+y\\
&2-y=y^2+y\\
&y^2+2y-2=0
\end{aligned}$$

For this quadratic equation, $a=1, b=2, c=-2$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
y&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
y&=\frac{-(2)\pm\sqrt{(2)^2-4(1)(-2)}}{2(1)}\\
y&=\frac{-2\pm\sqrt{4+8}}{2}\\
y&=\frac{-2\pm\sqrt{12}}{2}\\
y&=\frac{-2\pm2\sqrt{3}}{2}\\
y&=-1\pm\sqrt{3}
\end{aligned}$$

The exact solutions to the equation are $-1+\sqrt{3}$ or $-1-\sqrt{3}$. Note that neither of these solutions will cause the original equation to have a zero in the denominator, so they both work.

Concept Problem Revisited

To solve the equation $3x^2-5x+1=0$ algebraically, you can use the quadratic formula. For this quadratic equation, $a=3, b=-5, c=1$. Substitute these values into the quadratic formula and simplify.

$$\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
x &= \frac{-(-5) \pm \sqrt{(-5)^2-4(3)(1)}}{2(3)}\\
x &= \frac{5 \pm \sqrt{25-12}}{6}\\
x &= \frac{5 \pm \sqrt{13}}{6}
\end{aligned}$$

The solutions are $x = \frac{5 \pm \sqrt{13}}{6}$.

Vocabulary

Quadratic Formula

The ***quadratic formula*** is the formula $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ used to determine the solutions for a quadratic equation.

Guided Practice

1. For the following equation, rewrite as a quadratic equation and state the values for a, b and c:

$$\frac{2}{x-1} + \frac{3}{x+2} = 1$$

2. Solve the following quadratic equation using the quadratic formula:

$$6x^2 - 8x = 0$$

3. Find the approximate solutions to the following equation:

$$\frac{x+3}{2x-1} = \frac{2x+3}{x+5}$$

Answers:

1. Multiply by $(x-1)(x+2)$ to clear the fractions.

$$\begin{aligned}
&\frac{2}{x-1} + \frac{3}{x+2} = 1\\
&\frac{2}{x-1}(x-1)(x+2) + \frac{3}{x+2}(x-1)(x+2) = 1(x-1)(x+2)\\
&\frac{2}{\cancel{x-1}}(\cancel{x-1})(x+2) + \frac{3}{\cancel{x+2}}(x-1)(\cancel{x+2}) = 1(x^2+2x-1x-2)\\
&2(x+2)+3(x-1) = 1(x^2+x-2)\\
&2x+4+3x-3 = x^2+x-2\\
&x^2+x-2 = 5x+1\\
&x^2-4x-3 = 0
\end{aligned}$$

For this equation, $a=1, b=-4, c=-3$.

2. This equation does not have a 'c' term. The value of 'c' is 0. For this equation, $a=6, b=-8, c=0$.

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(6)(0)}}{2(6)}$$

$$x=\frac{8\pm\sqrt{64-0}}{12}$$
$$x=\frac{8\pm\sqrt{64}}{12}$$
$$x=\frac{-8\pm 8}{12}$$
$$x=\frac{8+8}{12}\ or\ x=\frac{8-8}{12}$$
$$\boxed{x=\frac{4}{3}\ or\ x=0}$$

3. Multiply by $(2x-1)(x+5)$ to clear the fractions.

$$\frac{x+3}{2x-1}=\frac{2x+3}{x+5}$$
$$\frac{x+3}{2x-1}(2x-1)(x+5)=\frac{2x+3}{x+5}(2x-1)(x+5)$$
$$\frac{x+3}{\cancel{2x-1}}(\cancel{2x-1})(x+5)=\frac{2x+3}{\cancel{x+5}}(2x-1)(\cancel{x+5})$$
$$(x+3)(x+5)=(2x+3)(2x-1)$$

$$x^2+5x+3x+15=4x^2-2x+6x-3$$
$$x^2+8x+15=4x^2+4x-3$$
$$-3x^2+4x+18=0$$

For this equation, $a=-3, b=4, c=18$.

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(4)\pm\sqrt{(4)^2-4(-3)(18)}}{2(-3)}$$

$$x=\frac{-4\pm\sqrt{16+216}}{-6}$$
$$x=\frac{-4\pm\sqrt{232}}{-6}$$
$$x=3.21\ and\ x=-1.87$$

The solutions to the quadratic equation to the nearest tenth are $x=3.2$ or $x=-1.9$.

Practice

State the value of a, b and c for each of the following quadratic equations.

1. $2x^2+7x-1=0$
2. $3x^2+2x=7$
3. $9x^2-7=4x$
4. $2x^2-7=0$
5. $4-2x^2=11x$

Determine the exact roots of the following quadratic equations using the quadratic formula.

6. $2x^2=8x-7$
7. $6y=2-y^2$
8. $1=8x+3x^2$
9. $2(n-2)(n+1)-(n+3)=0$
10. $\frac{2e}{e+1}-\frac{3}{e-1}=\frac{4}{e^2-1}$
11. $x^2-2x-5=0$
12. $\frac{m}{4}-\frac{m^2}{2}=-1$
13. $\frac{3}{y}-\frac{4}{y+2}=2$
14. $\frac{1}{2}x^2-\frac{x}{4}-1=0$
15. $3x^2+8x=1$

9.4 Applications of Quadratic Functions

Here you will consider real-world applications of quadratic functions.

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the maximum height of the rocket?

b) How long was the rocket in the air before hitting the ground?

c) At what time(s) will the rocket be at a height of 22 yd?

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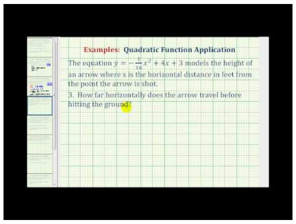
James Sousa: Quadratic Function Application- Horizontal Distance and Vertical Height



Examples: Quadratic Function Application

The equation $y = -\frac{1}{16}x^2 + 4x + 3$ models the height of an arrow where x is the horizontal distance in feet from the point the arrow is shot.

3. How far horizontally does the arrow travel before hitting the ground?

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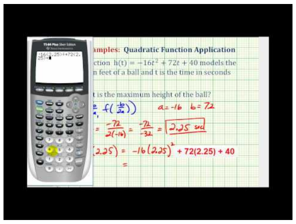
James Sousa: Quadratic Function Application- Time and Vertical Height



amples: Quadratic Function Application

ction $h(t) = -16t^2 + 72t + 40$ models the

n feet of a ball and t is the time in seconds

t is the maximum height of the ball?

$f\left(\frac{-b}{2a}\right)$ $a = -16$ $b = 72$

$= \frac{-72}{2(-16)} = \frac{-72}{-32} = \boxed{2.25 \text{ sec}}$

$(2.25) = -16(2.25)^2 + 72(2.25) + 40$

$=$

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Guidance

There are many real-world situations that deal with quadratics and parabolas. Throwing a ball, shooting a cannon, diving from a platform and hitting a golf ball are all examples of situations that can be modeled by quadratic functions.

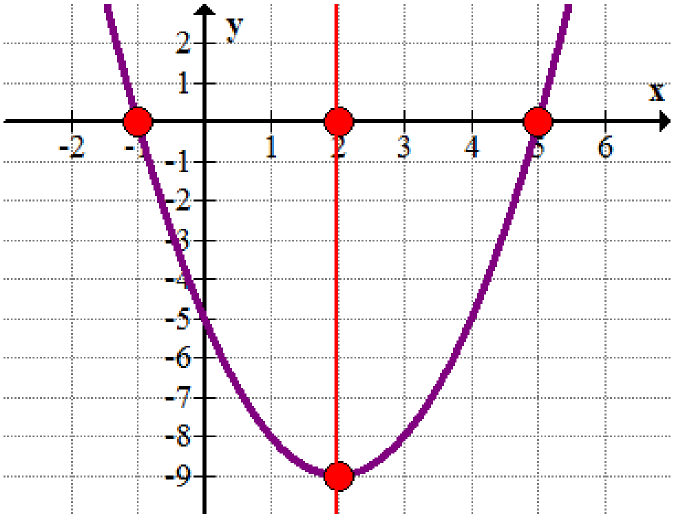
In many of these situations you will want to know the highest or lowest point of the parabola, which is known as the vertex. For example, consider that when you throw a football, the path it takes through the air is a parabola. Natural questions to ask are:

- "When does the football reach its maximum height?"
- "How high does the football get?"

If you know the equation for the function that models the situation, you can find the vertex. If the function is $f(x) = ax^2 + bx + c$, the x-coordinate of the vertex will be $-\frac{b}{2a}$. The y-coordinate of the vertex can be found by substituting the x-coordinate into the function. In the case of the football:

- The x-coordinate of the vertex will give you the time when the football is at its maximum height.
- The y-coordinate will give you the maximum height.

One way to understand where the $-\frac{b}{2a}$ comes from is to consider where the vertex is on a parabola.



Due to the symmetry of parabolas, the x-coordinate of the vertex is directly between the two x-intercepts. The two x-intercepts are, according to the quadratic formula:

$x = \frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}$

$x = \frac{-b}{2a} - \frac{\sqrt{b^2-4ac}}{2a}$

So, $x = -\frac{b}{2a}$ is in the middle. One x-intercept is $\frac{\sqrt{b^2-4ac}}{2a}$ to the right and the other x-intercept is $\frac{\sqrt{b^2-4ac}}{2a}$ to the left.

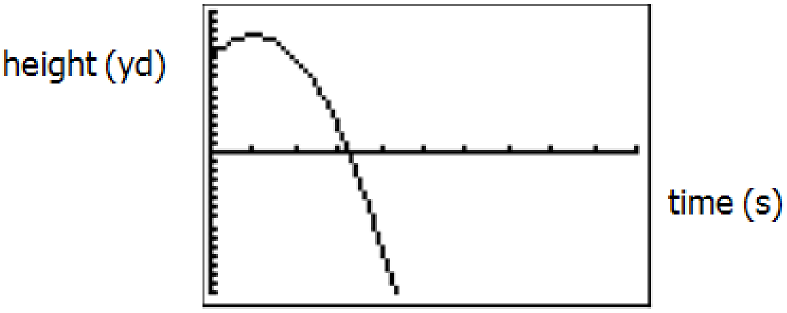
Example A

A toy rocket is fired into the air from the top of a barn. Its height (h) above the ground in yards after t seconds is given by the function $h(t) = -5t^2 + 10t + 20$.

a) What was the initial height of the rocket?

b) When did the rocket reach its maximum height?

Solution: Sketch a graph of the function. Your graphing calculator can be used to produce the graph.



a) The initial height of the rocket is the height from which it was fired. The time is zero.

$$\begin{aligned}
&h(t) = -5t^2 + 10t + 20\\
&h(t) = -5(0)^2 + 10(0) + 20\\
&\boxed{h(t) = 20\ yd}
\end{aligned}$$

The initial height of the toy rocket is 20 yards. This is the y-intercept of the graph. The y-intercept of a quadratic function written in general form is the value of 'c'.

b) The time at which the rocket reaches its maximum height is the x-coordinate of the vertex.

$$\begin{aligned}
&t = -\frac{b}{2a}\\
&t = -\frac{10}{2(-5)}\\
&\boxed{t = 1\ sec}
\end{aligned}$$

It takes the toy rocket 1 second to reach its maximum height.

Example B

The sum of a number and its square is 272. Find the number.

Solution: Let n represent the number. Write an equation to represent the problem.

$n^2 + n = 272$

You can solve this equation using a few different methods. Here, solve by completing the square.

$$n^2 + n + \frac{1}{4} = 272 + \frac{1}{4}$$

$$n^2 + n + \frac{1}{4} = \frac{1089}{4}$$

$$\left(n + \frac{1}{2}\right)^2 = \frac{1089}{4}$$

$$\begin{aligned}
&\sqrt{\left(n + \frac{1}{2}\right)^2} = \sqrt{\frac{1089}{4}}\\
&n + \frac{1}{2} = \pm\frac{33}{2}
\end{aligned}$$

$$\begin{aligned}
&n = \frac{32}{2}\ or\ n = -\frac{34}{2}\\
&n = 16\ or\ n = -17
\end{aligned}$$

These are both solutions to the problem. There are no restrictions listed in the problem regarding the solution.

Example C

The product of two consecutive positive odd integers is 195. Find the integers.

Solution: Let n represent the first positive odd integer. Let $n+2$ represent the second positive odd integer. Write an equation to represent the problem.

$$n(n+2) = 195$$
$$n^2+2n = 195$$

You can solve this equation with a few different methods. Here, use the quadratic formula.

$$n^2+2n-195 = 0$$

$$n = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$n = \frac{-(2) \pm \sqrt{(2)^2-4(1)(-195)}}{2(1)}$$

$$n = \frac{-2 \pm \sqrt{784}}{2}$$
$$n = \frac{-2 \pm 28}{2}$$

$$n = \frac{-2+28}{2} \text{ or } n = \frac{-2-28}{2}$$

$$n = 13 \text{ or } n = -15$$

There was a restriction on the solution presented in the problem. The solution must be an odd positive integer. Therefore, 13 is the solution you can use. The two positive odd integers are 13 and 15.

Concept Problem Revisited

a) The maximum height was reached by the rocket at one second as you found in Example A.

$$h(t) = -5t^2+10t+20$$
$$h(t) = -5(1)^2+10(1)+20$$
$$\boxed{h(t) = 25\ yd}$$

The maximum height reached by the rocket was 25 yd.

b) When the rocket hits the ground, its height will be zero.

$$\begin{aligned} h(t) &= -5t^2+10t+20 \\ 0 &= -5t^2+10t+20 \end{aligned}$$

Use the quadratic formula to solve for 't'. You have $a=-5, b=10, c=20$.

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$t=\frac{-(10)\pm\sqrt{(10)^2-4(-5)(20)}}{2(-5)}$$

$$\begin{aligned} t &= \frac{-10\pm10\sqrt{5}}{-10} \\ t &= 1\pm\sqrt{5} \end{aligned}$$

$$t=1+\sqrt{5}\ or\ t=1-\sqrt{5}$$

$$t=3.24\ s\ or\ t=-1.24\ s$$

$$\boxed{t=3.24\ s}$$

Accept this solution

$$\boxed{t=-1.24\ s}$$

Reject this solution. Time cannot be a negative quantity.

The toy rocket stayed in the air for approximately 3.24 seconds.

c) The rocket reached a maximum height of 25 yd at a time of 1 second. The rocket must reach a height of 22yd before and after one second. Remember the old saying: "What goes up must come down."

Use the quadratic formula to determine these times.

$$\begin{aligned} h(t) &= -5t^2+10t+20 \\ 22 &= -5t^2+10t+20 \end{aligned}$$

$$0 = -5t^2 + 10t - 2$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(10) \pm \sqrt{(10)^2 - 4(-5)(-2)}}{2(-5)}$$

$$t = \frac{5 + \sqrt{15}}{5} \; or \; t = \frac{5 - \sqrt{15}}{5}$$

$$t = 1.77\ s \; or \; t = 0.23\ s$$

$$\boxed{t = 1.77\ s}$$

Accept this solution

$$\boxed{t = 0.23\ s}$$

Accept this solution.

The rocket reached a height of 22 yd at 0.23 seconds on its way up and again at 1.77 seconds on its way down.

Vocabulary

Quadratic Equation

A ***quadratic equation*** is an equation of degree 2. The standard form of a quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$.

Quadratic Function

A ***quadratic function*** is a function that can be written in the form $f(x) = ax^2 + bx + c$ with $a \neq 0$. The graph of a quadratic function is a ***parabola***.

Completing the Square

Completing the square is a method used for solving quadratic equations. This method uses the principle of ***completing the square*** of an algebraic expression to make it a perfect square trinomial.

Quadratic Formula

The ***quadratic formula*** is the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ used to determine the solutions for a quadratic equation.

Guided Practice

1. A rectangular piece of cardboard measuring 40 in. by 30 in. is to be made into an open box with a base (bottom) of 900 in^2 by cutting equal squares from the four corners and then bending up the sides. Find, to the nearest tenth of an inch, the length of the side of the square that must be cut from each corner.

2. The local park has a rectangular flower bed that measures 10 feet by 15 feet. The caretaker plans on doubling its area by adding a strip of uniform width around the flower bed. Determine the width of the strip.

3. $h(t) = -4.9t^2 + 8t + 5$ represents Jeremiah's height (h) in meters above the water t seconds after he leaves the diving board.

i) What is the initial height of the diving board?

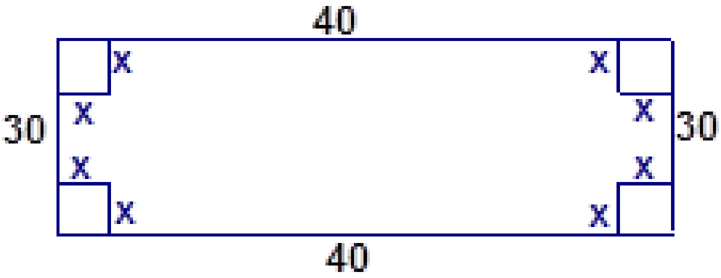
ii) At what time did Jeremiah reach his maximum height?

iii) What was Jeremiah's maximum height?

iv) How long was Jeremiah in the air?

Answers:

1. Sketch a diagram to represent the problem.



Let the variable x represent the side length of the square.

- $L = 40 - 2x$
- $W = 30 - 2x$

The area of a rectangle is the product of its length and its width. The area of the base of the rectangle must be 900 in^2, after the squares have been removed.

$$L \cdot W = Area$$
$$(40 - 2x)(30 - 2x) = 900$$

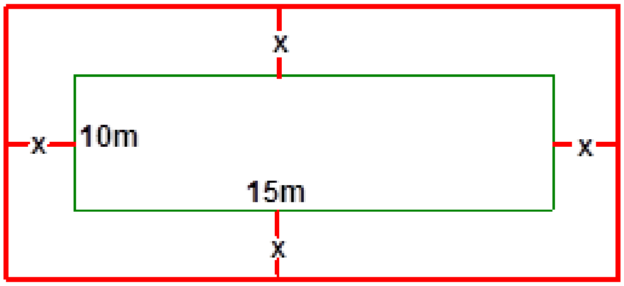
$$1200 - 80x - 60x + 4x^2 = 900$$
$$4x^2 - 140x + 1200 = 900$$

You can now solve using the quadratic formula.

$$x = 32.7 \text{ or } x = 2.3$$

The solution of 32.7 in must be rejected since it would cause the length and the width of the rectangle to result in negative values. The length of the side of the square that was cut from the cardboard was 2.3 inches.

2. Sketch a diagram to represent the problem.



Let the variable x, represent the side length of the uniform strip.

$$\begin{aligned} L &= 15+2x & L \cdot W &= Area \\ W &= 10+2x & (15)(10) &= 150\ ft^2 \end{aligned}$$

The area of a rectangle is the product of its length and its width. The area of the original flower bed is 150 ft^2. The new flower bed must be twice this area which is 300 ft^2.

$$\begin{aligned} L \cdot W &= Area \\ (15+2x)(10+2x) &= 300 \end{aligned}$$

$$\begin{aligned} 150+30x+20x+4x^2 &= 300 \\ 4x^2+50x+150 &= 300 \end{aligned}$$

You can now solve using the quadratic formula.

$$x = 2.5\ ft \text{ and } x = -15\ ft$$

$$\boxed{x = 2.5\ ft}$$

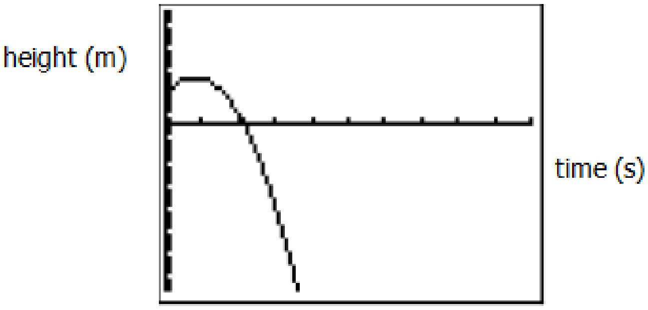
Accept this solution.

$$\boxed{x = -15\ ft}$$

Reject this solution. The width of the strip cannot be a negative value.

The width of the strip that is to be added to the flower bed is 2.5 feet.

3. Sketch a graph of the function.



i) The initial height of the diving board is when the time is zero.

$$\begin{aligned}
h(t) &= -4.9t^2 + 8t + 5 \\
h(t) &= -4.9(0)^2 + 8(0) + 5 \\
h(t) &= 0 = 0 + 5 \\
&\boxed{h(t) = 5\ m}
\end{aligned}$$

The initial height of the diving board is 5 m.

ii) The time at which Jeremiah reaches his maximum height is the x-coordinate of the vertex.

$$\begin{aligned}
t &= -\frac{b}{2a} \\
a &= -4.9 \\
b &= 8 \\
t &= -\frac{8}{2(-4.9)} \\
t &= \frac{-8}{-9.8} \\
&\boxed{t = 0.82\ sec}
\end{aligned}$$

It took Jeremiah 0.82 seconds to reach his maximum height.

iii) The maximum height was reached Jeremiah at 0.82 seconds.

$$\begin{aligned}
h(t) &= -4.9t^2 + 8t + 5 \\
h(t) &= -4.9(0.82)^2 + 8(0.82) + 5 \\
h(t) &= -3.29 + 6.56 + 5 \\
&\boxed{h(t) = 8.27\ m}
\end{aligned}$$

The maximum height reached by Jeremiah was 8.27 m.

iv) When Jeremiah hits the water, his height will be zero.

$$\begin{aligned}
h(t) &= -4.9t^2 + 8t + 5 \\
0 &= -4.9t^2 + 8t + 5
\end{aligned}$$

Use the quadratic formula to solve for 't'.

$$t = -0.48\ s\ or\ t = 2.12\ s$$

$$\boxed{t = 2.12\ s}$$

Accept this solution

$$\boxed{t = -0.48\ s}$$

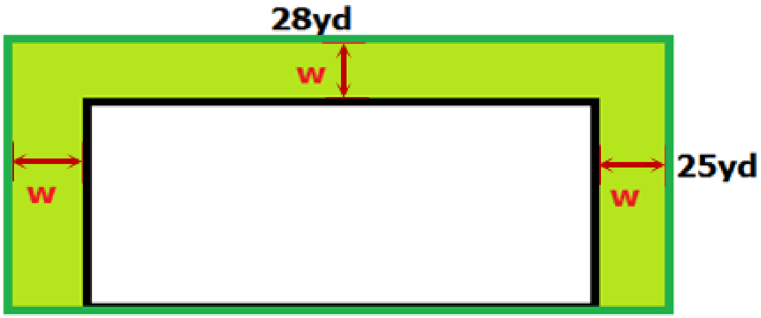
Reject this solution. Time cannot be a negative quantity.

Jeremiah stayed in the air for approximately 2.12 seconds.

Practice

Solve the following problems using your knowledge of quadratic functions.

1. The product of two consecutive even integers is 224. Find the integers.
2. The hypotenuse of a right triangle is 26 inches. The sum of the legs is 34 inches. Find the length of the legs of the triangle.
3. The product of two consecutive integers is 812. What are the integers?
4. The width of a rectangle is 3 inches longer than the length. The area of the rectangle is 674.7904 square inches. What are the dimensions of the rectangle?
5. The product of two consecutive odd integers is 3135. What are the integers?
6. Josie wants to landscape her rectangular back garden by planting shrubs and flowers along a border of uniform width as shown in the diagram. Determine the width of the border if the outside fence has dimensions of 28 yd by 25 yd and the remaining garden is to be $\frac{3}{4}$ of the original size.



7. Gregory ran the 1800 yard race last year but knows that if he could run 0.5 yd/s faster, he could reduce his time by 30 seconds. What was Gregory's time when he ran the race last year?

During a high school baseball tournament, Lexie hits a pitch and the baseball stays in the air for 4.42 seconds. The function describes the height over time, where h is its height, in yards, and t is the time, in seconds, from the instant the ball is hit.

$h = -5t^2 + 22t + 0.5$

8. Algebraically determine the maximum height the ball reaches.
9. When will the ball reach its maximum height?
10. How long will the ball be at a height of less than 20 meters while it is in the air?

A rock is thrown off a 75 meter high cliff into some water. The height of the rock relative to the cliff after t seconds is given by $h(t) = -5t^2 + 20t$.

11. Where will the rock be after five seconds?
12. How long before the rock reaches its maximum height?
13. When will the rock hit the water?

You jump off the end of a ski jump. Your height in meters relative to the height of the ski jump after t seconds is given by $h(t) = -5t^2 + 12t$.

14. How high will you be after 2 seconds? At this point are you going up or coming down?
15. If you spend 6.1 seconds in the air, how far below the end of the ski jump do you land? (What is the vertical distance?)

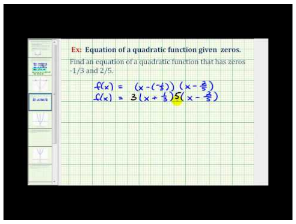
9.5 Roots to Determine a Quadratic Function

Here you will learn to find the equation of a quadratic function given its roots.

What quadratic function has roots of 2 and 7? Does more than one function have these roots?

Watch This

James Sousa: Find a Quadratic Function with Fractional Real Zeros



Ex: Equation of a quadratic function given zeros.

Find an equation of a quadratic function that has zeros -1/3 and 2/5.

$f(x) = (x-(-\frac{1}{3}))(x-\frac{2}{5})$

$f(x) = 3(x+\frac{1}{3})5(x-\frac{2}{5})$

MEDIA

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Guidance

If $x = 2$ and $x = 5$ are roots of a quadratic function, then $(x-2)$ and $(x-5)$ must have been factors of the quadratic equation. Therefore, a quadratic function with roots 2 and 5 is:

$$y = (x-2)(x-5)$$
$$y = x^2 - 7x + 10$$

Note that there are many other functions with roots of 2 and 5. These functions will all be multiples of the function above. The function below would also work:

$$y = 2(x-2)(x-5)$$
$$y = 2x^2 - 14x + 20$$

If you don't know the solutions of a quadratic equation, but you know the sum and the product of the solutions, you can find the equation. If r_1 and r_2 are the solutions to the quadratic equation $ax^2 + bx + c = 0$, then

$r_1 + r_2 = -\frac{b}{a}$ *and* $r_1 \times r_2 = \frac{c}{a}$.

The quadratic can be rewritten as:

$x^2 - (\text{sum of the solutions})x + (\text{product of the solutions}) = 0$

$x^2 - \left(-\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0$

Where did these sum and products come from? Consider the sum of the two solutions obtained from the quadratic formula:

$$\begin{aligned}
r_1+r_2 &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\\
r_1+r_2 &= -\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\\
r_1+r_2 &= -\frac{b}{2a}+\cancel{\frac{\sqrt{b^2-4ac}}{2a}}-\frac{b}{2a}-\cancel{\frac{\sqrt{b^2-4ac}}{2a}}\\
r_1+r_2 &= -\frac{b}{2a}-\frac{b}{2a}\\
r_1+r_2 &= -\frac{2b}{2a}\\
&\boxed{r_1+r_2=-\frac{b}{a}}
\end{aligned}$$

Now consider the product of the two solutions obtained from the quadratic formula:

$$\begin{aligned}
r_1\times r_2 &= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\times\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)\\
r_1\times r_2 &= \left(-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\times\left(-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)
\end{aligned}$$

$$\begin{aligned}
r_1\times r_2 &= \frac{b^2}{4a^2}-\frac{b^2-4ac}{4a^2}\\
r_1\times r_2 &= \frac{b^2-b^2+4ac}{4a^2}\\
r_1\times r_2 &= \frac{4ac}{4a^2}\\
&\boxed{r_1\times r_2=\frac{c}{a}}
\end{aligned}$$

You can use these ideas to determine a quadratic equation with solutions $2+\sqrt{3}$ and $2-\sqrt{3}$.

- The sum of the solutions is: $2+\sqrt{3}+2-\sqrt{3}=4$.
- The product of the solutions: $(2+\sqrt{3})(2-\sqrt{3})=4-3=1$.

Therefore, the equation is:

$x^2-(\text{sum of the solutions})x+(\text{product of the solutions})=0$

$x^2-4x+1=0$

Example A

Without solving, determine the sum and the product of the solutions for the following quadratic equations:

i) $x^2+3x+2=0$

ii) $3m^2+4m-3=0$

Solution: Remember that the sum of the solutions is $-\frac{b}{a}$ and the product of the solutions is $\frac{c}{a}$.

i) For $x^2+3x+2=0$, $a=1, b=3, c=2$. Therefore, the sum of the solutions is: $-\frac{3}{1}=-3$. The product of the solutions is $\frac{2}{1}=2$.

ii) For $3m^2+4m-3=0$, $a=3, b=4, c=-3$. Therefore, the sum of the solutions is: $-\frac{4}{3}$. The product of the solutions is $\frac{-3}{3}=-1$.

Example B

Find a quadratic function with the roots: $3 \pm \sqrt{5}$.

Solution: The solutions to the quadratic equation are:

$x=3+\sqrt{5}$ and $x=3-\sqrt{5}$

The factors are $(x-(3+\sqrt{5}))$ and $(x-(3-\sqrt{5}))$. One possible function in factored form is:

$$y=\left(x-3-\sqrt{5}\right)\left(x-3+\sqrt{5}\right)$$

Multiply and simplify:

$$y=x^2-3x+\sqrt{5}x-3x+9-3\sqrt{5}-\sqrt{5}x+3\sqrt{5}-\sqrt{25}$$

$$\begin{aligned}&y=x^2-3x+\cancel{\sqrt{5}x}-3x+9-\cancel{3\sqrt{5}}-\cancel{\sqrt{5}x}+\cancel{3\sqrt{5}}-5\\&y=x^2-3x-3x+9-5\\&\boxed{y=x^2-6x+4}\end{aligned}$$

Keep in mind that any multiple of the right side of the above function would also have the given roots.

Example C

Using the solutions indicated below; determine the quadratic equation by using the sum and the product of the solutions:

$3+2\sqrt{2}$ and $3-2\sqrt{2}$

Solution: The sum of the solutions is $3+2\sqrt{2}+3-2\sqrt{2}=6$.

The product of the solutions is $(3+2\sqrt{2})(3-2\sqrt{2})=9-8=1$.

$x^2-(\text{sum of the solutions})x+(\text{product of the solutions})=0$

$x^2-6x+1=0$

Concept Problem Revisited

What quadratic function has roots of 2 and 7? There are multiple functions with these roots. The basic example is $y=(x-2)(x-7)$ which is $y=x^2-9x+14$. However any function of the form $y=a(x-2)(x-7)$ with $a \neq 0$ would work.

Vocabulary

Roots of a Quadratic Function

The ***roots of a quadratic function*** are also the x-intercepts of the function. These are the values for the variable 'x' that will result in $y = 0$.

Product of the Roots

Product of the roots is an expression used to find the product of the roots of a given quadratic equation written in general form. The expression used to determine the ***product of the roots*** is:

$r_1 \times r_2 = \frac{c}{a}$

Sum of the Roots

Sum of the roots is an expression used to find the sum of the roots of a given quadratic equation written in general form. The expression used to determine the ***sum of the roots*** is:

$r_1 + r_2 = -\frac{b}{a}$

Guided Practice

1. By solving the given equation, find an equation whose solutions are each one less than the solutions to:

$$y^2 - 3y - 6 = 0$$

2. Without solving the given equation, find an equation whose solutions are the reciprocals of the solutions to:

$$2x^2 - 3x + 5 = 0$$

3. Without solving the given equation, find an equation whose solutions are the negatives of the solutions to:

$$m^2 - 4m + 9 = 0$$

Answers:

1. Determine the solutions of the quadratic equation with the quadratic formula. You should get that the solutions to the quadratic equation are:

$$y = \frac{3+\sqrt{33}}{2} \ or \ y = \frac{3-\sqrt{33}}{2}$$

The solutions of the new equation must be one less than each of the above solutions.

$$y = \frac{3}{2} + \frac{\sqrt{33}}{2} \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2}$$
$$y = \frac{3}{2} + \frac{\sqrt{33}}{2} - 1 \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2} - 1$$
$$y = \frac{3}{2} + \frac{\sqrt{33}}{2} - \frac{2}{2} \ or \ y = \frac{3}{2} - \frac{\sqrt{33}}{2} - \frac{2}{2}$$
$$y = \frac{1}{2} + \frac{\sqrt{33}}{2} \ or \ y = \frac{1}{2} - \frac{\sqrt{33}}{2}$$

The solutions of the new equation are:

$$y = \frac{1+\sqrt{33}}{2} \; or \; y = \frac{1-\sqrt{33}}{2}$$

The sum of the solutions is 1. The product of the solutions is -8. One possible quadratic equation is $y^2 - 1y - 8 = 0$.

2. The sum of the solutions is $\frac{3}{2}$. The product of the solutions is $\frac{5}{2}$. The solutions of the new equation must be the reciprocals of the solutions of the original equation. Therefore, the sum of the solutions of the new equation will be:

$$\begin{aligned}
R_1 + R_2 &= \frac{1}{r_1} + \frac{1}{r_2} \\
R_1 + R_2 &= \frac{1}{r_1}\left(\frac{r_2}{r_2}\right) + \frac{1}{r_2}\left(\frac{r_1}{r_1}\right) \\
R_1 + R_2 &= \frac{r_2}{r_1 r_2} + \frac{r_1}{r_1 r_2} \\
R_1 + R_2 &= \frac{r_2 + r_1}{r_1 r_2} \\
R_1 + R_2 &= \frac{\frac{3}{2}}{\frac{5}{2}} \\
R_1 + R_2 &= \frac{3}{2} \times \frac{2}{5} \\
&\boxed{R_1 + R_2 = \frac{3}{5}}
\end{aligned}$$

The product of the solutions of the new equation will be:

$$\begin{aligned}
R_1 \times R_2 &= \frac{1}{r_1} \times \frac{1}{r_2} \\
R_1 \times R_2 &= \frac{1}{r_1 r_2} \\
R_1 \times R_2 &= \frac{1}{\frac{5}{2}} \\
R_1 \times R_2 &= 1\left(\frac{2}{5}\right) \\
&\boxed{R_1 \times R_2 = \frac{2}{5}}
\end{aligned}$$

The new equation is:

$$\begin{aligned}
&x^2 - (r_1 + r_2)x + (r_1 \times r_2) = 0 \\
&x^2 - \left(\frac{3}{5}\right)x + \left(\frac{2}{5}\right) = 0 \\
&5\left(x^2 - \left(\frac{3}{5}\right)x + \left(\frac{2}{5}\right) = 0\right) \\
&\boxed{5x^2 - 3x + 2 = 0}
\end{aligned}$$

3. The sum of the solutions is 4 and the product of the solutions is 9. The solutions to the new equation must be negatives of the solutions of the original equation. Therefore, the sum and the product of the new solutions are:

$$\boxed{R_1+R_2=-r_1+(-r_2)=-(r_1+r_2)=-4} \text{ and } \boxed{R_1\times R_2=(-r_1)\times(-r_2)=r_1\times r_2=9}$$

The new equation is:

$$\begin{aligned}&m^2-(r_1+r_2)m+(r_1\times r_2)=0\\&m^2-(-4)m+(+9)=0\\&\boxed{m^2+4m+9=0}\end{aligned}$$

Practice

Without solving, determine the sum and the product of the roots of the following quadratic equations.

1. $2y^2-8y+3=0$
2. $3e^2-6e=4$
3. $0=14-12x+18x^2$
4. $5x^2+6=7x$
5. $2(2x-1)(x+5)=x^2+4$

For the following sums and products of the solutions, state one possible quadratic equation:

6. sum: 4; product: 3
7. sum: 0; product: –16
8. sum: –9; product: –7
9. sum: –6; product: –5
10. sum: $-\frac{2}{3}$; product: $\frac{5}{3}$

For the given roots, determine the factors of the quadratic function:

11. $-\frac{3}{2}$ and 5
12. $\frac{1}{4}$ and $\frac{3}{2}$
13. –5 and 3
14. $-\frac{5}{2}$ and $-\frac{4}{3}$
15. $\pm\frac{5}{2}$

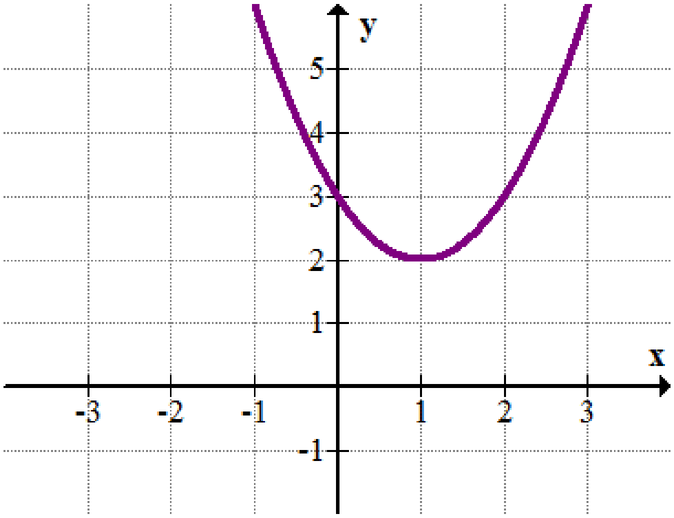
For the given roots, determine a potential quadratic function:

16. –2 and –4
17. –3 and $-\frac{1}{3}$
18. $2+\sqrt{3}$ and $2-\sqrt{3}$
19. $\pm 2\sqrt{5}$
20. $-3\pm\sqrt{7}$

9.6 Imaginary Numbers

Here you will learn about imaginary numbers.

The solutions to a quadratic equation show up as the x-intercepts of the corresponding quadratic function. The parabola $y = x^2 - 2x + 3$ is shown below.



What does this graph tell you about the solutions to $x^2 - 2x + 3 = 0$?

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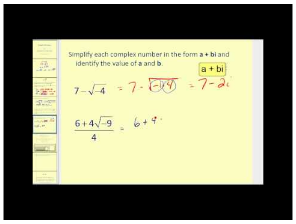
James Sousa: Complex Numbers



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Guidance

When humans first created the concept of numbers, they only had the counting numbers (whole numbers) $\{1, 2, 3, \ldots\}$ because numbers were meant to count physical objects. It took a long time (more than 700 years!) before even the concept of 0 was invented in the year 500 AD in India. Since then, humans have slowly been adding to our number system so that it can work for us. Fractions, decimals, and negative numbers have become an important part of our world. For a long time, it was accepted that the square root of negative numbers did not exist. In order for a solution to exist to the equation $x^2 = -1$, mathematicians invented a solution. This solution is called the imaginary number and is noted by the letter i:

$\sqrt{-1} = i$ and $i^2 = -1$

Imaginary numbers were not commonly accepted in mathematics until the 1700s, but since then they have become an important part of our number system and are especially important in physics. They are called imaginary because they cannot be found on a traditional number line of real numbers. The square root of any negative number can be written in terms of the imaginary number i:

- $\sqrt{-4} = \sqrt{4 \cdot -1} = \sqrt{4}\sqrt{-1} = 2i$
- $\sqrt{-5} = \sqrt{5 \cdot -1} = \sqrt{5}\sqrt{-1} = \sqrt{5}i$
- $\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16}\sqrt{-1} = 4i$

You can perform addition, subtraction, and multiplication with imaginary numbers just like regular numbers (you can also do division, but that is a bit more complicated and won't be considered here). When performing multiplication, remember that $i^2 = -1$. You should always express answers in such a way that the i does not have an exponent.

- $2i + 1 - 3i + 4$ simplifies to $-i + 5$ or $5 - i$
- $3i \cdot 2i$ can be multiplied to $6i^2 = 6(-1) = -6$

Numbers that are a combination of imaginary and real numbers, such as $5 - i$, are called ***complex numbers***. You will study complex numbers in much more depth in future courses.

Example A

Express $\sqrt{-49}$ as a simplified imaginary number.

Solution: $\sqrt{-49} = \sqrt{49 \cdot -1} = \sqrt{49}\sqrt{-1} = 7i$

Example B

Express $\sqrt{-40}$ as a simplified imaginary number.

Solution: $\sqrt{-40} = \sqrt{4 \cdot 10 \cdot -1} = \sqrt{4}\sqrt{10}\sqrt{-1} = 2\sqrt{10}i$

Example C

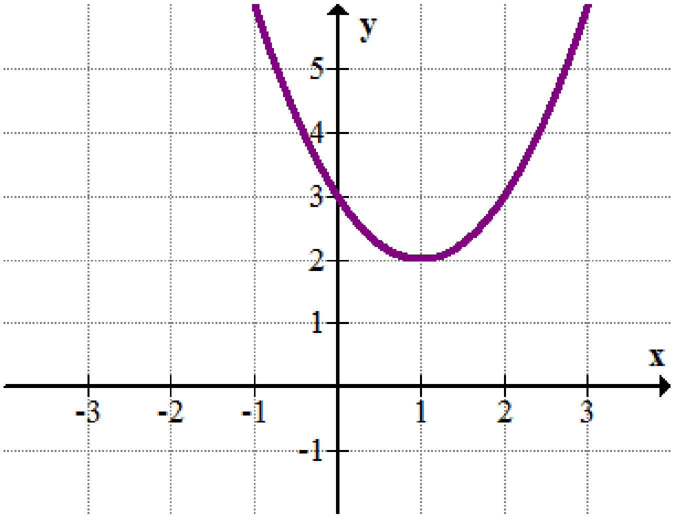
Simplify the following expression: $(4+3i)+(6-5i)$

Solution: Simplify by combining the real numbers with the real numbers and the imaginary numbers with the imaginary numbers:

$(4+3i)+(6-5i) = 10-2i$

Concept Problem Revisited

The parabola $y = x^2 - 2x + 3$ has no x-intercepts, as shown below.



This means that the solutions to the equation $x^2 - 2x + 3 = 0$ are not real numbers. The solutions are complex numbers. You can still find the solutions using the quadratic formula, but your result will be 2 complex number solutions.

Vocabulary

Complex Number

A ***complex number*** is a number in the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

Imaginary Number

An ***imaginary number*** is a number such that its square is a negative number.

$\sqrt{-16}$ is an imaginary number beacause its square is –16.

$\sqrt{-16} = i\sqrt{16} = 4i$

Guided Practice

Simplify each of the following:

1. $(5-3i)-(2+4i)$

2. $3i(4i^2-5i+3)$

3. $(7+2i)(3-i)$

4. $\sqrt{-12}$

Answers:

1. $(5-3i)-(2+4i) = 5-3i-2-4i = 3-7i$

2.

$$\begin{aligned}3i(4i^2-5i+3) &= 3i(4\cdot -1-5i+3)\\ &= 3i(-4-5i+3)\\ &= 3i(-1-5i)\\ &= -3i-15i^2\\ &= -3i-15(-1)\\ &= -3i+15\end{aligned}$$

3.

$$\begin{aligned}(7+2i)(3-i) =& 21-7i+6i-2i^2\\ &= 21-i-2i^2\\ &= 21-i-2(-1)\\ &= 21-i+2\\ &= 23-i\end{aligned}$$

4. $\sqrt{-12} = \sqrt{4\cdot 3\cdot -1} = 2i\sqrt{3}$

Practice

Express each as a simplified imaginary number.

1. $\sqrt{-300}$
2. $\sqrt{-32}$
3. $4\sqrt{-18}$
4. $\sqrt{-75}$
5. $\sqrt{-98}$

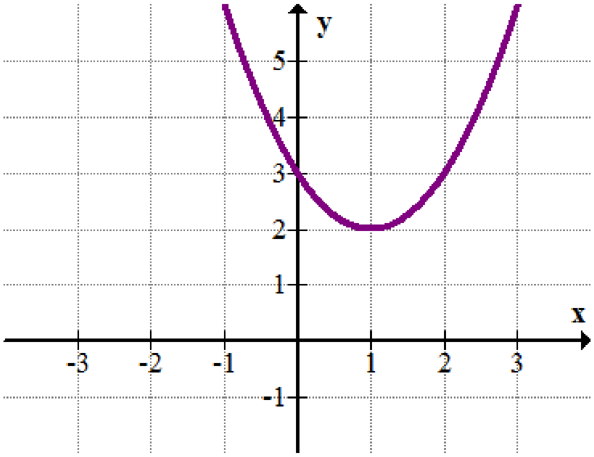
Simplify each of the following:

6. $(8+5i)-(12+8i)$
7. $(7+3i)(4-5i)$
8. $(2+i)(4-i)$
9. $3(5i-4)-2(6i-7)$
10. $5i(3i-2i^2+4)$
11. $(3+4i)+(11+6i)$
12. $(5+2i)(1-5i)$
13. $(1+i)(1-i)$
14. $2(6i-3)-4(2i+6)$
15. i^3
16. i^4
17. i^6
18. $5i(3i-2i^2+4)$

9.7 Complex Roots of Quadratic Functions

Here you'll learn how to find complex roots of a quadratic function and what it means when a function has complex roots.

The quadratic function $y = x^2 - 2x + 3$ (shown below) does not intersect the x-axis and therefore has no real roots. What are the complex roots of the function?



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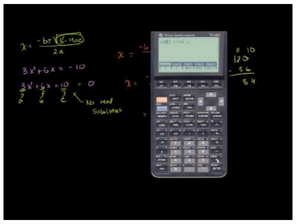
Khan Academy Using the Quadratic Formula



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Guidance

Recall that the imaginary number, i, is a number whose square is –1:

$i^2 = -1$ and $i = \sqrt{-1}$

The sum of a real number and an imaginary number is called a ***complex number***. Examples of complex numbers are $5+4i$ and $3-2i$. All complex numbers can be written in the form $a+bi$ where a and b are real numbers. Two important points:

- The set of real numbers is a subset of the set of complex numbers where $b = 0$. Examples of real numbers are $2, 7, \frac{1}{2}, -4.2$.
- The set of imaginary numbers is a subset of the set of complex numbers where $a = 0$. Examples of imaginary numbers are $i, -4i, \sqrt{2}i$.

This means that the set of complex numbers includes real numbers, imaginary numbers, and combinations of real and imaginary numbers.

When a quadratic function does not intersect the x-axis, it has complex roots. When solving for the roots of a function algebraically using the quadratic formula, you will end up with a negative under the square root symbol. With your knowledge of complex numbers, you can still state the complex roots of a function just like you would state the real roots of a function.

Example A

Solve the following quadratic equation for x.

$$\boxed{m^2-2m+5=0}$$

Solution: You can use the quadratic formula to solve. For this quadratic equation, $a=1, b=-2, c=5$.

$$\begin{aligned}
m &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
m &= \frac{-(-2) \pm \sqrt{(-2)^2-4(1)(5)}}{2(1)} \\
m &= \frac{2 \pm \sqrt{4-20}}{2} \\
m &= \frac{2 \pm \sqrt{-16}}{2} \qquad \sqrt{-16} = \sqrt{16} \times i = 4i \\
m &= \frac{2 \pm 4i}{2} \\
m &= 1 \pm 2i \\
m &= 1+2i \text{ or } m = 1-2i
\end{aligned}$$

There are no real solutions to the equation. The solutions to the quadratic equation are $1+2i$ *and* $1-2i$.

Example B

Solve the following equation by rewriting it as a quadratic and using the quadratic formula.

$\frac{3}{e+3}-\frac{2}{e+2}=1$

Solution: To rewrite as a quadratic equation, multiply each term by $(e+3)(e+2)$.

$$\begin{aligned}
&\frac{3}{e+3}(e+3)(e+2) - \frac{2}{e+2}(e+3)(e+2) = 1(e+3)(e+2) \\
&3(e+2)-2(e+3) = (e+3)(e+2)
\end{aligned}$$

Expand and simplify.

$$\begin{aligned}
&3e+6-2e-6 = e^2+2e+3e+6 \\
&e^2+4e+6=0
\end{aligned}$$

Solve using the quadratic formula. For this quadratic equation, $a = 1, b = 4, c = 6$.

$$\begin{aligned}
e &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
e &= \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(6)}}{2(1)} \\
e &= \frac{-4 \pm \sqrt{16 - 24}}{2} \\
e &= \frac{-4 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \times i = \sqrt{4 \cdot 2} \times i = 2i\sqrt{2} \\
e &= \frac{-4 \pm 2i\sqrt{2}}{2} \\
e &= -2 \pm i\sqrt{2} \\
e &= -2 + i\sqrt{2} \text{ or } e = -2 - i\sqrt{2}
\end{aligned}$$

There are no real solutions to the equation. The solutions to the equation are $-2 + i\sqrt{2}$ *and* $-2 - i\sqrt{2}$

Example C

Sketch the graph of the following quadratic function. What are the roots of the function?

$$y = x^2 - 4x + 5$$

Solution: Use your calculator or a table to make a sketch of the function. You should get the following:

As you can see, the quadratic function has no x-intercepts; therefore, the function has no real roots. To find the roots (which will be complex), you must use the quadratic formula.

For this quadratic function, $a = 1, b = -4, c = 5$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$
$$x = \frac{4 \pm \sqrt{16 - 20}}{2}$$
$$x = \frac{4 \pm \sqrt{-4}}{2} \qquad \sqrt{-4} = \sqrt{4} \times i = 2i$$
$$x = \frac{4 \pm 2i}{2}$$
$$x = 2 \pm i$$
$$x = 2 + i \; or \; x = 2 - i$$

The complex roots of the quadratic function are $2 + i$ *and* $2 - i$.

Concept Problem Revisited

To find the complex roots of the function $y = x^2 - 2x + 3$, you must use the quadratic formula.

For this quadratic function, $a = 1, b = -2, c = 3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$
$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$
$$x = \frac{2 \pm \sqrt{-8}}{2} \qquad \sqrt{-8} = \sqrt{8} \times i = 2\sqrt{2}i$$
$$x = \frac{2 \pm 2\sqrt{2}i}{2}$$
$$x = 1 \pm \sqrt{2}i$$

Vocabulary

Complex Number

A ***complex number*** is a number in the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

Imaginary Number

An ***imaginary number*** is a number such that its square is a negative number.
$\sqrt{-16}$ is an imaginary number beacause its square is –16.
$\sqrt{-16} = i\sqrt{16} = 4i$

Guided Practice

1. Solve the following quadratic equation. Express all solutions in simplest radical form.

$$2n^2+n=-4$$

2. Solve the following quadratic equation. Express all solutions in simplest radical form.

$$m^2+(m+1)^2+(m+2)^2=-1$$

3. Is it possible for a quadratic function to have exactly one complex root?

Answers:

1.

$$2n^2+n=-4$$

Set the equation equal to zero.

$$2n^2+n+4=0$$

Solve using the quadratic formula.

$$\begin{aligned}
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
n &= \frac{-(1)\pm\sqrt{(1)^2-4(2)(4)}}{2(2)}\\
n &= \frac{-1\pm\sqrt{1-32}}{4}\\
n &= \frac{-1\pm\sqrt{-31}}{4}\\
n &= \frac{-1\pm i\sqrt{31}}{4}
\end{aligned}$$

2.

$$m^2+(m+1)^2+(m+2)^2=-1$$

Expand and simplify.

$$\begin{aligned}
&m^2+(m+1)(m+1)+(m+2)(m+2)=-1\\
&m^2+m^2+m+m+1+m^2+2m+2m+4=-1\\
&3m^2+6m+5=-1
\end{aligned}$$

Write the equation in general form.

$$3m^2+6m+6=0$$

Divide by 3 to simplify the equation.

$$\begin{aligned}
\frac{3m^2}{3}+\frac{6m}{3}+\frac{6}{3}&=\frac{0}{3}\\
m^2+2m+2&=0
\end{aligned}$$

Solve using the quadratic formula:

$$\begin{aligned}
m&=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\\
m&=\frac{-(2)\pm\sqrt{(2)^2-4(1)(2)}}{2(1)}\\
m&=\frac{-2\pm\sqrt{4-8}}{2}\\
m&=\frac{-2\pm\sqrt{-4}}{2}\\
m&=\frac{-2\pm 2i}{2}\\
m&=-1\pm i
\end{aligned}$$

3. No, even in higher degree polynomials, complex roots will always come in pairs. Consider when you use the quadratic formula– if you have a negative under the square root symbol, both the + version and the - version of the two answers will end up being complex.

Practice

1. If a quadratic function has 2 x-intercepts, how many complex roots does it have? Explain.
2. If a quadratic function has no x-intercepts, how many complex roots does it have? Explain.
3. If a quadratic function has 1 x-intercept, how many complex roots does it have? Explain.
4. If you want to know whether a function has complex roots, which part of the quadratic formula is it important to focus on?
5. You solve a quadratic equation and get 2 complex solutions. How can you check your solutions?
6. In general, you can attempt to solve a quadratic equation by graphing, factoring, completing the square, or using the quadratic formula. If a quadratic equation has complex solutions, what methods do you have for solving the equation?

Solve the following quadratic equations. Express all solutions in simplest radical form.

7. $x^2+x+1=0$
8. $5y^2-8y=-6$
9. $2m^2-12m+19=0$
10. $-3x^2-2x=2$
11. $2x^2+4x=-11$
12. $-x^2+x-23=0$
13. $-3x^2+2x=14$
14. $x^2+5=-x$
15. $\frac{1}{2}d^2+4d=-12$

9.8 The Discriminant

Here you'll learn what the discriminant is and how to use it to help you to describe the roots and graph of a quadratic function.

Suppose you want to know whether the function $y = x^2 - 5x + 12$ has real roots. What part of the quadratic formula would you need to test in order to determine if the roots of the function are real or complex?

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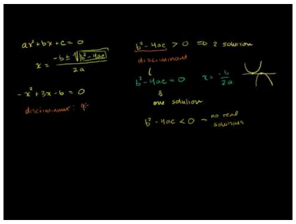
Khan Academy Discriminant of Quadratic Equations



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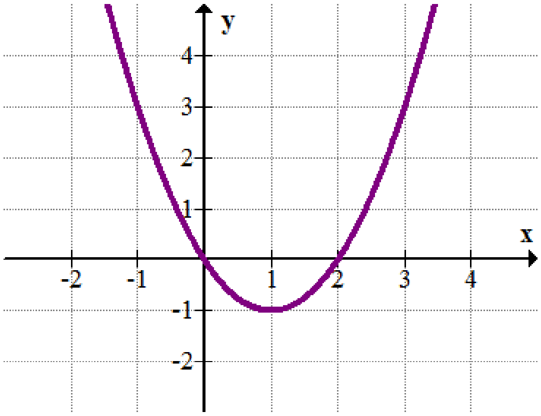
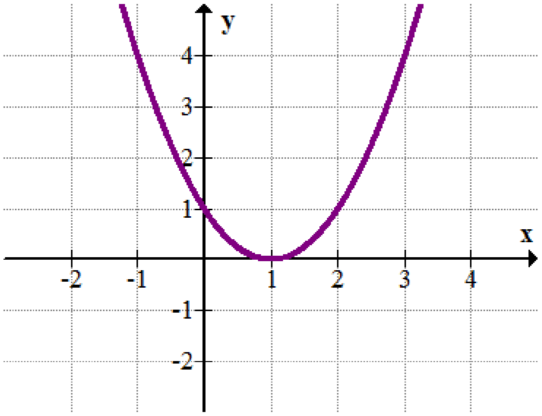
Guidance

Consider the potential roots of a quadratic function. There are three options:

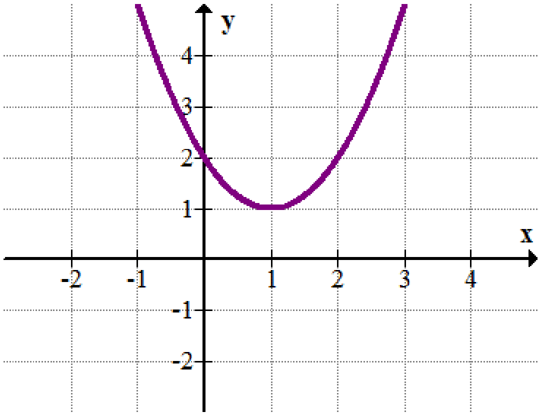
1. The function has 2 distinct real roots and 2 x-intercepts.



2. The function has 1 real root (of multiplicity 2) and 1 x-intercept.



3. The function has 0 real roots, 2 complex roots, and 0 x-intercepts.



The quadratic formula states that the two roots of a quadratic function $y = ax^2 + bx + c$ are:

$r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$

Under the radical sign is the expression $b^2 - 4ac$. This expression is known as the ***discriminant*** of the quadratic function and is important especially because when it is negative the roots of the function are complex. There are only three possible outcomes for the value of the discriminant. These outcomes are:

1. The value of the discriminant is positive ($b^2 - 4ac > 0$). This means the function has 2 distinct real roots and 2 x-intercepts.
2. The value of the discriminant is zero ($b^2 - 4ac = 0$). This means the function has 1 real root (of multiplicity 2) and 1 x-intercept.
3. The value of the discriminant is negative ($b^2 - 4ac < 0$). This means the function has 0 real roots, 2 complex roots, and 0 x-intercepts.

This discriminant is helpful when you are only looking to describe the graph of a quadratic function or its roots, but don't need to know its exact roots. If you need to know its exact roots, you will still have to use the complete quadratic formula.

Example A

If the discriminant of a quadratic function has the value shown below, determine if the function will have two distinct real roots, 1 real root of multiplicity 2, or two distinct complex roots.

a) 7

b) –3

c) $\frac{1}{2}$

d) 0

Solution:

a) $b^2 - 4ac > 0$ so the quadratic function will have two distinct real roots.

b) $b^2 - 4ac < 0$ so the quadratic function will have two distinct complex roots.

c) $b^2 - 4ac > 0$ so the quadratic function will have two distinct real roots.

d) $b^2 - 4ac = 0$ so the quadratic function will have 1 real root of multiplicity 2.

Example B

Given the function $y = ax^2 + bx + c$, how many times will the graph intersect the x-axis if the discriminant of its corresponding quadratic equation has the value:

a) 5

b) –6

c) 0

d) 0.2

Solution:

a) $b^2 - 4ac > 0$ so the parabola will cross the x-axis twice.

b) $b^2 - 4ac < 0$ so the parabola will not cross the x-axis.

c) $b^2 - 4ac = 0$ so the parabola will cross the x-axis once.

d) $b^2 - 4ac > 0$ so the parabola will cross the x-axis twice.

Example C

For each of the following quadratic equations, determine the value of the discriminant and use that value to describe the nature of the roots.

a) $4x^2 - 4x + 1 = 0$

b) $x^2 + 3x + 9 = 0$

c) $2x^2 + 3x - 4 = 0$

d) $9x^2 + 12x + 4 = 0$

Solution: Let D represent the discriminant.

a) The quadratic equation will have 1 real solution of multiplicity 2.

$$
\begin{aligned}
&D = b^2 - 4ac \\
&D = (-4)^2 - 4(4)(1) \\
&D = 16 - 16 \\
&\boxed{D = 0} \\
&b^2 - 4ac = 0
\end{aligned}
$$

b) The quadratic equation will have two distinct complex solutions and no real solutions.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (3)^2 - 4(1)(9)\\
&D = 9 - 36\\
&\boxed{D = -27}\\
&b^2 - 4ac < 0
\end{aligned}$$

c) The quadratic equation will have two distinct real solutions.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (3)^2 - 4(2)(-4)\\
&D = 9 + 32\\
&\boxed{D = 41}\\
&b^2 - 4ac > 0
\end{aligned}$$

d) The quadratic equation will have 1 real root of multiplicity 2.

$$\begin{aligned}
&D = b^2 - 4ac\\
&D = (12)^2 - 4(9)(4)\\
&D = 144 - 144\\
&\boxed{D = 0}\\
&b^2 - 4ac = 0
\end{aligned}$$

Concept Problem Revisited

In order to determine if the roots of $y = x^2 - 5x + 12$ are real or complex, you must find the discriminant, $b^2 - 4ac$. For this function:

$$\begin{aligned}
b^2 - 4ac &= (-5)^2 - 4(1)(12)\\
&= 25 - 48\\
&= -23
\end{aligned}$$

Because the discriminant is negative, this function has two distinct complex roots.

Vocabulary

Discriminant

The ***discriminant*** is the radicand of the quadratic formula and its value is used to describe the nature of the roots of a quadratic equation.

$b^2 - 4ac > 0 \rightarrow 2$ distinct real roots

$b^2 - 4ac = 0 \rightarrow$ 1 real root of multiplicity 2
$b^2 - 4ac < 0 \rightarrow$ 2 distinct complex roots and 0 real roots

Multiplicity

The ***multiplicity*** of a root or solution is the number of times that the same solution is obtained through factoring or the quadratic formula. When the discriminant is equal to zero, both solutions produced by the quadratic formula will be the same. There is only one solution, but it is said to have "multiplicity 2".

Guided Practice

1. Given the following quadratic equation, find the value of 'm' such that the equation will have 1 real solution of multiplicity 2.

$mx^2 + (m+8)x + 9 = 0$

2. Given the following quadratic equation, determine the nature of the solutions:

$4(y^2 - 5y + 5) = -5$

3. Given the following quadratic equation, find the value of 'm' such that the equation will have two distinct complex solutions.

$(m+1)e^2 - 2e - 3 = 0$

Answers:

1. Begin by determining the value of the discriminant.

$$b^2 - 4ac = (m+8)^2 - 4(m)(9)$$

Expand and Simplify

$$\begin{aligned}
&b^2 - 4ac = (m+8)(m+8) - 4(m)(9)\\
&b^2 - 4ac = m^2 + 8m + 8m + 64 - 36m\\
&b^2 - 4ac = m^2 - 20m + 64
\end{aligned}$$

If the equation has 1 real solution of multiplicity 2, the value of the discriminant must equal zero.

$$\begin{aligned}
&b^2 - 4ac = 0\\
&\therefore m^2 - 20m + 64 = 0
\end{aligned}$$

Factor the quadratic equation and solve for the variable 'm'.

$$\begin{aligned}
&(m-16)(m-4) = 0\\
&m - 16 = 0 \text{ or } m - 4 = 0\\
&m = 16 \text{ or } m = 4
\end{aligned}$$

The values of 'm' that would produce 1 solution of multiplicity 2 for the quadratic equation are $m = 16$ *or* $m = 4$.

2. Write the quadratic equation in general form.

$$4(y^2-5y+5)=-5$$

Apply the distributive property.

$$4y^2-20y+20=-5$$

Set the equation equal to zero.

$$\begin{aligned}&4y^2-20y+20+5=-5+5\\&4y^2-20y+25=0\end{aligned}$$

Determine the value of the discriminant for this quadratic equation.

$$\begin{aligned}&D=b^2-4ac\\&D=(-20)^2-4(4)(25)\\&D=400-400\\&\boxed{D=0}\\&b^2-4ac=0\end{aligned}$$

The quadratic equation will have 1 real solution of multiplicity 2.

3. Begin by determining the value of the discriminant.

$$b^2-4ac=(-2)^2-4(m+1)(-3)$$

Expand and Simplify

$$b^2-4ac=12m+16$$

If the equation has two distinct complex solutions the value of the discriminant must be less than zero.

$$\begin{aligned}&b^2-4ac<0\\&\therefore 12m+16<0\end{aligned}$$

Solve the inequality.

$$12m+16-16<0-16$$
$$12m<-16$$
$$\frac{12m}{12}<\frac{-16}{12}$$
$$\frac{\cancel{12}m}{\cancel{12}}<-\frac{16}{12}$$
$$\boxed{m<-\frac{4}{3}}$$

The value of 'm' that would produce two distinct complex solutions for the quadratic equation is $m<-\frac{4}{3}$.

Practice

If the discriminant of a quadratic equation has the value shown below, describe the nature of the solutions.

1. –14
2. 11
3. 0
4. –0.25
5. 124

State the nature of the solutions for each of the following quadratic equations.

6. $2x^2+7x-1=0$
7. $3x^2+2x=-7$
8. $-9x^2-7=4x$
9. $x^2-8x+16=0$
10. $4+2x^2=11x$

Determine the value(s) of 'm' that will produce the indicated solution for each of the following:

11. $y^2+(m+2)y+2m=0$; 1 real solution of multiplicity 2
12. $g^2+(m-1)g+1=0$; 2 real solutions
13. $3mx^2-3x+1=0$; 2 complex solutions
14. $x^2+4mx+1=0$; 1 real solution of multiplicity 2
15. $p^2+mp+16=0$; 2 real solutions

9.9 Radical Equations

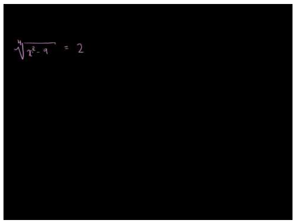
Here you will learn how to solve a radical equation.

Can you solve the following equation?

$x+\sqrt{x-2}=4$

Watch This

Khan Academy Radical Equation Examples



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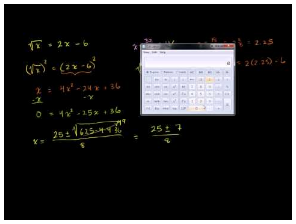
Khan Academy Extraneous Solutions to Radical Equations



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Guidance

A radical equation is an equation with a variable under a radical sign. The following are all examples of radical equations:

- $\sqrt{x}=5$
- $\sqrt{x-4}+5=0$
- $x+\sqrt{x-2}=4$
- $\sqrt{4x+5}-\sqrt{2x-6}=3$

Just like multiplication and division or addition and subtraction are inverse operations (they "undo" each other), squaring and finding a square root are inverse operations:

- $(\sqrt{5})^2=5$
- $\sqrt{(x+2)^2}=x+2$

Therefore, to eliminate a square root in an equation, isolate the square root part of the equation and then square both sides of the equation. If there are multiple square roots, you might have to go through this process multiple times. For example, consider the following equation:

$\sqrt{x-1}-5=0$

Step 1: Isolate the radical:

$$\sqrt{x-1}=5$$

Step 2: Square both sides of the equation.

$$\begin{aligned}\left(\sqrt{x-1}\right)^{2}&=(5)^{2}\\ x-1&=25\end{aligned}$$

Step 3: Solve the resulting equation.

$$x=26$$

Just like any equation, you can check your answer(s) by substituting them back into the original equation. Unlike other equations, ***it is very important to check your answers to radical equations.*** Due to the fact that the process of squaring produces positive numbers, sometimes you will end up with a solution to a radical equation that does not actually work in the original equation. These solutions are called "extraneous" and are not actually solutions. Therefore, ***you must always check your answers to radical equations***. Check the solution to the above equation by verifying that the left side (L.S.) of the equation equals the right side (R.S.) of the equation:

$$\begin{aligned}&\sqrt{x-1}-5=0\\ &L.S.=\sqrt{x-1}-5 \qquad R.S.=0\\ &L.S.=\sqrt{26-1}-5\\ &L.S.=\sqrt{25}-5\\ &L.S.=5-5\\ &L.S.=0\\ &L.S.=R.S.\end{aligned}$$

Therefore, the solution of 26 is correct.

Example A

Solve the following radical equation and verify the solution(s): $2\sqrt{x-1}-1=9$

Solution: Begin by isolating the radical:

$$\begin{aligned}2\sqrt{x-1}-1+1&=9+1\\ 2\sqrt{x-1}&=10\\ \frac{2\sqrt{x-1}}{2}&=\frac{10}{2}\\ \sqrt{x-1}&=5\end{aligned}$$

Now square both sides of the equation:

$$\begin{gathered}\left(\sqrt{x-1}\right)^{2}=(5)^{2}\\ x-1=25\end{gathered}$$

Solve the equation:

$$x=26$$

Finally, verify the result by substituting the value of 26 for 'x' into the original equation. If 26 is a solution to the equation, the left side (*L.S.*) will equal the right side (*R.S.*).

$$\begin{aligned}&2\sqrt{x-1}-1=9\\ &L.S.=2\sqrt{26-1}-1 \quad R.S.=9\\ &L.S.=2\sqrt{25}-1\\ &L.S.=2(5)-1\\ &L.S.=10-1\\ &L.S.=9\\ &L.S.=R.S.\end{aligned}$$

Therefore, the solution of 26 is correct.

Example B

Solve the following radical equation and verify the solution(s): $\sqrt{4x+5}-\sqrt{2x-6}=3$

Solution: When a radical equation has more than one radical, begin by writing the equation with one radical on each side of the equation. You will ultimately have to square the equation more than once throughout the solution process.

$$\sqrt{4x+5}=3+\sqrt{2x-6}$$

Square both sides of the equation:

$$\left(\sqrt{4x+5}\right)^{2}=\left(3+\sqrt{2x-6}\right)^{2}$$

Expand and simplify:

$$\begin{aligned}\left(\sqrt{4x+5}\right)^{2}&=\left(3+\sqrt{2x-6}\right)\left(3+\sqrt{2x-6}\right)\\ 4x+5&=9+3\sqrt{2x-6}+3\sqrt{2x-6}+2x-6\\ 4x+5&=3+6\sqrt{2x-6}+2x\end{aligned}$$

Isolate the radical:

$$2x+2=6\sqrt{2x-6}$$

Simplify the equation:

$$\frac{2x}{2}+\frac{2}{2}=\frac{6\sqrt{2x-6}}{2}$$
$$x+1=3\sqrt{2x-6}$$

Square both sides of the equation:

$$(x+1)^2=\left(3\sqrt{2x-6}\right)^2$$
$$x^2+x+x+1=9(2x-6)$$
$$x^2+2x+1=18x-54$$

The equation is quadratic. Write the equation in standard form:

$$x^2-16x+55=0$$

Solve the equation by factoring:

$$(x-11)(x-5)=0$$
$$x=11 \text{ or } x=5$$

Verify the results by first substituting the value of 11 for 'x' into the original equation and then substituting the value of 5 for 'x' into the original equation:

Verify $x=11$.
$\sqrt{4x+5}-\sqrt{2x-6}=3$
$L.S.=\sqrt{4x+5}-\sqrt{2x-6} \quad R.S.=3$
$L.S.=\sqrt{4(11)+5}-\sqrt{2(11)-6}$
$L.S.=\sqrt{49}-\sqrt{16}$
$L.S.=7-4$
$L.S.=3$
$R.S.=3$
$L.S.=R.S.$

Verify $x=5$.
$\sqrt{4x+5}-\sqrt{2x-6}=3$
$L.S.=\sqrt{4x+5}-\sqrt{2x-6} \quad R.S.=3$
$L.S.=\sqrt{4(5)+5}-\sqrt{2(5)-6}$
$L.S.=\sqrt{25}-\sqrt{4}$
$L.S.=5-2$
$L.S.=3$
$R.S.=3$
$L.S.=R.S.$

Therefore, the solutions of 11 and 5 are both correct.

Example C

Solve the following radical equation and verify the solution(s): $\sqrt{5x+6}-4=0$

Solution: Begin by isolating the radical:

$$\sqrt{5x+6}=4$$

Square both sides of the equation:

$$\begin{aligned}\left(\sqrt{5x+6}\right)^{2}&=(4)^{2}\\5x+6&=16\end{aligned}$$

Solve the equation:

$$\begin{aligned}5x&=10\\\frac{5x}{5}&=\frac{10}{5}\\x&=2\end{aligned}$$

Verify the result by substituting the value of 2 for 'x' into the original equation:

$$\begin{aligned}&\text{Verify } x=2.\\&\sqrt{5x+6}-4=0\\&L.S.=\ \sqrt{5x+6}-4\quad R.S.=0\\&L.S.=\ \sqrt{5(2)+6}-4\\&L.S.=\ \sqrt{10+6}-4\\&L.S.=\ \sqrt{16}-4\\&L.S.=4-4\\&L.S.=0\\&R.S.=0\\&L.S.=R.S.\end{aligned}$$

Therefore, the solution of 2 is correct.

Concept Problem Revisited

$x+\sqrt{x-2}=4$

Begin by isolating the radical:

$$\sqrt{x-2}=4-x$$

Square both sides of the equation:

$\left(\sqrt{x-2}\right)^{2}=(4-x)^{2}$

Expand and simplify:

$$x-2=16-8x+x^2$$

The equation is a quadratic. Write the equation in standard form:

$$x^2-9x+18=0$$

Solve the equation:

$$\begin{aligned}&(x-6)(x-3)=0\\&x-6=0 \text{ or } x-3=0\\&x=6 \text{ or } x=3\end{aligned}$$

Verify the results by first substituting the value of 6 for 'x' into the original equation and then substituting the value of 3 for 'x' into the original equation.

Verify $x=6$.	Verify $x=3$.
$x+\sqrt{x-2}=4$	$x+\sqrt{x-2}=4$
$L.S.=6+\sqrt{6-2} \quad R.S.=4$	$L.S.=3+\sqrt{3-2} \quad R.S.=4$
$L.S.=6+\sqrt{4}$	$L.S.=3+\sqrt{1}$
$L.S.=6+2$	$L.S.=3+1=4$
$L.S.=8$	$L.S.=4$
$R.S.=4$	$R.S.=4$
$L.S.\neq R.S.$	$L.S.=R.S.$

The value $x=6$ did not satisfy the original equation and is not a solution to the radical equation. It is called an *extraneous solution*. $x=3$ is a solution to the equation.

Vocabulary

Extraneous Solution

An ***extraneous solution*** is a solution to a radical equation that does not satisfy the original equation. Therefore, the solution is rejected as a solution to the equation and is called an ***extraneous solution***.

Radical Equation

A ***radical equation*** is an equation that has a variable under a radical sign.

Guided Practice

1. Verify whether or not $x = 3$ and $x = -5$ are solutions to the radical equation:

$2\sqrt{x+6} + \sqrt{2x+10} = 2$

2. Solve the following radical equation and verify the solution(s)to the equation.

$x - \sqrt{x-1} = 7$

3. Solve the following radical equation and verify the solution(s) to the equation.

$\sqrt{x+7} - \sqrt{x} = 1$

Answers:

1. Verify the results by first substituting the value of 3 for 'x' into the original equation and then substituting the value of -5 for 'x' into the original equation.

Verify $x = 3$.	Verify $x = -5$.
$2\sqrt{x+6} + \sqrt{2x+10} = 2$	$2\sqrt{x+6} + \sqrt{2x+10} = 2$
$L.S. = 2\sqrt{x+6} + \sqrt{2x+10} \quad R.S. = 2$	$L.S. = 2\sqrt{x+6} + \sqrt{2x+10} \quad R.S. = 2$
$L.S. = 2\sqrt{(3)+6} + \sqrt{2(3)+10}$	$L.S. = 2\sqrt{(-5)+6} + \sqrt{2(-5)+10}$
$L.S. = 2\sqrt{9} + \sqrt{16}$	$L.S. = 2(1) + \sqrt{0}$
$L.S. = 2(3) + 4$	$L.S. = 2(1) + 0$
$L.S. = 6 + 4$	$L.S. = 2 + 0$
$L.S. = 10$	$L.S. = 2$
$R.S. = 2$	$R.S. = 2$
$L.S. \neq R.S.$	$L.S. = R.S.$

The value $x = 3$ did not satisfy the original equation and is not a solution to the radical equation. It is an extraneous solution. $x = -5$ is a solution.

2. $-\sqrt{x-1} = 7 - x$

$$x - 1 = 49 - 14x + x^2$$

$$x^2 - 15x + 50 = 0$$

$$(x-10)(x-5) = 0$$
$$x = 10 \text{ or } x = 5$$

$$\begin{aligned}
&\text{Verify } x = 10. && \text{Verify } x = 5.\\
&x - \sqrt{x-1} = 7 && x - \sqrt{x-1} = 7\\
&L.S. = x - \sqrt{x-1} \quad R.S. = 7 && L.S. = x - \sqrt{x-1} \quad R.S. = 7\\
&L.S. = (10) - \sqrt{(10)-1} && L.S. = (5) - \sqrt{(5)-1}\\
&L.S. = 10 - \sqrt{9} && L.S. = 5 - \sqrt{4}\\
&L.S. = 10 - 3 && L.S. = 5 - 2\\
&L.S. = 7 && L.S. = 3\\
&R.S. = 7 && R.S. = 7\\
&L.S. = R.S. && L.S. \neq R.S.
\end{aligned}$$

The value $x = 5$ did not satisfy the original equation and is not a solution to the radical equation. It is an extraneous solution. $x = 10$ is a solution.

3. $\sqrt{x+7} - \sqrt{x} = 1$

$$\sqrt{x+7} = 1 + \sqrt{x}$$

$$\left(\sqrt{x+7}\right)^2 = \left(1 + \sqrt{x}\right)^2$$

$$x + 7 = 1 + 2\sqrt{x} + x$$

$$6 = 2\sqrt{x}$$

$$\begin{aligned}(6)^2 &= \left(2\sqrt{x}\right)^2\\ 36 &= 4x\end{aligned}$$

$$\begin{aligned}\frac{36}{4} &= \frac{4x}{4}\\ 9 &= x\end{aligned}$$

Verify $x = 9$.

$$\begin{aligned}
&\sqrt{x+7} - \sqrt{x} = 1 \\
&L.S. = \sqrt{x+7} - \sqrt{x} \quad R.S. = 1 \\
&L.S. = \sqrt{(9)+7} - \sqrt{9} \\
&L.S. = \sqrt{16} - \sqrt{9} \\
&L.S. = 4 - 3 \\
&L.S. = 1 \\
&L.S. = 1 \\
&R.S. = 1 \\
&L.S. = R.S.
\end{aligned}$$

$x = 9$ is a solution to the equation.

Practice

1. Is $x = 7$ a solution to $\sqrt{x+2} = -3$?
2. Is $x = 1$ a solution to $\sqrt{x^2+4x+4} - \sqrt{x^2+3x} = 1$?
3. Is $x = -3$ a solution to $\sqrt{x^2+4x+4} - \sqrt{x^2+3x} = 1$?
4. Is $x = 12$ a solution to $\sqrt{x+4} + 8 = x$?
5. Is $x = 5$ a solution to $\sqrt{x+4} + 8 = x$?
6. Is $x = 8$ a solution to $\sqrt{x+1} = 1 + \sqrt{x-4}$?
7. Is $x = 3$ a solution to $\sqrt{x+1} + \frac{2}{\sqrt{x+1}} = \sqrt{x+6}$?

Solve the following radical equations and verify the solution(s).

8. $x = 3 + \sqrt{x-1}$
9. $\sqrt{x+1} = 1 + \sqrt{x-4}$
10. $\sqrt{x} - \sqrt{x-16} = 2$
11. $\sqrt{3x-2} - 1 = \sqrt{2x-3}$
12. $5\sqrt{x-6} = x$
13. $x = 5 + \sqrt{x-4}$
14. $\sqrt{x+2} = 5 - \sqrt{x-3}$
15. $\sqrt{x} + \sqrt{x-9} = 5$

Summary

You learned that all quadratic equations have a corresponding quadratic function. Real solutions to quadratic equations are the x-intercepts of the quadratic function. If a quadratic equation has only complex solutions, the quadratic function will not have x-intercepts.

You also learned that there are four methods for solving quadratic equations:

1. Factoring and the zero product property (learned previously)
2. Graphing and looking for x-intercepts

3. Completing the square
4. The quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The advantage of the quadratic formula is that it will always work to give you solutions, even if the solutions are not real numbers.

If you want to determine whether the roots of a given quadratic function are real or complex, but you don't need to know specifically what the roots are, you can use the discriminant. The discriminant is the part of the quadratic formula under the square root symbol ($b^2 - 4ac$). If the discriminant is negative, the roots will be complex. If the discriminant is equal to zero, there will only be one root (of multiplicity 2). If the discriminant is positive, the roots will be real.

You also learned that radical equations are equations with variables under square roots. Radical equations can be solved by isolating the square root and squaring both sides. Sometimes radical equations will produce extraneous solutions, which are not really solutions, so it is important to always check your answers to radical equations.

CHAPTER 10 Geometric Transformations

Chapter Outline

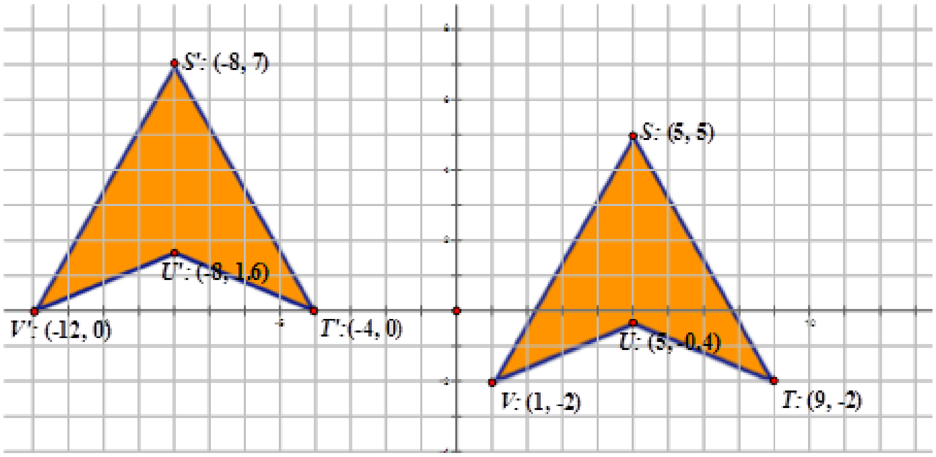
Introduction

Here you'll learn all about geometric transformations. You will learn about reflections, rotations, translations and dilations. You will also learn the distance formula and the midpoint formulas and see how algebra and geometry are connected.

10.1 Translations

Here you will learn to describe translations.

Karen looked at the image below and stated that the image was translated thirteen units backwards. Is she correct? Explain.



Watch This

First watch this video to learn about translations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10TranslationsA

Then watch this video to see some examples.

MEDIA

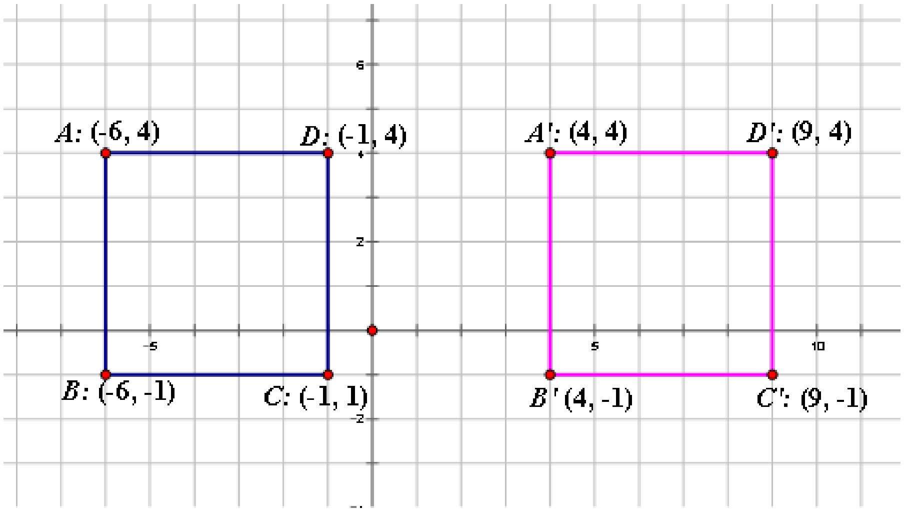
Click image to the left for more content.

CK-12 Foundation Chapter10TranslationsB

Guidance

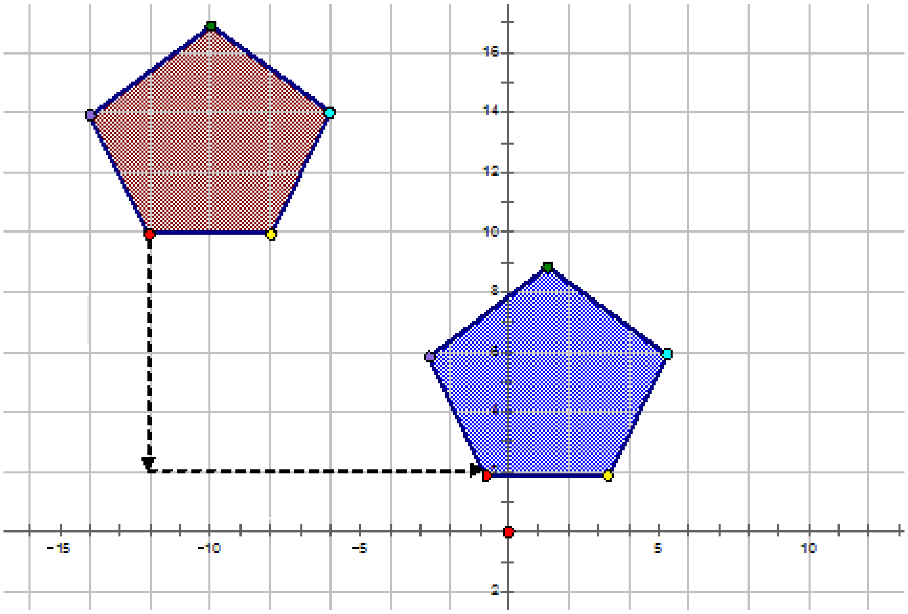
In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A translation is a type of transformation that moves each point in a figure the same distance in the same direction.

Translations are often referred to as slides. If you look at the picture below, you can see that the square *ABCD* is moved 10 units to the right. All points of the square have been moved 10 units to the right to make the translated image $(A'B'C'D')$. The original square $(ABCD)$ is called a **preimage**. The final square is called the **image.**



Example A

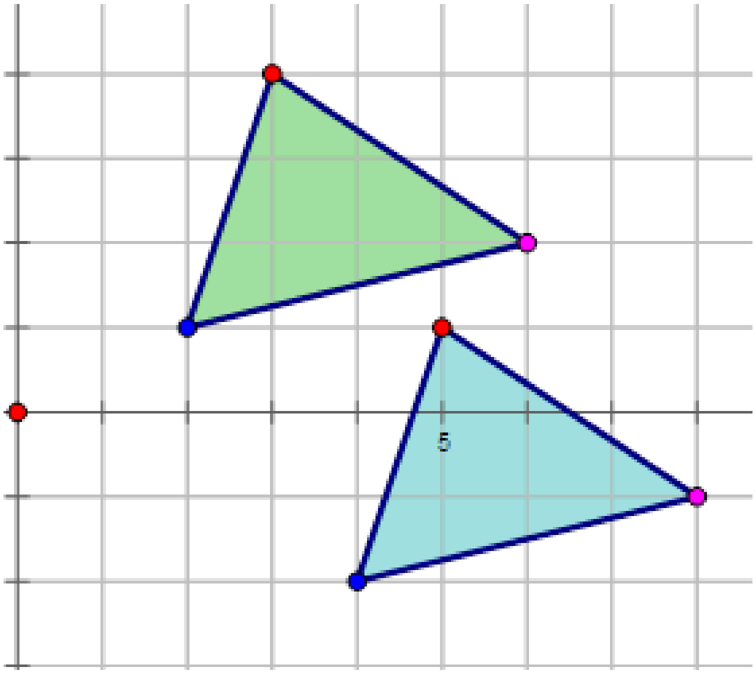
Describe the translation of the purple pentagon in the diagram below.



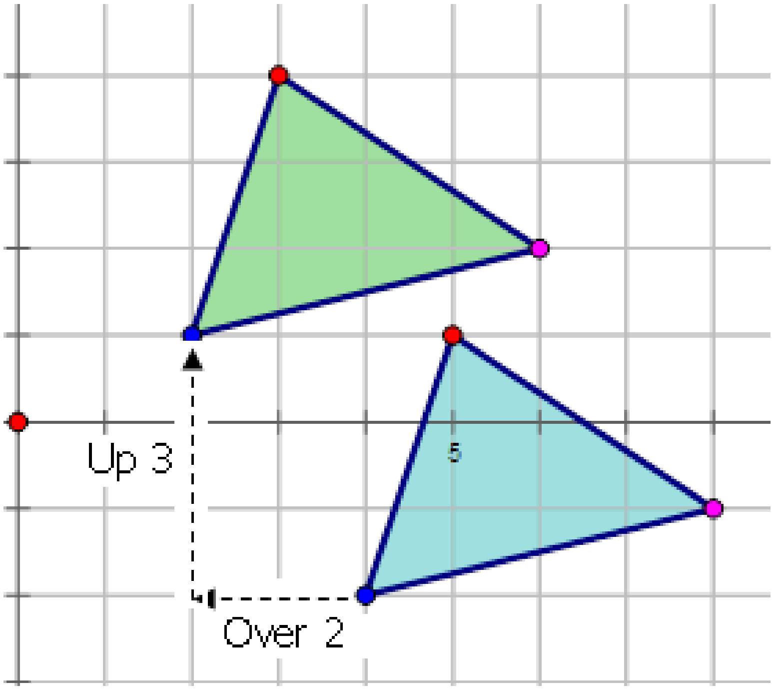
Solution: The pentagon is translated down 8 and over 11 to the right.

Example B

Describe the translation of the light blue triangle in the diagram to the right.

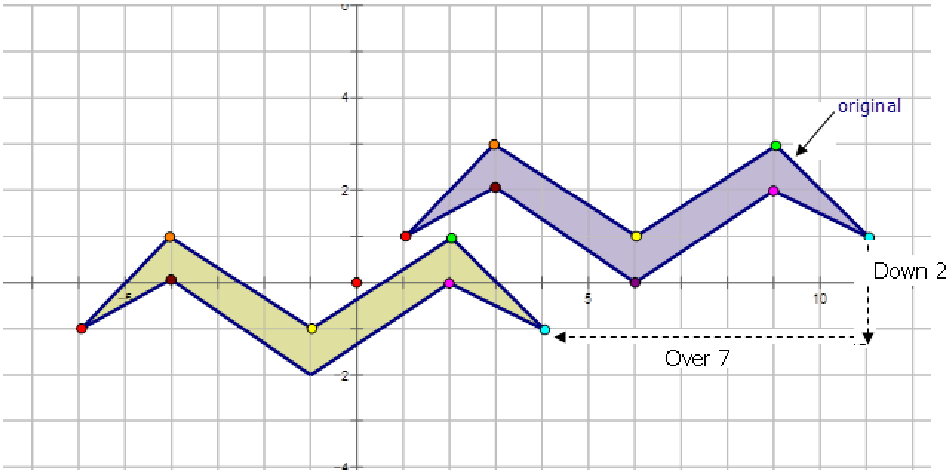


Solution: The blue triangle moves up 3 units and over 2 units to the left to make the green triangle image.



Example C

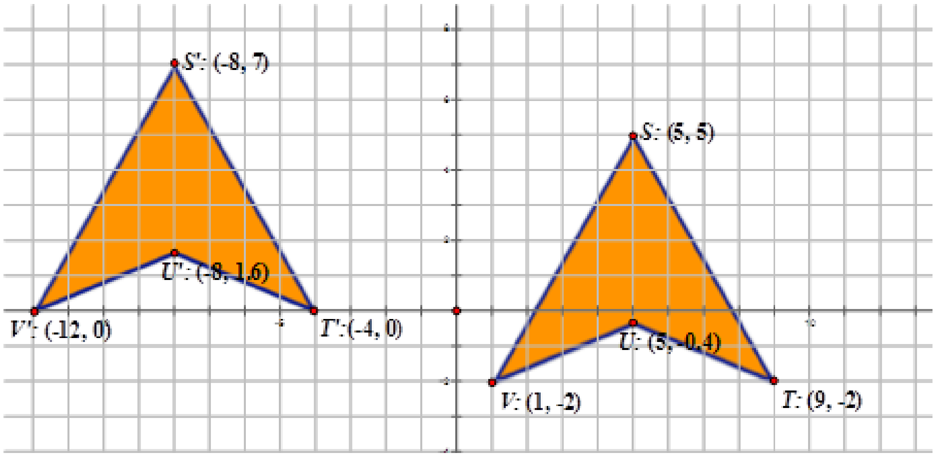
Describe the translation in the diagram below.



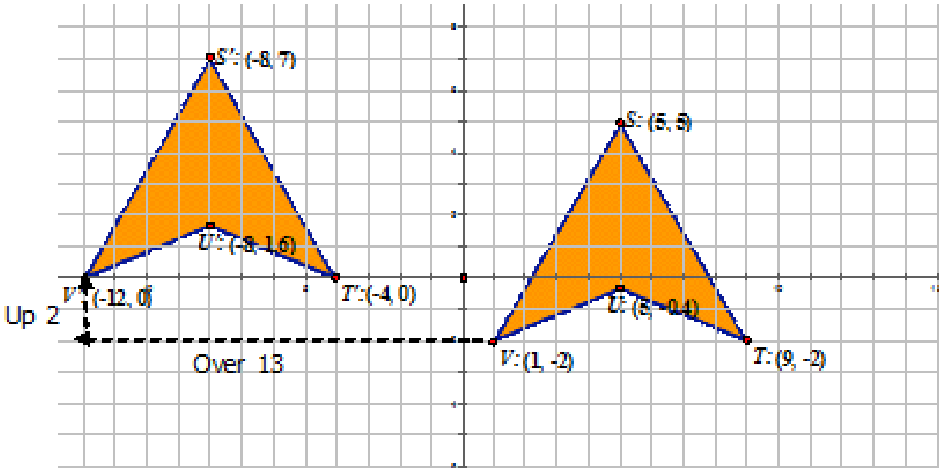
Solution: The original shape is translated down 2 and over 7 to the left.

Concept Problem Revisited

Karen looked at the image below and stated that the image was translated thirteen units backwards. Is she correct? Explain.



Karen is somewhat correct in that the translation is moving to the left (backwards). The proper way to describe the translation is to say that the image $STUV$ has moved 13 units to the left and 2 units up.



Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

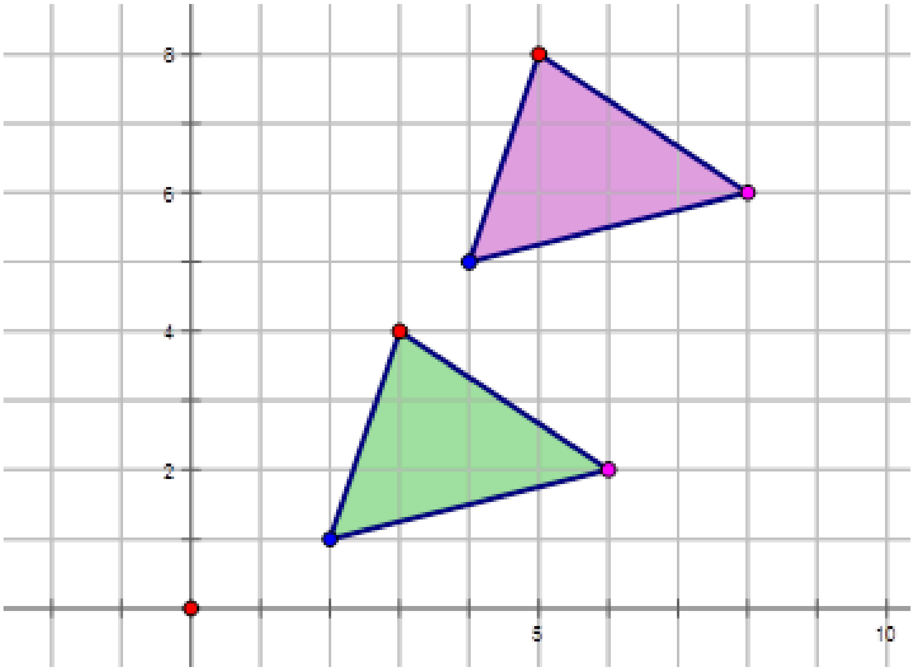
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Translation

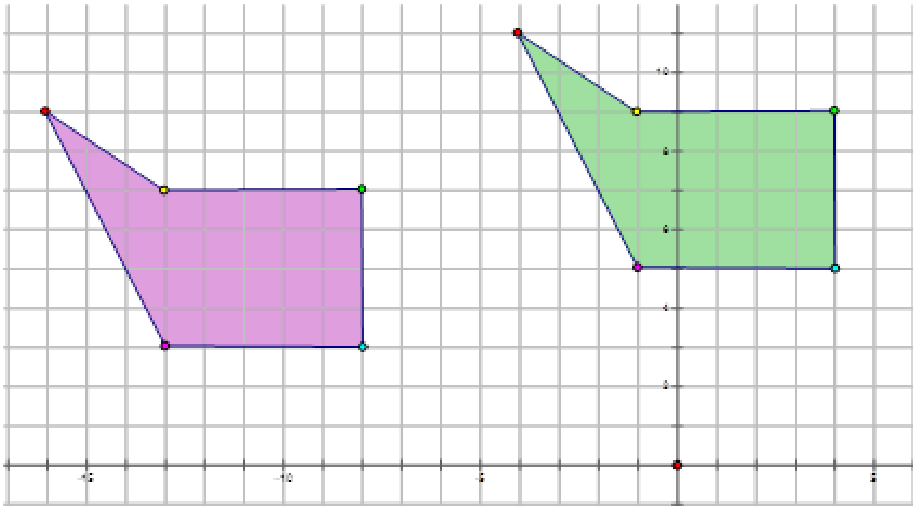
A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

Guided Practice

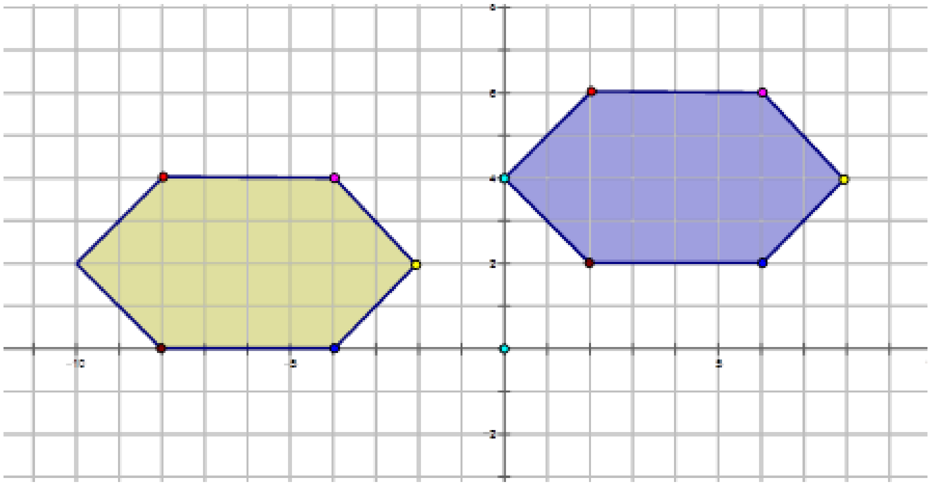
1. Describe the translation of the pink triangle in the diagram below.



2. Describe the translation of the purple polygon in the diagram below.

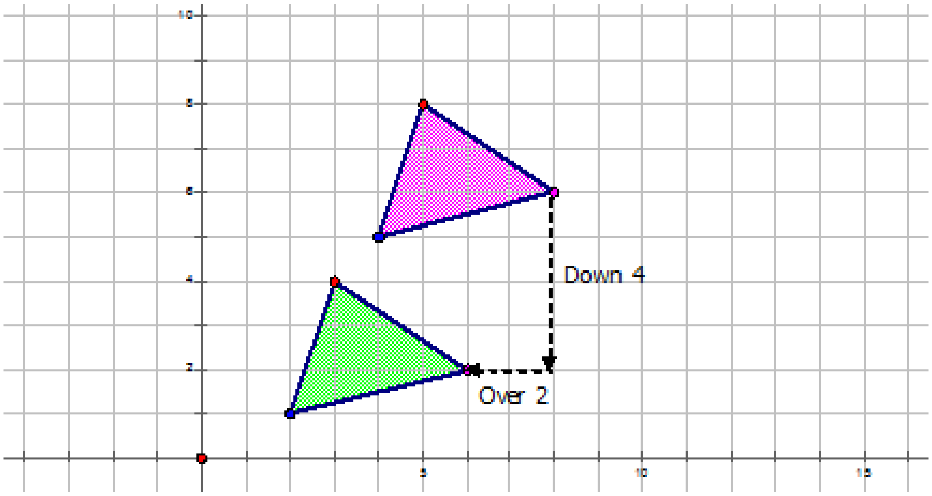


3. Describe the translation of the blue hexagon in the diagram below.

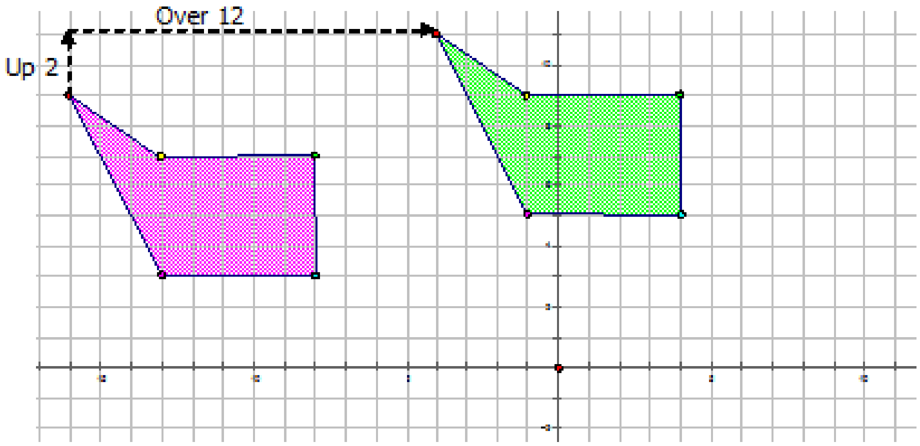


Answers:

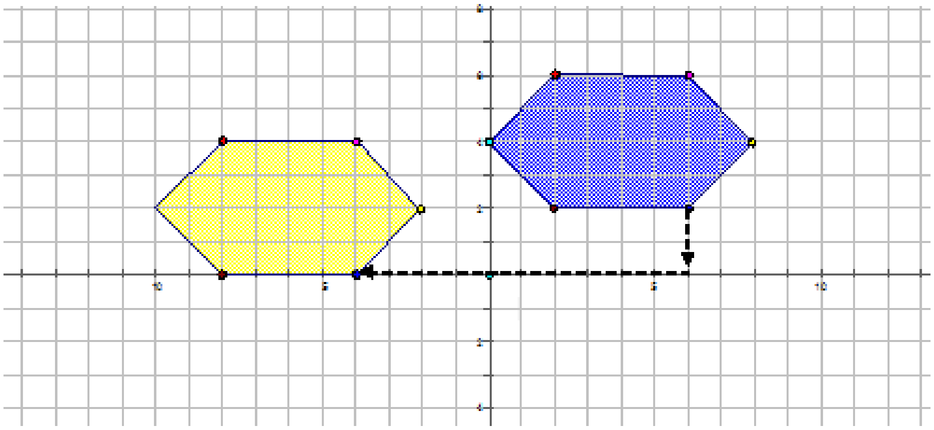
1. The pink triangle is translated down 4 and over 2 to the left.



2. The purple polygon is translated up 2 and over 12 to the right.

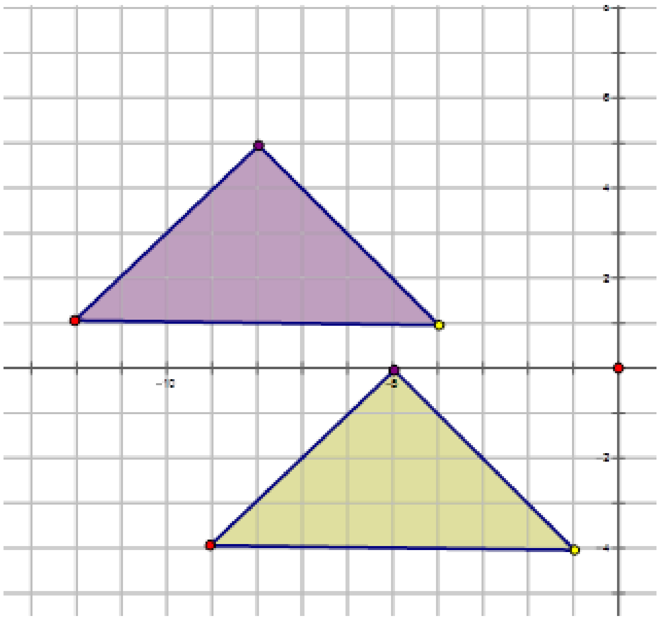


3. The blue hexagon is translated down 2 and over 10 to the left.



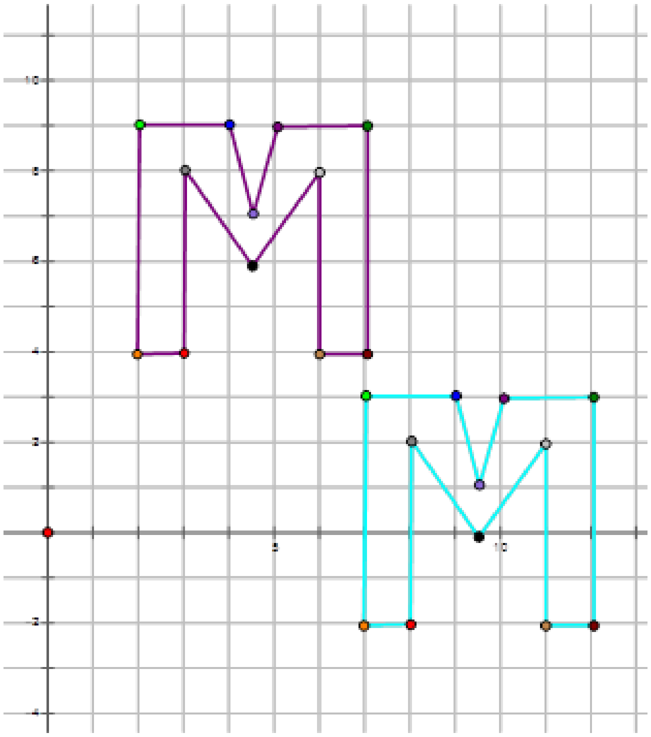
Practice

Describe the translation of the purple original figures in the diagrams:

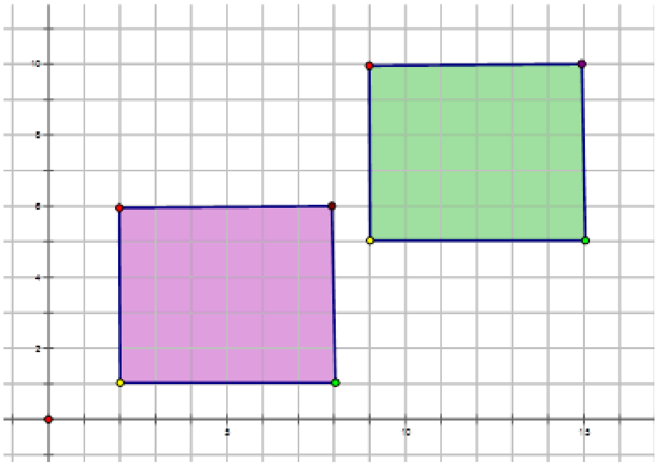


1.

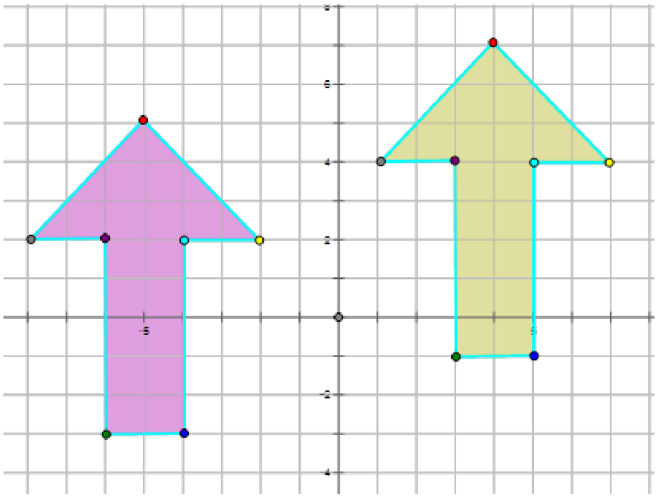
2.

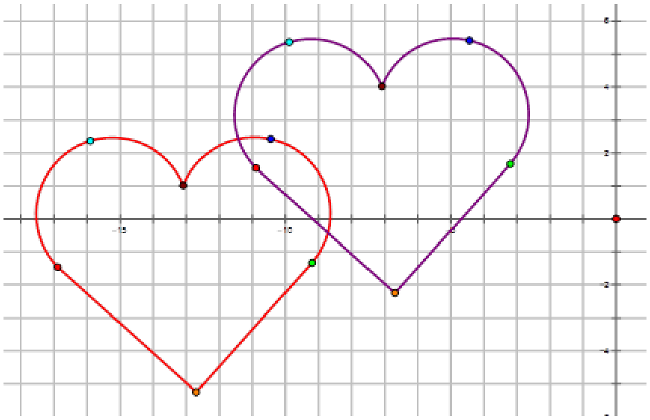


3.



4.

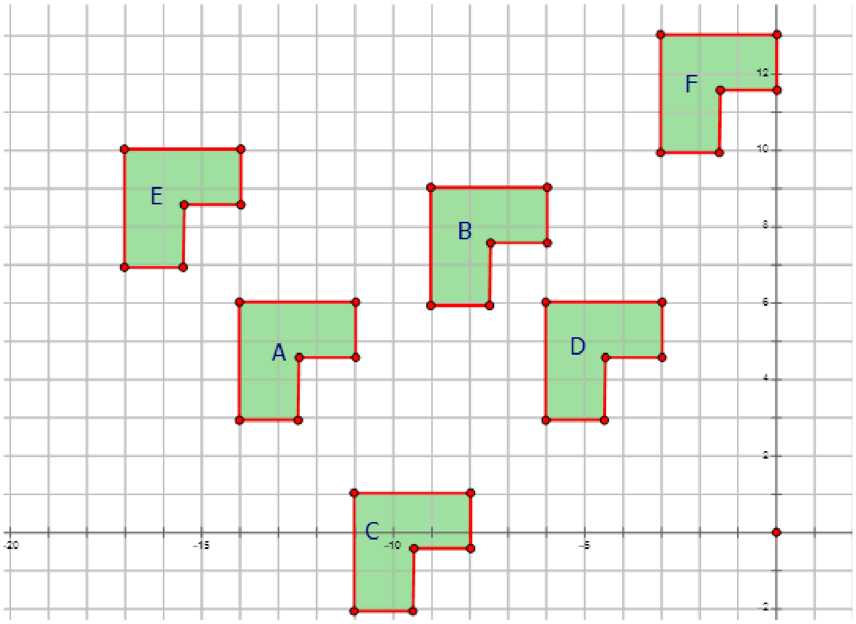




5.

Use the diagram below to describe the following translations:

6. A onto B
7. A onto C
8. A onto D
9. A onto E
10. A onto F



On a piece of graph paper, plot the points $A(2,3), B(6,3)$ and $C(6,1)$ to form $\triangle ABC$.

11. Translate the triangle 3 units to the right and 2 units down. Label this $\triangle A'B'C'$.
12. Translate $\triangle A'B'C'$ 3 units to the left and 4 units down. Label this $\triangle A''B''C''$.
13. Describe the translation necessary to bring $\triangle A''B''C''$ to $\triangle ABC$.

On a piece of graph paper, plot the points $D(1,5), E(2,3)$ and $F(1,0)$ to form $\triangle ABC$.

14. Translate the triangle 2 units to the left and 4 units down. Label this $\triangle D'E'F'$.
15. Translate $\triangle D'E'F'$ 5 units to the right and 2 units up. Label this $\triangle D''E''F''$.
16. Describe the translation necessary to bring $\triangle D''E''F''$ to $\triangle DEF$.

10.2 Graphs of Translations

Here you will learn how to graph a translation given a description of the translation.

Triangle ABC has coordinates $A(1,1), B(8,1)$ and $C(5,8)$. Draw the triangle on the Cartesian plane. Translate the triangle up 4 units and over 2 units to the right. State the coordinates of the resulting image.

Watch This

First watch this video to learn about graphs of translations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofTranslationsA

Then watch this video to see some examples.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofTranslationsB

Guidance

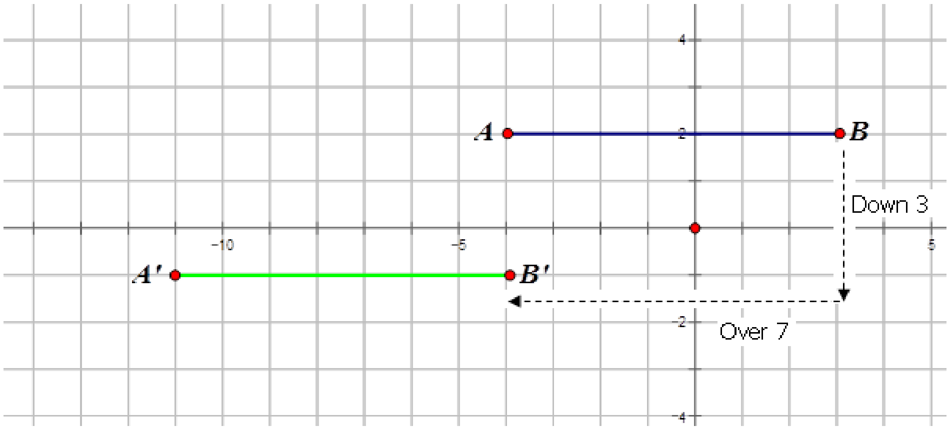
In geometry, a transformation is an operation that moves, flips, or changes a shape (called the preimage) to create a new shape (called the image). A translation is a type of transformation that moves each point in a figure the same distance in the same direction. Translations are often referred to as slides. When you perform a translation on a shape, the coordinates of that shape will change:

- translating up means you will add the translated unit to the y coordinate of the (x,y) points in the preimage
- translating down means you will subtract the translated unit from the y coordinate of the (x,y) points in the preimage
- translating right means you will add the translated unit to the x coordinate of the (x,y) points in the preimage
- translating left means you will subtract the translated unit from the x coordinate of the (x,y) points in the preimage

Example A

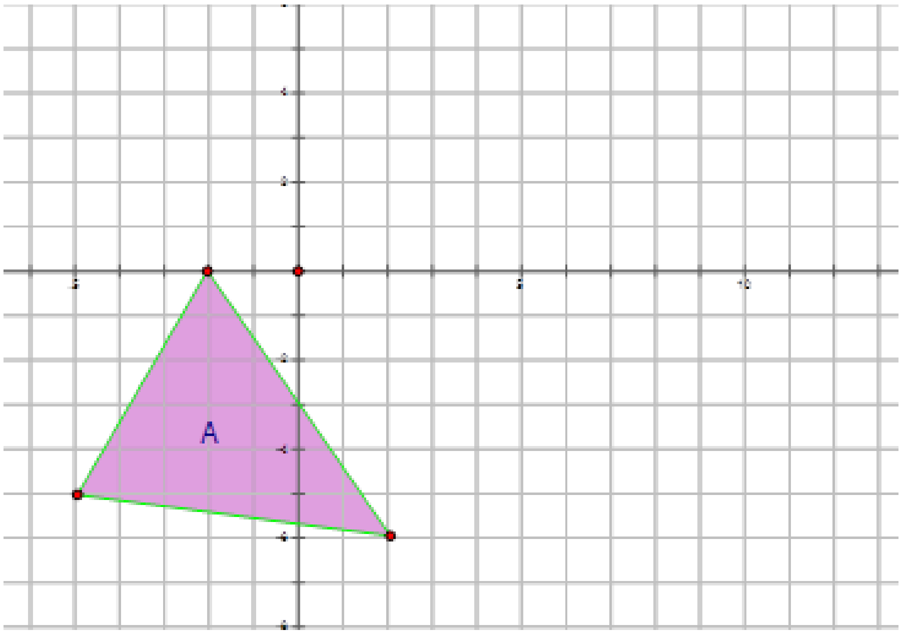
Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been translated 3 units down and 7 units to the left. Draw the preimage and image and properly label each.

Solution:

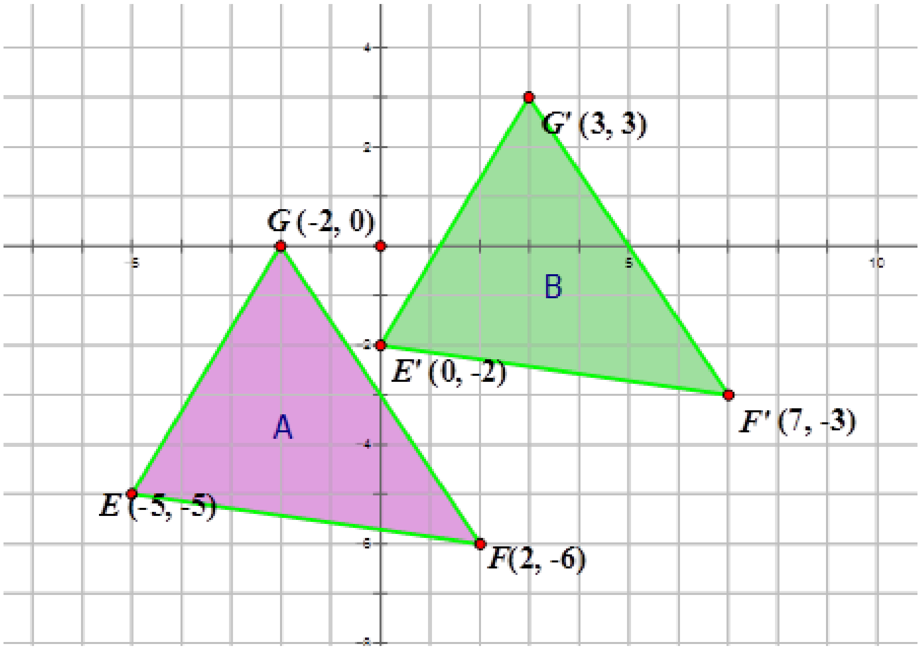


Example B

Triangle A is translated 3 units up and 5 units to the right to make triangle B. Find the coordinates of triangle B. On the diagram, draw and label triangle B.

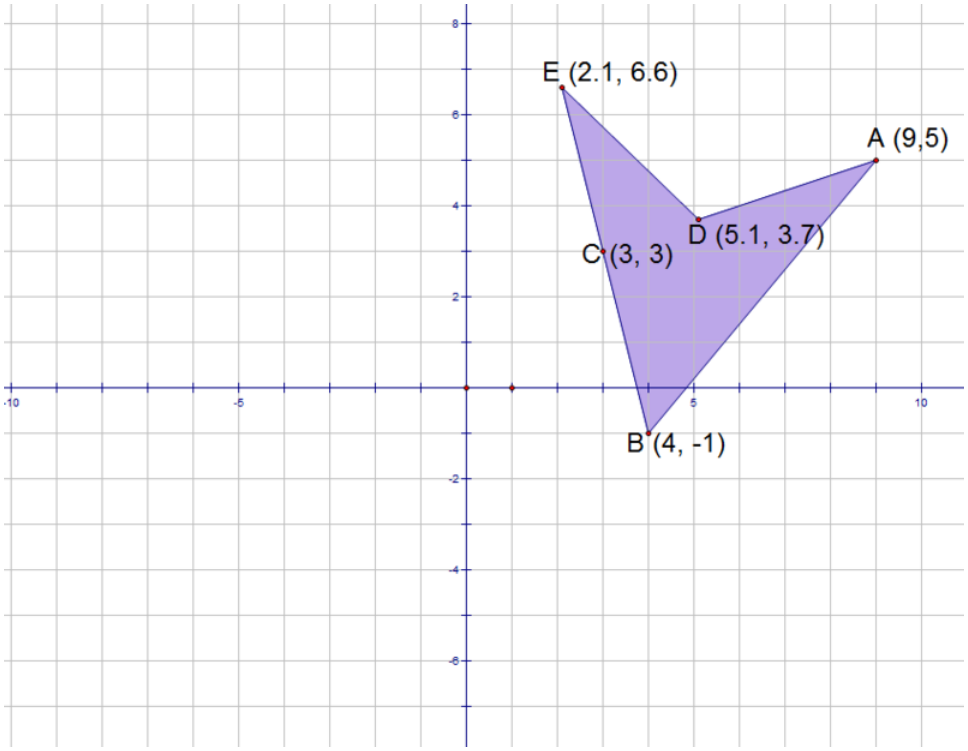


Solution:

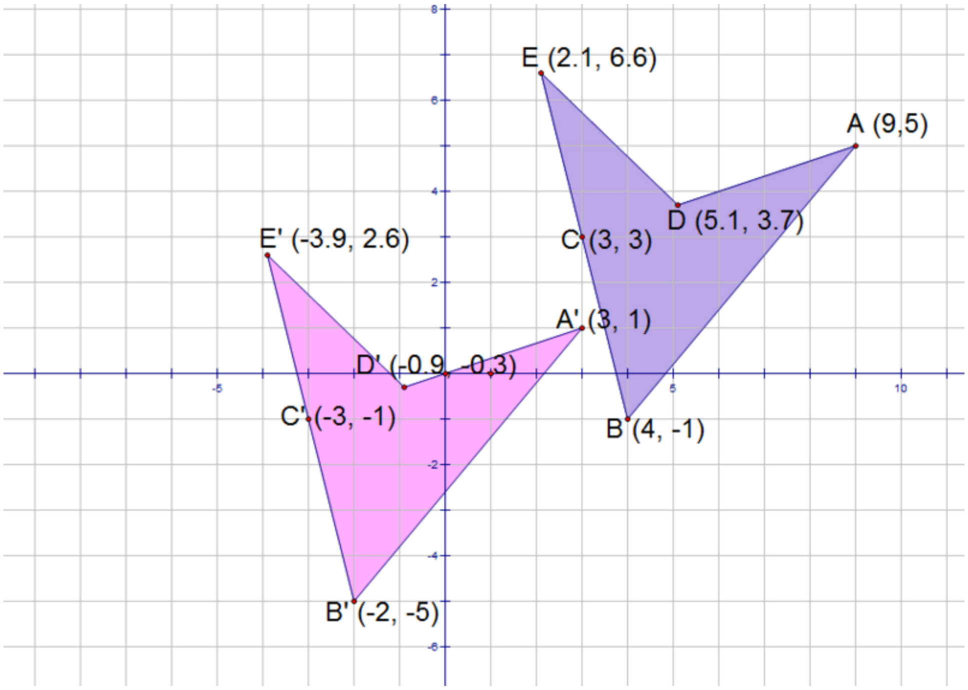


Example C

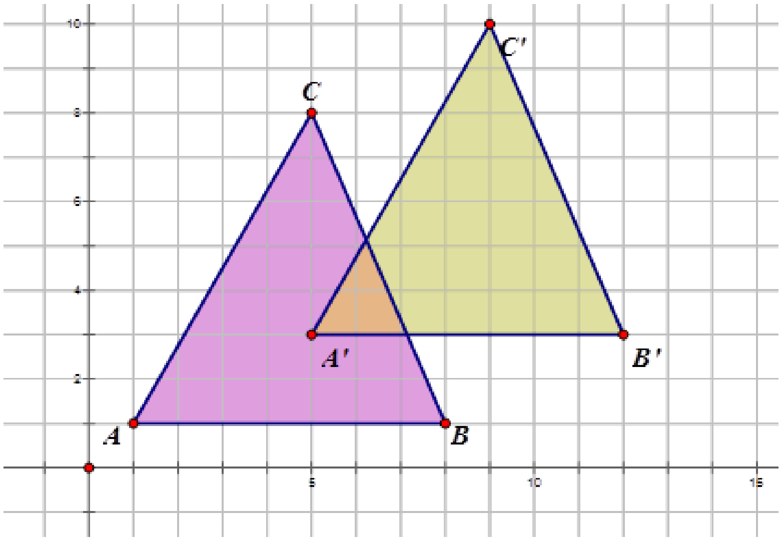
The following figure is translated 4 units down and 6 units to the left to make a translated image. Find the coordinates of the translated image. On the diagram, draw and label the image.



Solution:



Concept Problem Revisited



The coordinates of the new image are $A'(5,3), B'(12,3)$ and $C'(9,10)$.

Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

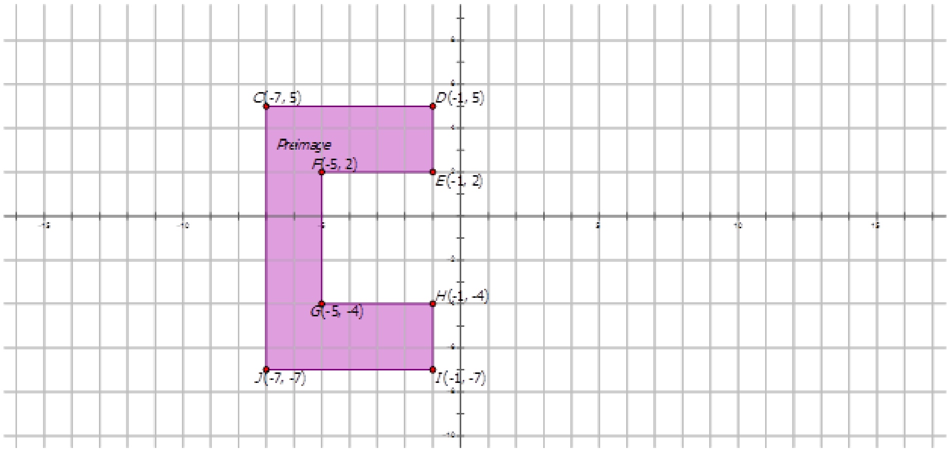
Translation

A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

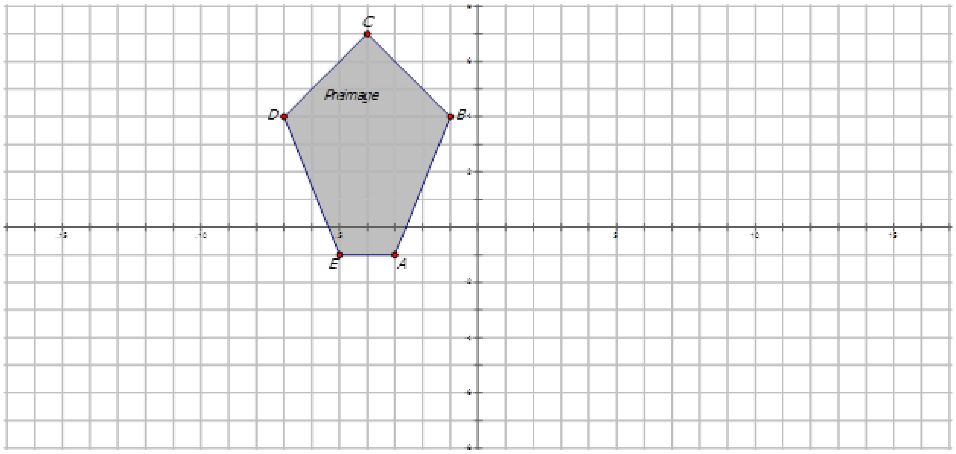
Guided Practice

1. Line $\overline{ST}$ drawn from (-3, -3) to (-3, 8) has been translated 4 units up and 3 units to the right. Draw the preimage and image and properly label each.

2. The polygon below has been translated 3 units down and 10 units to the right. Draw the translated image and properly label each.

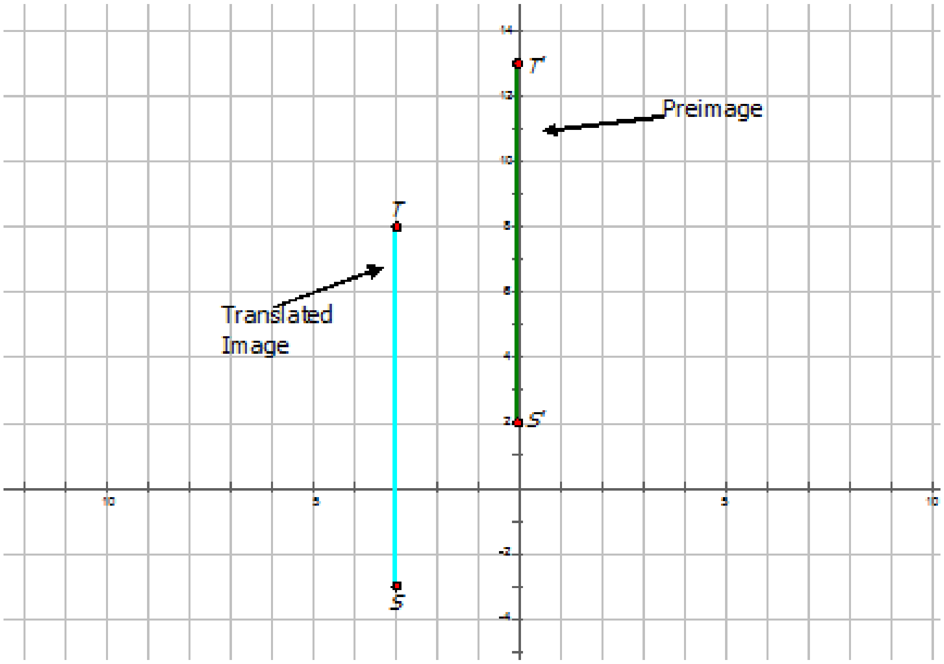


3. The purple pentagon is translated 5 units up and 8 units to the right to make the translated pentagon. Find the coordinates of the purple pentagon. On the diagram, draw and label the translated pentagon.

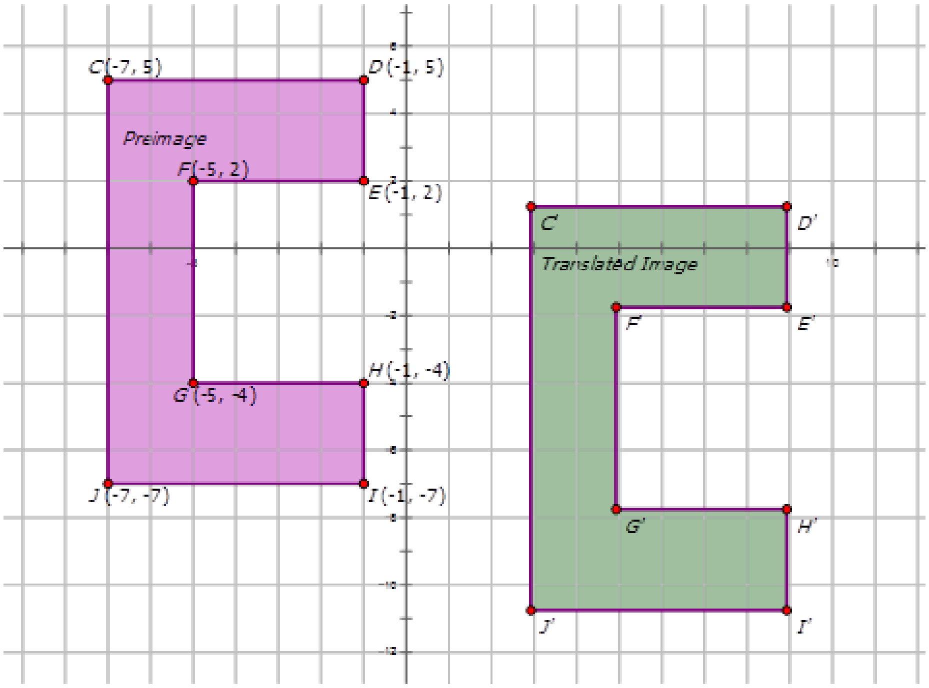


Answers:

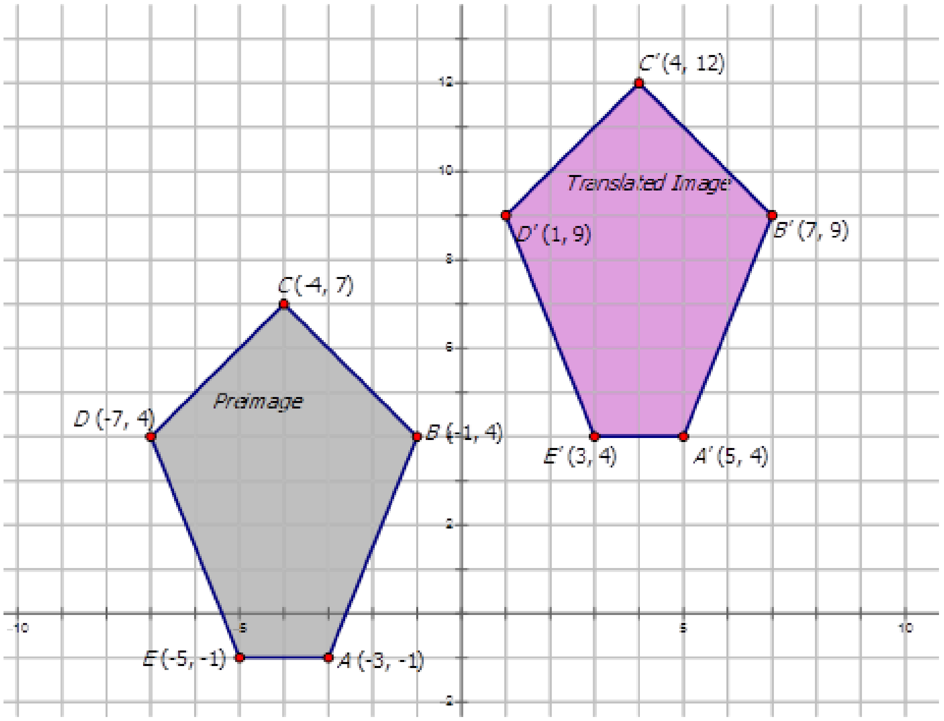
1.


T'
Preimage
T
Translated Image
S'
S

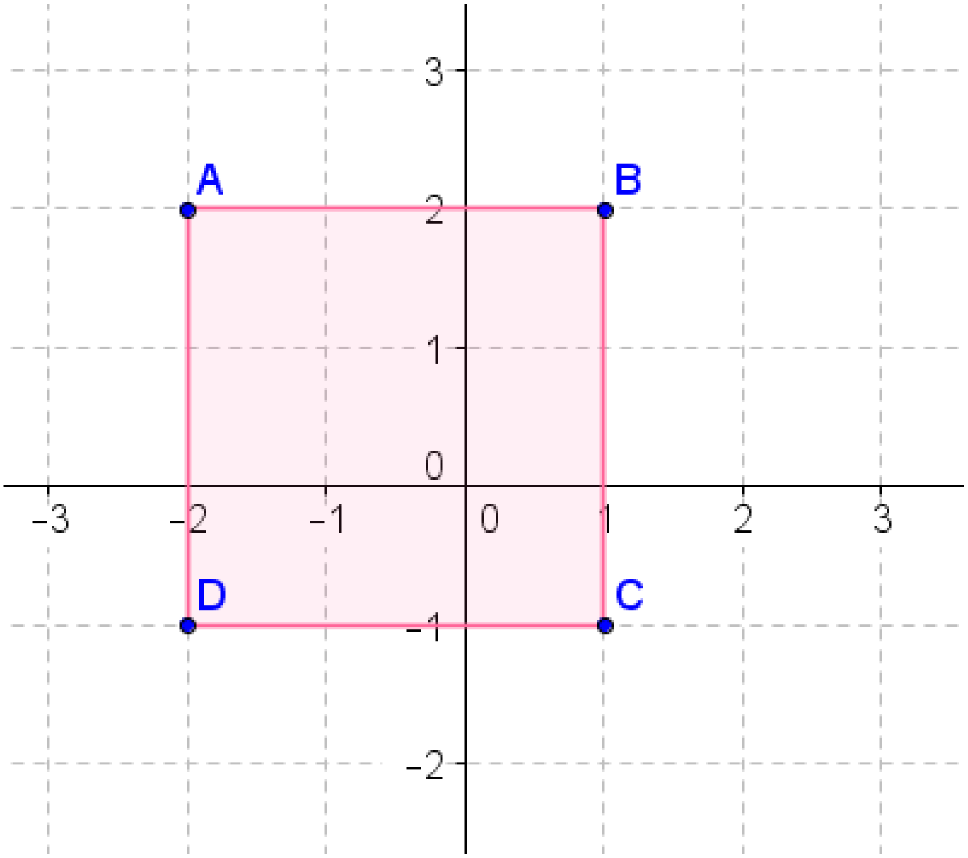
2.


C(-7, 5)
D(-1, 5)
Preimage
F(-5, 2)
E(-1, 2)
C'
D'
Translated Image
F'
E'
H(-1, -4)
G(-5, -4)
J(-7, -7)
I(-1, -7)
G'
H'
J'
I'

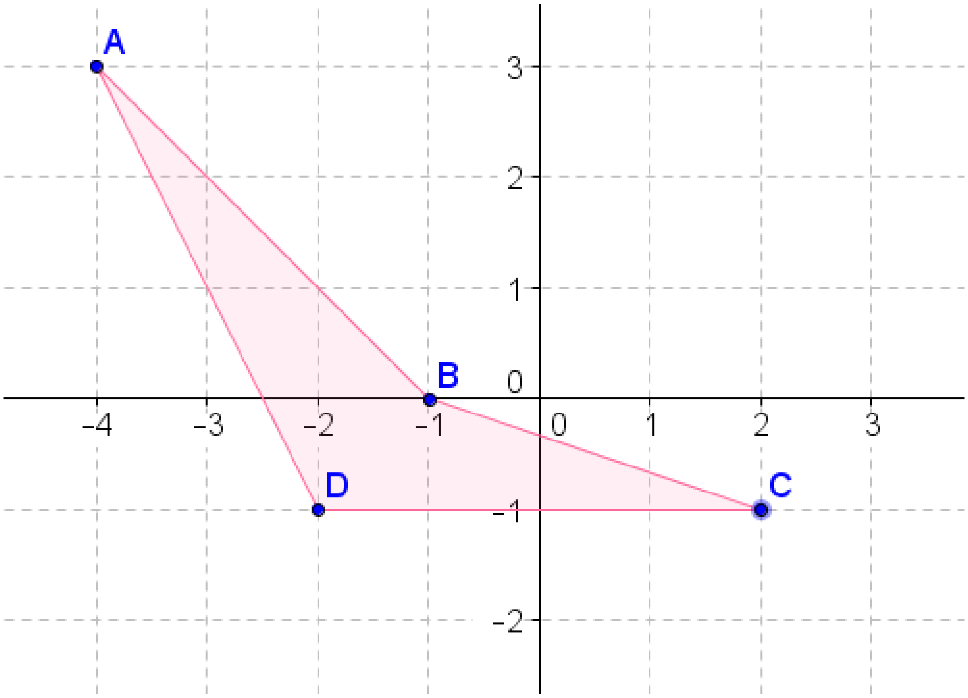
3.



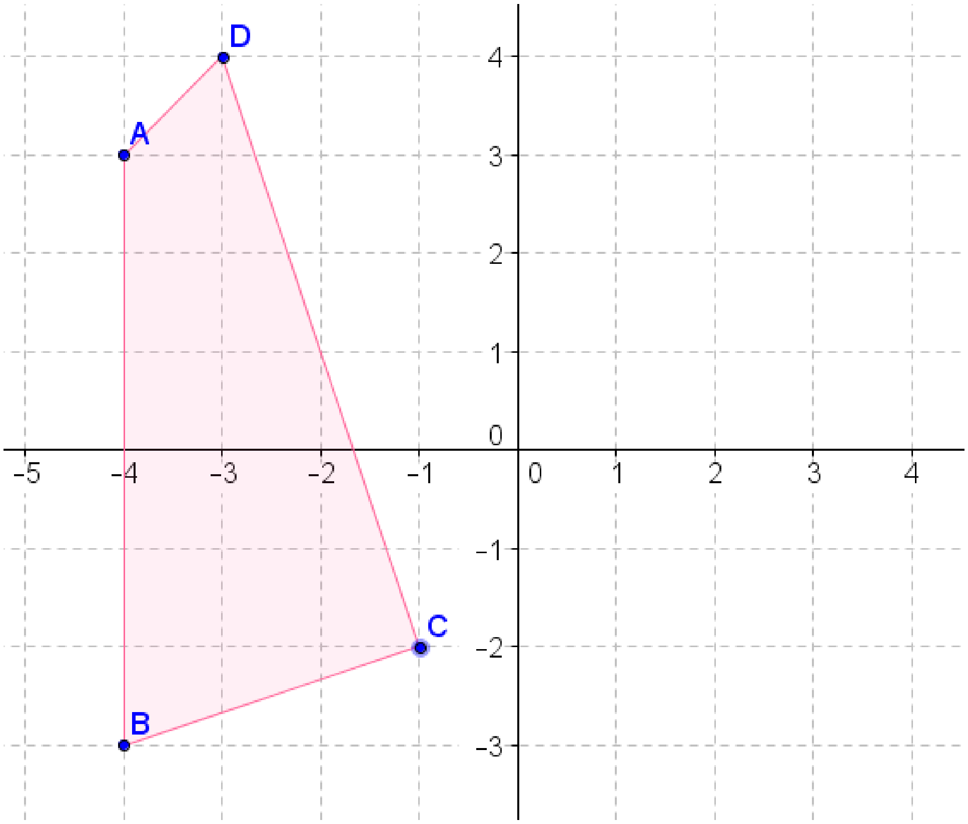
Practice



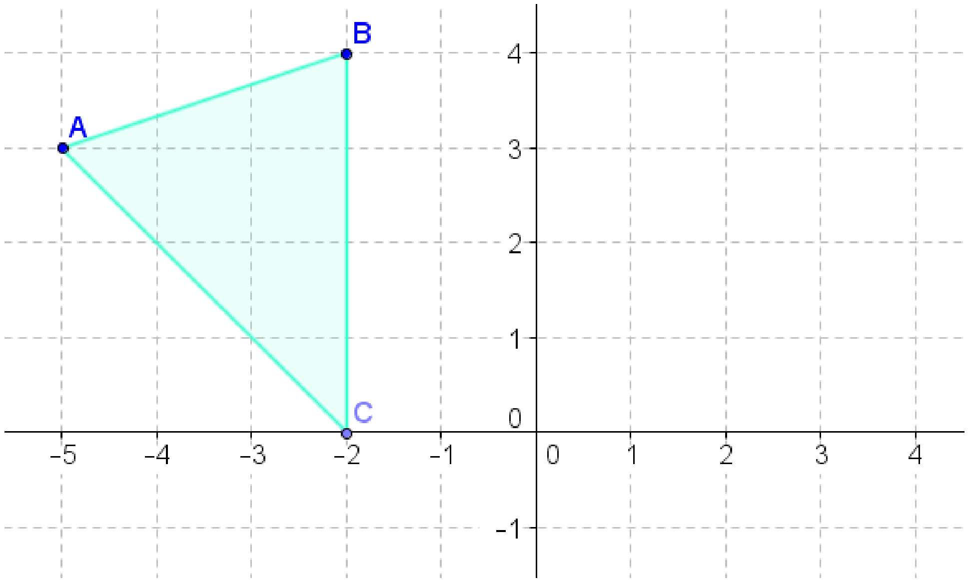
1. Translate the above figure 3 units to the right and 4 units down.
2. Translate the above figure 2 units to the left and 2 units up.



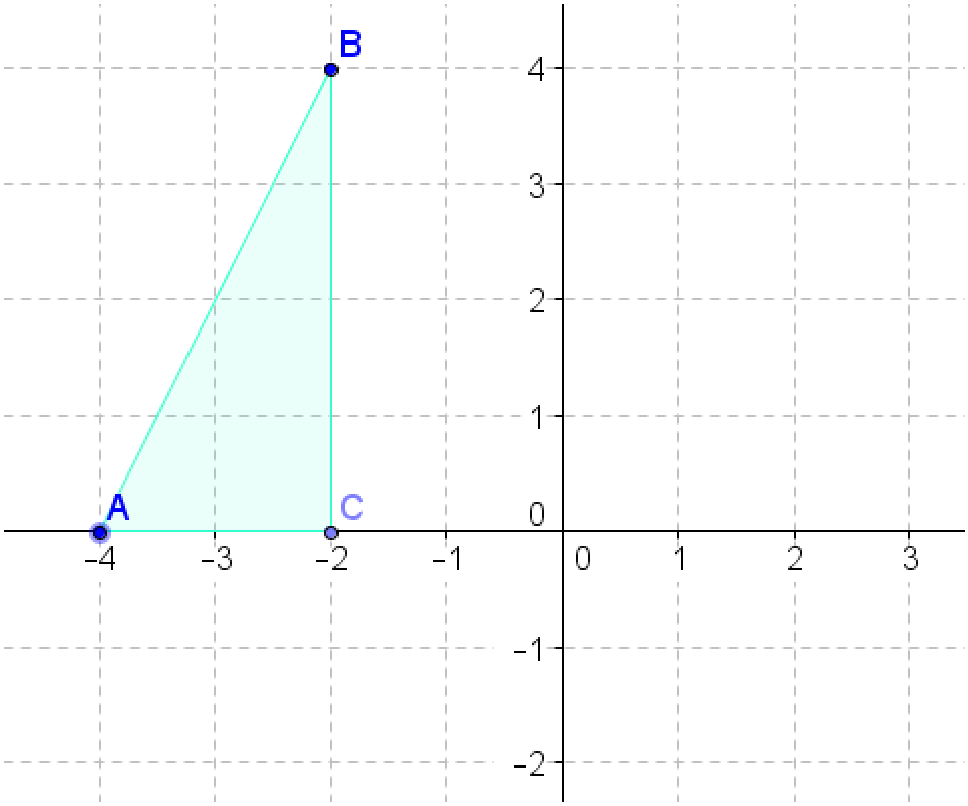
3. Translate the above figure 3 units to the right and 2 units down.
4. Translate the above figure 5 units to the left and 1 unit up.



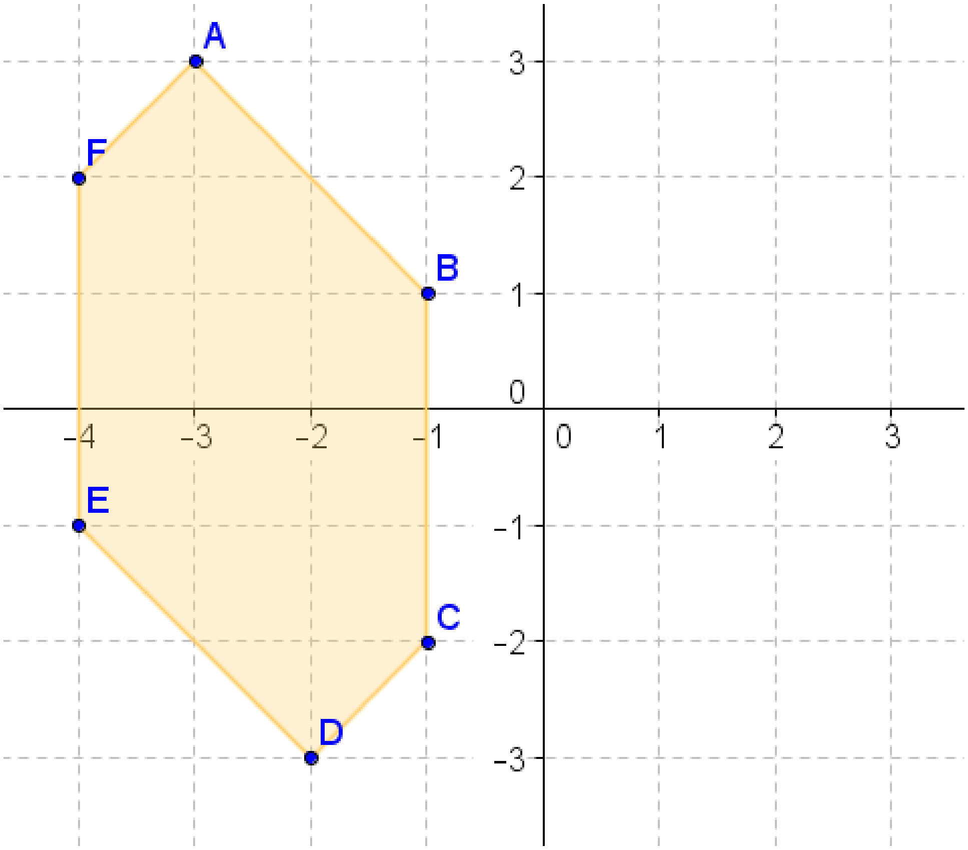
5. Translate the above figure 2 units to the right and 3 units down.
6. Translate the above figure 4 units to the left and 1 unit up.



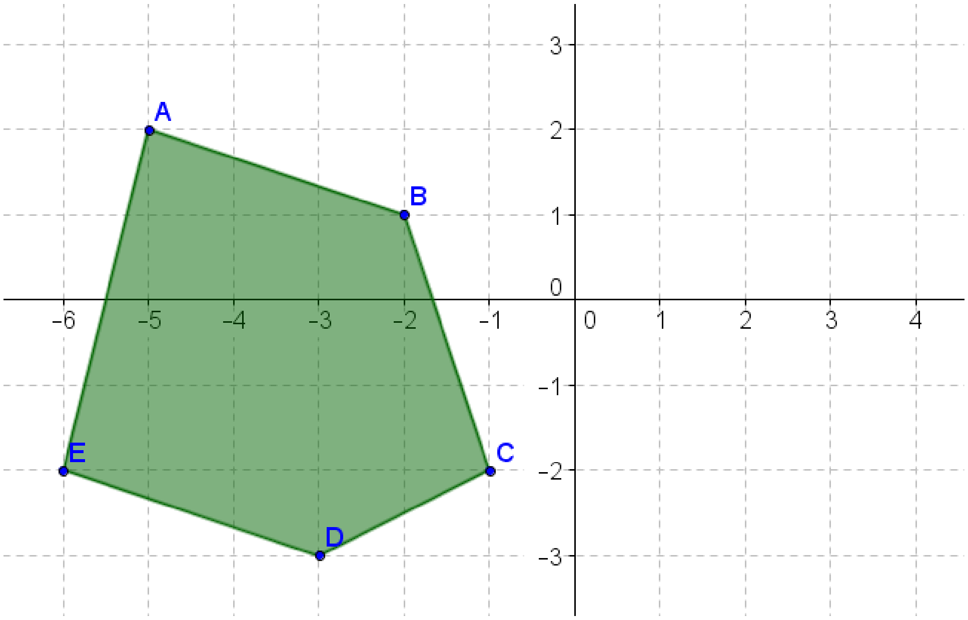
7. Translate the above figure 6 units to the right and 2 units down.
8. Translate the above figure 4 units to the left and 6 units up.



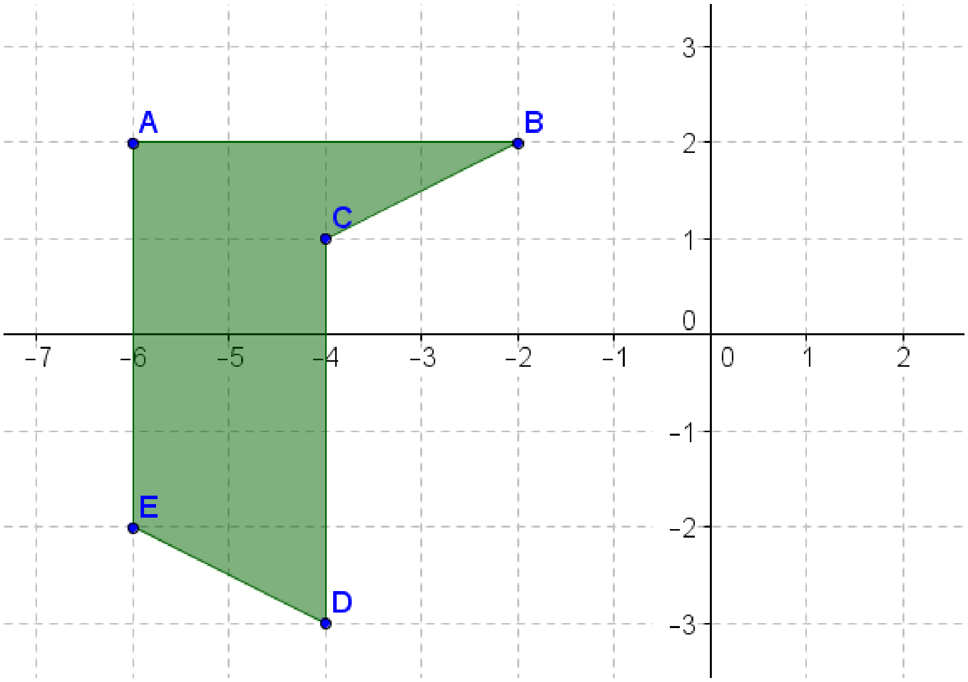
9. Translate the above figure 2 units to the right and 3 units down.
10. Translate the above figure 5 units to the left and 5 units up.



11. Translate the above figure 3 units to the right and 6 units down.
12. Translate the above figure 2 units to the left and 2 units up.



13. Translate the above figure 3 units to the right and 3 units down.
14. Translate the above figure 5 units to the left and 2 units up.

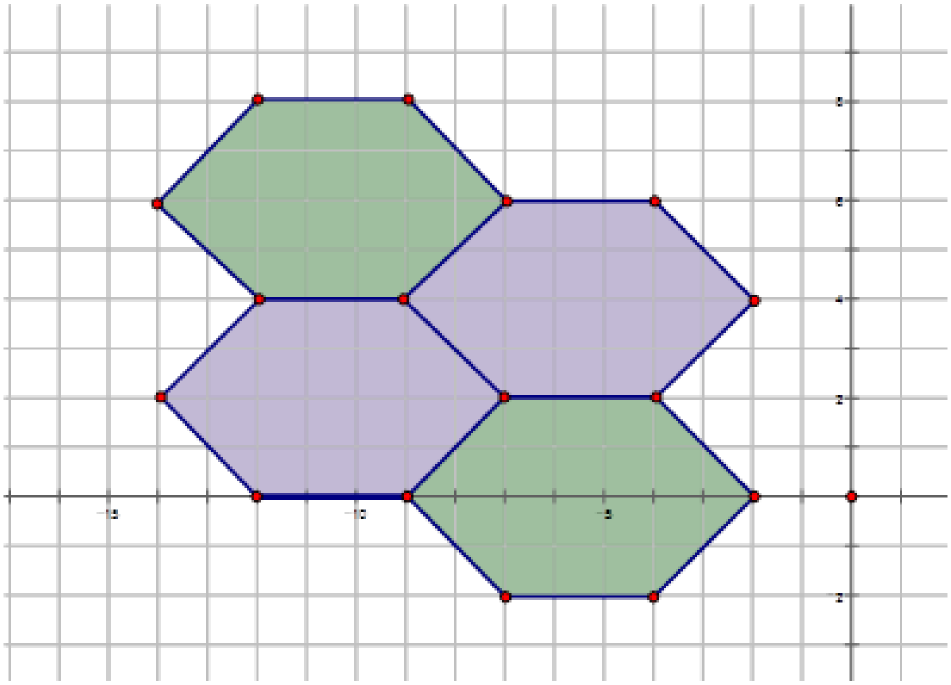


15. Translate the above figure 7 units to the right and 4 units down.
16. Translate the above figure 1 unit to the left and 2 units up.

10.3 Rules for Translations

Here you will learn the different notation used for translations.

The figure below shows a pattern of a floor tile. Write the mapping rule for the translation of the two blue floor tiles.



Watch This

First watch this video to learn about writing rules for translations.

Multimedia

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Click image to the left for more content.

CK-12 Foundation Chapter10RulesforTranslationsA

Then watch this video to see some examples.

Multimedia

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Click image to the left for more content.

CK-12 Foundation Chapter10RulesforTranslationsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape (called the preimage) to create a new shape (called the image). A translation is a type of transformation that moves each point in a figure the same distance in the same direction. Translations are often referred to as slides. You can describe a translation using words like "moved up 3 and over 5 to the left" or with notation. There are two types of notation to know.

1. One notation looks like $T_{(3,\ 5)}$. This notation tells you to add 3 to the x values and add 5 to the y values.
2. The second notation is a mapping rule of the form $(x,y) \rightarrow (x-7,y+5)$. This notation tells you that the x and y coordinates are translated to $x-7$ and $y+5$.

The mapping rule notation is the most common.

Example A

Sarah describes a translation as point P moving from $P(-2,2)$ to $P'(1,-1)$. Write the mapping rule to describe this translation for Sarah.

Solution: In general ,$P(x,y) \rightarrow P'(x+a,y+b)$.

In this case, $P(-2,2) \rightarrow P'(-2+a,2+b)$ or $P(-2,2) \rightarrow P'(1,-1)$

Therefore:

$$\begin{aligned} -2+a &= 1 \quad \textit{and} \quad & 2+b &= -1 \\ a &= 3 & b &= -3 \end{aligned}$$

The rule is:

$$(x,y) \rightarrow (x+3,y-3)$$

Example B

Mikah describes a translation as point D in a diagram moving from $D(1,-5)$ to $D'(-3,1)$. Write the mapping rule to describe this translation for Mikah.

Solution: In general ,$P(x,y) \rightarrow P'(x+a,y+b)$.

In this case, $D(1,-5) \rightarrow D'(1+a,-5+b)$ or $D(1,-5) \rightarrow D'(-3,1)$

Therefore:

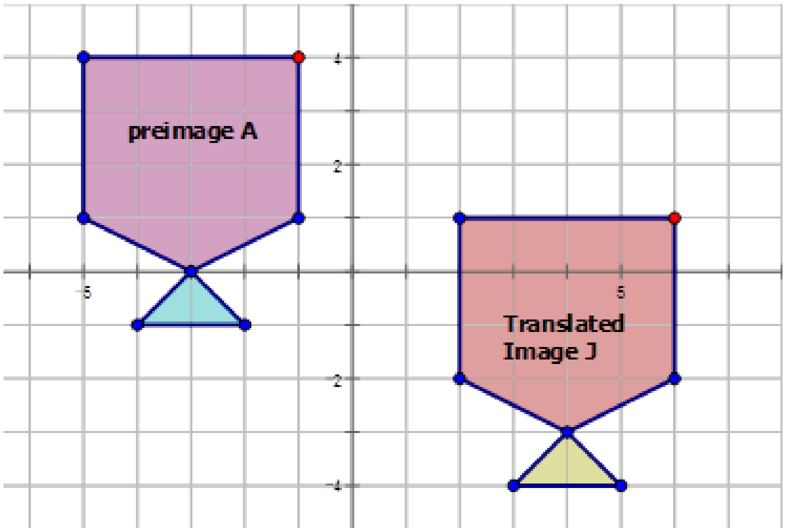
$$\begin{aligned} 1+a &= -3 \quad \textit{and} \quad & -5+b &= 1 \\ a &= -4 & b &= 6 \end{aligned}$$

The rule is:

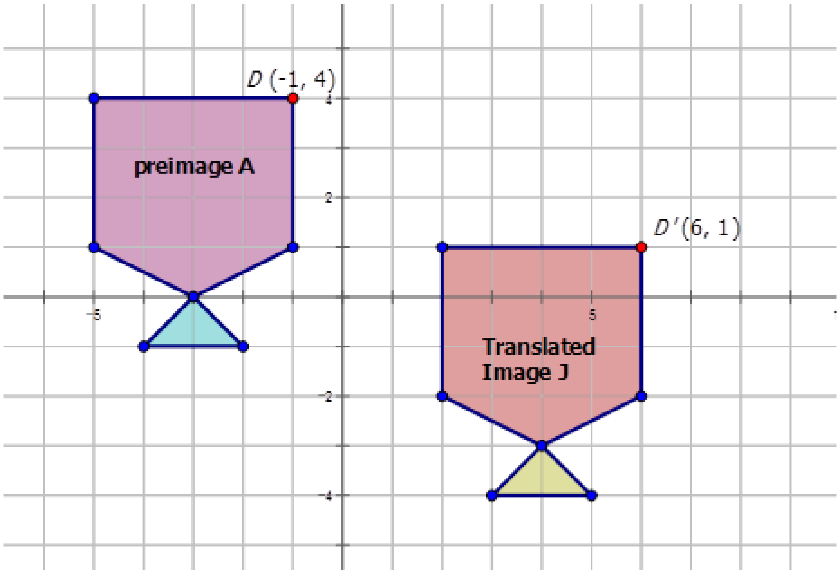
$$(x,y) \rightarrow (x-4,y+6)$$

Example C

Write the mapping rule that represents the translation of the preimage A to the translated image J in the diagram below.



Solution: First, pick a point in the diagram to use to see how it is translated.



$D:(-1,4)\ D':(6,1)$

$D(x,y)\rightarrow D'(x+a,y+b)$

So: $D(-1,4)\rightarrow D'(-1+a,4+b)$ or $D(-1,4)\rightarrow D'(6,1)$

Therefore:

$$\begin{aligned} -1+a&=6 \qquad \text{and} \qquad & 4+b&=1\\ a&=7 & b&=-3 \end{aligned}$$

The rule is:

$$(x,y)\rightarrow(x+7,y-3)$$

Vocabulary

Mapping Rule

A ***mapping rule*** has the following form $(x,y) \rightarrow (x-7,y+5)$ and tells you that the x and y coordinates are translated to $x-7$ and $y+5$.

Translation

A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

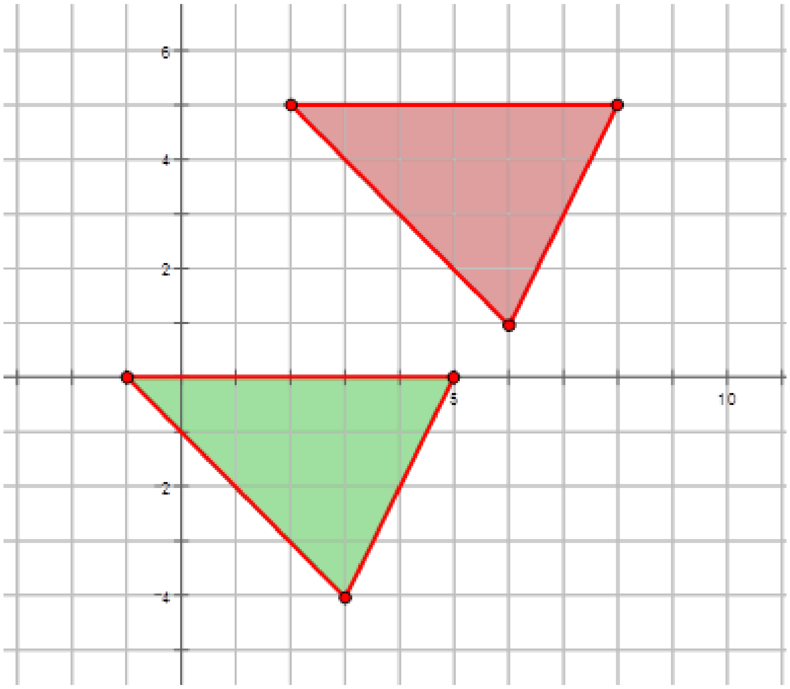
Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

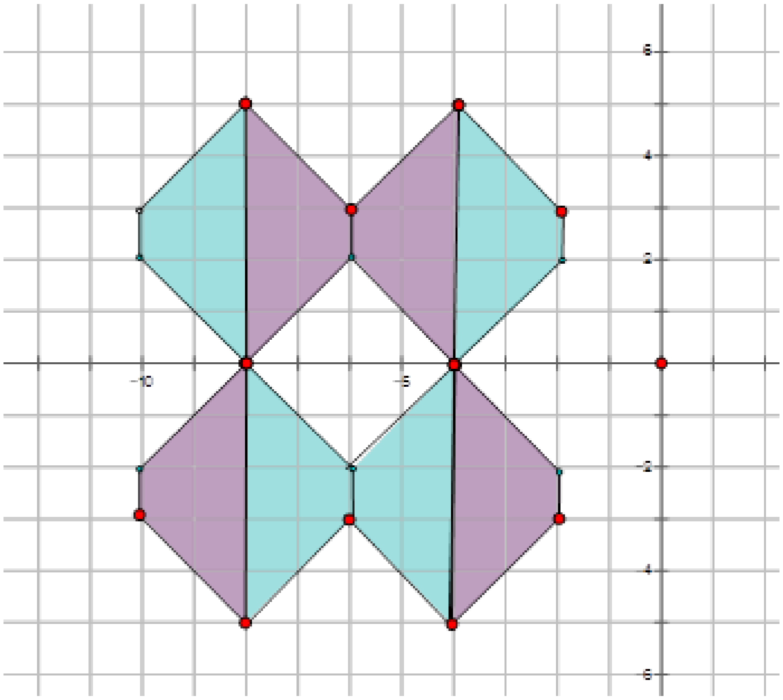
Guided Practice

1. Jack describes a translation as point J moving from $J(-2,6)$ to $J'(4,9)$. Write the mapping rule to describe this translation for Jack.

2. Write the mapping rule that represents the translation of the red triangle to the translated green triangle in the diagram below.



3. The following pattern is part of wallpaper found in a hotel lobby. Write the mapping rule that represents the translation of one blue trapezoid to a translated blue trapezoid shown in the diagram below.



Answers:

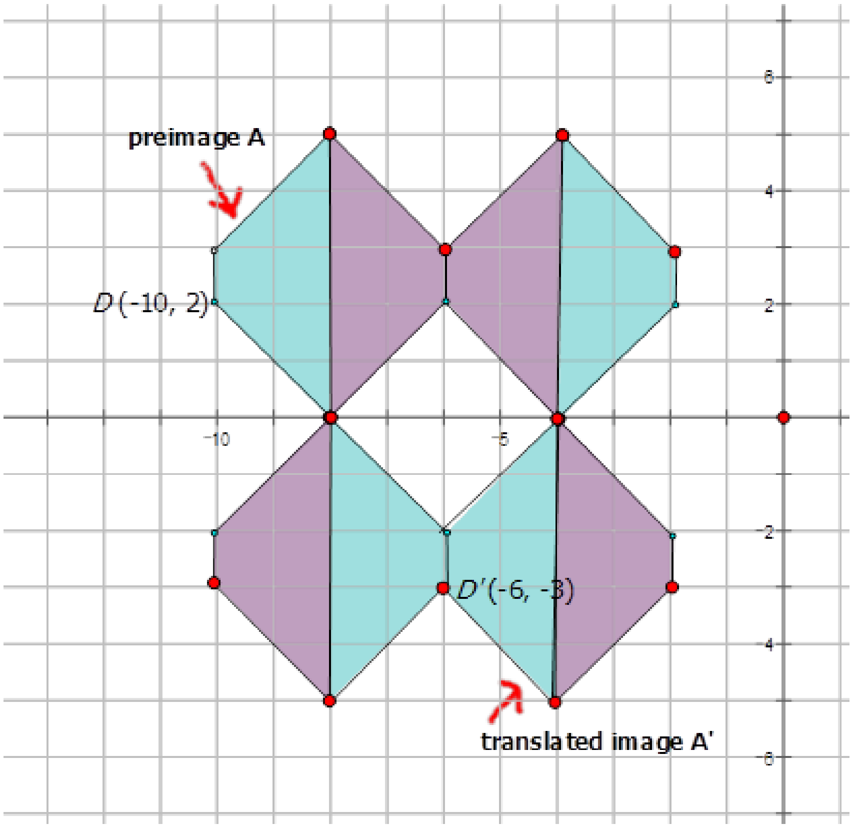
1.

$$(x,y) \rightarrow (x+6,y+3)$$

2.

$$(x,y) \rightarrow (x-3,y-5)$$

3. If you look closely at the diagram below, there two pairs of trapezoids that are translations of each other. Therefore you can choose one blue trapezoid that is a translation of the other and pick a point to find out how much the shape has moved to get to the translated position.



For those two trapezoids:

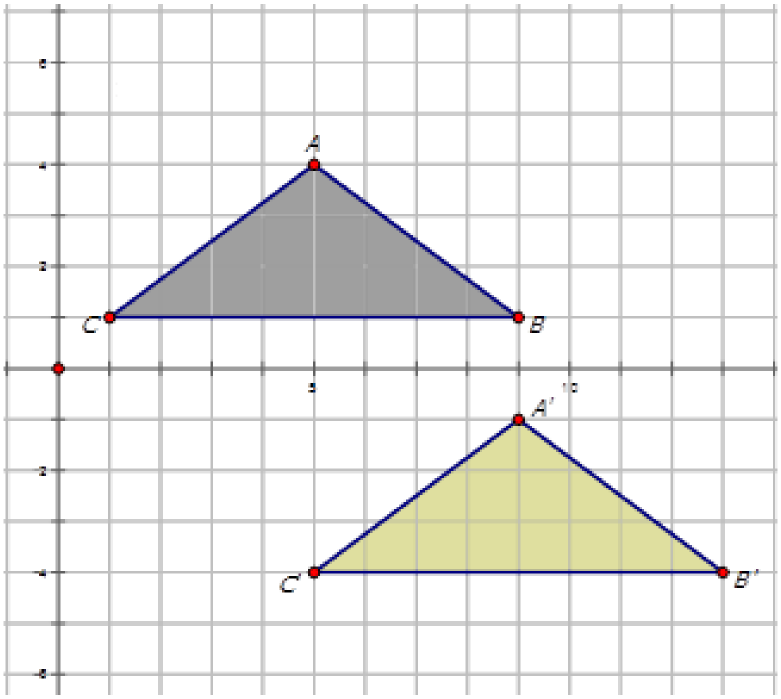
$$(x,y) \rightarrow (x+4, y-5)$$

Practice

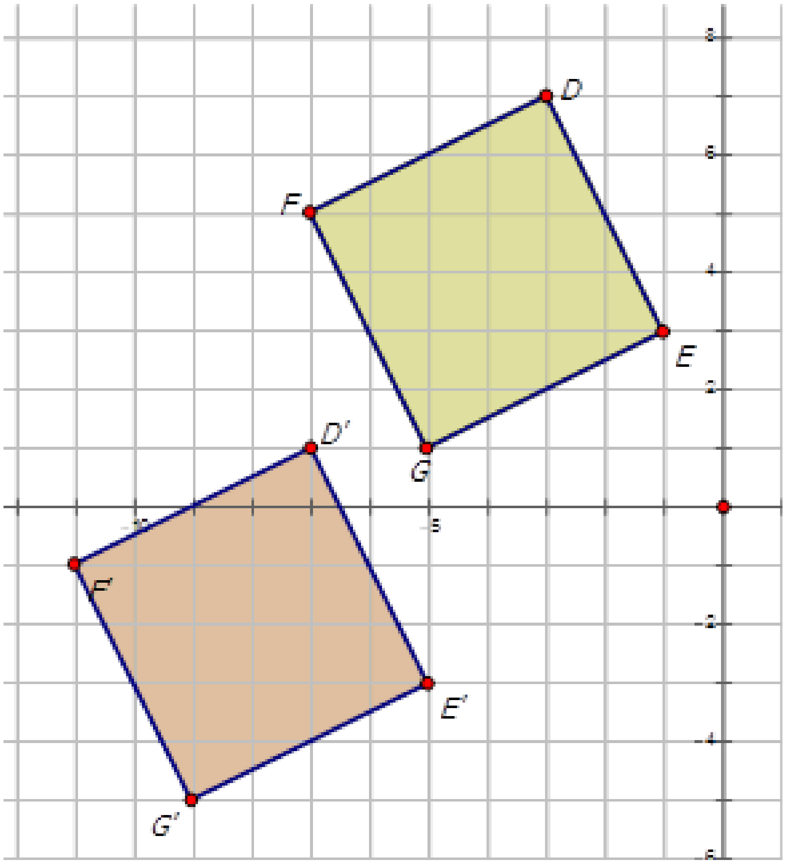
Write the mapping rule to describe the movement of the points in each of the translations below.

1. $S(1,5) \rightarrow S'(2,7)$
2. $W(-5,-1) \rightarrow W'(-3,1)$
3. $Q(2,-5) \rightarrow Q'(-6,3)$
4. $M(4,3) \rightarrow M'(-2,9)$
5. $B(-4,-2) \rightarrow B'(2,-2)$
6. $A(2,4) \rightarrow A'(2,6)$
7. $C(-5,-3) \rightarrow C'(-3,4)$
8. $D(4,-1) \rightarrow D'(-4,2)$
9. $Z(7,2) \rightarrow Z'(-3,6)$
10. $L(-3,-2) \rightarrow L'(4,-1)$

Write the mapping rule that represents the translation of the preimage to the image for each diagram below.

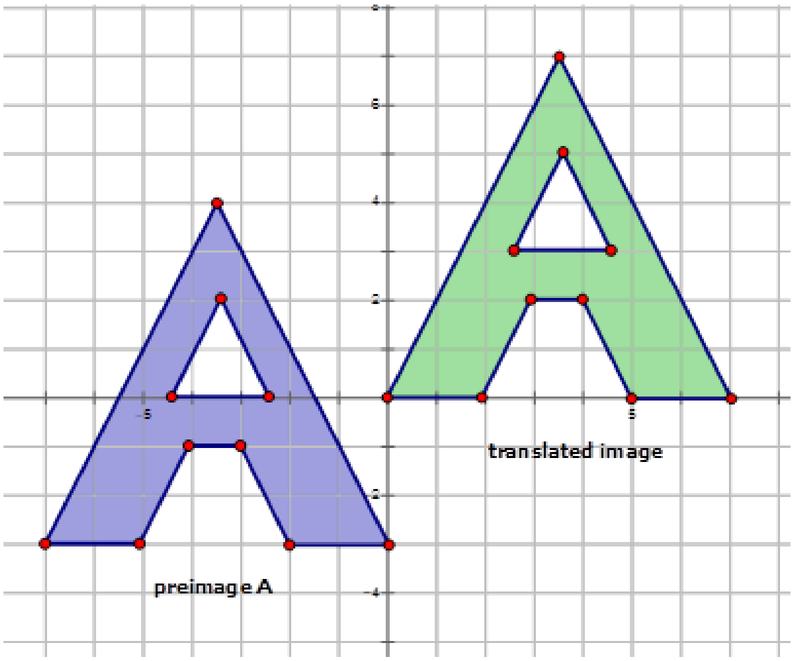


11.

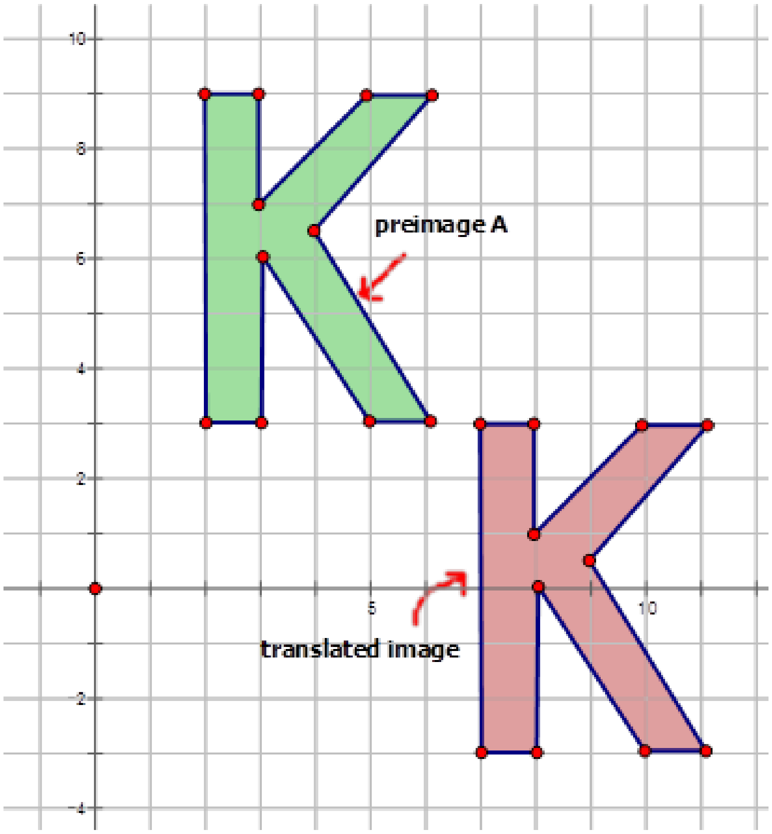


12.

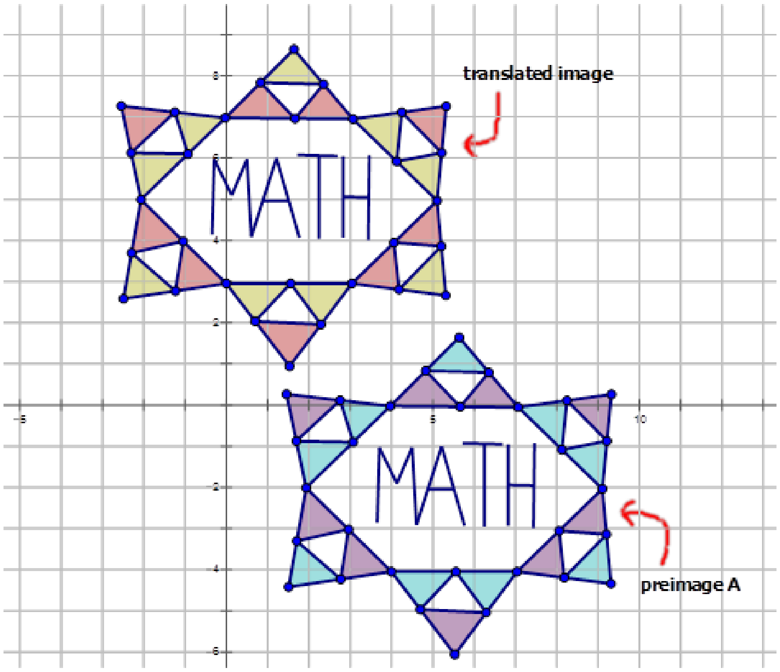
13.


preimage A
translated image
-5
5
2
4
6
-2
-4

14.


preimage A
translated image
10
8
6
4
2
-2
-4
5
10

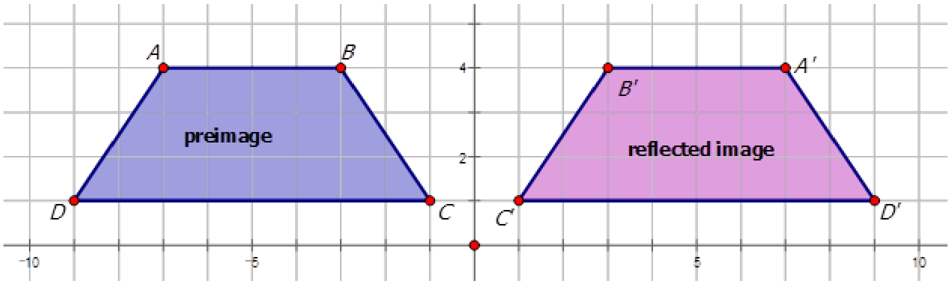
15.



10.4 Reflections

Here you'll learn about geometric reflections.

Scott looked at the image below and stated that the image was reflected about the y-axis. Is he correct? Explain.



Watch This

First watch this video to learn about reflections.

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Click image to the left for more content.

CK-12 Foundation Chapter10ReflectionsA

Then watch this video to see some examples.

Multimedia

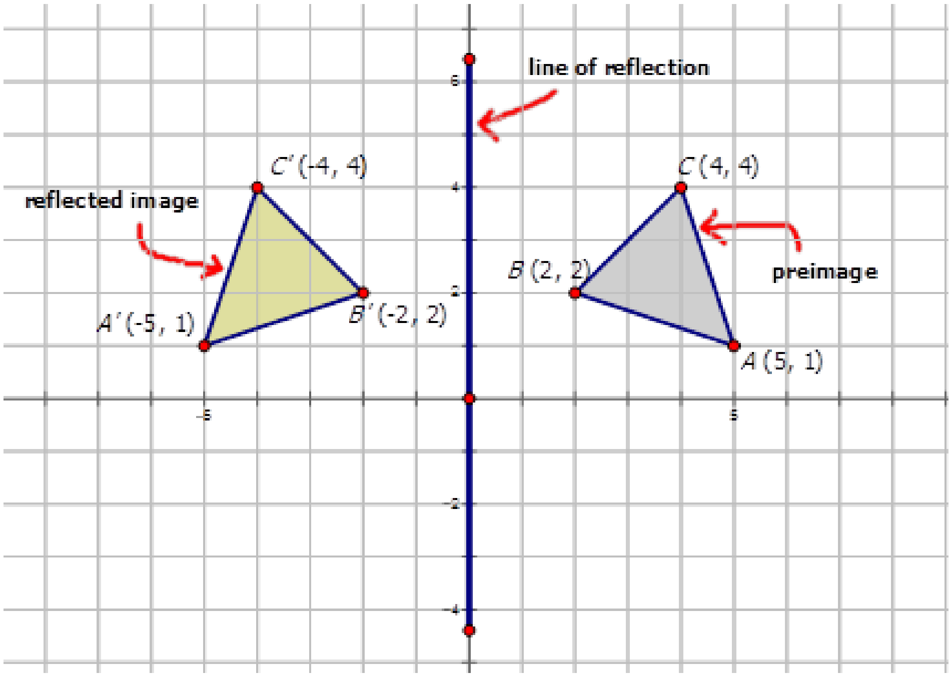
MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10ReflectionsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image).

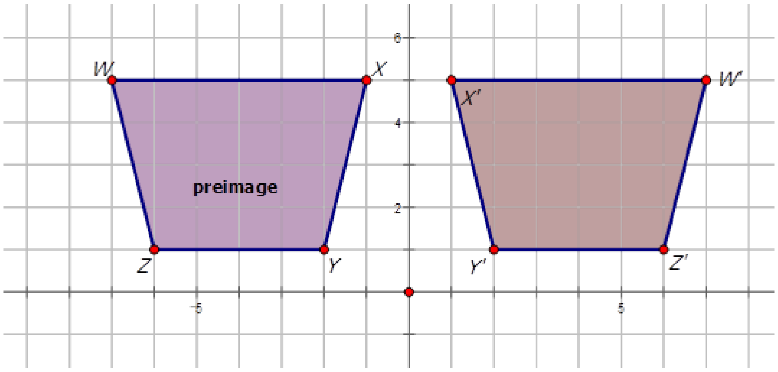


You can reflect a shape across any line, but the most common reflections are the following:

- reflections across the x-axis: y values are multiplied by -1.
- reflections across the y-axis: x values are multiplied by -1.
- reflections across the line $y = x$: x and y values switch places.
- reflections across the line $y = -x$. x and y values switch places and are multiplied by -1.

Example A

Describe the reflection shown in the diagram below.



Solution: The shape is reflected across the y-axis. Let's examine the points of the shapes.

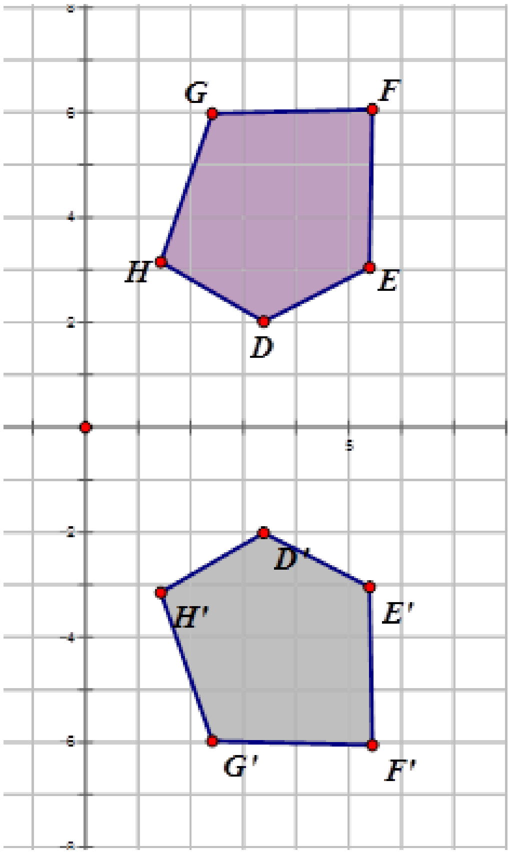
TABLE 10.1:

Points on $WXYZ$	$W(-7,5)$	$X(-1,5)$	$Y(-2,1)$	$Z(-6,1)$
Points on $W'X'Y'Z'$	$W'(7,8)$	$X'(1,5)$	$Y'(2,1)$	$Z'(6,1)$

In the table above, all of the x-coordinates are multiplied by -1. Whenever a shape is reflected across the y-axis, it's x-coordinates will be multiplied by -1.

Example B

Describe the reflection of the purple pentagon in the diagram below.



Solution: The pentagon is reflected across the x-axis. Let's examine the points of the pentagon.

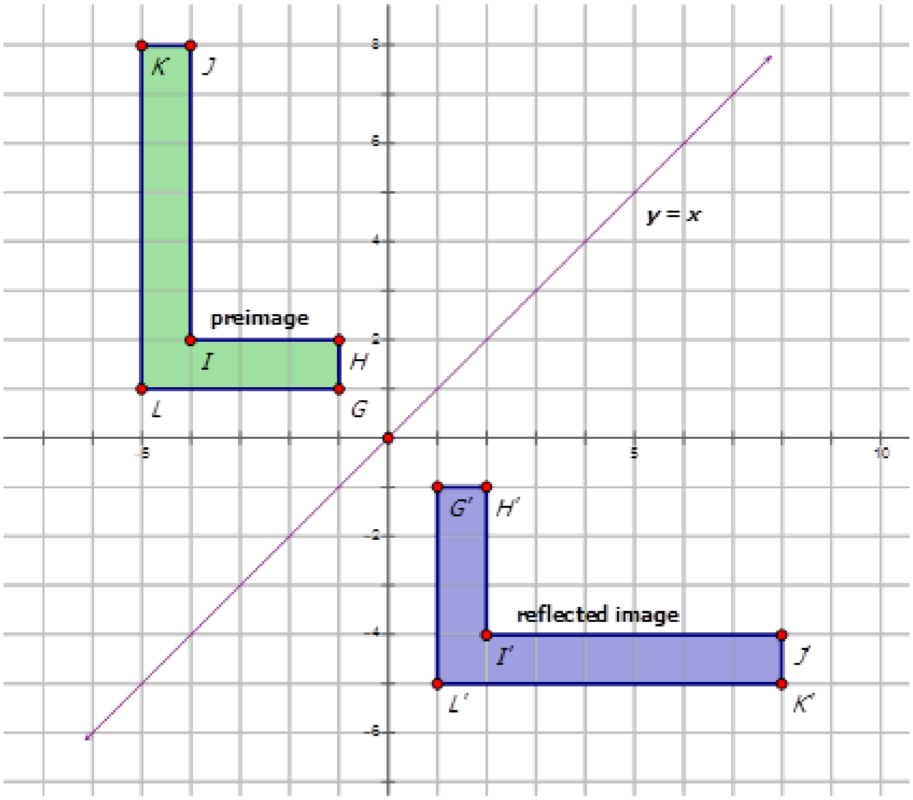
TABLE 10.2:

Points on $DEFGH$	$D(3.5,2)$	$E(5.4,3)$	$F(5.5,6)$	$G(2.3,6)$	$H(1.4,3.2)$
Points on $D'E'F'G'H'$	$D'(3.5,-2)$	$E'(5.4,-3)$	$F'(5.5,-6)$	$G'(2.3,-6)$	$H'(1.4,-3.2)$

In the table above, all of the x-coordinates are the same but the y-coordinates are multiplied by -1. This is what will happen anytime a shape is reflected across the x-axis.

Example C

Describe the reflection in the diagram below.



Solution: The shape is reflected across the line $y = x$. Let's examine the points of the preimage and the reflected image.

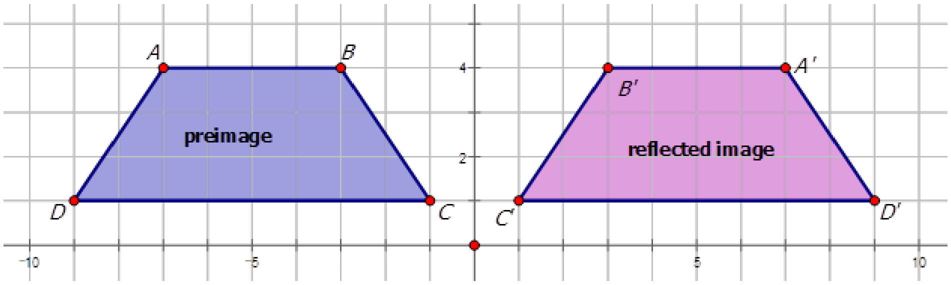
TABLE 10.3:

Points on $GHIJKL$	$G(-1,1)$	$H(-1,2)$	$I(-4,2)$	$J(-4,8)$	$K(-5,8)$	$L(-5,1)$
Points on $G'H'I'J'K'L'$	$G'(1,-1)$	$H'(2,-1)$	$I'(2,-4)$	$J'(8,-4)$	$K'(8,-5)$	$L'(1,-5)$

Notice that all of the points on the preimage reverse order (or interchange) to form the corresponding points on the reflected image. So for example the point G on the preimage is at (-1, 1) but the corresponding point G' on the reflected image is at (1, -1). The x values and the y values change places anytime a shape is reflected across the line $y = x$.

Concept Problem Revisited

Scott looked at the image below and stated that the image was reflected across the y-axis. Is he correct? Explain.



Scott is correct in that the preimage is reflected about the y-axis to form the translated image. You can tell this because all points are equidistant from the line of reflection. Let's examine the points of the trapezoid and see.

TABLE 10.4:

Point for $ABCD$	**Point for** $A'B'C'D'$
$A(-7,4)$	$A'(7,4)$
$B(-3,4)$	$B'(3,4)$
$C(-1,1)$	$C'(1,1)$
$D(-9,1)$	$D'(9,1)$

All of the y-coordinates for the reflected image are the same as their corresponding points in the preimage. However, the x-coordinates have been multiplied by -1.

Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

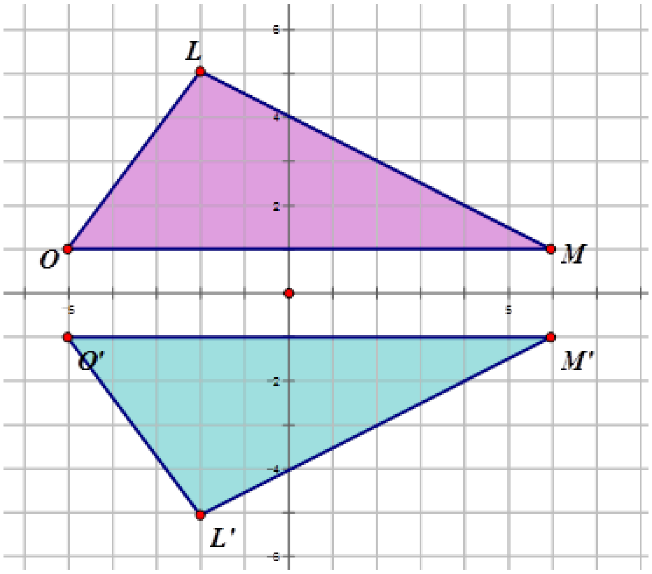
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Reflection

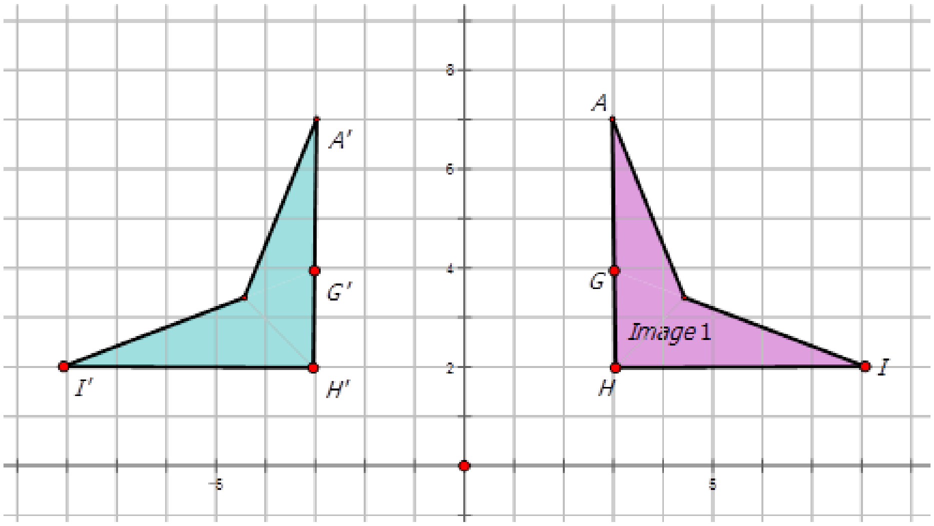
A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

Guided Practice

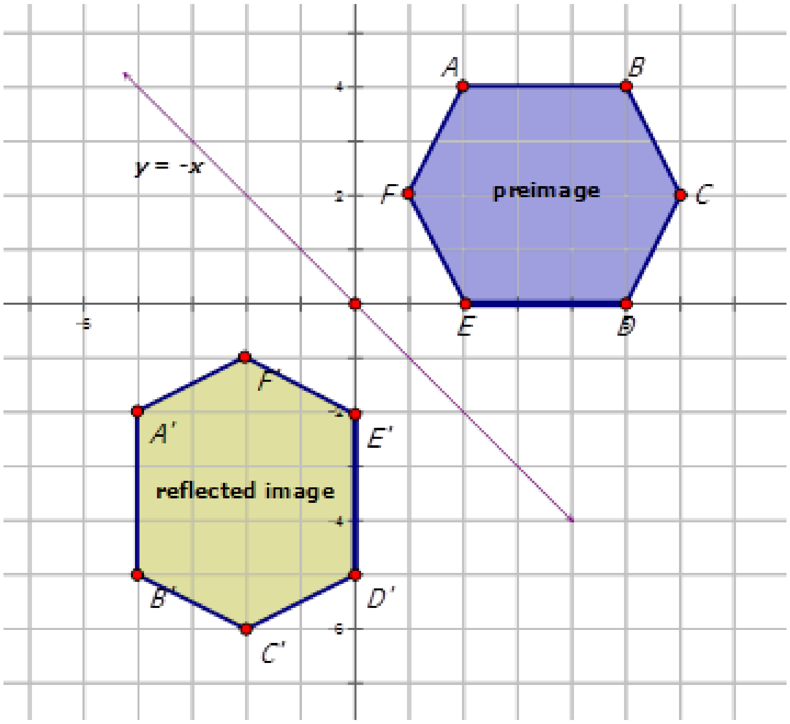
1. Describe the reflection of the pink triangle in the diagram below.



2. Describe the reflection of the purple polygon in the diagram below.



3. Describe the reflection of the blue hexagon in the diagram below.



Answers:

1. Examine the points of the preimage and the reflected image.

TABLE 10.5:

Points on *LMO*	$L(-2,5)$	$M(6,1)$	$O(-5,1)$
Points on $L'M'O'$	$L'(-2,-5)$	$M'(6,-1)$	$O'(-5,-1)$

Notice that all of the *y*-coordinates of the preimage (purple triangle) are multiplied by -1 to make the reflected image. The line of reflection is the *x*-axis.

2. Examine the points of the preimage and the reflected image.

TABLE 10.6:

Points on *AGHI*	$A(3,7)$	$G(3,4)$	$H(3,2)$	$I(8,2)$
Points on $A'G'H'I'$	$A'(-3,7)$	$G'(-3,4)$	$H'(-3,2)$	$I'(-8,2)$

Notice that all of the *x*-coordinates of the preimage (image 1) is multiplied by -1 to make the reflected image. The line of reflection is the *y*-axis.

3. Examine the points of the preimage and the reflected image.

TABLE 10.7:

Points on *ABCDEF*	$A(2,4)$	$B(5,4)$	$C(6,2)$	$D(5,0)$	$E(2,0)$	$F(1,2)$

TABLE 10.7: (continued)

Points on $A'B'C'D'E'F'$	$A'(-4,-2)$	$B'(-4,-5)$	$C'(-2,-6)$	$D'(0,-5)$	$E'(0,-2)$	$F'(-2,-1)$

Notice that both the x-coordinates and the y-coordinates of the preimage (image 1) change places to form the reflected image. As well the points are multiplied by -1. The line of reflection is the line $y = -x$.

Practice

If the following points were reflected across the x-axis, what would be the coordinates of the reflected points? Show these reflections on a graph.

1. (3, 1)
2. (4, -2)
3. (-5, 3)
4. (-6, 4)

If the following points were reflected across the y-axis, what would be the coordinates of the reflected points? Show these reflections on a graph.

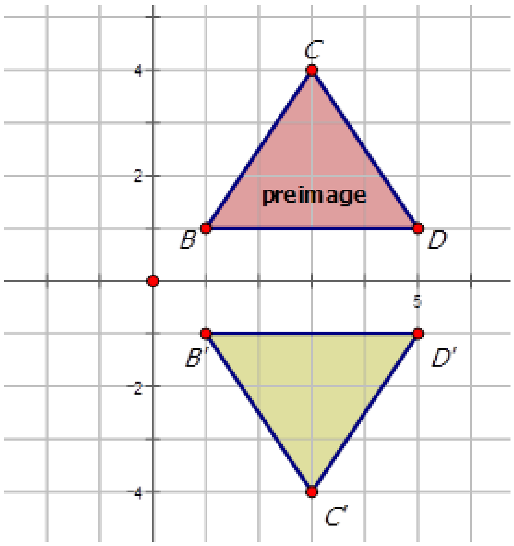
5. (-4, 3)
6. (5, -4)
7. (-5, -4)
8. (3, 3)

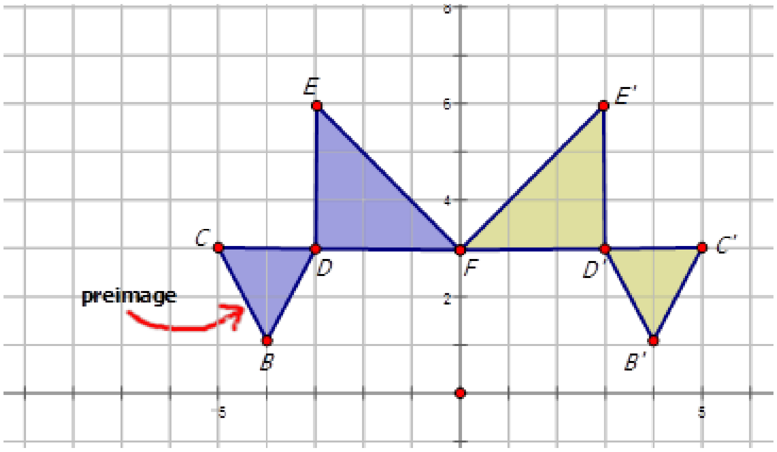
If the following points were reflected about the line $y = x$, what would be the coordinates of the reflected points? Show these reflections on a graph.

9. (3, 1)
10. (4, -2)
11. (-5, 3)
12. (-6, 4)

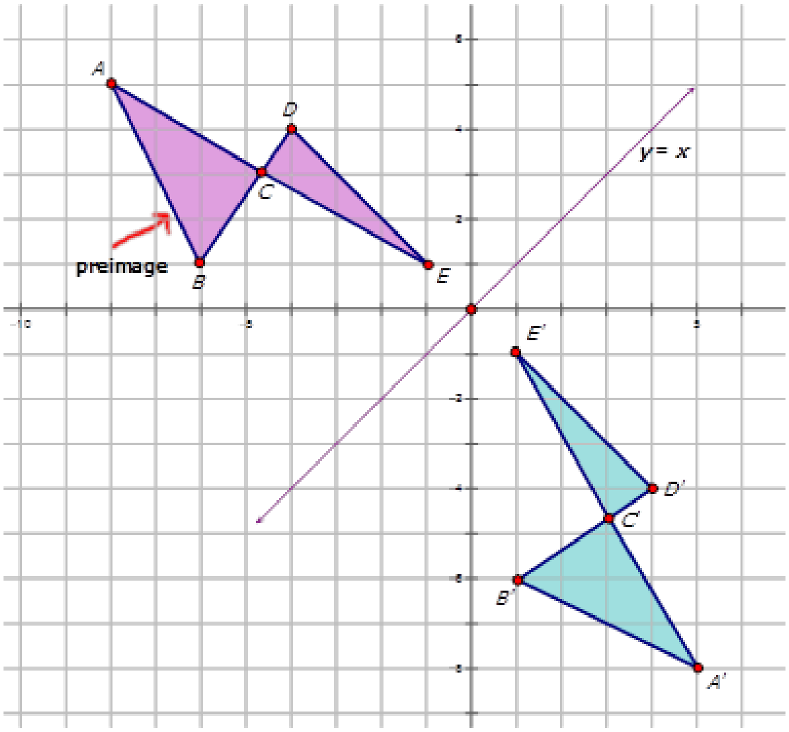
Describe the following reflections:

13.




E
E'
6
4
C
C'
D
F
D'
preimage
2
B
B'
-5
5

14.


6
A
D
4
y = x
C
2
preimage
B
E
-10
-5
5
E'
-2
-4
D'
C'
-6
B'
-8
A'

15.

10.5 Graphs of Reflections

Here you will learn how to reflect an image on a coordinate grid.

Triangle A has coordinates $E(-5,-5)$, $F(2,-6)$ and $G(-2,0)$. Draw the triangle on the Cartesian plane. Reflect the image across the y-axis. State the coordinates of the resulting image.

Watch This

First watch this video to learn about graphs of reflections.

Multimedia

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofReflectionsA

Then watch this video to see some examples.

Multimedia

MEDIA

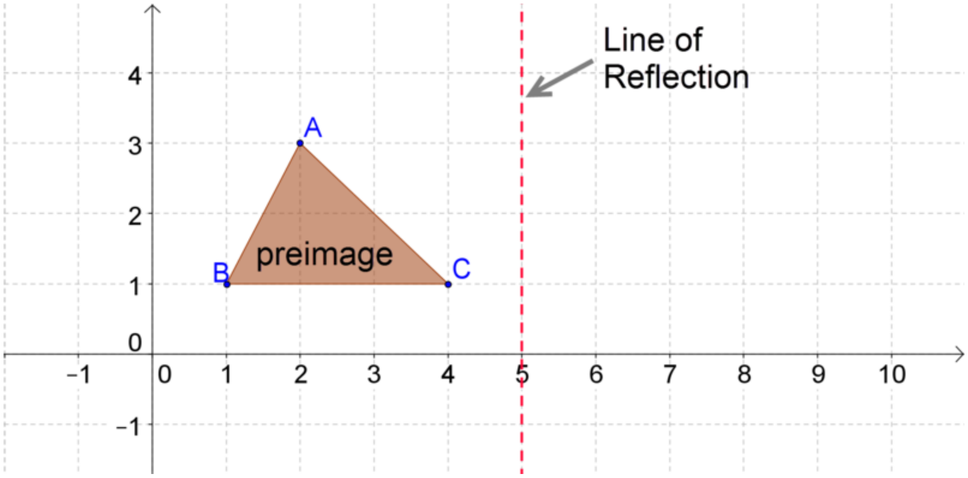
Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofReflectionsB

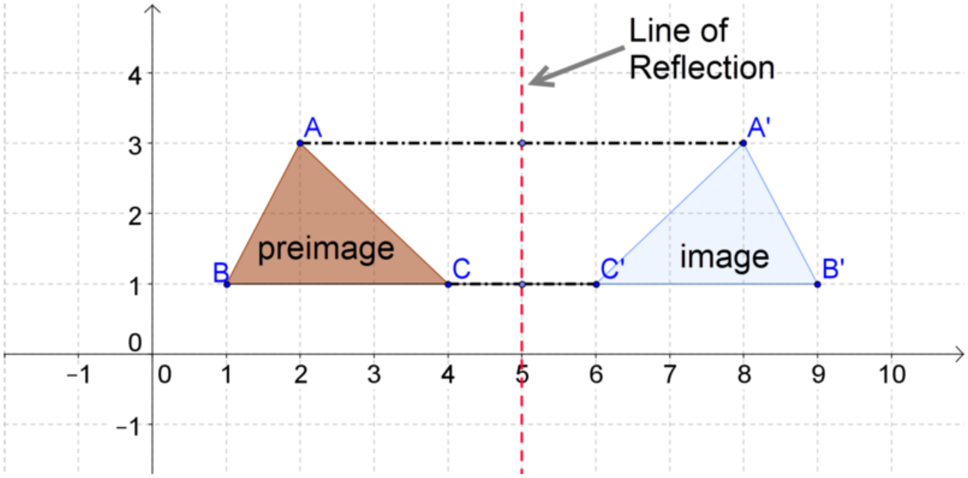
Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image).

To graph a reflection, you can visualize what would happen if you flipped the shape across the line.



Each point on the preimage will be the same distance from the line of reflection as it's corresponding point in the image. For example, for the pair of triangles below, both A and A' are 3 units away from the line of reflection.



For common reflections, you can also remember what happens to their coordinates:

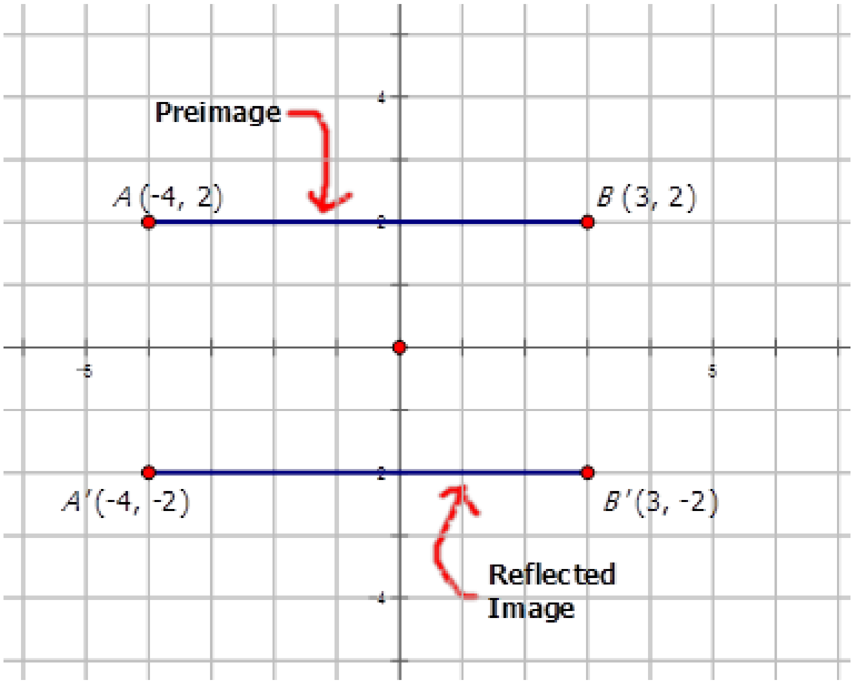
- reflections across the x-axis: y values are multiplied by -1.
- reflections across the y-axis: x values are multiplied by -1.
- reflections across the line $y = x$: x and y values switch places.
- reflections across the line $y = -x$. x and y values switch places and are multiplied by -1.

Knowing the rules above will allow you to recognize reflections even when a graph is not available.

Example A

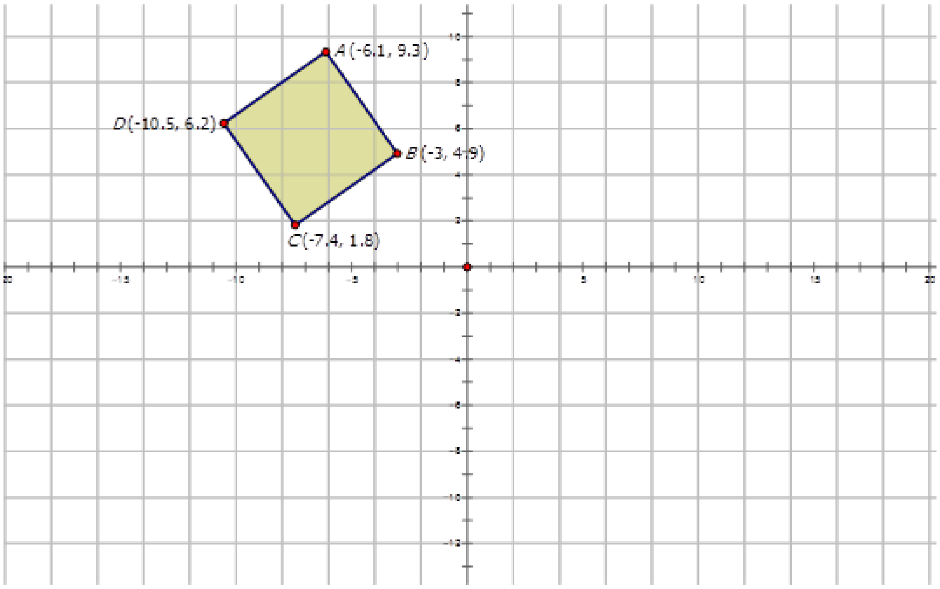
Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been reflected across the x-axis. Draw the preimage and image and properly label each.

Solution:

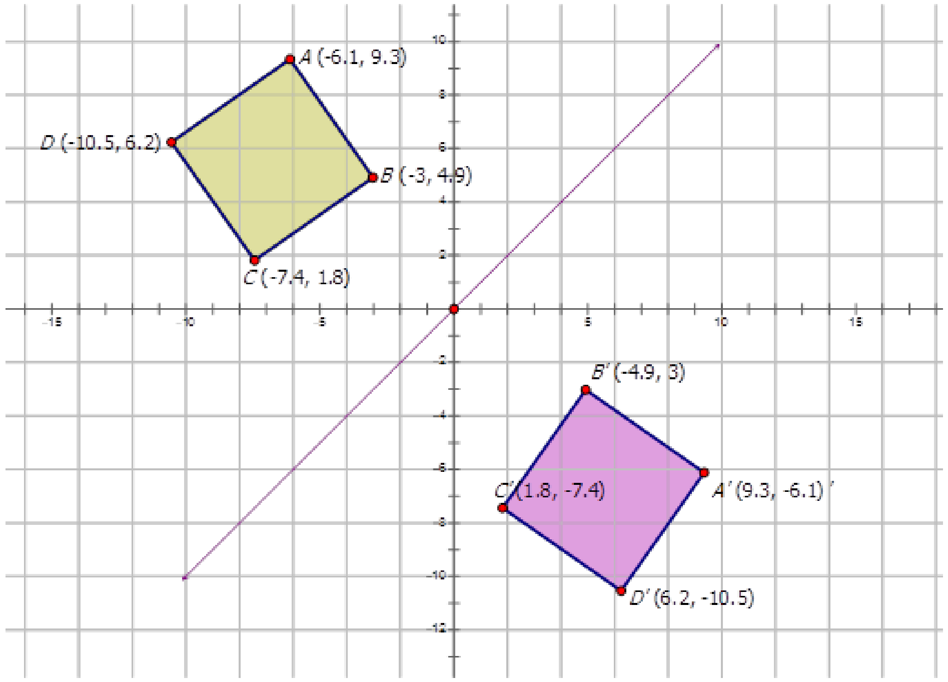


Example B

The diamond *ABCD* is reflected across the line $y = x$ to form the image $A'B'C'D'$. Find the coordinates of the reflected image. On the diagram, draw and label the reflected image.

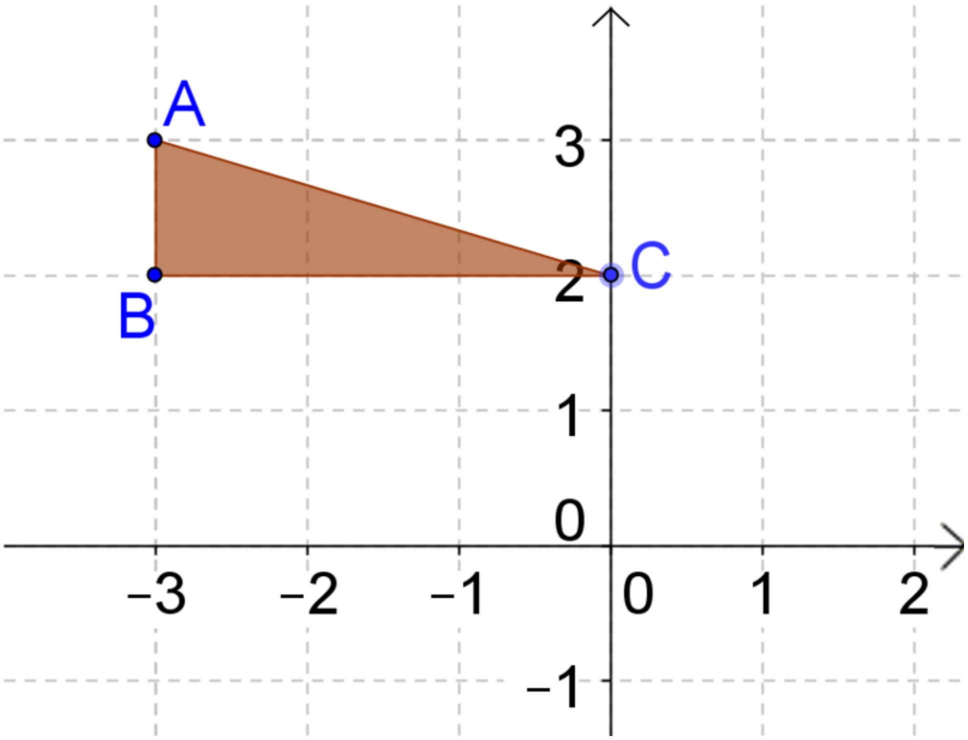


Solution:

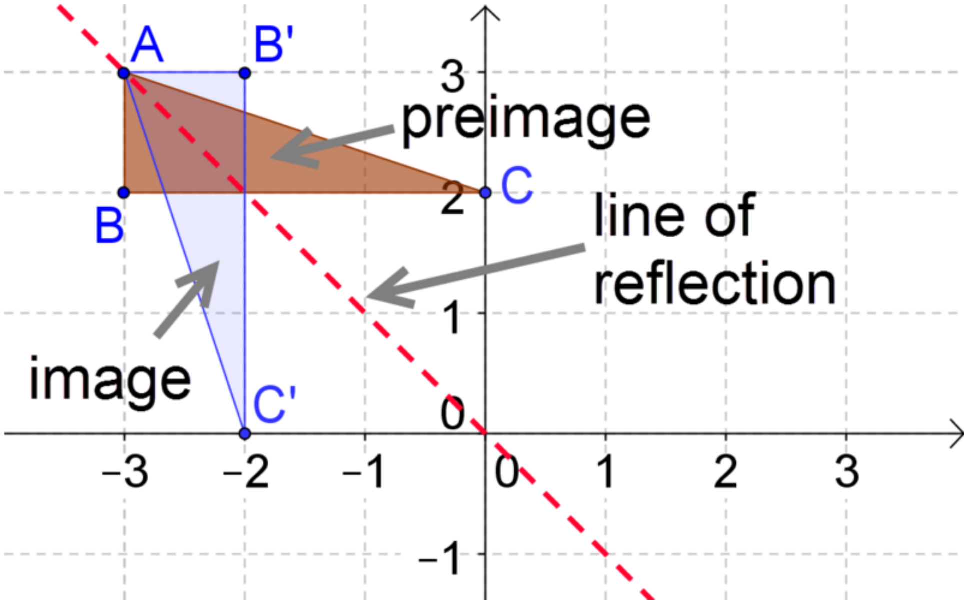


Example C

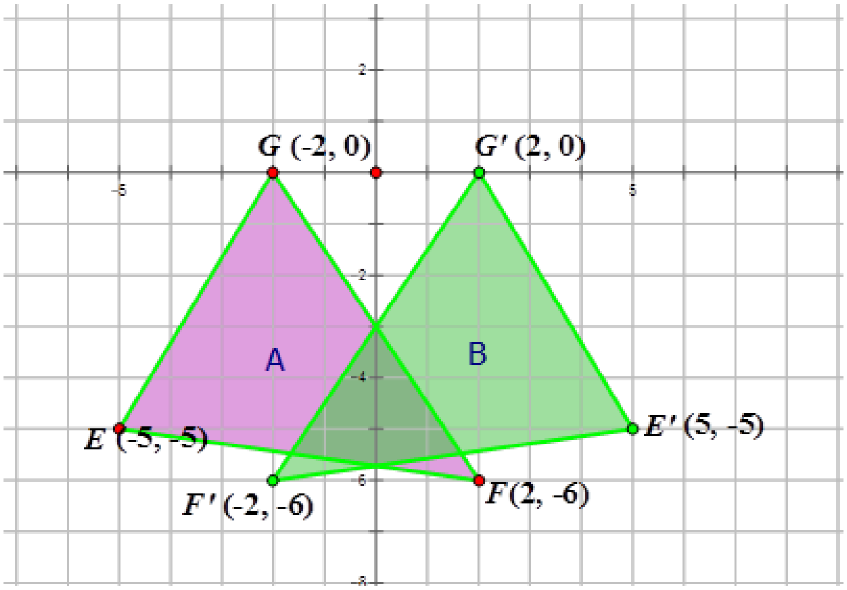
Triangle ABC is reflected across the line $y = -x$ to form the image $A'B'C'$. Draw and label the reflected image.



Solution:



Concept Problem Revisited



The coordinates of the new image (B) are $E'(5,-5)$, $F'(2,-6)$ and $G'(2,0)$.

Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Reflection

A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

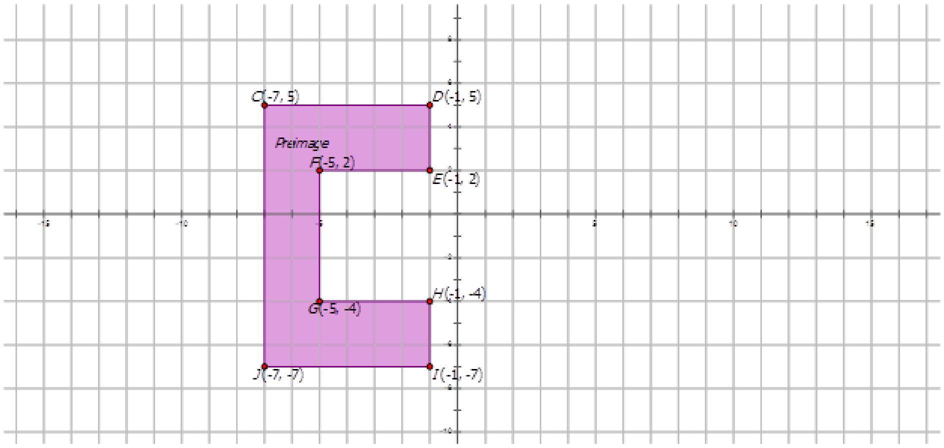
Line of Reflection

The ***line of reflection*** is the line that a shape reflects (flips) across when undergoing a reflection.

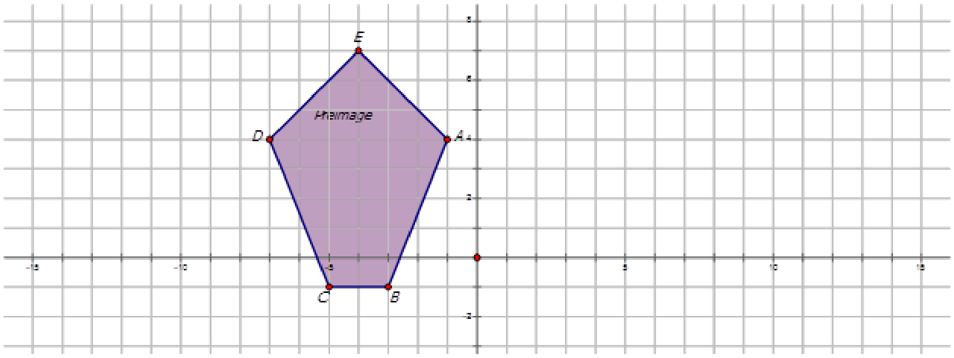
Guided Practice

1. Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has been reflected across the line $y = -x$. Draw the preimage and image and properly label each.

2. The polygon below has been reflected across the y-axis. Draw the reflected image and properly label each.

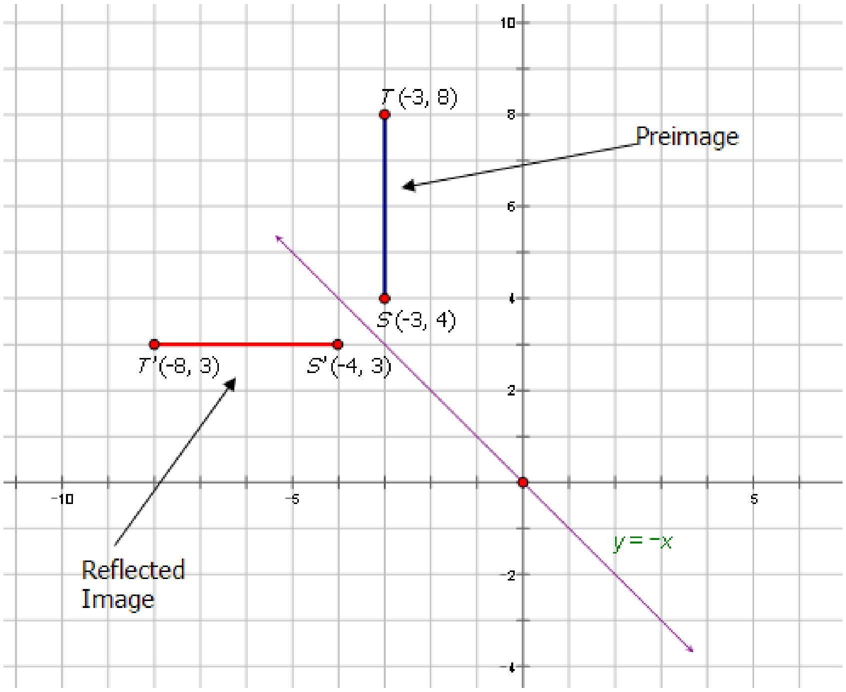


3. The purple pentagon is reflected across the $y-axis$ to make the new image. Find the coordinates of the purple pentagon. On the diagram, draw and label the reflected pentagon.

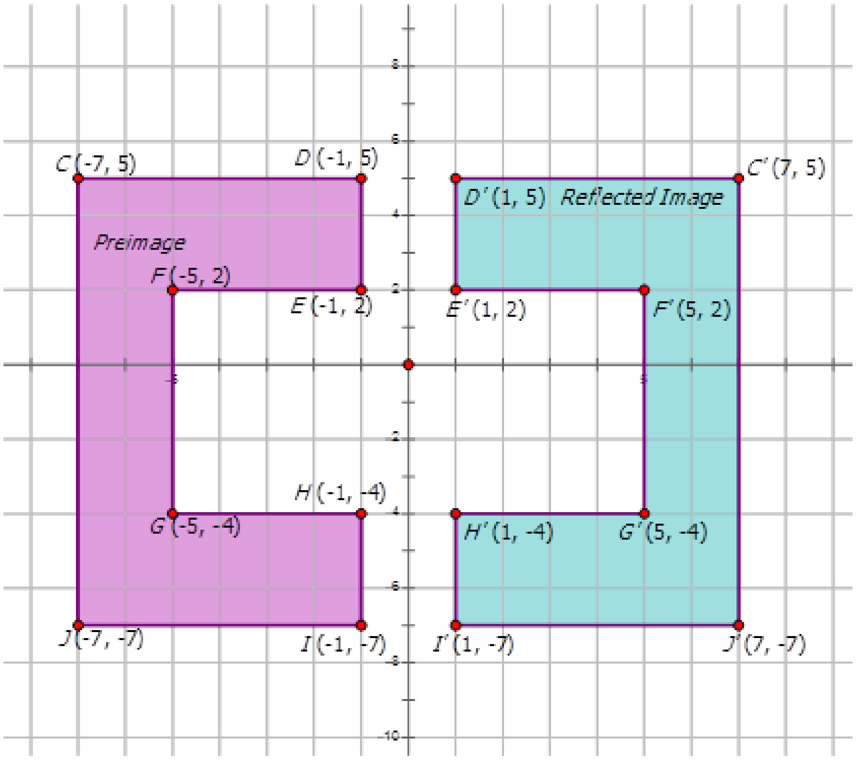


Answers:

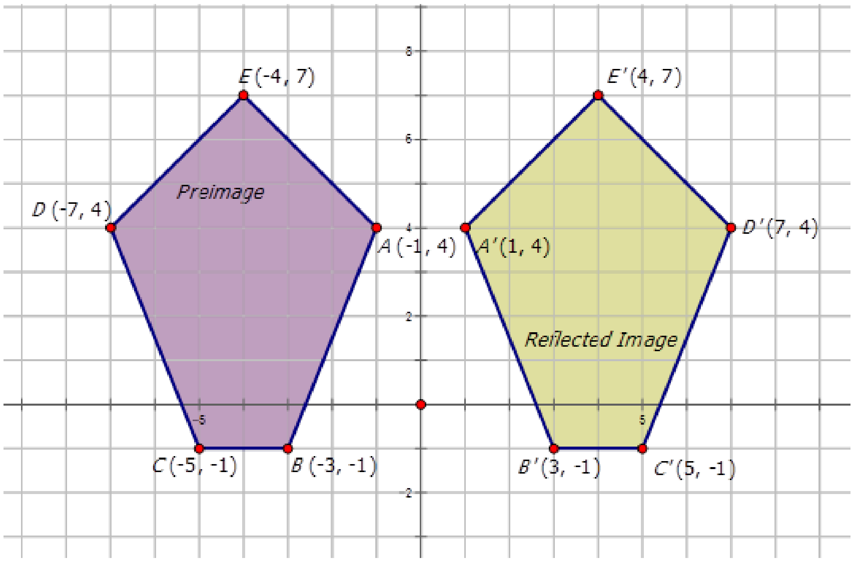
1.


T (-3, 8)
Preimage
S (-3, 4)
T'(-8, 3)
S'(-4, 3)
Reflected Image
y = −x
10
8
6
4
2
-2
-4
-10
-5
5

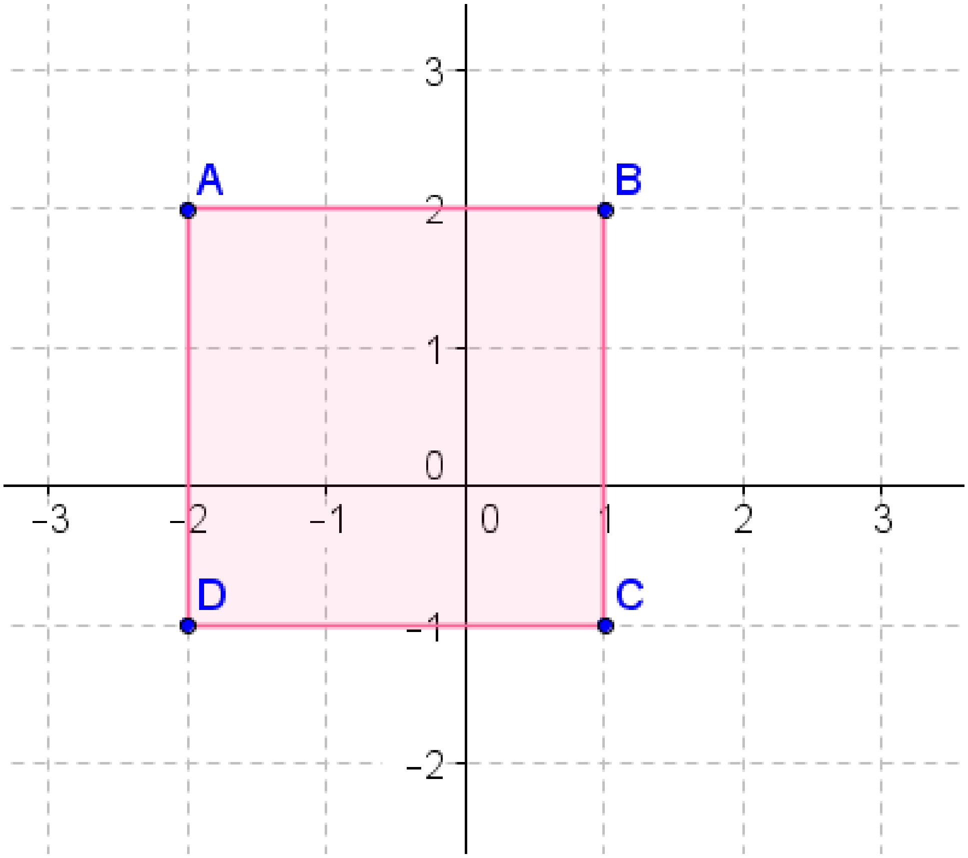
2.


C (-7, 5)
D (-1, 5)
Preimage
F (-5, 2)
E (-1, 2)
H (-1, -4)
G (-5, -4)
J (-7, -7)
I (-1, -7)
D' (1, 5)
Reflected Image
C' (7, 5)
E' (1, 2)
F' (5, 2)
H' (1, -4)
G' (5, -4)
I' (1, -7)
J' (7, -7)
8
6
4
2
-2
-4
-6
-8
-10
-5
5

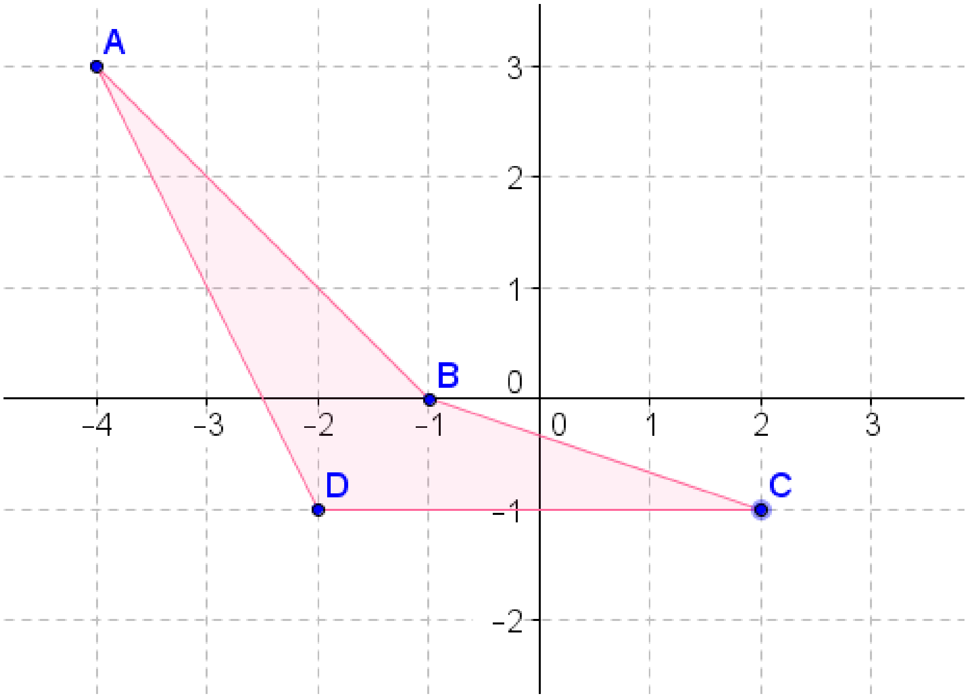
3.



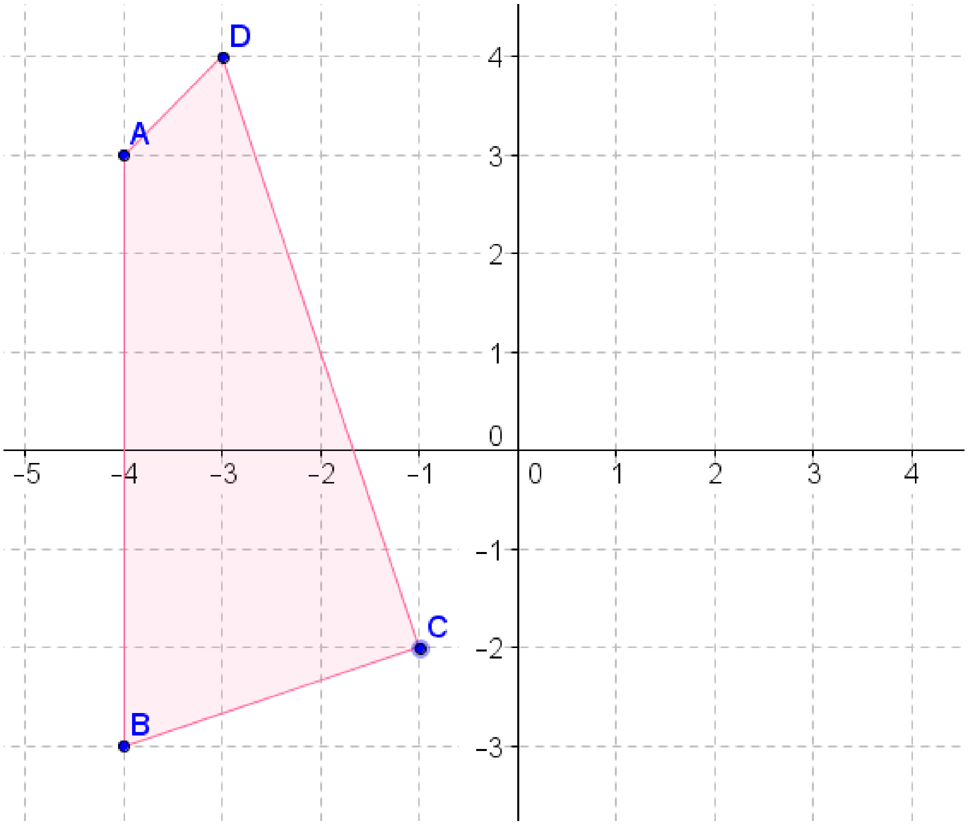
Practice



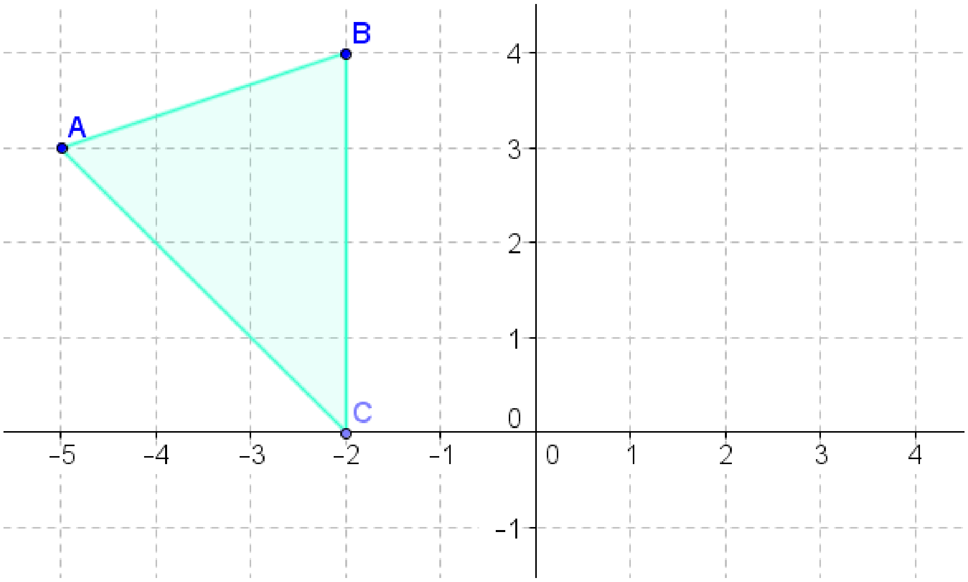
1. Reflect the above figure across the x-axis.
2. Reflect the above figure across the y-axis.
3. Reflect the above figure across the line $y = x$.



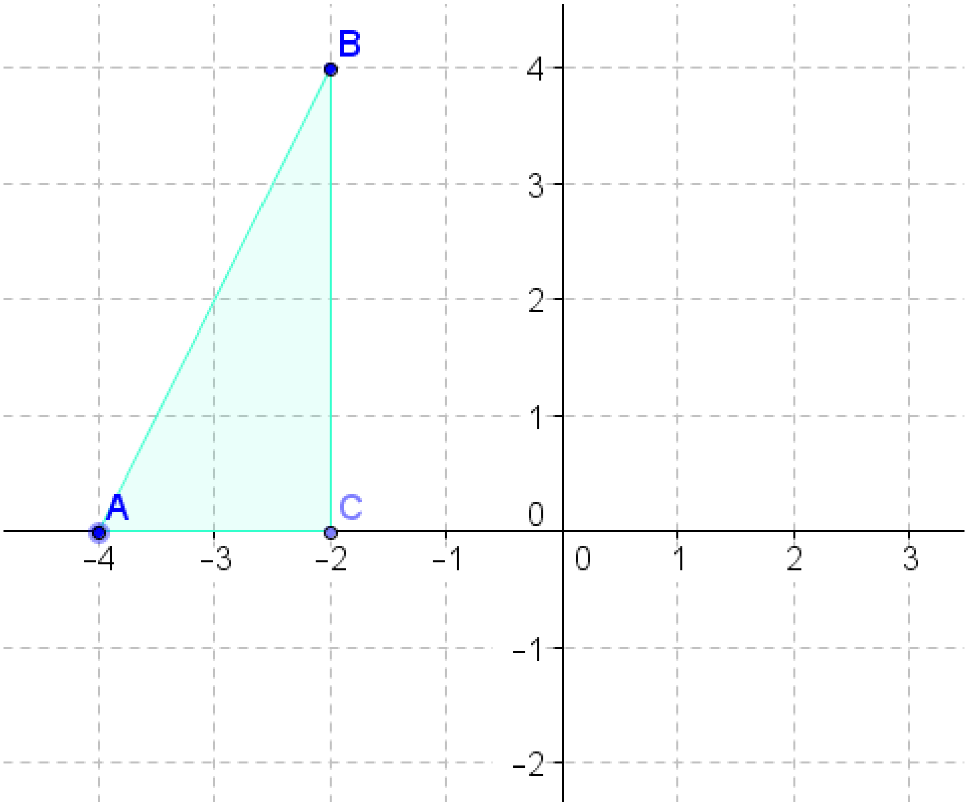
4. Reflect the above figure across the x-axis.
5. Reflect the above figure across the y-axis.
6. Reflect the above figure across the line $y = x$.



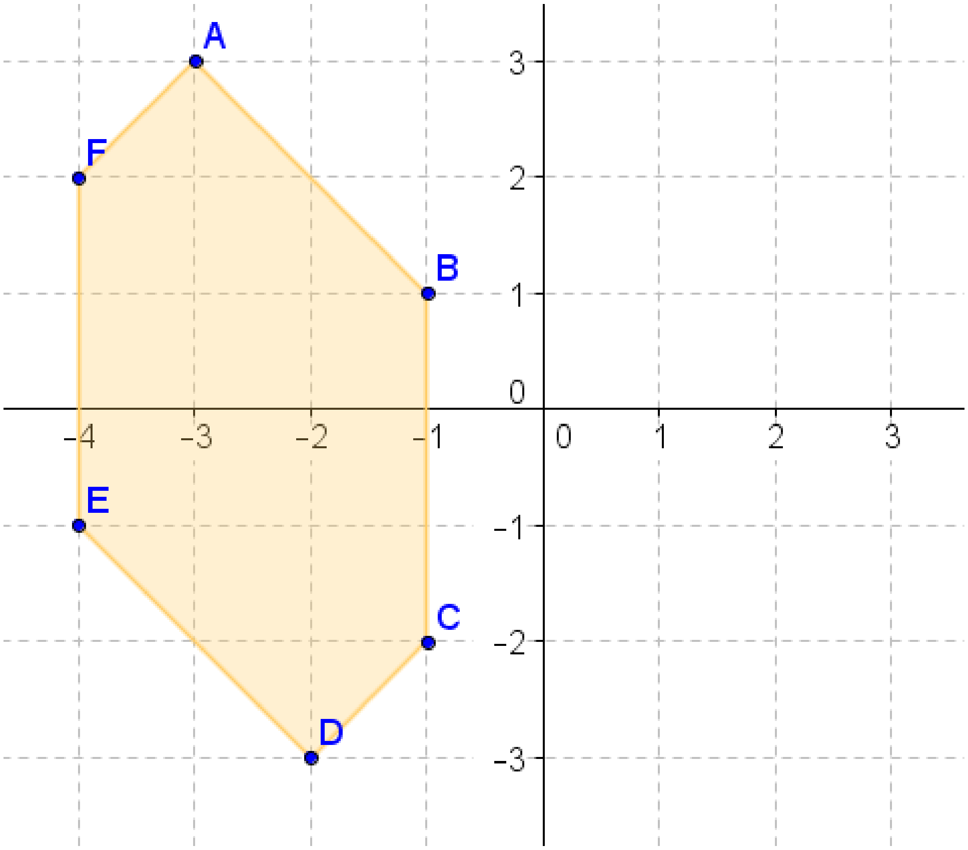
7. Reflect the above figure across the x-axis.
8. Reflect the above figure across the y-axis.
9. Reflect the above figure across the line $y = x$.



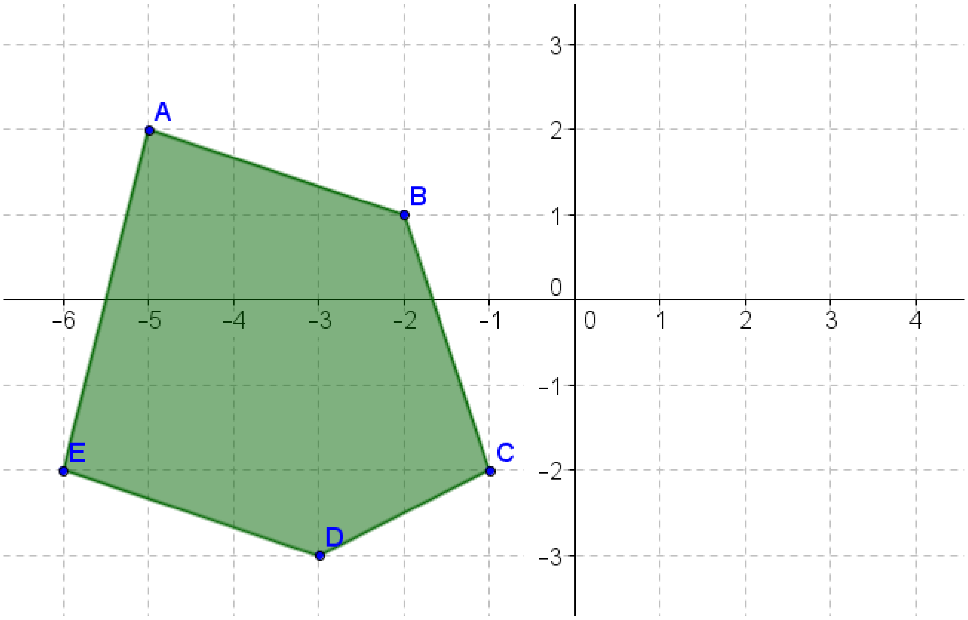
10. Reflect the above figure across the x-axis.
11. Reflect the above figure across the y-axis.
12. Reflect the above figure across the line $y = x$.



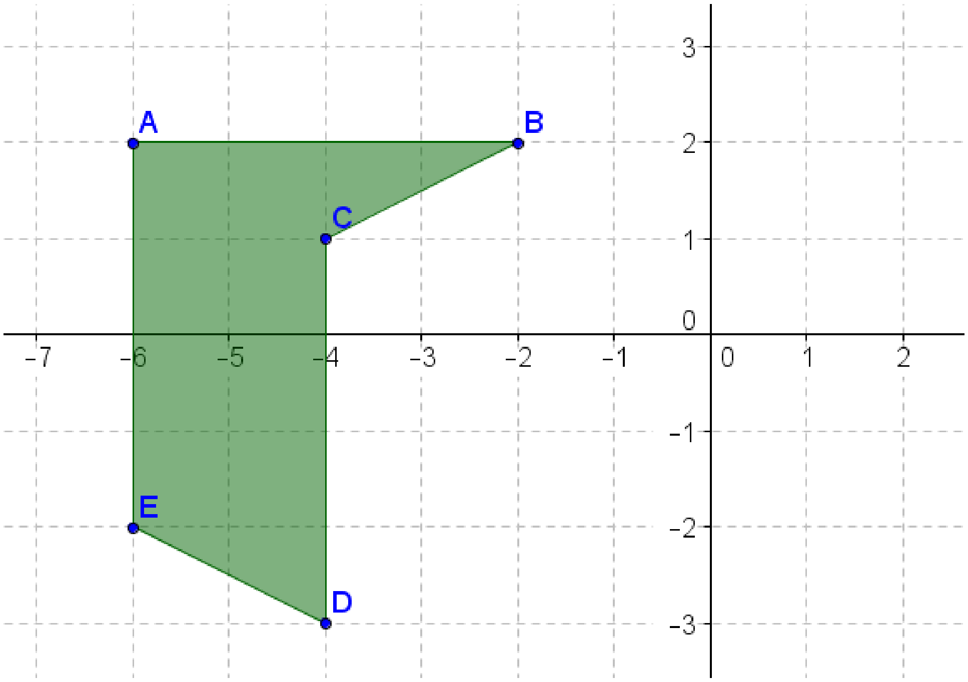
13. Reflect the above figure across the x-axis.
14. Reflect the above figure across the y-axis.
15. Reflect the above figure across the line $y = x$.



16. Reflect the above figure across the x-axis.
17. Reflect the above figure across the y-axis.
18. Reflect the above figure across the line $y = x$.



19. Reflect the above figure across the x-axis.
20. Reflect the above figure across the y-axis.
21. Reflect the above figure across the line $y = x$.

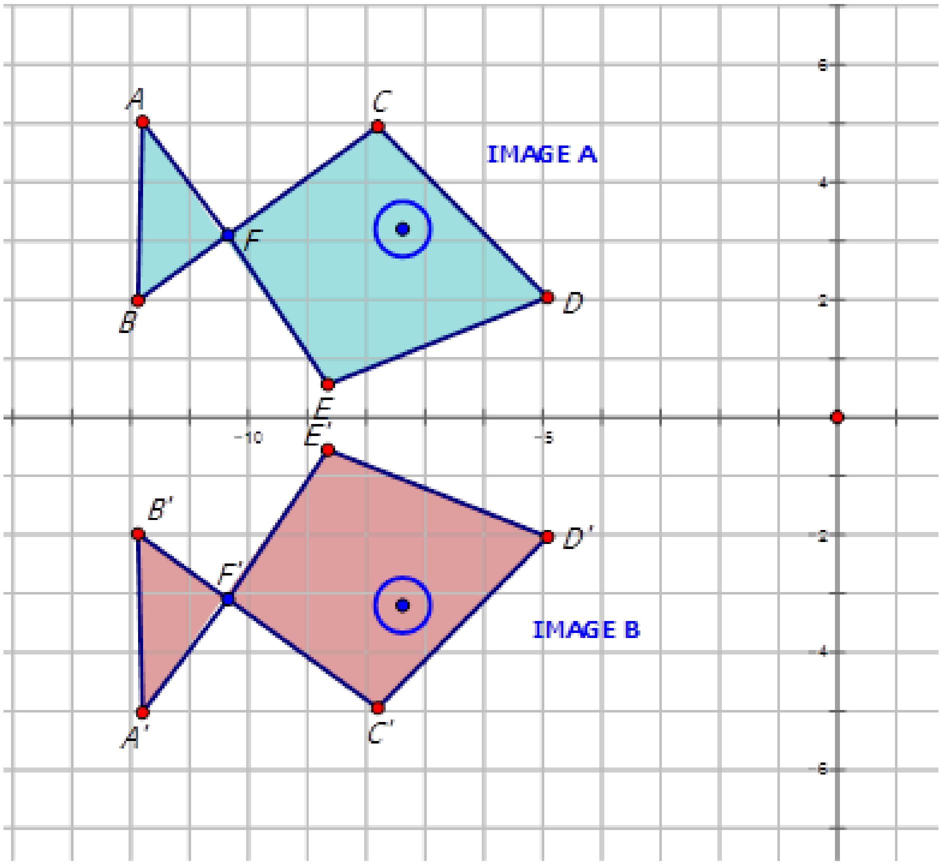


22. Reflect the above figure across the x-axis.
23. Reflect the above figure across the y-axis.
24. Reflect the above figure across the line $y = x$.

10.6 Rules for Reflections

Here you will learn notation for describing a reflection with a rule.

The figure below shows a pattern of two fish. Write the mapping rule for the reflection of Image A to Image B.



Watch This

First watch this video to learn about writing rules for reflections.

Multimedia

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Click image to the left for more content.

CK-12 Foundation Chapter10RulesforReflectionsA

Then watch this video to see some examples.

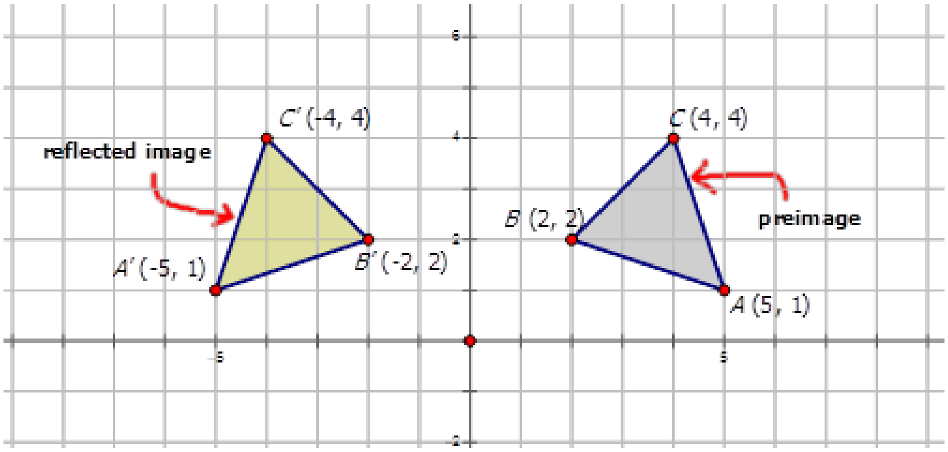
MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10RulesforReflectionsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A reflection is an example of a transformation that takes a shape (called the preimage) and flips it across a line (called the line of reflection) to create a new shape (called the image). By examining the coordinates of the reflected image, you can determine the line of reflection. The most common lines of reflection are the x-axis, the y-axis, or the lines $y = x$ or $y = -x$.



The preimage has been reflected across he y-axis. This means, all of the x-coordinates have been multiplied by -1. You can describe the reflection in words, or with the following notation:

$r_{y-axis}(x,y) \rightarrow (-x,y)$

Notice that the notation tells you exactly how each (x,y) point changes as a result of the transformation.

Example A

Find the image of the point (3, 2) that has undergone a reflection across

a) the x-axis,

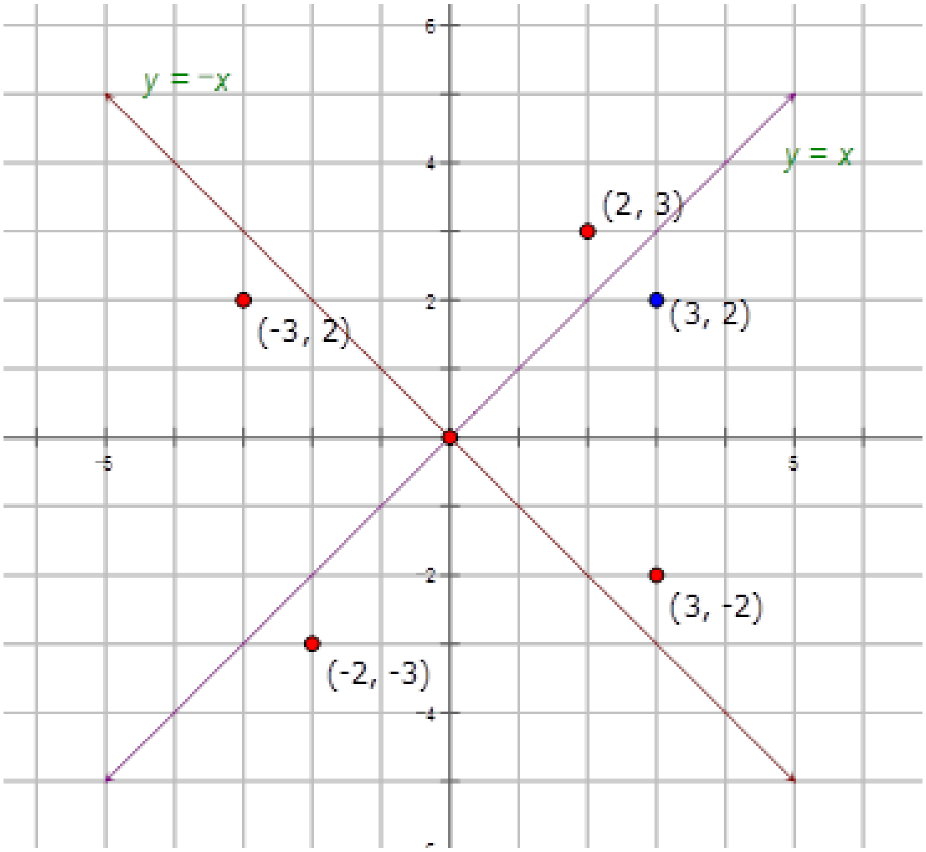
b) the y-axis,

c) the line $y = x$, and

d) the line $y = -x$.

Write the notation to describe the reflection.

Solution:



a) Reflection across the x-axis: $r_{x-axis}(3,2) \rightarrow (3,-2)$

b) Reflection across the y-axis: $r_{y-axis}(3,2) \rightarrow (3,-2)$

c) Reflection across the line $y = x$: $r_{y=x}(3,2) \rightarrow (2,3)$

d) Reflection across the line $y = -x$: $r_{y=-x}(3,2) \rightarrow (-2,-3)$

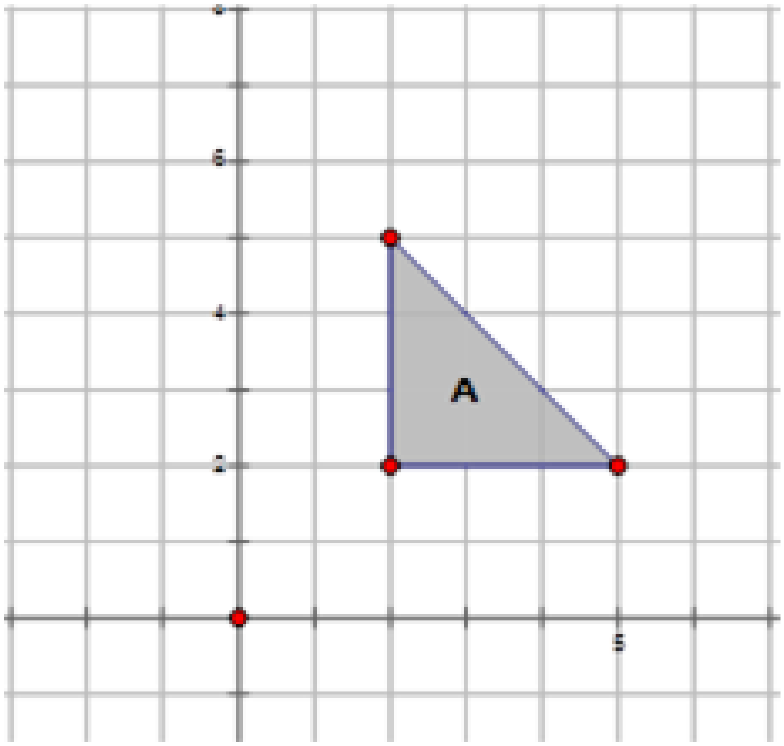
Example B

Reflect Image A in the diagram below:

a) Across the y-axis and label it B.

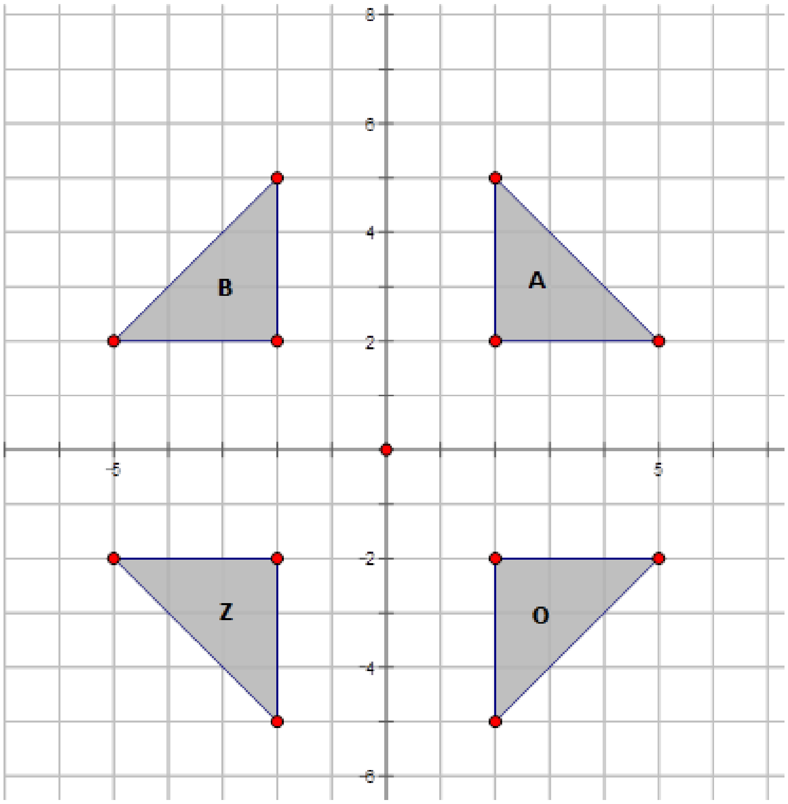
b) Across the x-axis and label it O.

c) Across the line $y = -x$ and label it Z.



Write notation for each to indicate the type of reflection.

Solution:



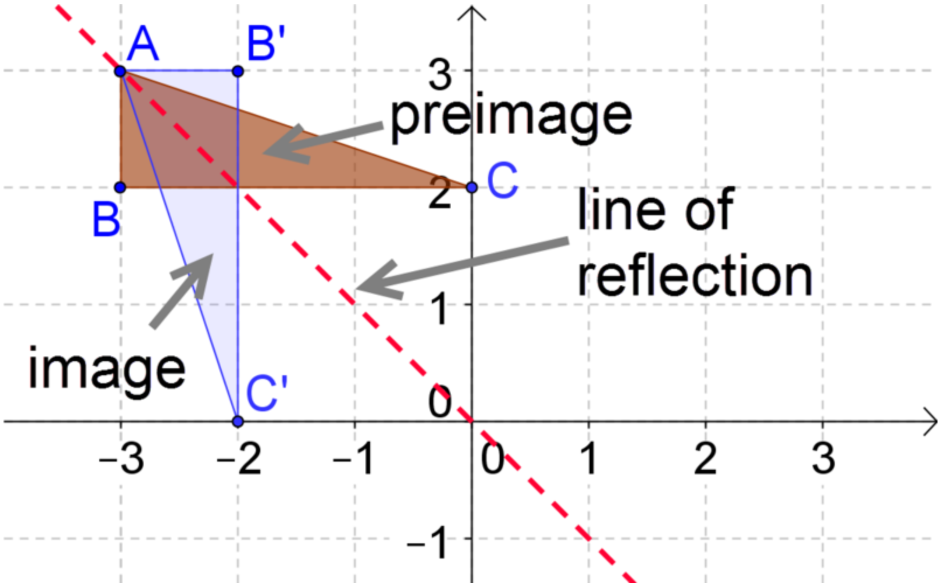
a) Reflection across the y-axis: $r_{y-axis}A \rightarrow B = r_{y-axis}(x,y) \rightarrow (-x,y)$

b) Reflection across the x-axis: $r_{x-axis}A \rightarrow O = r_{x-axis}(x,y) \rightarrow (x,-y)$

c) Reflection across the $y = -x$: $r_{y=-x}A \rightarrow Z = r_{y=-x}(x,y) \rightarrow (-y,-x)$

Example C

Write the notation that represents the reflection of the preimage to the image in the diagram below.

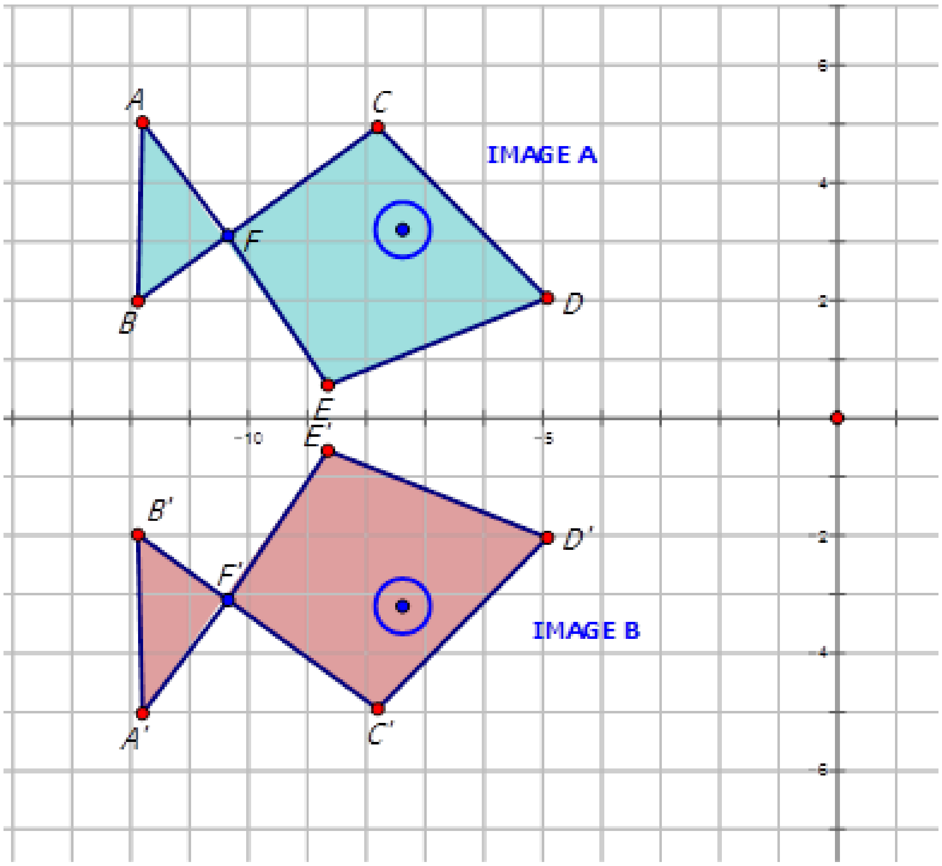


Solution:

This is a reflection across the line $y- = -x$. The notation is $r_{y=-x}(x,y) \rightarrow (-y,-x)$.

Concept Problem Revisited

The figure below shows a pattern of two fish. Write the mapping rule for the reflection of Image A to Image B.



To answer this question, look at the coordinate points for Image A and Image B.

TABLE 10.8:

Image A	$A(-11.8,5)$	$B(-11.8,2)$	$C(-7.8,5)$	$D(-4.9,2)$	$E(-8.7,0.5)$	$F(-10.4,3.1)$
Image B	$A'(-11.8,-5)$	$B'(-11.8,-2)$	$C'(-7.8,-5)$	$D'(-4.9,-2)$	$E'(-8.7,-0.5)$	$F'(-10.4,-3.1)$

Notice that all of the y-coordinates have changed sign. Therefore Image A has reflected across the x-axis. To write a rule for this reflection you would write: $r_{x-axis}(x,y) \rightarrow (x,-y)$.

Vocabulary

Notation Rule

A ***notation rule*** has the following form $r_{y-axis}A \rightarrow B = r_{y-axis}(x,y) \rightarrow (-x,y)$ and tells you that the image A has been reflected across the y-axis and the x-coordinates have been multiplied by -1.

Reflection

A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

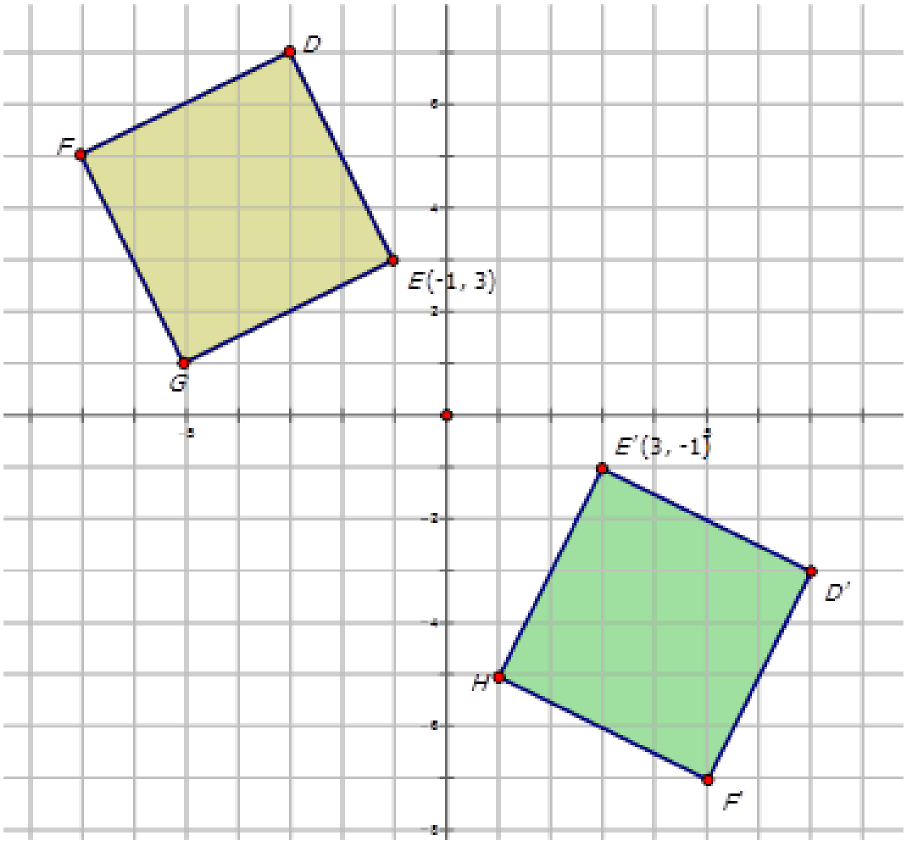
Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

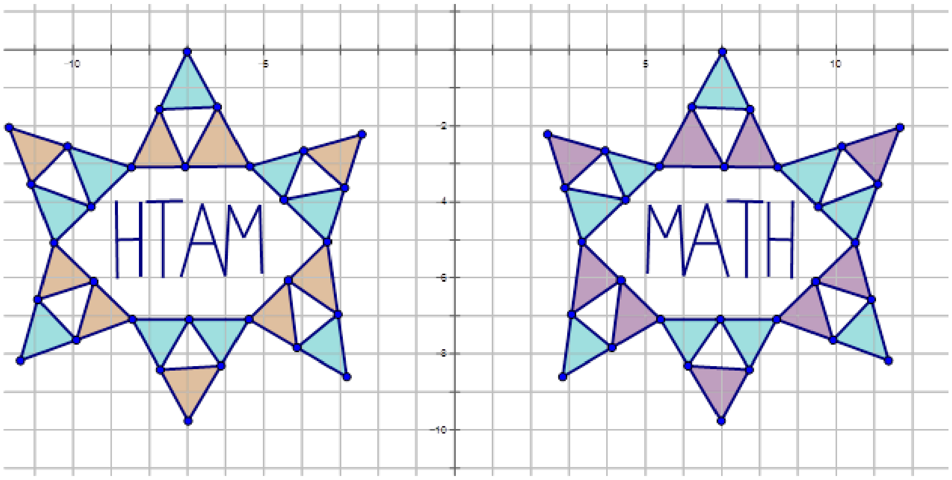
Guided Practice

1. Thomas describes a reflection as point J moving from $J(-2,6)$ to $J'(-2,-6)$. Write the notation to describe this reflection for Thomas.

2. Write the notation that represents the reflection of the yellow diamond to the reflected green diamond in the diagram below.



3. Karen was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



Answers:

1. $J:(-2,6) \quad J':(-2,-6)$

Since the y-coordinate is multiplied by -1 and the x-coordinate remains the same, this is a reflection in the x-axis. The notation is: $r_{x-axis}J \rightarrow J' = r_{x-axis}(-2,6) \rightarrow (-2,6)$

2. In order to write the notation to describe the reflection, choose one point on the preimage (the yellow diamond) and then the reflected point on the green diamond to see how the point has moved. Notice that point E is shown in the diagram:

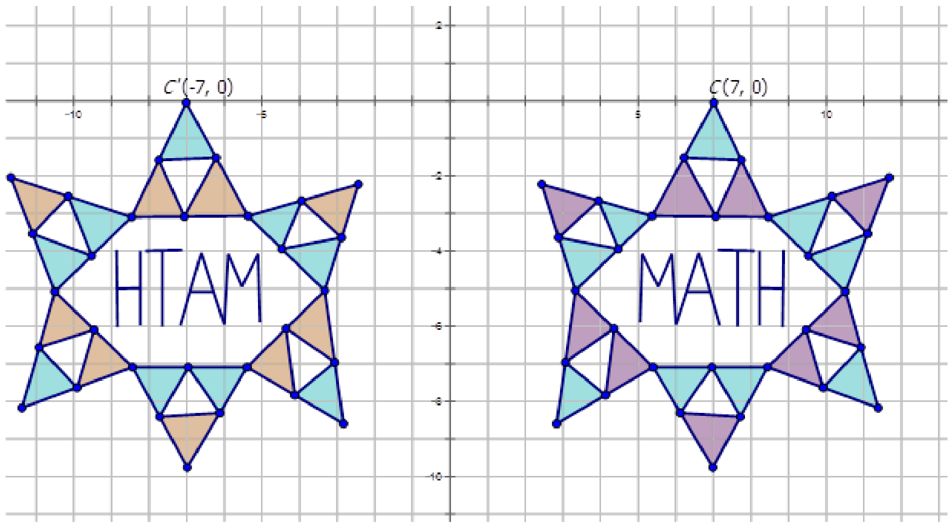
$E(-1,3) \rightarrow E'(3,-1)$

Since both x- and y-coordinates are reversed numbers, the reflection is in the line $y = x$. The notation for this reflection would be: $r_{y=x}(x,y) \rightarrow (y,x)$.

3. In order to write the notation to describe the transformation, choose one point on the preimage (purple and blue diagram) and then the transformed point on the orange and blue diagram to see how the point has moved. Notice that point A is shown in the diagram:

$C(7,0) \rightarrow C'(-7,0)$

Since both x-coordinates only are multiplied by -1, the transformation is a reflection is in y-axis. The notation for this reflection would be: $r_{y-axis}(x,y) \rightarrow (-x,y)$.

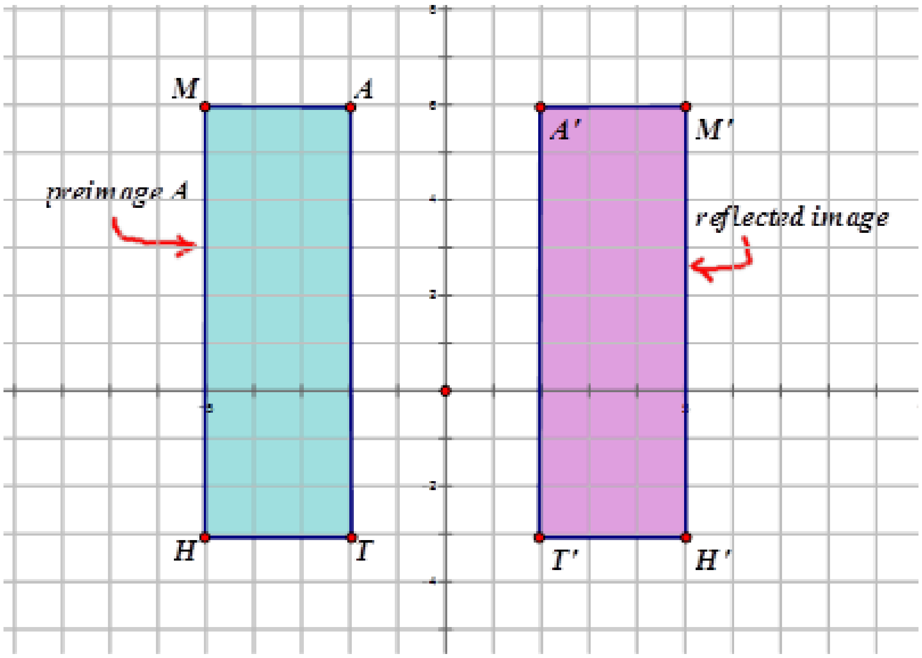


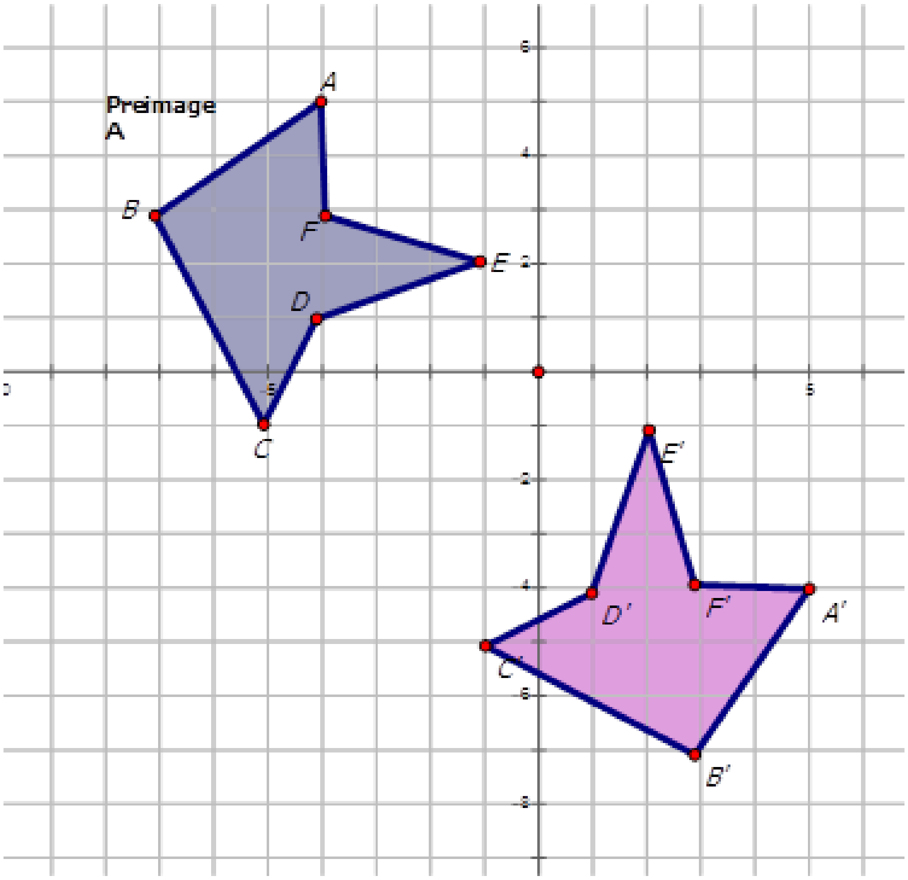
Practice

Write the notation to describe the movement of the points in each of the reflections below.

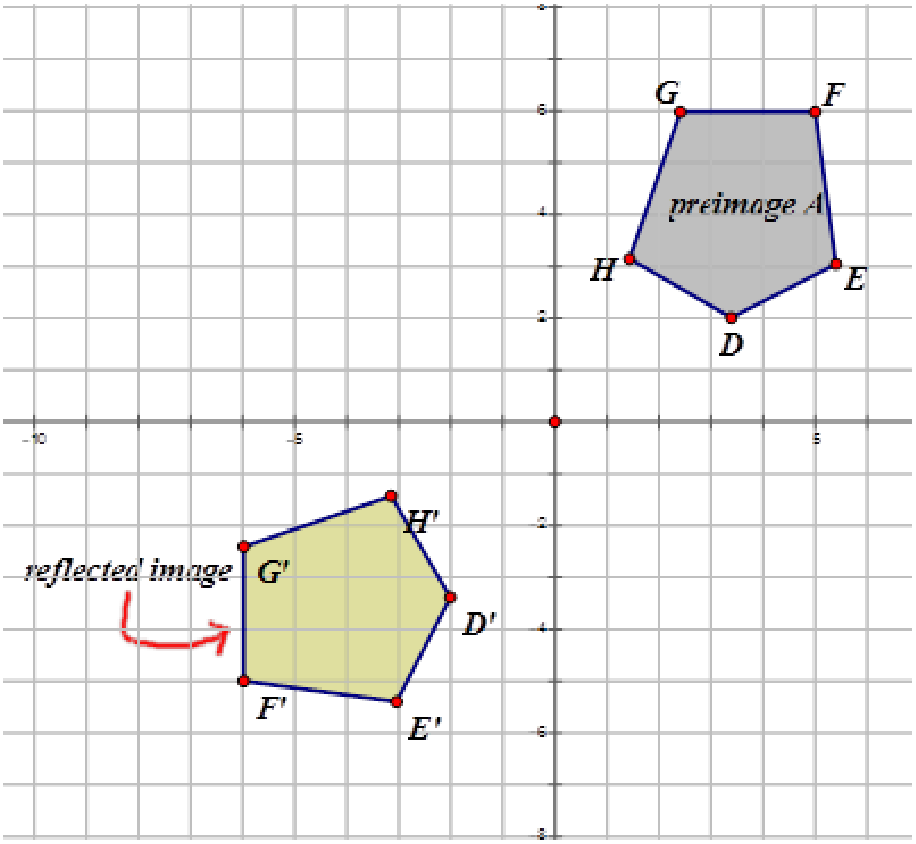
1. $S(1,5) \rightarrow S'(-1,5)$
2. $W(-5,-1) \rightarrow W'(5,-1)$
3. $Q(2,-5) \rightarrow Q'(2,5)$
4. $M(4,3) \rightarrow M'(-3,-4)$
5. $B(-4,-2) \rightarrow B'(-2,-4)$
6. $A(3,5) \rightarrow A'(-3,5)$
7. $C(1,2) \rightarrow C'(2,1)$
8. $D(2,-5) \rightarrow D'(5,-2)$
9. $E(3,1) \rightarrow E'(-3,1)$
10. $F(-4,2) \rightarrow F'(-4,-2)$
11. $G(1,3) \rightarrow G'(1,-3)$

Write the notation that represents the reflection of the preimage image for each diagram below.

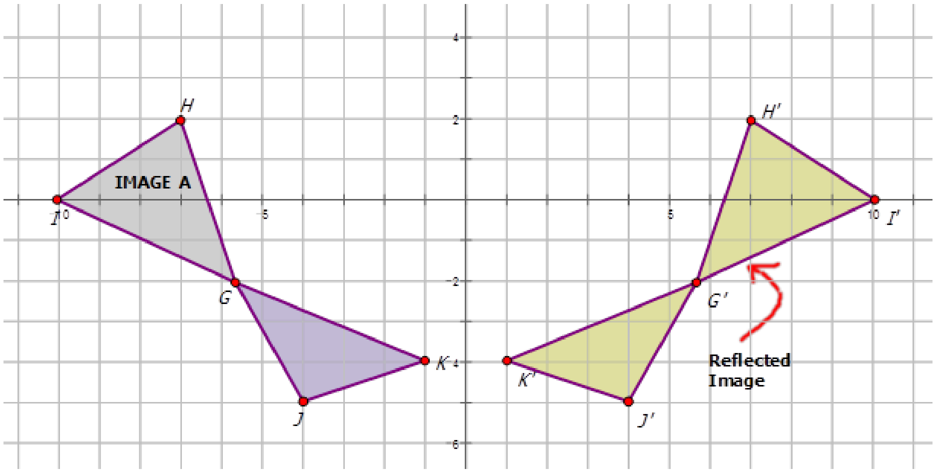
12. 


Preimage
A
A
B
C
D
E
F
A'
B'
C'
D'
E'
F'
6
4
2
-2
-4
-6
-8
-5
5

13.


G
F
preimage A
H
E
D
reflected image
G'
H'
D'
F'
E'
6
4
2
-2
-4
-6
-8
-10
-5
5

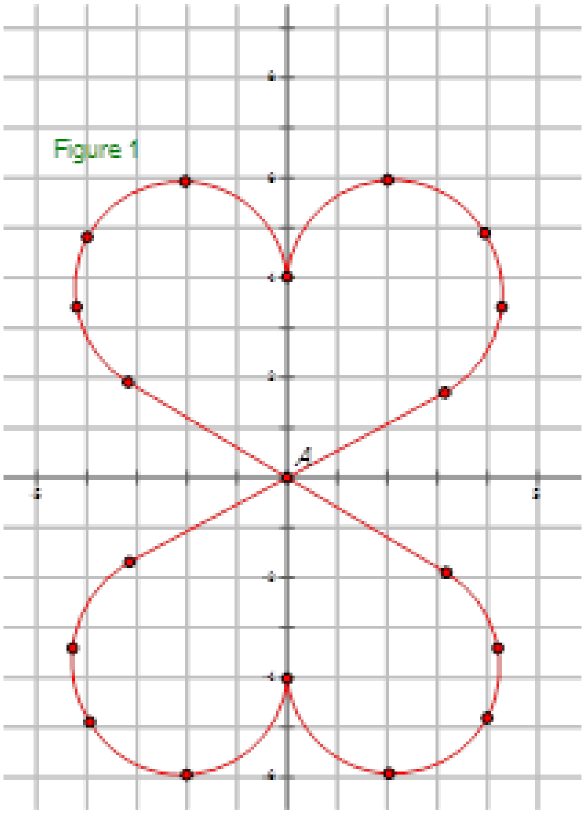
14.

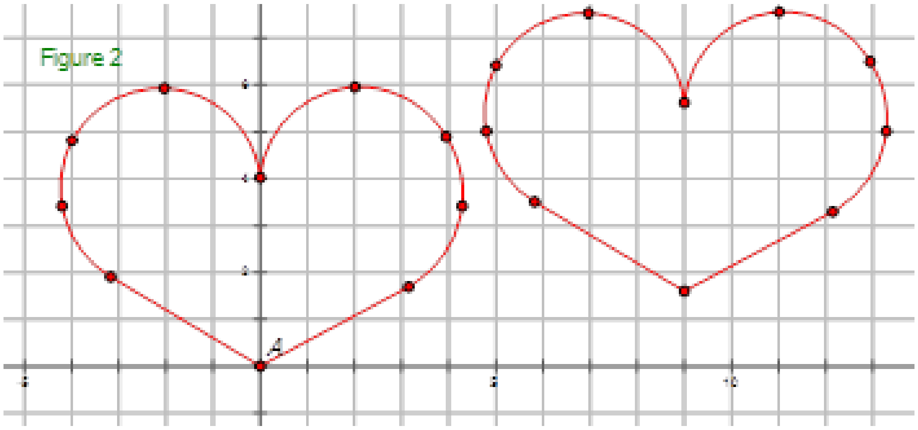
15. 

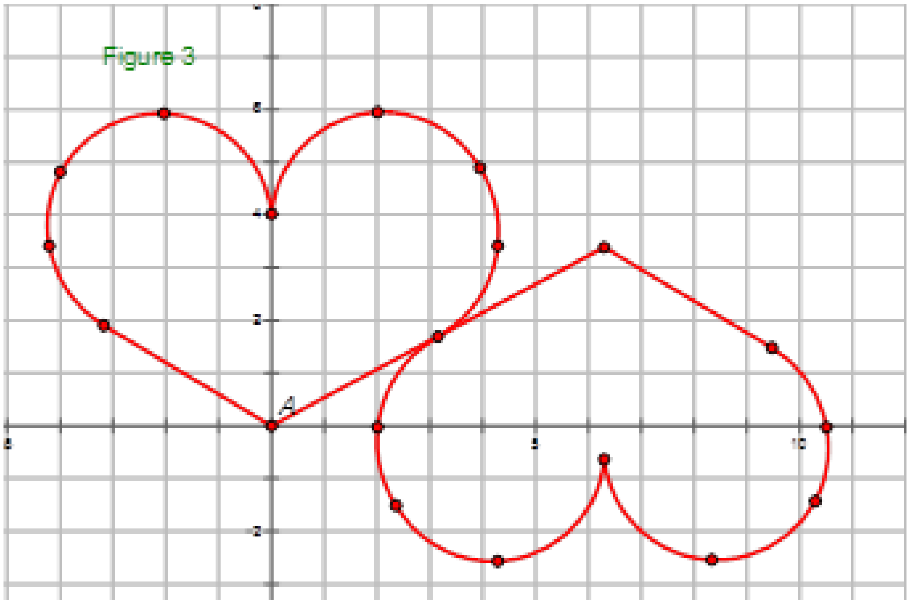
10.7 Rotations

Here you will learn about geometric rotations.

Which one of the following figures represents a rotation? Explain.







Watch This

First watch this video to learn about rotations.

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Click image to the left for more content.

CK-12 Foundation Chapter10RotationsA

Then watch this video to see some examples.

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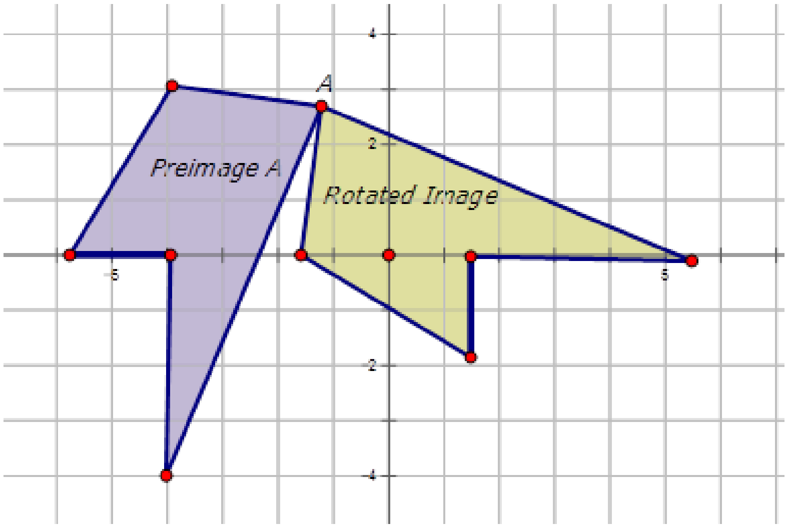
CK-12 Foundation Chapter10RotationsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees. The figure below shows that the Preimage A has been rotated 90° about point *A* to form the rotated image. Point *A* is the center of rotation.



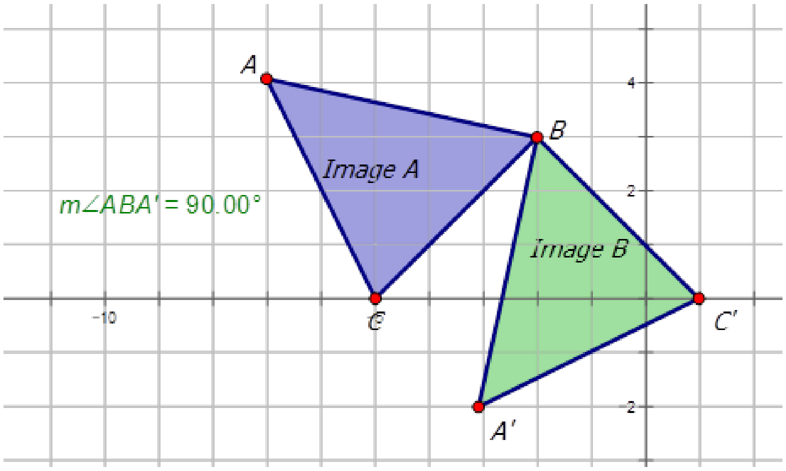
In order to describe a rotation, you need to state how many degrees the preimage rotated, the center of rotation, and the direction of the rotation (clockwise or counterclockwise). The most common center of rotation is the origin. The table below shows what happens to points when they have undergone a rotation about the origin. The angles are given as counterclockwise.

TABLE 10.9:

Center of Rotation	Angle of Rotation	Preimage (Point P)	Rotated Image (Point P')
(0, 0)	90° (or −270°)	(x,y)	$(-y,x)$
(0, 0)	180° (or −180°)	(x,y)	$(-x,-y)$
(0, 0)	270° (or −90°)	(x,y)	$(y,-x)$

Example A

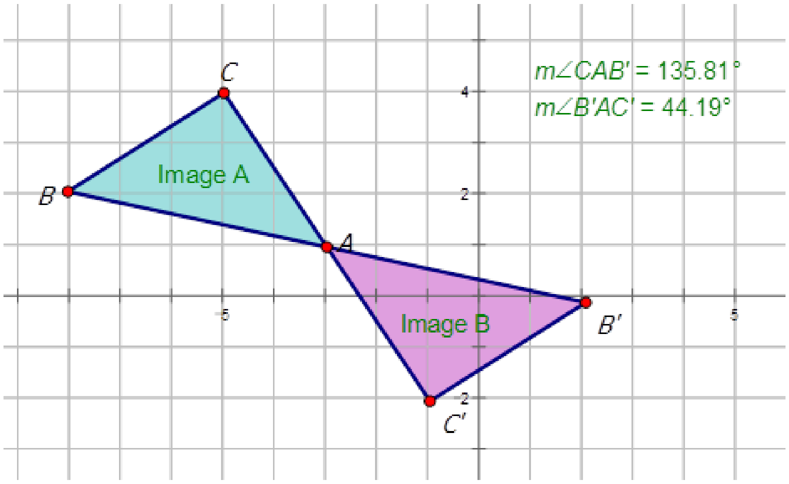
Describe the rotation of the blue triangle in the diagram below.



Solution: Looking at the angle measures, $\angle ABA' = 90^\circ$. Therefore the preimage, Image A, has been rotated 90° counterclockwise about the point B.

Example B

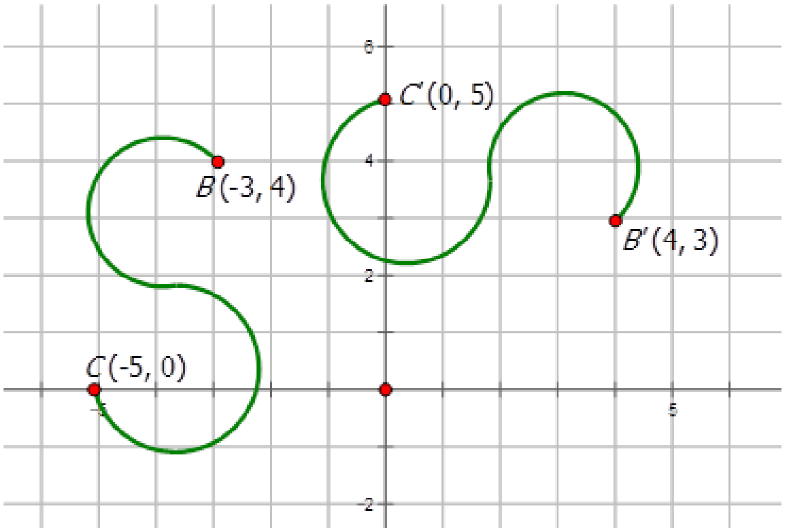
Describe the rotation of the triangles in the diagram below.



Solution: Looking at the angle measures, $\angle CAB' + \angle B'AC' = 180^\circ$. The triangle ABC has been rotated 180°CCW about the center of rotation Point A.

Example C

Describe the rotation in the diagram below.



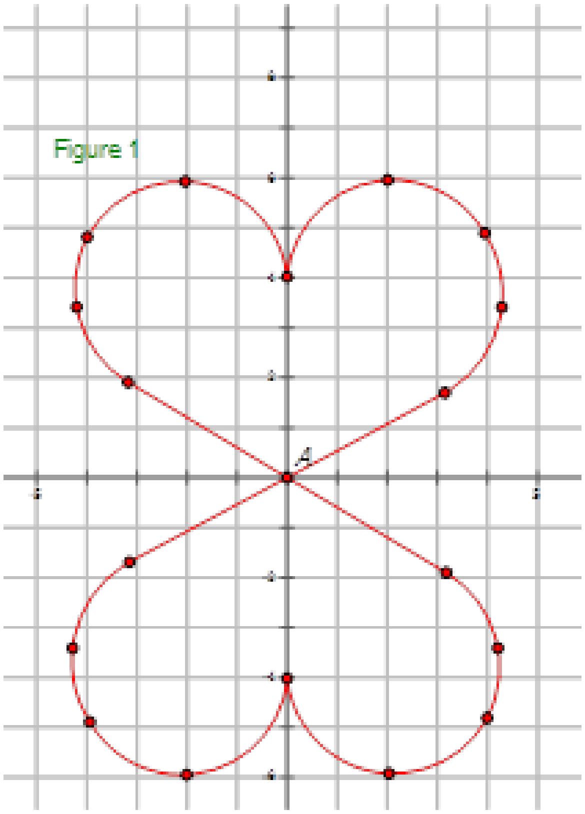
Solution: To describe the rotation in this diagram, look at the points indicated on the S shape.

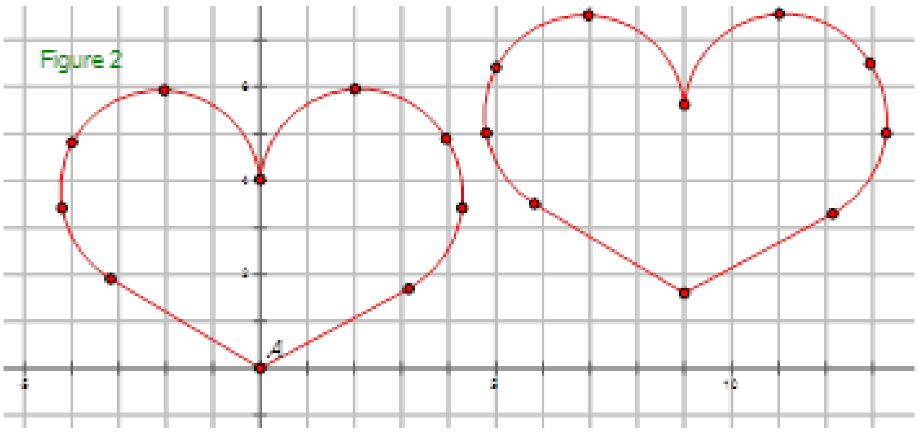
- Points BC: $B(-3,4)C(-5,0)$
- Points $B'C'$: $B'(4,3)C'(0,5)$

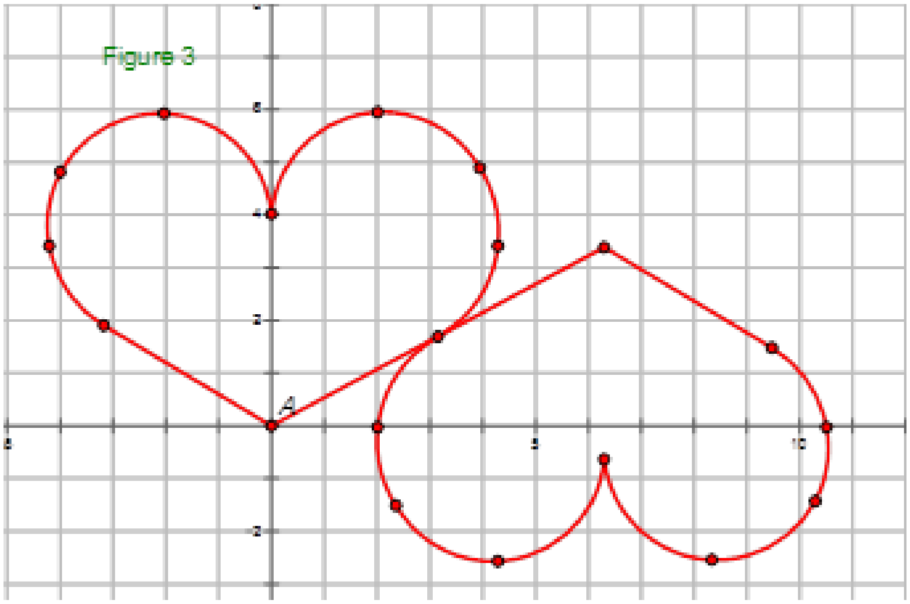
These points represent a rotation of 90° clockwise about the origin. Each coordinate point (x,y) has become the point $(y,-x)$.

Concept Problem Revisited

Which one of the following figures represents a rotation? Explain.







You know that a rotation is a transformation that turns a figure about a fixed point. This fixed point is the turn center or the center of rotation. In the figures above, Figure 1 and Figure 3 involve turning the heart about a fixed point. Figure 1 rotates the heart about the point A. Figure 3 rotates the heart about the point directly to the right of A. Figure 2 does a translation, not a rotation.

Vocabulary

Center of rotation

A ***center of rotation*** is the fixed point that a figure rotates about when undergoing a rotation.

Rotation

A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Image

In a transformation, the final figure is called the ***image***.

Preimage

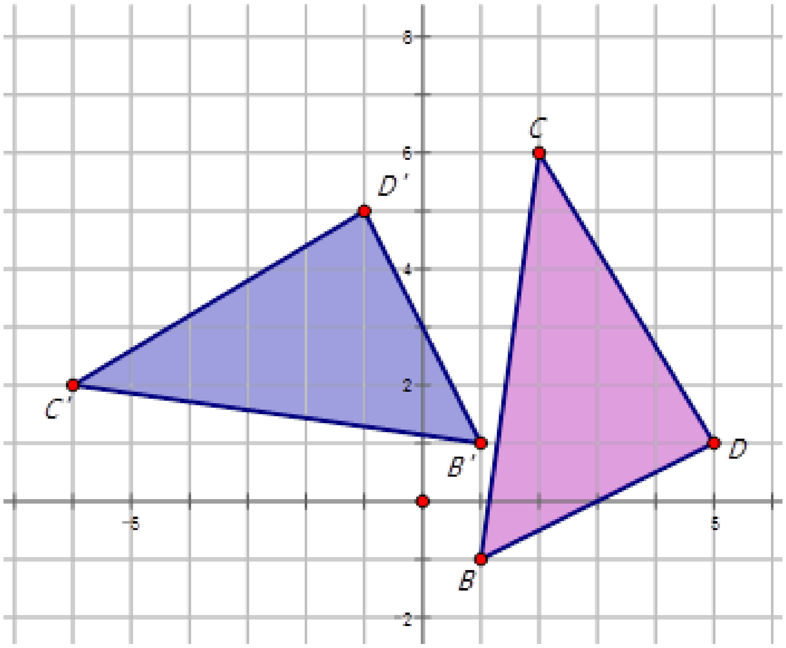
In a transformation, the original figure is called the ***preimage.***

Transformation

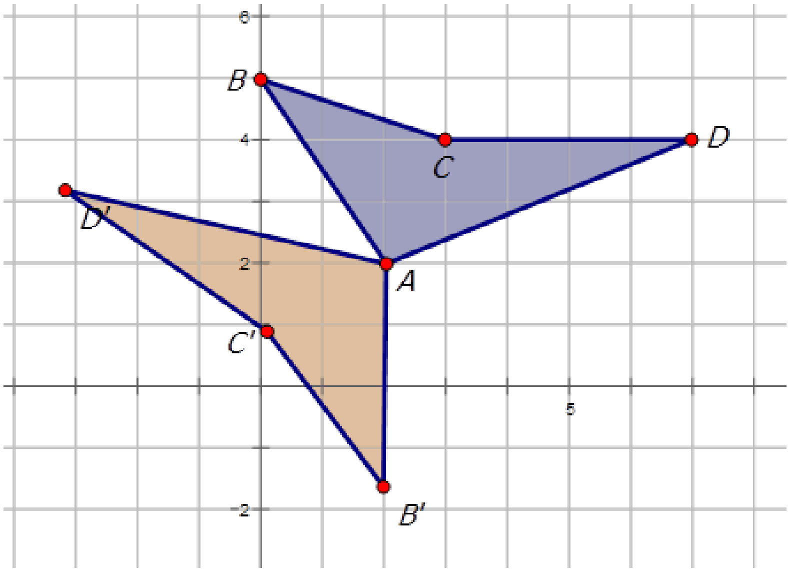
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Guided Practice

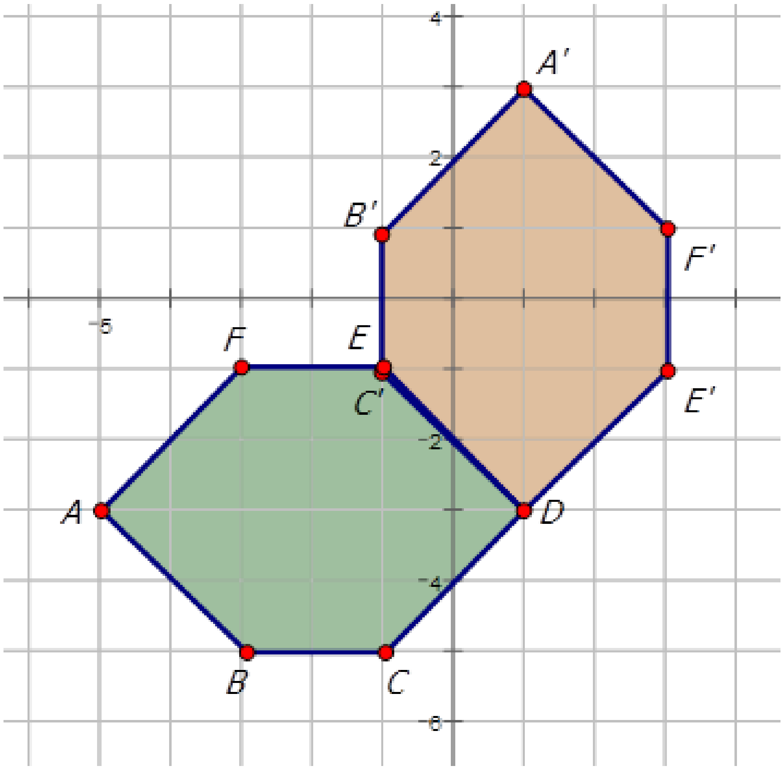
1. Describe the rotation of the pink triangle in the diagram below.



2. Describe the rotation of the blue polygon in the diagram below.



3. Describe the rotation of the green hexagon in the diagram below.



Answers:

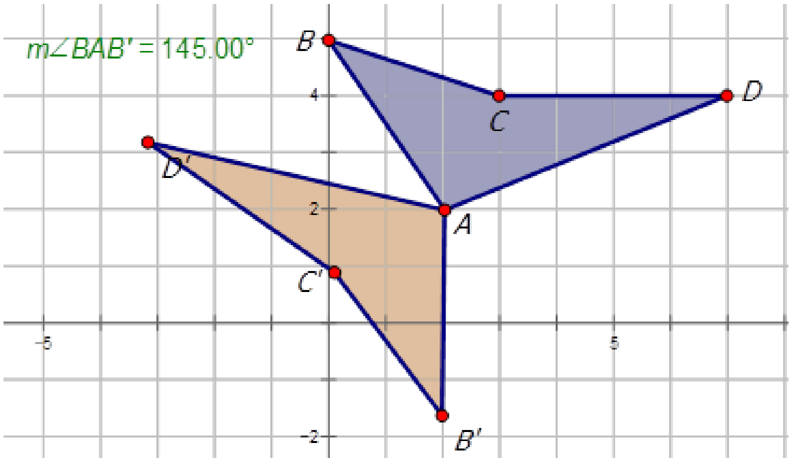
1. Examine the points of the preimage and the rotated image (the blue triangle).

TABLE 10.10:

Points on BCD	$B(1,-1)$	$C(2,6)$	$D(5,1)$
Points on $B'C'D'$	$B'(1,1)$	$C'(-6,2)$	$D'(-1,5)$

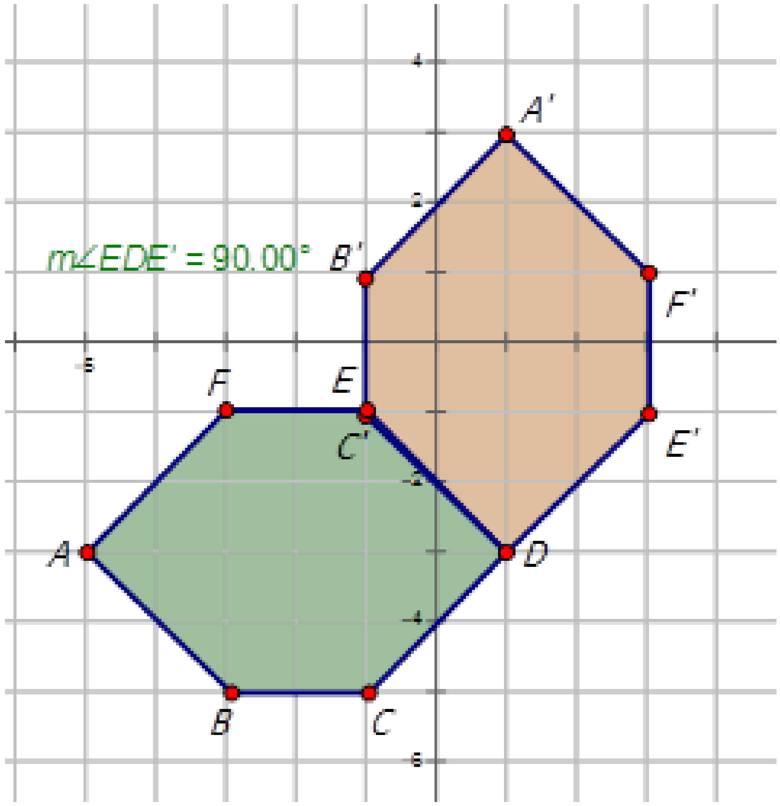
These points represent a rotation of 90°CW about the origin. Each coordinate point (x,y) has become the point $(-y,x)$.

2. For this image, look at the rotation. It is not rotated about the origin but rather about the point A. We can measure the angle of rotation:



The blue polygon is being rotated about the point *A* 145° clockwise. You would say that the blue polygon is rotated 145°CW to form the orange polygon.

3. For this image, look at the rotation. It is not rotated about the origin but rather about the point *A*. We can measure the angle of rotation:



The green polygon is being rotated about the point *D* 90° clockwise. You would say that the green hexagon is rotated 90°CW to form the orange hexagon.

Practice

If the following points were rotated about the origin with a 180°CCW rotation, what would be the coordinates of the rotated points?

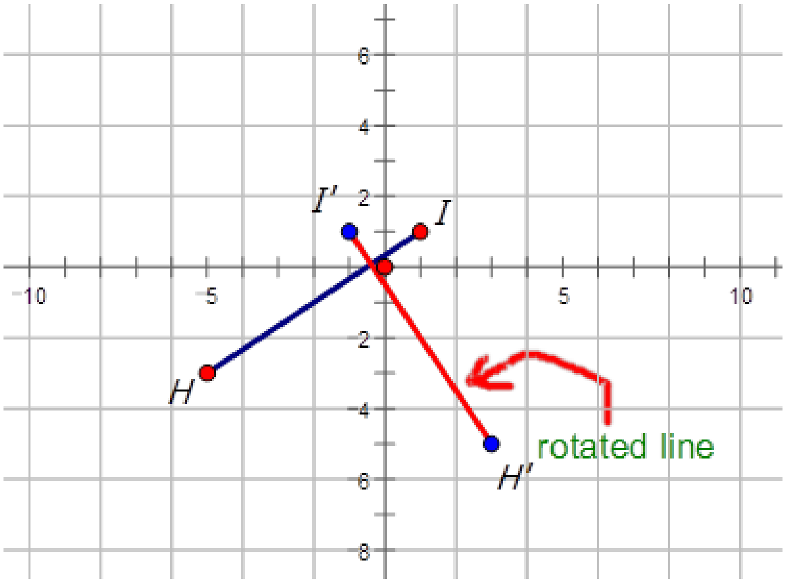
1. (3, 1)
2. (4, -2)
3. (-5, 3)
4. (-6, 4)
5. (-3, -3)

If the following points were rotated about the origin with a 90°CW rotation, what would be the coordinates of the rotated points?

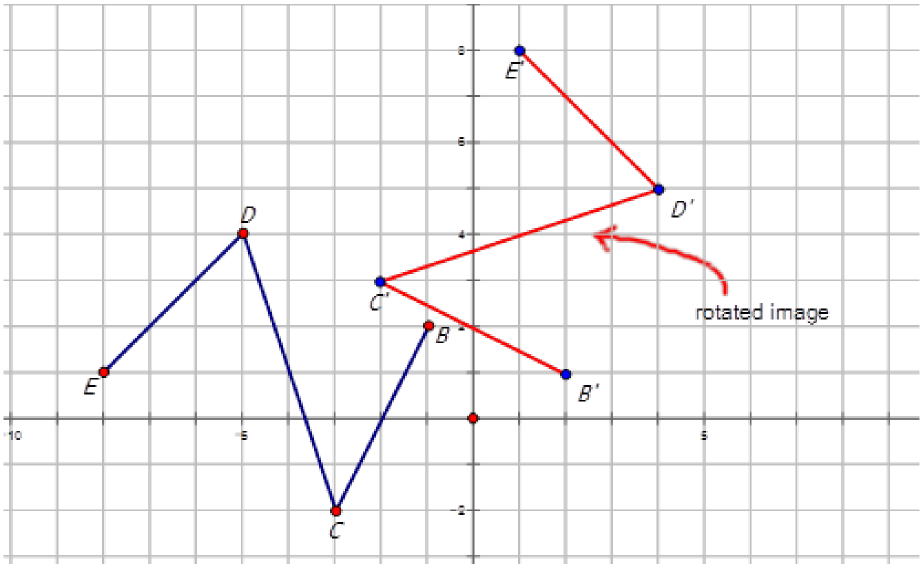
6. (-4, 3)
7. (5, -4)
8. (-5, -4)
9. (3, 3)

10. (-8, -9)

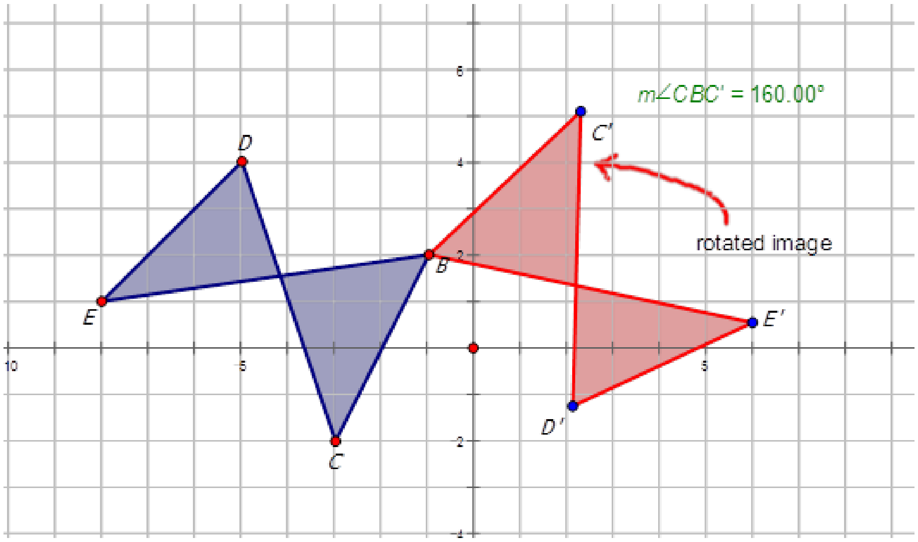
Describe the following rotations:

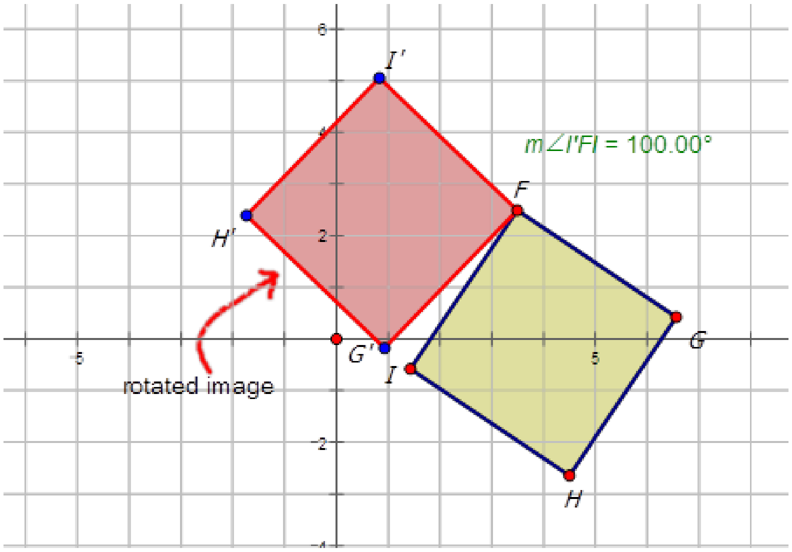


11.



12.

13. 

14. 

15. Why is it not necessary to specify the direction when rotating 180°?

10.8 Graphs of Rotations

Here you will learn how to graph a rotation.

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane. Rotate the image 110° counterclockwise about the point X. Show the resulting image.

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First watch this video to learn about graphs of rotations.

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CK-12 Foundation Chapter10GraphsofRotationsA

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Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees.

For now, in order to graph a rotation in general you will use geometry software. This will allow you to rotate any figure any number of degrees about any point. There are a few common rotations that are good to know how to do without geometry software, shown in the table below.

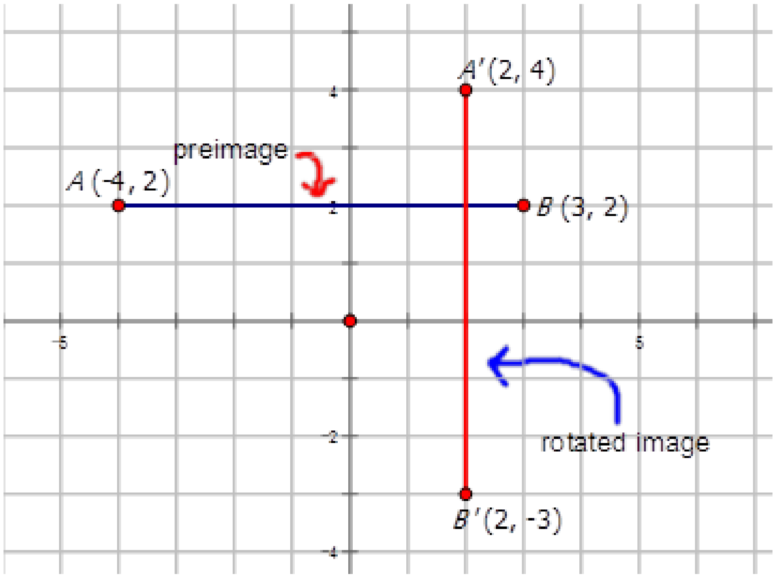
TABLE 10.11:

Center of Rotation	Angle of Rotation	Preimage (Point P)	Rotated Image (Point P')
(0, 0)	$90°$(or $-270°$)	(x,y)	$(-y,x)$
(0, 0)	$180°$(or $-180°$)	(x,y)	$(-x,-y)$
(0, 0)	$270°$(or $-90°$)	(x,y)	$(y,-x)$

Example A

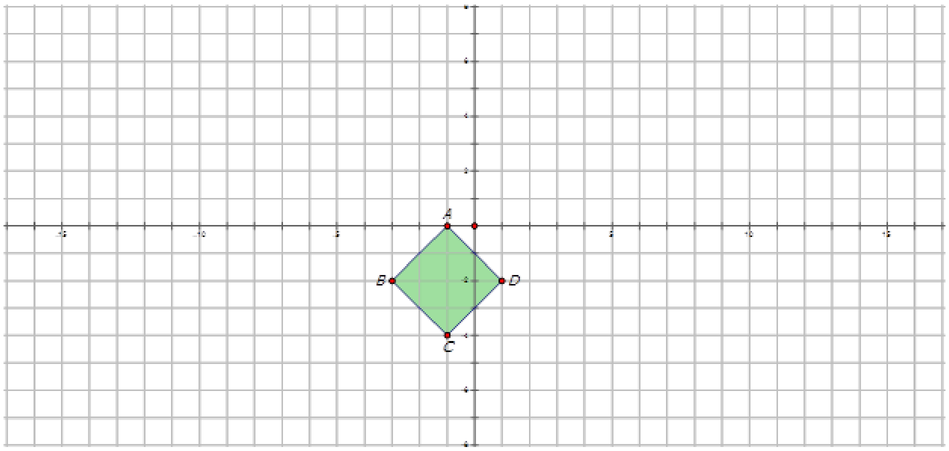
Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has been rotated about the origin at an angle of 90°CW. Draw the preimage and image and properly label each.

Solution:

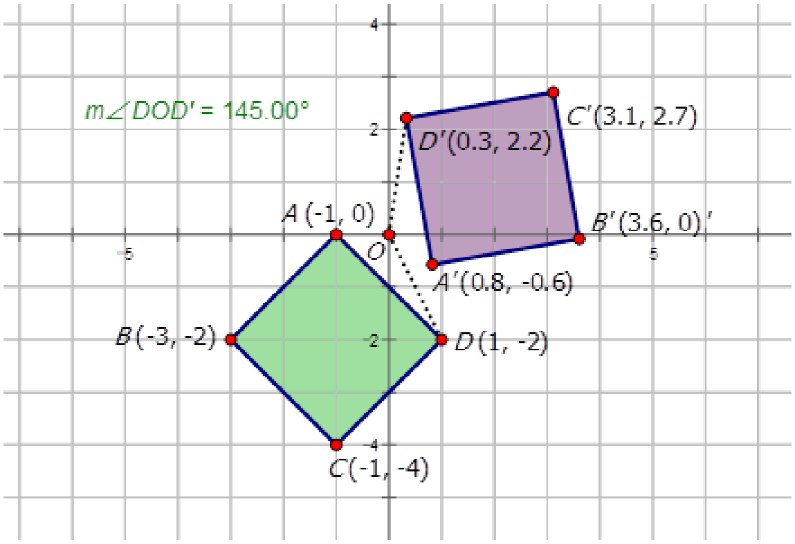


Example B

The diamond $ABCD$ is rotated 145°CCW about the origin to form the image $A'B'C'D'$. On the diagram, draw and label the rotated image.



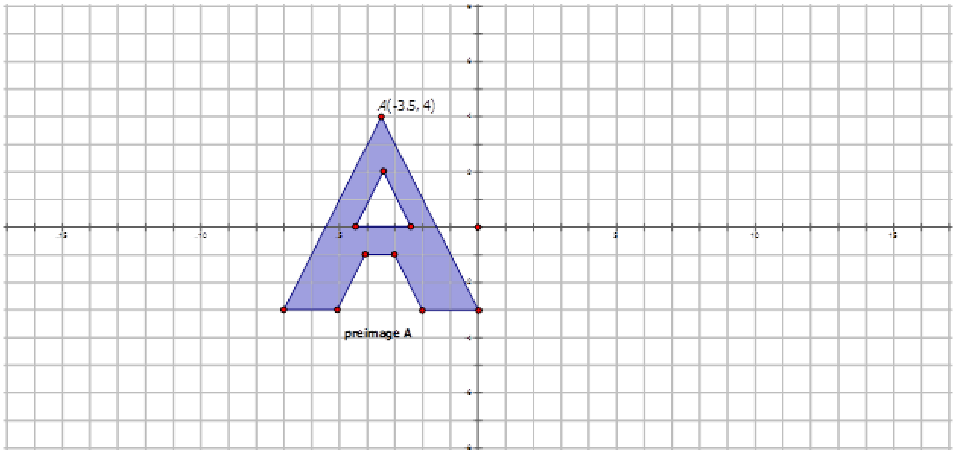
Solution:



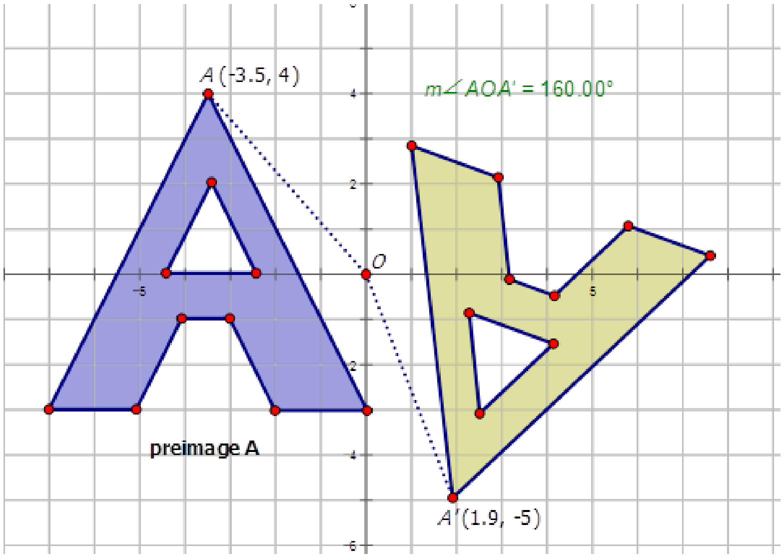
Notice the direction is counter-clockwise.

Example C

The following figure is rotated about the origin 200°CW to make a rotated image. On the diagram, draw and label the image.



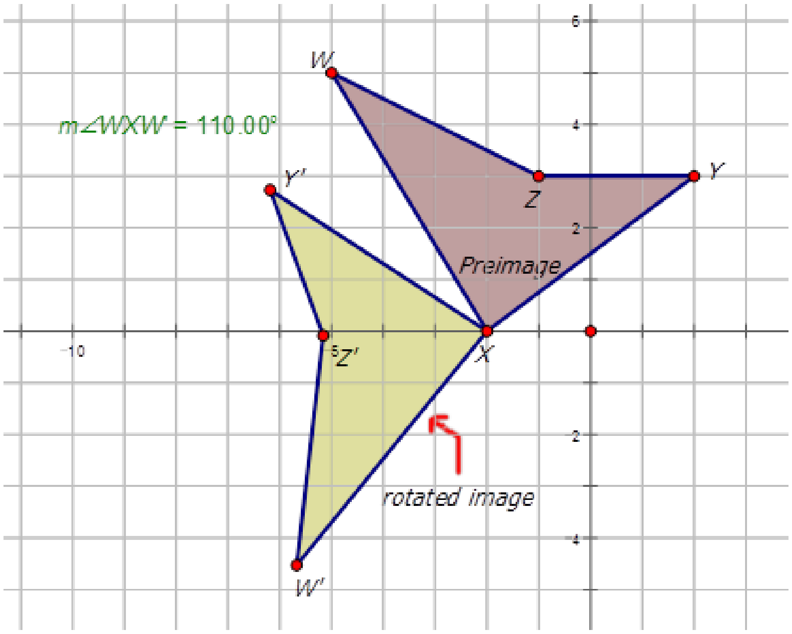
Solution:



Notice the direction of the rotation is counter-clockwise, therefore the angle of rotation is 160°.

Concept Problem Revisited

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane. Rotate the image 110° counterclockwise about the point X. Show the resulting image.



Vocabulary

Center of rotation

A ***center of rotation*** is the fixed point that a figure rotates about when undergoing a rotation.

Rotation

A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

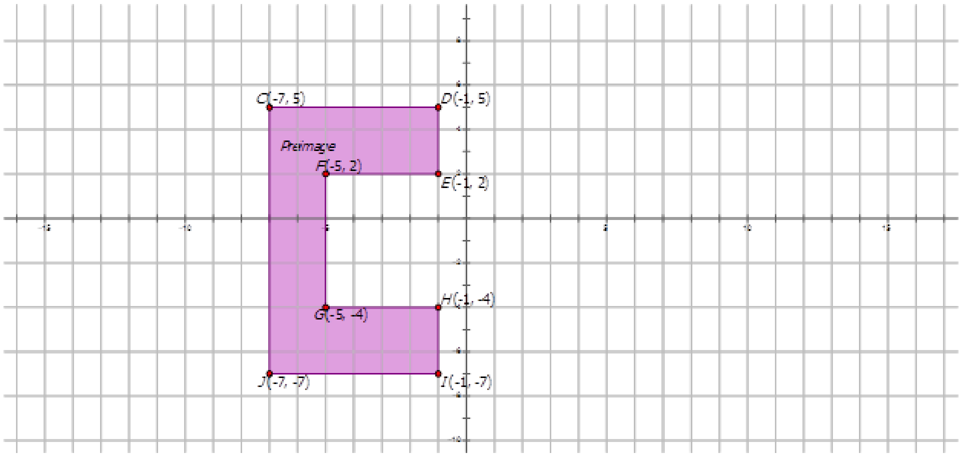
Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

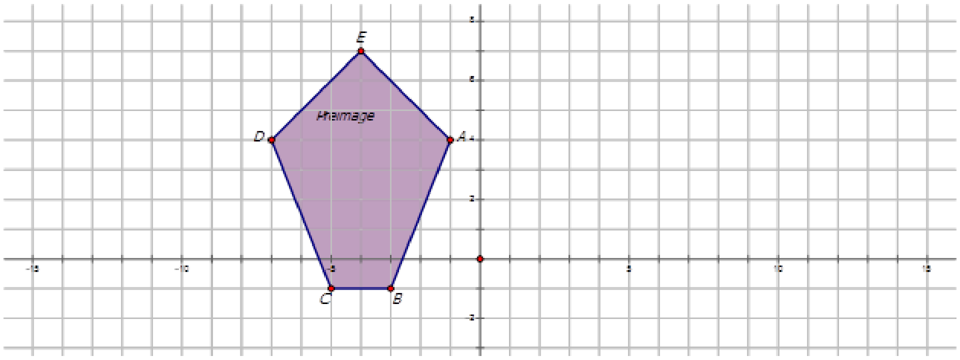
Guided Practice

1. Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has been rotated 60°CW about the point S. Draw the preimage and image and properly label each.

2. The polygon below has been rotated 155°CCW about the origin. Draw the rotated image and properly label each.

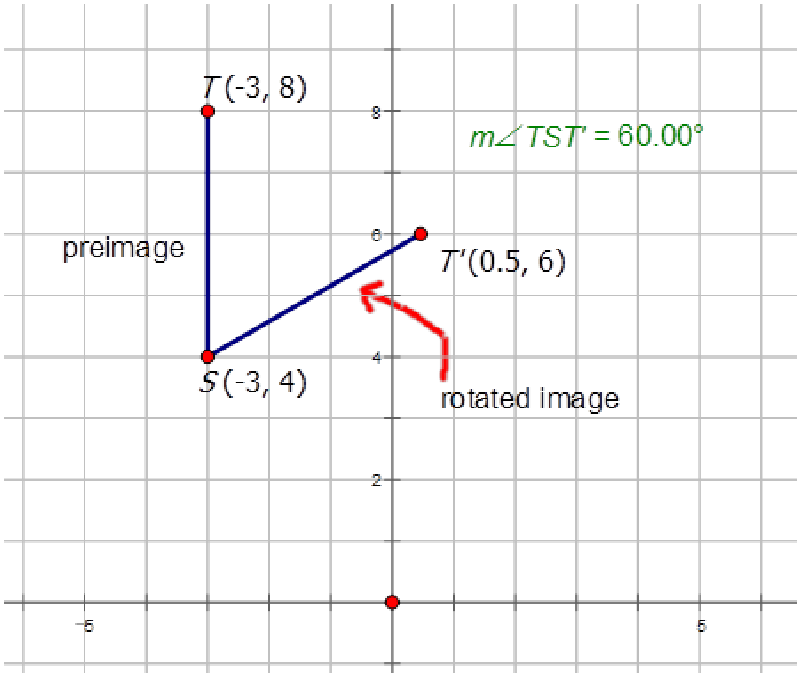


3. The purple pentagon is rotated about the point A 225°. Find the coordinates of the purple pentagon. On the diagram, draw and label the rotated pentagon.



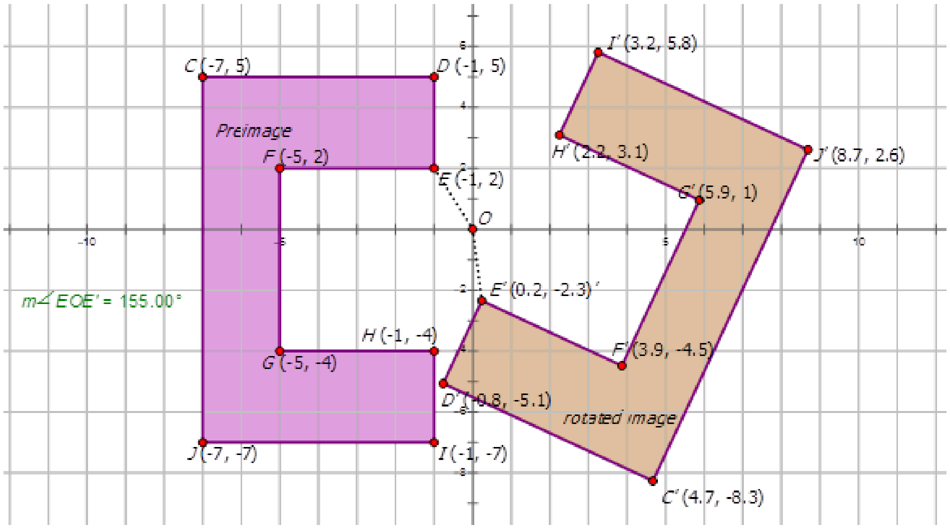
Answers:

1.



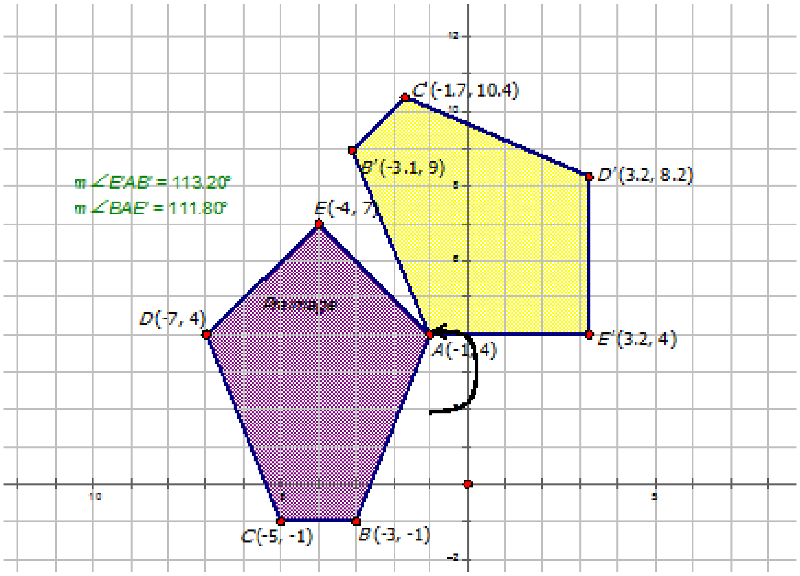
Notice the direction of the angle is clockwise, therefore the angle measure is 60°CW or $-60°$.

2.



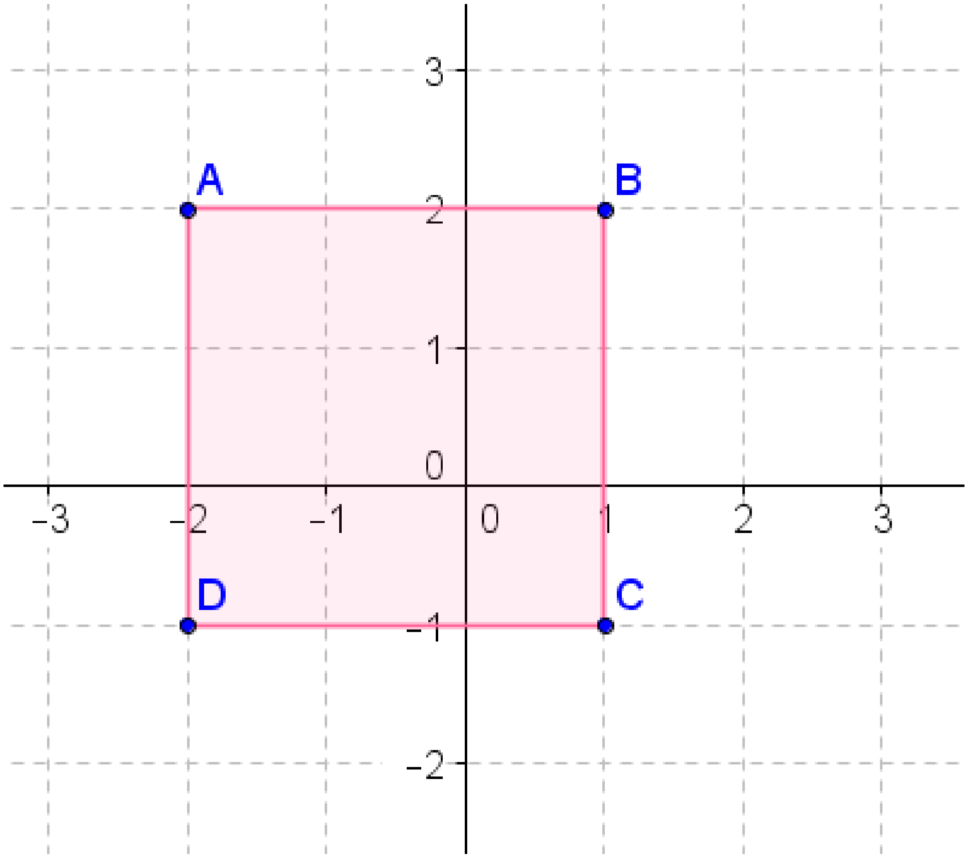
Notice the direction of the angle is counter-clockwise, therefore the angle measure is 155°CCW or 155°.

3.

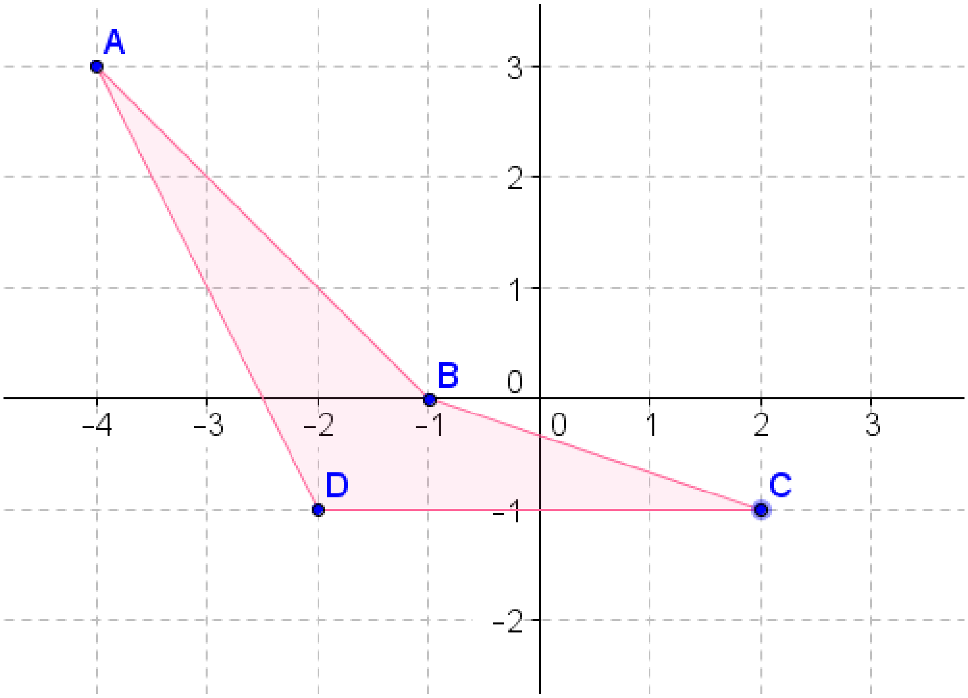


The measure of $\angle BAB' = m\angle BAE' + m\angle E'AB'$. Therefore $\angle BAB' = 111.80^\circ + 113.20^\circ$ or 225°. Notice the direction of the angle is counter-clockwise, therefore the angle measure is 225°CCW or 225°.

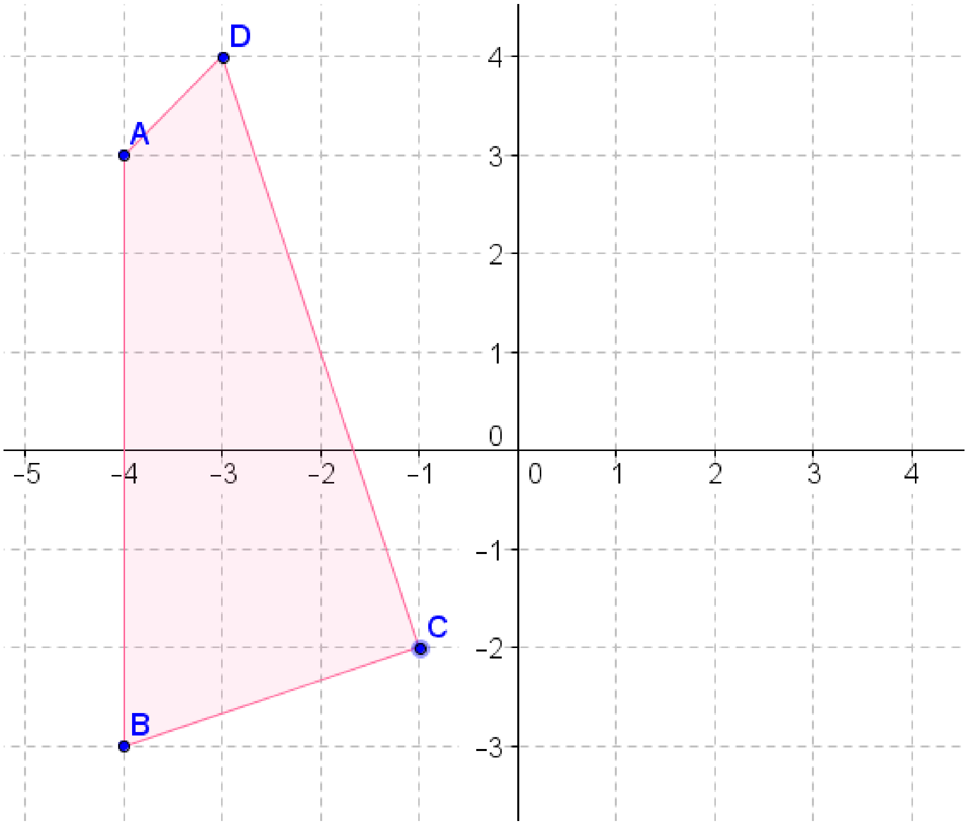
Practice



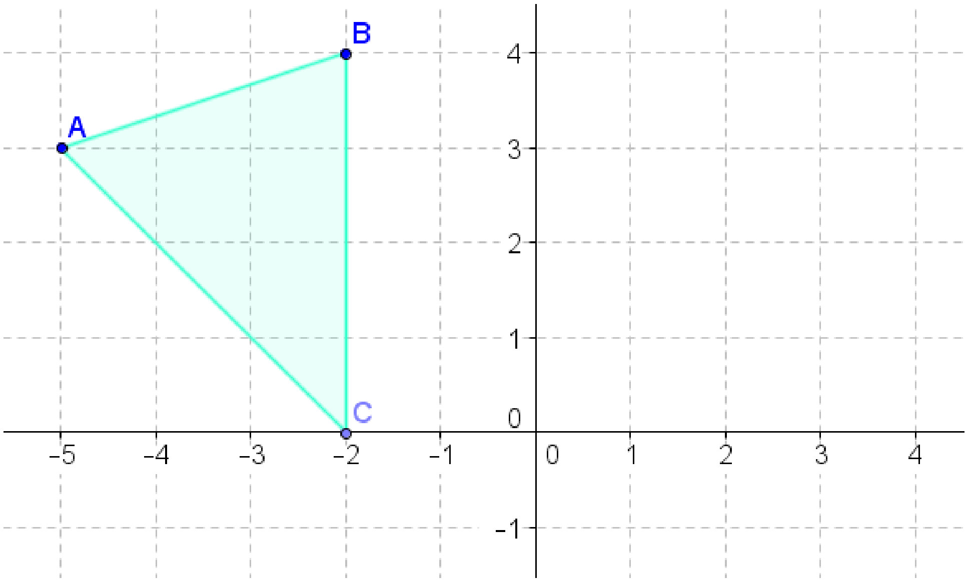
1. Rotate the above figure 90° clockwise about the origin.
2. Rotate the above figure 270° clockwise about the origin.
3. Rotate the above figure 180° about the origin.



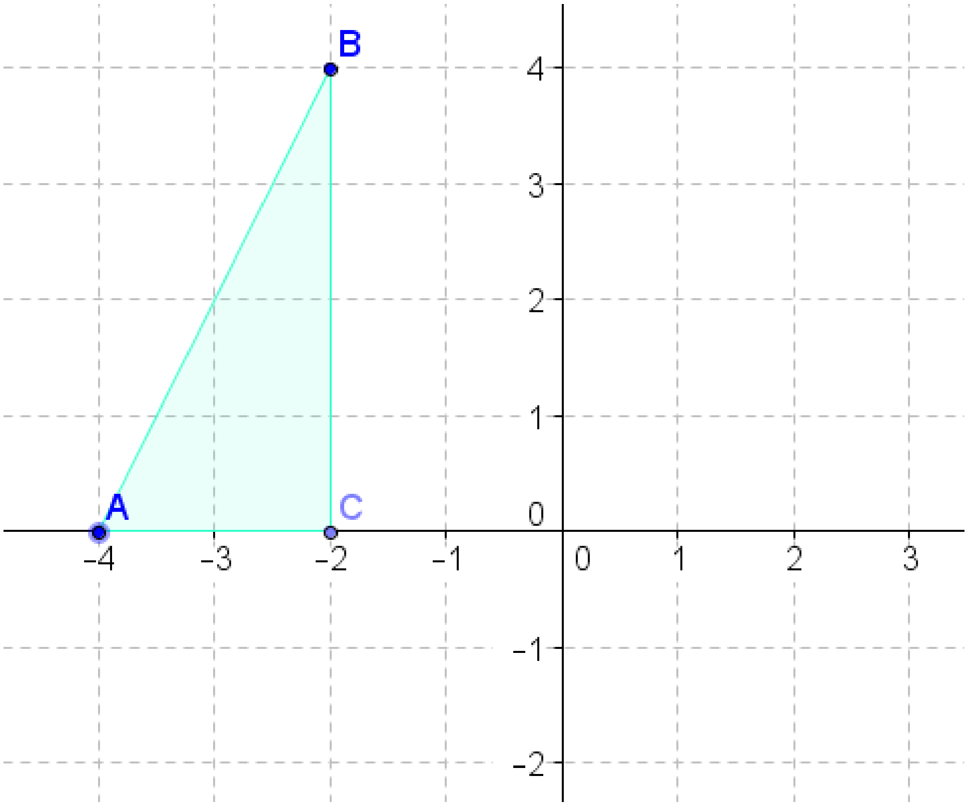
4. Rotate the above figure 90° counterclockwise about the origin.
5. Rotate the above figure 270° counterclockwise about the origin.
6. Rotate the above figure 180° about the origin.



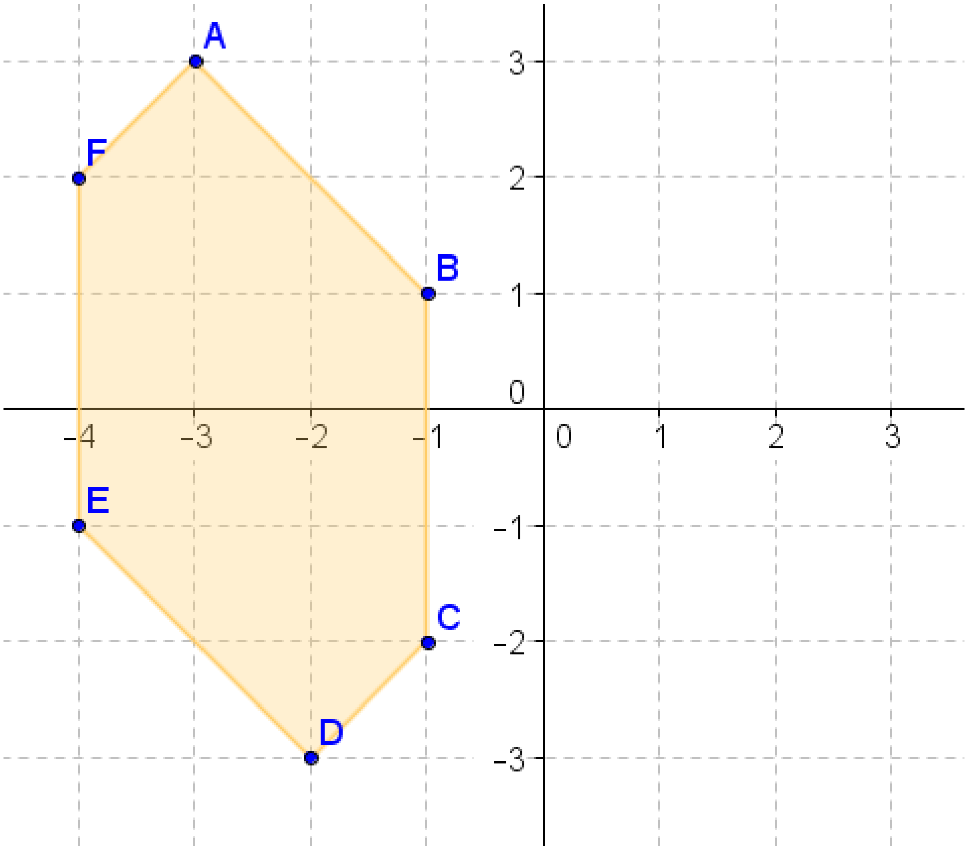
7. Rotate the above figure 90° clockwise about the origin.
8. Rotate the above figure 270° clockwise about the origin.
9. Rotate the above figure 180° about the origin.



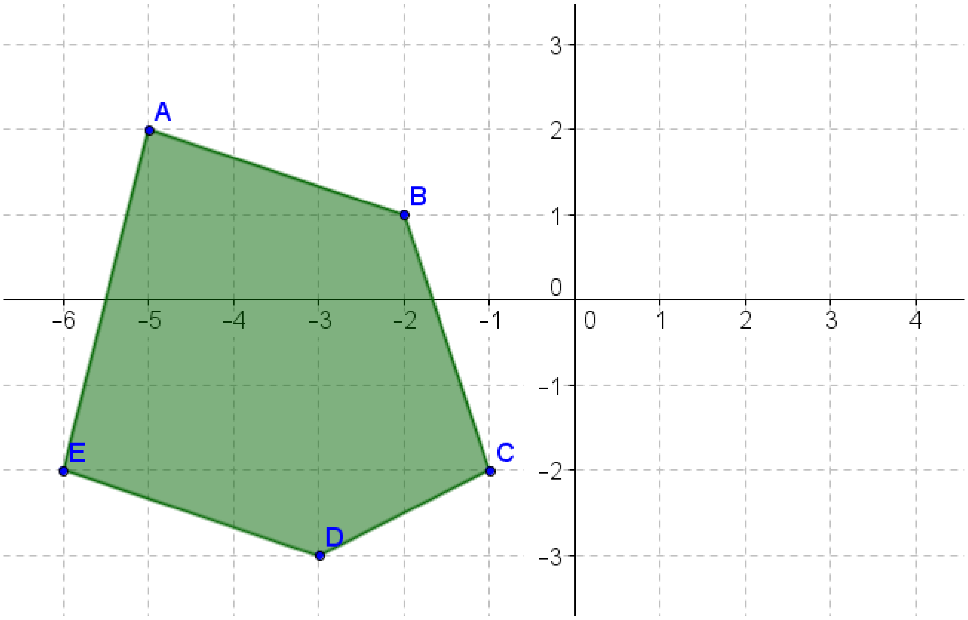
10. Rotate the above figure 90° counterclockwise about the origin.
11. Rotate the above figure 270° counterclockwise about the origin.
12. Rotate the above figure 180° about the origin.



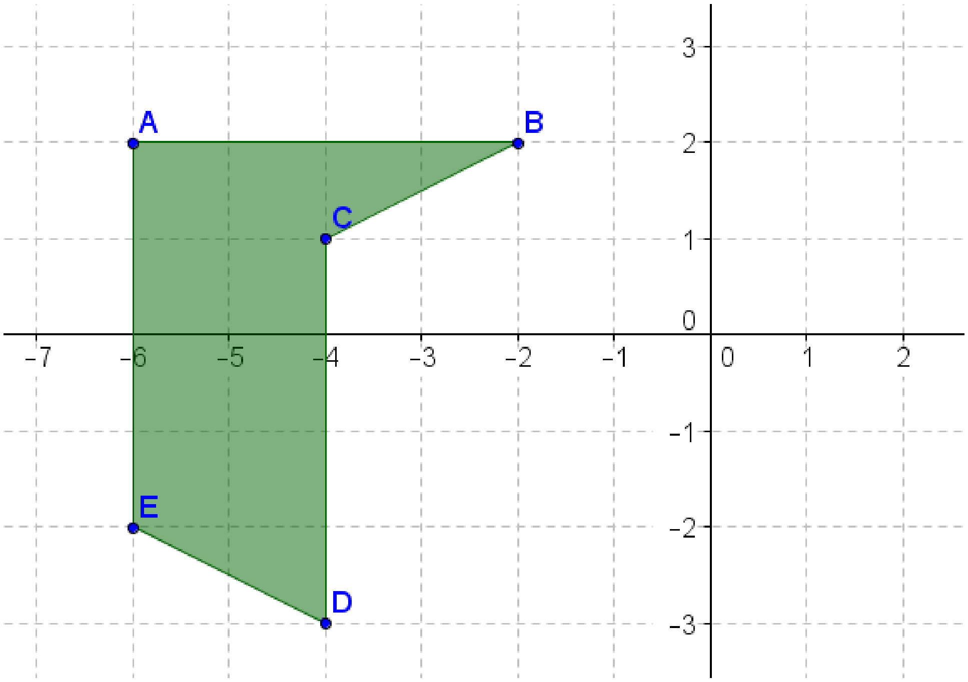
13. Rotate the above figure 90° clockwise about the origin.
14. Rotate the above figure 270° clockwise about the origin.
15. Rotate the above figure 180° about the origin.



16. Rotate the above figure 90° counterclockwise about the origin.
17. Rotate the above figure 270° counterclockwise about the origin.
18. Rotate the above figure 180° about the origin.



19. Rotate the above figure 90° clockwise about the origin.
20. Rotate the above figure 270° clockwise about the origin.
21. Rotate the above figure 180° about the origin.

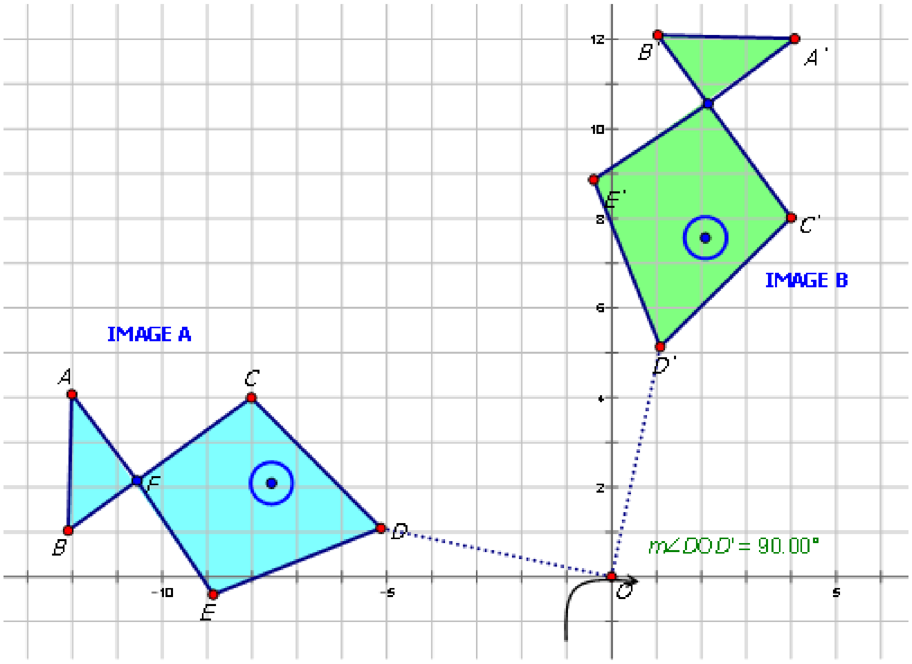


22. Rotate the above figure 90° counterclockwise about the origin.
23. Rotate the above figure 270° counterclockwise about the origin.
24. Rotate the above figure 180° about the origin.

10.9 Rules for Rotations

Here you will learn the notation used for rotations.

The figure below shows a pattern of two fish. Write the mapping rule for the rotation of Image A to Image B.



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First watch this video to learn about writing rules for rotations.

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Click image to the left for more content.

CK-12 Foundation Chapter10RulesforRotationsA

Then watch this video to see some examples.

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Click image to the left for more content.

CK-12 Foundation Chapter10RulesforRotationsB

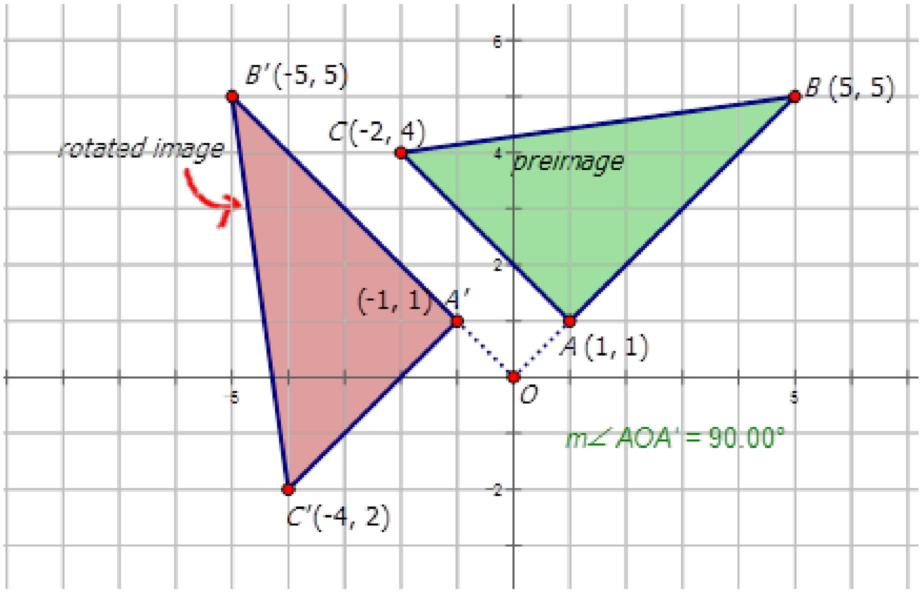
Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A rotation is an example of a transformation where a figure is rotated about a specific point (called the center of rotation), a certain number of degrees. Common rotations about the origin are shown below:

TABLE 10.12:

Center of Rotation	**Angle of Rotation**	**Preimage (Point P)**	**Rotated (Point P')**	**Image**	**Notation (Point P')**
(0, 0)	$90°$(or $-270°$)	(x,y)	$(-y,x)$		$(x,y) \rightarrow (-y,x)$
(0, 0)	$180°$(or $-180°$)	(x,y)	$(-x,-y)$		$(x,y) \rightarrow (-x,-y)$
(0, 0)	$270°$(or $-90°$)	(x,y)	$(y,-x)$		$(x,y) \rightarrow (y,-x)$

You can describe rotations in words, or with notation. Consider the image below:



Notice that the preimage is rotated about the origin 90°CCW. If you were to describe the rotated image using notation, you would write the following:

$$R_{90^\circ}(x,y) = (-y,x)$$

Example A

Find an image of the point (3, 2) that has undergone a clockwise rotation:

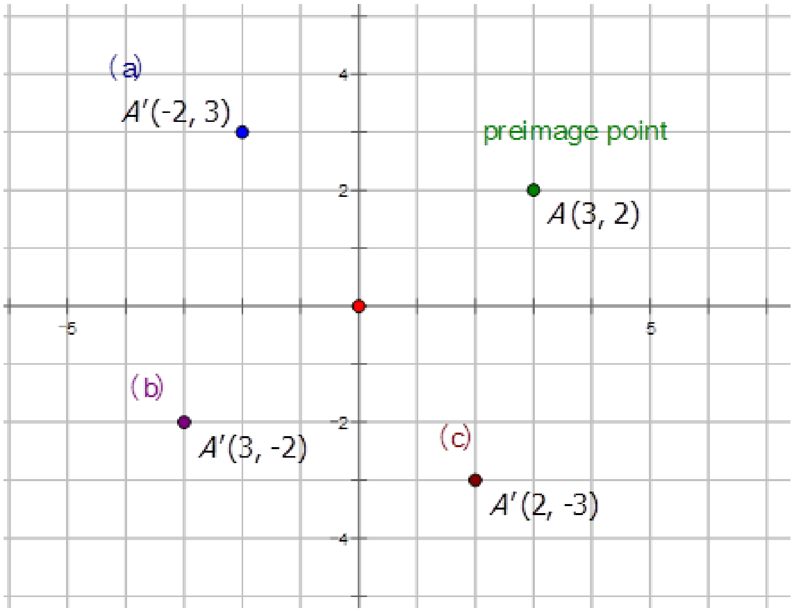
a) about the origin at 90°,

b) about the origin at 180°, and

c) about the origin at 270°.

Write the notation to describe the rotation.

Solution:



a) Rotation about the origin at 90° : $R_{90^\circ}(x,y) = (-y,x)$

b) Rotation about the origin at 180° : $R_{180^\circ}(x,y) = (-x,-y)$

c) Rotation about the origin at 270° : $R_{270^\circ}(x,y) = (y,-x)$

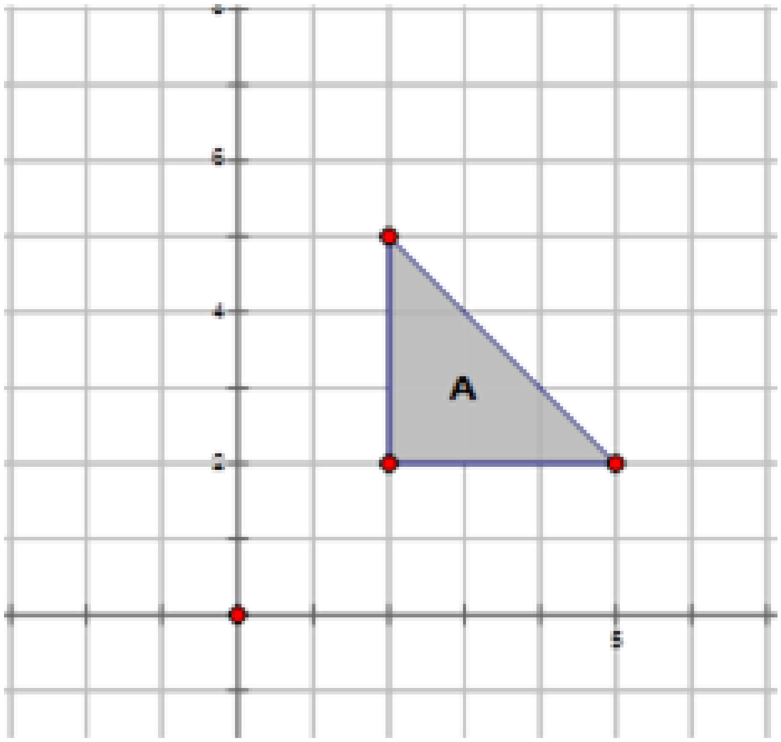
Example B

Rotate Image A in the diagram below:

a) about the origin at 90°, and label it *B*.

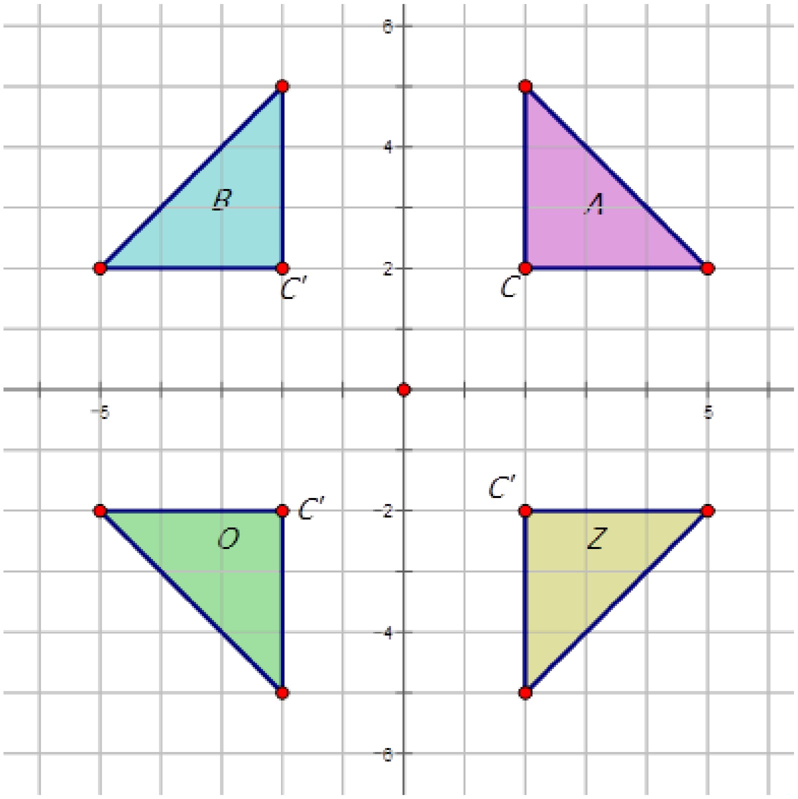
b) about the origin at 180°, and label it *O*.

c) about the origin at 270°, and label it *Z*.



Write notation for each to indicate the type of rotation.

Solution:



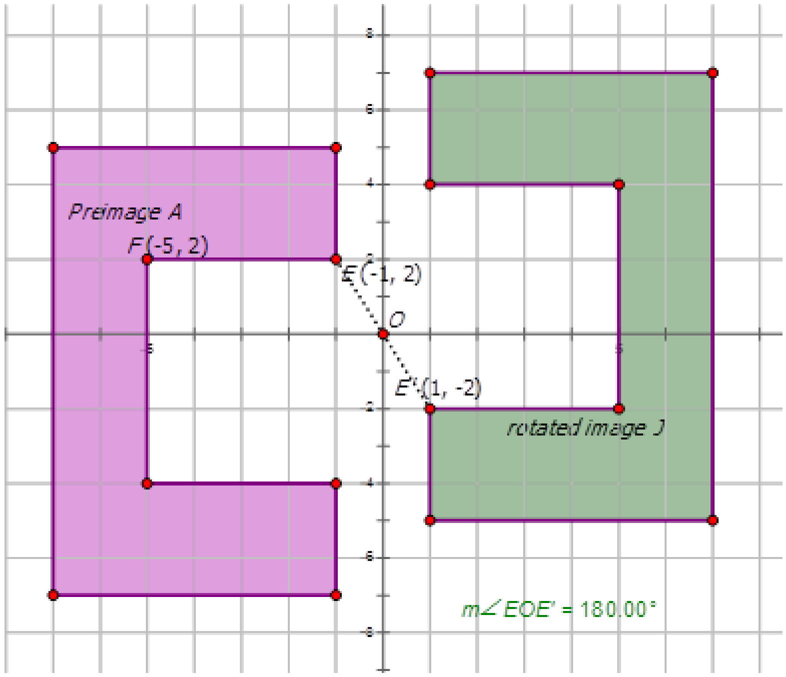
a) Rotation about the origin at 90°: $R_{90^\circ}A \rightarrow B = R_{90^\circ}(x,y) \rightarrow (-y,x)$

b) Rotation about the origin at 180°: $R_{180^\circ}A \rightarrow O = R_{180^\circ}(x,y) \rightarrow (-x,-y)$

c) Rotation about the origin at 270°: $R_{270^\circ}A \rightarrow Z = R_{270^\circ}(x,y) \rightarrow (y,-x)$

Example C

Write the notation that represents the rotation of the preimage A to the rotated image J in the diagram below.



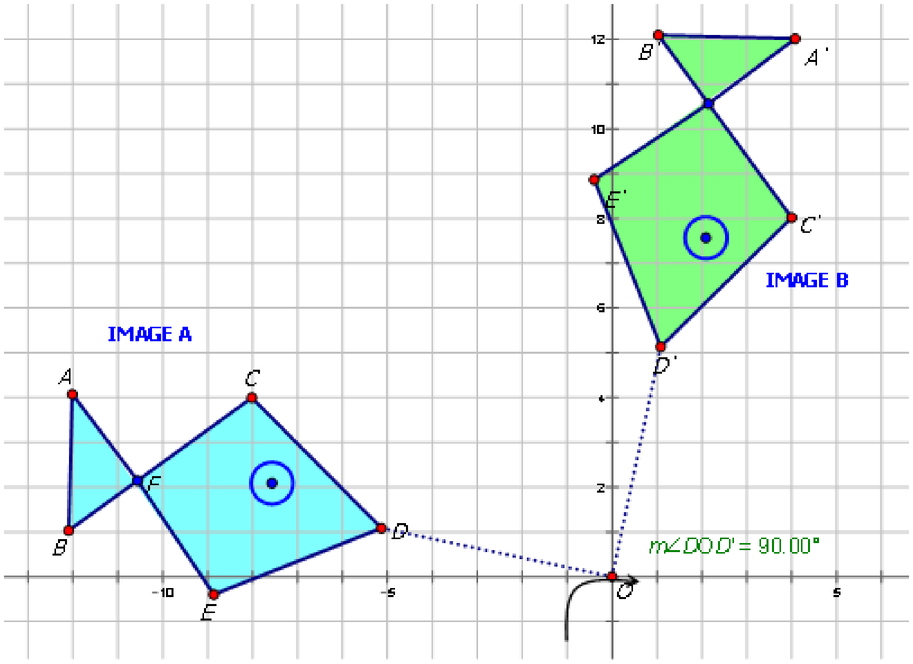
First, pick a point in the diagram to use to see how it is rotated.

$$E : (-1,2) \quad E' : (1,-2)$$

Notice how both the x- and y-coordinates are multiplied by -1. This indicates that the preimage A is reflected about the origin by 180°CCW to form the rotated image J. Therefore the notation is $R_{180^\circ}A \rightarrow J = R_{180^\circ}(x,y) \rightarrow (-x,-y)$.

Concept Problem Revisited

The figure below shows a pattern of two fish. Write the mapping rule for the rotation of Image A to Image B.



Notice that the angle measure is 90° and the direction is clockwise. Therefore the Image A has been rotated -90° to form Image B. To write a rule for this rotation you would write: $R_{270^\circ}(x,y) = (-y,x)$.

Vocabulary

Notation Rule

A ***notation rule*** has the following form $R_{180^\circ}A \rightarrow O = R_{180^\circ}(x,y) \rightarrow (-x,-y)$ and tells you that the image A has been rotated about the origin and both the x- and y-coordinates are multiplied by -1.

Center of rotation

A ***center of rotation*** is the fixed point that a figure rotates about when undergoing a rotation.

Rotation

A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

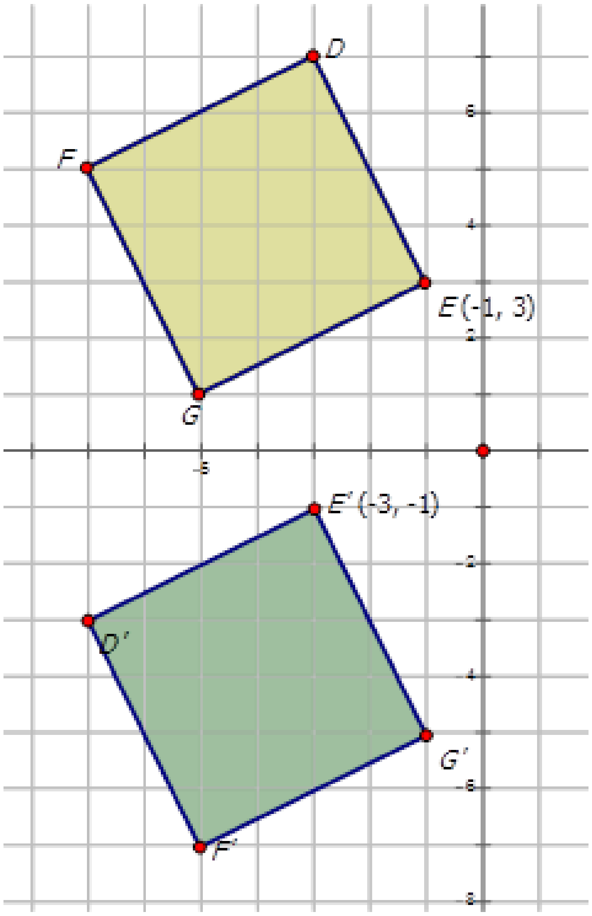
Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

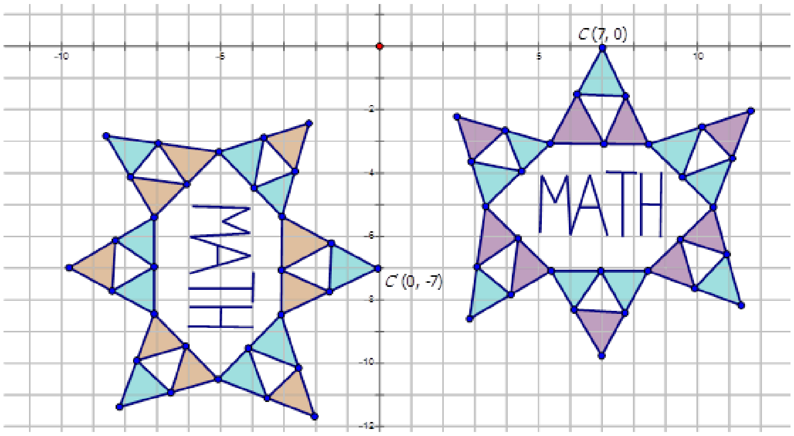
Guided Practice

1. Thomas describes a rotation as point J moving from $J(-2,6)$ to $J'(6,2)$. Write the notation to describe this rotation for Thomas.

2. Write the notation that represents the rotation of the yellow diamond to the rotated green diamond in the diagram below.

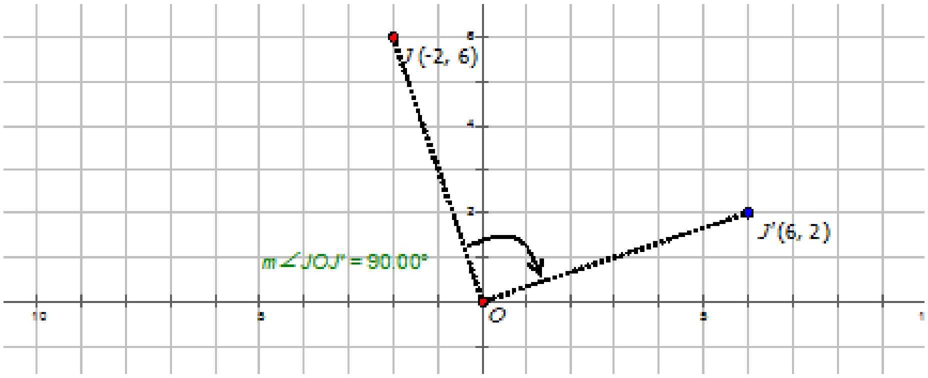


3. Karen was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



Answers:

1. $J : (-2,6) \quad J' : (6,2)$

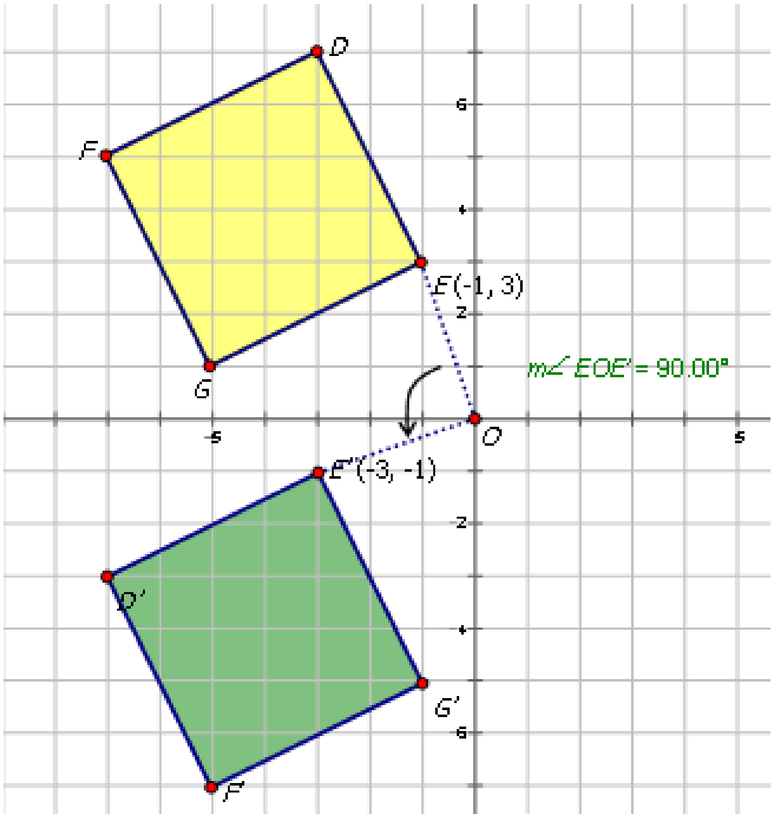


Since the x-coordinate is multiplied by -1, the y-coordinate remains the same, and finally the x- and y-coordinates change places, this is a rotation about the origin by 270° or $-90°$. The notation is: $R_{270^\circ}J \rightarrow J' = R_{270^\circ}(x,y) \rightarrow (y,-x)$

2. In order to write the notation to describe the rotation, choose one point on the preimage (the yellow diamond) and then the rotated point on the green diamond to see how the point has moved. Notice that point E is shown in the diagram:

$$E(-1,3) \rightarrow E'(-3,-1)$$

Since both x- and y-coordinates are reversed places and the y-coordinate has been multiplied by -1, the rotation is about the origin 90°. The notation for this rotation would be: $R_{90^\circ}(x,y) \rightarrow (-y,x)$.



3. In order to write the notation to describe the transformation, choose one point on the preimage (purple and blue

diagram) and then the transformed point on the orange and blue diagram to see how the point has moved. Notice that point C is shown in the diagram:

$$C(7,0) \rightarrow C'(0,-7)$$

Since the x-coordinates only are multiplied by -1, and then x- and y-coordinates change places, the transformation is a rotation is about the origin by 270°. The notation for this rotation would be: $R_{270^\circ}(x,y) \rightarrow (y,-x)$.

Practice

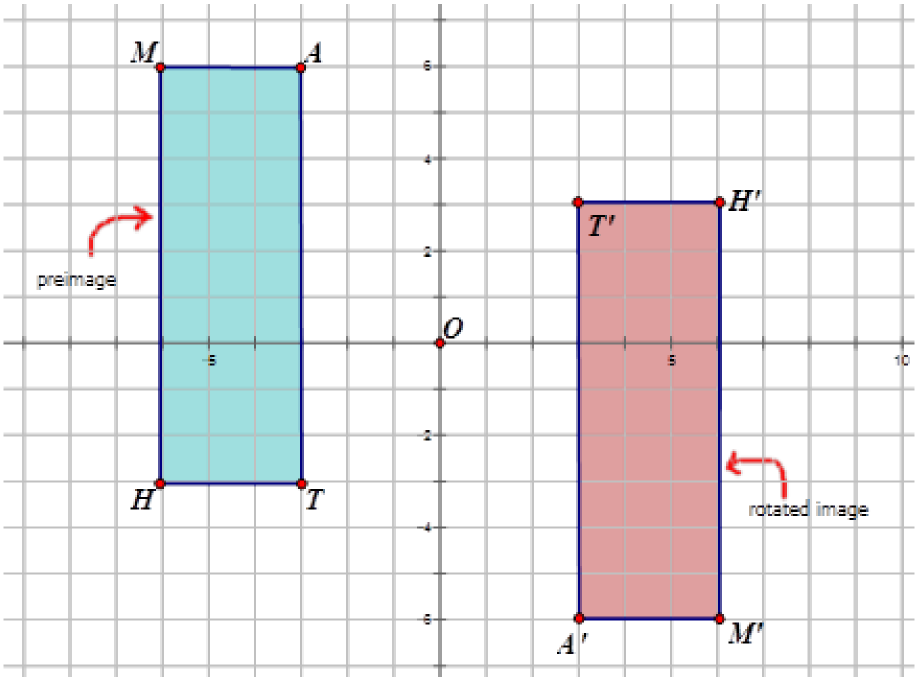
Complete the following table:

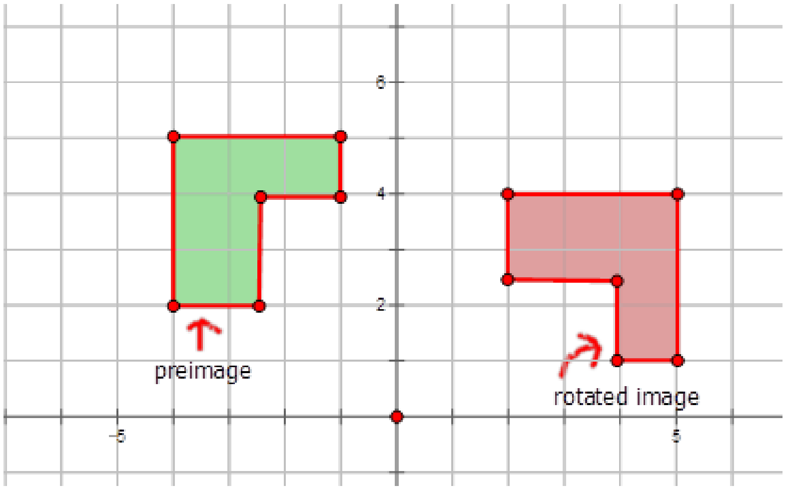
TABLE 10.13:

Starting Point	90° **Rotation**	180° **Rotation**	270° **Rotation**	360° **Rotation**
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				

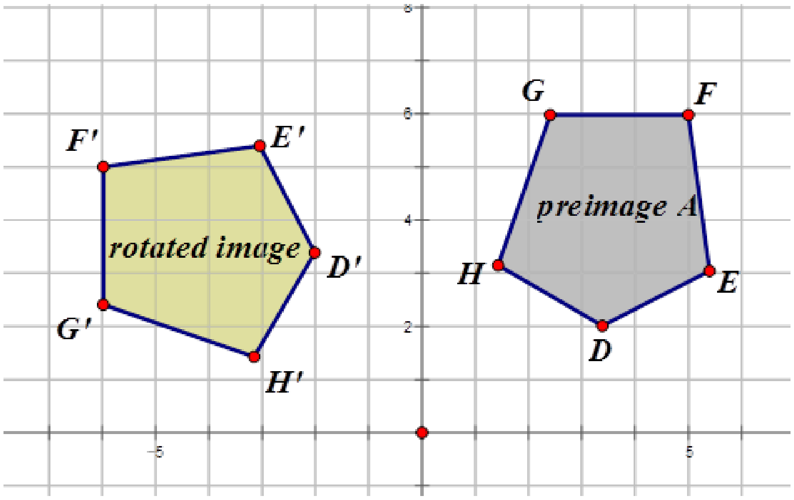
Write the notation that represents the rotation of the preimage to the image for each diagram below.

6.

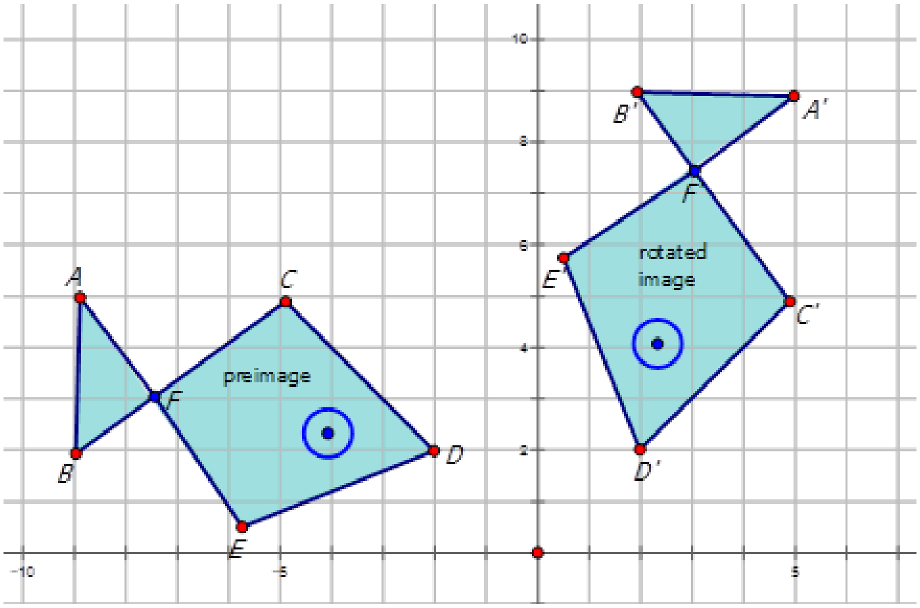



6
4
2
−5
5
preimage
rotated image

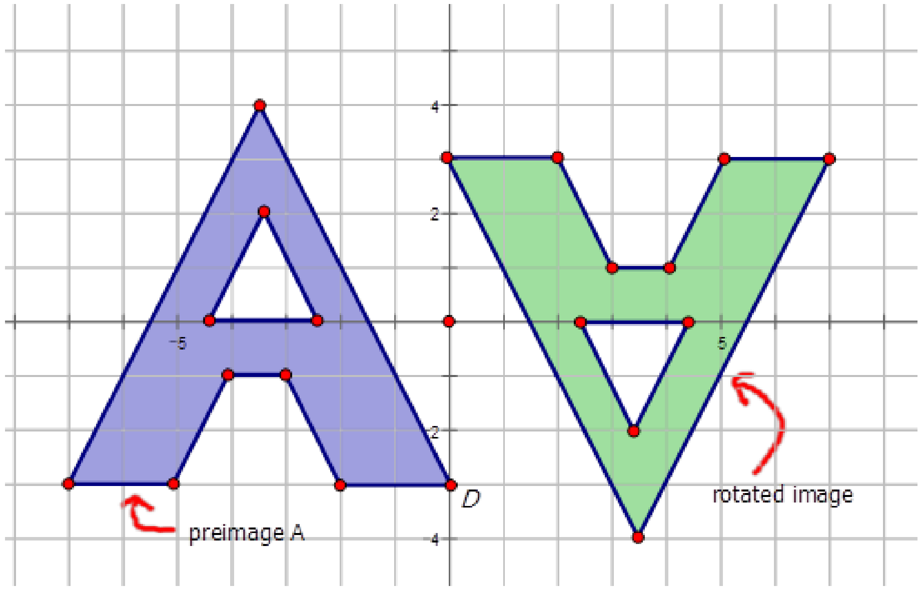
7.


8
6
4
2
−5
5
G
F
F'
E'
preimage A
rotated image
D'
H
E
G'
D
H'

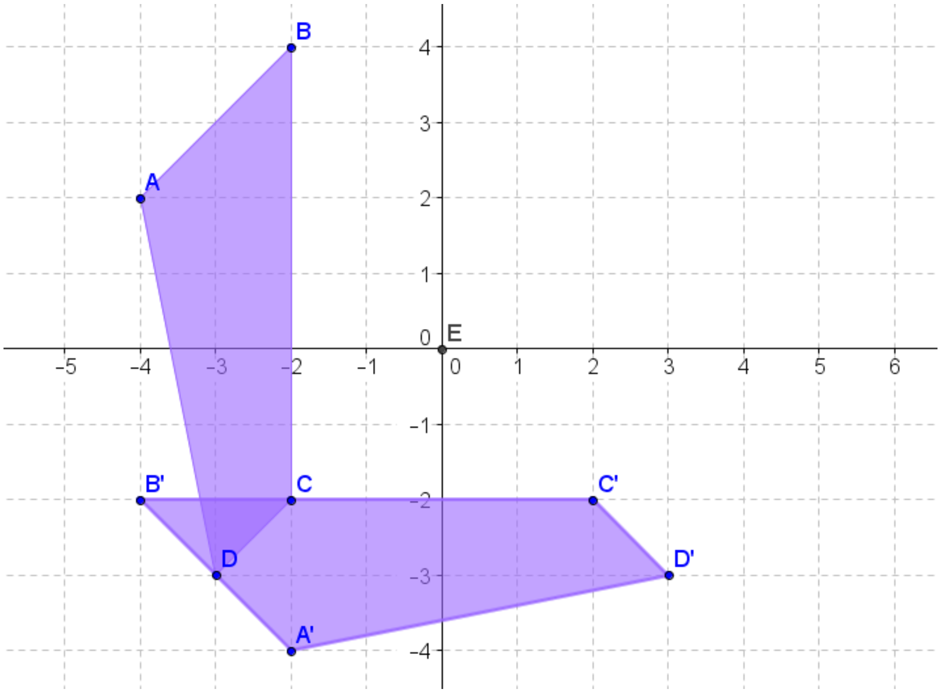
8.


10
8
6
4
2
−10
−5
5
B'
A'
F
rotated image
E'
C'
A
C
preimage
F
B
D
E
D'

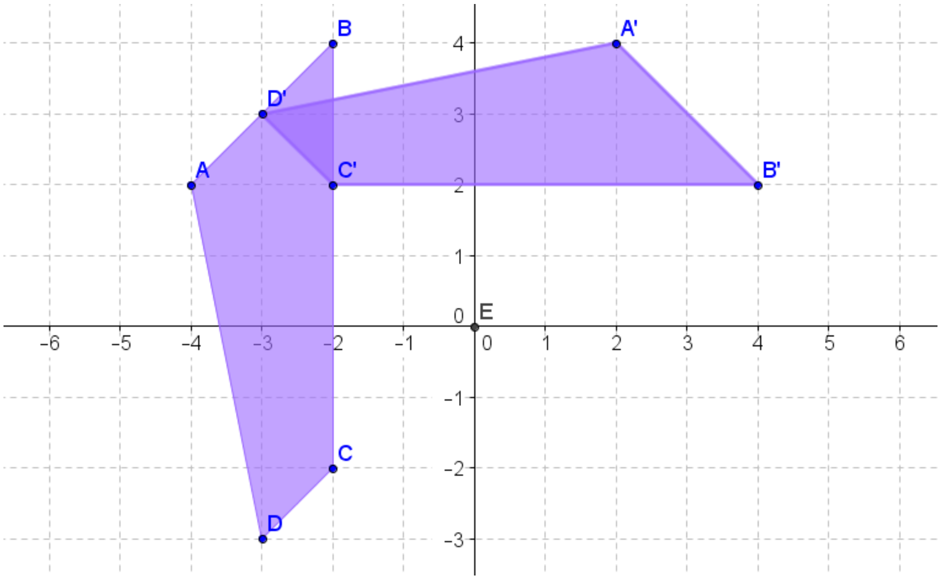
9.

10. 

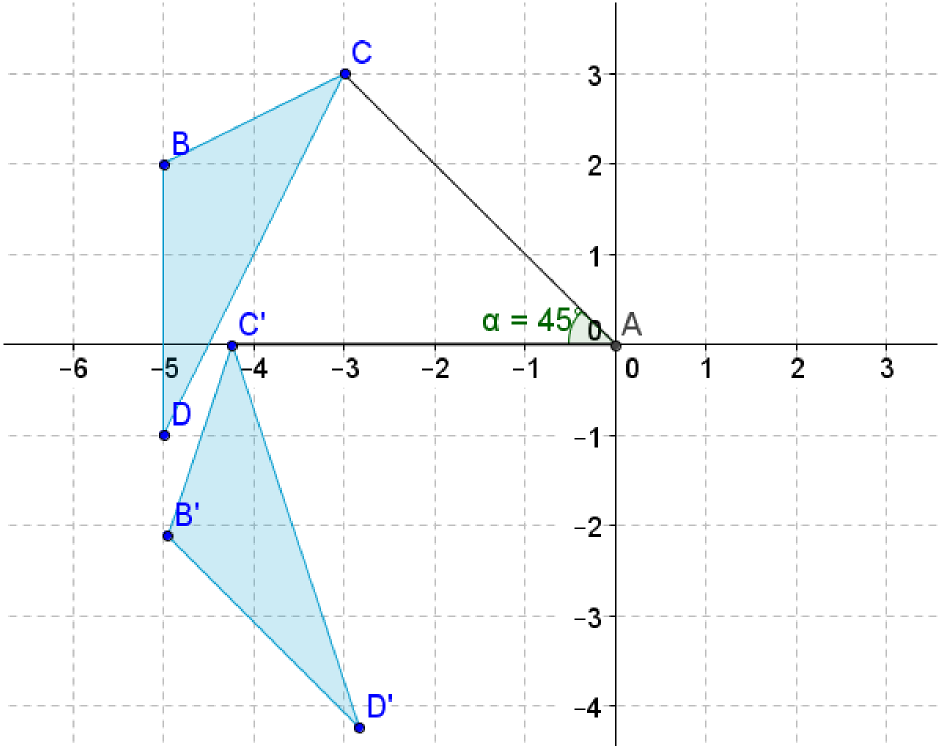
Write the notation that represents the rotation of the preimage to the image for each diagram below.

11. 

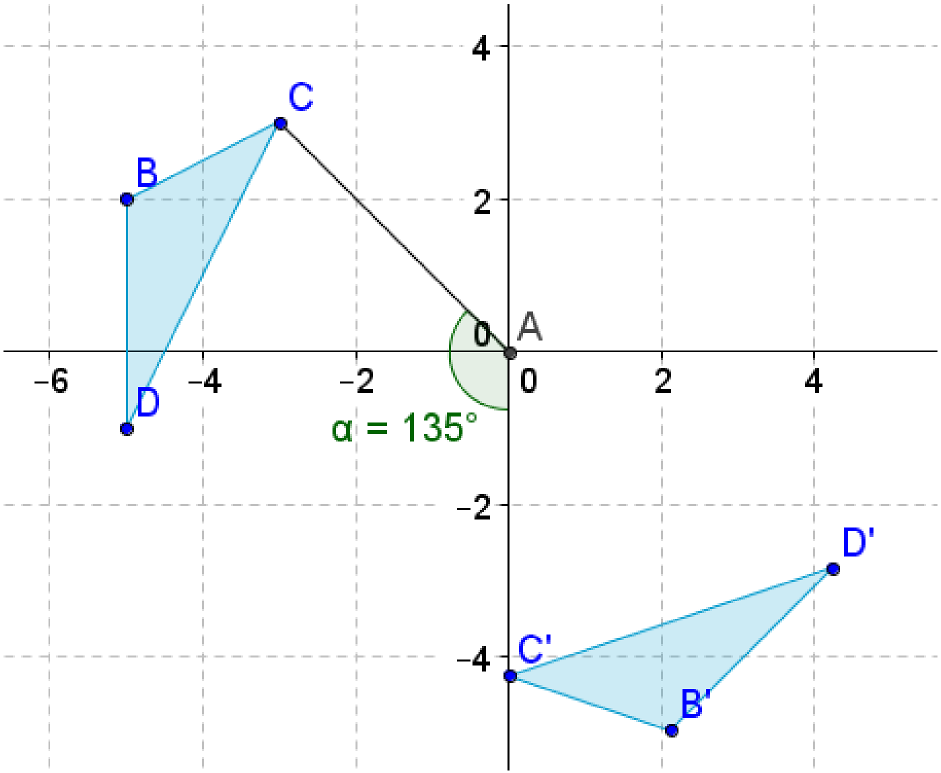
12.


A
B
C
D
A'
B'
C'
D'
E
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
-1
-2
-3
1
2
3
4

13.

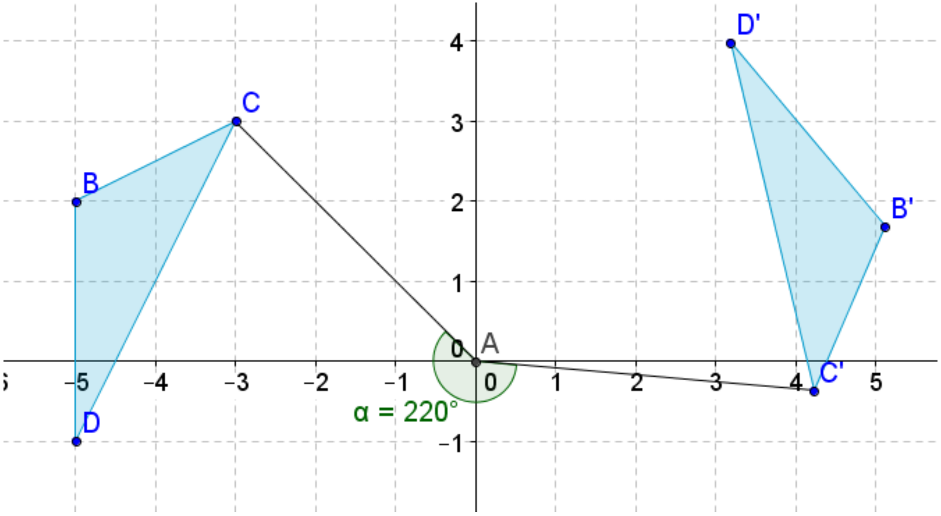

A
B
C
D
B'
C'
D'
α = 45°
-6
-5
-4
-3
-2
-1
0
1
2
3
-1
-2
-3
-4
1
2
3

14.



C
B
D
A
α = 135°
D'
C'
B'
4
2
0
−2
−4
−6
−4
−2
0
2
4

15.

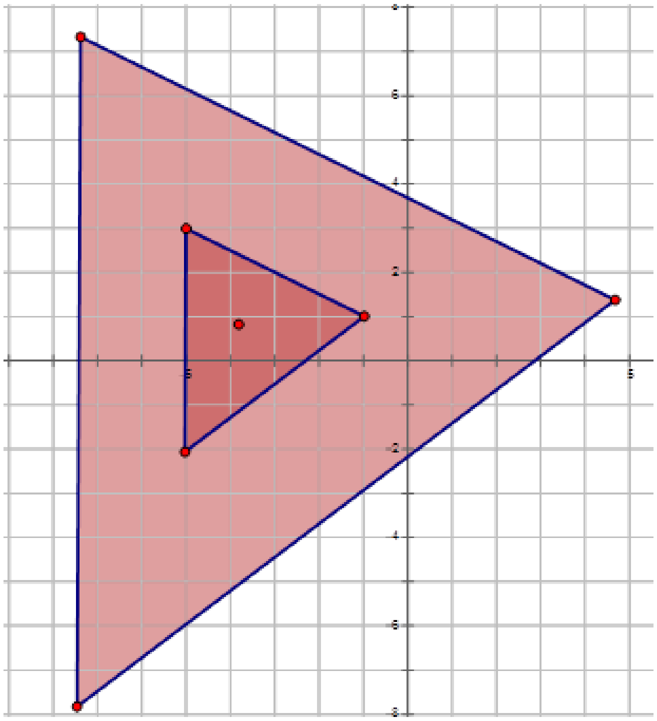


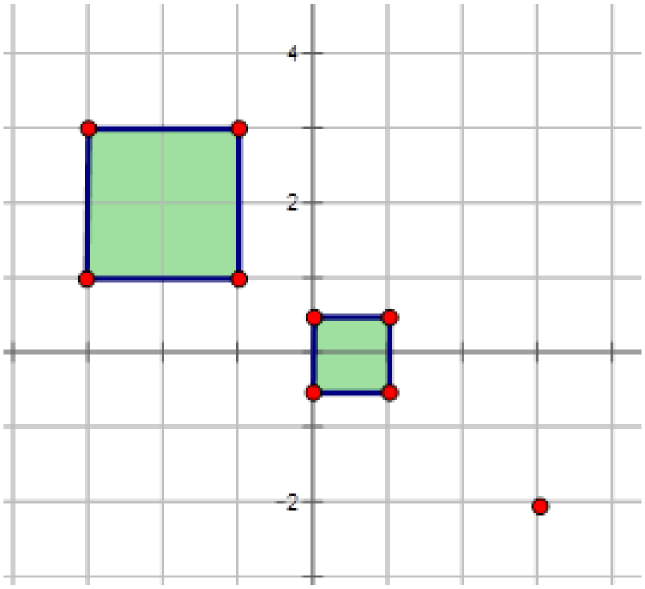
D'
C
B
B'
A
C'
D
α = 220°
4
3
2
1
0
−1
−5
−4
−3
−2
−1
0
1
2
3
4
5

10.10 Dilations

Here you will learn about geometric dilations.

Which one of the following figures represents a dilation? Explain.





Watch This

First watch this video to learn about dilations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10DilationsA

Then watch this video to see some examples.

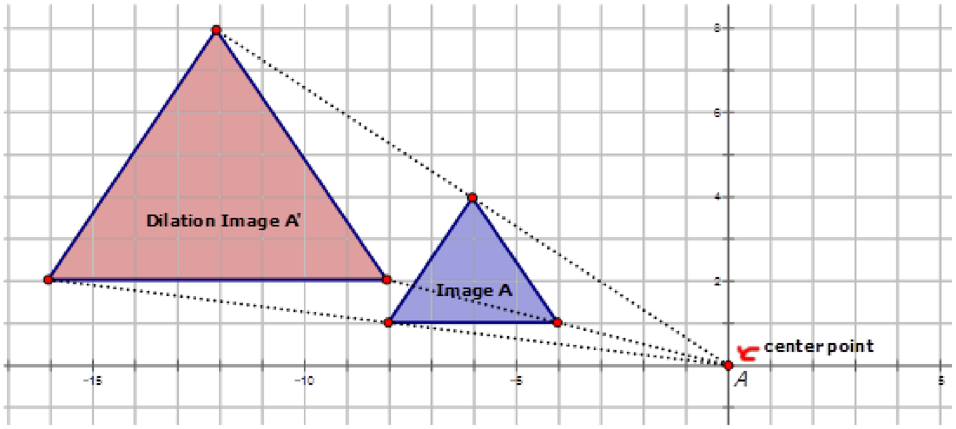
MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10DilationsB

Guidance

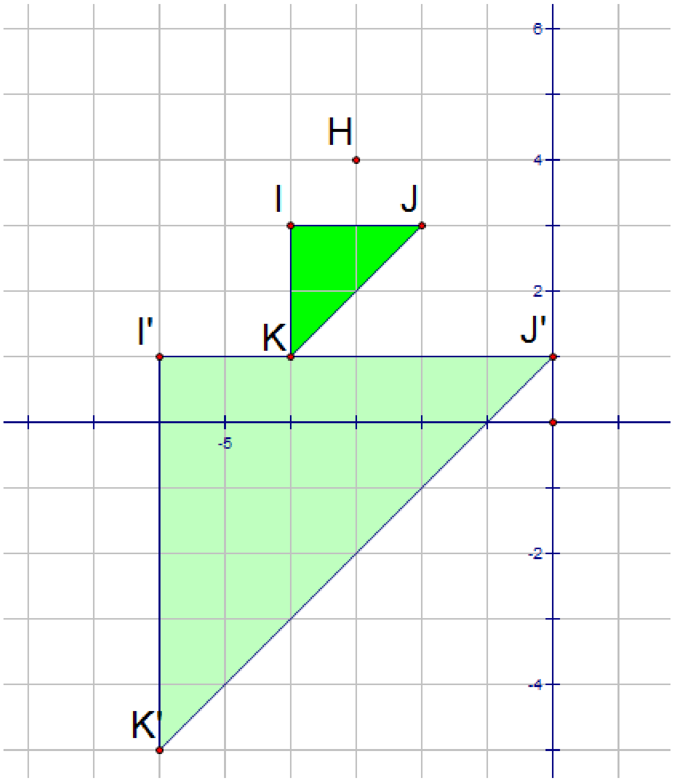
In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage. The figure below shows that the image A' is a dilation by a scale factor of 2.



Dilations also need a center point. The center point is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

Example A

Describe the dilation in the diagram below. The center of dilation is point H.



Solution:

Compare the lengths of corresponding sides to determine the scale factor. $\overline{IJ}$ is 2 units long and $\overline{I'J'}$ is 6 units long. $\frac{6}{2} = 3$, so the scale factor is 3. Therefore, the center point H is used to dilate $\triangle IJK$ to $\triangle I'J'K'$ by a factor of 3.

Example B

Using the measurement below and the scale factor, determine the measure of the dilated image.

$$m\overline{AB} = 15\ cm$$
$$r = \frac{1}{3}$$

Solution: You need to multiply the scale factor by the measure of AB in order to find the measurement of the dilated image $A'B'$.

$m\overline{A'B'} = (r)m\overline{AB}$

$$m\overline{A'B'} = \frac{1}{3}(15)$$
$$m\overline{A'B'} = 5\ cm$$

Example C

Using the measurement below and the scale factor, determine the measure of the preimage.

$$m\overline{H'I'} = 24\ cm$$
$$r = 3$$

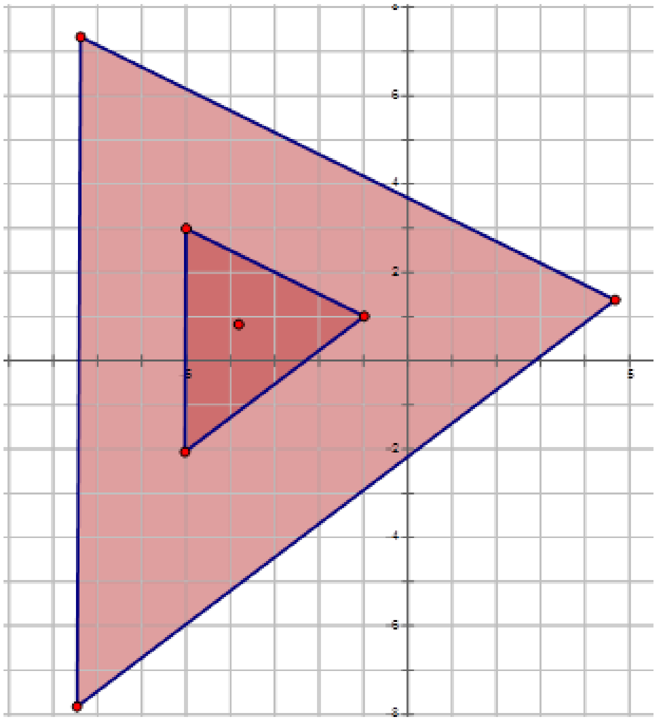
Solution: Here, you need to divide the scale factor by the measurement of $H'I'$ in order to find the measurement of the preimage HI.

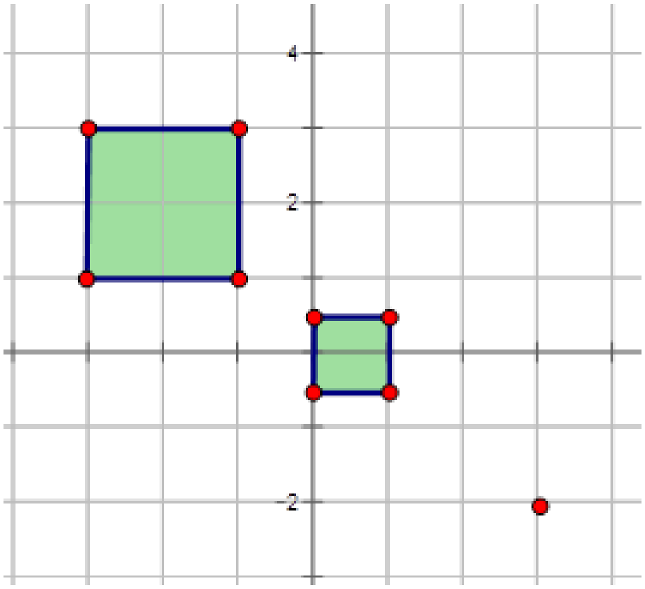
$m\overline{H'I'} = (r)m\overline{HI}$

$$24 = 2m\overline{HI}$$
$$m\overline{HI} = \frac{24}{2}$$
$$m\overline{HI} = 12\ cm$$

Concept Problem Revisited

Which one of the following figures represents a dilation? Explain.





You know that a dilation is a transformation that produces an image of the same shape but larger or smaller. Both of the figures above represent objects that involve dilations. In the figure with the triangles, the scale factor is 3.

The second figure with the squares also represents a dilation. In this figure, the center point $(3,-2)$ is used to dilate the small square by a factor of 2.

Vocabulary

Center Point

The ***center point*** is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Scale Factor

The ***scale factor*** determines how much bigger or smaller the dilation image will be compared to the preimage. The scale factor often uses the symbol r.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

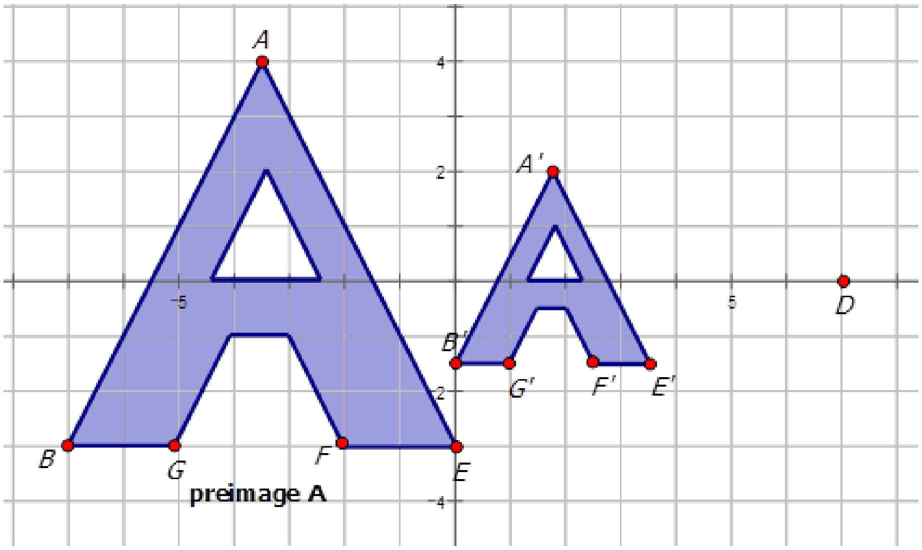
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Guided Practice

1. Using the measurement below and the scale factor, determine the measure of the preimage.

$$m\overline{T'U'} = 12\ cm$$
$$r = 4\ cm$$

2. Describe the dilation in the diagram below.



3. Quadrilateral $STUV$ has vertices $S(-1,3), T(2,0), U(-2,-1)$, and $V(-3,1)$. The quadrilateral undergoes a dilation about the origin with a scale factor of $\frac{8}{5}$. Sketch the preimage and the dilation image.

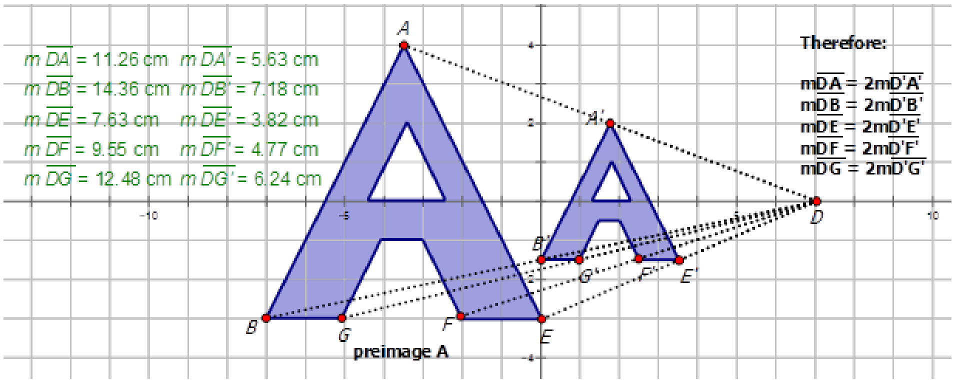
Answers:

1. Here, you need to divide the scale factor by the measurement of $H'I'$ in order to find the measurement of the preimage HI.

$m\overline{T'U'} = |r|m\overline{TU}$

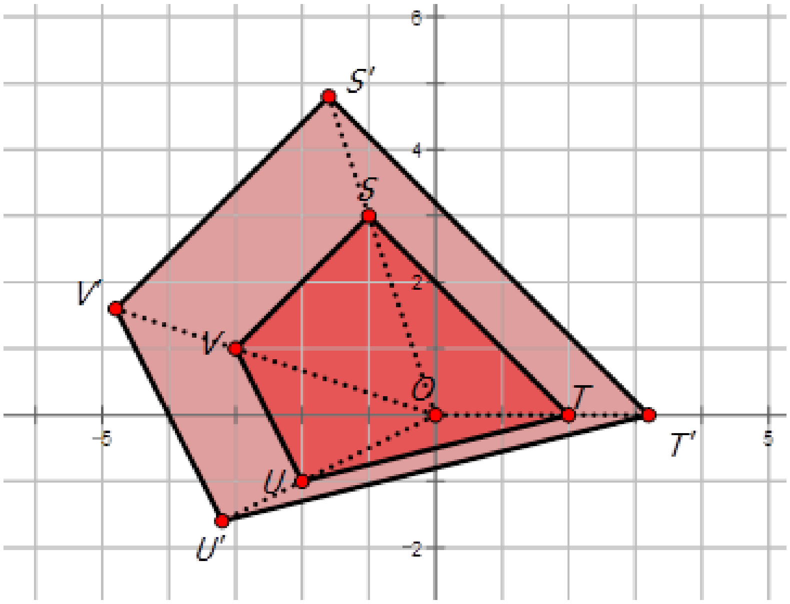
$$12 = 2m\overline{TU}$$
$$m\overline{TU} = \frac{12}{4}$$
$$m\overline{TU} = 3\ cm$$

2. Look at the diagram below:



In the figure, the center point D is used to dilate the A by a factor of $\frac{1}{2}$.

3. Look at the diagram below:



Practice

Find the measure of the dilation image given the following information:

1.

$$m\overline{AB} = 12\ cm$$
$$r = 2$$

2.

$$m\overline{CD} = 25\ cm$$
$$r = \frac{1}{5}$$

3.

$$m\overline{EF} = 18\ cm$$
$$r = \frac{2}{3}$$

4.

$$m\overline{GH} = 18\ cm$$
$$r = 3$$

5.

$$m\overline{IJ} = 100\ cm$$
$$r = \frac{1}{10}$$

Find the measure of the preimage given the following information:

6.

$$m\overline{K'L'} = 48\ cm$$
$$r = 4$$

7.

$$m\overline{M'N'} = 32\ cm$$
$$r = 4$$

8.

$$m\overline{O'P'} = 36\ cm$$
$$r = 6$$

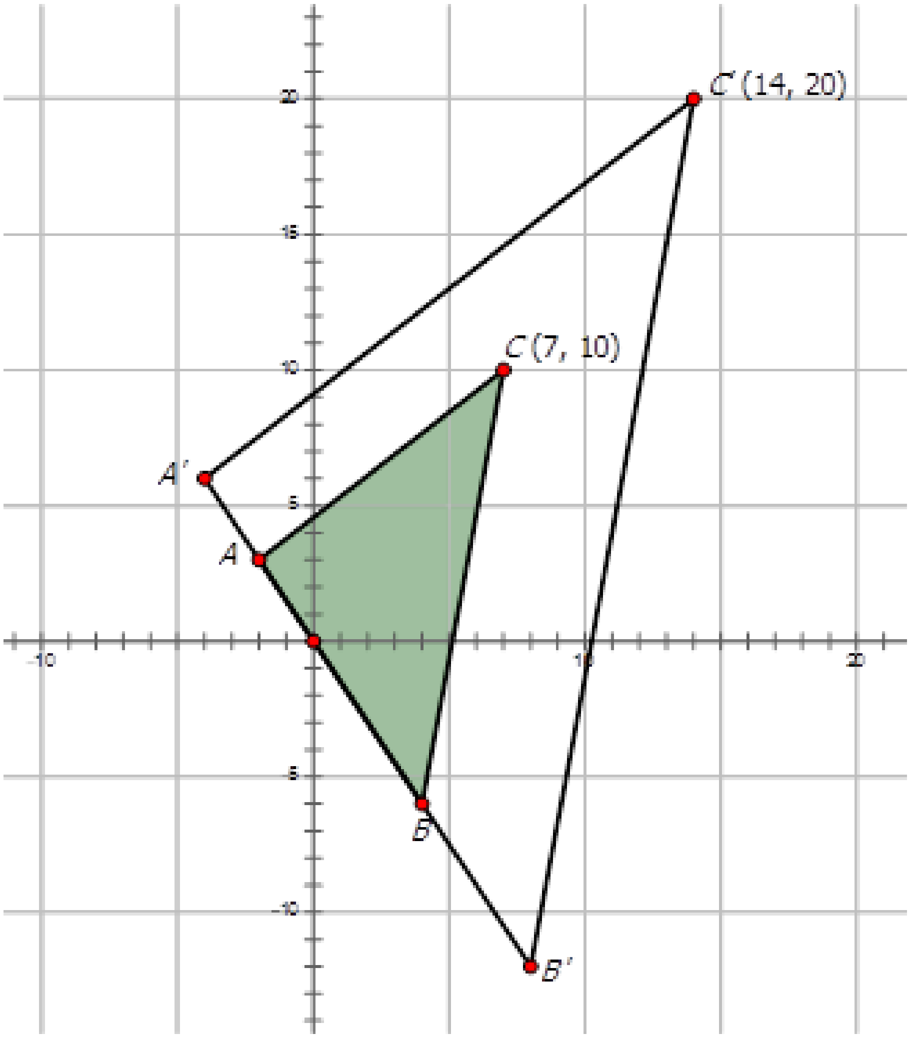
9.

$$m\overline{Q'R'} = 20\ cm$$
$$r = \frac{1}{4}$$

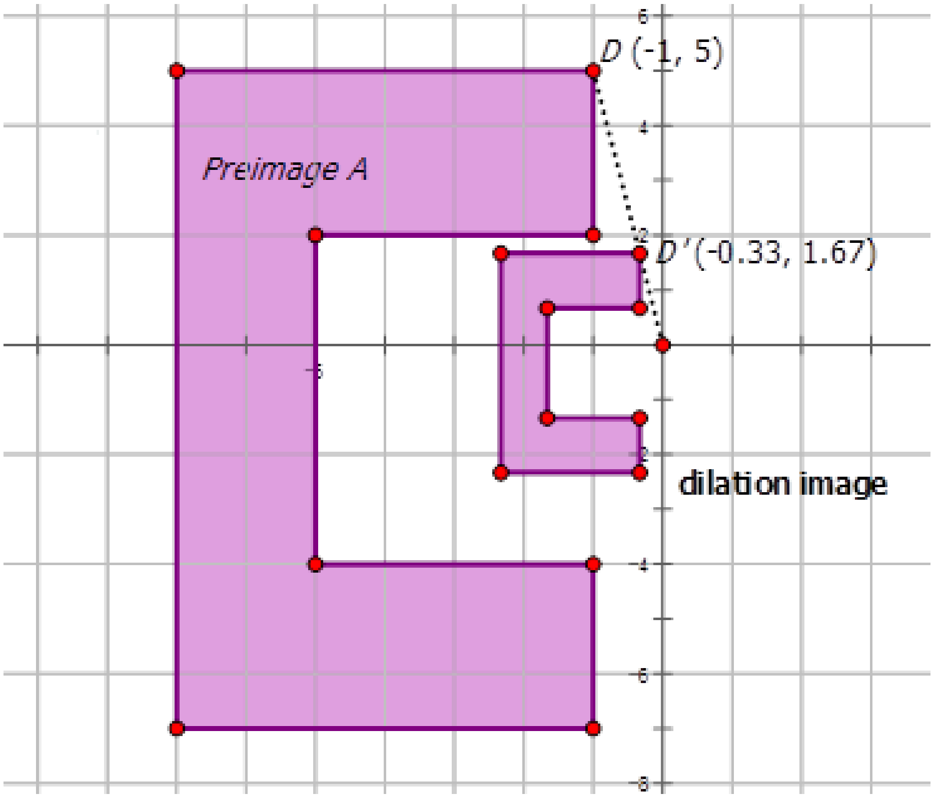
10.

$$m\overline{S'T'} = 40\ cm$$
$$r = \frac{4}{5}$$

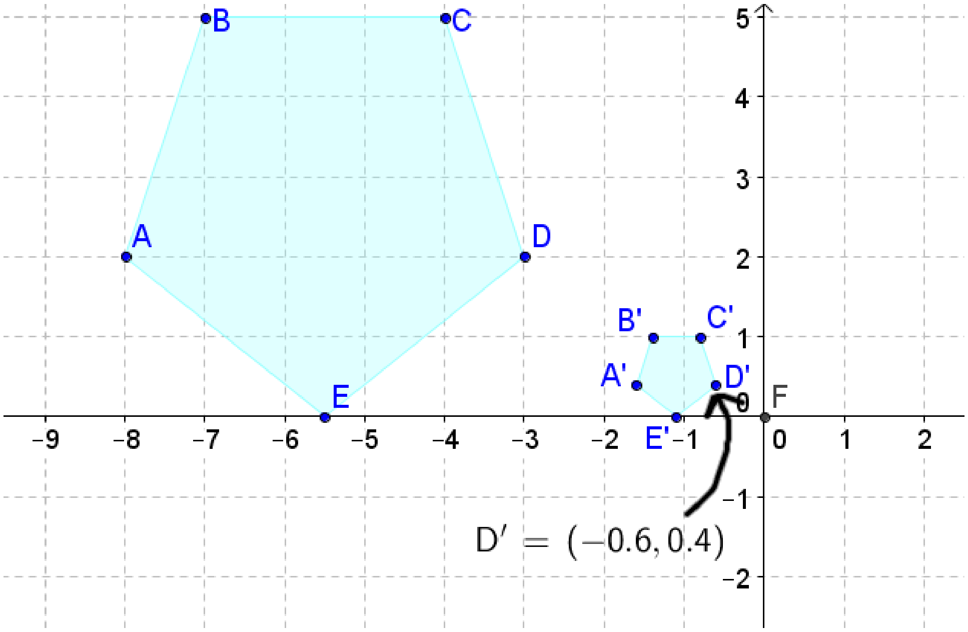
Describe the following dilations:

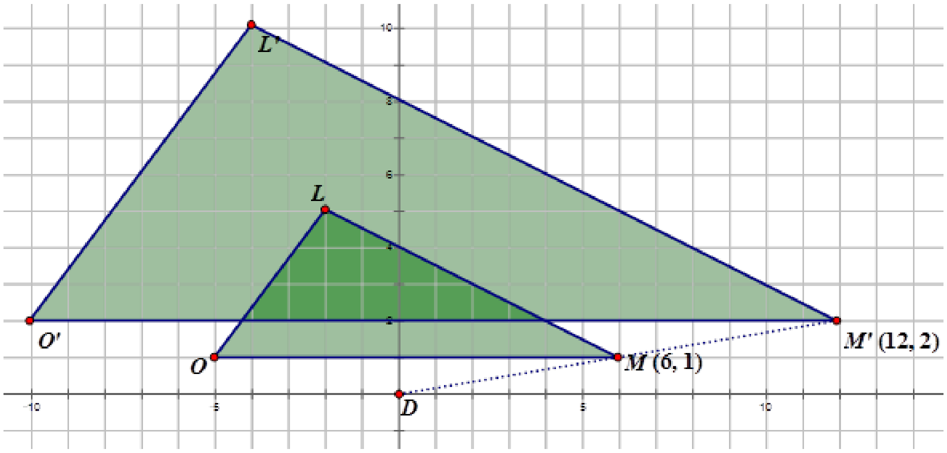

C′ (14, 20)
C (7, 10)
A′
A
B
B′
20
15
10
5
-5
-10
-10
10
20

11.

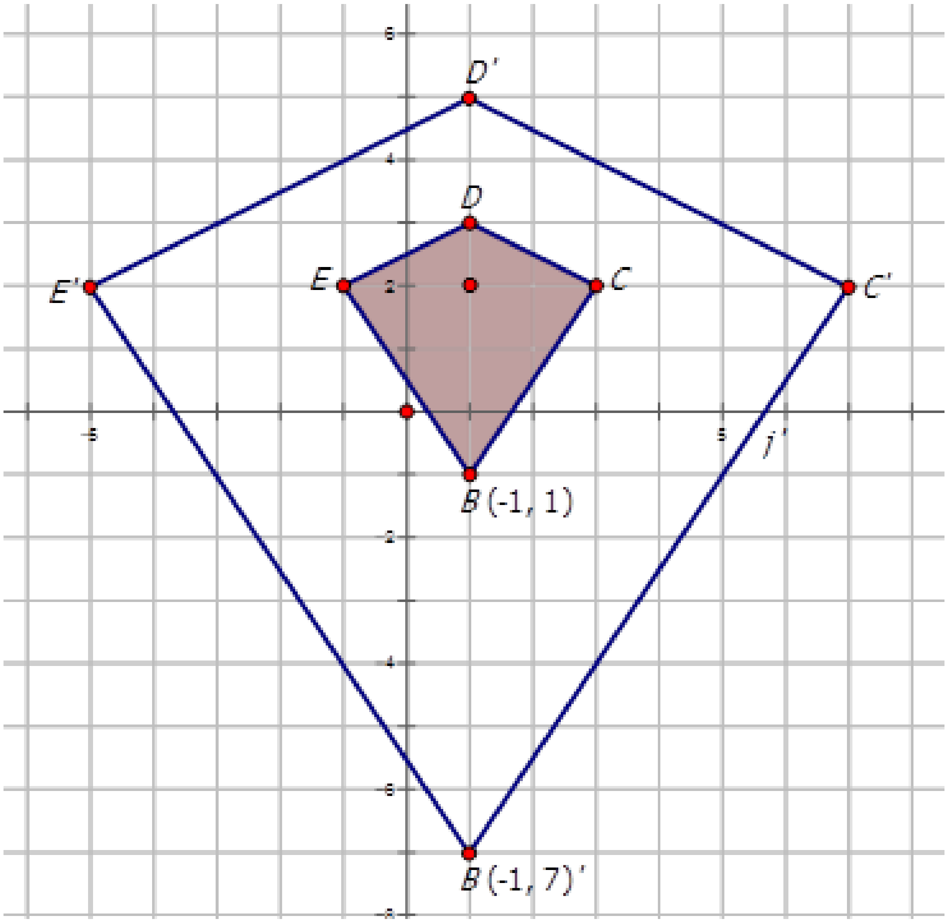

D (-1, 5)
Preimage A
D′(-0.33, 1.67)
dilation image
6
4
2
-2
-4
-6
-8
-5

12.

13. 
B
C
A
D
E
B'
C'
A'
D'
E'
F
D′ = (−0.6, 0.4)
5
4
3
2
1
0
−1
−2
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
1
2

14. 
L'
L
O'
O
M (6, 1)
M' (12, 2)
D
10
8
6
4
2
-10
-5
5
10

15.



10.11 Graphs of Dilations

Here you will learn how to graph a dilation given a description of the dilation.

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$. Show the resulting image.

Watch This

First watch this video to learn about graphs of dilations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofDilationsA

Then watch this video to see some examples.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10GraphsofDilationsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage.

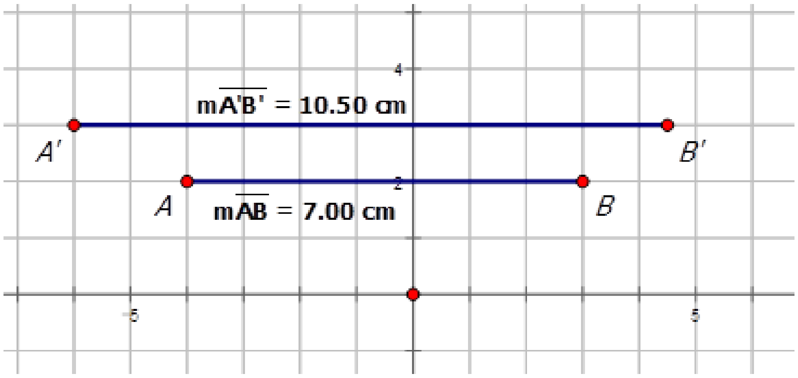
In order to graph a dilation, use the center of dilation and the scale factor. Find the distance between a point on the preimage and the center of dilation. Multiply this length by the scale factor. The corresponding point on the image will be this distance away from the center of dilation in the same direction as the original point.

If you compare the length of a side on the preimage to the length of the corresponding side on the image, the length of the side on the image will be the length of the side on the preimage multiplied by the scale factor.

Example A

Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has undergone a dilation about the origin to produce $A'(-6,3)$ and $B'(4.5,3)$. Draw the preimage and dilation image and determine the scale factor.

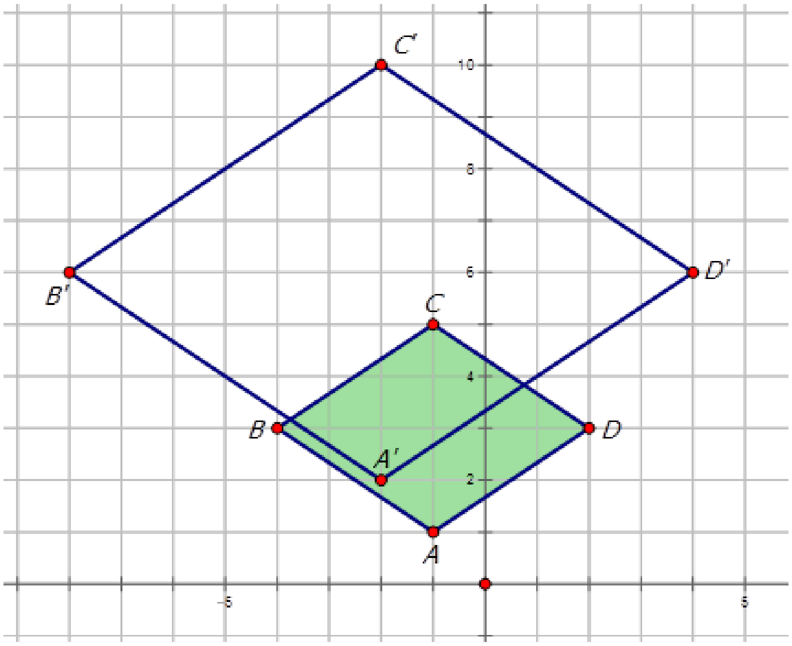
Solution:



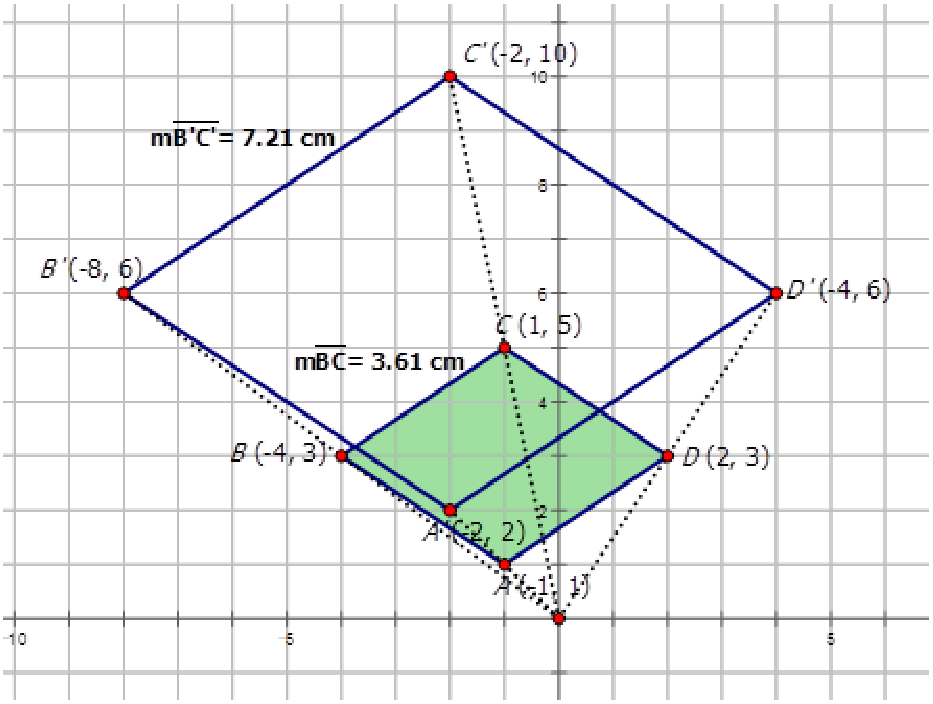
$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{10.5}{7.0}$$
$$scale\ factor = \frac{3}{2}$$

Example B

The diamond $ABCD$ undergoes a dilation about the origin to form the image $A'B'C'D'$. Find the coordinates of the dilation image. Using the diagram, determine the scale factor.



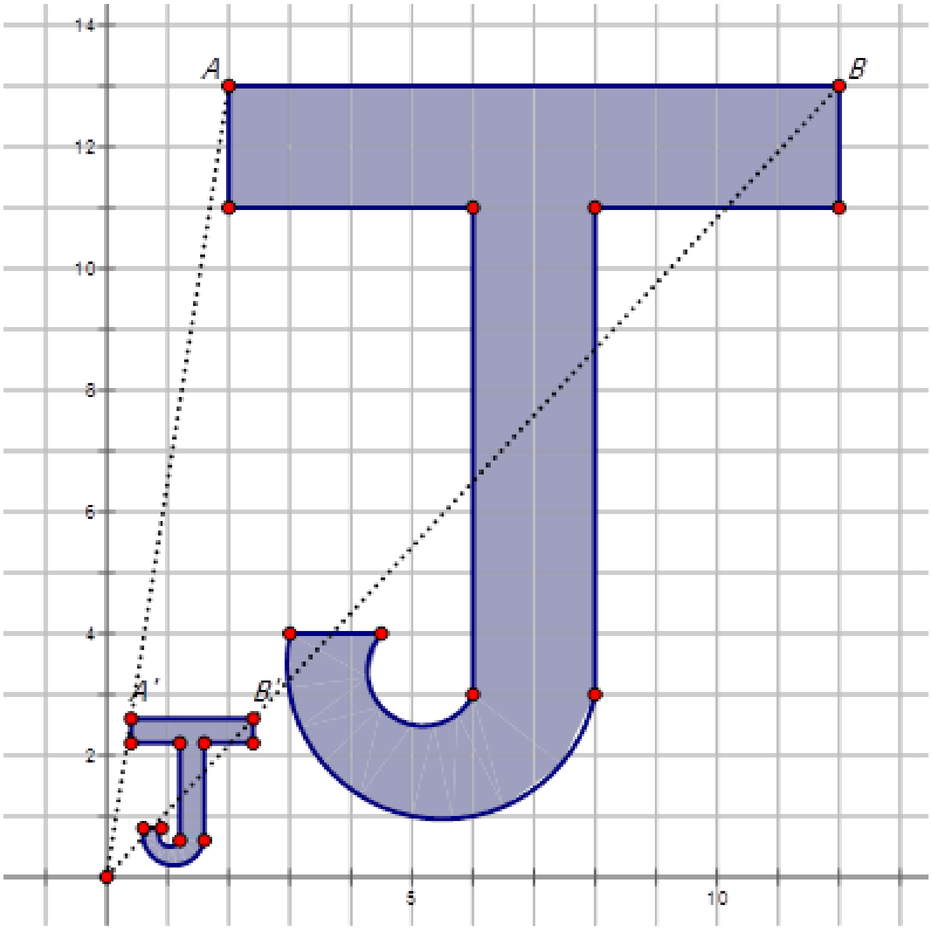
Solution:



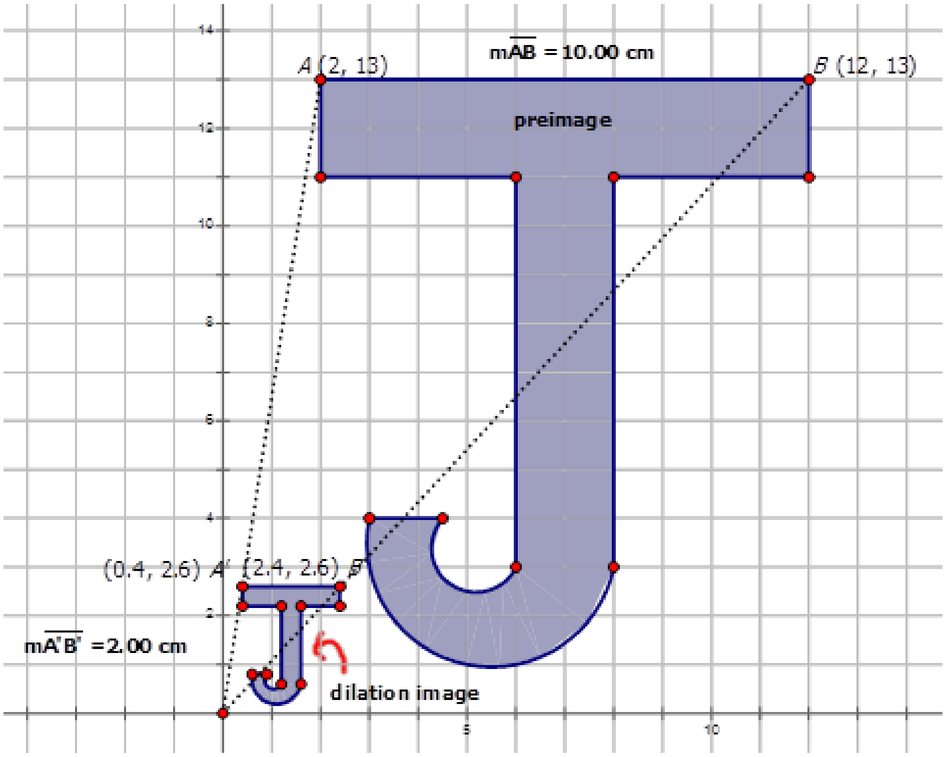
$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scalefactor = \frac{7.21}{3.61}$$
$$scalefactor = 2$$

Example C

The diagram below undergoes a dilation about the origin to form the dilation image. Find the coordinates of A and B and A' and B' of the dilation image. Using the diagram, determine the scale factor.

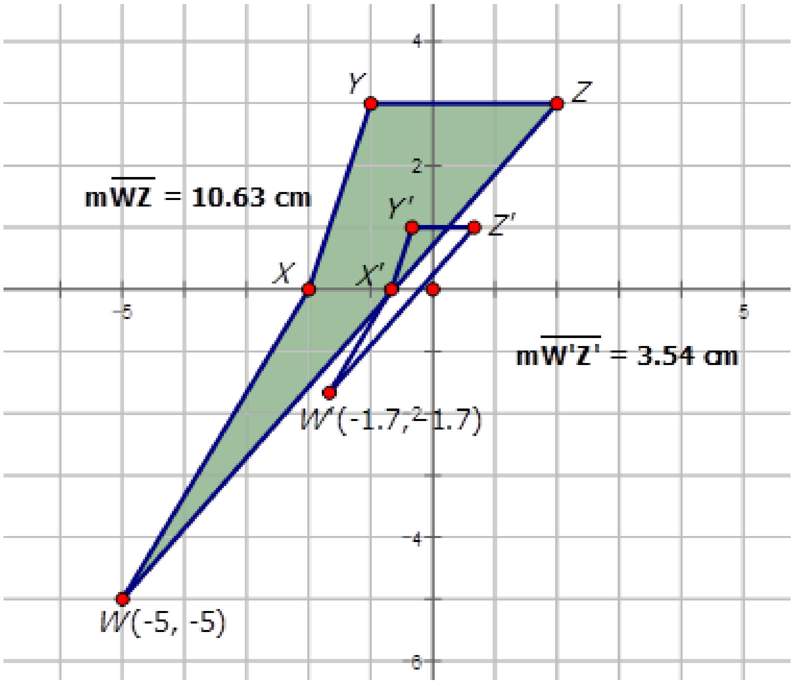

14
12
10
8
6
4
2
5
10
A
B
A'
B'

Solution:


14
12
10
8
6
4
2
5
10
A (2, 13)
B (12, 13)
mAB = 10.00 cm
preimage
(0.4, 2.6) A'
(2.4, 2.6) B'
mA'B' = 2.00 cm
dilation image

$$\begin{aligned} scalefactor &= \frac{dilation\ image\ length}{preimage\ length} \\ scale\ factor &= \frac{2.00}{10.00} \\ scale\ factor &= \frac{1}{5} \end{aligned}$$

Concept Problem Revisited



Test to see if the dilation is correct by determining the scale factor.

$$\begin{aligned} scale\ factor &= \frac{dilation\ image\ length}{preimage\ length} \\ scale\ factor &= \frac{10.63}{3.54} \\ scale\ factor &= 3 \end{aligned}$$

Vocabulary

Center Point

The ***center point*** is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Scale Factor

The ***scale factor*** determines how much bigger or smaller the dilation image will be compared to the preimage. The scale factor often uses the symbol *r*.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

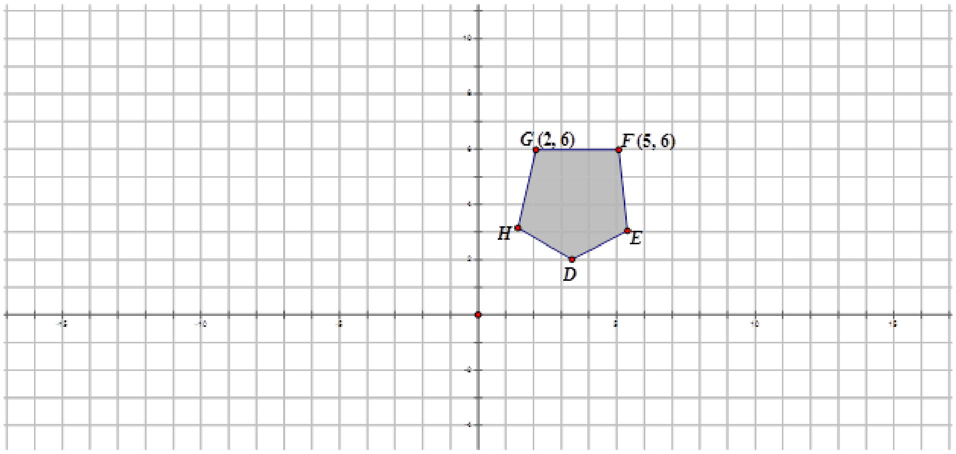
Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Guided Practice

1. Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a dilation of scale factor 3 about the point $A(1,6)$. Draw the preimage and image and properly label each.

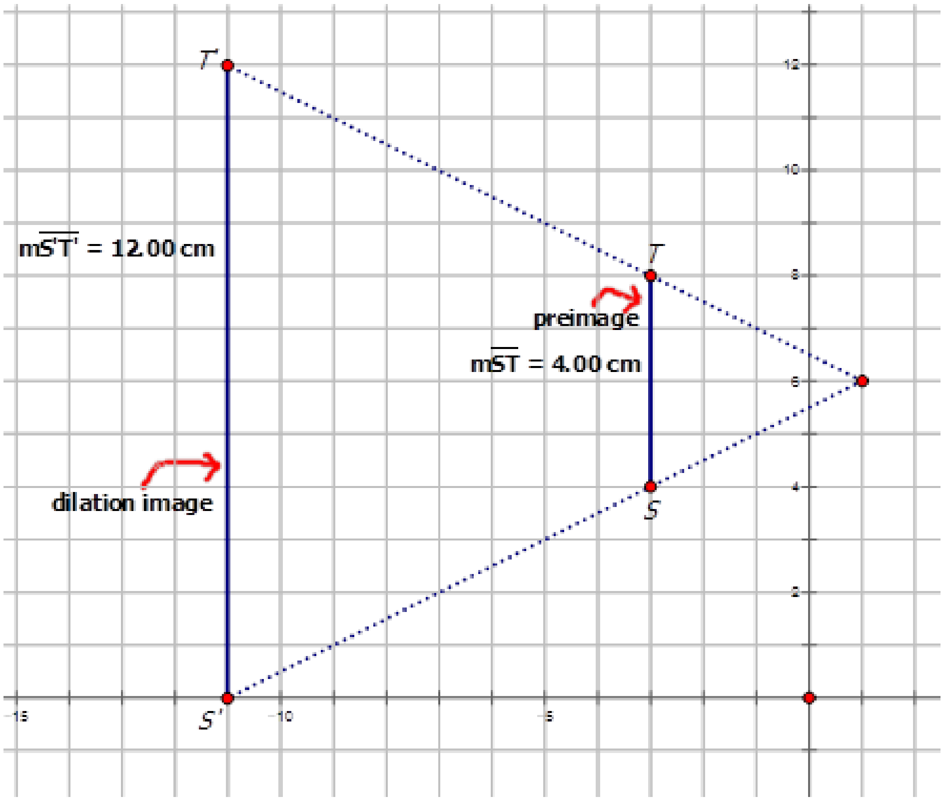
2. The polygon below has undergone a dilation about the origin with a scale factor of $\frac{5}{3}$. Draw the dilation image and properly label each.



3. The triangle with vertices $J(-5,-2), K(-1,4)$ and $L(1,-3)$ has undergone a dilation of scale factor $\frac{1}{2}$. about the center point L. Draw and label the dilation image and the preimage then check the scale factor.

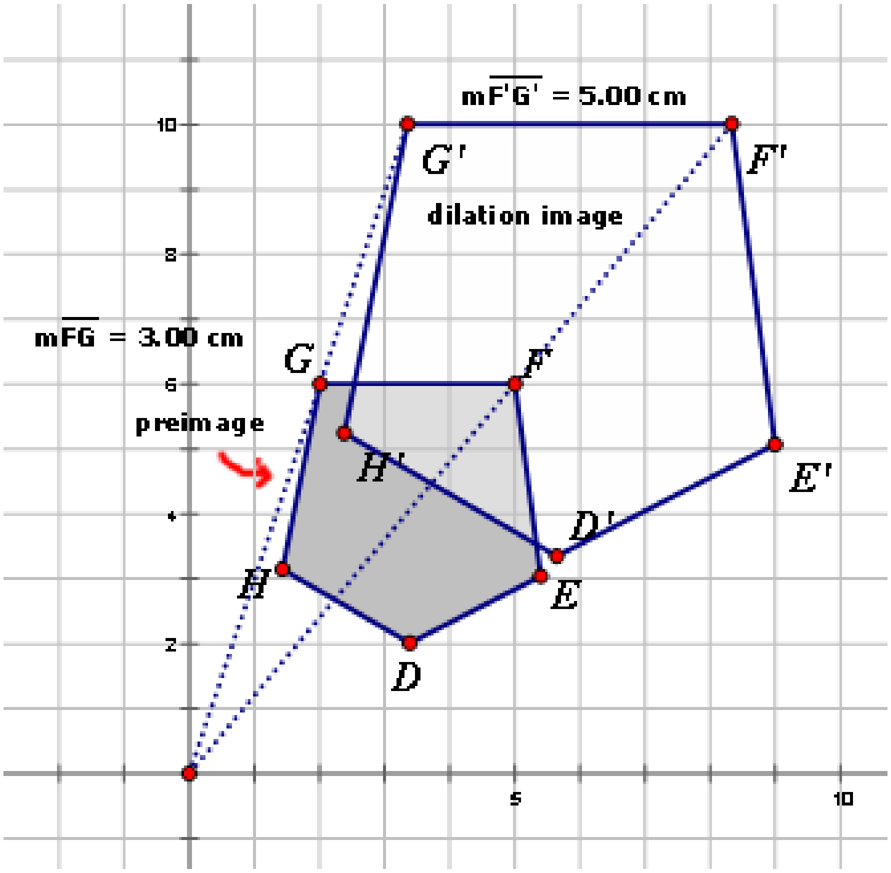
Answers:

1.



$$
\begin{aligned}
scale\ factor &= \frac{dilation\ image\ length}{preimage\ length}\\
scale\ factor &= \frac{12.00}{4.00}\\
scale\ factor &= 3
\end{aligned}
$$

2.

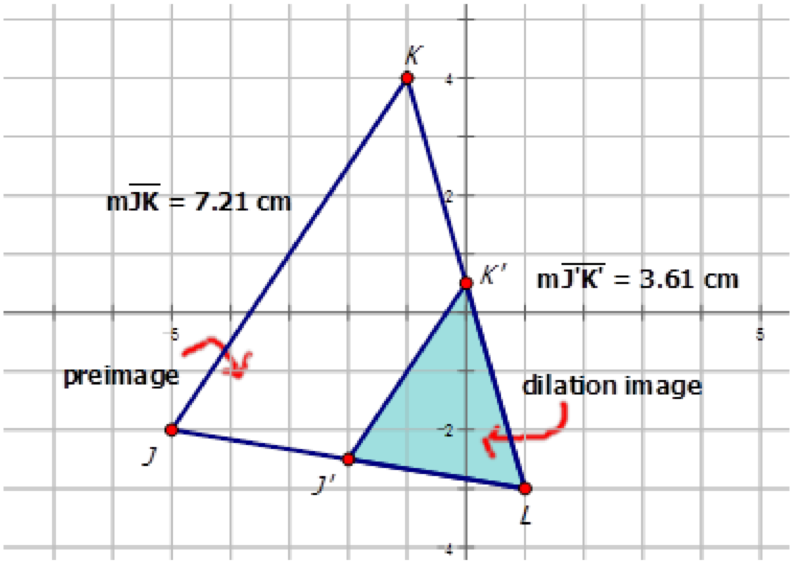


$$scale factor = \frac{dilation\ image\ length}{preimage\ length}$$

$$scale\ factor = \frac{5.00}{3.00}$$

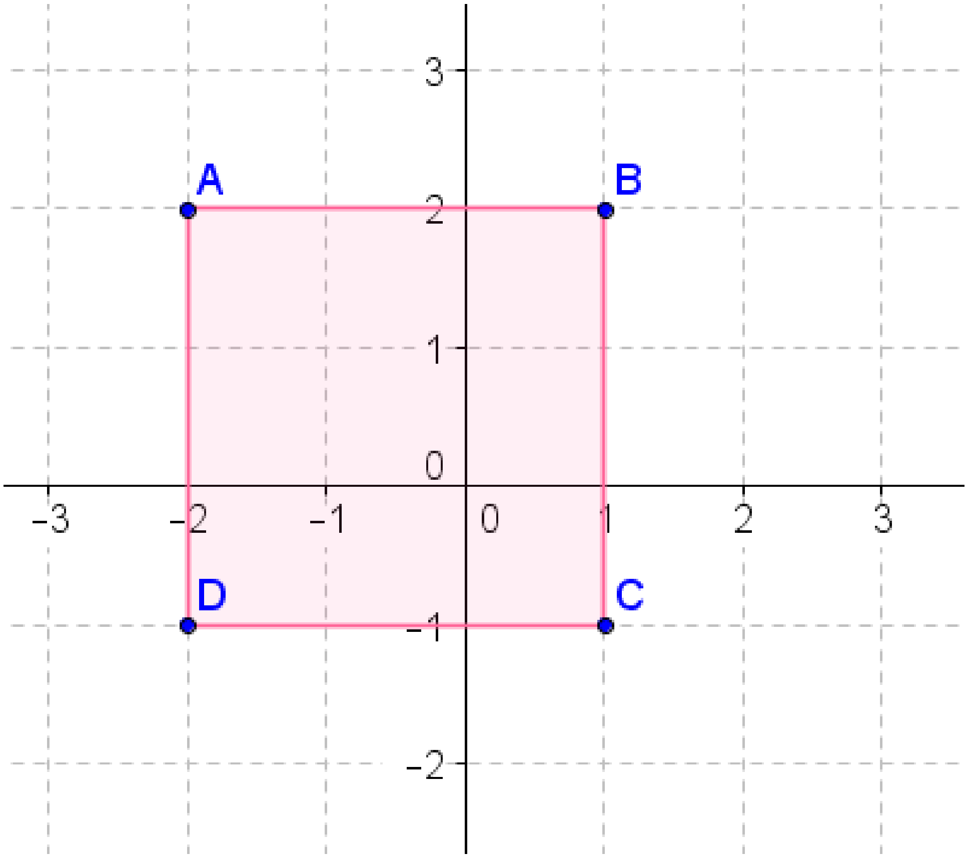
$$scale\ factor = \frac{5}{3}$$

3.

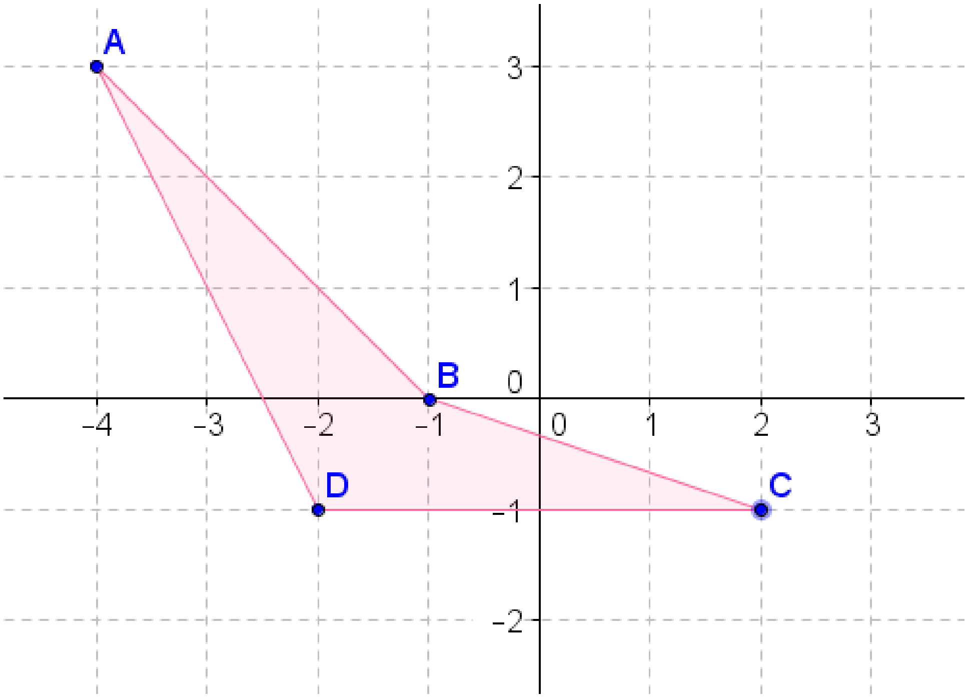


$$scale\ factor = \frac{dilation\ image\ length}{preimage\ length}$$
$$scale\ factor = \frac{7.21}{3.61}$$
$$scale\ factor = \frac{1}{2}$$

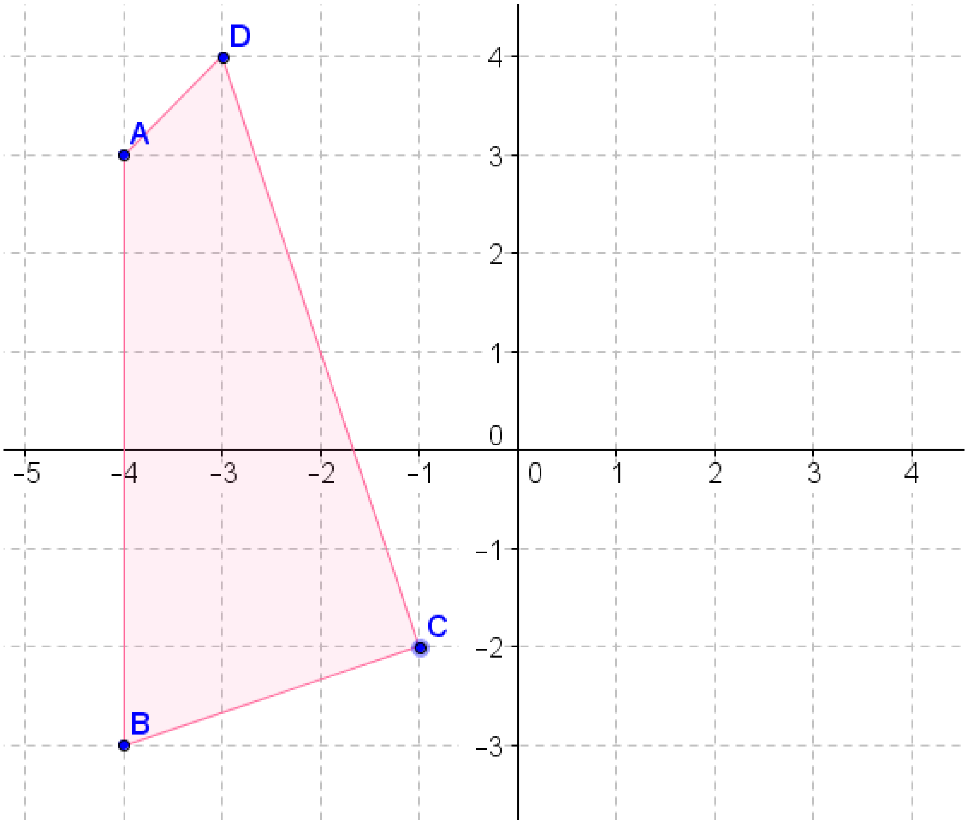
Practice



1. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
2. Dilate the above figure by a factor of 2 about point D.

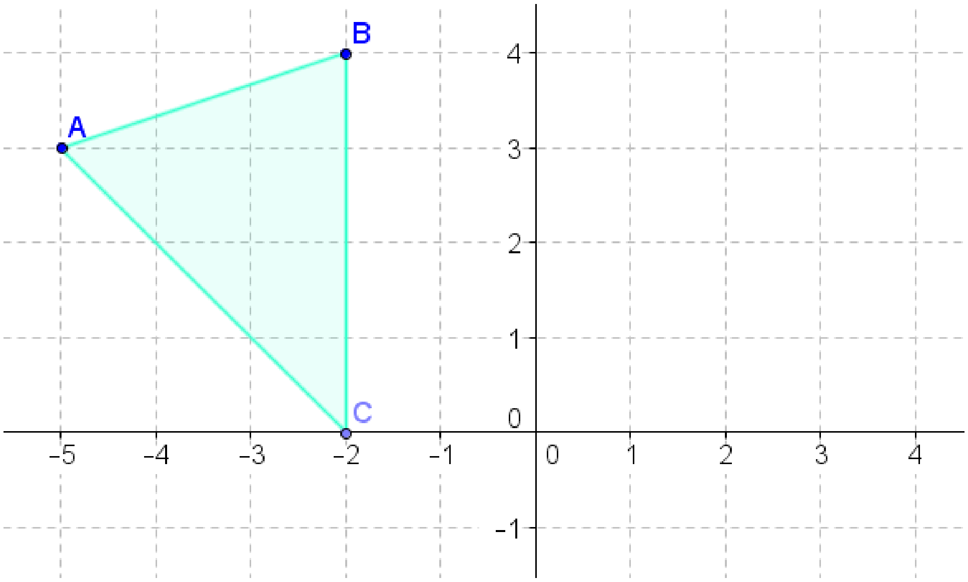


3. Dilate the above figure by a factor of 3 about the origin.
4. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.

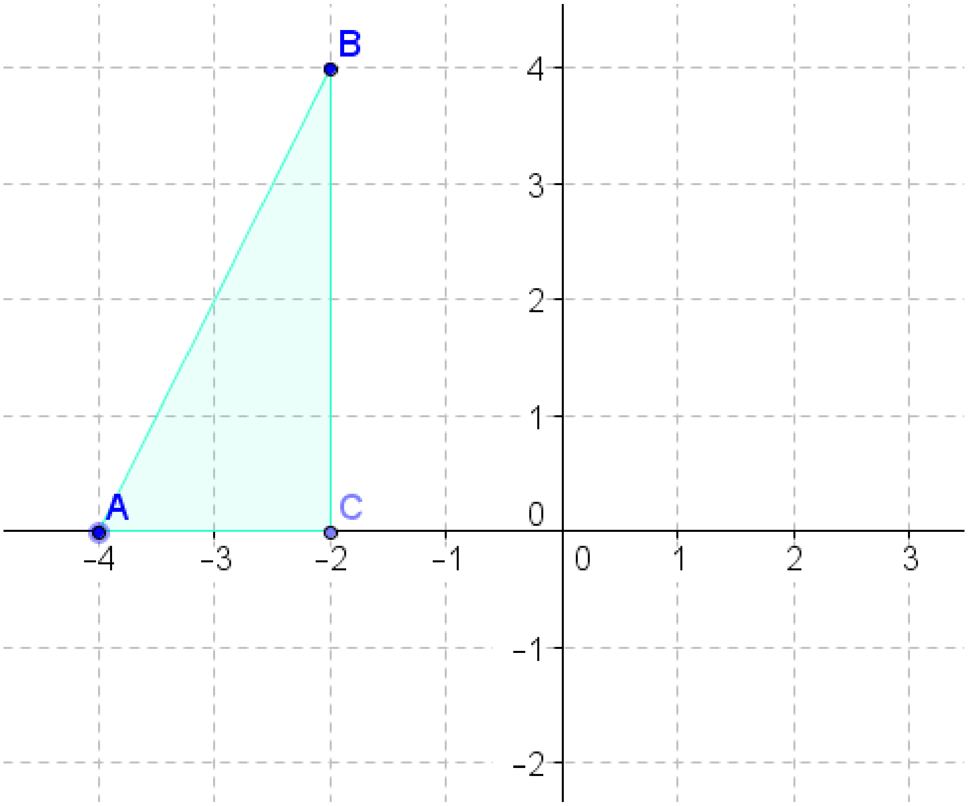


5. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.

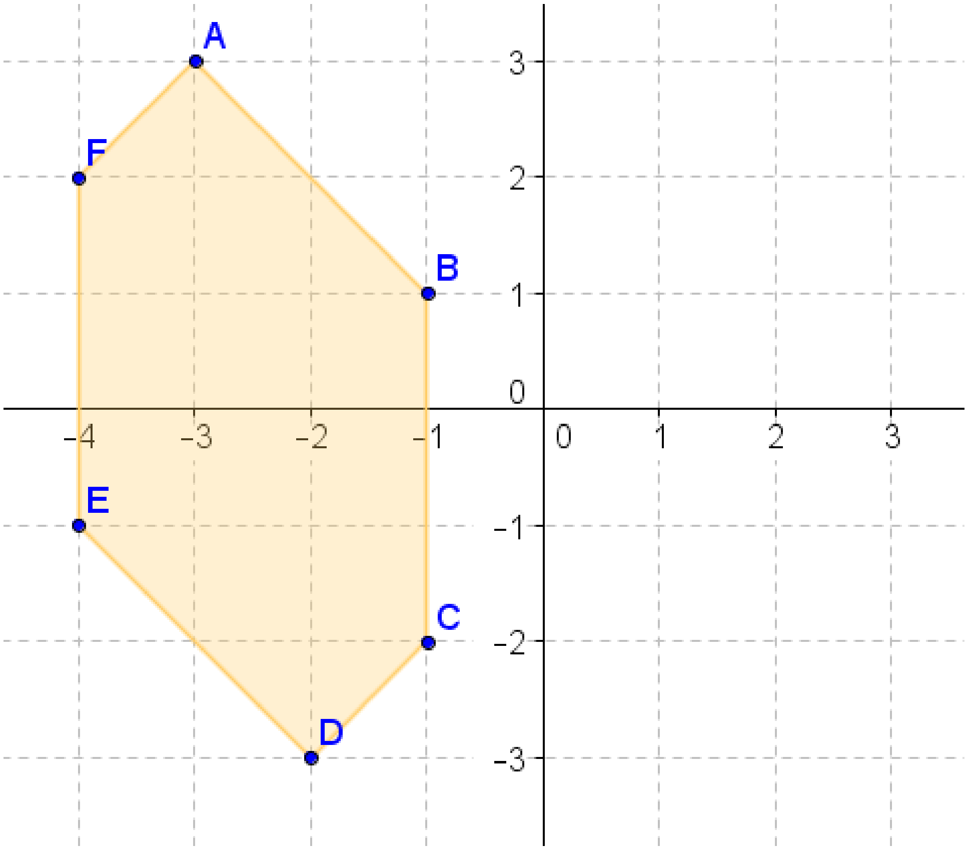
6. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.



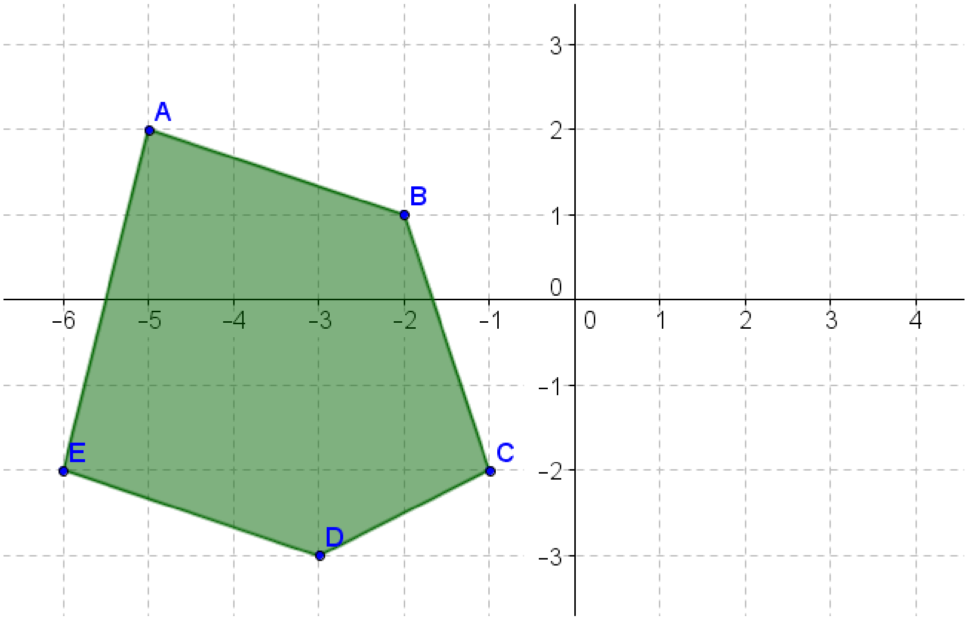
7. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
8. Dilate the above figure by a factor of $\frac{1}{4}$ about point C.



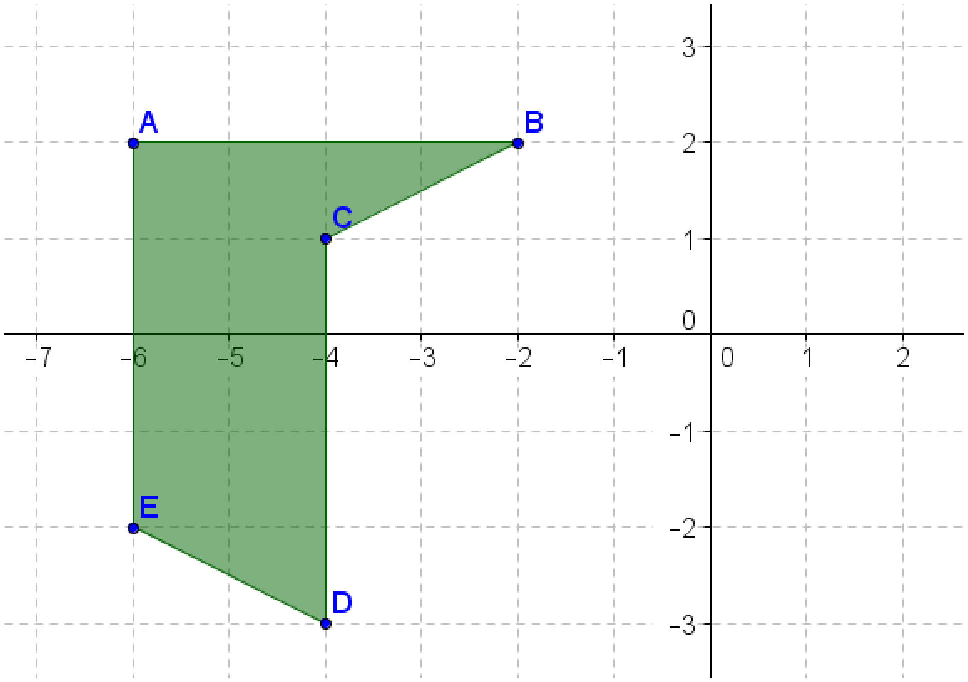
9. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
10. Dilate the above figure by a factor of 2 about point A.



11. Dilate the above figure by a factor of 2 about the origin.
12. Dilate the above figure by a factor of $\frac{1}{2}$ about point D.



13. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
14. Dilate the above figure by a factor of 3 about point D.

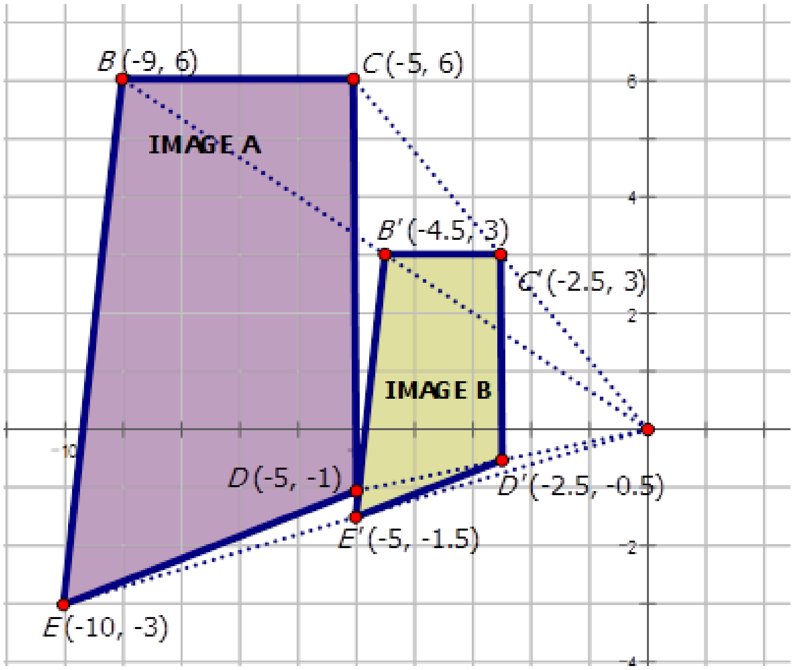


15. Dilate the above figure by a factor of $\frac{1}{2}$ about the origin.
16. Dilate the above figure by a factor of $\frac{1}{2}$ about point C.

10.12 Rules for Dilations

Here you will learn the notation for describing a dilation.

The figure below shows a dilation of two trapezoids. Write the mapping rule for the dilation of Image A to Image B.



Watch This

First watch this video to learn about writing rules for dilations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10RulesforDilationsA

Then watch this video to see some examples.

MEDIA

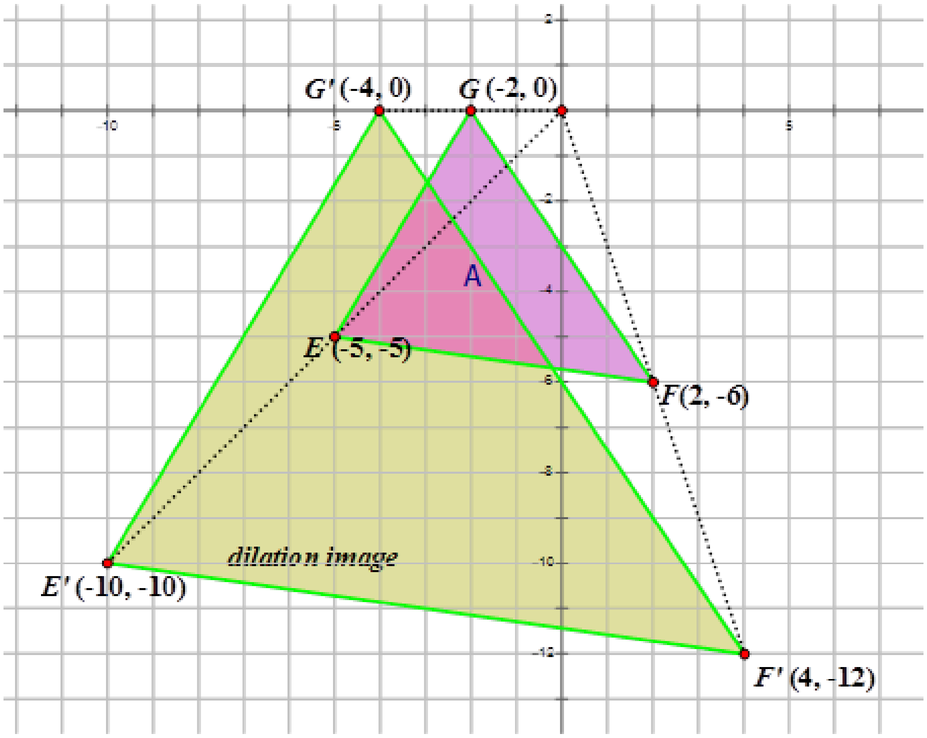
Click image to the left for more content.

CK-12 Foundation Chapter10RulesforDilationsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A dilation is a type of transformation that enlarges or reduces a figure (called the preimage) to create a new figure (called the image). The scale factor, r, determines how much bigger or smaller the dilation image will be compared to the preimage.

Look at the diagram below:



The Image A has undergone a dilation about the origin with a scale factor of 2. Notice that the points in the dilation image are all double the coordinate points in the preimage. A dilation with a scale factor k about the origin can be described using the following notation:

$$D_k(x,y) = (kx,ky)$$

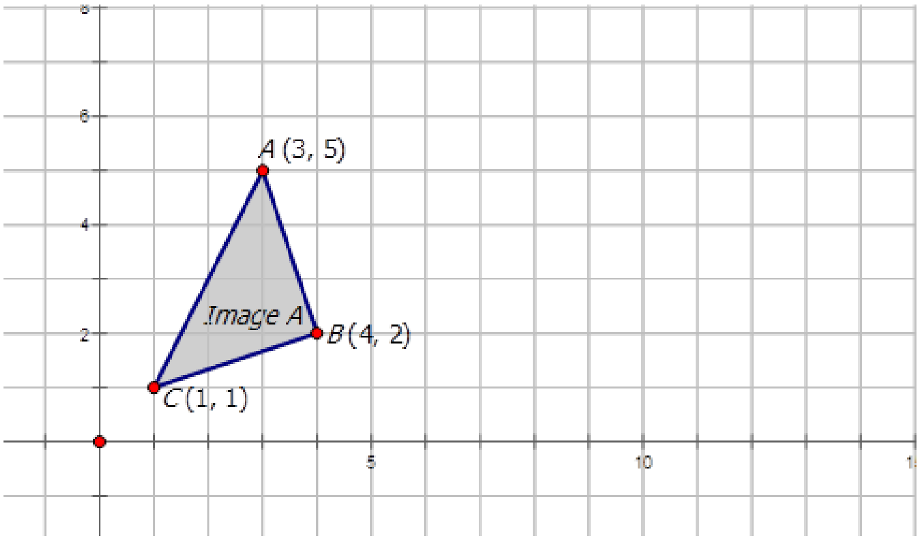
k will always be a value that is greater than 0.

TABLE 10.14:

Scale Factor, k	Size change for preimage
$k > 1$	Dilation image is larger than preimage
$0 < k < 1$	Dilation image is smaller than preimage
$k = 1$	Dilation image is the same size as the preimage

Example A

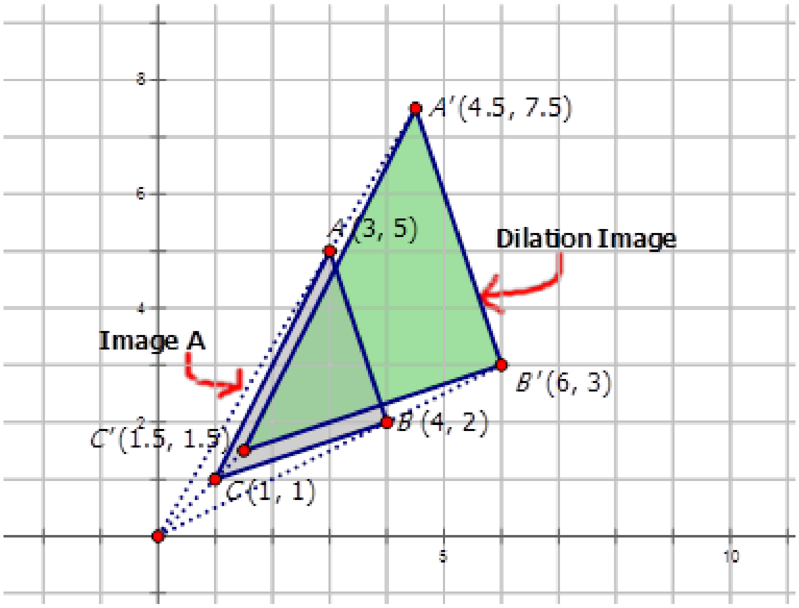
The mapping rule for the dilation applied to the triangle below is $(x,y) \rightarrow (1.5x, 1.5y)$. Draw the dilation image.



Solution: With a scale factor of 1.5, each coordinate point will be multiplied by 1.5.

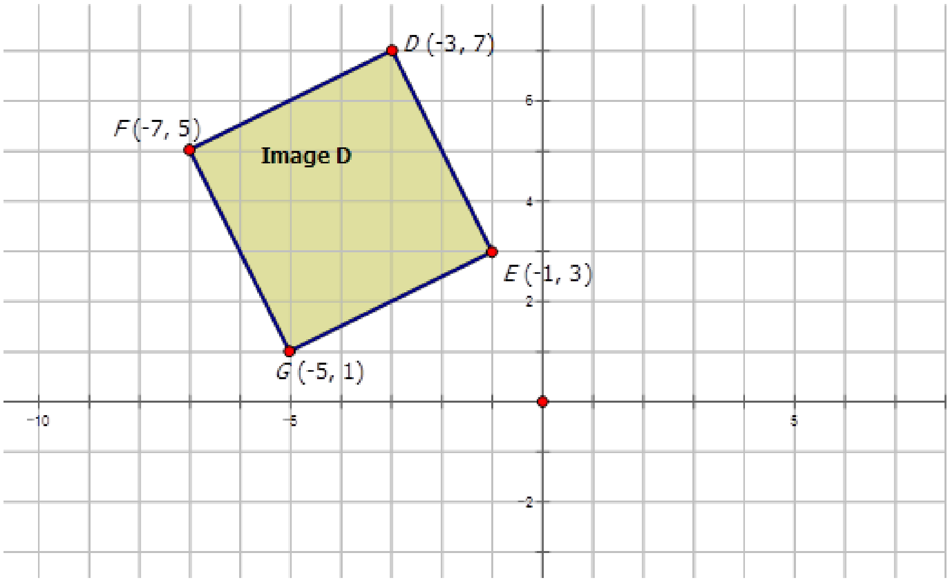
Image A	$A(3,5)$	$B(4,2)$	$C(1,1)$
Dilation Image	$A'(4.5,7.5)$	$B'(6,3)$	$C'(1.5,1.5)$

The dilation image looks like the following:



Example B

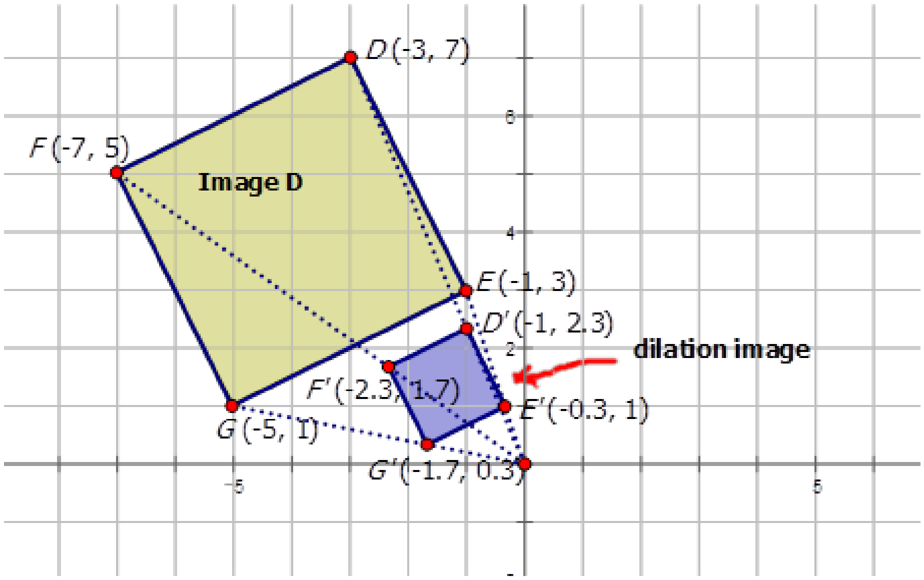
The mapping rule for the dilation applied to the diagram below is $(x,y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$. Draw the dilation image.



Solution: With a scale factor of $\frac{1}{3}$, each coordinate point will be multiplied by $\frac{1}{3}$.

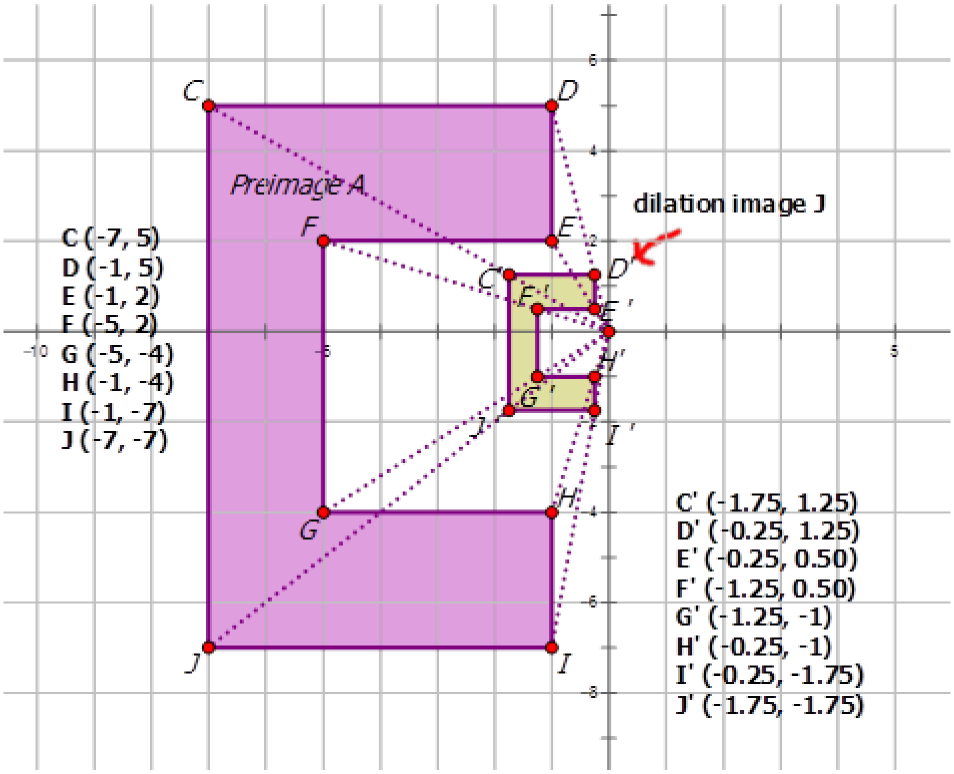
Image D	$D(-3,7)$	$E(-1,3)$	$F(-7,5)$	$G(-5,1)$
Dilation Image	$D'(-1,2.3)$	$E'(-0.3,1)$	$F'(-2.3,1.7)$	$G'(-1.7,0.3)$

The dilation image looks like the following:



Example C

Write the notation that represents the dilation of the preimage A to the dilation image J in the diagram below.

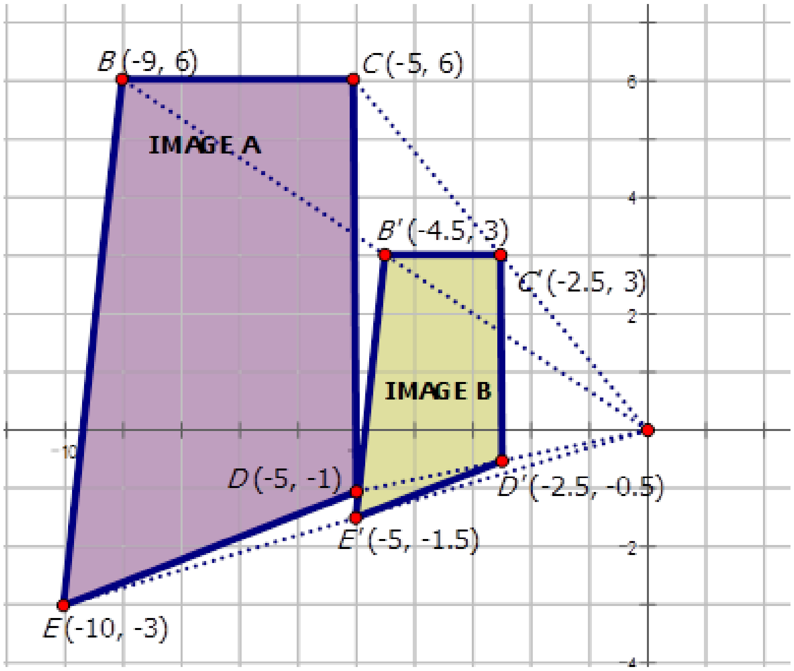


Solution: First, pick a point in the diagram to use to see how it has been affected by the dilation.

$C : (-7, 5) \quad C' : (-1.75, 1.25)$

Notice how both the x- and y-coordinates are multiplied by $\frac{1}{4}$. This indicates that the preimage A undergoes a dilation about the origin by a scale factor of $\frac{1}{4}$ to form the dilation image J. Therefore the mapping notation is $(x, y) \rightarrow \left(\frac{1}{4}x, \frac{1}{4}y\right)$.

Concept Problem Revisited



Look at the points in each image:

Image A	$B(-9,6)$	$C(-5,6)$	$D(-5,-1)$	$E(-10,-3)$
Image B	$B'(-4.5,3)$	$C'(-2.5,3)$	$D'(-2.5,-0.5)$	$E'(-5,-1.5)$

Notice that the coordinate points in Image B (the dilation image) are $\frac{1}{2}$ that found in Image A. Therefore the Image A undergoes a dilation about the origin of scale factor $\frac{1}{2}$. To write a mapping rule for this dilation you would write: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

Vocabulary

Notation Rule

A ***notation rule*** has the following form $D_k(x,y) = (kx,ky)$ and tells you that the preimage has undergone a dilation about the origin by scale factor k. If k is greater than one, the dilation image will be larger than the preimage. If k is between 0 and 1, the dilation image will be smaller than the preimage. If k is equal to 1, you will have a dilation image that is congruent to the preimage. The mapping rule corresponding to a dilation notation would be: $(x,y) \rightarrow (kx,ky)$

Center Point

The ***center point*** is the center of the dilation. You use the center point to measure the distances to the preimage and the dilation image. It is these distances that determine the scale factor.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Scale Factor

The ***scale factor*** determines how much bigger or smaller the dilation image will be compared to the preimage. The scale factor often uses the symbol r.

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

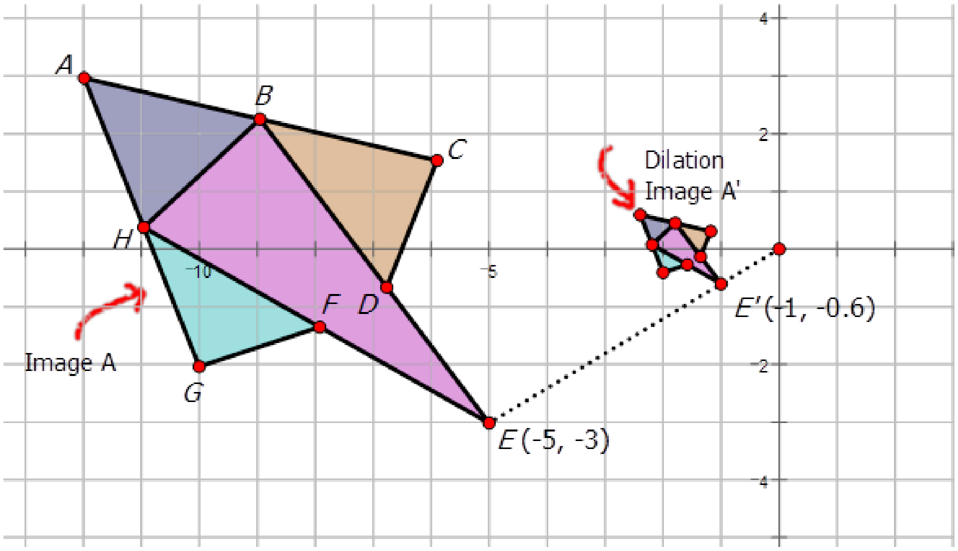
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Guided Practice

1. Thomas describes a dilation of point JT with vertices $J(-2,6)$ to $T(6,2)$ to point $J'T'$ with vertices $J'(-4,12)$ and $T'(12,4)$. Write the notation to describe this dilation for Thomas.

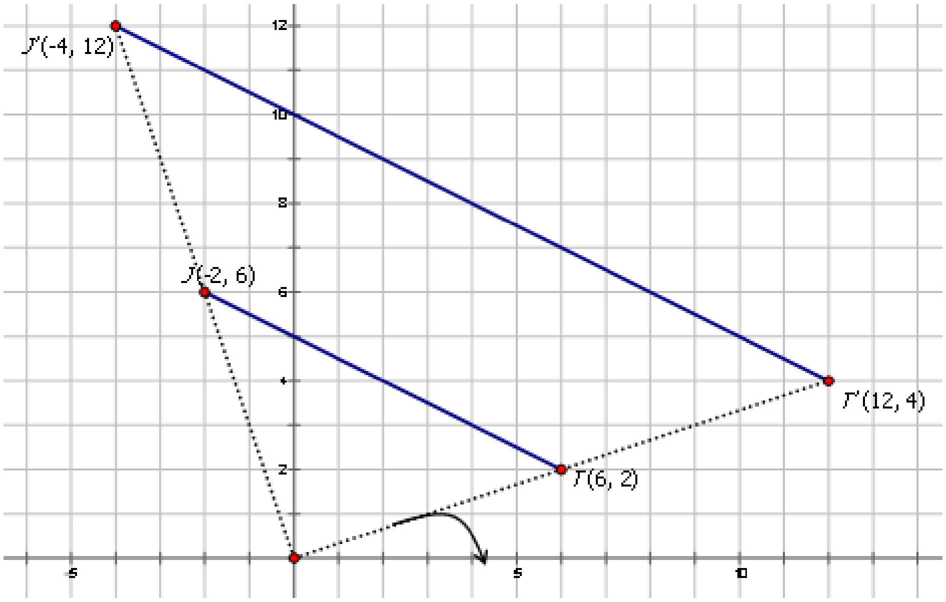
2. Given the points $A(12,8)$ and $B(8,4)$ on a line undergoing a dilation to produce $A'(6,4)$ and $B'(4,2)$, write the notation that represents the dilation.

3. Janet was playing around with a drawing program on her computer. She created the following diagrams and then wanted to determine the transformations. Write the notation rule that represents the transformation of the purple and blue diagram to the orange and blue diagram.



Answers:

1.



Since the x- and y-coordinates are each multiplied by 2, the *scale factor* is 2. The mapping notation is: $(x,y) \rightarrow (2x,2y)$

2. In order to write the notation to describe the dilation, choose one point on the preimage and then the corresponding point on the dilation image to see how the point has moved. Notice that point EA is:

$$A(12,8) \rightarrow A'(6,4)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{2}$, the dilation is about the origin has a scale factor of $\frac{1}{2}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{2}x, \frac{1}{2}y\right)$.

3. In order to write the notation to describe the dilation, choose one point on the preimage A and then the corresponding point on the dilation image A' to see how the point has changed. Notice that point E is shown in the diagram:

$$E(-5,-3) \rightarrow E'(-1,-0.6)$$

Since both x- and y-coordinates are multiplied by $\frac{1}{5}$, the dilation is about the origin has a scale factor of $\frac{1}{5}$. The notation for this dilation would be: $(x,y) \rightarrow \left(\frac{1}{5}x, \frac{1}{5}y\right)$.

Practice

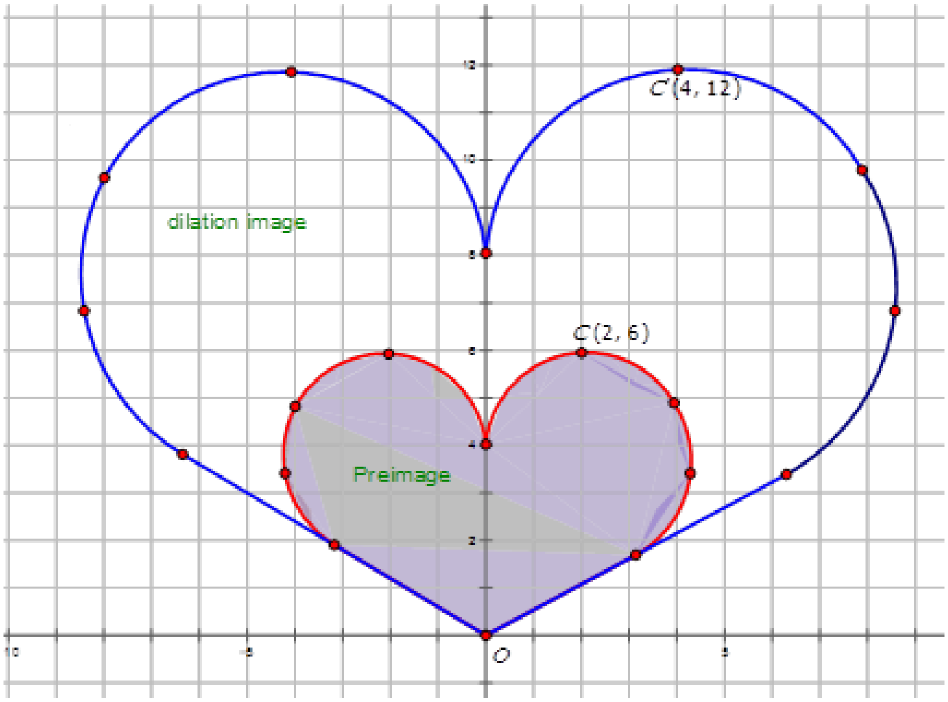
Complete the following table. Assume that the center of dilation is the origin.

TABLE 10.15:

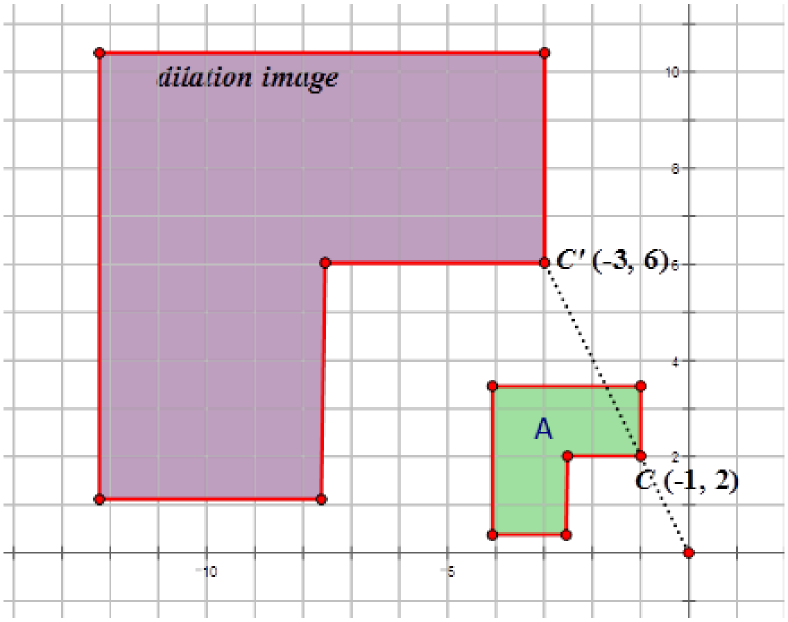
Starting Point	D_2	D_5	$D_{\frac{1}{2}}$	$D_{\frac{3}{4}}$
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				
6. (9, 4)				
7. (-1, 3)				
8. (-5, 2)				
9. (2, 6)				
10. (-5, 7)				

Write the notation that represents the dilation of the preimage to the image for each diagram below.

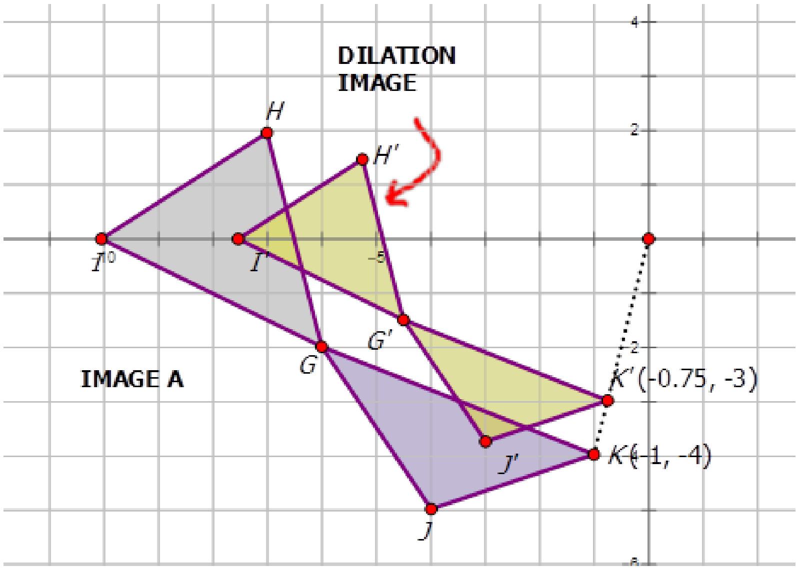
11.



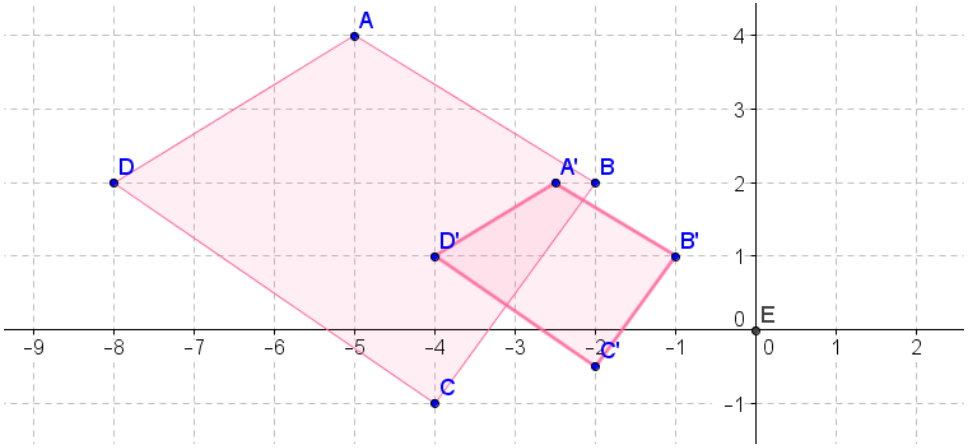
12.


dilation image
C′ (-3, 6)
A
C (-1, 2)
10
8
6
4
2
-10
-5

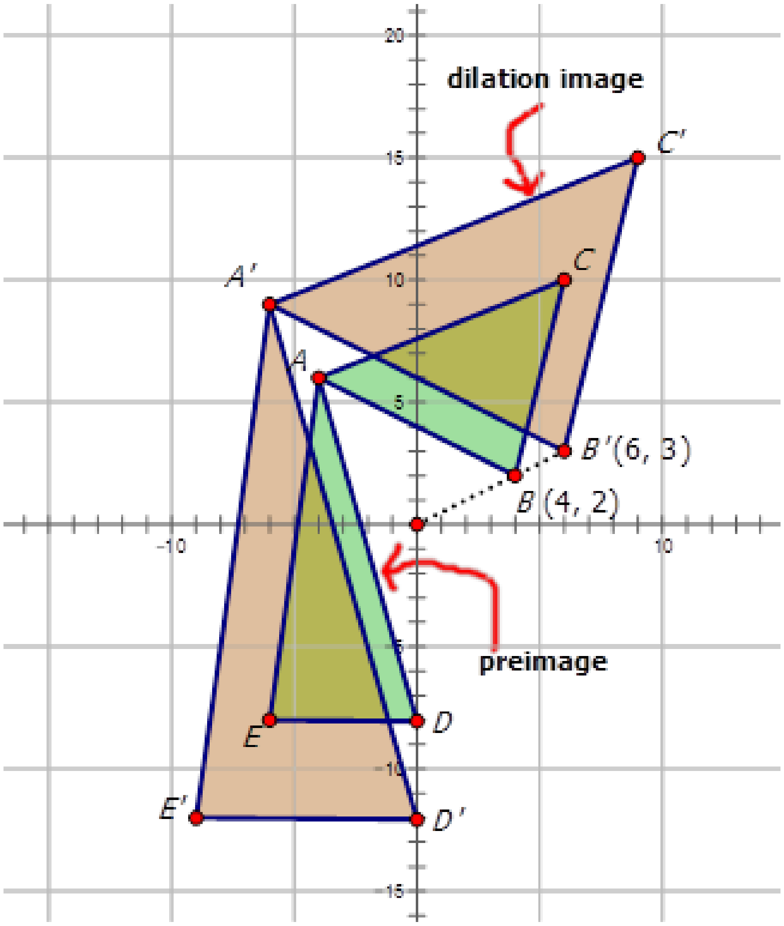
13.


DILATION IMAGE
H
H′
I
I′
G′
G
IMAGE A
K′(-0.75, -3)
K(-1, -4)
J′
J
4
2
-10
-5
-2
-6

14.


A
D
A′
B
D′
B′
C′
C
E
4
3
2
1
0
-1
-9
-8
-7
-6
-5
-4
-3
-2
-1
0
1
2

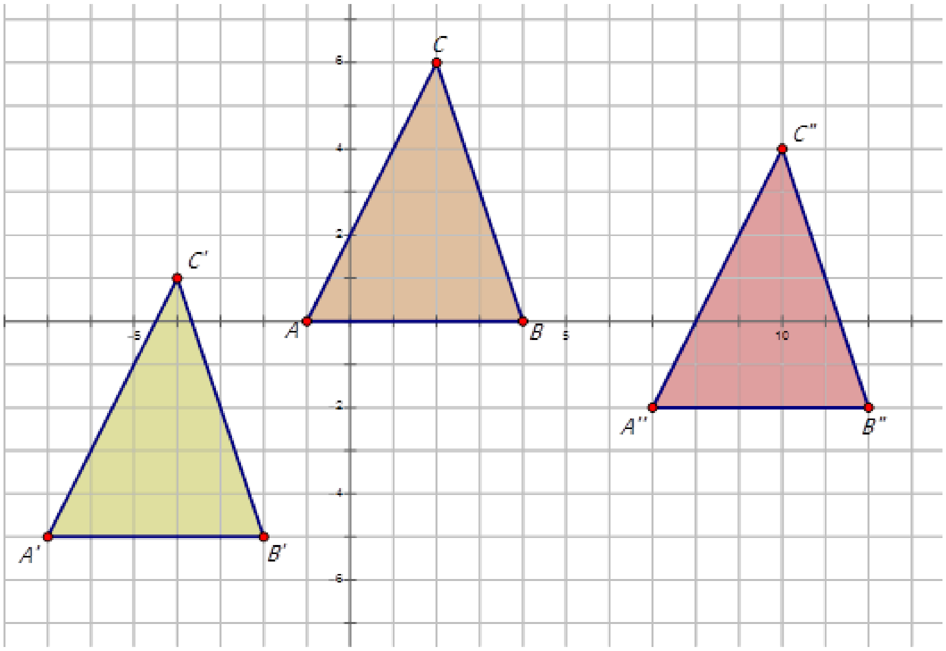
15.


dilation image
preimage
A'
C'
C
A
B'(6, 3)
B (4, 2)
D
E
E'
D'
20
15
10
5
-10
10
-5
-10
-15

10.13 Composite Transformations

Here you will learn about composite transformations.

Look at the following diagram. It involves two translations. Identify the two translations of triangle *ABC*.



Watch This

First watch this video to learn about composite transformations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10CompositeTransformationsA

Then watch this video to see some examples.

MEDIA

Click image to the left for more content.

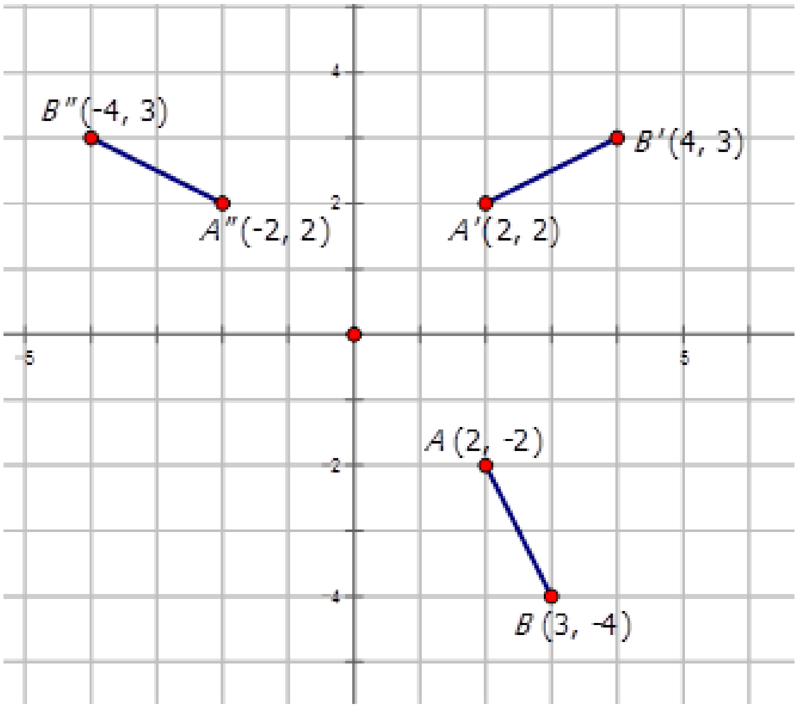
CK-12 Foundation Chapter10CompositeTransformationsB

Guidance

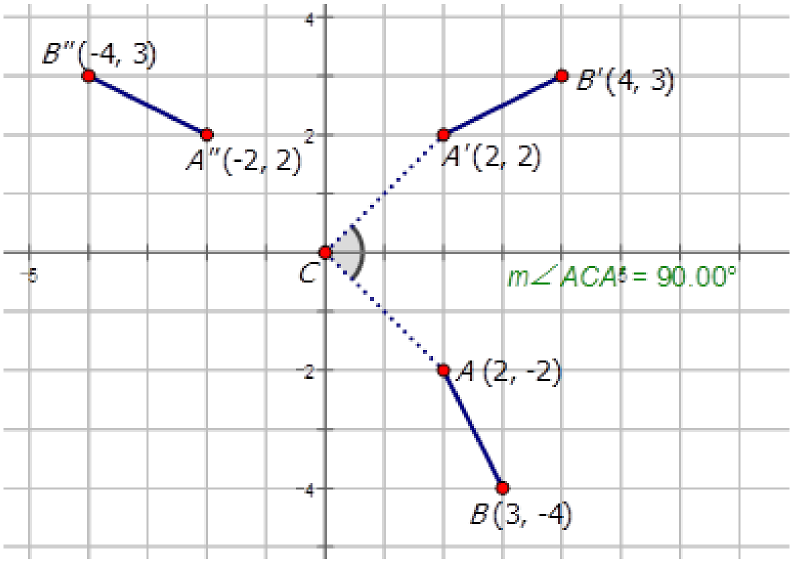
In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image).

Example A

Describe the transformations in the diagram below. The transformations involve a reflection and a rotation.



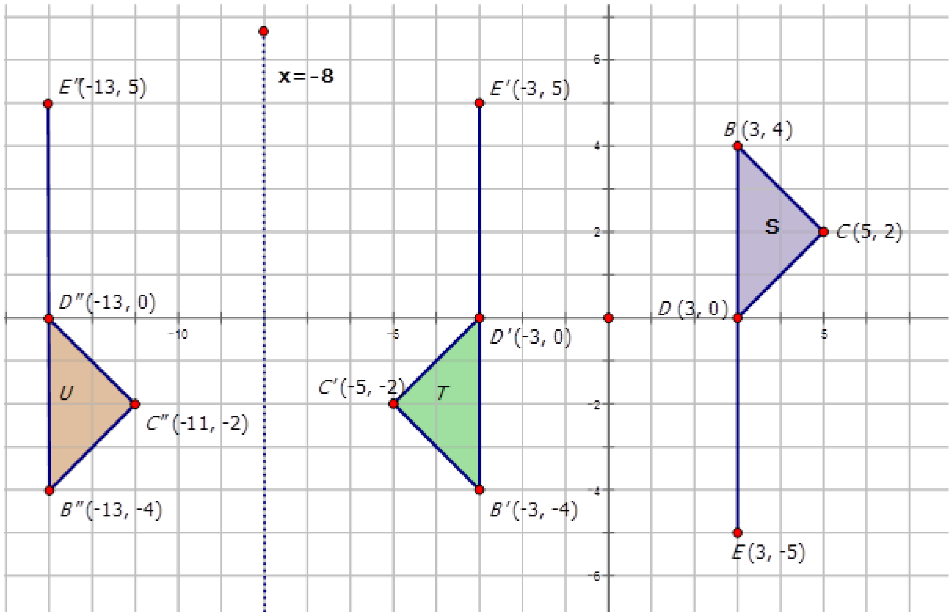
Solution: First line AB is rotated about the origin by 90°CCW.



Then the line $A'B'$ is reflected about the y-axis to produce line $A''B''$.

Example B

Describe the transformations in the diagram below.



Solution: The flag in diagram S is rotated about the origin 180° to produce flag T. You know this because if you look at one point you notice that both x- and y-coordinate points is multiplied by -1 which is consistent with a 180° rotation about the origin. Flag T is then reflected about the line $x = -8$ to produce Flag U.

Example C

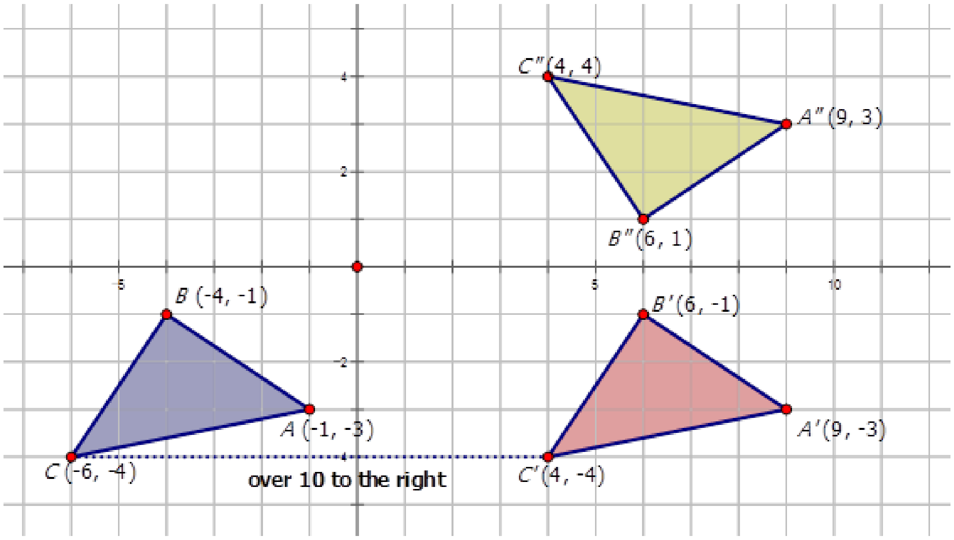
Triangle ABC where the vertices of ΔABC are $A(-1,-3)$, $B(-4,-1)$, and $C(-6,-4)$ undergoes a composition of transformations described as:

a) a translation 10 units to the right, then

b) a reflection in the x-axis.

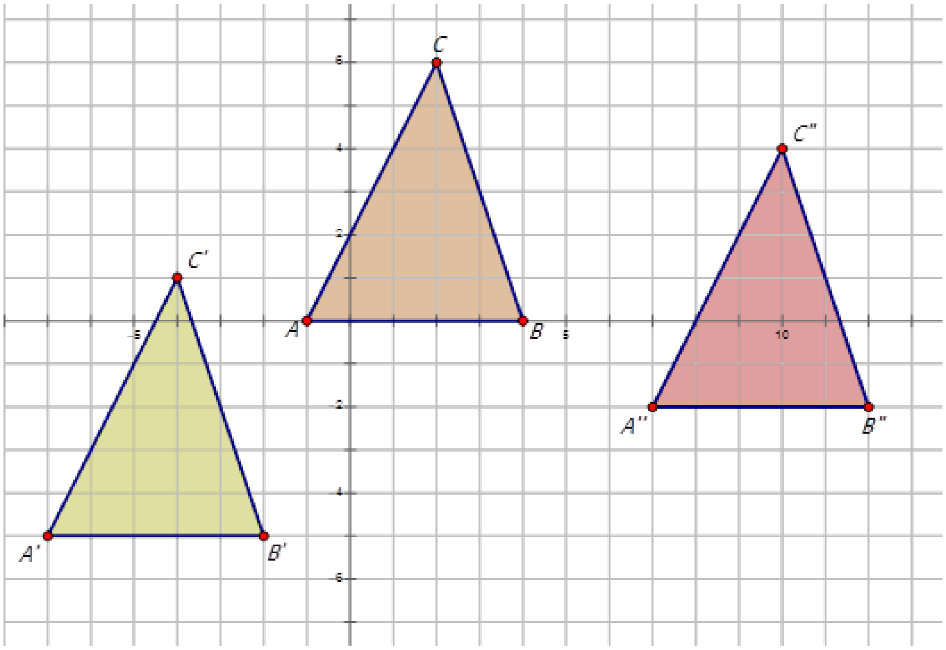
Draw the diagram to represent this composition of transformations. What are the vertices of the triangle after both transformations are applied?

Solution:

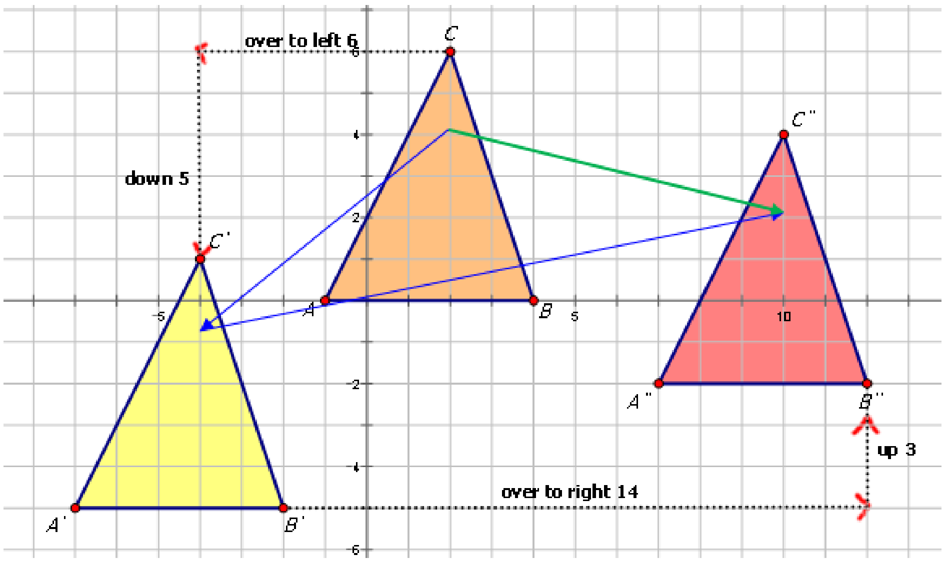


Triangle $A''B''C''$ is the final triangle after all transformations are applied. It has vertices of $A''(9,3)$, $B''(6,1)$, and $C''(4,4)$.

Concept Problem Revisited



ΔABC moves over 6 to the left and down 5 to produce $\Delta A'B'C'$. Then $\Delta A'B'C'$ moves over 14 to the right and up 3 to produce $\Delta A''B''C''$. These translations are represented by the blue arrows in the diagram.



All together ΔABC moves over 8 to the right and down 2 to produce $\Delta A''B''C''$. The total translations for this movement are seen by the green arrow in the diagram above.

Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Translation

A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

Rotation

A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Reflection

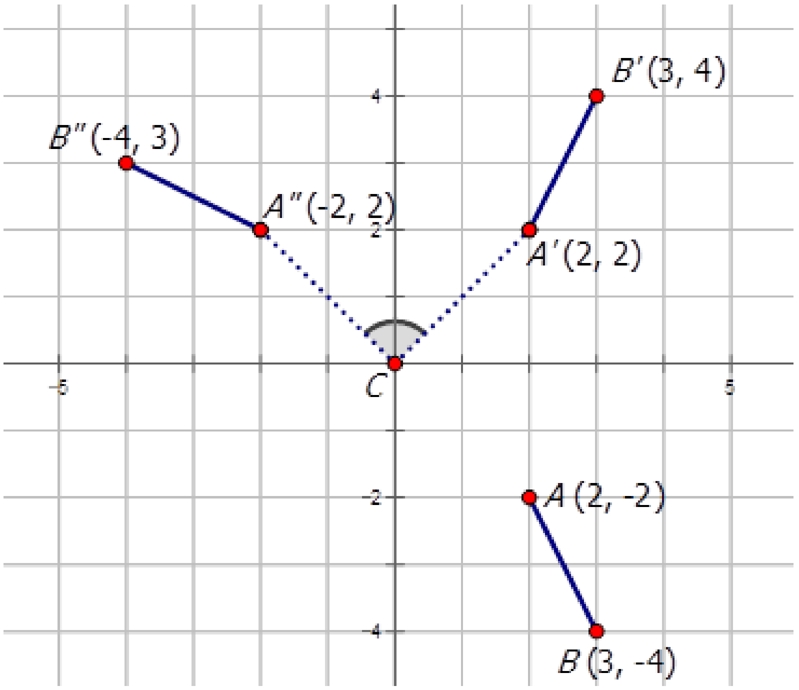
A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

Composite Transformation

A ***composite transformation*** is when two or more transformations are combined to form a new image from the preimage.

Guided Practice

1. Describe the transformations in the diagram below. The transformations involve a rotation and a reflection.



2. Triangle XYZ has coordinates $X(1,2)$, $Y(-3,6)$ and $Z(4,5)$.The triangle undergoes a rotation of 2 units to the right and 1 unit down to form triangle $X'Y'Z'$. Triangle $X'Y'Z'$ is then reflected about the y-axis to form triangle $X''Y''Z''$. Draw the diagram of this composite transformation and determine the vertices for triangle $X''Y''Z''$.

3. The coordinates of the vertices of ΔJAK are $J(1,6)$, $B(2,9)$, and $C(7,10)$.

a) Draw and label ΔJAK.

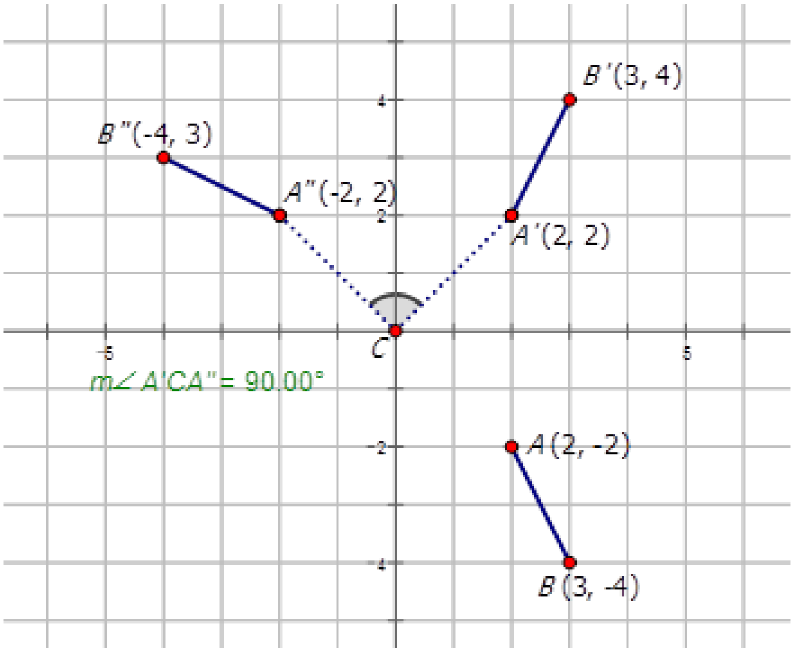
b) ΔJAK is reflected over the line $y = x$. Graph and state the coordinates of $\Delta J'A'K'$.

c) $\Delta J'A'K'$ is then reflected about the x-axis. Graph and state the coordinates of $\Delta J''A''K''$.

d) $\Delta J''A''K''$undergoes a translation of 5 units to the left and 3 units up. Graph and state the coordinates of $\Delta J'''A'''K'''$.

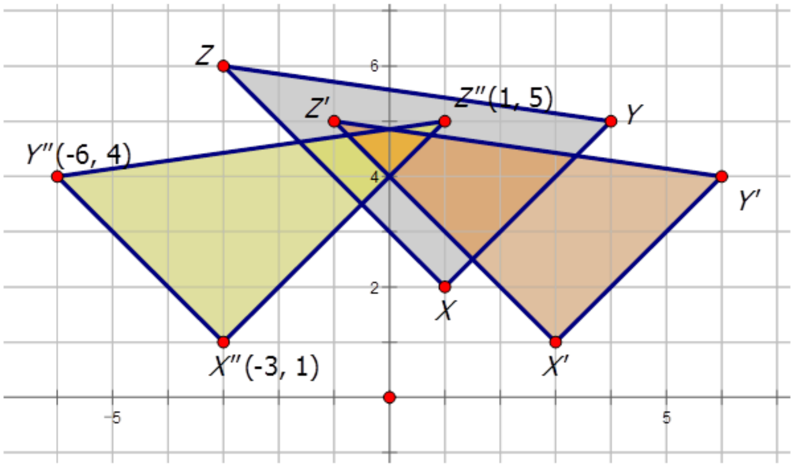
Answers:

1. The transformations involve a reflection and a rotation. First line AB is reflected about the y-axis to produce line $A'B'$.

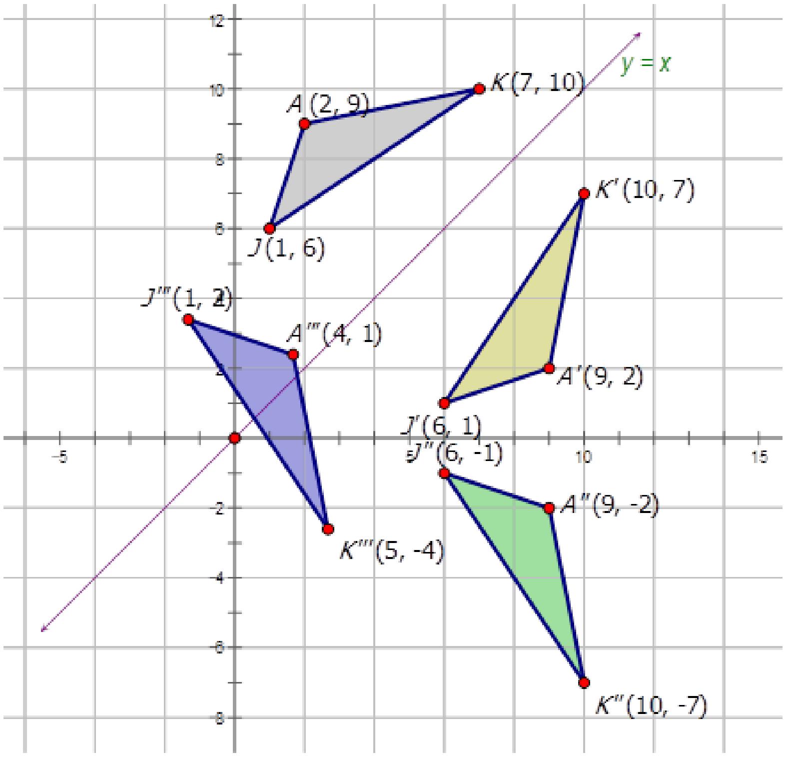


Then the line $A'B'$ is rotated about the origin by 90°CCW to produce line $A''B''$.

2.



3.

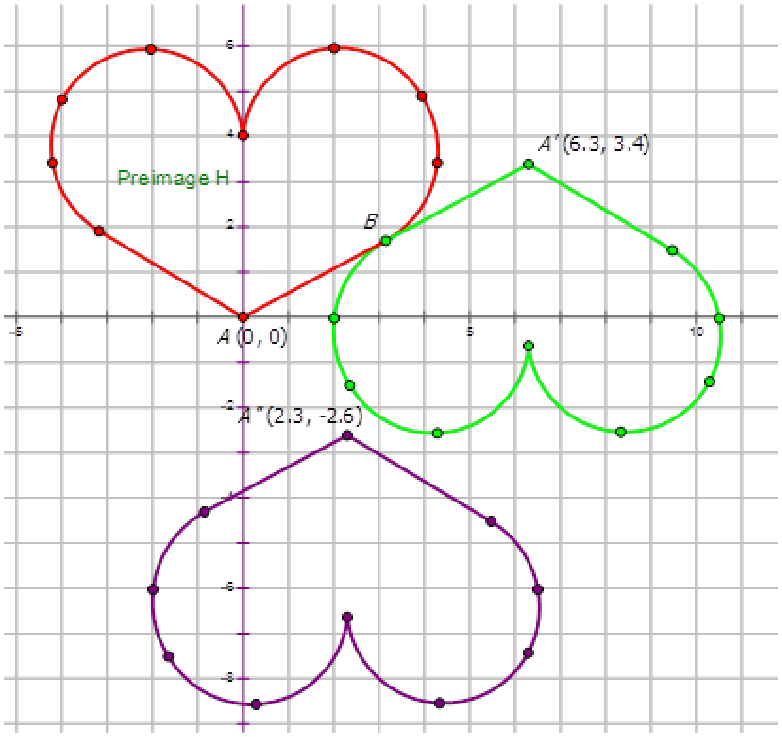


Practice

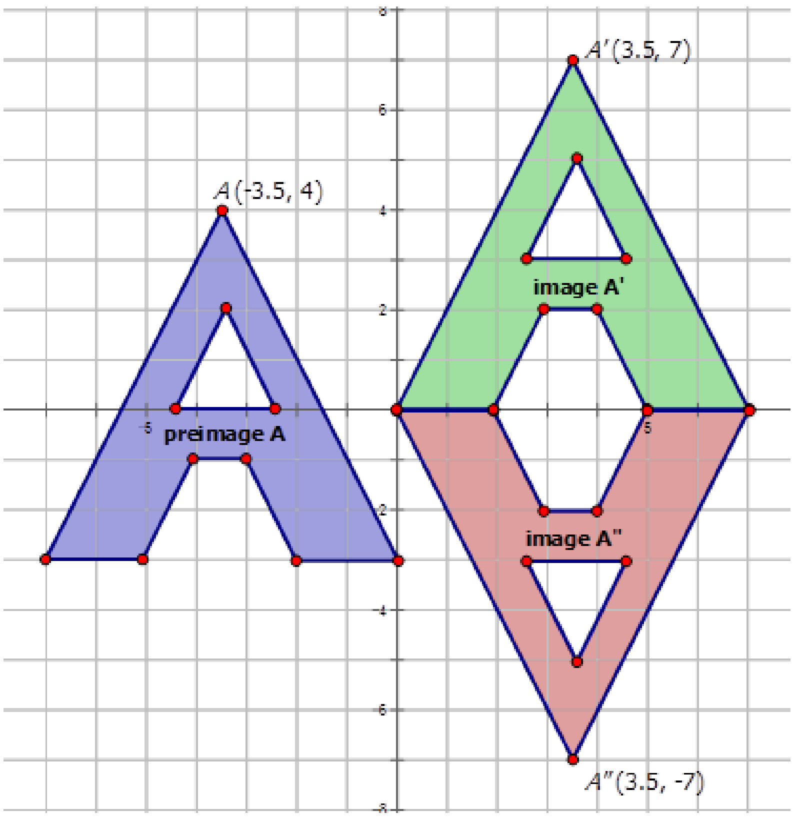
1. A point X has coordinates (-1, -8). The point is reflected across the y-axis to form X'. X' is translated over 4 to the right and up 6 to form X''. What are the coordinates of X' and X''?
2. A point A has coordinates (2, -3). The point is translated over 3 to the left and up 5 to form A'. A' is reflected across the x-axis to form A''. What are the coordinates of A' and A''?
3. A point P has coordinates (5, -6). The point is reflected across the line $y = -x$ to form P'. P' is rotated about the origin 90°CW to form P''. What are the coordinates of P' and P''?
4. Line JT has coordinates $J(-2,-5)$ and $T(2,3)$. The segment is rotated about the origin 180° to form $J'T'$. $J'T'$ is translated over 6 to the right and down 3 to form $J''T''$. What are the coordinates of $J'T'$ and $J''T''$?
5. Line SK has coordinates $S(-1,-8)$ and $K(1,2)$. The segment is translated over 3 to the right and up 3 to form $S'K'$. $S'K'$ is rotated about the origin 90°CCW to form $S''K''$. What are the coordinates of $S'K'$ and $S''K''$?
6. A point K has coordinates (-1, 4). The point is reflected across the line $y = x$ to form K'. K' is rotated about the origin 270°CW to form K''. What are the coordinates of K' and K''?

Describe the following composite transformations:

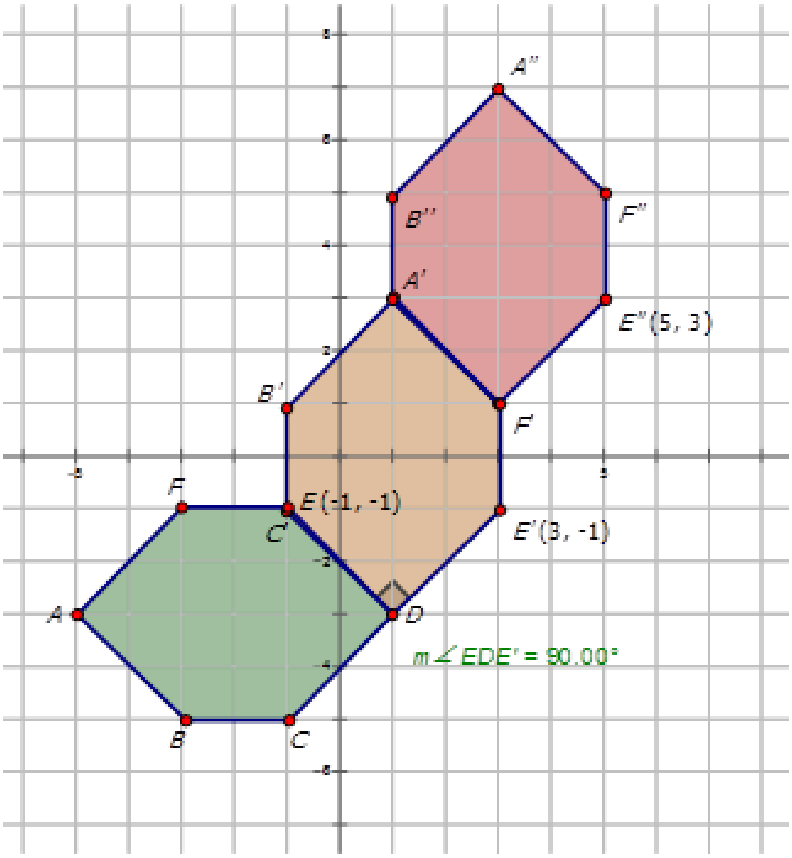
7.


A'(6.3, 3.4)
Preimage H
B
A (0, 0)
A''(2.3, -2.6)
6
4
2
-2
-4
-6
-8
-5
5
10

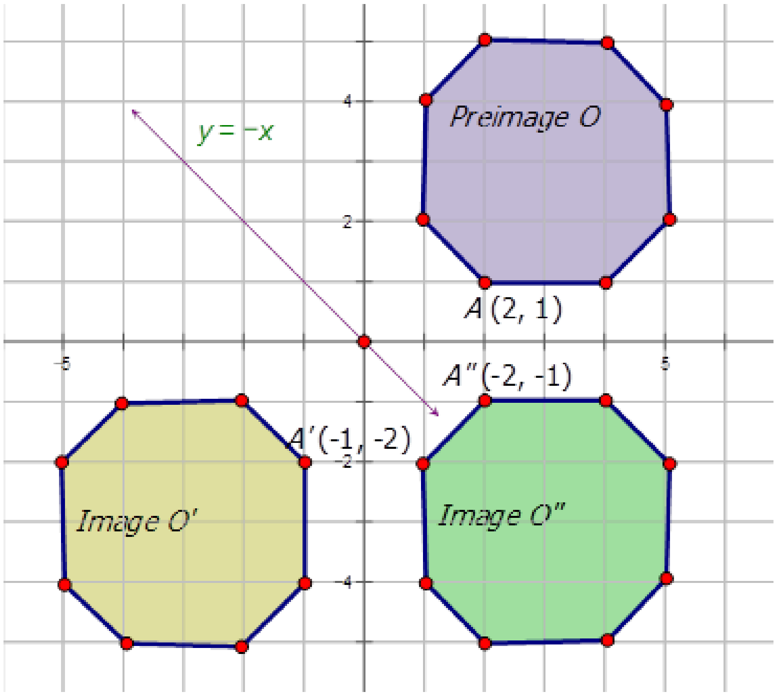
8.

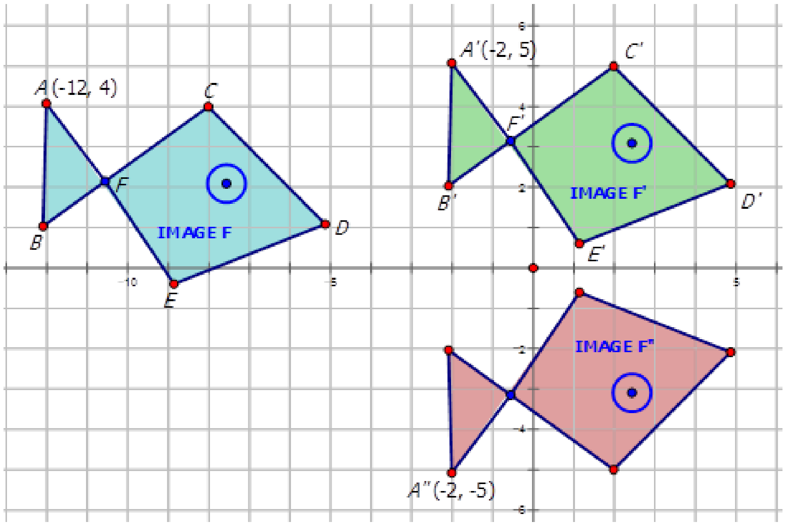

A'(3.5, 7)
A(-3.5, 4)
image A'
preimage A
image A''
A''(3.5, -7)
8
6
4
2
-2
-4
-6
-8
-5
5

9.


A''
B''
F''
A'
E''(5, 3)
B'
F'
F
E(-1, -1)
C'
E'(3, -1)
A
D
m∠EDE' = 90.00°
B
C

10.


y = −x
Preimage O
A (2, 1)
A''(-2, -1)
A'(-1, -2)
Image O'
Image O''
4
2
-2
-4
-5
5

11. 

12. Explore what happens when you reflect a shape twice, over a pair of parallel lines. What one transformation could have been performed to achieve the same result?
13. Explore what happens when you reflect a shape twice, over a pair of intersecting lines. What one transformation could have been performed to achieve the same result?
14. Explore what happens when you reflect a shape over the x-axis and then the y-axis. What one transformation could have been performed to achieve the same result?
15. A composition of a reflection and a translation is often called a glide reflection. Make up an example of a glide reflection. Why do you think it's called a **glide** reflection?

10.14 Order of Composite Transformations

Here you will investigate whether or not the order that transformations are performed matters when doing a composite transformation.

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$ and then is translated 4 units to the right and 5 units down. Show the resulting image.

Watch This

First watch this video to learn about the order of composite transformations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10OrderofCompositeTransformationsA

Then watch this video to see some examples.

MEDIA

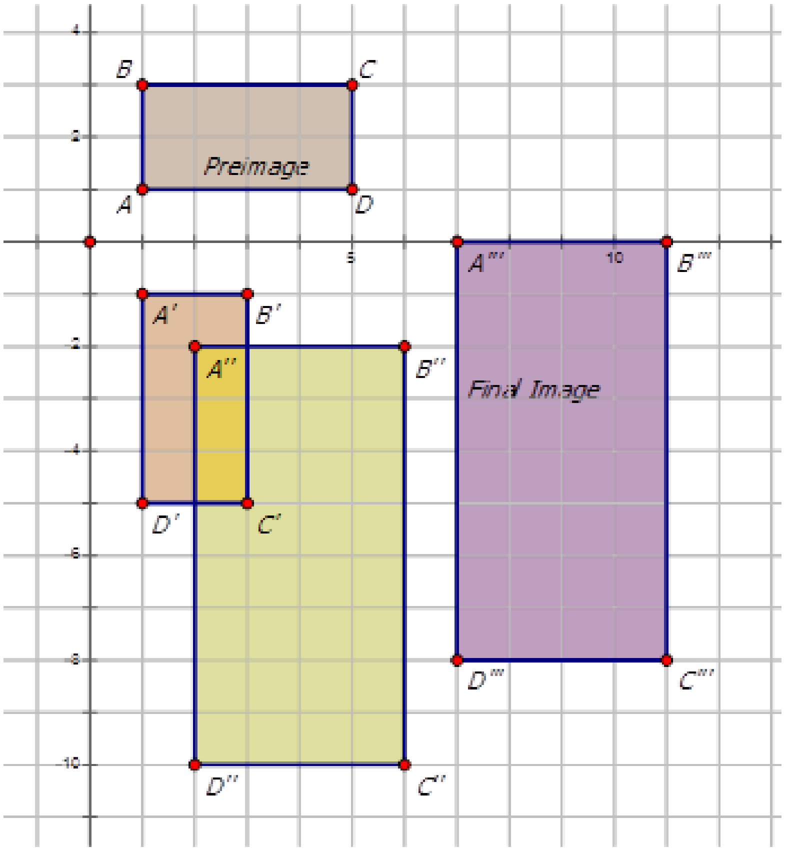
Click image to the left for more content.

CK-12 Foundation Chapter10OrderofCompositeTransformationsB

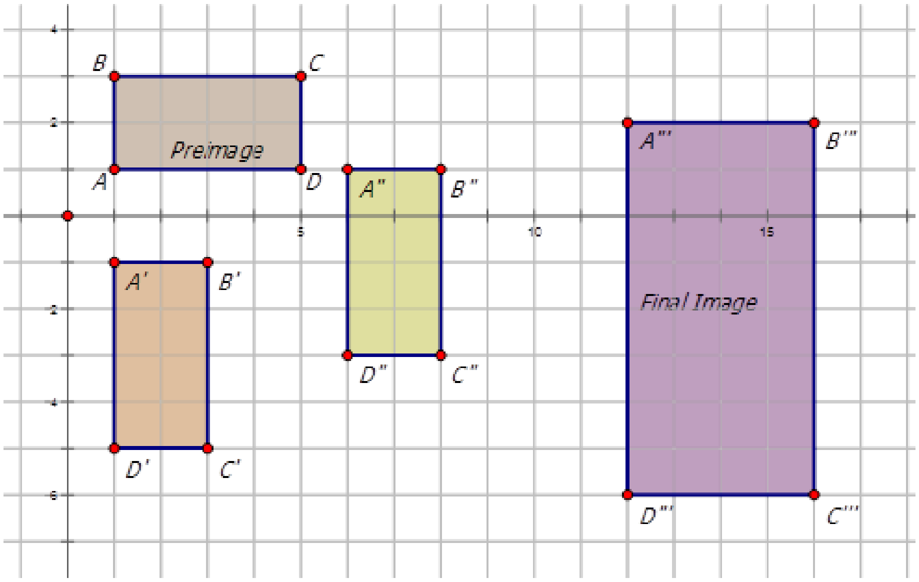
Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image).

Imagine if you rotate, then dilate, and then translate a rectangle of vertices $A(1,1), B(1,3), C(5,3)$, and $D(5,1)$. You would end up with a diagram similar to that found below:



If you take the same preimage and rotate, translate it, and finally dilate it, you could end up with the following diagram:

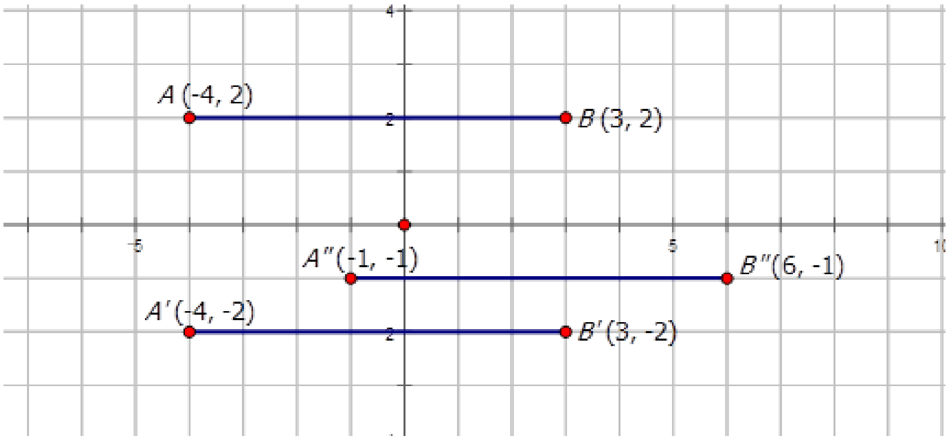


Therefore, the order is important when performing a composite transformation. Remember that the composite transformation involves a series of one or more transformations in which each transformation after the first is performed on the image that was transformed. Only the first transformation will be performed on the initial preimage.

Example A

Line $\overline{AB}$ drawn from (-4, 2) to (3, 2) has undergone a reflection across the x-axis. It then undergoes a translation up one unit and over 3 units to the right to produce $A''B''$. Draw a diagram to represent this composite transformation and indicate the vertices for each transformation.

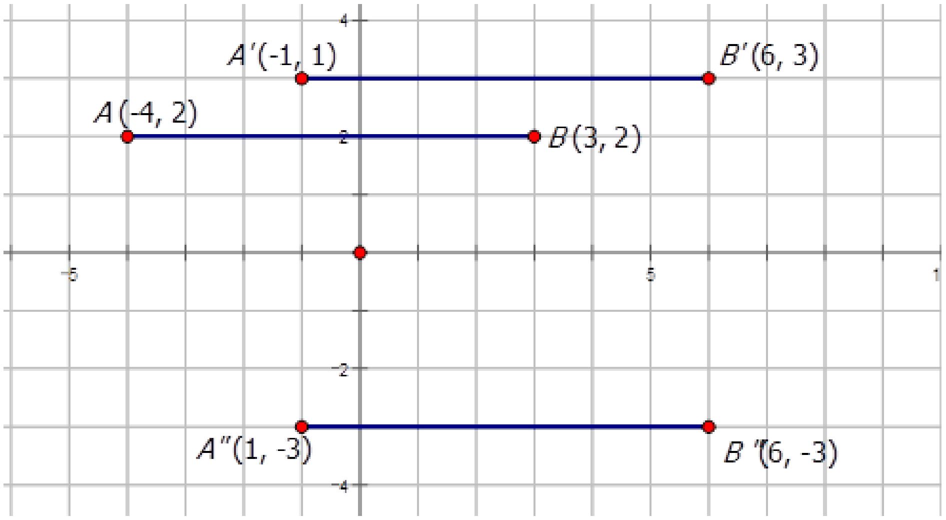
Solution:



Example B

For the composite transformation in Example A, suppose the preimage AB undergoes a translation up one unit and over 3 units to the right and then undergoes a reflection across the x-axis. Does the order matter?

Solution:

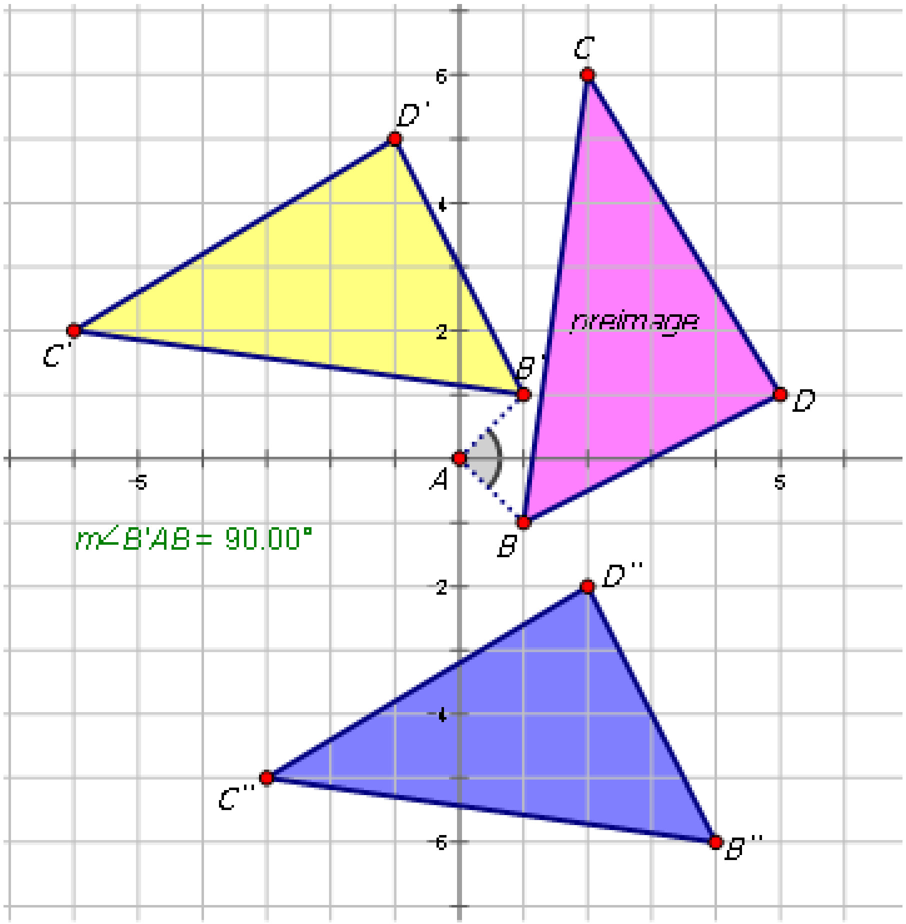


For this example $A''B''$ is not the same as $A''B''$ from the previous example (example A). Therefore order does matter.

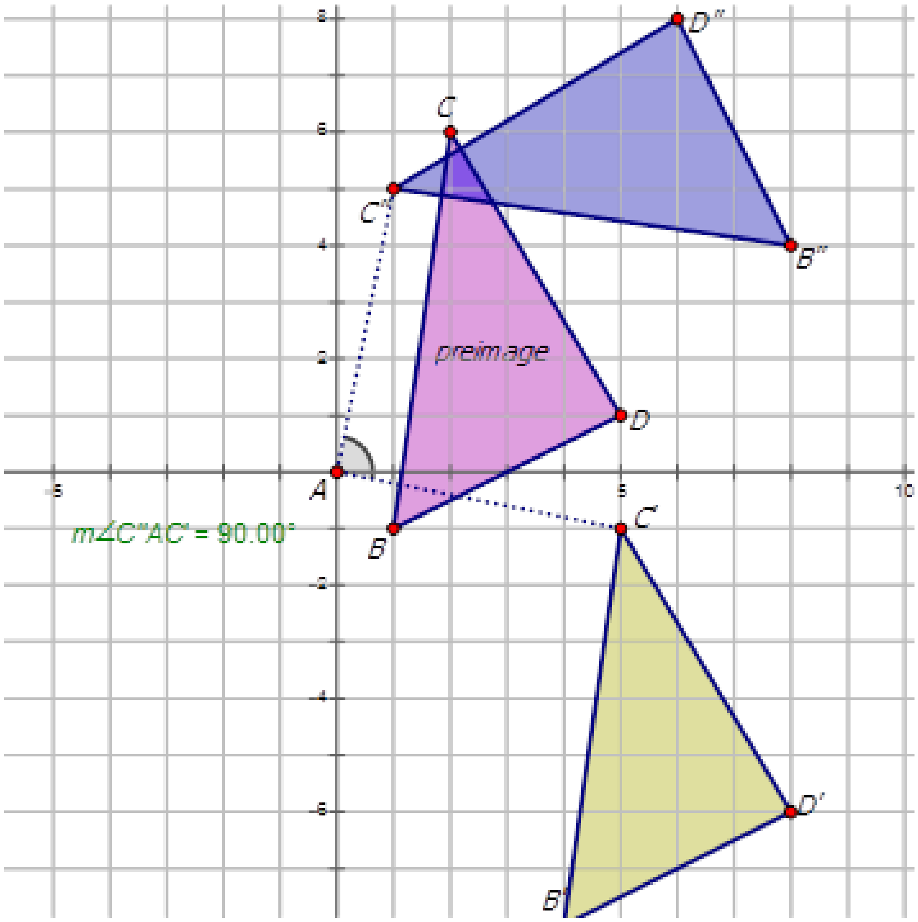
Example C

Triangle BCD is rotated 90°CCW about the origin. The resulting figure is then translated over 3 to the right and down 7. Does order matter?

Order: Rotation then Translation



Order: Translation then Rotation

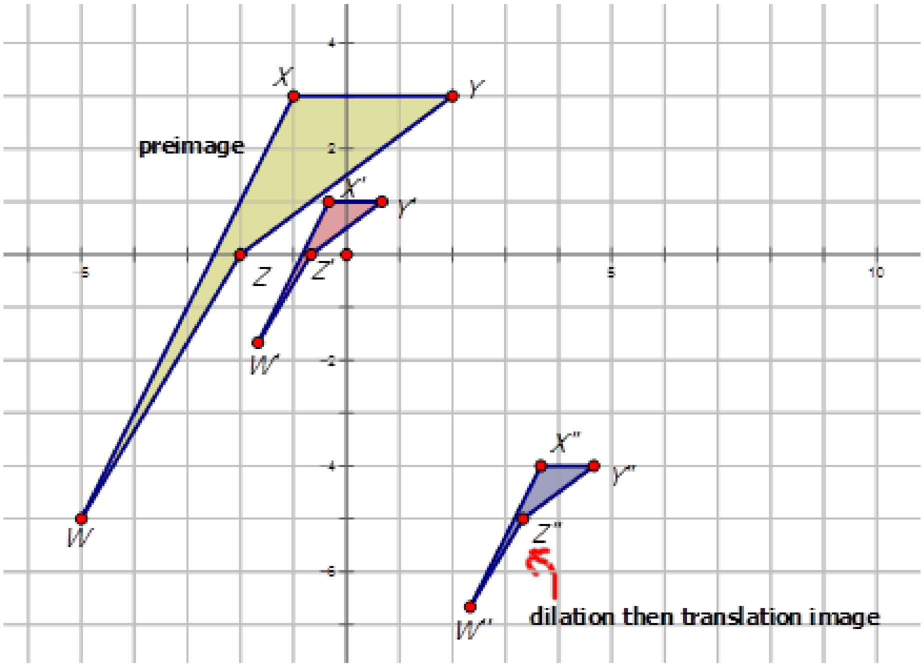


Solution: The blue triangle represents the final image after the composite transformation. In this example, order does matter as the blue triangles do not end up in the same locations.

Concept Problem Revisited

Quadrilateral $WXYZ$ has coordinates $W(-5,-5), X(-2,0), Y(2,3)$ and $Z(-1,3)$. Draw the quadrilateral on the Cartesian plane.

The quadrilateral undergoes a dilation centered at the origin of scale factor $\frac{1}{3}$ and then is translated 4 units to the right and 5 units down. Show the resulting image.



Vocabulary

Image
In a transformation, the final figure is called the ***image***.

Preimage
In a transformation, the original figure is called the ***preimage.***

Transformation
A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Dilation
A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Translation
A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

Rotation
A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Reflection
A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

Composite Transformation
A ***composite transformation*** is when two or more transformations are combined to form a new image from the preimage.

Guided Practice

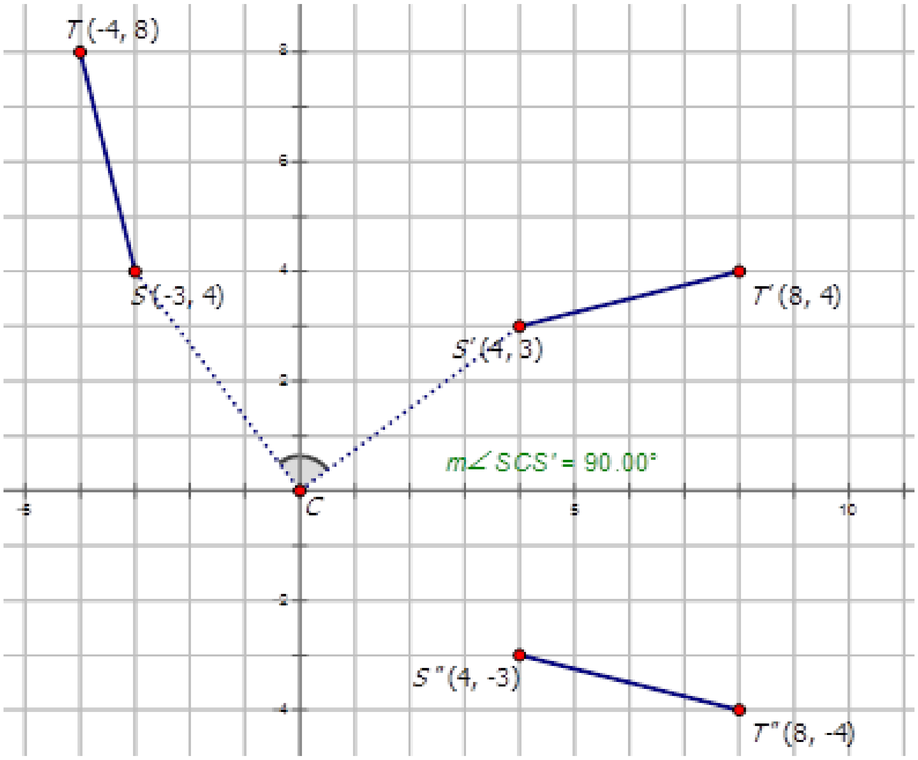
1. Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a rotation about the origin at 90°CW and then a reflection in the x-axis. Draw a diagram with labeled vertices to represent this composite transformation.

2. Line $\overline{ST}$ drawn from (-3, 4) to (-3, 8) has undergone a reflection in the x-axis and then a rotation about the origin at 90°CW. Draw a diagram with labeled vertices to represent this composite transformation. Is the graph the same as the diagram in #1?

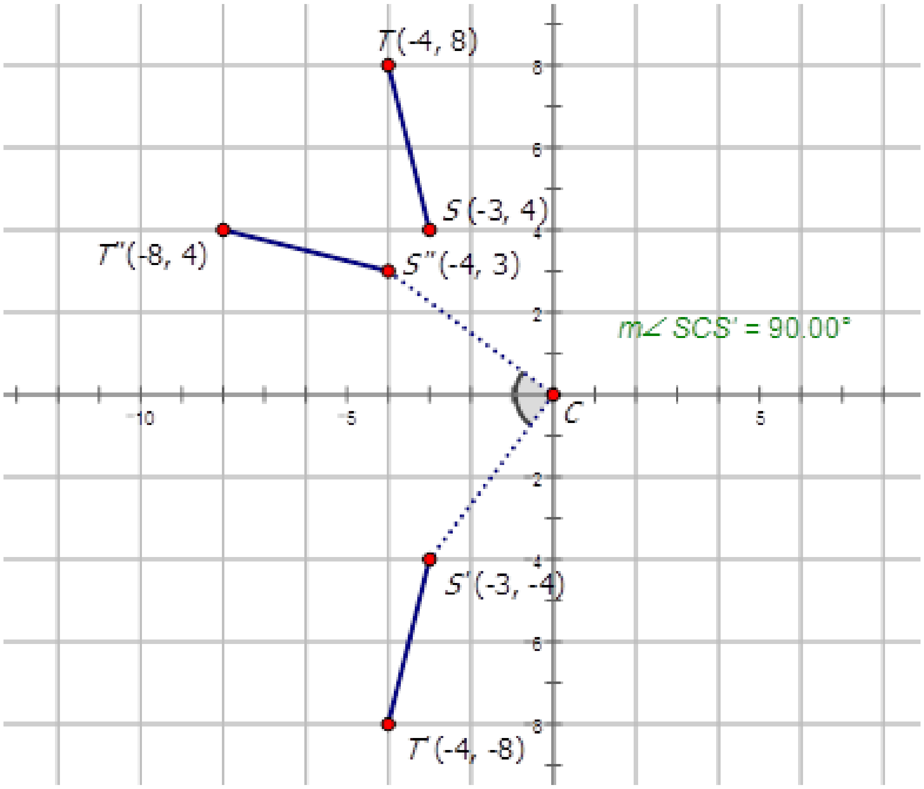
3. The triangle with vertices $J(-5,-2), K(-1,4)$ and $L(1,-3)$ has undergone a transformation of up 4 and over to the right 4 and then a reflection in the x-axis. Draw and label the composite transformation. Does order matter?

Answers:

1.

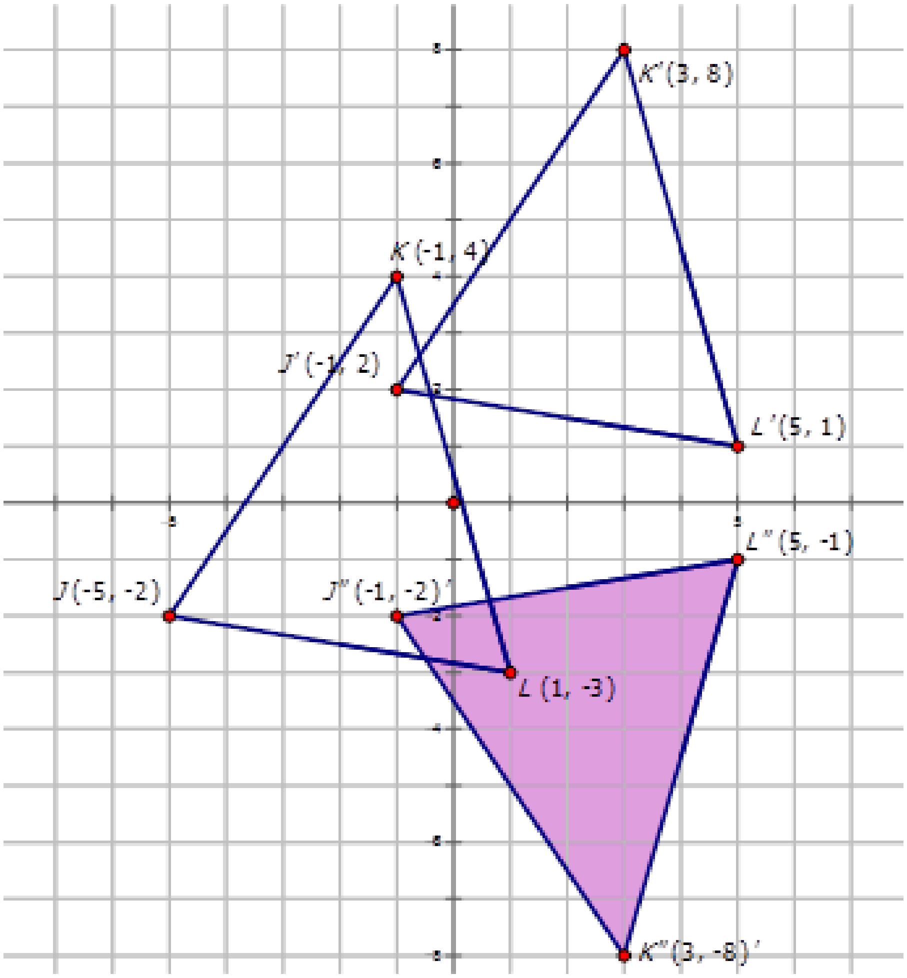


2.

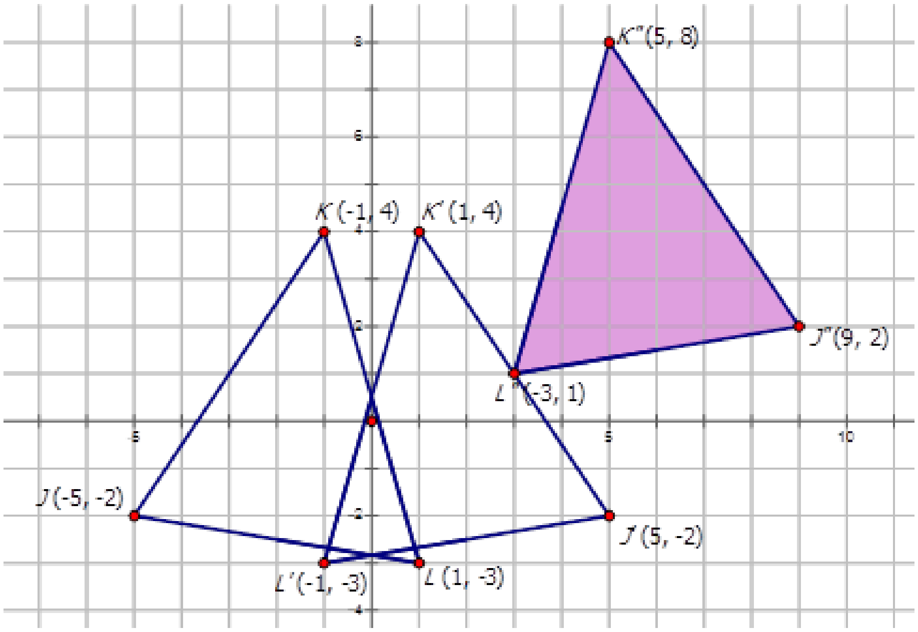


If you compare the graph above to that found in Question 1, you see that the final transformation image $S''T''$ has different coordinates than the image $S''T''$ in question 2. Therefore order does matter.

3.



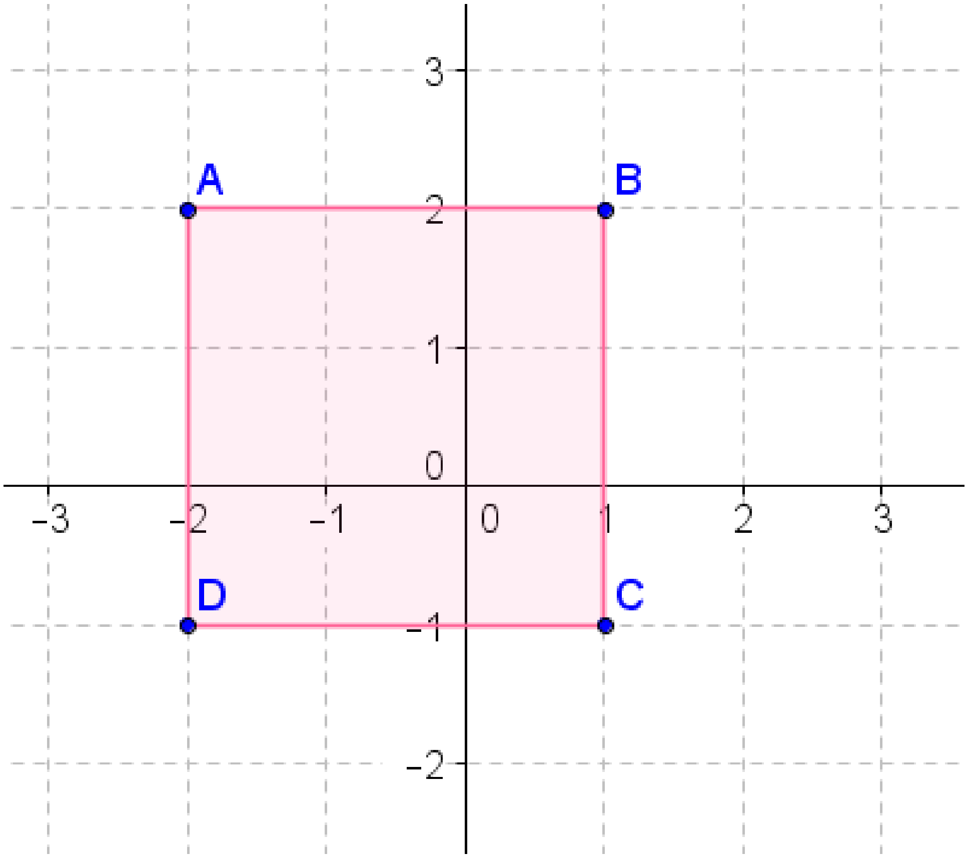
Order: Translation then Reflection



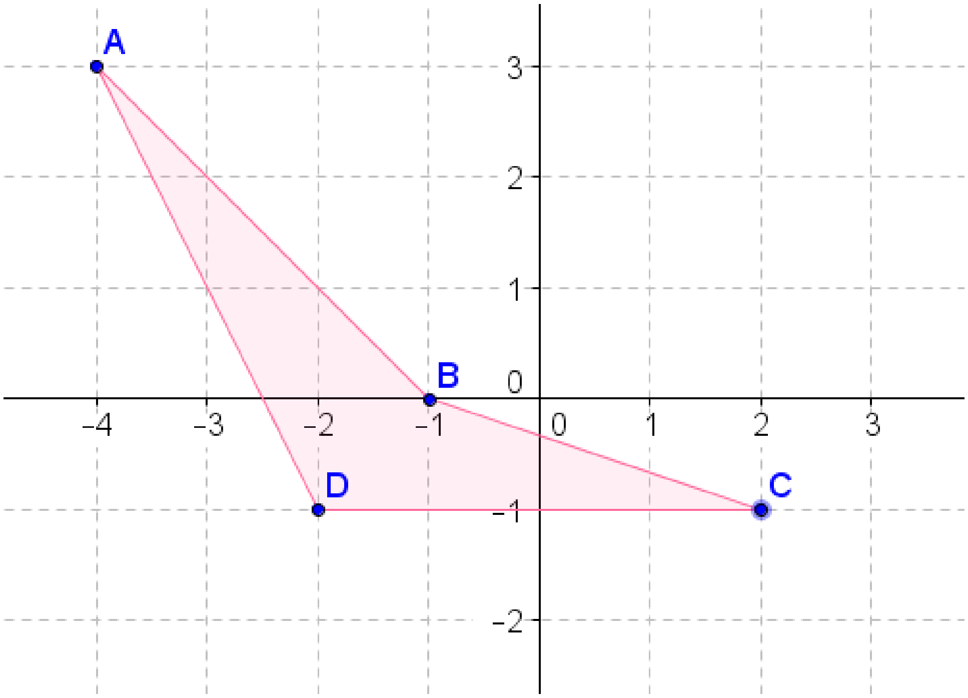
Order: Reflection then Transformation

In this problem, order did matter. The final image after the composite transformation changed when the order changed.

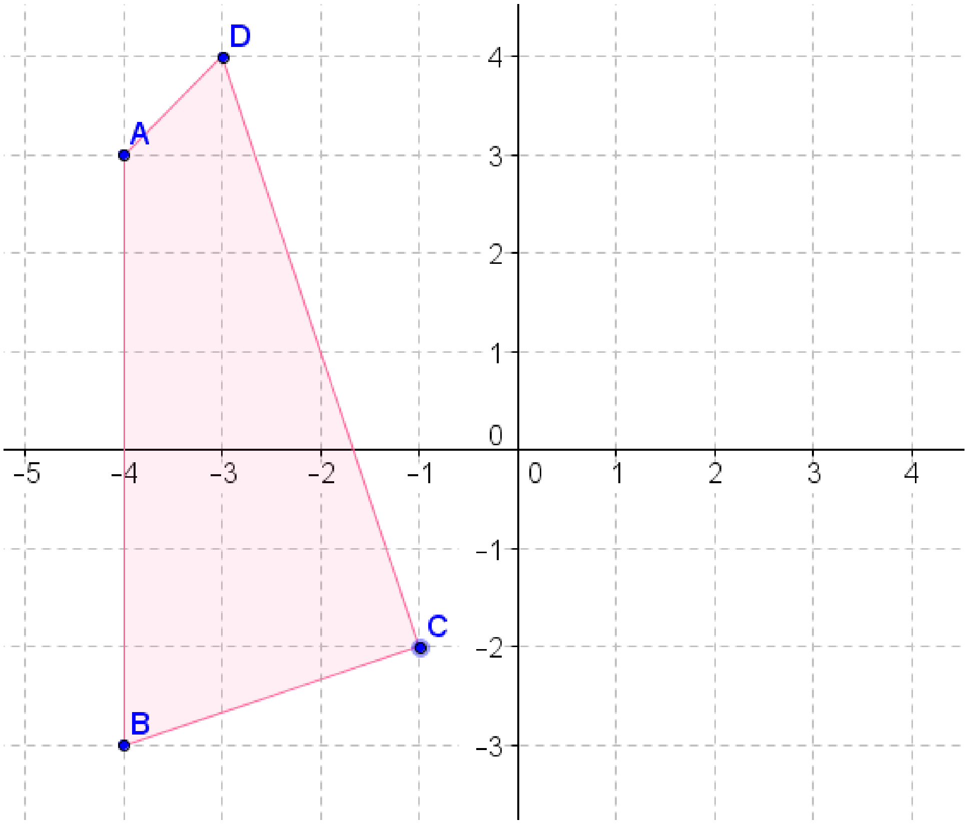
Practice



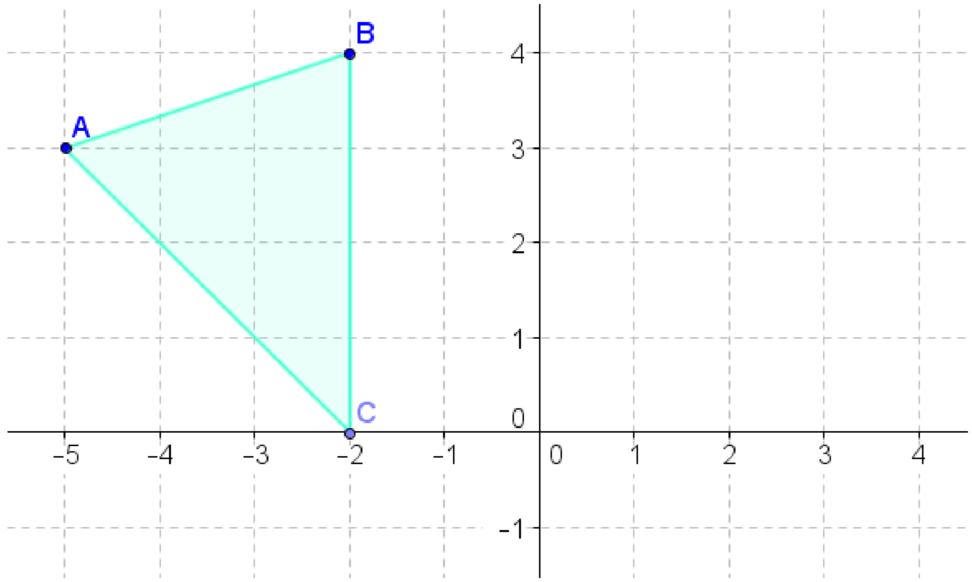
1. Reflect the above figure across the x-axis and then rotate it 90° CW about the origin.
2. Translate the above figure 2 units to the left and 2 units up and then reflect it across the line $y = x$



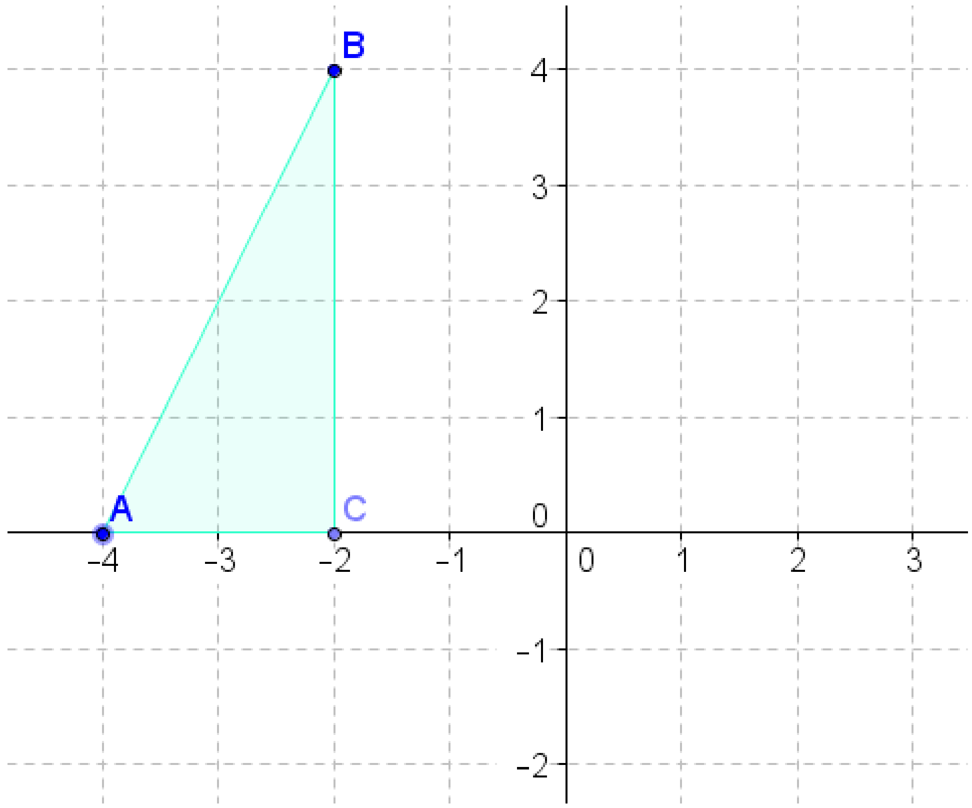
3. Reflect the above figure across the y-axis and then reflect it across the x-axis.
4. What single transformation would have produced the same result as in #3?



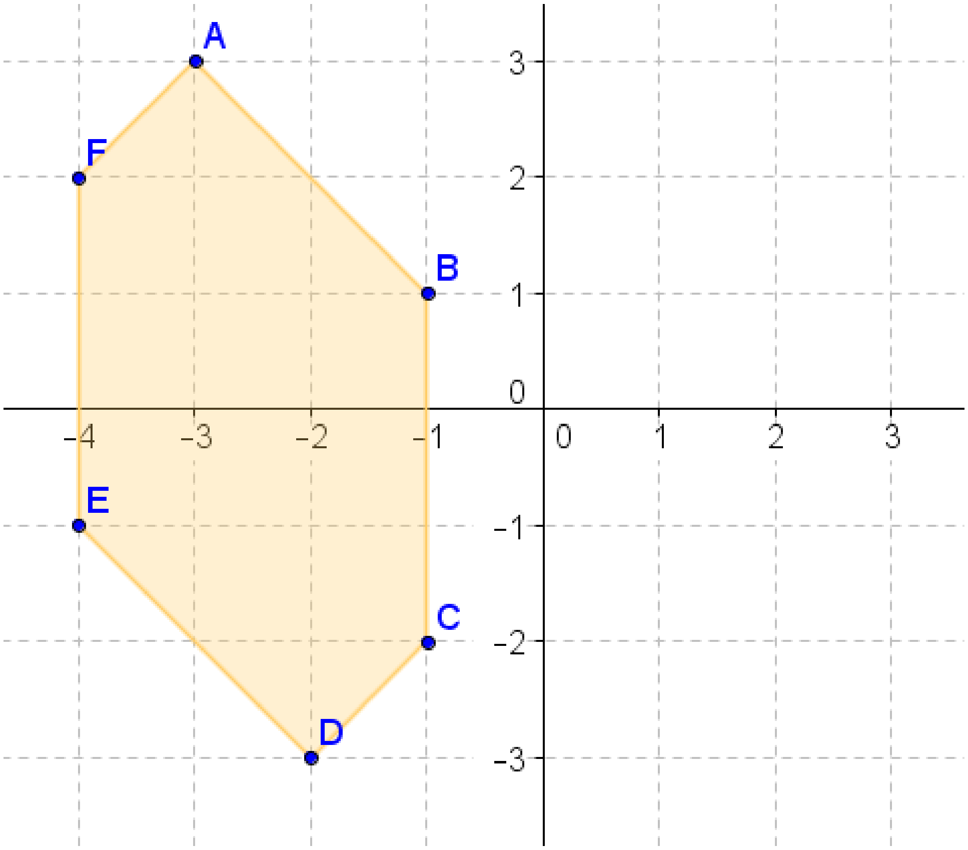
5. Reflect the above figure across the line $x = 2$ and then across the line $x = 8$.
6. What single transformation would have produced the same result a in #5?



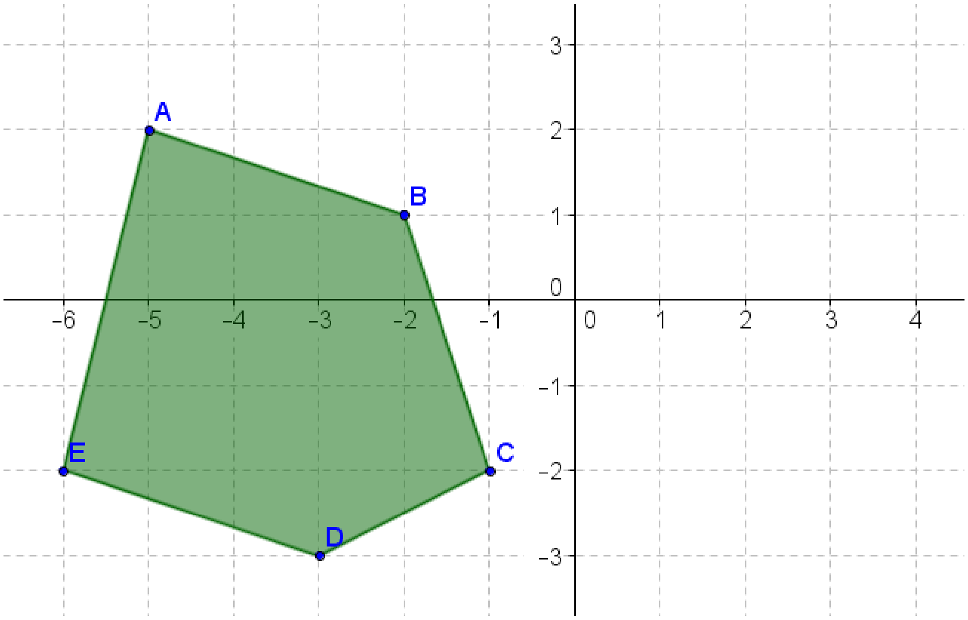
7. Reflect the above figure across the line y=x and then across the x-axis.
8. What single transformation would have produced the same result a in #7?



9. Translate the above figure 2 units to the right and 3 units down and then reflect it across the y-axis.
10. Rotate the above figure 270° CCW about the origin and then translate it over 1 unit to the right and down 1 unit.



11. Reflect the above figure across the line $y = -x$ and then translate it 2 units to the left and 3 units down.
12. Translate the above figure 2 units to the left and 3 units down and then reflect it across the line $y = -x$.

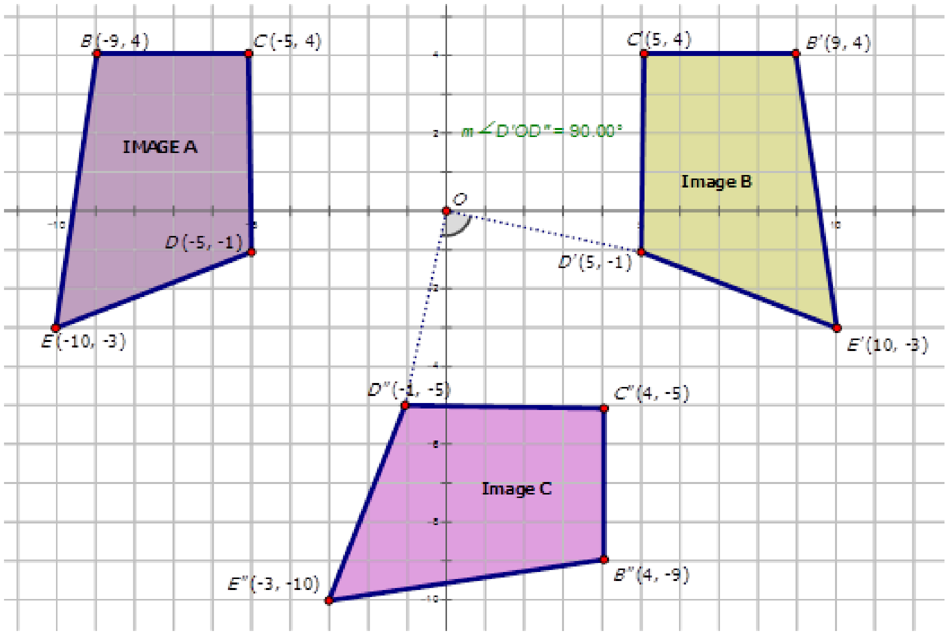


13. Translate the above figure 3 units to the right and 4 units down and then rotate it about the origin 90° CW.
14. Rotate the above figure about the origin 90° CW and then translate it 3 units to the right and 4 units down.
15. How did your result to #13 compare to your result to #14?

10.15 Notation for Composite Transformations

Here you will learn notation for describing a composite transformation.

The figure below shows a composite transformation of a trapezoid. Write the mapping rule for the composite transformation.



Watch This

First watch this video to learn about notation for composite transformations.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10NotationforCompositeTransformationsA

Then watch this video to see some examples.

MEDIA

Click image to the left for more content.

CK-12 Foundation Chapter10NotationforCompositeTransformationsB

Guidance

In geometry, a transformation is an operation that moves, flips, or changes a shape to create a new shape. A composite transformation is when two or more transformations are performed on a figure (called the preimage) to produce a new figure (called the image). The order of transformations performed in a composite transformation matters.

To describe a composite transformation using notation, state each of the transformations that make up the composite transformation and link them with the symbol $\circ$. The transformations are performed in order from right to left. Recall the following notation for translations, reflections, and rotations:

- Translation: $T_{a,b} : (x,y) \rightarrow (x+a, y+b)$ is a translation of a units to the right and b units up.
- Reflection: $r_{y-axis}(x,y) \rightarrow (-x,y)$.
- Rotation: $R_{90^\circ}(x,y) = (-y,x)$

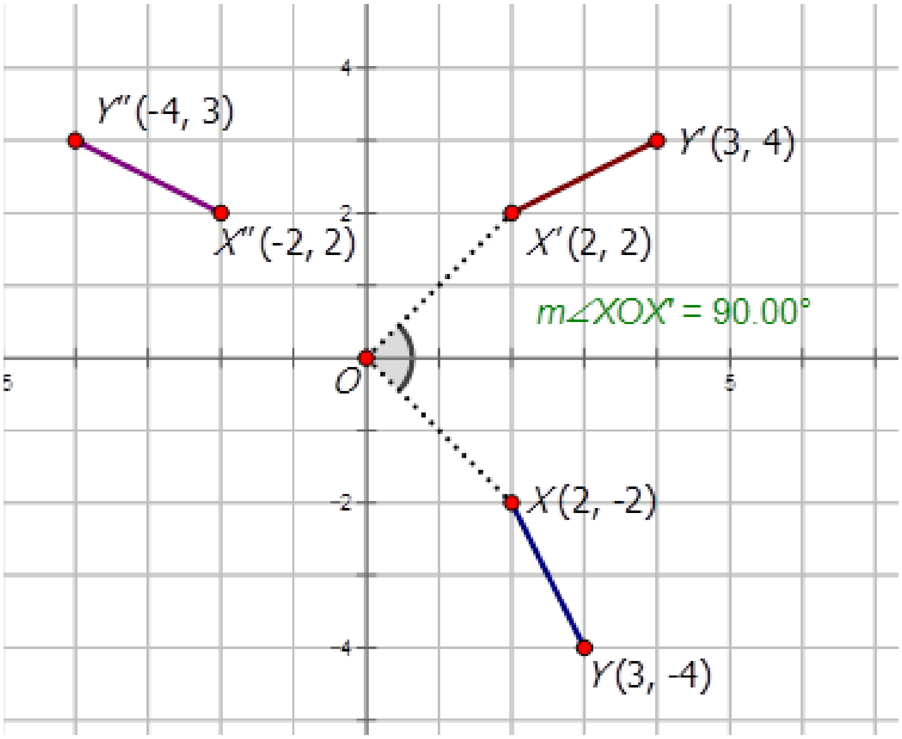
Example A

Graph the line XY given that $X(2,-2)$ and $Y(3,-4)$. Also graph the composite image that satisfies the rule

$$r_{y-axis} \circ R_{90^\circ}$$

.

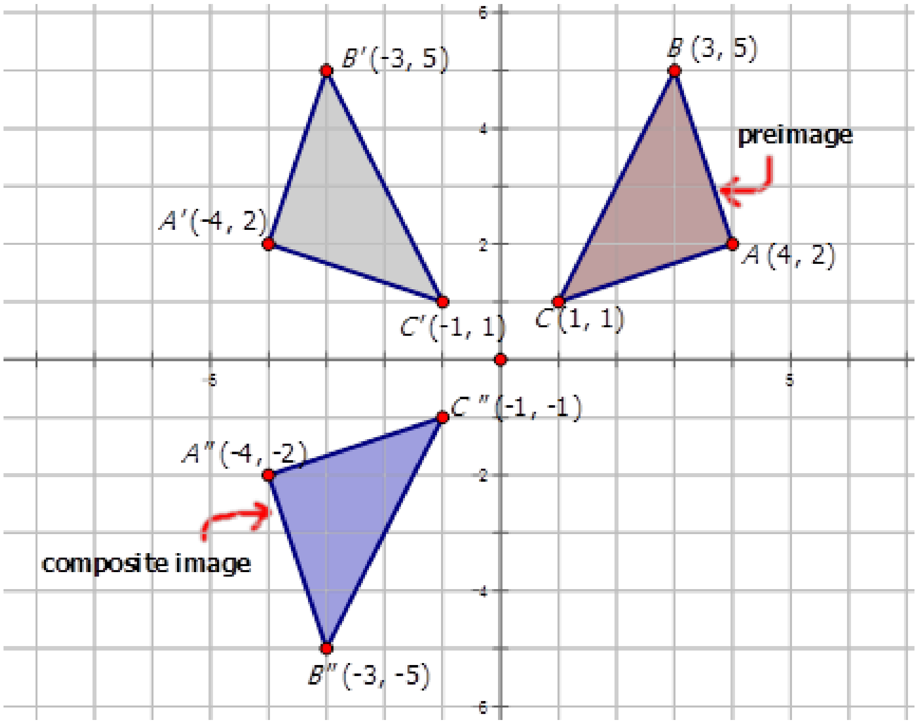
Solution: The first translation is a 90°CCW turn about the origin to produce $X'Y'$. The second translation is a reflection about the y-axis to produce $X''Y''$.



Example B

Image A with vertices $A(3,5), B(4,2)$ and $C(1,1)$ undergoes a composite transformation with mapping rule $r_{x-axis} \circ r_{y-axis}$. Draw the preimage and the composite image and show the vertices of the composite image.

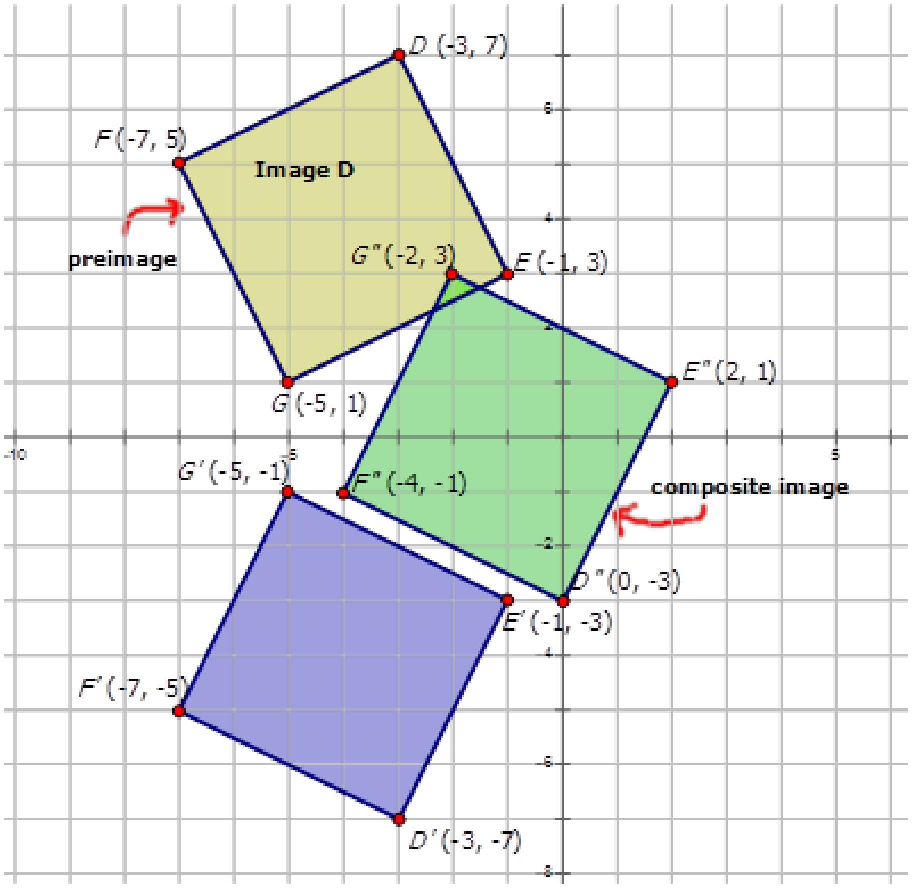
Solution:



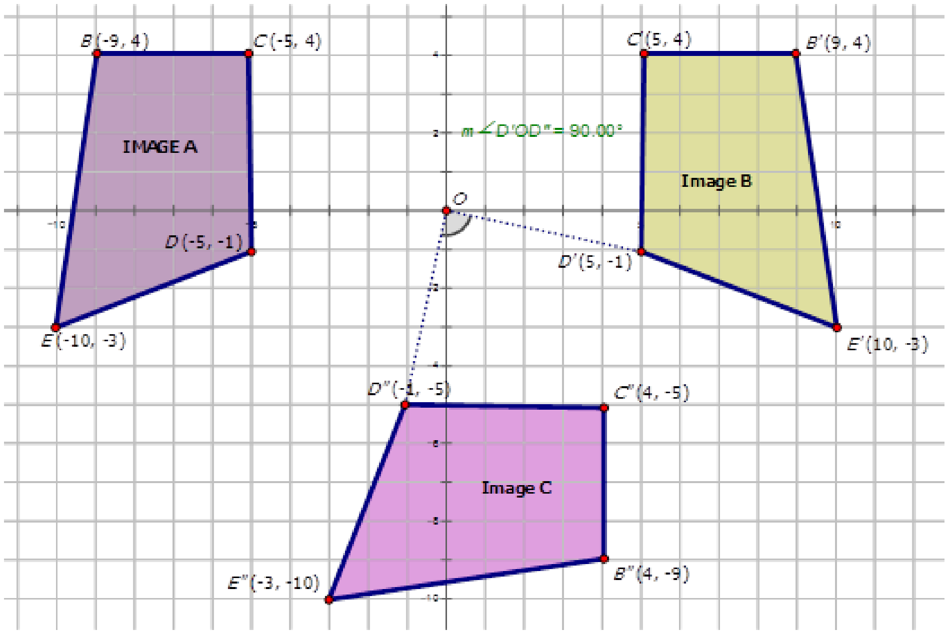
Example C

Image D with vertices $D(-3,7), E(-1,3), F(-7,5)$ and $G(-5,1)$ undergoes a composite transformation with mapping rule $T_{3,4} \circ r_{x-axis}$. Draw the preimage and the composite image and show the vertices of the composite image.

Solution:



Concept Problem Revisited



The transformation from Image A to Image B is a reflection across the y-axis. The notation for this is r_{y-axis}. The transformation for image B to form image C is a rotation about the origin of 90°CW. The notation for this transformation is R_{270°. Therefore, the notation to describe the transformation of Image A to Image C is

$$R_{270^\circ} \circ r_{y-axis}$$

.

Vocabulary

Image

In a transformation, the final figure is called the ***image***.

Preimage

In a transformation, the original figure is called the ***preimage.***

Transformation

A ***transformation*** is an operation that is performed on a shape that moves or changes it in some way. There are four types of transformations: translations, reflections, dilations and rotations.

Dilation

A ***dilation*** is a transformation that enlarges or reduces the size of a figure.

Translation

A ***translation*** is an example of a transformation that moves each point of a shape the same distance and in the same direction. Translations are also known as **slides**.

Rotation

A ***rotation*** is a transformation that rotates (turns) an image a certain amount about a certain point.

Reflection

A ***reflection*** is an example of a transformation that flips each point of a shape over the same line.

Composite Transformation

A ***composite transformation*** is when two or more transformations are combined to form a new image from the preimage.

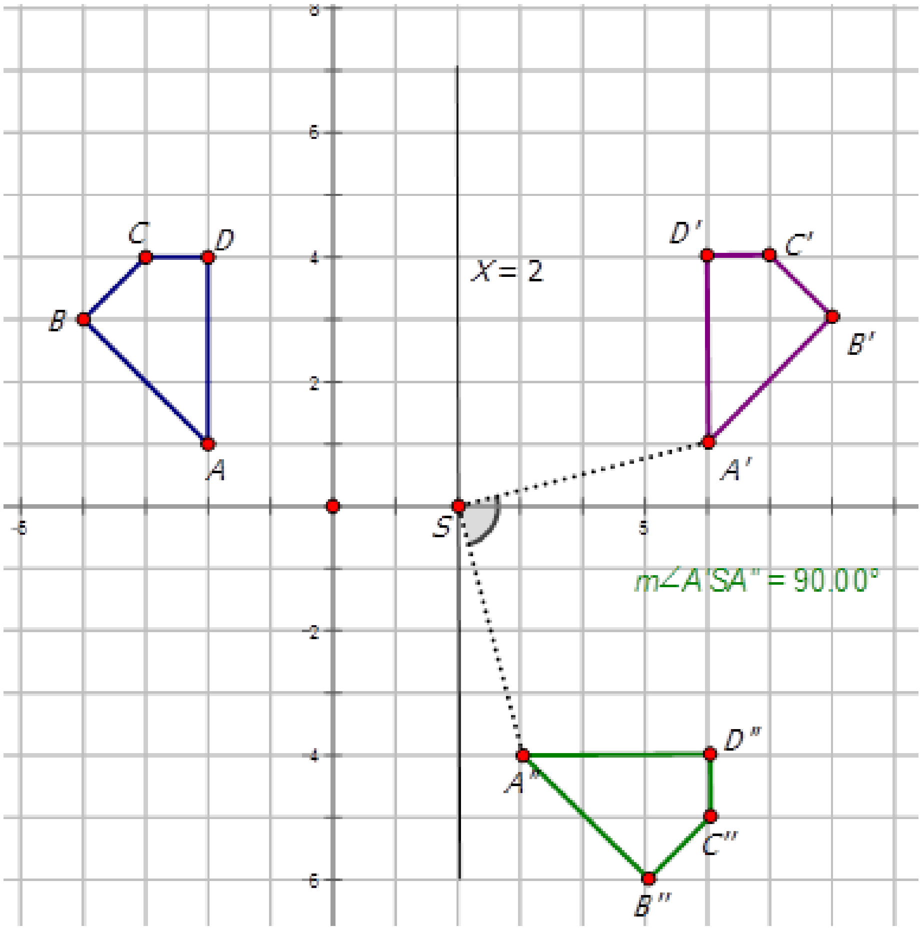
Guided Practice

1. Graph the line XY given that $X(2,-2)$ and $Y(3,-4)$. Also graph the composite image that satisfies the rule

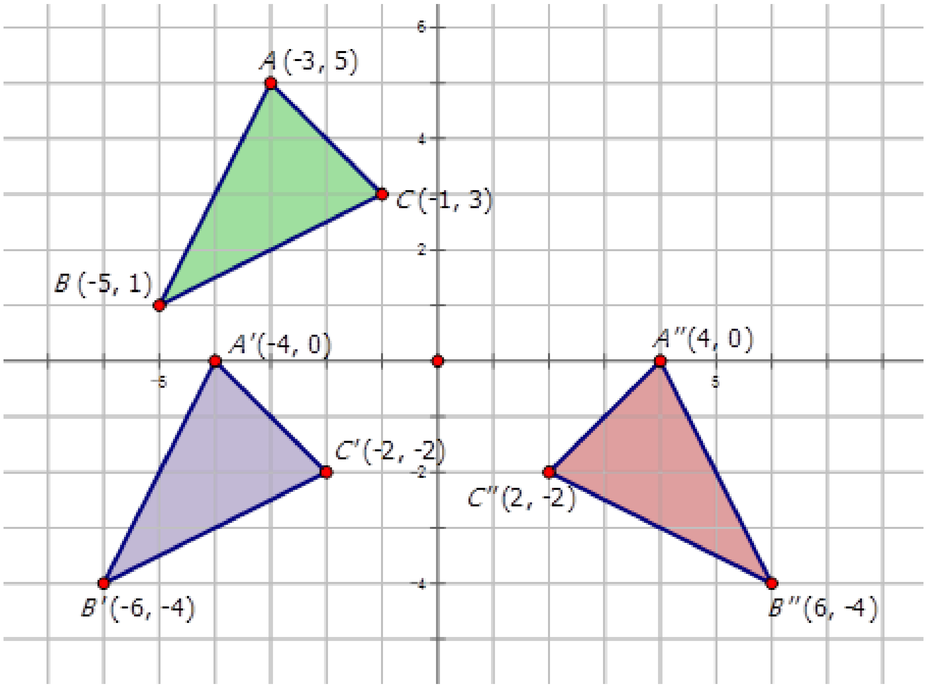
$$R_{90^\circ} \circ r_{y-axis}$$

.

2. Describe the composite transformations in the diagram below and write the notation to represent the transformation of figure $ABCD$ to $A''B''C''D''$.

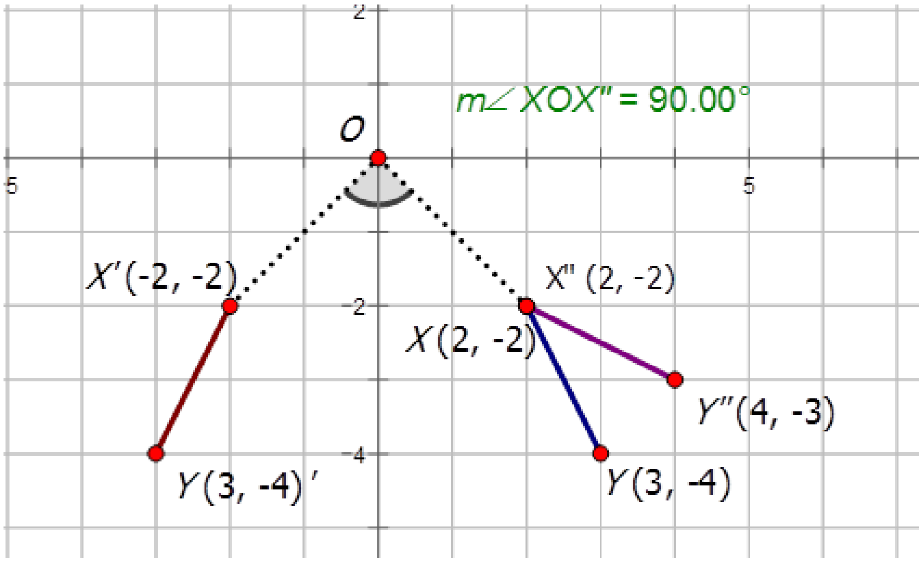


3. Describe the composite transformations in the diagram below and write the notation to represent the transformation of figure ABC to $A''B''C''$.



Answers:

1. The first transformation is a reflection about the y-axis to produce $X'Y'$. The second transformation is a 90°CCW turn about the origin to produce $X''Y''$.



2. There are two transformations shown in the diagram. The first transformation is a reflection about the line $X = 2$ to produce $A'B'C'D'$. The second transformation is a 90°CW (or 270°CCW) rotation about the point (2, 0) to produce the figure $A''B''C''D''$. Notation for this composite transformation is:

$$R_{270^\circ} \circ r_{x=2}$$

3. There are two transformations shown in the diagram. The first transformation is a translation of 1 unit to the left and 5 units down to produce $A'B'C'$. The second reflection in the y-axis to produce the figure $A''B''C''$. Notation for this composite transformation is:

$$r_{y-axis} \circ T_{-1,-5}$$

Practice

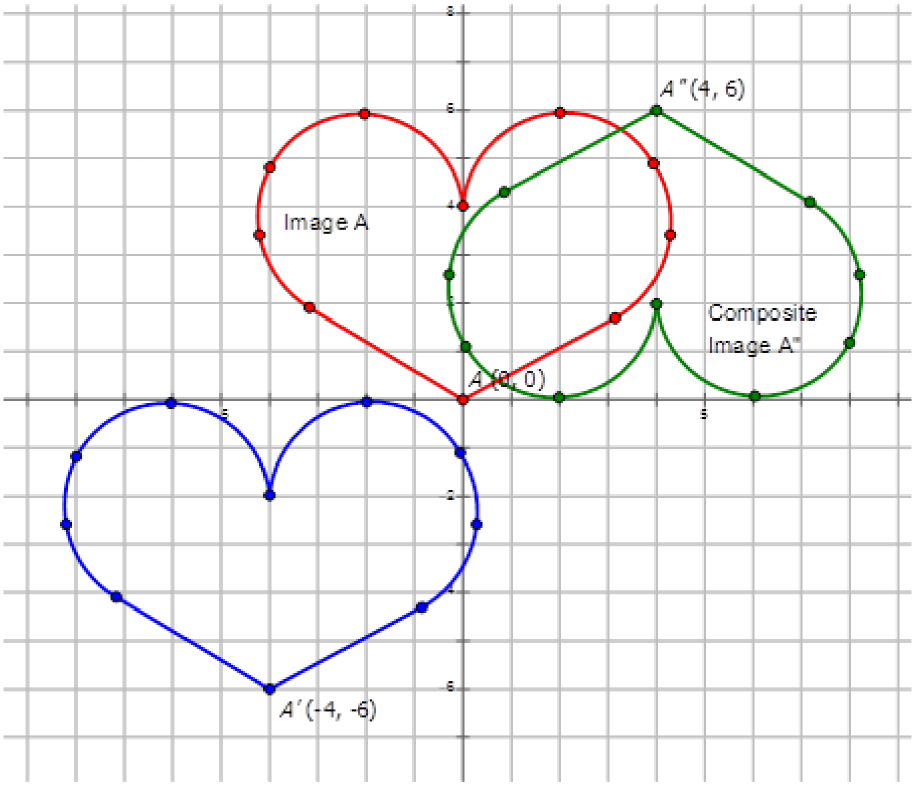
Complete the following table:

TABLE 10.16:

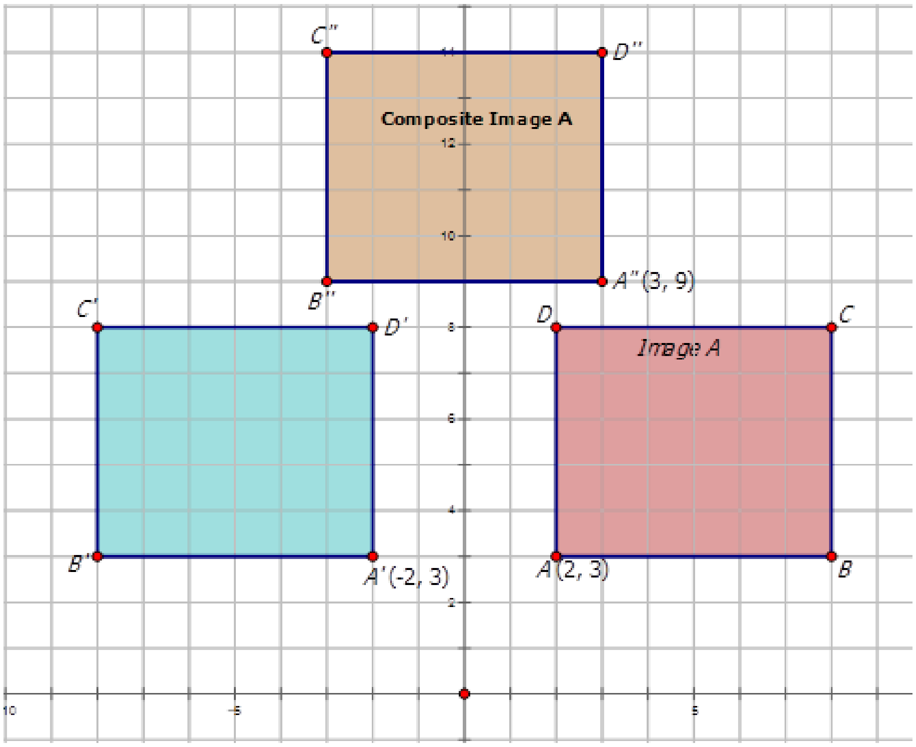
Starting Point	$T_{3,-4} \circ R_{90^\circ}$	$r_{x-axis} \circ r_{y-axis}$	$T_{1,6} \circ r_{x-axis}$	$r_{y-axis} \circ R_{180^\circ}$
1. (1, 4)				
2. (4, 2)				
3. (2, 0)				
4. (-1, 2)				
5. (-2, -3)				
6. (4, -1)				
7. (3, -2)				
8. (5, 4)				
9. (-3, 7)				
10. (0, 0)				

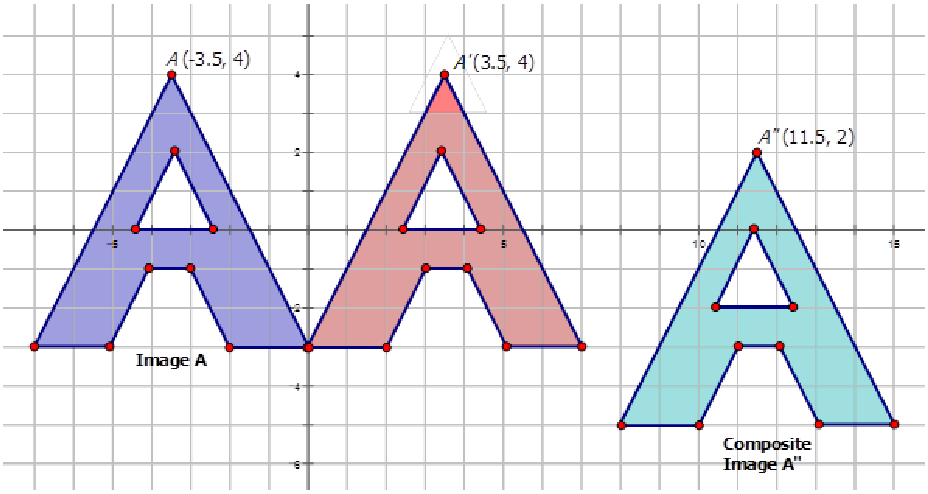
Write the notation that represents the composite transformation of the preimage A to the composite images in the diagrams below.

11.

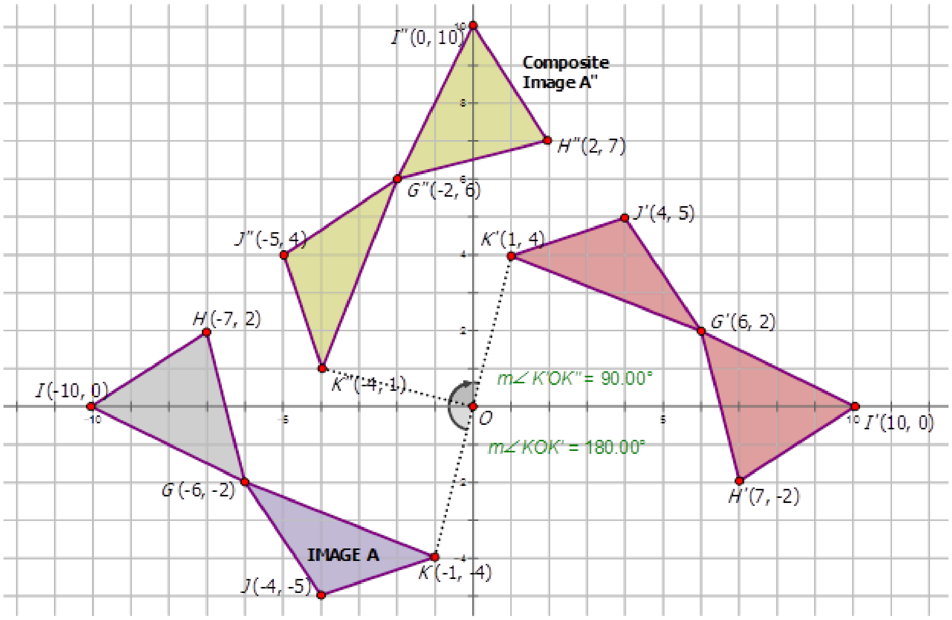


12.

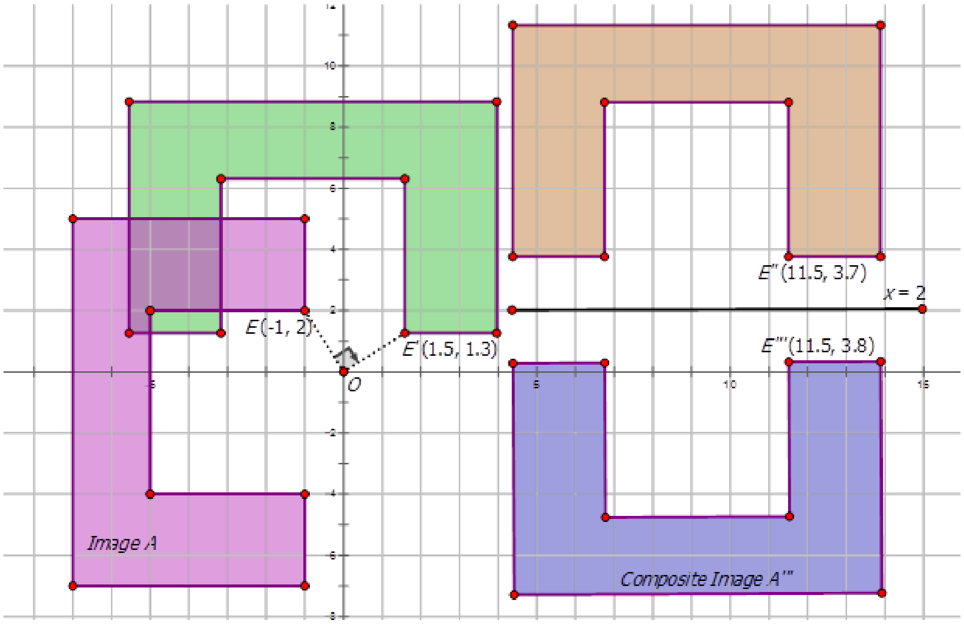



A (-3.5, 4)
A'(3.5, 4)
A''(11.5, 2)
Image A
Composite Image A''
4
2
-5
5
10
15
-2
-4
-6

13.


I''(0, 10)
Composite Image A''
H''(2, 7)
G''(-2, 6)
J''(-5, 4)
K'(1, 4)
J'(4, 5)
H(-7, 2)
G'(6, 2)
I(-10, 0)
K''(-4, 1)
m∠K'OK'' = 90.00°
O
I'(10, 0)
m∠KOK' = 180.00°
G(-6, -2)
H'(7, -2)
IMAGE A
K(-1, -4)
J(-4, -5)

14.

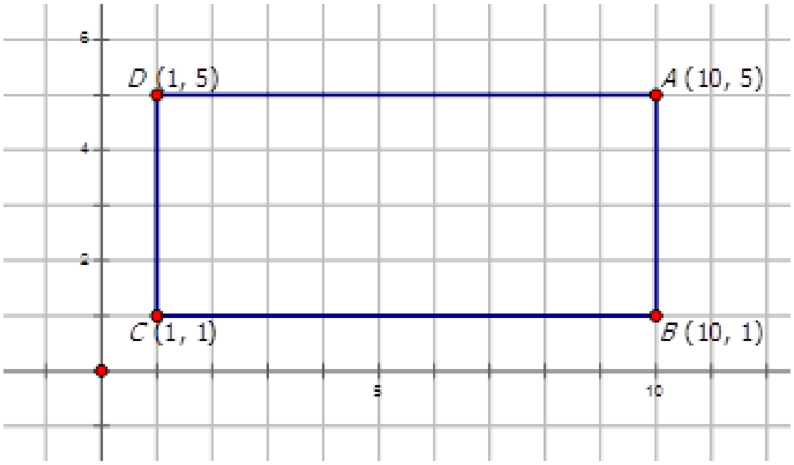

E''(11.5, 3.7)
x = 2
E(-1, 2)
E'(1.5, 1.3)
E'''(11.5, 3.8)
O
Image A
Composite Image A'''
10
8
6
4
2
-2
-4
-6
-8
5
15

15.

10.16 The Midpoint Formula

Here you will learn how to find the midpoint of a line segment.

Find the midpoints for the diagram below and then draw the lines of reflection.



Watch This

First watch this video to learn about the midpoint formula.

Multimedia

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Click image to the left for more content.

CK-12 Foundation Chapter10TheMidpointFormulaA

Then watch this video to see some examples.

Multimedia

MEDIA

Click image to the left for more content.

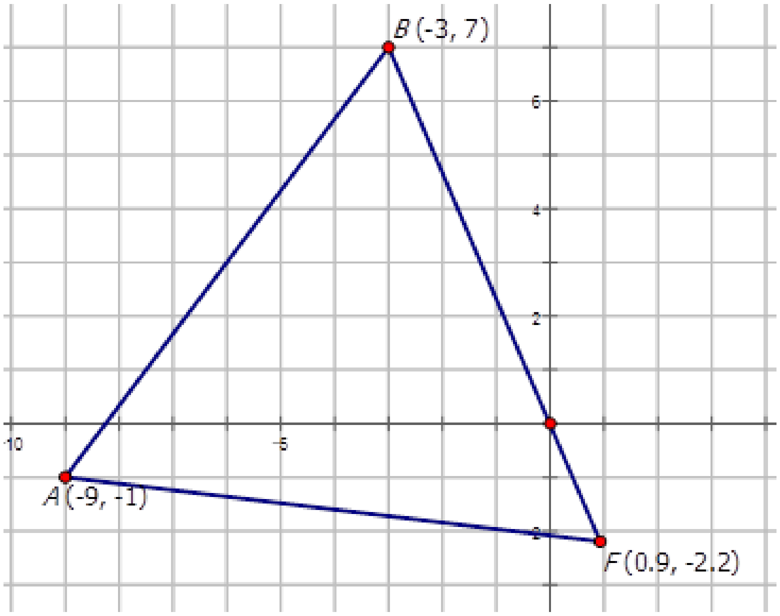
CK-12 Foundation Chapter10TheMidpointFormulaB

Guidance

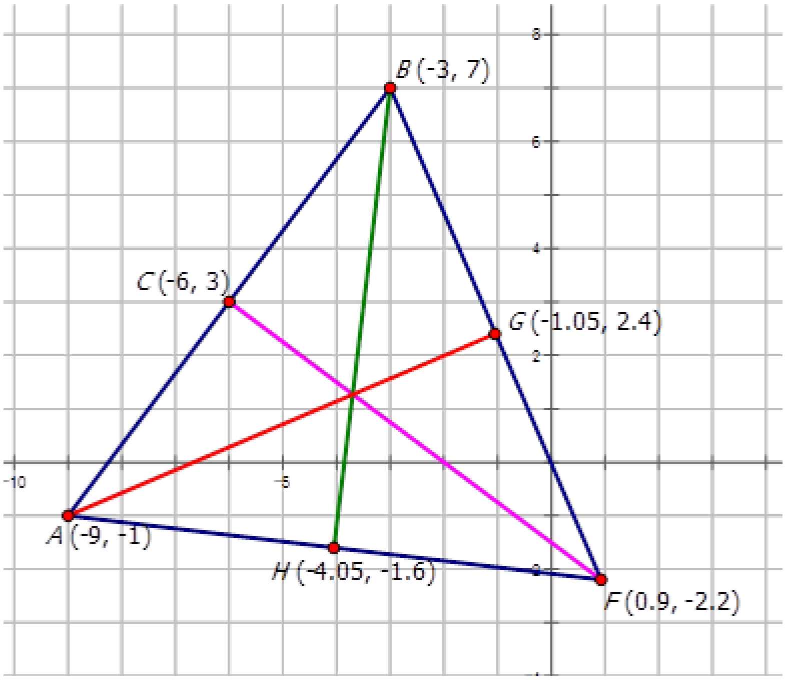
The midpoint of a line segment is the point exactly in the middle of the two endpoints. In order to calculate the coordinates of the midpoint, find the average of the two endpoints:

$$M = \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$$

Sometimes midpoints can help you to find lines of reflection (lines of symmetry) in shapes. Look at the equilateral triangle in the diagram below.



In an equilateral triangle there are three lines of symmetry. The lines of symmetry connect each vertex to the midpoint on the opposite side.

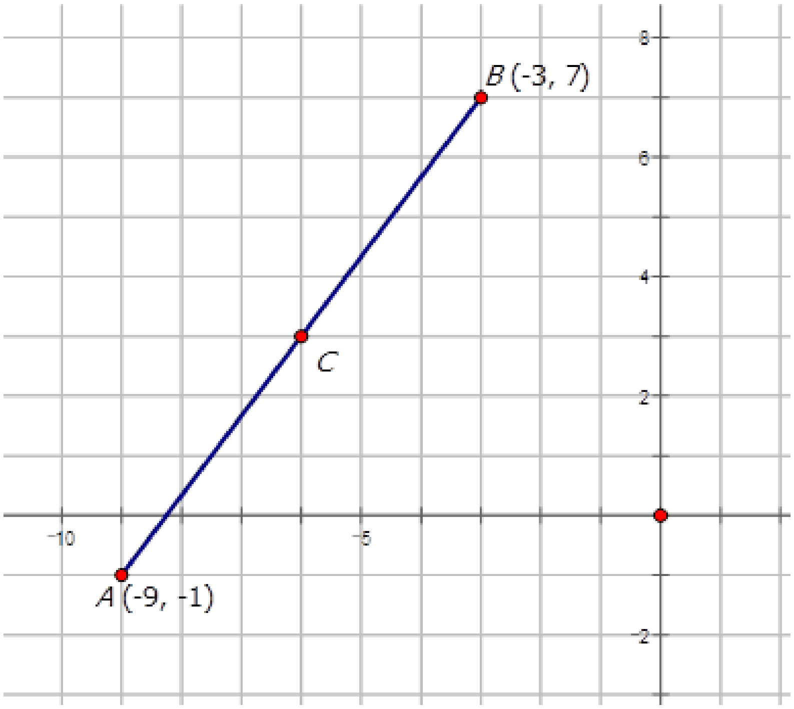


C is the mid-point of AB, G is the midpoint of BF, and H is the midpoint of AF. The lines AG, FC, and BH are all lines of symmetry or lines of reflection.

Keep in mind that not all midpoints will create lines of symmetry!

Example A

In the diagram below, C is the midpoint between $A(-9,-1)$ and $B(-3,7)$. Find the coordinates of C.



Solution:

$$M_{AB} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{AB} = \left(\frac{-9+-3}{2}, \frac{-1+7}{2}\right)$$
$$M_{AB} = \left(\frac{-12}{2}, \frac{6}{2}\right)$$
$$M_{AB} = (-6,3)$$

Example B

Find the coordinates of point T on the line ST knowing that S has coordinates (-3, 8) and the midpoint is (12, 1).

Solution: Look at the midpoint formula:

$$M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

For this problem, if you let point T have the coordinates x_1 and y_1, then you need to find x_1 and y_1 using the midpoint formula.

$$M_{ST} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$(12,1) = \left(\frac{-3+x_1}{2}, \frac{8+y_1}{2}\right)$$

Next you need to separate the x-coordinate formula and the y-coordinate formula to solve for your unknowns.

$$12 = \frac{-3+x_1}{2} \quad 1 = \frac{8+y_1}{2}$$

Now multiply each of the equations by 2 in order to get rid of the fraction.

$$24 = -3+x_1 \quad 2 = 8+y_1$$

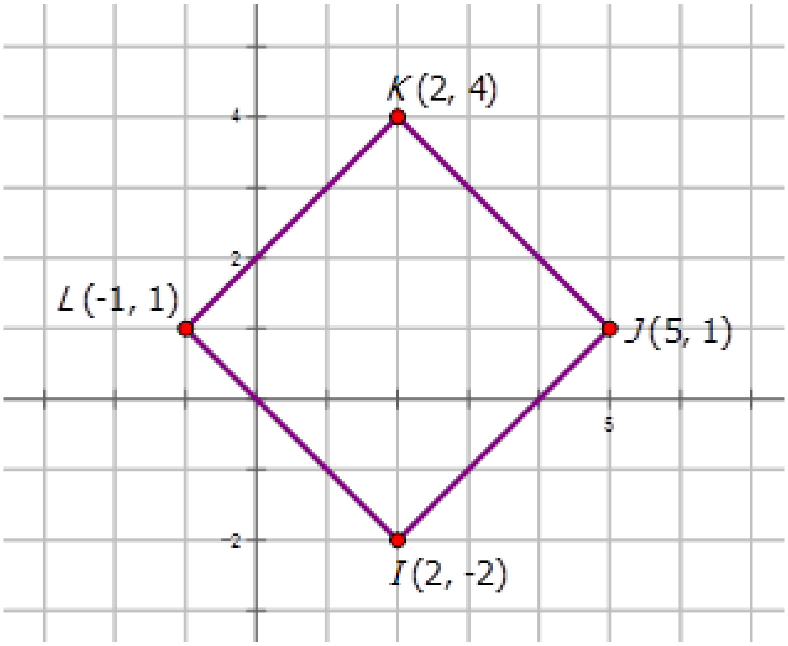
Now you can solve for x_1 and y_1.

$$27 = x_1 \quad -6 = y_1$$

Therefore the point T in the line ST has coordinates (27, -6).

Example C

Find the midpoints for the digram below in order to draw the lines of reflection (or the line of symmetry).



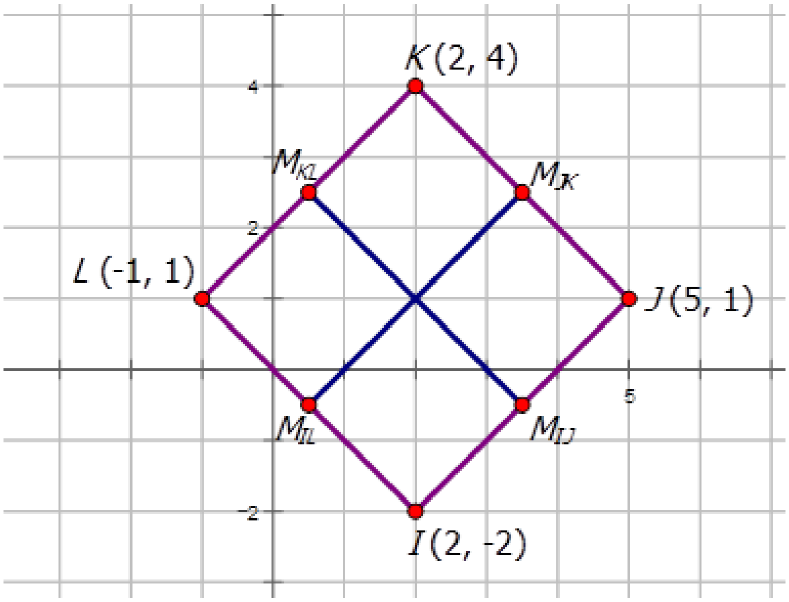
Solution:

$$M_{IL} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{IL} = \left(\frac{2+-1}{2}, \frac{-2+1}{2}\right)$$
$$M_{IL} = \left(\frac{1}{2}, \frac{-1}{2}\right)$$

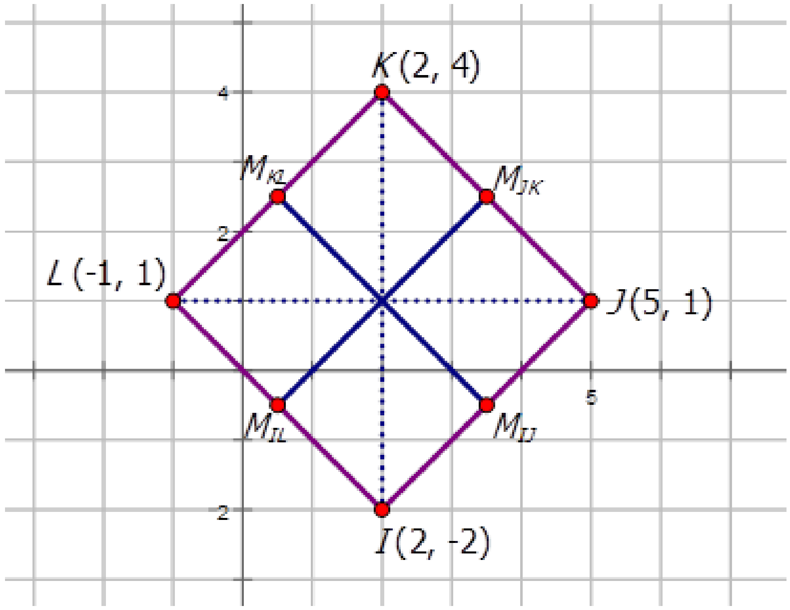
$$M_{IJ} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{IJ} = \left(\frac{2+5}{2}, \frac{-2+1}{2}\right)$$
$$M_{IJ} = \left(\frac{7}{2}, \frac{-1}{2}\right)$$
$$M_{IJ} = (3.5, -0.5)$$

$$M_{JK} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{JK} = \left(\frac{5+2}{2}, \frac{4+1}{2}\right)$$
$$M_{JK} = \left(\frac{7}{2}, \frac{5}{2}\right)$$
$$M_{JK} = (3.5, 2.5)$$

$$M_{KL} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{KL} = \left(\frac{2+-1}{2}, \frac{1+4}{2}\right)$$
$$M_{KL} = \left(\frac{1}{2}, \frac{5}{2}\right)$$
$$M_{KL} = (0.5, 2.5)$$

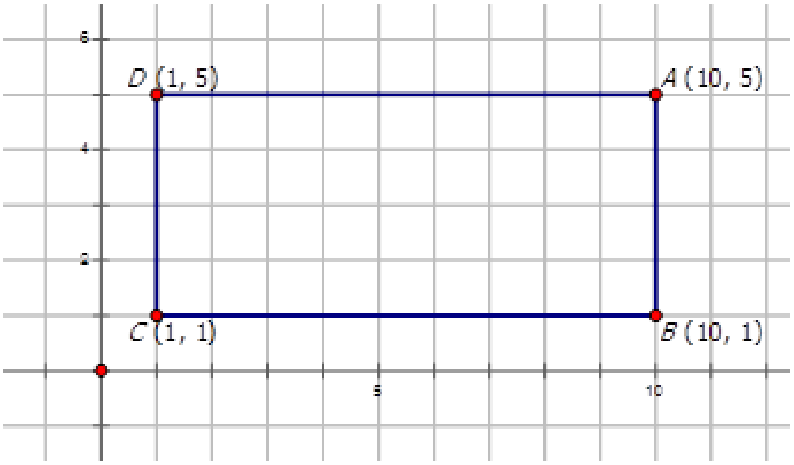


As seen in the graph above, a square has two lines of symmetry drawn from the mid-points of the opposite sides. A square actually has two more lines of symmetry that are the diagonals of the square.



Concept Problem Revisited

Find the midpoints for the diagram below in order to draw the lines of reflection.



$$M_{AB} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

$$M_{AB} = \left(\frac{10+10}{2}, \frac{1+5}{2}\right)$$

$$M_{AB} = \left(\frac{20}{2}, \frac{6}{2}\right)$$

$$M_{AB} = (10, 3)$$

$$M_{AD} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

$$M_{AD} = \left(\frac{10+1}{2}, \frac{5+5}{2}\right)$$

$$M_{AD} = \left(\frac{11}{2}, \frac{10}{2}\right)$$

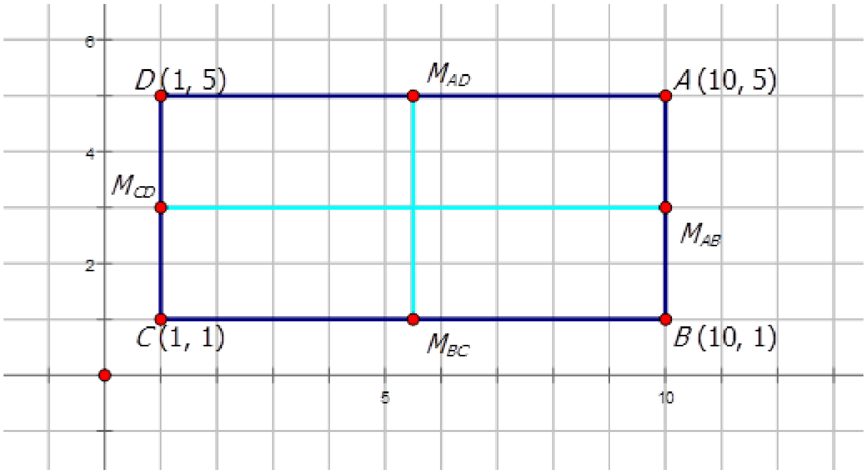
$$M_{AD} = (5.5, 5)$$

$$M_{BC} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right) \qquad M_{CD} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

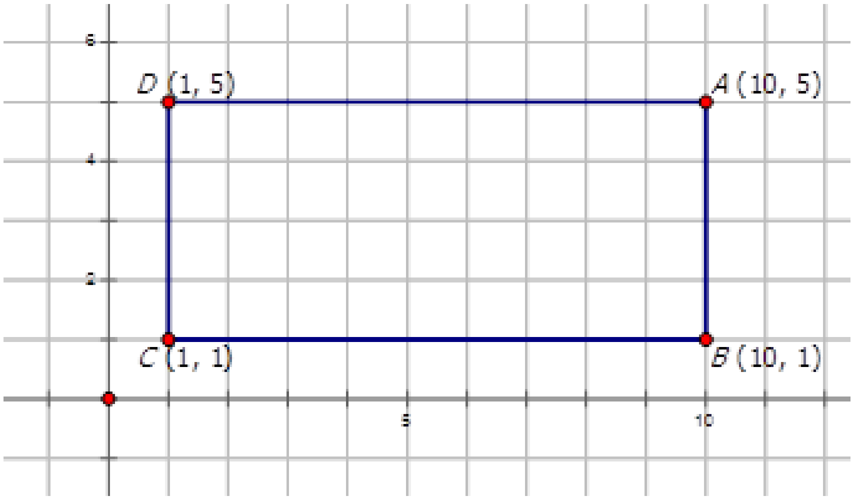
$$M_{BC} = \left(\frac{10+1}{2}, \frac{1+1}{2}\right) \qquad M_{CD} = \left(\frac{1+1}{2}, \frac{1+5}{2}\right)$$

$$M_{BC} = \left(\frac{11}{2}, \frac{2}{2}\right) \qquad M_{CD} = \left(\frac{2}{2}, \frac{6}{2}\right)$$

$$M_{BC} = (5.5, 1) \qquad M_{CD} = (1, 3)$$



As seen in the graph above, a rectangle has two lines of symmetry.



Vocabulary

Line of Symmetry

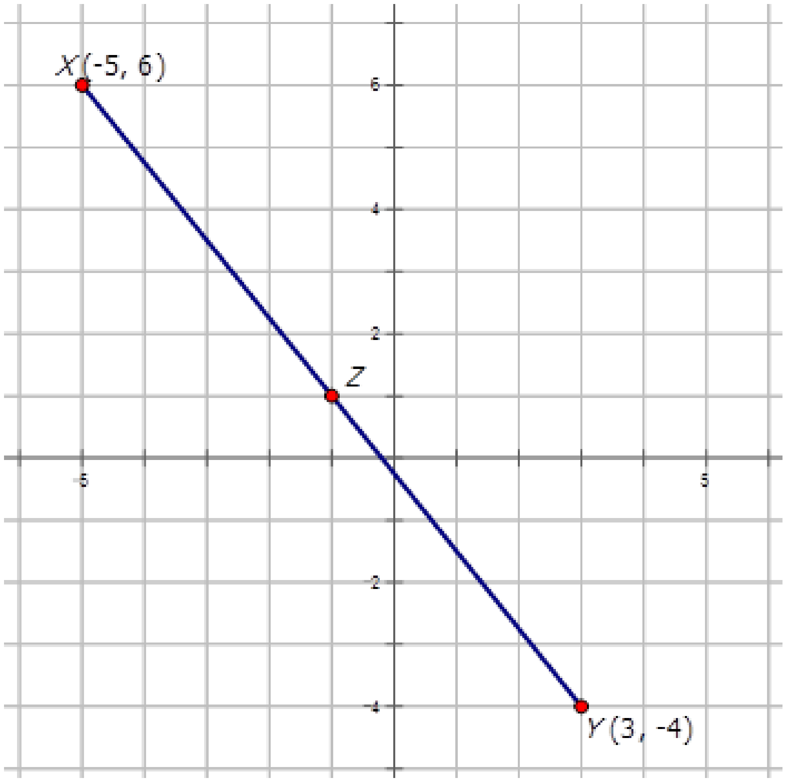
The ***line of symmetry*** (or the line of reflection) is the line drawn so that each of the halves that result from drawing a line of symmetry is congruent (the same size and shape).

Midpoint

The ***midpoint*** of a line segment is the point exactly in the middle of the two endpoints. The midpoint is the average of the two endpoints in a segment: $M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$

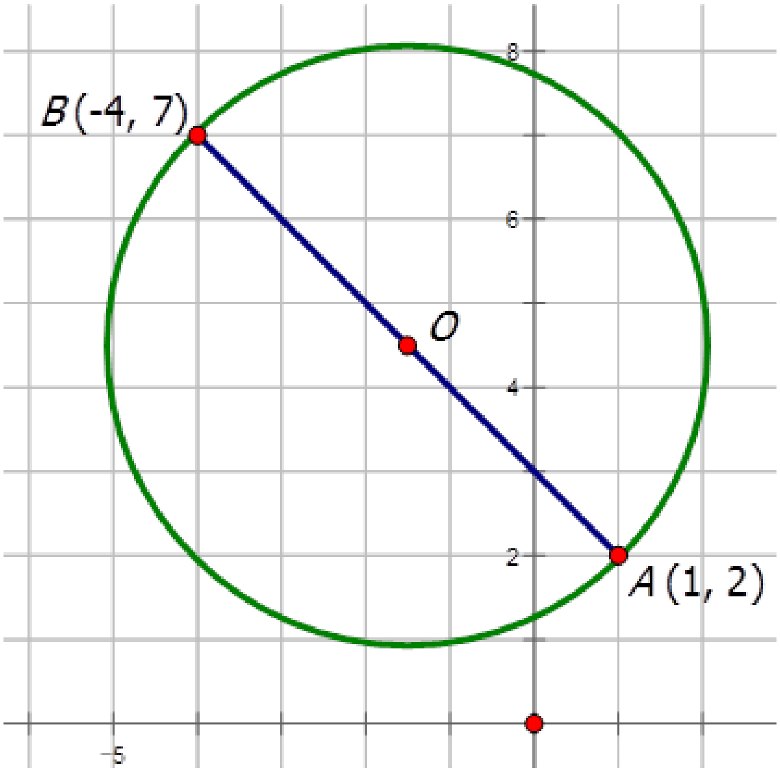
Guided Practice

1. In the diagram below, Z is the midpoint between $X(-5,6)$ and $Y(3,-4)$. Find the coordinates of Z.



2. Find the coordinates of point K on the line JK knowing that J has coordinates (-2, 5) and the midpoint is (10, 1).

3. A diameter is drawn in the circle as shown in the diagram below. What are the coordinates for the center of the circle, O?



Answers:

1.

$$M_{XY} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$M_{XY} = \left(\frac{-5+3}{2}, \frac{-4+6}{2}\right)$$
$$M_{XY} = \left(\frac{-2}{2}, \frac{2}{2}\right)$$
$$M_{XY} = (-1,1)$$

2. Let point K have the coordinates x_1 and y_1, then find x_1 and y_1 using the midpoint formula.

$$M_{JK} = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$
$$(10,1) = \left(\frac{-2+x_1}{2}, \frac{5+y_1}{2}\right)$$

Next you need to separate the x-coordinate formula and the y-coordinate formula to solve for your unknowns.

$$10 = \frac{-2+x_1}{2} \quad 1 = \frac{5+y_1}{2}$$

Now multiply each of the equations by 2 in order to get rid of the fraction.

$$20 = -2 + x_1 \quad 2 = 5 + y_1$$

Now you can solve for x_1 and y_1.

$$22 = x_1 \quad -3 = y_1$$

Therefore the point K in the line JK has coordinates (22, -3).

3.

$$\begin{aligned}
M_{AB} &= \left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)\\
M_{AB} &= \left(\frac{1 + -4}{2}, \frac{2 + 7}{2}\right)\\
M_{AB} &= \left(\frac{-3}{2}, \frac{9}{2}\right)\\
M_{AB} &= (-1.5, 4.5)
\end{aligned}$$

Practice

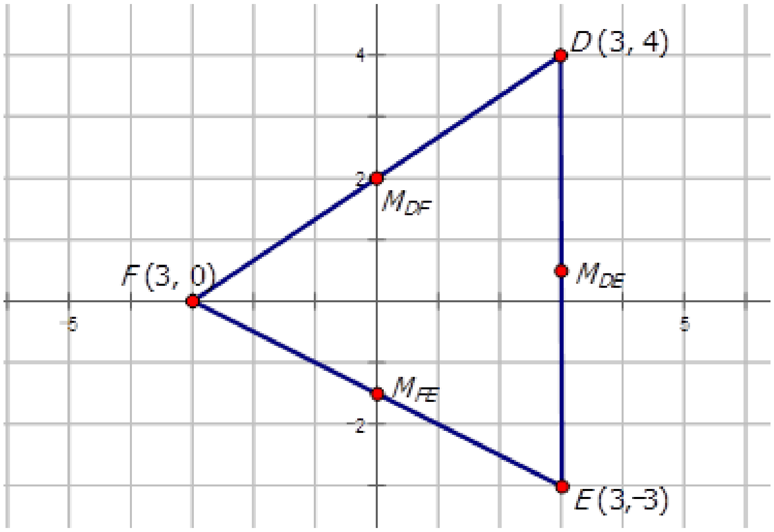
Find the mid-point for each line below given the endpoints:

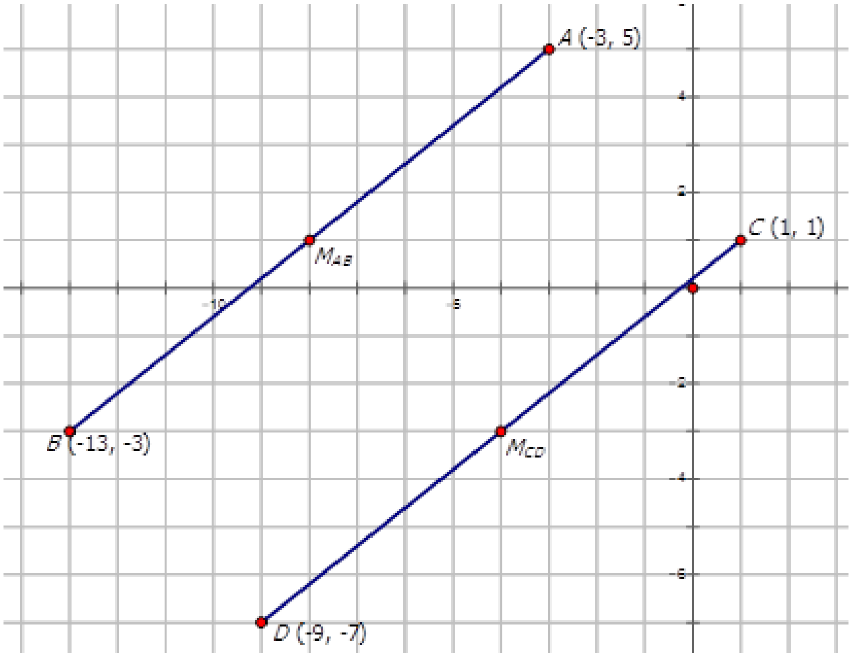
1. Line AB given $A(5,7)$ and $B(3,9)$.
2. Line BC given $B(3,8)$ and $C(5,2)$.
3. Line CD given $C(4,6)$ and $D(3,5)$.
4. Line DE given $D(9,11)$ and $E(2,2)$.
5. Line EF given $E(1,1)$ and $F(8,7)$.
6. Line FG given $F(1,8)$ and $G(1,4)$.

For the following lines, one endpoint is given and then the mid-point. Find the other endpoint.

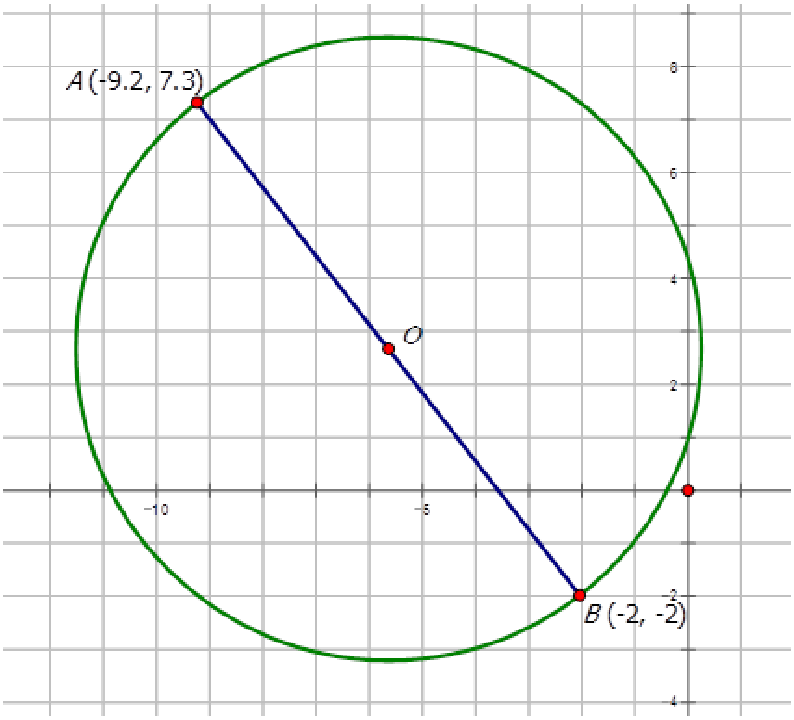
7. Line AB given $A(3,-5)$ and $M_{AB}(7,7)$.
8. Line BC given $B(2,4)$ and $M_{BC}(4,9)$.
9. Line CD given $C(-2,6)$ and $M_{CD}(1,1)$.
10. Line DE given $D(2,9)$ and $M_{DE}(8,2)$.
11. Line EF given $E(-6,-5)$ and $M_{EF}(-2,6)$.

For each of the diagrams below, find the midpoints.

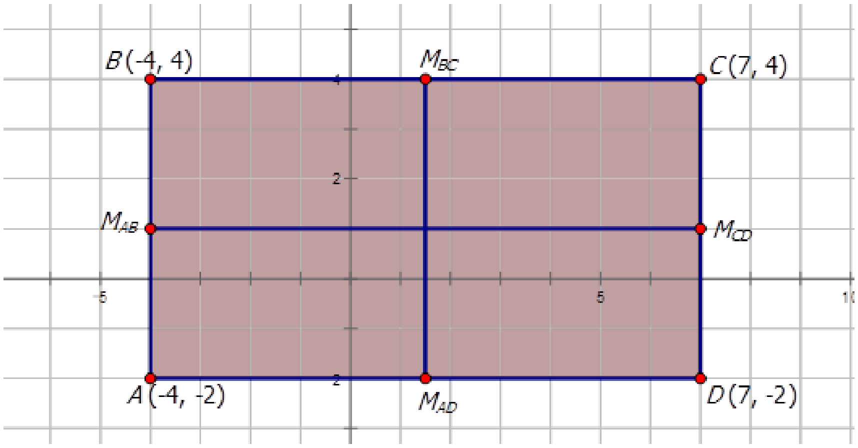
12. 
D (3, 4)
M_DF
F (3, 0)
M_DE
M_FE
E (3,−3)
4
2
-2
-5
5

13. 
A (-3, 5)
M_AB
C (1, 1)
B (-13, -3)
M_CD
D (-9, -7)
4
2
-2
-4
-6
-10
-5

14.



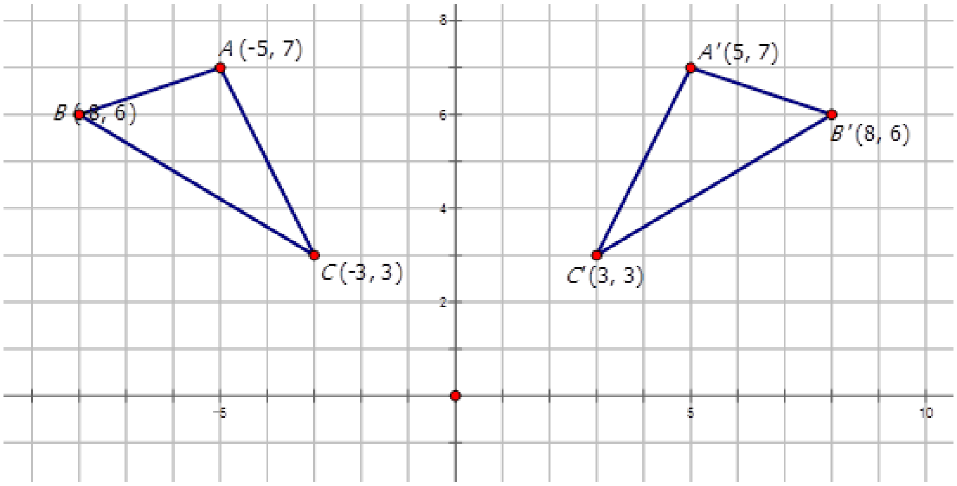
15.



10.17 The Distance Formula

Here you will learn the distance formula and how to use it to determine whether or not two line segments are congruent.

Triangle ABC has vertices $A(-5,7), B(-8,6)$ and $C(-3,3)$. The triangle is reflected about the y-axis to form triangle $A'B'C'$. Assuming that $\angle A = \angle A', \angle B = \angle B', and\ \angle C = \angle C'$, prove the two triangles are congruent.



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First watch this video to learn about the distance formula.

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CK-12 Foundation Chapter10TheDistanceFormulaA

Then watch this video to see some examples.

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Click image to the left for more content.

CK-12 Foundation Chapter10TheDistanceFormulaB

Guidance

Two shapes are **congruent** if they are exactly the same shape and exactly the same size. In congruent shapes, all corresponding sides will be the same length and all corresponding angles will be the same measure. Translations, reflections, and rotations all create congruent shapes.

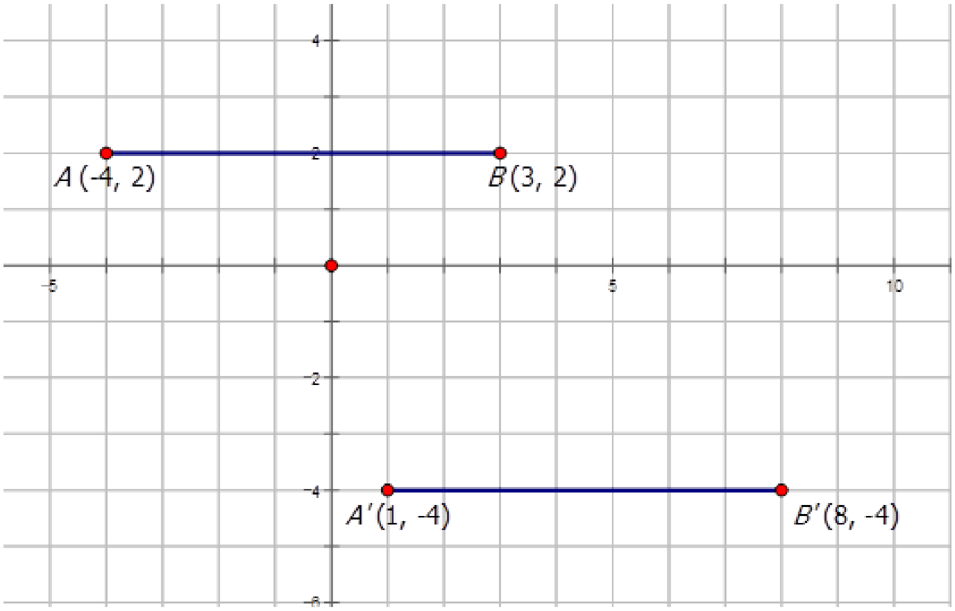
If you want to determine whether two segments are the same length, you could try to use a ruler. Unfortunately, it's hard to be very precise with a ruler. You could also use geometry software, but that is not always available. If the segments are on the coordinate plane and you know their endpoints, you can use the distance formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The distance formula helps justify congruence by proving that the sides of the preimage have the same length as the sides of the transformed image. The distance formula is derived using the Pythagorean Theorem, which you will learn more about in geometry.

Example A

Line segment AB is translated 5 units to the right and 6 units down to produce line $A'B'$. The diagram below shows the endpoints of lines AB and $A'B'$. Prove the two line segments are congruent.



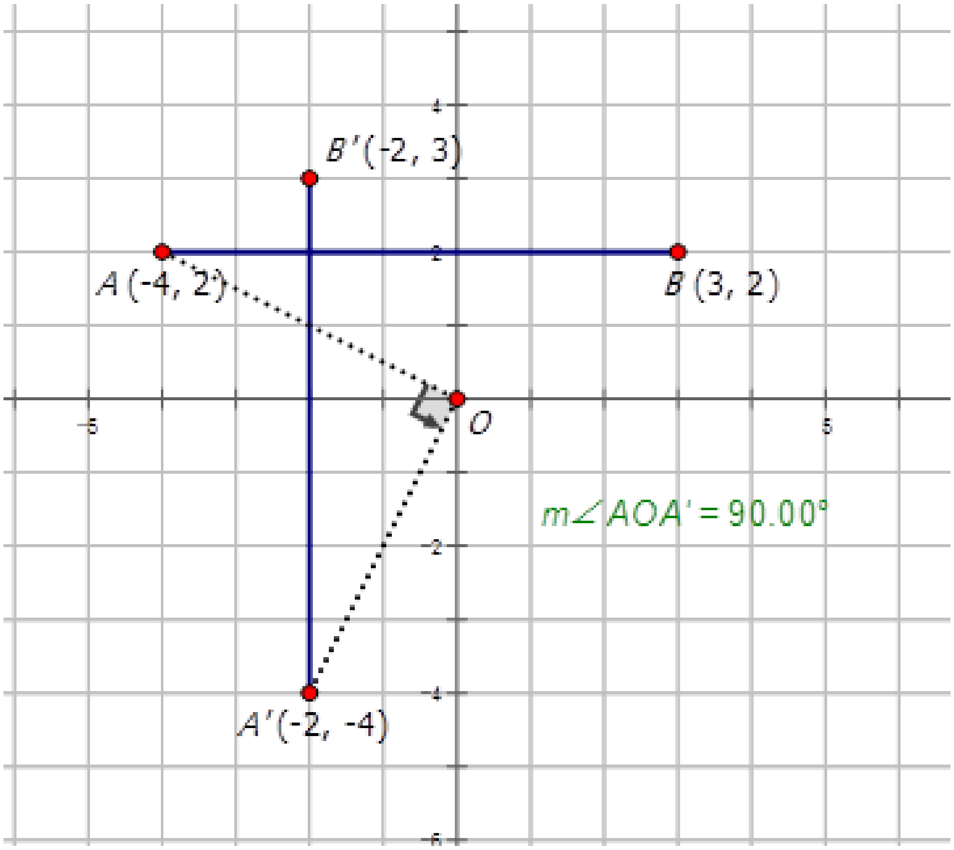
Solution:

$$\begin{aligned} d_{AB} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d_{AB} &= \sqrt{(-4-3)^2 + (2-2)^2} \\ d_{AB} &= \sqrt{(-7)^2 + (0)^2} \\ d_{AB} &= \sqrt{49+0} \\ d_{AB} &= \sqrt{49} \\ d_{AB} &= 7\ cm \end{aligned}$$

$$\begin{aligned} d_{A'B'} &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ d_{A'B'} &= \sqrt{(1-8)^2 + (-4-(-4))^2} \\ d_{A'B'} &= \sqrt{(-7)^2 + (0)^2} \\ d_{A'B'} &= \sqrt{49+0} \\ d_{A'B'} &= \sqrt{49} \\ d_{A'B'} &= 7\ cm \end{aligned}$$

Example B

Line segment AB has been rotated about the origin 90°CCW to produce $A'B'$. The diagram below shows the lines AB and $A'B'$. Prove the two line segments are congruent.

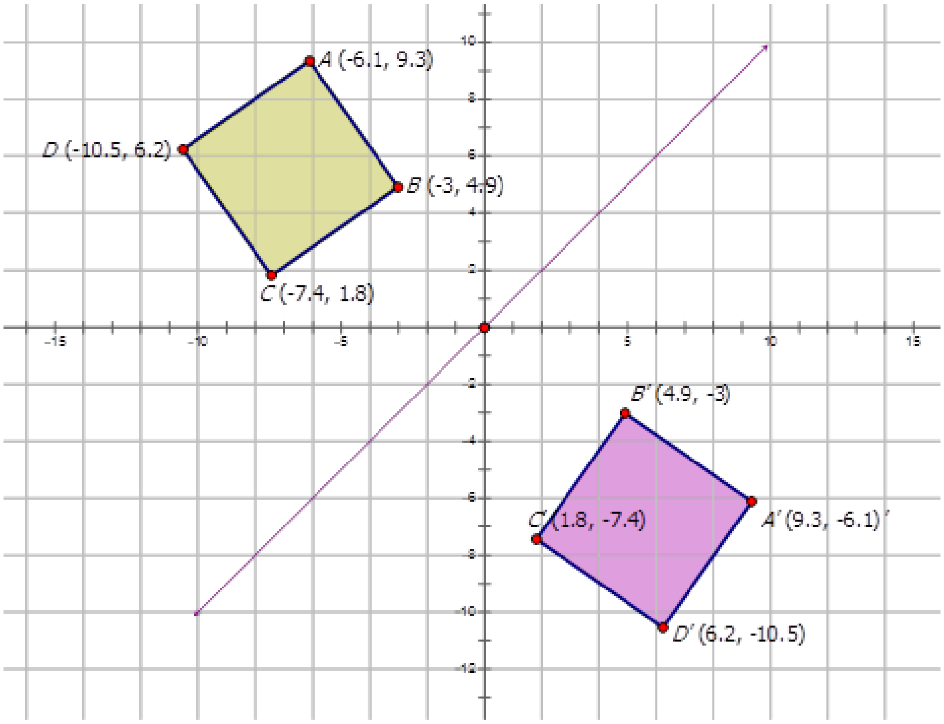


Solution:

$$\begin{aligned} d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} & d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\ d_{AB} &= \sqrt{(-4-3)^2+(2-2)^2} & d_{A'B'} &= \sqrt{(-2-(-2))^2+(-4-3)^2} \\ d_{AB} &= \sqrt{(-7)^2+(0)^2} & d_{A'B'} &= \sqrt{(0)^2+(-7)^2} \\ d_{AB} &= \sqrt{49+0} & d_{A'B'} &= \sqrt{0+49} \\ d_{AB} &= \sqrt{49} & d_{A'B'} &= \sqrt{49} \\ d_{AB} &= 7\ cm & d_{A'B'} &= 7\ cm \end{aligned}$$

Example C

The square $ABCD$ has been reflected about the line $y = x$ to produce $A'B'C'D'$ as shown in the diagram below. Prove the two are congruent.



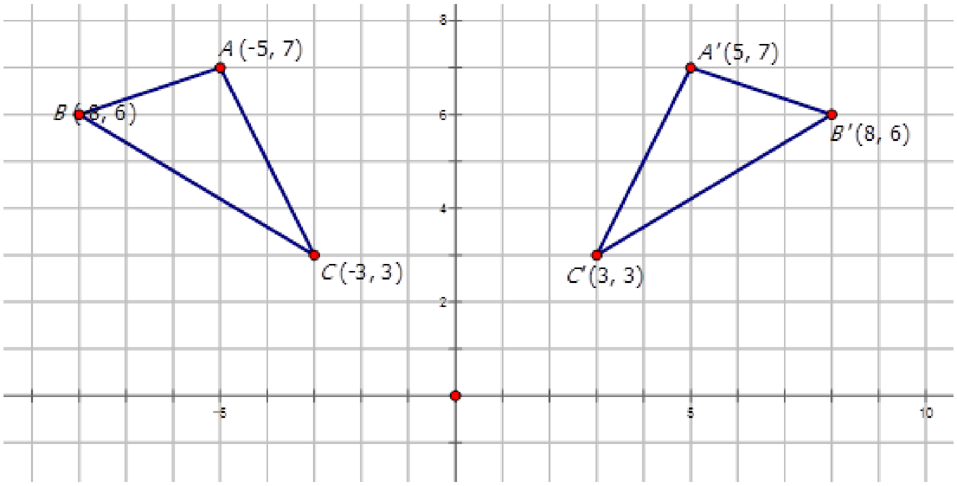
Solution: Since the figures are squares, you can conclude that all angles are the same and equal to 90°. You can also conclude that for each square, all the sides are the same length. Therefore, all you need to verify is that $m\overline{AB} = m\overline{A'B'}$.

$$\begin{aligned}
d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{AB} &= \sqrt{(-6.1-(-3))^2+(9.3-4.9)^2} \\
d_{AB} &= \sqrt{(-3.1)^2+(4.4)^2} \\
d_{AB} &= \sqrt{9.61+19.36} \\
d_{AB} &= \sqrt{28.97} \\
d_{AB} &= 5.38\ cm
\end{aligned}$$

$$\begin{aligned}
d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'B'} &= \sqrt{(9.3-4.9)^2+(-6.1-(-3))^2} \\
d_{A'B'} &= \sqrt{(4.4)^2+(-3.1)^2} \\
d_{A'B'} &= \sqrt{19.36+9.61} \\
d_{A'B'} &= \sqrt{28.97} \\
d_{A'B'} &= 5.38\ cm
\end{aligned}$$

Since $m\overline{AB} = m\overline{A'B'}$ and both shapes are squares, all 8 sides must be the same length. Therefore, the two squares are congruent.

Concept Problem Revisited



To prove congruence, prove that $m\overline{AB} = m\overline{A'B'}, m\overline{AC} = m\overline{A'C'}$, *and* $m\overline{BC} = m\overline{B'C'}$.

$$
\begin{aligned}
d_{AB} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{AB} &= \sqrt{(-5-(-8))^2+(7-6)^2} \\
d_{AB} &= \sqrt{(3)^2+(1)^2} \\
d_{AB} &= \sqrt{9+1} \\
d_{AB} &= \sqrt{10} \\
d_{AB} &= 3.16\ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'B'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'B'} &= \sqrt{(5-8)^2+(7-6)^2} \\
d_{A'B'} &= \sqrt{(-3)^2+(1)^2} \\
d_{A'B'} &= \sqrt{9+1} \\
d_{A'B'} &= \sqrt{10} \\
d_{A'B'} &= 3.16\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{AC} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{AC} &= \sqrt{(-5-(-3))^2+(7-3)^2} \\
d_{AC} &= \sqrt{(-2)^2+(4)^2} \\
d_{AC} &= \sqrt{4+16} \\
d_{AC} &= \sqrt{20} \\
d_{AC} &= 4.47\ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'C'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'C'} &= \sqrt{(5-3)^2+(7-3)^2} \\
d_{A'C'} &= \sqrt{(2)^2+(4)^2} \\
d_{A'C'} &= \sqrt{4+16} \\
d_{A'C'} &= \sqrt{20} \\
d_{A'C'} &= 4.72\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{BC} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{BC} &= \sqrt{(-8-(-3))^2+(6-3)^2} \\
d_{BC} &= \sqrt{(-5)^2+(3)^2} \\
d_{BC} &= \sqrt{25+9} \\
d_{BC} &= \sqrt{34} \\
d_{BC} &= 5.83\ cm
\end{aligned}
\qquad
\begin{aligned}
d_{A'C'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} \\
d_{A'C'} &= \sqrt{(8-3)^2+(6-3)^2} \\
d_{A'C'} &= \sqrt{(5)^2+(3)^2} \\
d_{A'C'} &= \sqrt{25+9} \\
d_{A'C'} &= \sqrt{34} \\
d_{A'C'} &= 5.83\ cm
\end{aligned}
$$

It is given that $\angle A = \angle A'$, $\angle B = \angle B'$, *and* $\angle C = \angle C'$, and the distance formula proved that $m\overline{AB} = m\overline{A'B'}$, $m\overline{AC} = m\overline{A'C'}$, *and* $m\overline{BC} = m\overline{B'C'}$. Therefore the two triangles are congruent.

Vocabulary

Congruent

Two shapes or segments are ***congruent*** if they are exactly the same shape and size.

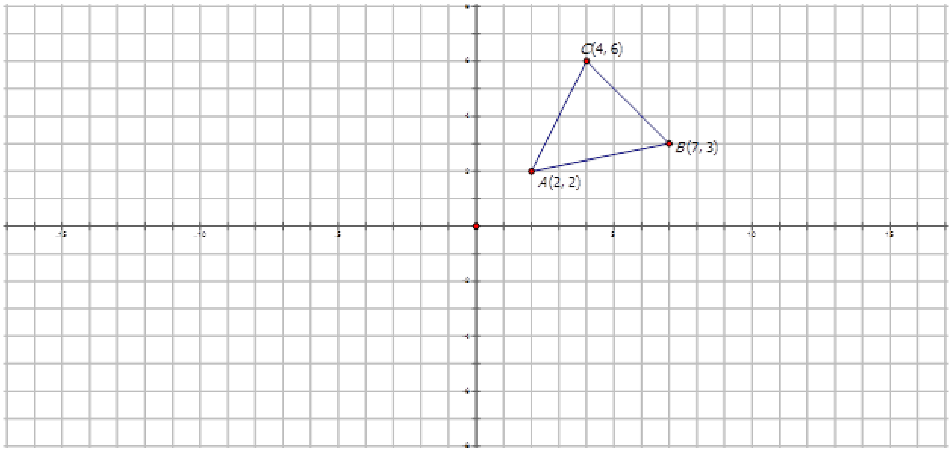
Distance Formula

The ***distance formula***, which finds the distance between two points, is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

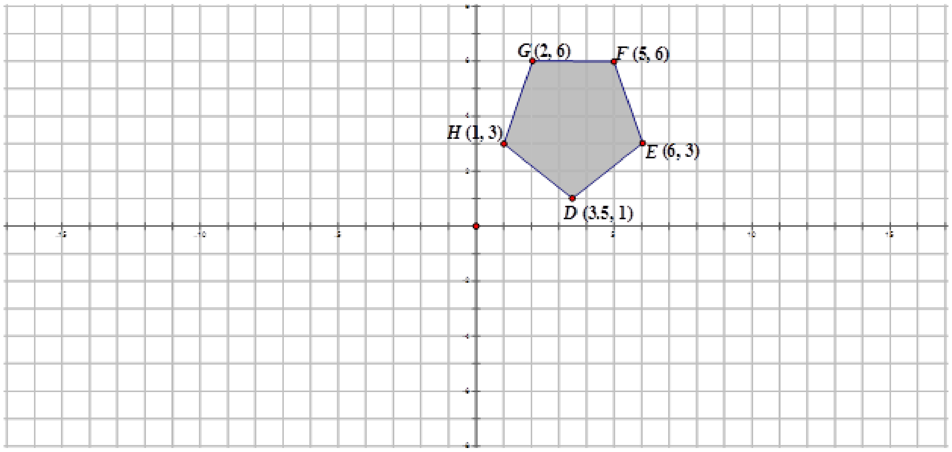
Guided Practice

1. Line segment $\overline{ST}$ drawn from $S(-3,4)$ to $T(-3,8)$ has undergone a reflection in the y-axis to produce Line $S'T'$ drawn from $S'(3,4)$ to $T'(4,8)$. Draw the preimage and image and prove the two lines are congruent.

2. The triangle below has undergone a rotation of 90°CW about the origin. Given that all of the angles are equal, draw the transformed image and prove the two figures are congruent.

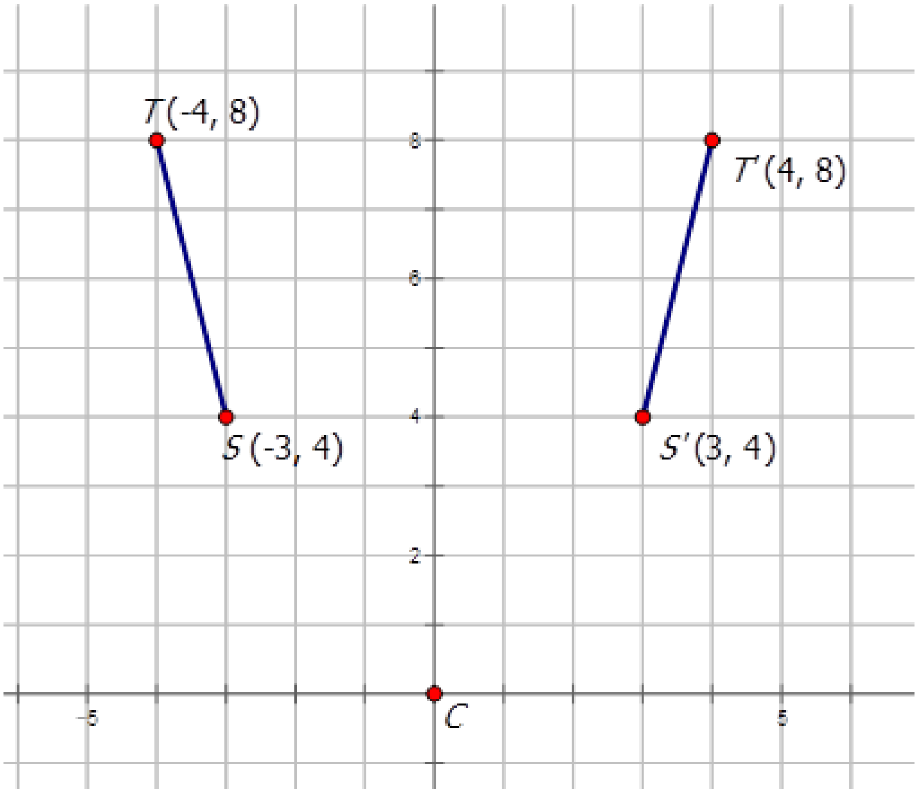


3. The polygon below has undergone a translation of 7 units to the left and 1 unit up. Given that all of the angles are equal, draw the transformed image and prove the two figures are congruent.



Answers:

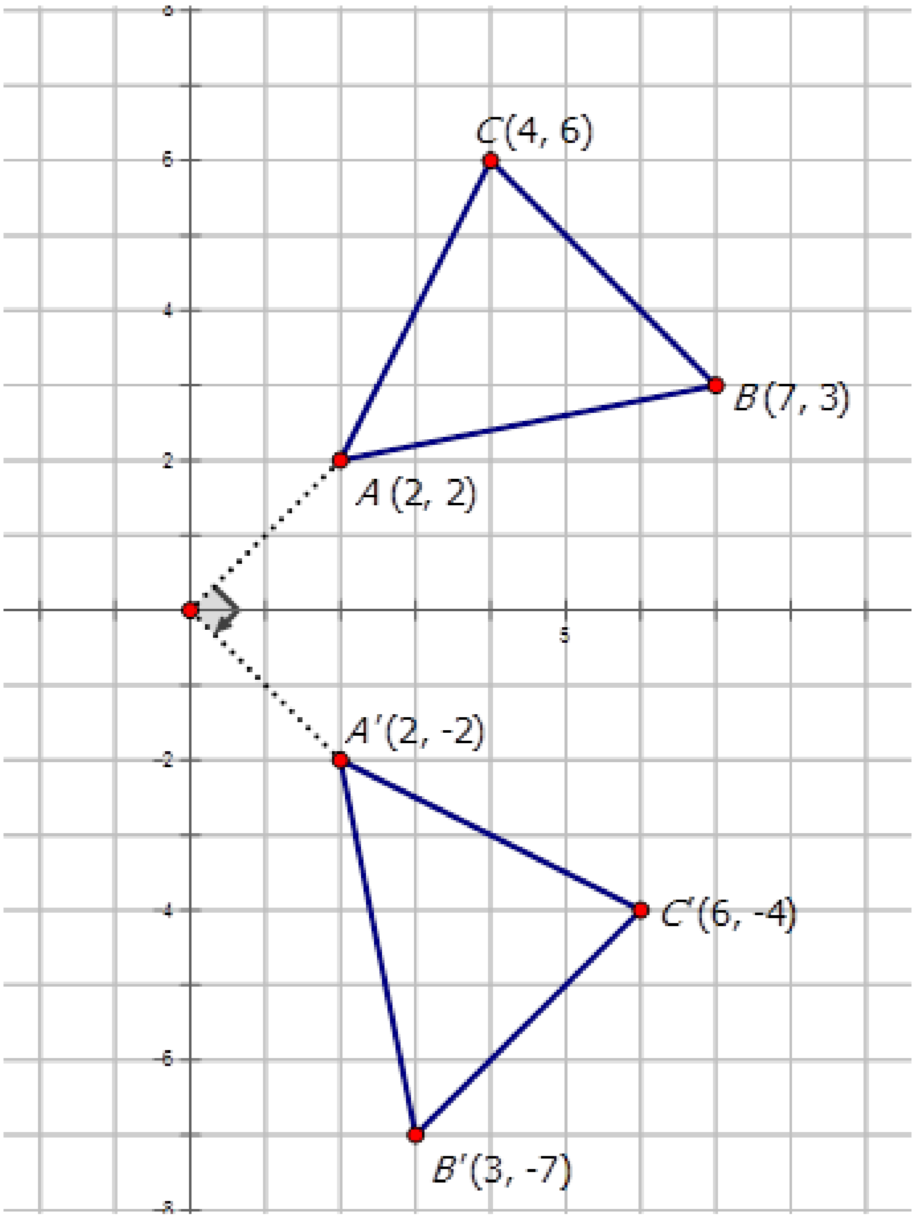
1.



$$
\begin{aligned}
d_{ST} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{ST} &= \sqrt{(-3-(-4))^2+(4-8)^2}\\
d_{ST} &= \sqrt{(1)^2+(-4)^2}\\
d_{ST} &= \sqrt{1+16}\\
d_{ST} &= \sqrt{17}\\
d_{ST} &= 4.12\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{S'T'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{S'T'} &= \sqrt{(3-4)^2+(4-8)^2}\\
d_{S'T'} &= \sqrt{(-1)^2+(-4)^2}\\
d_{S'T'} &= \sqrt{1+16}\\
d_{S'T'} &= \sqrt{17}\\
d_{S'T'} &= 4.12\ cm
\end{aligned}
$$

2.



$$d_{AB} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{AB} = \sqrt{(2-7)^2+(2-3)^2}$$
$$d_{AB} = \sqrt{(-5)^2+(-1)^2}$$
$$d_{AB} = \sqrt{25+1}$$
$$d_{AB} = \sqrt{26}$$
$$d_{AB} = 5.10\ cm$$

$$d_{A'B'} = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$d_{A'B'} = \sqrt{(3-2)^2+(-7-(-2))^2}$$
$$d_{A'B'} = \sqrt{(1)^2+(-5)^2}$$
$$d_{A'B'} = \sqrt{1+25}$$
$$d_{A'B'} = \sqrt{26}$$
$$d_{A'B'} = 5.10\ cm$$

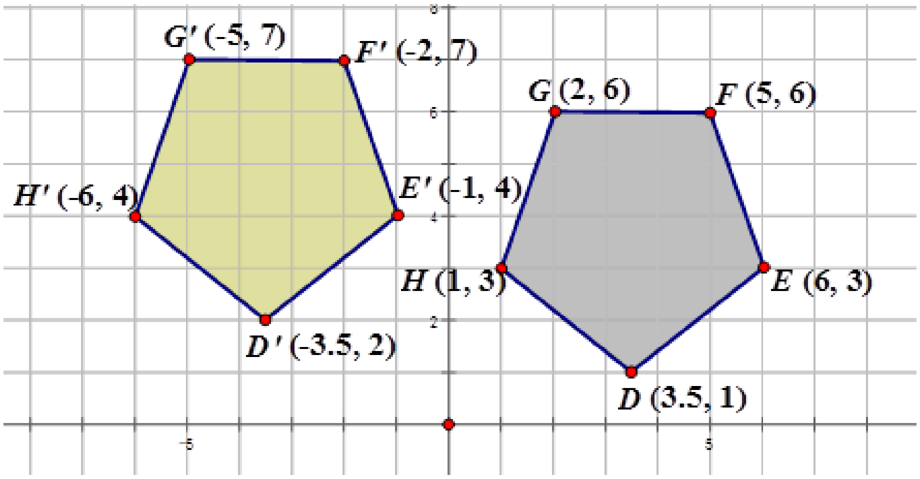
$$\begin{aligned}
d_{AC} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{AC} &= \sqrt{(2-4)^2+(2-6)^2}\\
d_{AC} &= \sqrt{(-2)^2+(-4)^2}\\
d_{AC} &= \sqrt{4+16}\\
d_{AC} &= \sqrt{20}\\
d_{AC} &= 4.47\ cm
\end{aligned}$$

$$\begin{aligned}
d_{A'C'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{A'C'} &= \sqrt{(2-6)^2+(-2-(-4))^2}\\
d_{A'C'} &= \sqrt{(-4)^2+(2)^2}\\
d_{A'C'} &= \sqrt{16+4}\\
d_{A'C'} &= \sqrt{20}\\
d_{A'C'} &= 4.72\ cm
\end{aligned}$$

$$\begin{aligned}
d_{BC} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{BC} &= \sqrt{(7-4)^2+(3-6)^2}\\
d_{BC} &= \sqrt{(3)^2+(-3)^2}\\
d_{BC} &= \sqrt{9+9}\\
d_{BC} &= \sqrt{18}\\
d_{BC} &= 4.24\ cm
\end{aligned}$$

$$\begin{aligned}
d_{B'C'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{B'C'} &= \sqrt{(3-6)^2+(-7-(-4))^2}\\
d_{B'C'} &= \sqrt{(-3)^2+(-3)^2}\\
d_{B'C'} &= \sqrt{9+9}\\
d_{B'C'} &= \sqrt{18}\\
d_{B'C'} &= 4.24\ cm
\end{aligned}$$

3.



$$\begin{aligned}
d_{DE} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{DE} &= \sqrt{(3.5-6)^2+(1-3)^2}\\
d_{DE} &= \sqrt{(-2.5)^2+(-2)^2}\\
d_{DE} &= \sqrt{6.25+4}\\
d_{DE} &= \sqrt{10.25}\\
d_{DE} &= 3.20\ cm
\end{aligned}$$

$$\begin{aligned}
d_{D'E'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{D'E'} &= \sqrt{(-3.5-(-1))^2+(2-4)^2}\\
d_{D'E'} &= \sqrt{(-2.5)^2+(-2)^2}\\
d_{D'E'} &= \sqrt{6.25+4}\\
d_{D'E'} &= \sqrt{10.25}\\
d_{D'E'} &= 3.20\ cm
\end{aligned}$$

$$
\begin{aligned}
d_{EF} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{EF} &= \sqrt{(6-5)^2+(3-6)^2}\\
d_{EF} &= \sqrt{(1)^2+(-3)^2}\\
d_{EF} &= \sqrt{1+9}\\
d_{EF} &= \sqrt{10}\\
d_{EF} &= 3.16\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{E'F'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{E'F'} &= \sqrt{(-1-(-2))^2+(4-7)^2}\\
d_{E'F'} &= \sqrt{(1)^2+(-3)^2}\\
d_{E'F'} &= \sqrt{1+9}\\
d_{E'F'} &= \sqrt{10}\\
d_{E'F'} &= 3.16\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{FG} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{FG} &= \sqrt{(5-2)^2+(6-6)^2}\\
d_{FG} &= \sqrt{(3)^2+(0)^2}\\
d_{FG} &= \sqrt{9+0}\\
d_{FG} &= \sqrt{9}\\
d_{FG} &= 3.00\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{F'G'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{F'G'} &= \sqrt{(-2-(-5))^2+(7-7)^2}\\
d_{F'G'} &= \sqrt{(3)^2+(0)^2}\\
d_{F'G'} &= \sqrt{9+0}\\
d_{F'G'} &= \sqrt{9}\\
d_{F'G'} &= 3.00\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{GH} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{GH} &= \sqrt{(2-1)^2+(6-3)^2}\\
d_{GH} &= \sqrt{(1)^2+(3)^2}\\
d_{GH} &= \sqrt{1+9}\\
d_{GH} &= \sqrt{10}\\
d_{GH} &= 3.16\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{G'H'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{G'H'} &= \sqrt{(-5-(-6))^2+(7-4)^2}\\
d_{G'H'} &= \sqrt{(1)^2+(3)^2}\\
d_{G'H'} &= \sqrt{1+9}\\
d_{G'H'} &= \sqrt{10}\\
d_{G'H'} &= 3.16\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{HD} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{HD} &= \sqrt{(1-3.5)^2+(3-1)^2}\\
d_{HD} &= \sqrt{(-2.5)^2+(2)^2}\\
d_{HD} &= \sqrt{6.25+4}\\
d_{HD} &= \sqrt{10.25}\\
d_{HD} &= 3.20\ cm
\end{aligned}
$$

$$
\begin{aligned}
d_{H'D'} &= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\\
d_{H'D'} &= \sqrt{(-6-(-3.5))^2+(4-2)^2}\\
d_{H'D'} &= \sqrt{(-2.5)^2+(2)^2}\\
d_{H'D'} &= \sqrt{6.25+4}\\
d_{H'D'} &= \sqrt{10.25}\\
d_{H'D'} &= 3.20\ cm
\end{aligned}
$$

Practice

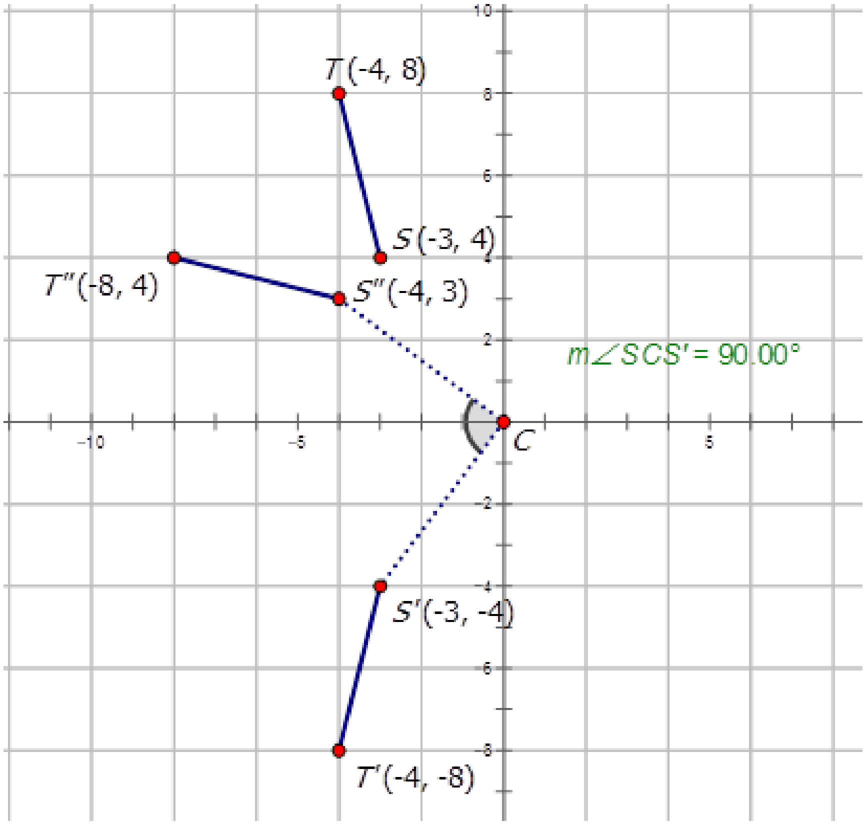
Find the length of each line segment below given its endpoints. Leave all answers in simplest radical form.

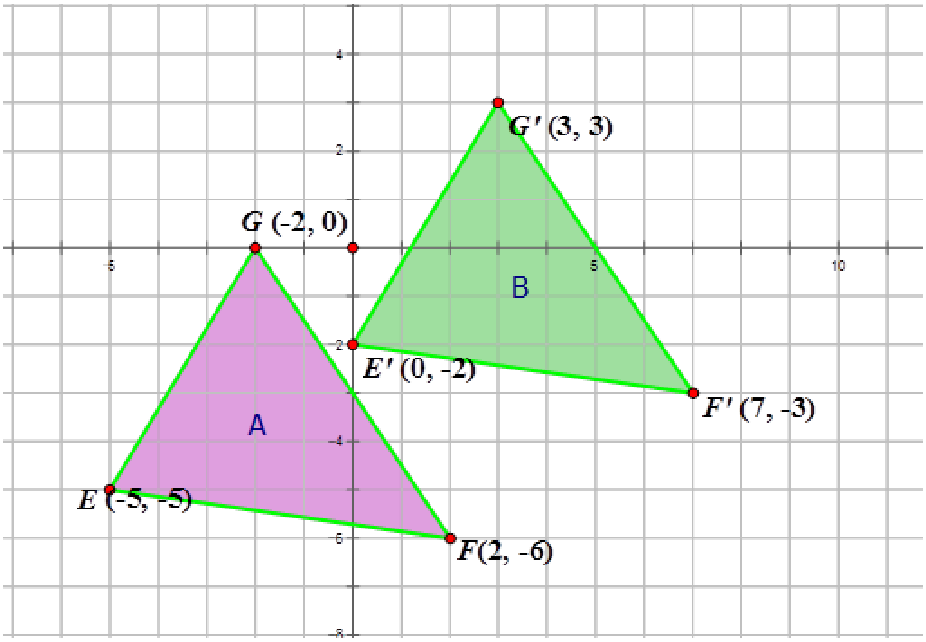
1. Line segment AB given $A(5,7)$ and $B(3,9)$.

2. Line segment BC given $B(3,8)$ and $C(5,2)$.
3. Line segment CD given $C(4,6)$ and $D(3,5)$.
4. Line segment DE given $D(9,11)$ and $E(2,2)$.
5. Line segment EF given $E(1,1)$ and $F(8,7)$.
6. Line segment FG given $F(3,6)$ and $G(2,4)$.
7. Line segment GH given $G(-2,4)$ and $H(6,-1)$.
8. Line segment HI given $H(1,-5)$ and $I(3,3)$.
9. Line segment IJ given $I(3.4,7)$ and $J(1,6)$.
10. Line segment JK given $J(6,-3)$ and $K(-2,4)$.
11. Line segment KL given $K(-3,-3)$ and $L(2,-1)$.

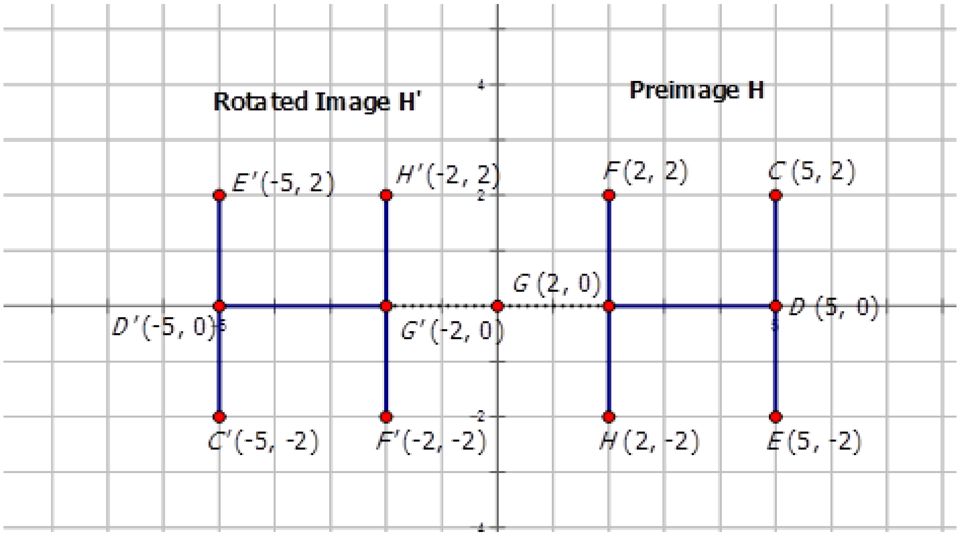
For each of the diagrams below, assume the corresponding angles are congruent. Find the lengths of the line segments to prove congruence.

12.



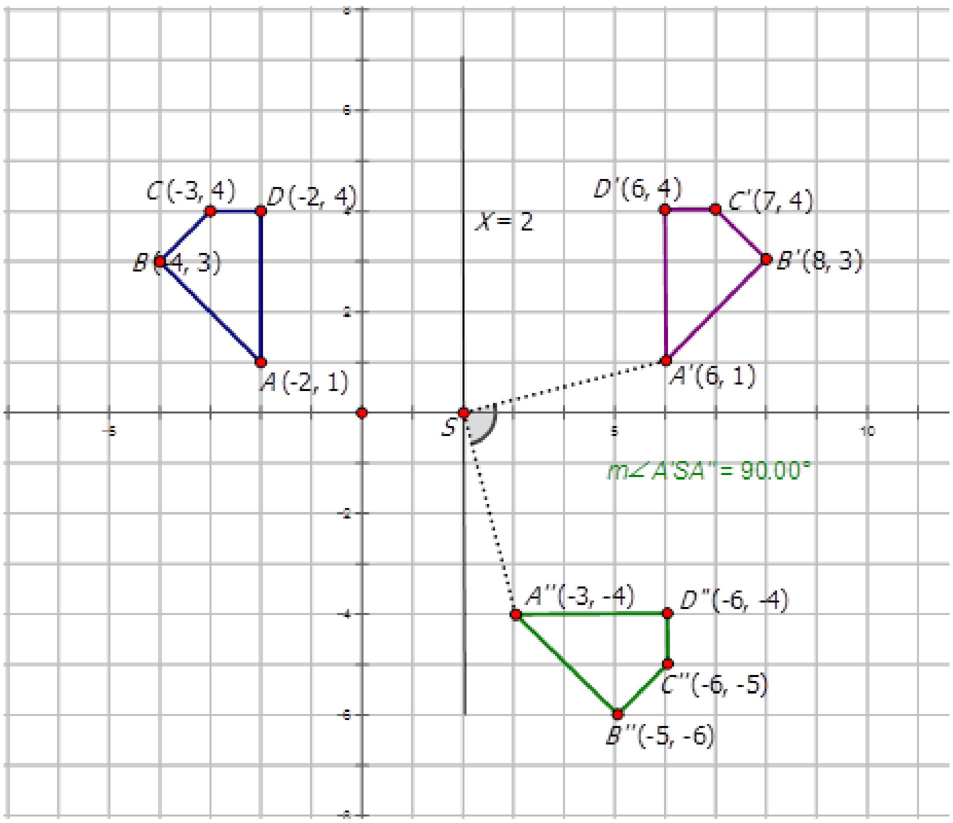

G′ (3, 3)
G (-2, 0)
B
E′ (0, -2)
F′ (7, -3)
A
E (-5, -5)
F(2, -6)
-5
5
10
4
2
-2
-4
-6
-8

13.


Rotated Image H′
Preimage H
E′(-5, 2)
H′(-2, 2)
F(2, 2)
C(5, 2)
G(2, 0)
D′(-5, 0)
G′(-2, 0)
D(5, 0)
C′(-5, -2)
F′(-2, -2)
H(2, -2)
E(5, -2)
4
2
-2
-4

14.

15.



Summary

You learned that there are four geometric transformations. Translations, reflections, and rotations all produce congruent shapes. Congruent shapes are exactly the same shape and size. Translations are slides, reflections are flips, and rotations are turns.

The fourth geometric transformation is the dilation. A dilation produces a shape that is an enlargement or reduction of the preimage.

You also learned that two or more transformations can be performed in sequence. The result is called a composite transformation.

Finally, you learned the midpoint formula and the distance formula. The midpoint of a line segment is the point exactly in the middle of the two endpoints. Sometimes midpoints can help you to find lines of reflection (also known as lines of symmetry). The distance formula helps you to calculate the length of line segments. The distance formula is useful for determining whether or not the corresponding sides of shapes are the same length. This can help you to determine whether one shape has been transformed to create another shape.